# The $K$-Theory of Vector bundles with Endomorphisms over a Scheme

Dongyuan Yao*

December 7, 1995

**Abstract**

Using Thomason and Trobaugh's Localization theorem [Th-Tr] for the $K$-theory of a scheme, we study the $K$-theory of the category of vector bundles with endomorphisms over a scheme. The results generalize those of D. Grayson [Gr1] when the scheme is affine. We also give an example showing that the Mayer-Vietoris sequence does not hold for the $K$-theory of vector bundles with endomorphisms , which indicates that the $K$-theory of vector bundles with endomorphisms is a global theory (not determined by local data).

## §1 Introduction

Let $A$ be a commutative ring (always with a unit). Let $\mathcal{E}nd\,(A)$ denote the exact category of all pairs $(P, f)$ where $P$ is a finitely generated projective $A$-module and $f$ is an endomorphism of $P$. The forget functor $(P, f) \to P$ induces a map $K(\mathcal{E}nd\,(A)) \to K(A)$, which is split surjective. Let $End\,_i(A)$ denote the kernel of $K_i(\mathcal{E}nd\,(A)) \to K_i(A)$

Let $A[T]$ be the polynomial ring over A with one variable $T$. Form the ring $R = (1+TA[T])^{-1}A[T]$. $R$ is an augmented $A$-algebra and $A \hookrightarrow R \xrightarrow{T=0} A$ gives the splitting map. Let $EK_i(A) = \text{cokernel}\,(K_i(A) \to K_i(K))$. D. Grayson ([Gr1]) shows that we have isomorphisms for all $i$:

$$EK_i(A) \cong End\,_{i-1}(A).$$

*Key Words: Localization theorem, Endomorphisms, Scheme, $K$-theory, Mayer-Vietoris sequence.

The key player in his proof is the localization theorem in [Gr2] for the $K$-theory of schemes under very restrictive conditions. The lack of a general localization theorem then prevented pushing the proof to (not affine) schemes. Now with Thomason and Trobaugh's powerful localization theorem ([Th-Tr]), this can be done without much difficulty.

Let $X$ be a scheme. Let $\mathcal{E}nd\,(X)$ denote the exact category of all pairs $(\mathcal{F}, f)$ where $\mathcal{F}$ is a vector bundle on $X$ (= coherent local free $\mathcal{O}_X$-module) and $f$ is an endomorphism of $\mathcal{F}$. Morphisms in $\mathcal{E}nd\,(X)$ are those that commute with the endomorphisms and a short sequence in $\mathcal{E}nd\,(X)$ is exact iff the underlying sequence of vector bundles is exact. The forget map $(\mathcal{F}, f) \to \mathcal{F}$ gives a functor from $\mathcal{E}nd\,(X)$ to the category of all vector bundles over $X$. This forget functor is clearly splitting with the splitting injection $\mathcal{F} \to (\mathcal{F}, 0)$. Let

$$End_{\,i}(X) = \ker(K_i(\mathcal{E}nd\,(X)) \to K_i(X)).$$

($K_i(X)$ here was called $K_i^{Naive}(X)$ in [Th-Tr], but under the conditions we will use later, they are the same.)

Given a scheme $X$, let $\tilde{S}$ be the multiplicative set of all polynomials of the form $\tilde{g}(T) = 1 + a_1T + \cdots + a_nT^n$ where all $a_i \in \Gamma(\mathcal{O}_X, X)$. We form a new scheme $\tilde{X} = \tilde{S}^{-1}X[T]$ in the following way:

Let $X[T] = X \times \mathrm{Spec}\,(Z[T])$. Locally for any affine open subscheme $U$ of $X$, $U = \mathrm{Spec}\,(A)$, denote $\tilde{S}_U$ = the image of $\tilde{S}$ under the restriction map $\Gamma(\mathcal{O}_X, X)[T] \to \Gamma(\mathcal{O}_X, U)[T] = A[T]$. Let $\tilde{U} = \mathrm{Spec}\,(\tilde{S}_U^{-1}A[T])$. Clearly these local data can glue up and form the scheme $\tilde{X}$.

We have the splitting injective map $\varphi : X \to \tilde{X}$ which is induced locally by the surjective splitting ring map $\tilde{S}_U^{-1}A[T] \to A$ by setting $T = 0$.

Let $EK_i(X) = \ker\,(K_i(\tilde{X}) \xrightarrow{\varphi^*} K_i(X))$.

**Theorem 1.1 :** Let $X$ be a quasi-compact scheme with an ample family of line bundles. Then with the above notations, we have

$$End_{\,i-1}(X) \cong EK_i(X).$$

The proof is in §4.

The identification of the $K$-theory of vector bundles with endomorphisms as in Theorem 1.1, in particular the fact that the polynomials in $\tilde{S}$ are required to have all coefficients to be global sections on $X$, indicates that the $K$-theory of endomorphisms may not have a good local-global property. This is indeed the case. Take $X = P^1_k$, the projective line over a field $k$. $X = U_1 \cup U_2$ where $U_1 =$ Spec $k[U]$ and $U_2 =$ Spec $k[T]$ glue up along $U \to T^{-1}$. Then we do NOT have the exact Mayer-Victoris sequence

$$\cdots \to K_i(\mathcal{E}nd\,(P^1_k)) \to K_i(\mathcal{E}nd\,(U_1)) \bigoplus K_i(\mathcal{E}nd\,(U_2)) \to K_i(\mathcal{E}nd\,(U_1 \cap U_2) \to \cdots$$

The details are in §4.

As in affine cases, let $\mathcal{N}i\ell\,(X)$ denote the exact category of all pairs $(\mathcal{F}, f)$ where $\mathcal{F}$ is a vector bundle on $X$ and $f$ is a nilpotent endomorphism of $\mathcal{F}$, and let $Nil_i(X) = \ker(K_i(\mathcal{N}i\ell\,(X)) \to K_i(X))$. Let $NK_i(X) = \ker(K_i(X[T]) \to K_i(X))$.

**Theorem 1.2 :** Let $X$ be a quasi-compact scheme with an ample family of line bundles. Then for all $i$ we have isomorphisms

$$Nil_{i-1}(X) \cong NK_i(X).$$

The proof is in §4.

**Acknowledge :** I am in debt to R. Thomason for many helpful e-mail correspondences. He also suggested to me another way to look at the K-theory of vector bundles with endomorphisms which is not explored in this paper:

Suppose $X$ is a quasi-compact scheme which has an ample family of line bundles. Then we have a natural homotopy equivalence of the K-theory spectra:

$$K(\mathcal{E}nd\,(X)) \cong \lim_{\leftarrow, Y} K(X[T], Y)$$

where $Y$ runs over all the closed subschemes of $X[T]$ which are finite over $X$.

## §2 Preliminary

We need some preparation so that we can draw the suitable localization theorem we need. The results presented here can be regarded as a continuation of the discussion in [Th-Tr] Appendix B.

Given a scheme $X$, we have two categories to consider. One is the category of all $\mathcal{O}_X$-modules, denoted by $\mathcal{O}_X$-Mod; the other is the category of all quasi-coherent $\mathcal{O}_X$-modules, denoted by $Q\text{coh}\,(X)$. $\mathcal{O}_X$-Mod always has enough injectives and arbitrary products, but it seems unknown if $Q\text{coh}\,(X)$ has enough injectives and arbitrary products.

Assume $X$ is quasi-compact and quasi-separated, then $Q\text{coh}\,(X)$ also has enough injectives. This can be seen as follows: Let $X = \cup U_i$ be a finite affine open covering of $X$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then we have an injection in $Q\text{coh}\,(X)$:

$$\mathcal{F} \to \bigoplus_i j_{i*}(\mathcal{F}|U_i), \quad \text{where} \quad j_i{:}U_i \hookrightarrow X.$$

Let $\Gamma(\mathcal{F}, U_i) \to I_i$ be an embedding as $\Gamma(\mathcal{O}_X, U_i)$-module where $I_i$ is injective. Then we have injections

$$\mathcal{F} \to \bigoplus_i j_{i*}(\mathcal{F}|U_i) \to \bigoplus_i j_{i*}(\tilde{I}_i),$$

where each $j_{i*}(\tilde{I}_i)$ is injective in $Q\text{coh}\,(X)$ since $(j_i^*, j_{i*})$ is an adjoint pair.

However when $Q\text{coh}\,(X)$ and $\mathcal{O}_X$-Mod both have enough injectives, the inclusion $Q\text{coh}\,(X) \subset \mathcal{O}_X$-Mod may not send injectives in $Q\text{coh}\,(X)$ to injectives in $\mathcal{O}_X$-Mod, and injectives in $Q\text{coh}\,(X)$ may not even be flasque, in general. So when we have a quasi-coherent $\mathcal{O}_X$-module or a complex of quasi-coherent $\mathcal{O}_X$-modules and consider various cohomology theories, we can take injective resolutions from $Q\text{coh}\,(X)$ or injective resolutions from $\mathcal{O}_X$-Mod respectively and the results may not be the same. We need to distinguish them. For a functor $G$, we will use $R_{Q\text{coh}}G$ to denote the right derived functor by taking injective resolutions from $Q\text{coh}\,(X)$ and use $RG$ as usual to denote the right derived functor by taking injective resolutions from $\mathcal{O}_X$-Mod. It is in our interest to see under what circumstances $R_{Q\text{coh}}G$ and $RG$ are the same.

**Definition** ([Th-Tr]) Let $X$ be a scheme. $X$ is called semi-seperated if $X$ has a basis (in Zariski topology) $\mathcal{B}$ such that each $U \in \mathcal{B}$ is affine and for any $U, V \in \mathcal{B}, U \cap V$ is affine again.

$X$ is semi-separated if and only if there is an open affine covering $X = \cup U_\alpha$ such that each $U_\alpha \cap U_\beta$ is affine again. Such a covering is called a semi-separated covering.

A map $f : X \to Y$ between two schemes is called semi-separated if for any affine scheme $Z$ and map $Z \to Y, Z \times_Y X$ is always semi-separated.

If $X$ and $Y$ are semi-separated, then any map between them is semi-separated.

Clearly separatedness implies semi-separatedness and semi-separatedness implies quasi-separatedness.

A complex of $\mathcal{O}_X$-modules is called pseudo-coherent if locally it is quasi-isomorphic to a bounded above complex of vector bundles. In particular, an $\mathcal{O}_X$-module is pseudo-coherent iff it is quasi-coherent and locally it has a resolution by vector bundles.

For two $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, as usual $\mathcal{H}om\,(\mathcal{F}, \mathcal{G})$ will denote the sheaf $\{U \to \mathrm{Hom}_{\mathcal{O}_\mathcal{U}}(\mathcal{F}|_U, \mathcal{G}|_U)\}$. If $\mathcal{F}$ is pseudo-coherent and $\mathcal{G}$ is quasi-coherent, then $\mathcal{H}om\,(\mathcal{F}, \mathcal{G})$ is quasi-coherent.

If $\mathcal{I}$ is injective in $\mathcal{O}_X$-Mod, then for any open subset $U \subset X, \mathcal{E}|_U$ is injective in $\mathcal{O}_U$-Mod. So we have

$$R\,\mathcal{H}om\,(\mathcal{F}, \mathcal{E})|_U \cong R\,\mathcal{H}om\,(\mathcal{F}|_U, \mathcal{E}|_U)$$

for any $\mathcal{O}_X$-module $\mathcal{F}$ and any complex $\mathcal{E}$ in $D^+(\mathcal{O}_X\text{-Mod})$.

If $\mathcal{I}$ is injective in $Q\mathrm{coh}\,(X)$, then $\mathcal{I}|_U$ may not be injective in $Q\mathrm{coh}\,(U)$ in general. So for $\mathcal{F}$ pseudo-coherent and $\mathcal{E} \in D^+(Q\mathrm{coh}\,(X))$, we may not have

$$R_{Q\mathrm{coh}}\,\mathcal{H}om\,(\mathcal{F}, \mathcal{E})|_U \cong R_{Q\mathrm{coh}}\,\mathcal{H}om\,(\mathcal{F}|_U, \mathcal{E}|_U).$$

**Lemma 2.1 :** Let $X$ be a quasi-compact and semi-separated scheme and $\mathcal{F}$ be a pseudo-coherent $\mathcal{O}_X$-module. Then for any $\mathcal{E} \in D^+(Q\mathrm{coh}\,(X))$,

$$R_{Q\mathrm{coh}}\,\mathcal{H}om\,(\mathcal{F}, \mathcal{E}) \cong R\ \mathcal{H}om\,(\mathcal{F}, \mathcal{E}).$$

**Corollary 2.2 :** Let $X$ be quasi-compact and semi-separated. For any pseudo-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and any $\mathcal{E} \in D^+(Q\mathrm{coh}\,(X))$, and any quasi-compact open subscheme $U \subset X$, we have

$$R_{Q\mathrm{coh}}\,\mathcal{H}om\,(\mathcal{F}, \mathcal{E})|_U \cong R_{Q\mathrm{coh}}\,\mathcal{H}om\,(\mathcal{F}|_U, \mathcal{E}|_U).$$

**Proof :** By Lemma 2.1, we have

$$\begin{aligned} R_{Q\mathrm{coh}}\,\mathcal{H}(\mathcal{F}, \mathcal{E})|_U &\cong R\ \mathcal{H}om\,(\mathcal{F}, \mathcal{E})|_U \cong R\ \mathcal{H}om\,(\mathcal{F}|_U, \mathcal{E}|_U) \\ &\cong R_{Q\mathrm{coh}}\,\mathcal{H}om\,(\mathcal{F}|_U, \mathcal{E}|_U). \end{aligned}$$

**Proof of Lemma 2.1 :** First let's assume $X$ is affine with $X = \text{Spec}\,(A)$. Let $\mathcal{E} = \tilde{E} \to \mathcal{I} = \tilde{I}$ be an injective resolution in $Q\text{coh}\,(X)$ where $E$ and $I$ are complexes of $\mathcal{A}$-modules and each $I^i$ is injective. Since $\mathcal{F} = \tilde{F}$ is pseudo-coherent, $\mathcal{F}$ has a resolution $\mathcal{P} = \tilde{P} \to \tilde{F} = \mathcal{F}$ where each $P^i$ is a finitely generated free $A$-module. Then

$$\begin{aligned} R_{Q\text{coh}}\mathcal{H}om\,(\mathcal{F},\mathcal{E}) &= \mathcal{H}om\,(\mathcal{F},\mathcal{I}) \\ &= \widetilde{\text{Hom}_A}(F,I) \cong \widetilde{\text{Hom}_A}(P,I) \cong \widetilde{\text{Hom}_A}(P,E) \\ &= \mathcal{H}om\,(\mathcal{P},\mathcal{E}). \end{aligned}$$

Now let $\mathcal{E} \to \mathcal{J}$ be an injective resolution in $\mathcal{O}_X$-Mod. Then

$$R\,\mathcal{H}om\,(\mathcal{F},\mathcal{E}) = \mathcal{H}om\,(\mathcal{F},\mathcal{J}) \cong \mathcal{H}om\,(\mathcal{P},\mathcal{J}) \cong \mathcal{H}om\,(\mathcal{P},\mathcal{E})$$

where the last quasi-isomorphism is from the fact that each $\mathcal{P}$ is a finite copies of $\mathcal{O}_X$. So we have

$$R_{Q\text{coh}}\mathcal{H}om\,(\mathcal{F},\mathcal{E}) \cong \mathcal{H}om\,(\mathcal{P},\mathcal{E}) \cong R\,\mathcal{H}om\,(\mathcal{F},\mathcal{E}).$$

Now for $X$ may not be affine, let $X = \cup X_i$ be a finite semi-separated open covering of $X$. For $\mathcal{E} \in D^+(Q\text{coh}\,(X))$, consider the Čech resolution with respect to the covering:

$$\mathcal{E} \to \bigoplus j_{i*}j_i^*\mathcal{E} \to \bigoplus_{i<j} j_{ij*}j_{ij}{}^*\mathcal{E} \to \cdots$$

where $j_I : X_{i_1} \cap \cdots \cap X_{i_r} \to X$ is the inclusion with $I = \{i_1, \cdots i_r\}$. By the assumption each $X_{i_1} \cap \cdots \cap X_{i_r}$ is affine.

Take a Cartan-Eilenberg resolution $\mathcal{I}$ (see, e.g., [Wei]) for the Čech resolution using injections in $Q\text{coh}\,(X)$ (this can be done because the category of chain complexes of $\mathcal{O}_X$-modules has enough injectives). Then Tot $\mathcal{I}$ is an injective resolution of $\mathcal{E}$ in $Q\text{coh}\,(X)$ and

$$R_{Q\text{coh}}\mathcal{H}om\,(\mathcal{F},\mathcal{E}) = \mathcal{H}om\,(\mathcal{F},\text{Tot }\mathcal{I}) = \text{Tot }\mathcal{H}om\,(\mathcal{F},\mathcal{I}).$$

Similarly, take a Cartan-Eilenberg resolution $\mathcal{J}$ for the Čech resolution using injectives in $\mathcal{O}_X$-Mod, then

$$R\,\mathcal{H}om\,(\mathcal{F},\mathcal{E}) = \mathcal{H}om\,(\mathcal{F},\text{Tot }\mathcal{J}) = \text{Tot }\mathcal{H}om\,(\mathcal{F},\mathcal{J}).$$

The total complexes of the double complexes $\mathcal{H}om(\mathcal{F}, \mathcal{I})$ and $\mathcal{H}om(\mathcal{F}, \mathcal{J})$ have convergent spectral sequences. So it will suffice to show that they have isomorphic $E^2$-terms.

For the both double complexes, we take the filtration such that the vertical cohomology comes first and the horizontal cohomology comes second, i.e.

$$\begin{aligned} E_2^{p,q}(\mathcal{I}) &= H_h^p H_v^q(\mathcal{H}om(\mathcal{F}, \mathcal{I})) \Longrightarrow R^{p+q}_{Q\mathrm{coh}}\mathcal{H}om(\mathcal{F}, \mathcal{E}) \\ E_2^{p,q}(\mathcal{J}) &= H_h^p H_v^q(\mathcal{H}om(\mathcal{F}, \mathcal{J})) \Longrightarrow R^{p+q}\mathcal{H}om(\mathcal{F}, \mathcal{E}) \end{aligned}$$

Since each vertical complex $\mathcal{I}^p$(or $\mathcal{I}^p$) of $\mathcal{I}$(or $\mathcal{J}$) is an injective resolution in $Q\mathrm{coh}(X)$ ( or $\mathcal{O}_X$-Mod) of some $\oplus {j_I}_* {j_I}^* \mathcal{E}$, we have

$$\begin{aligned} H_v^q(\mathcal{H}om(\mathcal{F}, \mathcal{I}^p)) &= R^q_{Q\mathrm{coh}}\mathcal{H}om(\mathcal{F}, \bigoplus j_{I*}j_I^*\mathcal{E}) \\ &= \bigoplus R^q_{Q\mathrm{coh}}\mathcal{H}om(\mathcal{F}, j_{I*}j_I^*\mathcal{E}) \\ &\cong \bigoplus j_{I*}R^q_{Q\mathrm{coh}}\mathcal{H}om(j_I^*\mathcal{F}, j_I^*\mathcal{E}) \\ &\cong \bigoplus j_{I*}R^q\mathcal{H}om(j_I^*\mathcal{F}, j_I^*\mathcal{E}) \\ &\cong \bigoplus R^q\mathcal{H}om(\mathcal{F}, j_{I*}j_I^*\mathcal{E}) \\ &= R^q\mathcal{H}om(\mathcal{F}, \bigoplus j_{I*}j_I^*\mathcal{E}) = H_v^q(\mathcal{H}om(\mathcal{F}, \mathcal{J}^p)). \end{aligned}$$

So $E_2^{p,q}(\mathcal{I}) \cong E_2^{p,q}(\mathcal{J})$.

## §3. A localization theorem

The localization theorm we will prove below is a generalization of a localization theorem in [Gr2] where the open subscheme was required to be afine.

We say a scheme has an ample family of line bundles if non-zero loci of global sections of line bundles on $X$ form a basis for $X$. Examples of schemes having an ample family of line bundles include affine schemes, quasi-projective schemes over a ring, or more general, quasi-projective schemes over a scheme that has an ample family of line bundles.

When $X$ has an ample family of line bundles, $X$ is also semi-separated and any quasi-coherent $\mathcal{O}_X$-module has a resolution by locally free $\mathcal{O}_X$-modules.

Let $X$ be a quasi-compact scheme with an ample family of line bundles. Let $\mathcal{I}$ be the quasi-coherent ideal of $\mathcal{O}_X$-module such that for any $x \in X$ $\mathcal{I}_x$ is a principle ideal of $\mathcal{O}_{X,x}$ generated by a non-zero devisor. Let $Y = V(\mathcal{I})$

be the closed subscheme of $X$ determined by $\mathcal{I}$. In another word, $Y$ is a regular immersion of codim=1. Let $U = X - Y$.

Let $\mathcal{H}_{\mathcal{I}}(X)$ denote the category of all pseudo-coherent $\mathcal{O}_X$-modules which are supported in $Y$ and have torsion dimension $\leq 1$. That is, a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is in $\mathcal{H}_{\mathcal{I}}(X)$ iff $\mathcal{F}|_U = 0$ and for each point $x \in X$, there is an open neighborhood $W$ and an exact sequence

$$\mathcal{O}_X^{n_1}|_W \to \mathcal{O}_X^{n_2}|_W \to \mathcal{F}|_W \to 0.$$

Clearly $\mathcal{H}_{\mathcal{I}}(X)$ is an exact category and $\mathcal{O}_X/\mathcal{I}^d \in \mathcal{H}_{\mathcal{I}}(X)$ for all positive integers $d$ since locally $\mathcal{I} = (s)$ and

$$0 \to \mathcal{O}_X \xrightarrow{s^d} \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}^d \to 0$$

is exact.

**Theorem 3.1 :** With the above notations and assumptions, we have a homotopy fibration of K-theory

$$K(\mathcal{H}_{\mathcal{I}}(X)) \to K(X) \to K(U).$$

(the K-theory spectra in the fibration sequence are already made into non-connnected K-theory spectra, i.e., have been extended to negative degrees of K-groups. This is needed to abtain a genuine homotopy fibration. Such a non-connected K-theory spectrum is denoted by $K^B$ in [Th-Tr]. For details, see [Th-Tr] §6.)

**Proof :** By the Localization Theorem 6.8 in [Th-Tr], we have a homotopy fibration sequence

$$K(\underline{\underline{P}}) \to K(X) \to K(U),$$

where $\underline{\underline{P}}$ is the category of all perfect complexes of quasi-coherent $\mathcal{O}_X$-modules which are acyclic when restricted to $U$ (a perfect complex is a complex of $\mathcal{O}_X$-modules which is locally quasi-isomorphic to a bounded complex of vector bundles). $\underline{\underline{P}}$ is a category with cofibrations and weak equivalences where cofibrations are termwise split monomorphisms with quotients still in $\underline{\underline{P}}$ and weak equivalences are quasi-isomorphisms. So we need to show that $\underline{\underline{P}}$ and $\mathcal{H}_{\mathcal{I}}(X)$ have equivalenct K-theory.

Let $\underline{\underline{P}}_0$ be the full subcategory of $\underline{\underline{P}}$ of all those perfect complexes $\mathcal{E}$ such that $\mathcal{E}|_U = 0$. Let $\underline{\underline{P}}_1$ be the full subcategory of $\underline{\underline{P}}$ of all bounded complexes

$\mathcal{E}$ such that each term $\mathcal{E}^i \in \mathcal{H}_{\mathcal{I}}(X)$. $\underline{\underline{P}}_0$ and $\underline{\underline{P}}_1$ are both subcategory with cofibrations and weak equivalences and $\underline{\underline{P}}_1 \subset \underline{\underline{P}}_0 \subset \underline{\underline{P}}$.

By Gillet-Waldhausen's theorem ( see, e.g., [Th-Tr] Theorem 1.11.7) that the K-theory of an exact category is equivalent to the K-theory of the category of bounded chain complexes of objects in the exact category, we have $K(\mathcal{H}_{\mathcal{I}}(X)) \cong K(\underline{\underline{P}}_1)$.

We need to show that $K(\underline{\underline{P}}_1) \cong K(\underline{\underline{P}}_0) \cong K(\underline{\underline{P}})$. We apply [Th-Tr] Theorem 1.9.8.

Let $W^{-1}\underline{\underline{P}}$ denotethe quotient category of $\underline{\underline{P}}$ by formally inverting all weak equivalences, i.e., inverting all quasi-isomorphisms. $W^{-1}\underline{\underline{P}}$ has the same objects as $\underline{\underline{P}}$ does. A morphism in $W^{-1}\underline{\underline{P}}$ is an equivalance class of diagrams $\mathcal{E} \overset{\sim}{\leftarrow} \mathcal{G} \to \mathcal{F}$ such that the left one is a quasi-isomorphism, i.e., $W^{-1}\underline{\underline{P}}$ admits a "calculus of fractions" (if the reader is not familiar with the construction of $W^{-1}\underline{\underline{P}}$, a quick read is [Th-Tr] §1.9.6). The Theorem 1.9.8 in [Th-Tr] says that if $W^{-1}\underline{\underline{P}}_1 \subset W^{-1}\underline{\underline{P}}_0$ is an equivalence of categories, then $K(\underline{\underline{P}}_1) \cong K(\underline{\underline{P}}_0)$, and the same is for $\underline{\underline{P}}_0 \subset \underline{\underline{P}}$.

To show that $W^{-1}\underline{\underline{P}}_1$ is equivalent to $W^{-1}\underline{\underline{P}}_0$, we need to show that for any $\mathcal{E} \in W^{-1}\underline{\underline{P}}_0$ there is an $\mathcal{L} \in W^{-1}\underline{\underline{P}}_1$ and a quasi-isomorphism between them, and $W^{-1}\underline{\underline{P}}_1 \subset W^{-1}\underline{\underline{P}}_0$ is full and faithful.

We apply [Th-Tr] Lemma 1.9.5 which gives an inductive construction of $\mathcal{L}$. Let $\underline{\underline{A}}$ denote the full subcategory of $Q\mathrm{coh}\,(X)$ of all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ such that $\mathcal{F}|_U = 0$. If $\mathcal{F} \in \underline{\underline{A}}$ and $\mathcal{F}$ is of finite type, then $\mathcal{F}$ is annihilated by $\mathcal{I}^d$ for some $d$ surficient large. Since $X$ has an ample family of line bundle, there is a vector bundle $\mathcal{V}$ and a surjection $\mathcal{V} \to \mathcal{F}$. So we have a surjection $\mathcal{V} \otimes \mathcal{O}_X/\mathcal{I}^d \to \mathcal{F}$. Let $\underline{\underline{D}}$ be the full subcategory of $\mathcal{A}$ of all $\mathcal{O}_X$-modules of the form $\mathcal{V} \otimes \mathcal{O}_X/\mathcal{I}^d$ where $\mathcal{V}$ is a vector bundle and $d$ is any positive integer. Clearly $\underline{\underline{D}} \subset \mathcal{H}_{\mathcal{I}}(X)$. Apply [Th-Tr] Lemma 1.9.5, for each perfect complex $\mathcal{E} \in \underline{\underline{P}}_0$, there is a bounded above complex $\mathcal{L}'$ of objects in $\underline{\underline{D}}$ and a quasi-isomorphism $\mathcal{L}' \to \mathcal{E}$. Since $\mathcal{E}$ is perfect, $\mathcal{L}'$ can be trancated into a bounded complex $\mathcal{L} \in \underline{\underline{P}}_1$ and $\mathcal{L} \to \mathcal{E}$ is still a quasi-isomorphism.

To see that $W^{-1}\underline{\underline{P}}_1 \subset W^{-1}\underline{\underline{P}}_0$ is full and faithful, let $\mathcal{E}, \mathcal{F} \in W^{-1}\underline{\underline{P}}_1$ and a morphism $f : \mathcal{E} \to \mathcal{F}$ in $W^{-1}\underline{\underline{P}}_0$ be represented by

$$\mathcal{E} \leftarrow \mathcal{G} \to \mathcal{F}.$$

From the above there is an $\mathcal{L} \in W^{-1}\underline{\underline{P}}_1$ and a quasi-isomorphism $\mathcal{L} \to \mathcal{G}$.

Then $f$ can be represented by

$$\mathcal{E} \leftarrow \mathcal{L} \rightarrow \mathcal{F}.$$

So $W^{-1}\underline{\underline{P}}_1 \subset W^{-1}\underline{\underline{P}}_0$ is full. That $W^{-1}\underline{\underline{P}}_1 \subset W^{-1}\underline{\underline{P}}_0$ is faithful is proved in a similar way. So we have the equivalence $W^{-1}\underline{\underline{P}}_1 \simeq W^{-1}\underline{\underline{P}}_0$ and $K(\underline{\underline{P}}_1) \cong K(\underline{\underline{P}}_0)$.

Now we work on the equivalence from $W^{-1}\underline{\underline{P}}_0$ to $W^{-1}\underline{\underline{P}}$. Let $\mathcal{E} \in \underline{\underline{P}}$ be such that each $\mathcal{E}^i$ is injective in $\mathcal{O}_X$. Define

$$\varphi(\mathcal{E}) = \lim_n \mathcal{H}om\,(\mathcal{O}_X/\mathcal{I}^n, \mathcal{E}).$$

We want to show that $\varphi(\mathcal{E}) \in \underline{\underline{P}}_0$ and $\varphi(\mathcal{E}) \rightarrow \mathcal{E}$ is a quasi-isomorphism.

For any $x \in X$, let $W$ be an open affine neighborhood of $x$ in $X$ such that $\mathcal{I}|_W = \widetilde{sA}$, where $s \in \Gamma(\mathcal{O}_X, W) = A$ is a non-zero-divisor. So we have the exact sequence

$$0 \rightarrow \mathcal{O}_X|_W \xrightarrow{s^n} \mathcal{O}_X|_W \rightarrow \mathcal{O}_X/\mathcal{I}^n|_W \rightarrow 0$$

Since $\mathcal{E}$ is injective in $Q\mathrm{coh}\,(X)$, we have

$$\mathcal{H}om\,(\mathcal{O}_X/\mathcal{I}^n, \mathcal{E}) = R_{Q\mathrm{coh}}\,\mathcal{H}om\,(\mathcal{O}_X/\mathcal{I}^n, \mathcal{E}).$$

By Corollary 2.2 in §2, we have a quasi-isomorphism

$$\mathcal{H}om\,(\mathcal{O}_X/\mathcal{I}^n, \mathcal{E})|_W \xrightarrow{\sim} R_{Q\mathrm{coh}}\,\mathcal{H}om\,(\mathcal{O}_X/\mathcal{I}^n|_W, \mathcal{E}|_W).$$

Let $\mathcal{E}|_W \rightarrow \mathcal{J} = \tilde{J}$ be an injective resolution in $Q\mathrm{coh}\,(W)$ where each $J^i$ is an injective $A$-module. Then

$$R_{Q\mathrm{coh}}\,\mathcal{H}om\,(\mathcal{O}_X/\mathcal{I}^n|_W, \mathcal{E}|_W) \xrightarrow{\sim} \mathcal{H}om\,(\mathcal{O}_X/\mathcal{I}^n\_W, \mathcal{J}) = \widetilde{Hom(A/s^nA, J)}.$$

Taking limit along $n$ for the short exact sequences of complexes

$$0 \rightarrow \widetilde{Hom(A/s^nA, J)} \rightarrow \widetilde{Hom(A, J)} = \mathcal{J} \xrightarrow{s^{n*}} \widetilde{Hom(A/, J)} = \mathcal{J} \rightarrow 0.$$

we have

$$0 \rightarrow \lim_n \mathcal{H}om\,(\mathcal{O}_X/\mathcal{I}^n|_W, \mathcal{J}) \rightarrow \mathcal{J} \rightarrow s^{-1}\mathcal{J} \rightarrow 0.$$

So we have quasi-isomorphisms

$$\varphi(\mathcal{E})|_W = \lim_n \mathcal{H}om\,(\mathcal{O}_X/\mathcal{I}^n, \mathcal{E})|_W \xrightarrow{\sim} \lim_n R_{Q\mathrm{coh}}\mathcal{H}om\,(\mathcal{O}_X/\mathcal{I}^n|_W, \mathcal{E}|_W)$$

$$\xrightarrow{\sim} \lim_n \mathcal{H}om\,(\mathcal{O}_X/\mathcal{I}^n|_W, \mathcal{J}) \xrightarrow{\sim} \mathcal{J}.$$

The last quasi-isomorphism is because $s^{-1}\mathcal{J}$ is acyclic. The composition factors as

$$\varphi(\mathcal{E})|_W \to \mathcal{H}om\,(\mathcal{O}_X, \mathcal{E})|_W = \mathcal{E}|_W \xrightarrow{\sim} \mathcal{J}.$$

So $\varphi(\mathcal{E}) \to \mathcal{E}$ is also a quasi-isomorphism . Thus $\varphi(\mathcal{E})$ is also perfect. Clearly $\varphi(\mathcal{E})|_U = 0$ since $\varphi(\mathcal{E})$ is just the $\mathcal{I}$-torsion part of $\mathcal{E}$.

Now for any $\mathcal{E} \in \underline{\underline{P}}$, Let $\mathcal{E} \to \mathcal{J}$ be an injective resolution. Then

$$\mathcal{E} \to \mathcal{J} \leftarrow \varphi(\mathcal{J})$$

gives an isomorphism in $W^{-1}\underline{\underline{P}}$ and $\varphi(\mathcal{J}) \in W^{-1}\underline{\underline{P}}_0$.

To see that $W^{-1}\underline{\underline{P}}_0 \subset W^{-1}\underline{\underline{P}}$ is full and faithful, let $\mathcal{E}, \mathcal{F} \in W^{-1}\underline{\underline{P}}_0$ and a morphism $f : \mathcal{E} \to \mathcal{F}$ in $W^{-1}\underline{\underline{P}}$ be represented by

$$\mathcal{E} \to \mathcal{G} \leftarrow \mathcal{F}.$$

Let $\mathcal{G} \xrightarrow{\sim} \mathcal{J}$ be an injective resolution in $Q\mathrm{coh}\,(X)$. Then $f$ can be represented by

$$\mathcal{E} \to \mathcal{J} \leftarrow \mathcal{F}.$$

Since $\mathcal{E}$ and $\mathcal{F}$ are in $\underline{\underline{P}}_0$, so $\mathcal{E}|_U = 0$ and $\mathcal{F}|_U = 0$, i.e., $\mathcal{E}$ and $\mathcal{F}$ are both $\mathcal{I}$-torsion. So the maps $\mathcal{E} \to \mathcal{J}$ and $\mathcal{F} \to \mathcal{J}$ factors through $\mathcal{E} \to \varphi(\mathcal{J}) \to \mathcal{J}$ and $\mathcal{F} \to \varphi(\mathcal{J}) \to \mathcal{J}$. So $f$ can be represented by

$$\mathcal{E} \to \varphi(\mathcal{J}) \leftarrow \mathcal{F}.$$

Thus $W^{-1}\underline{\underline{P}}_0$ is full in $W^{-1}\underline{\underline{P}}$. The faithfulness of $W^{-1}\underline{\underline{P}}_0 \subset W^{-1}\underline{\underline{P}}$ is also easy to see. So we have the desired equivalence $W^{-1}\underline{\underline{P}}_0 \cong W^{-1}\underline{\underline{P}}$ and $K(\underline{\underline{P}}_0) \cong K(\underline{\underline{P}})$.

**Theorem 3.2 (Excision) :** Let $X$ be a quasi-compact scheme with an ample family of line bundles, $\mathcal{I}$ be a quasi-coherent ideal which is locally principly generated by a non-zero-divisor, $Y$ be the closed subscheme determined by $\mathcal{I}$. If $W$ is an open subscheme of $X$ such that $Y \in W$, then

$$K(\mathcal{H}_{\mathcal{I}}(X)) \cong K(\mathcal{H}_{\mathcal{I}|_W}(W)).$$

**Proof :** Let $\underline{\underline{P}}$ be the category of all perfect complexes on $X$ which when restricted on $X - Y$ are acyclic. We showed in the proof of Theorem 3.1 that $K(\underline{\underline{P}}) \cong K(\mathcal{H}_{\mathcal{I}}(X))$. Let $\underline{\underline{Q}}$ be the category of all perfect complexes on $W$ which when restricted on $W - Y$ are acyclic. Then we also have $K(\underline{\underline{Q}}) \cong K(\mathcal{H}_{\mathcal{I}|_W}(W))$. The excision theorem in [Th-Tr] Proposition 3.1.9 says that $K(\underline{\underline{P}}) \cong K(\underline{\underline{Q}})$, so we have

$$K(\mathcal{H}_{\mathcal{I}}(X)) \cong K(\mathcal{H}_{\mathcal{I}|_W}(W)).$$

## §4. Proofs and examples

The proofs for Theorem 1.1 and 1.2 model after the proof given in [Gr1] for affine cases.

Let $X$ be a quasi-compact scheme with an ample family of line bundles. We may assume that $X$ is connected (otherwise we may just consider each connected component seperately). Denote $X[U] = X \times \mathrm{Spec}(Z[U])$ where $Z[U]$ is the polynomial ring over integers with one variable $U$. $X[U]$ is also quasi-compact with an ample family of line bundles. Clearly $\Gamma(\mathcal{O}_{X[U]}, X[U]) = \Gamma(\mathcal{O}_X, X)[U]$. So each global section of $\mathcal{O}_{X[U]}$ is a polynomial $g(U) = a_nU^n + \cdots + a_1U + a_0$ where $a_0, \cdots, a_n$ are all global sections of $\mathcal{O}_X$. For an $\mathcal{O}_X$-module $\mathcal{F}$, we let $\mathcal{F}[U]$ denote the $\mathcal{O}_{X[U]}$-module $\pi^*(\mathcal{F})$ where $\pi : X[U] \to X$ is the projection.

**Lemma 4.1 :** Let $\mathcal{F}$ be a vector bundle on $X$ and $f$ be an endomorphism of $\mathcal{F}$. Then there is a monic polynomial $p(U) \in \Gamma(\mathcal{O}_X, X)[U]$ such that $p(f) = 0$.

**Proof :** Consider the map $U - f : \mathcal{F}[U] \to \mathcal{F}[U]$. Assume the rank of $\mathcal{F}$ is $n$. Let

$$p(U) = det(U - f) = \wedge^n(U - f) : \wedge^n\mathcal{F}[U] \to \wedge^n\mathcal{F}[U].$$

So $p(U) \in Hom(\wedge^n\mathcal{F}[U], \wedge^n\mathcal{F}[U]) = Hom(\wedge^n\mathcal{F}, \wedge^n\mathcal{F})[U] = \Gamma(\mathcal{O}_X, X)[U]$. Clearly $p(U)$ is monic and $p(f) = 0$ because locally it is so by Hamilton-Cayley theorem (see, e.g., see [Ba]).

Let $S$ be a multiplicatively cloced set of monic polynomials in $\Gamma(\mathcal{O}_X, X)[U]$ and $U \in S$. For each $g(U) = U^n + a_{n-1}U^{n-1} + \cdots + a_0 \in S$, let $\tilde{g}(T) = T^n g(T^{-1}) = 1 + a_{n-1}T + \cdots + a_0T^n \in \Gamma(\mathcal{O}_X, X)[T]$. Let $\tilde{S} = \{\tilde{g} \mid g \in S\}$.

Then $\tilde{S}$ is also a multiplicative closed set of polynomials in $\Gamma(\mathcal{O}_X, X)[T]$. Form the new scheme $\widetilde{X^S} = \tilde{S}^{-1}X[T]$ in the way as described in §1. Let $E^S K_i(X) = \ker(K_i(\widetilde{X^S}) \to K_i(X))$.

Let $\mathcal{E}nd^{\,S}(X)$ be the category of all pairs $(\mathcal{F}, f)$ where $\mathcal{F}$ is a vector bundle on $X$ and $f$ is an endomorphism of $\mathcal{F}$ such that there is a polynomial $g(U) \in S$ such that $g(f) = 0$. Let $End^{\,S}_i(X) = \ker(K_i(\mathcal{E}nd^{\,S}(X)) \to K_i(X))$.

**Theorem 4.2 :** With the notations and assumptions as above we have

$$End^{\,S}_{i-1}(X) \cong E^S K_i(X) \quad \text{for all } i.$$

In particular, when $S =$ the set of all monic polynomials in $\Gamma(\mathcal{O}_X, X)[U]$, we get the Theorem 1.1; when $S =$ the set of all powers of $U$, we get the Theorem 1.2.

**Proof :** Let $P^1_X$ be the projective line over $X$. $P^1_X$ has a standerd covering $P^1_X = X[U] \cup X[T]$ gluing along $X[U, U^{-1}] \to X[T, T^{-1}]$ by sending $U \to T^{-1}$.

For each $g \in S$, let $\mathcal{I}$ be the principle ideal sheaf on $P^1_X$ such that $\mathcal{I}|_{X[U]} = (Ug)$ and $\mathcal{I}|_{X[T]} = (\tilde{g})$. Let $Y$ be the closed subscheme determined by $\mathcal{I}$. Then $P^1_X - Y = X[T] - V(\tilde{g}) = X[T, \tilde{g}^{-1}]$.

By the localization theorem 3.1, we have a homotopy fibration of K-theories:

$$K(\mathcal{H}_{\mathcal{I}}(P^1_X)) \to K(P^1_X) \to K(X[T, \tilde{g}^{-1}]).$$

Since $\tilde{g}$ has the constant term $= 1$, we have $V(T) \cap V(\tilde{g}) = \phi$, so $Y \subset X[U]$. By the excision Theorem 3.2, we have a homotopy equivalence

$$K(\mathcal{H}_{\mathcal{I}}(P^1_X)) \cong K(\mathcal{H}_{(Ug)}(X[U])).$$

Then we have a homotopy fibration sequence

$$K(\mathcal{H}_{(Ug)}(X[U])) \to K(P^1_X) \to K(X[T, \tilde{g}^{-1}]).$$

Taking the limit, we have the homotopy fibration sequence

$$K(\bigcup_{g \in S} \mathcal{H}_{(Ug)}(X[U])) \to K(P^1_X) \to K(\widetilde{X^S}).$$

Since $U \in S$, we see that

$$\bigcup_{g \in S} \mathcal{H}_{(Ug)}(X[U]) = \bigcup_{g \in S} \mathcal{H}_{(g)}(X[U])).$$

Next we need to identify $\bigcup_{g\in S}\mathcal{H}_{(g)}(X[U])$ with $\mathcal{E}nd^{\,S}(X)$. But this is a local property, and the same proof for affine case works as well here. The rest of the proof goes parallel to the proof of the affine case, and we refer the reader to [Gr1] .

Using the inedtification of the K-theory of endomorphisms in Theorem 1.1, in the example below we will demonstrate that the K-theory of endomorphisms on a schemes does not have the usual Mayer-Vietoris exact sequence with respect an open covering. This phenomenon indicates that the K-theory of endomorphisms is a rather global theory than a local one.

**Example 4.3 :** Let $X = P^1_k$ be the projective line over a field $k$ and $X = X_0 \cup X_1$ where $X_0 = \mathrm{Spec}\,(k[x])$ and $X_1 = \mathrm{Spec}\,(k[y])$ is the standerd covering gluing along $x \to y^{-1}$. We claim that the diagram

$$\begin{array}{ccc} K(\mathcal{E}nd\,(X)) & \to & K(\mathcal{E}nd\,(X_0)) \\ \downarrow & & \downarrow \\ K(\mathcal{E}nd\,(X_1)) & \to & K(\mathcal{E}nd\,(X_0 \cap X_1)) \end{array}$$

does not form a homotopy cartisian sqare.

Suppose it does.

By [Th-Tr] Theorem 8.1, the diagram

$$\begin{array}{ccc} K(X) & \to & K(X_0) \\ \downarrow & & \downarrow \\ K(X_1) & \to & K(X_0 \cap X_1) \end{array}$$

is a homotopy cartisian square. Since $K(\mathcal{E}nd\,(X)) = K(X) \times K(\widetilde{\mathcal{E}nd}\,(X))$ is a natural splitting, where $\pi_i(K(\widetilde{\mathcal{E}nd}\,(X))) = End_i(X)$, then the diagram

$$\begin{array}{ccc} K(\widetilde{\mathcal{E}nd}\,(X)) & \to & K(\widetilde{\mathcal{E}nd}\,(X_0)) \\ \downarrow & & \downarrow \\ K(\widetilde{\mathcal{E}nd}\,(X_1)) & \to & K(\widetilde{\mathcal{E}nd\,(X_0}\cap X_1)) \end{array}$$

is a homotopy cartisian square. Meanwhile $K(\tilde{X}) = K(X) \times EK(X)$ is also a natural splitting, where $\pi_i(EK(X)) = EK_i(X)$, and $K(\widetilde{\mathcal{E}nd}\,(X)) \cong \Omega EK(X)$. So the diagram

$$\begin{array}{ccc} EK(X) & \to & EK(X_0) \\ \downarrow & & \downarrow \\ EK(X_1) & \to & EK(X_0 \cap X_1) \end{array}$$

is also a homotopy cartisian square. So we have the long exact sequence of K-groups:

$$\cdots \to EK_1(X) \to EK_1(X_0) \oplus EK_1(X_1)$$
$$\to EK_1(X_0 \cap X_1) \to EK_0(X) \to \cdots \qquad (4.3.0)$$

Next we show that the above sequence 4.3.0 can not be exact, thus a contradiction.

Since $\Gamma(\mathcal{O}_X, X) = \Gamma(\mathcal{O}_{P^1_k}, P^1_k) = k$, we have $\tilde{X} = P^1_B = \tilde{X}_0 \cup \tilde{X}_1$ with $\tilde{X}_0 = \mathrm{Spec}\,(B_0)$ and $\tilde{X}_1 = \mathrm{Spec}\,(B_1)$ where

$B = (1+Tk[T])^{-1}k[T]$, $B_0 = (1+Tk[x,T])^{-1}k[x,T]$, $B_1 = (1+Tk[y,T])^{-1}k[y,T]$,

and $\widetilde{X_0 \cup X_1} = \tilde{X}_0 \cap \tilde{X}_1 = \mathrm{Spec}\,(B_{01})$ where

$$B_{01} = (1 + Tk[x, x^{-1}, T])^{-1}k[x, x^{-1}, T].$$

Since $\ker(B_0 \stackrel{T=0}{\to} k[x]) = TB_0$ is in radical of $(B_0)$, we have

$$EK_1(X_0) = \ker(K_1(B_0) \to K_1(k[x])) = 1 + TB_0.$$

Similarly we have

$$EK_1(X_1) = \ker(K_1(B_1) \to K_1(k[y])) = 1 + TB_1,$$

$$EK_1(X_0 \cap X_1) = \ker(K_1(B_{01}) \to K_1(k[x, x^{-1}])) = 1 + TB_{01}.$$

Since $B$ is local noetherian, we have $EK_0(k) = \ker(K_0(B) \to K_0(k)) = 0$.

From the calculation in [Qu], we have isomorphisms

$$K_i(X) = K_i(P^1_k) = K_i(k) \oplus K_i(k),$$

$$K_i(\tilde{X}) = K_i(P^1_B) = K_i(B) \oplus K_i(B).$$

So $EK_0(X) = \ker(K_0(\tilde{X}) \to K_0(X)) = 0$. Then the sequence 4.3.0 becomes

$$\cdots \to (1 + TB_0) \times (1 + TB_1) \stackrel{\alpha}{\to} 1 + TB_{01} \to 0.$$

We claim that $1 + (x + x^{-1})T \in 1 + TB_{01}$ is not in the image of $\alpha$, so the sequence 4.3.0 can not be exact.

That $1+(x+x^{-1})T$ is in the image of $\alpha$ means that there are

$$\frac{f_1(x,T)}{f_2(x,T)} \in B_0, \quad \text{and} \quad \frac{g_1(y,T)}{g_2(y,T)} \in B_1 \text{ such that}$$

$$1+(x+x^{-1})T = (1+\frac{f_1(x,T)}{f_2(x,T)})(1+\frac{g_1(x^{-1},T)}{g_2(x^{-1},T)}).$$

So

$$x+x^{-1} = \frac{f_1(x,T)}{f_2(x,T)} + \frac{g_1(x^{-1},T)}{g_2(x^{-1},T)} + \frac{f_1(x,T)}{f_2(x,T)}\frac{g_1(x^{-1},T)}{g_2(x^{-1},T)}.$$

We may assume that g.c.d($f_1, f_2$)=1 and g.c.d($g_1, g_2$)=1.

Let $\deg_x(f(x,T))$ denote the degree of $f(x,T)$ in $x$.

First we claim that

$$\deg_x(f_1(x,T)) = \deg_x(f_2(x,T)) + 1 \text{ and}$$

$$\deg_y(g_1(y,T)) = \deg_y(g_2(x,T)) + 1.$$

Assume

$$f_1 = a_0(T) + a_1(T)x + \cdots + a_n(T)x^n, f_2 = b_0(T) + b_1(T)x + \cdots + b_m(T)x^m,$$

$$g_1 = c_0(T) + a_1(T)x^{-1} + \cdots + c_p(T)x^{-p}, g_2 = d_0(T) + d_1(T)x^{-1} + \cdots + d_q(T)x^{-q}.$$

Since $g_2 \in 1+Tk[y,T]$, we see that $d_0(T)$ has its constant term $=1$. Let $x \to \infty$ (the same as comparing the terms having the highest powers in $x$), then

$$x \sim \frac{f_1}{f_2} + \frac{g_1}{g_2} + T\left(\frac{f_1}{f_2}\frac{g_1}{g_2}\right) \sim \frac{a_n(T)x^n}{b_m(T)x^m} + \frac{c_0(T)}{d_0(T)} + T\left(\frac{a_n(T)x^n}{b_m(T)x^m}\frac{c_0(T)}{d_0(T)}\right)$$

$$= \frac{c_0(T)}{d_0(T)} + \frac{a_n(T)x^n}{b_m(T)x^m}\left(1+\frac{Tc_0(T)}{d_0(T)}\right) = \frac{c_0(T)}{d_0(T)} + \frac{a_n(T)x^n}{b_m(T)x^m}\left(\frac{d_0(T)+Tc_0(T)}{d_0(T)}\right).$$

Since $d_0(T)$ has its constant term $=1$, $d_0(T)+Tc_0(T) \neq 0$, so we must have $n = m+1$. By symmetry, we have $p = q+1$.

By symmetry, we may assume that $q \geq m$. Write

$$x^p g_1(x^{-1},T) = h_1(x,T), \quad x^q g_2(x^{-1},T) = h_2(x,T),$$

Then

$$x + x^{-1} = \frac{f_1}{f_2} + \frac{h_1}{xh_2} + T\frac{f_1h_1}{f_2xh_2},$$

$$x^2 + 1 = \frac{xf_1}{f_2} + \frac{h_1}{h_2} + T\frac{f_1h_1}{f_2h_2} = \frac{xf_1h_2 + h_1f_2 + Tf_1h_1}{f_2xh_2},$$

$$(x^2+1)f_2h_2 - h_1f_2 = xf_1h_2 + Tf_1h_1$$

$$f_2((x^2+1)h_2 - h_1) = f_1(xh_2 + Th_1)$$

Since g.c.d($f_1, f_2$)=1, we have

$$f_1|(x^2+1)h_2 - h_1, \quad f_2|xh_2 + Th_1.$$

Let $(x^2+1)h_2 - h_1 = f_1r_1, xh_2 + Th_1 = f_2r_2$. We have $f_2f_1r_1 = f_1f_2r_2$, So $r_1 = r_2 = r$. Solving the linear system

$$\begin{aligned} (x^2+1)h_2 - h_1 &= f_1r \\ xh_2 + Th_1 &= f_2r, \end{aligned}$$

we get

$$h_1 = \frac{(-xf_1 + (x^2+1)f_2)r}{(x^2+1)T + x}, \quad h_2 = \frac{(Tf_1 + f_2)r}{(x^2+1)T + x}.$$

Since g.c.d($h_1, h_2$)=1, clear $(x^2+1)T + x$ is irreducible, and $\deg_x(h_2) = q \geq m = \deg_x(f_2)$, we have

$$h_1 = -xf_1 + (x^2+1)f_2, \quad h_2 = Tf_1 + f_2.$$

Since $f_2 \in 1 + Tk[x,T], h_2 = x^q g_2(x^{-1}, T)$ with $g_2(y,T) \in 1 + Tk[y,T]$, we have $x^q = h_2(x,0) = 0 + f_2(x,0) = 1$. So $q = 0, m = 0, f_2 = 1, h_2 = 1$ and

$$f_1 = a_0(T) + a_1(T)x, \quad g_1 = b_0(T) + b_1(T)x^{-1}.$$

Then

$$\begin{aligned} &x + x^{-1} = f_1 + f_2 + Tf_1f_2 \\ &= (a_0 + a_1x) + (b_0 + b_1x^{-1}) + T(a_0 + a_1x)(b_0 + b_1x^{-1}) \\ &= (a_0 + b_0 + a_0b_0 + a_1b_1T) + (a_1 + a_1b_0T)x + (b_1 + a_0b_1)x^{-1}. \end{aligned}$$

So $a_0 = 0, b_0 = 0, a_1 = 1, b_1 = 1$, and $T = 0$ which is not possible.

**References**

[Ba] H. Bass, Algebraic K-theory, Benjamin, New York, 1968
[Gr1] D. Grayson, The K-theory of Endomorphisms, J. of Algebra 48(1977), 439-446
[Gr2] D. Grayson, Higher algebraic K-theory II (after Quillen), Algebraic K-Theory: Evanston 1976, Springer Lect. Notes Math 551(1976), 217-240.
[Qu] D. Quillen, Higher algebraic K-theory I, Higher K-theories, Springer Lect. Notes Math 341(1973), 85-147.
[Th-Tr] R. Thomason and T.Trobaugh, Higher algebraic K-theory of schemes and of derived categories, The Grothendieck Festschrift III, Progress in Math. 88, Birkhäuser 1990, 247-435.
[Wa] F. Waldhausen, Algebraic K-theory of spaces, Algebraic and Geometric Topology, Springer Lect. Notes Math. 1126(1985), 318-419.
[Wei] C. Weibel, An introduction to homological algebra, Combridge University Press, 1994

Department of Mathematics, Washington University in St. Louis, St. Louis, MO 63130

e-mail: yao@math.wustl.edu