Controlled Algebraic K-Theory of Integral Group Ring of $SL(3, \mathbb{Z})$

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Abstract. We calculate the lower Controlled Algebraic K-theory of any finitely generated infinite subgroup of $SL(3, \mathbb{Z})$, the group of 3×3 integral matrices of determinant 1.

Key words. Topological pseudo-isotopy, Quinn's spectral sequence, Virtually (infinite) cyclic groups.

1 Introduction

Let $SL(3, \mathbb{Z})$ denote the group of 3×3 integral matrices of determinant 1, and Γ denote any finitely generated infinite subgroup of $SL(3, \mathbb{Z})$. In this paper we calculate the lower controlled algebraic K-theory of $\mathbb{Z}\Gamma$ as defined by Farrell and Jones (cf. [4]). We refer the reader to [4] (Section 1 and Appendix) for various definitions.

Let X denote any CW-complex and let $\mathcal{C}(X)$ denote the class of virtually

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cyclic subgroups of $\pi_1(X)$. A group G is virtually cyclic if it is either a finite group, or there is an exact sequence

$$0 \longrightarrow T \longrightarrow G \longrightarrow F \longrightarrow 0$$

where T is an infinite cyclic group and F is a finite group. Let A denote a universal $(\pi_1(X), \mathcal{C}(X))$- space and $\tilde{X}$ be the universal covering space of X. (See [4] , p. 250 for the definition of universal $(\pi_1(X), \mathcal{C}(X))$- space .) Let $\pi_1(X) \times \tilde{X} \times A \longrightarrow \tilde{X} \times A$ denote the diagonal action, then $\rho : \mathcal{E}(X) \longrightarrow \mathcal{B}(X)$ and $f : \mathcal{E}(X) \longrightarrow X$ are defined to be the quotient of the standard projections $\tilde{X} \times A \longrightarrow A$ and $\tilde{X} \times A \longrightarrow \tilde{X}$ under the $\pi_1(X)$- actions. Let $\mathcal{F}_*()$ denote any of the Ω-spectra valued functors as in [4] (p. 251-253). Then Farrell and Jones conjecture that,

$$\mathcal{F}_*(f) o \mathcal{A}_* : \mathbb{H}_*(\mathcal{B}(X), \mathcal{F}_*(\rho)) \longrightarrow \mathcal{F}_*(X)$$

is an equivalence of the Ω- spectra, where $\mathcal{A}_*$ is an assembly map for the simplicially stratified fibration $\rho : \mathcal{E}(X) \longrightarrow \mathcal{B}(X)$ and $\mathcal{F}_*(f)$ is the image of $f : \mathcal{E}(X) \longrightarrow X$ under the functor $\mathcal{F}_*()$. Farrell and Jones verify this conjecture for the functor $\mathcal{F}_*() = \mathcal{P}_*()$, the topological pseudoisotopy Ω-spectrum (cf. [4], Th. 2.1) when $\pi_1(X)$ is a co-compact discrete subgroup of a virtually connected Lie group. In this paper we calculate $H_i(\mathcal{B}(X), \mathcal{F}_*(\rho))$, where $i = -1, 0$ and 1, for the case when $X = K(\Gamma, 1)$ and $\mathcal{F}_*() = \mathcal{P}_{*+2}()$. Our main result is the following in which T^k denotes the free abelian group of rank k.

Theorem 1.1 *For Γ a finitely generated infinite subgroup of $SL(3, \mathbb{Z})$ and $\mathcal{F}_*() = \mathcal{P}_{*+2}()$, $H_i(\mathcal{E}/\Gamma, \mathcal{F}_*(p))$ vanishes for $i = 0, 1$, and is T^k for $i = -1$, where k is the number of distinct conjugacy classes of D_6 (the dihedral group of order 12) in Γ . In particular, for $\Gamma = SL(3, \mathbb{Z})$, $H_i(\mathcal{E}/\Gamma, \mathcal{F}_*(p))$ vanishes for $i = 0, 1$, and is infinite cyclic for $i = -1$.*

We prove the above theorem for the special case when $\Gamma = SL(3, \mathbb{Z})$; the proof of the general case will be evident from this. Because of the simpler nature of the virtually (infinite) cyclic subgroups of $SL(3, \mathbb{Z})$, we first reduce the problem of calculating $\mathbb{H}_*(\mathcal{B}(X), \mathcal{F}_*(\rho))$ to calculating $\mathbb{H}_*(\mathcal{B}'(X), \mathcal{F}_*(\rho'))$, where $\mathcal{B}'(X)$ and ρ' are defined as above except that instead of $\mathcal{C}(X)$, we now consider $\mathcal{C}'(X)$, which is the class of finite subgroups of $SL(3, \mathbb{Z})$. Also , A is replaced by A' which is a universal $(\pi_1(X), \mathcal{C}'(X))$- space. Now using an explicit description of the space A' as constructed by Soule (cf. [13]), complete determination of finite subgroups of $SL(3, \mathbb{Z})$ up to conjugacy by Tahara (cf. [14]) and the knowledge of K-theory of finite groups which occur in $SL(3, \mathbb{Z})$ (cf. [2], [10], [12]), we complete the calculation.

2 Reduction to finite subgroups

We again refer the reader to [4] for various definitions. Let Γ be a discrete group and $\mathcal{C}$ and $\mathcal{C}'$ be two full classes of subgroups of Γ with $\mathcal{C}' \subset \mathcal{C}$. Let $\mathcal{E}$ and $\mathcal{E}'$ be universal $(\Gamma, \mathcal{C})$ and $(\Gamma, \mathcal{C}')$- spaces respectively and let E be a universal Γ-space. Let

$$\begin{aligned}
\mathcal{A}_{\mathcal{C}} \quad &: \mathbb{H}_*(\mathcal{E}/\Gamma, \mathcal{F}_*(p)) \longrightarrow \mathcal{F}_*(E/\Gamma) \\
\mathcal{A}_{\mathcal{C}'} \quad &: \mathbb{H}_*(\mathcal{E}'/\Gamma, \mathcal{F}_*(p')) \longrightarrow \mathcal{F}_*(E/\Gamma) \\
\mathcal{A}_{\mathcal{C}',\mathcal{C}} \quad &: \mathbb{H}_*(\mathcal{E}'/\Gamma, \mathcal{F}_*(p')) \longrightarrow \mathbb{H}_*(\mathcal{E}/\Gamma, \mathcal{F}_*(p))
\end{aligned}$$

be as defined in [4]. Here p,p' denote the simplicially stratified fibrations $\mathcal{E} \times_\Gamma E \longrightarrow \mathcal{E}/\Gamma$, $\mathcal{E}' \times_\Gamma E \longrightarrow \mathcal{E}'/\Gamma$ respectively. For $S \in \mathcal{C}$, define $\mathcal{C}'_S$ to be the class of subgroups of S such that $G \in \mathcal{C}'_S$ if and only if $G \in \mathcal{C}'$ and $G \subset S$ and let $\mathcal{E}'_S$ be a universal $(S, \mathcal{E}'_S)$- space. Then

$$\mathcal{A}_{\mathcal{C}'_S} : \mathbb{H}_*(\mathcal{E}'_S/S, \mathcal{F}_*(p'_S) \longrightarrow \mathcal{F}_*(E_S/S)$$

is similarly defined.

We state the following theorem from [4] (cf. [4], Th. A. 10).

Theorem 2.1 *The relative assembly map $A_{\mathcal{C}',\mathcal{C}}$ is an equivalence of Ω-spectra, provided for each $S \in \mathcal{C}$ we have that $\mathcal{A}_{\mathcal{C}'_S}$ is an equivalence of Ω-spectra.*

We formulate a lemma for our purpose.

Lemma 2.2 *The relative assembly map $\mathcal{A}_{\mathcal{C}',\mathcal{C}}$ is an isomorphism at the H_i-level for all $i \leq 0$ and surjective for $i = 1$, provided $\mathcal{A}_{\mathcal{C}'_S}$ is an isomorphism at the H_i-level for all $i \leq 1$.*

Proof: Follows from the proof of the Theorem 2.1 and the Five-lemma.□

Before we check the hypothesis of the above lemma for $\Gamma = SL(3,\mathbb{Z})$, we recall Quinn's spectral sequence (cf. [4], Lemma 1.4.2, [11], Appendix).

Lemma 2.3 *Let $f : E \longrightarrow X$ be a simplicially stratified fibration. Then there is a spectral sequence with ${E^2}_{p,q} = H_p(X, \pi_q\mathcal{F}_*(f))$ which abuts to $H_{p+q}(X, \mathcal{F}_*(f))$. Here $\pi_j\mathcal{F}_*(f)$ denote the stratified system of groups $\{\pi_j\mathcal{F}_*(f^{-1}(x)) : x \in X\}$ over X.*

Now, let $\Gamma = SL(3,\mathbb{Z})$ and $\mathcal{C}$ and $\mathcal{C}'$ denote class of virtually cyclic and finite subgroups of Γ respectively. Let $\mathcal{F}_*() = \mathcal{P}_{*+2}()$ where $\mathcal{P}_*()$ is topological pseudoisotopy functor (cf. [6] for the definition of $\mathcal{P}_*()$). Note that by Anderson-Hsiang's result (cf. [1]) $\pi_i\mathcal{P}_{*+2}(X) = K_i(\mathbb{Z}(\pi_1 X))$ for $i \leq -1$, $\tilde{K}_0(\mathbb{Z}(\pi_1 X))$ for $i = 0$, and $Wh(\pi_1 X)$ for $i = 1$. We will check the hypothesis of Lemma 2.2 in this case. If $S \in \mathcal{C}$ is finite subgroup of Γ, then $\mathcal{C}'_S =$ all subgroups of S; hence $\mathcal{E}'_S$ can be taken to be a point. This yields that $\mathcal{A}_{\mathcal{C}'_S}$ is a weak homotopy equivalence; in particular $\mathcal{A}_{\mathcal{C}'_S}$ induces an isomorphism for all i when $S \in \mathcal{C}$ is finite. Now let $S \in \mathcal{C}$ be a virtually infinite cyclic subgroup of Γ. But any virtually infinite cyclic subgroup is either of the type $F \rtimes_\alpha T$ where F is a finite subgroup of Γ, or it maps onto D_∞, the infinite dihedral group (cf. [5], Lemma 2.5). (Here T denotes infinite cyclic group and $F \rtimes_\alpha T$ denotes semi-direct product of F and T where T acts on F by an automorphism α of F.) Using this fact and classification of finite subgroups of Γ (cf. [14]), an elementary calculation shows that following are all the virtually infinite cyclic subgroups of Γ up to isomorphism

$$T,\ T \times T_2,\ D_\infty,\ D_\infty \times T_2,\ T \rtimes_\alpha T_4. \qquad (*)$$

Here T_n denotes finite cyclic group of order n and the action of α on T is non-trivial. The above classification of virtually infinite cyclic subgroup of Γ is obtained as follows. First, to identify the groups of the type $F \rtimes_\alpha T$ which occur in Γ we observe that since F is a finite group, $F \rtimes_\alpha T$ contains $F \times nT$ for some positive integer n. Now, a case by case checking yields that for F a non-trivial finite subgroup of Γ (which is completely classified up to conjugacy by Tahara cf. [14]) , the centralizer of F contains an element of infinite order only if $F = T_2$. Next, to identify the groups which map onto

D_∞, we proceed as follows. Let S map onto D_∞ with non-trivial kernel F, i.e.; we are given following exact sequence,

$$0 \longrightarrow F \longrightarrow S \longrightarrow D_\infty \longrightarrow 0.$$

Now, this exact sequence gives rise to another exact sequence

$$0 \longrightarrow F \longrightarrow S' \longrightarrow T \longrightarrow 0$$

where T is infinite cyclic subgroup of index two in D_∞ and S' is just the inverse image of T under the map $S \longrightarrow D_\infty$. The sequence $0 \longrightarrow F \longrightarrow S' \longrightarrow T \longrightarrow 0$ splits since T is free . Hence F is isomorphic to T_2 and S' is isomorphic to $T \times T_2$ by the classification of the groups of the type $F \rtimes_\alpha T$ in Γ. Since S' is a subgroup of index two in S, we have the following exact sequence

$$0 \longrightarrow S' \longrightarrow S \longrightarrow T_2 \longrightarrow 0.$$

An elementary argument shows that in the above exact sequence S has to be isomorphic to one of the following groups

$$T \times T_2,\ T \times T_2 \times T_2,\ D_\infty \times T_2,\ T \rtimes_\alpha T_4$$

The first two groups in the above list are ruled out since S has to map onto D_∞. And it is a fact that the other two groups in the above list do occur as subgroups of Γ.

It is also a fact that $K_i(\mathbb{Z}F)$ (for $i \leq -1$),$\tilde{K}_0(\mathbb{Z}F)$ and $Wh(F)$ all vanish when F is any finite group occuring as a subgroup of any of the groups in the list $(*)$. This fact is proven in [2], [12], [10], [8]. A second fact is that $K_i(\mathbb{Z}S)$ (for $i \leq -1$),$\tilde{K}_0(\mathbb{Z}S)$ and $Wh(S)$ also all vanish where S is any one of the groups in the list $(*)$. This is well known for the first two groups in this list (cf. [7], p. 43,Remark). The vanishing of $K_i(\mathbb{Z}S)$ (for $i \leq -1$) for all groups in the list follows from [5] (cf. Th. 2.1) using the first fact. The argument showing that $\tilde{K}_0(\mathbb{Z}S) = Wh(S) = 0$ for the last three groups in the list $(*)$ is the following and uses [5] (cf. Th. 2.6) together with Lemma 2.3 applied to the simplicially stratified fibration $\rho_E : E \longrightarrow X$ of [5] (cf. Sect. 2). The key point is that the fundamental group G of a fiber of ρ_E is either the group $T \times T_2$ or some finite subgroup of S. In either case $K_i(\mathbb{Z}G)$ for $i \leq -1$,$\tilde{K}_0(\mathbb{Z}G)$ and $Wh(G)$ all vanish. Hence, the spectral sequence shows that $H_i(X, \mathcal{F}_*(\rho_E)) = 0$ for $i = 0, 1$. But [5] (cf. Th 2.6) says that these groups maps onto $\tilde{K}_0(\mathbb{Z}S)$ and $Wh(S)$ respectively. Once again using the spectral sequence of Lemma 2.3 together with the above two facts, we see that $\mathcal{A}_{\mathcal{C}'_S}$ is an isomorphism for all $i \leq 1$. The hypothesis of Lemma 2.2 is consequently satisfied. We therefore conclude the following when $\Gamma = SL(3, \mathbb{Z})$,$\mathcal{F}_*() = \mathcal{P}_{*+2}()$ and the classes $\mathcal{C}$ and $\mathcal{C}'$ are as above.

Proposition 2.4 *The relative assembly map $\mathcal{A}_{\mathcal{C}',\mathcal{C}}$ is an isomorphism at the H_i-level for all $i \leq 0$ and surjective for $i = 1$.*

3 Calculation of $H_i(\mathcal{E}'/\Gamma, \mathcal{F}_*(p'))$

In this section we calculate $H_i(\mathcal{E}'/\Gamma, \mathcal{F}_*(p'))$ for $\mathcal{F}_*() = \mathcal{P}_{*+2}()$ and $-1 \leq i \leq 1$. Here $\mathcal{E}'$ is a universal $(\Gamma, \mathcal{C}')$- space, $\mathcal{C}'$ is the class of finite subgroup of $\Gamma = SL(3, \mathbb{Z})$ and p' is the simplicially stratified fibration as described earlier. In [13] Soule constructs a simplicial complex X_3 on which Γ acts simplicially with all the istropy group being finite groups. We now check that the fixed point set for any finite group in Γ is non-empty and contractible. Using Soule's notation, let X_1 denote the space of positive definite symmetric matrices mod scalars. Since X_1 can be given a Riemannain metric of nonpositive curvature, and Γ acts on it by isometries, we see that the fixed point set in X_1 for any finite subgroup of Γ is non-empty by Cartan's theorem (cf. [9], p. 75, Theorem 13.5). It is in fact contractible (cf. [9], p. 82, Theorem 14.6). Since X_3 is a retract of X_1 by a Γ-equivariant retraction (cf. [13]), we conclude that the fixed point set in X_3 for any finite subgroup of Γ is non-empty and contractible.

Hence, using a result of Connolly and Koz'niewski (cf. [3]), we can identify $\mathcal{E}'$ with the space X_3 as constructed by Soule (cf. [13]). We first calculate $H_0(\mathcal{E}'/\Gamma, \mathcal{F}_*(p'))$. From Lemma 2.3 there is spectral sequence with ${E^2}_{s,t} = H_s(\mathcal{E}'/\Gamma, \pi_t\mathcal{F}_*(p'))$ with $s+t = 0$ which converges to $H_0(\mathcal{E}'/\Gamma, \mathcal{F}_*(p'))$. By Carter's results (cf. [2]), $H_s(\mathcal{E}'/\Gamma, \pi_t\mathcal{F}_*(p')) = 0$ if $t \leq -2$. Hence the possible terms of the spectral sequence with ${E^2}_{s,t} \neq 0$ and $s + t = 0$ are ${E^2}_{0,0} = H_0(\mathcal{E}'/\Gamma, \pi_0\mathcal{F}_*(p'))$ and ${E^2}_{1,-1} = H_1(\mathcal{E}'/\Gamma, \pi_{-1}\mathcal{F}_*(p'))$. But these groups vanish as well since in the triangulation of the space $\mathcal{E}'$ the groups which occur as the isotropy group of vertices under the action of Γ are D_2, D_4, D_6, S_3 and S_4. (Here D_n denote the dihedral group of order $2n$

and S_n denotes the symmetric group on n letters.) For all of them, $\tilde{K}_0$ vanishes(cf. [12]). Furthermore, the groups which occur as the stabilizer of a 1- simplex of $\mathcal{E}'$ under the action of Γ are D_2, D_4 and S_3. For these groups, K_{-1} vanishes as well (cf. [2]). It follows that

$$H_0(\mathcal{E}'/\Gamma, \mathcal{F}_*(p')) = 0.$$

While calculating $H_{-1}(\mathcal{E}'/\Gamma, \mathcal{F}_*(p'))$, we note as above that the only possibly non-zero term of the spectral sequence is ${E^2}_{0,-1} = H_0(\mathcal{E}'/\Gamma, \pi_{-1}\mathcal{F}_*(p'))$. Now since the groups which occur as stabilizer of 1-simplices of $\mathcal{E}'$ under the action of Γ are D_2, D_4 and S_3 and for all of them K_{-1} vanishes (cf.[2]), we see that $H_0(\mathcal{E}'/\Gamma, \pi_{-1}\mathcal{F}_*(p')) = \bigoplus K_{-1}(\mathbb{Z}F)$. In this equation F varies over the stabilizers of the vertices of $\mathcal{E}'$ under the action of Γ with $K_{-1}(\mathbb{Z}F)$ occuring in the summand as many times as the number of conjugacy classes of the subgroup F in Γ. Now note that K_{-1} vanishes for all the groups occuring as the stabilizer of the vertices of $\mathcal{E}'$ under the action of Γ except for $F = D_6$, in fact $K_{-1}(\mathbb{Z}D_6) = T$, the infinite cyclic group (cf.[2]). Furthermore there is only one conjugacy class of subgroup D_6 of Γ. Hence,

$$E^2_{0,-1} = H_0(\mathcal{E}'/\Gamma, \pi_{-1}\mathcal{F}_*(p')) = T.$$

It follows that $H_{-1}(\mathcal{E}'/\Gamma, \mathcal{F}_*(p')) = T$. A similar calculation gives that $H_1(\mathcal{E}'/\Gamma, \mathcal{F}_*(p')) = 0$. This combined with Proposition 2.4 completes the proof of Theorem 1.1 for the case $\Gamma = SL(3, \mathbb{Z})$.

The proof for the arbitrary Γ case differs only in the calculation of K_{-1} where $E^2_{0,-1} = T^k$, and k is the number of distinct conjugacy classes of D_6 in Γ; hence $H_{-1}(\mathcal{E}'/\Gamma, \mathcal{F}_*(p')) = T^k$.

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