THE REDUCTION MAP FOR THE ÉTALE K-THEORY OF A CURVE

G.Banaszak, W.Gajda, B.Kahn and P.Krasoń

Abstract. In the present work, we investigate the reduction map on the étale K-theory of a curve defined over a global field. We prove that on the even-dimensional K-groups the map has finite kernel and reduce the odd-dimensional case to a conjecture of Jannsen.

1. Introduction

The algebraic K-groups of arithmetic schemes are expected to carry deep arithmetic and geometric information, according to Beilinson's striking conjectures which relate them to special values of L-functions. Yet still little is known on their structure. In the case of the ring of integers $\mathcal{O}_K$ of a number field K, Quillen proved that the groups $K_i(\mathcal{O}_K)$ are finitely generated and Borel computed their ranks. In [AB] D.Arlettaz and the first author investigated the kernel of the natural map

$$K_i(\mathcal{O}_K) \to \prod_{\mathfrak{p} \subseteq \mathcal{O}_K} K_i(\kappa_{\mathfrak{p}}) \tag{1.1}$$

where $\mathfrak{p}$ runs through the maximal ideals of $\mathcal{O}_K$, and proved that this kernel is finite. This is trivial for i even, since then $K_i(\mathcal{O}_K)$ itself is finite; the difficulty is for i odd. We note that the same proof yields the same result for rings of integers of global fields of positive characteristic.

In this paper, we consider the next-dimensional case, that of a curve X over a global field K. Fix an odd prime number l different from the characteristic of K. Let S be a finite set of places of K containing the places above l and the places where X has bad reduction, and let $\mathcal{O}_S$ denote the ring of S-integers in K. Let $\mathcal{X}$ be a smooth, proper model of X over $Spec\, \mathcal{O}_S$. Let $K_i^{ét}(\mathcal{X})$ denote the i-th l-adic étale K-group of $\mathcal{X}$, in the sense of Dwyer-Friedlander [DF]. Our main result is then:

Theorem. *For a finite place $v \notin S$, let $\mathcal{X}_v$ be the fibre of $\mathcal{X}$ above v. Then the kernel of the reduction map*

$$K_i^{ét}(\mathcal{X}) \to \prod_{v \notin S} K_i^{ét}(\mathcal{X}_v)$$

is finite for i even.

Notice the shift from odd to even from the case of rings of integers to that of curves. We expect this theorem to hold when replacing étale K-groups by ordinary

K-groups, but have too little information on the former to be able to do that. In [AB], this was made possible by the Soulé-Dwyer-Friedlander theorem that the maps $K_i(\mathcal{O}_K) \to K_i^{\acute{e}t}(\mathcal{O}_K)$ have finite kernel; however, nothing is known on this kernel in the case of $\mathcal{X}$.

The proof of the theorem is not very difficult. We prove a somewhat finer result, namely that an element of infinite order in $K_i^{\acute{e}t}(\mathcal{X})$ goes by the reduction map to a collection of elements of arbitrary large orders in the factors $K_i^{\acute{e}t}(\mathcal{X}_v)$ (this implies the theorem because $K_i^{\acute{e}t}(\mathcal{X})$ is a finitely generated $\mathbb{Z}_l$-module). To do this, we reduce through the Dwyer-Friedlander spectral sequence (relating the étale K-groups of $\mathcal{X}$ to its continuous étale cohomology) and the Leray spectral sequence for the morphism $\mathcal{X} \to Spec\, \mathcal{O}_S$ (relating the latter to S-ramified Galois cohomology of K) to a similar statement which replaces the étale K-groups by $H^1(G_S, T_l(J)(n))$ and $H^1(\kappa_v, T_l(J)(n))$. Here G_S denotes Galois group of the maximal S-ramified extenison of K, J is the Jacobian variety of $\overline{X} = X \times_K \overline{K}$, $T_l(J)$ is its Tate module and n is a Tate twist. The result then follows from a result of Serre [Se] (compare [J1]) followed by a Chebotarev density argument, as in [AB].

This paper is organized as follows. In section 2 we collect étale K-theory computations using the two spectral sequences mentioned above. In section 3, we prove the reduction result on $H^1(G(\bar{K}/K), T_l(A)(k))$ for any abelian variety A defined over K. In section 4 we prove the theorem. Finally, in section 5, we discuss the case of odd-dimensional étale K-groups.

We would like to thank Soulé for kindly pointing out Serre's paper [Se], which helped us to simplify and generalize the earlier proofs we had. The first and the fourth authors would like to thank the Swiss National Science Foundation and the University of Lausanne for an invitation in January 1995.

Convention. In all this paper, l is an *odd* prime.

2. COHOMOLOGICAL AND THE ÉTALE K-THEORETIC COMPUTATION

In this section we collect some facts about the continuous étale cohomology groups of $\mathcal{X}$ with values in the l-adic sheaf $\mathbb{Z}_l(k)$ as well as its l-adic étale K-groups. We briefly recall the definition of the former groups, [J2, p. 216]. Let $(\mathcal{F}_m)$ be a projective system of $\mathbb{Z}/l^m$-sheaves on $\mathcal{X}$. The functor which sends $(\mathcal{F}_m)$ to $\varprojlim H^0(\mathcal{X}, (\mathcal{F}_m))$ is left exact and we define $H^i_{cts}(\mathcal{X}, (\mathcal{F}_m))$ to be its ith right derived functor. When the projective system is $\mathbb{Z}_l(k)$, we denote these groups by $H^i_{cts}(\mathcal{X}, \mathbb{Z}_l(k))$. We will need the following properties of continous cohomology.

(a) There is an exact sequence [J2, p.216, (3.1)]:

$$0 \to \varprojlim{}^1 H^{i-1}(\mathcal{X}, \mathcal{F}_m) \to H^i_{cts}(\mathcal{X}, (\mathcal{F}_m)) \to \varprojlim H^i(\mathcal{X}, \mathcal{F}_m) \to 0. \tag{2.1}$$

In particular, if $H^i(\mathcal{X}, \mathcal{F}_m)$ are finite for any i and m, then by the Mittag-Leffler condition $H^i_{cts}(\mathcal{X}, (\mathcal{F}_m)) \cong \varprojlim H^i(\mathcal{X}, \mathcal{F}_m)$.

(b) Let $f : \mathcal{X} \to Spec\, \mathcal{O}_S$ be the natural map. Then there is a spectral sequence [J2, p. 219, (3.10)]:

$$E_2^{r,s} = H^r_{cts}(\mathcal{O}_S, R^s f_*(\mathcal{F}_m)) \Rightarrow H^{r+s}_{cts}(\mathcal{X}, (\mathcal{F}_m)) \tag{2.2}$$

Note that the projective system $R^j f_*(\mathcal{F}_m)$ above is isomorphic to $(R^j f_*\mathcal{F}_m)$, [J2, Prop. 1.2, p. 210]. ¿From now on $\mathcal{F}_m$ will be one of the sheaves: $\mathbb{Z}/l^m(k)$ or $R^j f_*\mathbb{Z}/l^m(k)$.

(c) There is the Dwyer-Friedlander spectral sequence, which calculates étale K-theory in terms of continuous cohomology, [DF, Prop. 5.1, p.260]:

$$E_2^{r,s} = H^r_{cts}(\mathcal{X}, \mathbb{Z}_l(s/2)) \Rightarrow K^{et}_{s-r}(\mathcal{X}) \tag{2.3}$$

where $\mathbb{Z}_l(s/2)$ is defined as 0 if s is odd.

Lemma 2.1. *(a) If A is a locally constant constructible l-torsion étale sheaf on $Spec\,\mathcal{O}_S$, then the groups $H^i(\mathcal{O}_S; A)$ are finite for $i = 0, 1, 2$ and vanish for $i > 2$.*
(b) If A is a locally constant constructible $\mathbb{Z}_l$-sheaf on $Spec\,\mathcal{O}_S$, then the groups $H^i_{cts}(\mathcal{O}_S; A)$ are finitely generated $\mathbb{Z}_l$-modules for $i = 0, 1, 2$ and vanish for $i > 2$.

Proof. (a) Since A is locally constant, there is an étale covering $\mathcal{U}' = Spec\,\mathcal{O}'_S$ of $Spec\,\mathcal{O}_S$ such that $A(-1)$ is constant over $\mathcal{U}'$ and the field of fractions F' of $\mathcal{O}'_S$ is a finite Galois extension of K. Using the Hochschild-Serre spectral sequence of the covering we reduce the proof to the case of a constant sheaf. Without loss of generality we may assume that $A = \mathbb{Z}/l^m(1)$. The Kummer exact sequence:

$$0 \longrightarrow \mathbb{Z}/l^m(1) \longrightarrow \mathbb{G}_m \xrightarrow{l^m} \mathbb{G}_m \longrightarrow 0 \tag{2.4}$$

gives two exact sequences:

$$0 \to \mathcal{O}_S^*/\mathcal{O}_S^{*l^m} \to H^1(\mathcal{O}_S; \mathbb{Z}/l^m(1)) \to Pic(\mathcal{O}_S)[l^m] \to 0$$

$$0 \to Pic(\mathcal{O}_S)/l^m \to H^2(\mathcal{O}_S; \mathbb{Z}/l^m(1)) \to Br(\mathcal{O}_S)[l^m] \to 0.$$

Since kernels and cokernels in the sequences are finite [CF], it follows that the groups $H^i(\mathcal{O}_S; \mathbb{Z}/l^m(1))$, for $i = 1, 2$ are finite. The groups vanish for $i > 2$ because the cohomological dimension of $\mathcal{O}_S$ is 2 since l is odd, compare [DF, p.273].
(b) It follows from (a) by the exact sequence [J2, p. 219, (3.10)] and the Mittag-Leffler condition. □

We will apply the Leray spectral sequence (2.2) to compute $H^i_{cts}(\mathcal{X}, \mathbb{Z}_l(k))$, but first we need some basic facts about the sheaves $R^j f_*\mathbb{Z}/l^m(k)$ and $R^j f_*\mathbb{Z}_l(k)$.

Lemma 2.2. *Let $\bar{\mathcal{X}}_v = \mathcal{X} \times_{\mathcal{O}_S} \bar{\kappa}_v$ and $\bar{X} = X \times_{\mathcal{O}_S} \bar{K}$. Let x be a point of $Spec\,\mathcal{O}_S$ and $\bar{x}$ be the geometric point over x. Then the following isomorphisms hold.*

$$R^0 f_*\mathbb{Z}/l^m(k) \cong \mathbb{Z}/l^m(k)$$

$$R^1 f_*\mathbb{Z}/l^m(k)_{\bar{x}} \cong \begin{cases} J[l^m] \otimes \mathbb{Z}/l^m(k-1), & for\ x = Spec\,K \\ J_v[l^m] \otimes \mathbb{Z}/l^m(k-1), & for\ x = Spec\,\kappa_v \end{cases}$$

$$R^2 f_*\mathbb{Z}/l^m(k) \cong \mathbb{Z}/l^m(k-1)$$

$$R^j f_*\mathbb{Z}/l^m(k) = 0,\ for\, j > 2.$$

Here, J_v is the Jacobian variety of $\bar{\mathcal{X}}_v$. In particular, the sheaves $R^j f_* \mathbb{Z}/l^m(k)$ are locally constant and constructible.

Proof. The first isomorphism is obvious. We explain the second isomorphism in the case when $x = Spec\, \kappa_v$. The other case goes the same way. By proper base change [M1, VI, Cor. 2.5] we have:

$$R^1 f_* \mathbb{Z}/l^m(k)_{\bar{x}} \cong H^1(\bar{\mathcal{X}}_v, \mathbb{Z}/l^m(k)).$$

Using the Kummer sequence (2.4) and the identification $H^1(\bar{\mathcal{X}}_v, \mathbb{G}_m) \cong Pic(\bar{\mathcal{X}}_v)$ we obtain an isomorphism:

$$H^1(\bar{\mathcal{X}}_v, \mathbb{Z}/l^m(1)) \cong Pic(\bar{\mathcal{X}}_v)[l^m] \cong J_v[l^m]. \tag{2.5}$$

Hence:

$$H^1(\bar{\mathcal{X}}_v; \mathbb{Z}/l^m(k)) \cong H^1(\bar{\mathcal{X}}_v; \mathbb{Z}/l^m(1)) \otimes \mathbb{Z}/l^m(k-1) \cong J_v[l^m] \otimes \mathbb{Z}/l^m(k-1)$$

The third isomorphism of the lemma is the trace map isomorphism of sheaves [SGA 4, Exposé XVIII, Section 1.1]: $R^2 f_* \mathbb{Z}/l^m(k) \cong \mathbb{Z}/l^m(k-1)$. As far as the last equality is concerned, we observe that the sheaf in question has all stalks trivial. Consider for example the geometric point:

$$Spec\, \mathcal{O}_S \longleftarrow Spec\, K \longleftarrow Spec\, \bar{K} = \bar{x},$$

then we have by proper base change [M1, VI, Th. 1.1]:

$$R^j f_* \mathbb{Z}/l^m(k)_{\bar{x}} \cong H^j(\bar{X}, \mathbb{Z}/l^m(k)) = 0 \tag{2.6}$$

for $j > 2$. ☐

Lemma 2.3. *We have the following isomorphisms*

$$R^0 f_* \mathbb{Z}_l(k) \cong \mathbb{Z}_l(k)$$

$$R^1 f_* \mathbb{Z}_l(k)_{\bar{x}} \cong \begin{cases} T_l(J) \otimes \mathbb{Z}_l(k-1), & for\ x = Spec\, K \\ T_l(J_v) \otimes \mathbb{Z}_l(k-1), & for\ x = Spec\, \kappa_v \end{cases}$$

$$R^2 f_* \mathbb{Z}_l(k) \cong \mathbb{Z}_l(k-1)$$

$$R^j f_* \mathbb{Z}_l(k) = 0,\ for\, j > 2.$$

In particular, the $R^j f_* \mathbb{Z}_l(k)$ are locally constant and constructible $\mathbb{Z}_l$-sheaves.

Proof. Lemma 2.3 follows from Lemma 2.2. ☐

The Leray spectral sequence of the map $f : \mathcal{X} \to Spec\, \mathcal{O}_S$ gives the following useful description of the groups $H^i_{cts}(\mathcal{X}; \mathbb{Z}_l(k))$ in terms of Galois cohomology.

Proposition 2.4.

(a)
$$H^0_{cts}(\mathcal{X};\mathbb{Z}_l(k)) \cong H^0_{cts}(\mathcal{O}_S;\mathbb{Z}_l(k)) = 0$$
$$H^1_{cts}(\mathcal{X};\mathbb{Z}_l(k)) \cong H^1_{cts}(\mathcal{O}_S;\mathbb{Z}_l(k)) \cong H^1(G_S;\mathbb{Z}_l(k))$$

(b) There are exact sequences:
$$0 \to H^2(G_S,\mathbb{Z}_l(k)) \to H^2_{cts}(\mathcal{X},\mathbb{Z}_l(k)) \to H^1(G_S,T_l(J)(k-1)) \to 0$$
$$0 \to H^2(G_S,T_l(J)(k-1)) \to H^3_{cts}(\mathcal{X},\mathbb{Z}_l(k)) \to H^1(G_S,\mathbb{Z}_l(k-1)) \to 0.$$

(c)
$$H^4_{cts}(\mathcal{X},\mathbb{Z}_l(k)) \cong H^2(G_S,\mathbb{Z}_l(k-1))$$
$$H^i_{cts}(\mathcal{X},\mathbb{Z}_l(k)) = 0 \, for \, i > 4.$$

Proof. (a) For the group H^0_{cts} the Leray spectral sequence gives immediately:
$$H^0_{cts}(\mathcal{X},\mathbb{Z}_l(k)) \cong H^0(\mathcal{O}_S,R^0f_*\mathbb{Z}_l(k)) \cong H^0(\mathcal{O}_S,\mathbb{Z}_l(k)) = \mathbb{Z}_l(k)^{G_S} = 0.$$
As for H^1_{cts}, we have the low terms exact sequence:

(2.7) $$0 \to H^1_{cts}(\mathcal{O}_S;\mathbb{Z}_l(k)) \to H^1_{cts}(\mathcal{X},\mathbb{Z}_l(k)) \to H^0_{cts}(\mathcal{O}_S;R^1f_*\mathbb{Z}_l(k)) \to \dots$$

On the other hand, for any i:

(2.8) $$H^i_{cts}(\mathcal{O}_S;R^jf_*\mathbb{Z}_l(k)) \cong H^i(G_S;R^jf_*\mathbb{Z}_l(k)_{\bar{x}}) \cong H^i(G_S;H^j(\bar{X};\mathbb{Z}_l(k))).$$

To prove (2.8) one first checks the same isomorphisms for the étale cohomology with coefficients in $R^jf_*\mathbb{Z}/l^m(k)$ which hold true by [M2, Prop. 2.9, p. 209] and [M1, Prop 1.13, p. 88], respectively. The isomorphisms (2.8) follow then by Lemma 2.3 and the Mittag-Leffler condition applied to the sequence (2.1) (where we put $\mathcal{F}_m = R^jf_*\mathbb{Z}/l^m(k)$). In particular, (2.8) implies: $H^0_{cts}(\mathcal{O}_S;R^1f_*\mathbb{Z}_l(k)) \cong H^0(G_S;H^1(\bar{X};\mathbb{Z}_l(k)) \cong H^0(G_S;T_l(J)(k-1))$ which vanishes by a result of Suslin [Su, Prop 2.4, p. 11]. This proves the second isomorphism in (a).

(b) We have: $H^0_{cts}(\mathcal{O}_S,R^2f_*\mathbb{Z}_l(k)) \cong H^0(\mathcal{O}_S;\mathbb{Z}_l(k-1)) = 0$ by Lemma 2.3 and (a) above. The Leray spectral sequence and (2.8) imply that there is an exact sequence:
$$0 \to H^2_{cts}(\mathcal{O}_S;R^0f_*\mathbb{Z}_l(k)) \to H^2_{cts}(\mathcal{X},\mathbb{Z}_l(k)) \to H^1(G_S,T_l(J)(k-1)) \to 0.$$
It also follows by (2.8) that the kernel in the last sequence is: $H^2(G_S;R^0f_*\mathbb{Z}_l(k)) \cong H^2(G_S;\mathbb{Z}_l(k))$. It gives the first exact sequence in (b). To prove that the second one is exact it is enough to check that in the Leray spectral sequence (2.2) (with $\mathcal{F}_m = R^sf_*\mathbb{Z}/l^m(k)$) the groups $E_2^{3,0}$ and $E_2^{0,3}$ vanish. For $E_2^{3,0}$ the exact sequence (2.1) gives :
$$0 \to \varprojlim{}^1 H^2(\mathcal{O}_S,\mathbb{Z}/l^m(k)) \to H^3_{cts}(\mathcal{O}_S,\mathbb{Z}_l(k)) \to \varprojlim H^3(\mathcal{O}_S,\mathbb{Z}/l^m(k)) \to 0.$$
The lim^1 vanishes by the Mittag-Leffler condition and Lemma 2.1 (a). Since $cd_l(\mathcal{O}_S) = 2$, this shows that $E_2^{3,0} = 0$. For $E_2^{0,3} = H^0_{cts}(\mathcal{O}_S,R^3f_*\mathbb{Z}_l(k)) = 0$ by part (a) and Lemma 2.3.

(c) The same argument as in the above proof of part (b) shows that all groups $E_2^{r,s} = H^r_{cts}(\mathcal{O}_S,R^sf_*\mathbb{Z}_l(k))$ for $r+s=4$ vanish except for $E_2^{2,2}$. Since $E_2^{2,2} \cong H^2(G_S;\mathbb{Z}_l(k-1))$ by (2.8) and Lemma 2.3, this proves the first isomorphism in part (c). The second isomorphism in (c) follows in the same way by Lemma 2.3. ☐

The Leray spectral sequence of the map $f_v : \mathcal{X}_v \to Spec\,\kappa_v$ gives the following Galois-theoretic description of the groups $H^i_{cts}(\mathcal{X}_v;\mathbb{Z}_l(k))$.

Proposition 2.5.

(a)
$$H^0_{cts}(\mathcal{X}_v;\mathbb{Z}_l(k)) = 0$$
$$H^1_{cts}(\mathcal{X}_v;\mathbb{Z}_l(k)) \cong H^1(\kappa_v;\mathbb{Z}_l(k))$$

(b)
$$H^2_{cts}(\mathcal{X}_v,\mathbb{Z}_l(k)) \cong H^1(\kappa_v, T_l(J_v)(k-1))$$

(c)
$$H^3_{cts}(\mathcal{X}_v,\mathbb{Z}_l(k)) \cong H^1(\kappa_v,\mathbb{Z}_l(k-1))$$
$$H^i_{cts}(\mathcal{X}_v,\mathbb{Z}_l(k)) = 0\, for\, i > 3.$$

Proof. One can prove the proposition by reasoning as in the proof of Proposition 2.4 given above. In particular, the isomorphism:

$$H^1_{cts}(\mathcal{X}_v;\mathbb{Z}_l(k)) \cong H^1(\kappa_v;\mathbb{Z}_l(k))$$

follows from the low terms exact sequence of the Leray spectral sequence, since by the result of Suslin [Su, Prop. 2.4] the group $H^0(\kappa_v; R^1 f_{v*}\mathbb{Z}_l(k))$ vanishes. We leave details of the proof to the reader. □

Proposition 2.6.
(a) There are exact sequences:

$$0 \to H^4_{cts}(\mathcal{X},\mathbb{Z}_l(n+2)) \to K^{et}_{2n}(\mathcal{X}) \to H^2_{cts}(\mathcal{X},\mathbb{Z}_l(n+1)) \to 0$$
$$0 \to H^3_{cts}(\mathcal{X},\mathbb{Z}_l(n+2)) \to K^{et}_{2n+1}(\mathcal{X}) \to H^1_{cts}(\mathcal{X},\mathbb{Z}_l(n+1)) \to 0.$$

(b) There is an isomorphism and an exact sequence:

$$K^{et}_{2n}(\mathcal{X}_v) \cong H^2_{cts}(\mathcal{X}_v;\mathbb{Z}_l(n+1))$$
$$0 \to H^3_{cts}(\mathcal{X}_v,\mathbb{Z}_l(n+2)) \to K^{et}_{2n+1}(\mathcal{X}_v) \to H^1_{cts}(\mathcal{X}_v,\mathbb{Z}_l(n+1)) \to 0.$$

Proof. (a) The exact sequences exist by the Dwyer-Friedlander spectral sequence (2.3) and Proposition 2.4.

(b) It follows in the same way as (a) since $cd_l(\mathcal{X}_v) = 3$, compare [DF, p. 273]. □

3. The Key Proposition

Proposition 3.1.
Assume that A is an abelian variety defined over a global field K. Let l be an odd rational prime, $l \neq char(K)$ and $k > 0$. Let S be a finite set of primes of K which contains primes of bad reduction for A and primes over l. Let $G_K = G(\bar{K}/K)$. Fix a choice of the decomposition group G_v for every place of K. Finally, let $x \in H^1(G_K, T_l(A)(k))$ be a nontorsion element. Given $M_1 = l^{m_1}$ a fixed power of l, there exist infinitely many primes $v \notin S$ such that x restricts in $H^1(G_v, T_l(A_v)(k))$ to an element of order at least M_1.

Since the group $H^1(G_K, T_l(A)(k))$ is a finitely generated $\mathbb{Z}_l$-module, we have the following corollary.

Corollary 3.2.
For A, l, k as in Proposition 3.1 the kernel of the reduction map:

$$H^1(G_K, T_l(A)(k)) \to \prod_{v \notin S} H^1(G_v, T_l(A_v)(k))$$

is a finite group.

When K is a number field, Corollary 3.2 is a special case of a theorem of Jannsen [J1, Th. 3(a), p.337]. We refine his argument to get the more precise statement of Proposition 3.1.

Proof of Proposition 3.1
Let M be a power of l which we specify below. Consider the Galois representation $\rho : G_K \to Aut(T_l(A)(k)/M)$. Let $G = Im\rho$ and define L to be the smallest extension of K such that $T_l(A)(k)/M \cong A[M](k)$ becomes a trivial module over L. Then L/K is a finite extension with Galois group $Im(G \to Aut(A[M](k)))$. Consider the following commutative diagram.

$$\begin{tikzcd}
H^1(G_K, T_l(A)(k))/M \arrow[r] \arrow[d, "h_1"'] & H^1(G_v, T_l(A_v)(k))/M \arrow[d] \\
H^1(G_K, A[M](k)) \arrow[r] \arrow[d, "h_2"'] & H^1(G_v, A_v[M](k)) \arrow[d] \\
H^1(G_L, A[M](k)) \arrow[r] \arrow[d, "h_3"', "\cong"] & H^1(G_w, A_w[M](k)) \arrow[d, "="] \\
Hom(G_L, A[M](k)) \arrow[r] \arrow[d, "h_4"', "\cong"] & Hom(G_w, A_w[M](k)) \arrow[d] \\
Hom(G_L^{ab}, A[M](k)) \arrow[r, "h_5"] & Hom(G(\bar{\kappa}_w/\kappa_w), A_w[M](k))
\end{tikzcd} \tag{3.1}$$

The horizontal arrows in the diagram (3.1) are induced by the restrictions. We describe the vertical maps. The map h_1 is the injection from the exact sequence which we obtain by taking the long cohomology exact sequence associated with the exact sequence of Galois modules:

$$0 \to T_l(A)(k) \to T_l(A)(k) \to A[M](k) \to 0. \tag{3.2}$$

The arrow labeled h_2 is the restriction map. The map h_3 is the natural isomorphism obtained by the definition of L while h_4 exists because the group $A[M](k)$ is abelian. The first three vertical maps on the right hand side of the diagram (3.1) are defined in a similar way. The lowest vertical map on the right is obtained by the Néron-Ogg-Shafarevich criterion [ST, Th.1, p.493] which implies that the inertia group at w acts trivially on $A_w[M](k)$. It is so because by definition S contains all primes of bad reduction for A.

Lemma 3.3. *$ker\, h_2$ is a finite group of order bounded independently of M.*

Proof of lemma 3.3.
By the inflation-restriction sequence:

$$0 \to H^1(G(L/K), A[M](k)) \to H^1(G_K, A[M](k)) \to H^1(G_L, A[M](k)) \tag{3.3}$$

the kernel of h_2 is $H^1(G(L/K), A[M](k))$. On the other hand, a theorem of Serre [Se, Cor., p. 734] implies that the groups $H^i(G, T_l(A)(k))$ are finite for any i and k. Strictly speaking, Serre only proves this for $k = 0$, but his argument also applies to the general case (compare Jannsen's discussion [J1, bottom of page 338]). By (3.2) we obtain an exact sequence:

$$0 \to H^1(G, T_l(A)(k))/M \to H^1(G, A[M](k)) \to H^2(G, T_l(A)(k))[M] \to 0 \tag{3.4}$$

Hence the group $H^1(G, T_l(A)(k)/M) \cong H^1(G, A[M](k))$ is finite and its order is bounded independently of M. Since $G(L/K)$ is a quotient of G, the injectivity of the inflation map: $H^1(G(L/K), A[M](k)) \to H^1(G, A[M](k))$ finishes the proof of the lemma. $\square$

Consider the nontorsion element $x \in H^1(G_K, T_l(A)(k))$. Let l^s be the largest power of l such that $x = l^s y$ for an $y \in H^1(G_K, T_l(A)(k))$. Such an l^s exists since $H^1(G_K, T_l(A)(k))$ is a finitely generated $\mathbb{Z}_l$-module. By lemma 3.3, let l^t be an upper bound independent of M for the order of $ker h_2$. We choose $M \geq M_1 l^{s+t}$. Let $\bar{x}$ be the image of x in $H^1(G_K, T_l(A)(k))/M$. Since the maps h_1, h_3 and h_4 are injective and $l^t = \# ker h_2$, the element $\bar{x}$ is sent by the composition of the left vertical maps of (3.1) to a $\phi \in Hom(G(\bar{K}/L), A[M](k))$ of order at least M_1. By the Chebotarev density theorem [L, ch. VIII, th. 10 p. 169], there exist infinitely many primes $w \notin S$ such that the map h_5 preserves the order of ϕ. Hence, for those w the element $\bar{x}$ is sent by the composition of the left vertical arrows in (3.1) and the lowest horizontal to an element whose order is at least M_1 in $Hom(G(\bar{\kappa}_w/\kappa_w), A_w[M](k))$. The commutativity of diagram (3.1) implies that the restriction of x in $H^1(G_v, T_l(A)(k))$ is of order at least M_1 for the primes $v = w \cap \mathcal{O}_K$. $\square$

4. The Main Theorem

Theorem 4.1. *Assume that X is a smooth, proper and geometrically irreducible curve defined over a global field K. Let l be an odd rational prime, $l \neq char(K)$. Let $n > 0$ and let S be a finite set of primes of K which contains primes of bad reduction for X and primes lying over l. Let $\mathcal{X}$ be the smooth and proper model of X over the ring of S-integers of K. Assume that $x \in K_{2n}^{et}(\mathcal{X})$ is a nontorsion element. Given $M_1 = l^{m_1}$ a fixed power of l, there exist infinitely many primes $v \notin S$ such that x reduces in $K_{2n}^{et}(\mathcal{X}_v)$ to an element of order at least M_1.*

Proof. Consider the pullback diagram:

(4.1)
$$\begin{tikzcd}
\mathcal{X}_v \arrow[r, "i_v"] \arrow[d, "f_v"'] & \mathcal{X} \arrow[d, "f"'] \\
Spec\, \kappa_v \arrow[r, "j_v"] & Spec\, \mathcal{O}_S
\end{tikzcd}$$

On continuous cohomology the map $i_v : \mathcal{X}_v \to \mathcal{X}$ induces:

$$H^i_{cts}(\mathcal{X}, \mathbb{Z}_l(n+1)) \to H^i_{cts}(\mathcal{X}_v, i_v^* \mathbb{Z}_l(n+1)) \tag{4.2}$$

By Proosition 2.6, Proposition 2.4 (c) and the isomorphism $i_v^*\mathbb{Z}_l(n+1) \cong \mathbb{Z}_l(n+1)$, there is the following commutative diagram in which the lower horizontal arrow is the map (4.2) for $i = 2$.

(4.3)
$$\begin{tikzcd}
0 \arrow[d] & \\
H^2(G_S, \mathbb{Z}_l(n+1)) \arrow[r] \arrow[d] & 0 \arrow[d] \\
K^{et}_{2n}(\mathcal{X}) \arrow[r] \arrow[d] & K^{et}_{2n}(\mathcal{X}_v) \arrow[d, "\cong"'] \\
H^2_{cts}(\mathcal{X}; \mathbb{Z}_l(n+1)) \arrow[r] \arrow[d] & H^2_{cts}(\mathcal{X}_v, \mathbb{Z}_l(n+1)) \\
0 &
\end{tikzcd}$$

Lemma 4.2. *The group $H^2(G_S, \mathbb{Z}_l(k))$ is finite for every $k > 1$.*

Proof. This is a well-known fact. By Soulé [So1, Th. 4 and remark] the higher Chern class map

$$K_{2k-2}(\mathcal{O}_{K,S}, \mathbb{Z}/l^n) \to H^2(\mathcal{O}_{K,S}, \mathbb{Z}/l^n(k))$$

has cokernel killed by $k!$. When K is a number field the group $K_{2k-2}(\mathcal{O}_{K,S})$ is finite because it is finitely generated by [Q] and torsion by [B], moreover $K_{2k-3}(\mathcal{O}_{K,S})_{\text{tors}}$ is finite because $K_{2k-3}(\mathcal{O}_{K,S})$ is finitely generated. When K is a function field this is the same by [G] and [H]. Hence the order of the finite group $H^2(\mathcal{O}_{K,S}, \mathbb{Z}/l^n(k))$ is bounded independently of n. The claim follows from this, (2.8), Mittag-Leffler and the finiteness of $H^1(\mathcal{O}_{K,S}, \mathbb{Z}/l^n(k))$ (lemma 2.1 (b)). □

On the other hand, the map j_v induces the reduction map:

$$H^1_{cts}(\mathcal{O}_S, R^1 f_* \mathbb{Z}_l(k)) \to H^1_{cts}(\kappa_v, j_v^* R^1 f_* \mathbb{Z}_l(k)). \tag{4.4}$$

By the proper base change theorem [Mi, Cor. 2.3, p. 233] applied to the pullback (4.1) for $v \notin S$:

$$j_v^* R^1 f_* \mathbb{Z}_l(k) \cong R^1 f_{v_*} i_v^* \mathbb{Z}_l(k). \tag{4.5}$$

Since $i_v^*\mathbb{Z}_l(k) \cong \mathbb{Z}_l(k)$, the reduction map (4.4) can be written as follows:

$$H^1_{cts}(\mathcal{O}_S, R^1 f_* \mathbb{Z}_l(k)) \to H^1_{cts}(\kappa_v, R^1 f_{v*} \mathbb{Z}_l(k)) \tag{4.6}$$

Moreover, Proposition 2.4 (b), Proposition (2.5) (b) and the isomorphisms (2.8) give the following commutative diagram with the maps (4.2) and (4.6).

$$\begin{tikzcd}
0 \arrow[d] & \\
H^2(G_S, \mathbb{Z}_l(n+1)) \arrow[r] \arrow[d] & 0 \arrow[d] \\
H^2_{cts}(\mathcal{X}, \mathbb{Z}_l(n+1)) \arrow[r] \arrow[d] & H^2_{cts}(\mathcal{X}_v, \mathbb{Z}_l(n+1)) \arrow[d, "\cong"'] \\
H^1_{cts}(\mathcal{O}_S, R^1 f_* \mathbb{Z}_l(n+1)) \arrow[r] \arrow[d] & H^1_{cts}(\kappa_v, R^1 f_{v*} \mathbb{Z}_l(n+1)) \\
0 &
\end{tikzcd} \tag{4.7}$$

By the naturality of the isomorphisms (2.8) we have the following commutative diagram with the reduction map (4.6).

$$\begin{tikzcd}
H^1_{cts}(\mathcal{O}_S, R^1 f_* \mathbb{Z}_l(n+1)) \arrow[r] \arrow[d, "\cong"'] & H^1_{cts}(\kappa_v, R^1 f_{v*} \mathbb{Z}_l(n+1)) \arrow[d, "\cong"'] \\
H^1(G_S, T_l(J)(n)) \arrow[r] & H^1(\kappa_v, T_l(J_v)(n))
\end{tikzcd} \tag{4.8}$$

Here J and J_v denote the Jacobian varieties of $\bar{X}$ and $\bar{X}_v$, respectively.

Since by Lemma 4.2 the group $H^2(G_S, \mathbb{Z}_l(n+1))$ is finite, the nontorsion element $x \in K^{et}_{2n}(\mathcal{X})$ is sent by the composition of the lower vertical arrows in the left columns of the diagrams (4.3) and (4.7) to a nontorsion element of the group $H^1_{cts}(\mathcal{O}_S, R^1 f_* \mathbb{Z}_l(n+1)) \cong H^1(G_S, T_l(J)(n))$. The last group injects by the inflation map into $H^1(G_K, T_l(J)(n))$. Theorem 4.1 follows by Proposition 3.1 applied for $A = J$ and by the commutativity of the inflation and restriction maps:

$$\begin{tikzcd}
H^1(G_S, T_l(J)(n)) \arrow[r] \arrow[d] & H^1(\kappa_v, T_l(J_v)(n)) \arrow[d] \\
H^1(G_K, T_l(J)(n)) \arrow[r] & H^1(G_v, T_l(J_v)(n))
\end{tikzcd} \tag{4.9}$$

□

Theorem 4.3. *For X, l and n as in Theorem 4.1 the kernel of the reduction map:*

$$K^{et}_{2n}(\mathcal{X}) \to \prod_{p \notin S} K^{et}_{2n}(\mathcal{X}_v)$$

is a finite group.

Proof. It follows from Proposition 2.6, Proposition 2.4 (b) and Lemma 2.1 (b) that $K^{et}_{2n}(\mathcal{X})$ is a finitely generated $\mathbb{Z}_l$-module. Now, Theorem 4.3 is an immediate corollary of Theorem 4.1. □

5. Remarks on the odd dimensional case

Let us consider the reduction map for the odd dimensional étale K-theory.

$$K_{2n+1}^{et}(\mathcal{X}) \to \prod_{v \notin S} K_{2n+1}^{et}(\mathcal{X}_v) \tag{5.1}$$

By Proposition 2.6 (a) we have the exact sequence:

$$0 \to H^3_{cts}(\mathcal{X};\, \mathbb{Z}_l(n+2)) \to K_{2n+1}^{et}(\mathcal{X}) \to H^1_{cts}(\mathcal{X};\, \mathbb{Z}_l(n+1)) \to 0. \tag{5.2}$$

By Proposition 2.4 (b) there is another exact sequence:

$$0 \to H^2(G_S, T_l(J)(n+1)) \to H^3_{cts}(\mathcal{X}, \mathbb{Z}_l(n+2)) \to H^1(G_S, \mathbb{Z}_l(n+1)) \to 0. \tag{5.3}$$

It is a special case of a conjecture of Jannsen [J1, p. 317] that the group $H^2(G_S, T_l(J)(n+1))$ is finite for any curve X and $n > 0$. If the conjecture holds true, then one can prove an analog of our main theorem for the odd dimensional étale K-theory using the argument of this paper. As for today, the conjecture of Jannsen was proved only in the case when X is an elliptic curve with complex multiplication and l is a regular prime for X by Wingberg [Wi]. Hence for these X and l's Theorem 4.1 is also true for the reduction map on the odd-dimensional étale K-groups of $\mathcal{X}$. This follows from the following unconditional statement:

Proposition 5.1. *For any curve X, the kernel of the reduction map:*

$$H^1_{cts}(\mathcal{X};\, \mathbb{Z}_l(n+1)) \to \prod_{v \notin S} H^1(\mathcal{X}_v;\, \mathbb{Z}_l(n+1))$$

is a finite group.

Proof. Note that by (2.1) and by the Mittag-Leffler condition $H^1_{cts}(-;\mathbb{Z}_l(n+1)) = H^1(-;\mathbb{Z}_l(n+1))$ (the l-adic cohomology). To prove the proposition we use a result of Suslin [Su, Cor. 2.7, p. 12] which brings the claim to the situation which was considered in [AB]. Note that Suslin proves it for $n = 1$ but the proof goes the same way for $n \neq 0$. Consider the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & H^1(\mathcal{O}_S;\, \mathbb{Z}_l(n+1)) \arrow[r, "\cong"] \arrow[d, "\cong"'] & H^1(K;\, \mathbb{Z}_l(n+1)) \arrow[d, "\cong"'] \\ 0 \arrow[r] & H^1(\mathcal{X};\, \mathbb{Z}_l(n+1)) \arrow[r, "\cong"] & H^1(X;\, \mathbb{Z}_l(n+1)) \end{tikzcd} \tag{5.4}$$

All maps in the diagram (5.4) are injective because they come from short exact sequences of low terms associated with appropriate Leray spectral sequences. The top horizontal arrow is also surjective which follows by the localization exact sequence for étale K-theory with coefficients and the étale analogue of [So1, Th. 3, p. 274]. Hence the indicated maps in the diagram (5.4) are isomorphisms because the composition of injective maps:

$$H^1(K;\, \mathbb{Z}_l(n+1)) \to H^1(X;\, \mathbb{Z}_l(n+1)) \to H^1(K(X);\, \mathbb{Z}_l(n+1)) \tag{5.5}$$

is an isomorphism by [Su, loc.cit.]. Here $K(X)$ denotes the function field of X. Again using Suslin's result, we see that the map:

$$H^1(\kappa_v;\, \mathbb{Z}_l(n+1)) \to H^1(\mathcal{X}_v;\, \mathbb{Z}_l(n+1)) \tag{5.6}$$

is an isomorphism for any $v \notin S$. Hence the map from the statement is the reduction map:

$$H^1(\mathcal{O}_S;\, \mathbb{Z}_l(n+1)) \to \prod_{v \notin S} H^1(\kappa_v;\, \mathbb{Z}_l(n+1)) \tag{5.7}$$

which by [AB, Prop 1] has a finite kernel. □

References

[AB] D.Arlettaz and G.Banaszak, *On the non-torsion elements in the algebraic K-theory of rings of integers*, J. reine angew.Math. (to appear).

[B] A.Borel, *Stable real cohomology of arithmetic groups*, Ann. Sci. Ec. Norm. Sup. **7** (1974), 235–272.

[CF] J.W.S.Cassels and A.Fröhlich, *Algebraic Number Theory, Proccedings of an instructional conference organized by the London Mathematical Society*, Academic Press, 1967.

[DF] W.Dwyer and E.Friedlander, *Étale K-theory and arithmetic*, Trans. Amer. Math. Soc. **292** (1985), 247–280.

[G] D.Grayson, *Finite generation of K-groups of a curve over a finite field (after D.Quillen)*, Lecture Notes in Math. **551** (1982), 217–240.

[H] G.Harder, *Die Kohomologie S-arithmetisches Gruppen über Funktionenkörpen*, Invent. Math. **42** (1977), 135–175.

[J1] U.Jannsen, *On the l-adic cohomology of varieties over number fields and its Galois cohomology*, Galois groups over $\mathbb{Q}$. Proceedings of a Workshop Held March 23-27 1987, Y.Ihara, K.Ribet and J-P.Serre (eds), MSRI Publications **16**, Springer Verlag (1989), 315–353.

[J2] U.Jannsen, *Continuous étale cohomology*, Math. Ann. **280** (1988), 207–245.

[L] S.Lang, *Algebraic number theory*, Addison-Wesley, New York, 1970.

[M1] J.S.Milne, *Étale cohomology*, Princeton University Press, 1980.

[M2] J.S.Milne, *Arithmetic Duality Theorems*, Perspectives in Math., vol.1, Academic Press, 1986.

[Q] D.Quillen, *Finite generation of the groups K_i of rings of integers*, Lecture Notes in Math. **341** (1973), 179–198.

[Se] J.P.Serre, *Sur les groupes de congruence des variétés abéliennes, II*, Izvestija Akademii Nauk CCCP **35** (1971), 731–735.

[ST] J.P.Serre and J.Tate, *Good reduction of abelian varieties*, Ann. Math. **88** (1968), 492–517.

[So1] C.Soulé, *K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale*, Invent. Math. **55** (1979), 251–295.

[So2] C.Soulé, *p-adic K-theory of elliptic curves*, Duke Math. Jour. **54** (1987), 249–269.

[Su] A.Suslin, *Torsion in K_2 of fields*, K-theory **1** (1987), 5–29.

[Wi] K.Wingberg, *On the étale K-theory of an elliptic curve with complex multiplication for regular primes*, Canad. Math. Bull. **33 (2)** (1990), 145–150.

Department of Mathematics, Adam Mickiewicz University, Poznań, Poland
E-mail address: BANASZAK@math.amu.edu.pl

Department of Mathematics, Adam Mickiewicz University, Poznań, Poland
E-mail address: GAJDA@math.amu.edu.pl

Institut de Mathématiques de Jussieu, Université Paris 7, Case 7012, 75251 Paris Cedex 05, France
E-mail address: KAHN@mathp7.jussieu.fr

Department of Mathematics, Szczecin University, 70-451 Szczecin, Poland
E-mail address: KRASON@uoo.univ.szczecin.pl