Homology of bi-Grassmannian complexes.

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For the last decade Grassmannian complexes have been playing an important role in the investigation of the homology of general linear groups. (See, for example, [5] and also [6], where complexes similar to Grassmannian play a crucial role.) It can be explained by both their quite geometric structure (These are complexes constructed by considering rational points of an open subset of Grassmannian variety over given field) and the fact that their homology is a coproduct of the relative homology of GL_n 's. The natural generalisation of Grassmannian complexes are bi-Grassmannian complexes $(G(\star, \star))$. These are also being used rather widely in research related to the theory of motivic cohomology and polylogarithms (See [2]). However, their homology was still uncalculated.

The following conjecture was formulated by A. Suslin several years ago. Let F be an infinite field, $G(\star, \star)$ be bi-Grassmannian complex over F . Then

$$H_k(G(\star, \star)) = \coprod_{0 \leq p \leq (k-1)/2} H_{k-2p-1}(GL(F)).$$

Subsequently Suslin found that this conjecture doesn't hold for homology with integral coefficients.

This paper is devoted to the proof of this assertion with rational coefficients.

Also we prove a similar formula for truncated complex consisting only of the several bottom rows of bi-Grassmannian complex up to the n -th one. This formula is valid in case in which $n!$ is invertible in a coefficient ring.

I'd like to thank professor A.Suslin who introduced me to this problem and gave a lot of very valuable advice during my work.

1. DEFINITION OF BI-GRASSMANNIAN COMPLEXES.

Consider an infinite field F . Denote by $Gr(p, q)$ the set of all planes of dimension p in F^{p+q} . Denote by $\tilde{G}(p, q)$ the subset of $Gr(p, q)$ consisting of planes $V \subset F^{p+q}$ defined by the following equivalent conditions:

- a) V intersects any coordinate plane of codimension $\leq p$ properly.
- b) The intersection of V and any coordinate plane of codimension p is 0.

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c) If $\pi : F^{p+q} \twoheadrightarrow F^p$ is any coordinate projection then the image of V coincides with F^p .

Let us verify the equivalence of these conditions.

b) $\Leftrightarrow$ c) There is an one to one correspondence between coordinate projections $F^{p+q} \twoheadrightarrow F^p$ and the coordinate planes of codimension p : $\pi \leftrightarrow W = \text{Ker}(\pi)$. On the other hand we have: $\dim \pi(V) = \dim V - \dim(V \cap \text{Ker}(\pi))$, therefore $\pi(V) = F^p \Leftrightarrow V \cap W = 0$.

a) $\Rightarrow$ b) Evidently.

b) $\Rightarrow$ a) Let U be a coordinate subspace of codimension $k \leq p$. Choose on U $p - k$ coordinate functions $x_1, x_2, \dots, x_{p-k}$ and let $W = \{u \in U : x_1(u) = x_2(u) = \dots = x_{p-k}(u) = 0\}$. Then W is a coordinate subspace in F^{p+q} and $\text{codim } W = p$. Therefore, $0 = V \cap W = V \cap U|_{x_i=0}$, that implies $\dim(V \cap U) \leq p - k = p + (p + q - k) - (p + q) = \dim V + \dim U - \dim F^{p+q}$. Thus, the intersection of V and U has the right dimension. $\square$

Introduce two families of maps between the sets $\tilde{G}(p, q)$. The first ones are projection operators $d_i : \tilde{G}(p, q) \rightarrow \tilde{G}(p, q - 1)$, where d_i is the projection onto the i -th coordinate plane of codimension 1. The other family consists of the "intersection operators" $\partial_j : \tilde{G}(p, q) \rightarrow \tilde{G}(p - 1, q)$. More precisely, consider the linear maps:

$$F^{p+q-1} \xrightarrow{\xi_i} F^{p+q} \xrightarrow{\pi_i} F^{p+q-1}$$

$$\xi_i(x_1, \dots, x_{p+q-1}) = (x_1, \dots, x_{i-1}, 0, x_i, \dots, x_{p+q-1})$$

$$\pi_i(x_1, \dots, x_{p+q}) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{p+q}).$$

If $V \in \tilde{G}(p, q)$ then set $d_i(V) = \pi_i(V)$, $\partial_i(V) = \xi_i^{-1}(V)$.

The previous discussion shows that these operators are well-defined.

Lemma 1.1.

$$\partial_j d_i = \begin{cases} d_{i-1} \partial_j, & \text{if } j < i \\ d_i \partial_{j+1}, & \text{if } j \geq i. \end{cases}$$

Proof. Let $W = \partial_j d_i V$, then $W = \xi_j^{-1}(\pi_i(V))$ and for any $w \in W$ there exists such $y \in V$ that $(w_1, \dots, w_{j-1}, 0, w_j, \dots, w_{p+q-2}) = (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_{p+q})$ ($\xi_j(w) = \pi_i(y)$). The last equality implies that

$$\begin{cases} y_j = 0, & \text{if } j < i \\ y_{j+1} = 0, & \text{if } j \geq i. \end{cases}$$

Omitting y_j in the vector y in the first case or y_{j+1} in the second one, we get a vector z such that $\xi_j(z) = y$ (respectively $\xi_{j+1}(z) = y$). On the other hand we have $\pi_{i-1}(z) = w$ ($\pi_i(z) = w$), therefore $w \in \pi_{i-1}(\xi_j^{-1}(V))$ ($w \in \pi_i(\xi_{j+1}^{-1}(V))$), i.e. $\partial_j d_i(V) \subset d_{i-1} \partial_j(V)$ (resp. $\partial_j d_i(V) \subset d_i \partial_{j+1}(V)$). But dimensions of these spaces are the same, therefore, both inclusions are equalities. $\square$

Further we will denote by $\mathbb{A}$ a commutative ring with unit. We often denote the groups $H(\dots, \mathbb{A})$ by $H(\dots)$. We also will denote by $G(p, q)$ a free $\mathbb{A}$ -module $\mathbb{A}[\tilde{G}(p, q)]$ generated by elements of $\tilde{G}(p, q)$. Supply the bigraded module $G(\star, \star)$ by two operations

$$d : G(p, q) \rightarrow G(p, q-1), \quad d = \sum_i (-1)^i d_i$$

and

$$\partial : G(p, q) \rightarrow G(p-1, q), \quad \partial = \sum_i (-1)^i \partial_i,$$

where d_i and ∂_j are induced by the corresponding maps on $\tilde{G}(p, q)$.

Lemma 1.2. *The following equalities hold: $d^2 = 0$, $\partial^2 = 0$, $d\partial = -\partial d$.*

Proof. We will verify only the last one. The others can be checked in the same way, using the fact that ∂_i and d_i satisfy the simplicial relations.

$$\begin{aligned} \partial d &= \sum_{j=1}^{p+q-1} \sum_{i=1}^{p+q} (-1)^{i+j} \partial_j d_i = \sum_{1 \leq i \leq j < p+q} (-1)^{i+j} d_i \partial_{j+1} + \sum_{1 \leq j < i \leq p+q} (-1)^{i+j} d_{i-1} \partial_j = \\ &= \sum_{i=1}^{p+q-1} \sum_{j=i+1}^{p+q} (-1)^{i+j-1} d_i \partial_j + \sum_{i=1}^{p+q-1} \sum_{j=1}^i (-1)^{i+j+1} d_i \partial_j = -d\partial. \end{aligned}$$

□

Definition 1.3. *We call the following bicomplex $G(\star, \star)$ the “bi-Grassmannian complex”.*

$$\begin{array}{ccccccc} & \dots & & \dots & & \dots & \\ & \partial \downarrow & & \partial \downarrow & & \partial \downarrow & \\ G(2, 1) & \xleftarrow{d} & G(2, 2) & \xleftarrow{d} & G(2, 3) & \xleftarrow{d} & \dots \\ & \partial \downarrow & & \partial \downarrow & & \partial \downarrow & \\ G(1, 1) & \xleftarrow{d} & G(1, 2) & \xleftarrow{d} & G(1, 3) & \xleftarrow{d} & \dots \\ & \partial \downarrow & & \partial \downarrow & & \partial \downarrow & \\ G(0, 1) & \xleftarrow{d} & G(0, 2) & \xleftarrow{d} & G(0, 3) & \xleftarrow{d} & \dots \end{array}$$

Here differentials d and ∂ are defined as above. Further we fix some integer $n \geq 0$ and consider the n -truncated bi-Grassmannian complex $G^n(\star, \star)$ consisting of rows of $G(\star, \star)$ from the 0-th up to the n -th one.

Now we can formulate the main result of the paper.

Theorem 1.4 (The Main Theorem.). *Let F be an infinite field, $G^n(\star, \star)$ be the n -truncated bi-Grassmannian complex over F . Assume that $n!$ is invertible in $\mathbb{A}$. Then*

$$H_k(G^n(\star, \star), \mathbb{A}) = \coprod_{0 \leq p \leq n/2} H_{k-2p-1}(GL_{n-2p}(F), \mathbb{A}).$$

Corollary 1.5. *In the same notation*

$$H_k(G(\star, \star), \mathbb{Q}) = \coprod_{0 \leq p} H_{k-2p-1}(GL(F), \mathbb{Q}).$$

The bicomplex defined above being quite geometrical is absolutely inconvenient for any calculations. In the next section we introduce some algebraic objects and a purely algebraically defined bicomplex which is quasiisomorphic to the bi-Grassmannian complex.

2. SOME AUXILIARY OBJECTS.

Let $A = (v_1, v_2, \dots, v_n)$ be a $m \times n$ -matrix over an infinite field F and let $k \leq \min(m, n)$ be a nonnegative integer. We will say that A is $W(k)$ -matrix if any set of $\min(m, n)$ columns of A containing the columns $v_{n-k+1}, v_{n-k+2}, \dots, v_n$ is linearly independent. If such a matrix is a $W(0)$ -matrix then we say that it has columns in general position. Note, also, that if $m \geq n$ then A is a $W(k)$ -matrix iff the columns of A are linearly independent (so that, in particular this condition does not really depend on k).

For any $k \leq \min(p, q)$ set $WP^{(k)}(p, q)$ to be a free $\mathbb{A}$ -module generated by all $p \times q$ $W(k)$ -matrices. Consider a complex $WP^{(k)}(p, \star)$ having the modules $WP^{(k)}(p, q)$ in dimensions $q > p$ and 0 elsewhere. A differential operator $d^{(k)}$ is given by the formula:

$$d^{(k)} : WP^{(k)}(p, q) \rightarrow WP^{(k)}(p, q-1), \quad d^{(k)}(A) = \sum_{i=1}^{q-k} (-1)^i (v_1, v_2, \dots, \hat{v}_i, \dots, v_q).$$

The general linear group $GL_p(F)$ acts on $WP^{(k)}(p, q)$ by left multiplication. This action commutes with the differential $d^{(k)}$ and gives each group $WP^{(k)}(p, q)$ a structure of a left GL_p -module. Therefore, we can consider a factor complex ${}_{GL_p}WP^{(k)}(p, \star)$. There is a canonical morphism of complexes

$$\phi_k : {}_{GL_{p+k}}WP^{(k)}(p+k, \star)[k] \rightarrow {}_{GL_p}WP^{(0)}(p, \star)$$

given as follows. In each orbit U of action of GL_{p+k} on the basis set of the module $WP^{(k)}(p+k, q+k)$ (the set of all $(p+k) \times (q+k)$ $W(k)$ -matrices) we can choose an element having the form

$$\begin{pmatrix} M_{p \times q} & 0 \\ \star & I_k \end{pmatrix}$$

Moreover, it's clear that the GL_p -orbit of M depends only on U , so that we can define a map ϕ_k using the formula $\phi_k(U) = M \bmod GL_p$.

Theorem 2.1. *The map $\phi_k : {}_{GL_{p+k}}WP^{(k)}(p+k, \star)[k] \rightarrow {}_{GL_p}WP^{(0)}(p, \star)$ is a quasiisomorphism .*

Lemma 2.2. *Denote the complex*

$$0 \leftarrow WP^{(k)}(p+k, k) \xleftarrow{d^{(k)}} \dots \xleftarrow{d^{(k)}} WP^{(k)}(p+k, p+k)$$

by $R^{(k)}(p+k, \star)$. This complex is acyclic up to dimension $p+k-1$ and the complex $WP^{(k)}(p+k, \star)[1]$ is a free GL_{p+k} -resolution of the complex $R^{(k)}(p+k, \star)$.

Proof. Obviously, GL_{p+k} -modules $WP^{(k)}(p+k, q)$ are free provided that $q \geq p+k$. Thus, we should just check that the sequence

$$\begin{aligned} 0 \leftarrow WP^{(k)}(p+k, k) \xleftarrow{d^{(k)}} \dots \xleftarrow{d^{(k)}} WP^{(k)}(p+k, p+k) \xleftarrow{d^{(k)}} \\ \xleftarrow{d^{(k)}} WP^{(k)}(p+k, p+k+1) \xleftarrow{d^{(k)}} \dots \end{aligned}$$

is exact. Let $w = \sum_j a_j(v_{1j}, \dots, v_{qj}) \in WP^{(k)}(p+k, q)$ be a cycle. Since the field F is infinite we can choose a general enough vector v_0 such that

$$y = - \sum_j a_j(v_0, v_{1j}, \dots, v_{qj}) \in WP^{(k)}(p+k, q+1).$$

Then

$$dy = \sum_j a_j(v_{1j}, \dots, v_{qj}) + \sum_j a_j \sum_{i=1}^{q-k} (-1)^{i+1} (v_0, v_{1j}, \dots, \hat{v}_{ij}, \dots, v_{qj}) = w.$$

□

Proof of theorem 2.1.

Consider a map $\gamma : {}_{GL_p}WP^{(0)}(p, q) \rightarrow {}_{GL_{p+k}}WP^{(k)}(p+k, q+k)$ given by the formula $M \mapsto \begin{pmatrix} M & 0 \\ 0 & I_k \end{pmatrix}$. We want to show that this map induces an isomorphism

$$\gamma_\star : H_l({}_{GL_p}WP^{(0)}(p, \star)) \simeq H_{l+k}({}_{GL_{p+k}}WP^{(k)}(p+k, \star)).$$

To do so note that the same map γ defines also a homomorphism of complexes $\gamma : R^{(0)}(p, \star) \rightarrow R^{(k)}(p+k, \star)[k]$. Furthermore, the groups GL_p and GL_{p+k} act on $R^{(0)}$ and $R^{(k)}$ respectively and the map γ is compatible with the group embedding $GL_p \hookrightarrow GL_{p+k}$. Thus, γ induces a homomorphism of spectral sequences $\gamma : E \rightarrow \bar{E}[0, k]$, where E (resp. $\bar{E}$) is a hyperhomology spectral sequence corresponding to the action of GL_p on $R^{(0)}$ (resp. GL_{p+k} on $R^{(k)}$).

$$E_{mn}^1 = H_m(GL_p, R^{(0)}(p, n)) \Rightarrow H_{m+n}(GL_p, R^{(0)}(p, \star))$$

$$\bar{E}_{mn}^1 = H_m(GL_{p+k}, R^{(k)}(p+k, n)) \Rightarrow H_{m+n}(GL_{p+k}, R^{(k)}(p+k, \star)).$$

Lemma 2.2 shows that the homomorphism under consideration coincides with the induced map on E^∞ -terms:

$$\begin{aligned} \gamma_* : E_l^\infty &= H_l(GL_p, R^{(0)}(p, \star)) \simeq H_{l+1}(GL_p W P^{(0)}(p, \star)) \rightarrow \\ &\rightarrow H_{l+k+1}(GL_{p+k} W P^{(k)}(p+k, \star)) \simeq H_{l+k}(GL_{p+k}, R^{(k)}(p+k, \star)) = \bar{E}_{l+k}^\infty. \end{aligned}$$

It's sufficient now to show that γ induces an isomorphism on E^1 -terms. Namely, let's show that for $n \leq p$

$$H_m(GL_p, R^{(0)}(p, n)) \simeq H_m(GL_{p-n}) \simeq H_m(GL_{p+k}, R^{(k)}(p+k, n+k)).$$

The group GL_{p+k} acts transitively on the canonical basis of $WP^{(k)}(p+k, n+k)$. The stabiliser of the element $(e_{p-n+1}, e_{p-n+2}, \dots, e_{p+k})$ is the affine subgroup $\text{Aff}_{p-n, n+k} \stackrel{\text{def}}{=} \begin{pmatrix} GL_{p-n} & 0 \\ \star & I_{n+k} \end{pmatrix}$. Using Shapiro's lemma (see, for example, [1]) we get an isomorphism $H_m(GL_{p+k}, R^{(k)}(p+k, n+k)) \simeq H_m(\text{Aff}_{p-n, n+k})$, but the theorem 1.11[5] asserts that this group is isomorphic to $H_m(GL_{p-n})$. (Setting $k=0$ we have: $H_m(GL_p, R^{(0)}(p, n)) \simeq H_m(GL_{p-n})$.) It's easy to verify that the resulting isomorphism $H_m(GL_p, R^{(0)}(p, n)) \simeq H_m(GL_{p+k}, R^{(k)}(p+k, n+k))$ is induced by the map γ . The isomorphism of E^1 -terms of the spectral sequences gives required isomorphism on the limits. Since $\phi_k \gamma = \text{id}$ and γ is a quasiisomorphism we conclude that ϕ_k is a quasiisomorphism as well. $\square$

Introduce a right action of the symmetric group Σ_k on the complex $WP^{(k)}(p, \star)$. Let $v = (v_1, \dots, v_{q-k}, v_{q-k+1}, \dots, v_q) \in WP^{(k)}(p, q)$ and $\sigma \in \Sigma_k$. We set

$$v\sigma = (v_1, \dots, v_{q-k}, v_{q-k+\sigma^{-1}(1)}, v_{q-k+\sigma^{-1}(2)}, \dots, v_{q-k+\sigma^{-1}(k)}).$$

Evidently, this action commutes with the left action of the group GL_p . We will say that vectors $v_{q-k+1}, v_{q-k+2}, \dots, v_q$ are on the right-hand side of the matrix v and sometimes separate these vectors in formulas by the sign “|”. Note that the group Σ_k acts exclusively on the right-hand vectors whereas the face operators $\partial_i^{(k)}$ act only on vectors on the left-hand side. This allows us to consider factor complexes $WP^{(k)}(p, \star)_{\Sigma_k}$ and $GL_p WP^{(k)}(p, \star)_{\Sigma_k}$.

Lemma 2.3. *If $k!$ is invertible in $\mathbb{A}$ then the map ϕ_k induces a canonical quasi-isomorphism*

$$\phi_{\star_k} : GL_{p+k} WP^{(k)}(p+k, \star)_{\Sigma_k}[k] \rightarrow GL_p WP^{(0)}(p, \star).$$

Proof. One can easily see that if $g \in GL_{p+k} WP^{(k)}(p+k, q+k)$ and $\sigma \in \Sigma_k$ then $\phi_k(g) = \phi_k(g\sigma)$. Therefore the following diagram commutes

$$\begin{array}{ccc} GL_{p+k} WP^{(k)}(p+k, \star)[k] & \xrightarrow{\phi_k} & GL_p WP^{(0)}(p, \star) \\ \downarrow & & \parallel \\ GL_{p+k} WP^{(k)}(p+k, \star)_{\Sigma_k}[k] & \xrightarrow{\phi_{\star_k}} & GL_p WP^{(0)}(p, \star)_{\Sigma_k} \end{array}$$

The second row is the result of application of Σ_k -coinvariant functor to the first one. This functor is exact in the category of $\mathbb{A}$ -modules because $H_i(\Sigma_k, \mathbb{A}) = 0$,

provided that $i > 0$ (see [1]). Theorem 2.1 tells us that ϕ_k is a quasiisomorphism. Therefore, $\phi_{\star_k}$ is a quasiisomorphism too. $\square$

From now on we assume that n is a fixed nonnegative integer and that $n!$ is invertible in the ring $\mathbb{A}$.

Set

$$S(p, q) \stackrel{\text{def}}{=} \begin{cases} WP^{(n-p)}(n, q)_{\Sigma_{n-p}}, & 0 \leq p \leq n, \quad p + q \geq n \\ 0, & \text{otherwise.} \end{cases}$$

Consider a bigraded module $S(\star, \star)$ which has $S(p, q)$ in dimension (p, q) if $q > n$ and 0 elsewhere. Supply it by two operators d and ∂ , where $d : S(p, q) \rightarrow S(p, q-1)$ is induced by the differential operator $d^{(n-p)}$ above and the operator $\partial : S(p, q) \rightarrow S(p-1, q)$ is given by the formula $\partial = \sum_{i=1}^{p+q-n} (-1)^i \partial_i$, where

$$\partial_i(v_1, \dots, v_{p+q-n} | v_{p+q-n+1}, \dots, v_q) = (v_1, \dots, \hat{v}_i, \dots, v_{p+q-n} | v_{p+q-n+1}, \dots, v_q, v_i).$$

Lemma 2.4. $S(\star, \star)$ is a bicomplex.

Proof. We have to verify that $\partial^2 = 0$, $d\partial = -\partial d$ and $d^2 = 0$. We will check the first equality. The other ones can be checked in the similar way. Let $x = (v_1, v_2, \dots, v_q) \in S(p, q)$. We have:

$$\begin{aligned} \partial^2(x) &= \sum_{i=1}^{q+p-n-1} (-1)^i \partial_i \left(\sum_{j=1}^{q+p-n} (-1)^j (v_1, \dots, \hat{v}_j, \dots | \dots, v_q, v_j) \right) = \\ &= \sum_{1 \leq i < j \leq q+p-n} \left((-1)^{i+j} (v_1, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots | \dots, v_q, v_j, v_i) + \right. \\ &\quad \left. (-1)^{i+j-1} (v_1, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots | \dots, v_q, v_i, v_j) \right) = 0. \end{aligned}$$

$\square$

Let's come back to the bi-Grassmannian complexes. We are going to construct a quasiisomorphism between $_{GL_n} S(\star, \star)$ and the bi-Grassmannian complex $G^n(\star, \star)$. First, consider a map $\psi : WP^{(0)}(p, p+q) \rightarrow G(p, q)$

$$\psi \begin{pmatrix} a_{11} & \dots & a_{1,p+q} \\ \vdots & & \vdots \\ a_{p1} & \dots & a_{p,p+q} \end{pmatrix} = [\text{plane with basis } (a_{11}, \dots, a_{1,p+q}), \dots, (a_{p1}, \dots, a_{p,p+q})].$$

(Recall, that $WP^{(0)}(p, p+q)$ is generated by matrices having columns in general position.) One can easily see that this map ψ defines an isomorphism $_{GL_p} WP^{(0)}(p, p+q) \xrightarrow{\psi} G(p, q)$.

Combining it with $\phi_{\star_{n-p}}$ we get a map:

$$_{GL_n} S(p, n+q) \xrightarrow{\phi_{\star_{n-p}}} _{GL_p} WP^{(0)}(p, p+q) \xrightarrow{\psi} G(p, q).$$

Proposition 2.5. The composition map $\psi \phi_{\star_{n-p}}$ defines a quasiisomorphism of bi-complexes

$$_{GL_n} S(\star, \star)[0, n] \xrightarrow{\sim} G^n(\star, \star).$$

The proof is straightforwardly implied by lemma 2.3 and will be omitted. The previous proposition shows that we can calculate homology of $_{GL_n}S(\star, \star)$ instead of homology of the complex $G^n(\star, \star)$. The following section is devoted to this calculation.

3. PROOF OF THE MAIN THEOREM.

Lemma 3.1. *If $i > 0$, $q > n$, then $H_i(GL_n, S(p, q)) = 0$.*

Proof. The group $S(p, q)$ has a canonical basis consisting of the orbits of action of Σ_{n-p} on $n \times q$ $W(n-p)$ -matrices. The group GL_n acts on the canonical basis of $S(p, q)$ by permutations. In each orbit β of its action we can choose an element which can be presented by the matrix

$$A_\beta = \begin{pmatrix} a_{11} & \dots & a_{1,q-n} & 1 & 0 & \dots & 0 \\ a_{21} & \dots & a_{2,q-n} & 0 & 1 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & \dots & a_{n,q-n} & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Shapiro's Lemma tells us that

$$H_i(GL_n, S(p, q)) \simeq \coprod_{\text{all orbits}} H_i(\text{Stab } A_\beta)$$

But for any β $\text{Stab } A_\beta$ is isomorphic to a subgroup of $\Sigma_{(n-p)}$. Since the order of this group is a divisor of $n!$ we conclude that each group $H_i(\text{Stab } A_\beta)$ is trivial. $\square$

Corollary 3.2. *The natural map $H_\star(GL_n, H_{n+1}S(p, \star)) \rightarrow H_\star(_{GL_n}S(p, \star))[n+1]$ is an isomorphism.*

Proof. Consider two hyperhomology spectral sequences of GL_n with coefficients in the complex $S(p, \star)$. The E^2 -term of the first spectral sequence is concentrated in the $p+1$ -th column and has the form $E_{l,p+1}^2 = H_l(GL_n, H_{p+1}(S(p, \star)))$. The $\tilde{E}^1$ -term of the second one is concentrated in the 0-th row and has the form $\tilde{E}_{0l}^1 = H_0(GL_n, S(p, l)) = _{GL_n}S(p, l)$ ($l > n$). Computation of the $\tilde{E}^2$ -term gives the assertion of the corollary. $\square$

Lemma 3.3. *For any $0 \leq p \leq n$ the sequence*

$$0 \leftarrow S(p, 0) \leftarrow S(p, 1) \leftarrow \dots \leftarrow S(p, n) \leftarrow S(p, n+1) \leftarrow \dots$$

is exact.

Proof. It follows from lemma 2.2 and exactness of the Σ_{n-p} -coinvariant functor in our conditions. $\square$

Let's introduce a bicomplex $\bar{S}(\star, \star)$ given as follows:

$$\begin{array}{ccccccc}
0 & \longleftarrow & S(n, 0) & \xleftarrow{d} & S(n, 1) & \xleftarrow{d} & \dots \xleftarrow{d} S(n, n) \\
& & \downarrow & & \partial \downarrow & & \partial \downarrow \\
& & 0 & \longleftarrow & S(n-1, 1) & \xleftarrow{d} & \dots \xleftarrow{d} S(n-1, n) \\
& & & & \downarrow & & \partial \downarrow \\
& & & & 0 & \longleftarrow & \dots \xleftarrow{d} S(n-2, n) \\
& & & & & & \vdots \\
& & & & & & 0 \longleftarrow S(0, n) \\
& & & & & & \downarrow \\
& & & & & & 0
\end{array}$$

The differentials d and ∂ are defined here in the same way as in the bicomplex $S(\star, \star)$. There is a canonical morphism of bicomplexes $S(\star, \star)[0, 1] \rightarrow \bar{S}(\star, \star)$ induced by the differential d . Using lemma 3.3, it's easy to verify that this map is a quasiisomorphism. This fact together with corollary 3.2 gives us the following proposition.

Proposition 3.4.

$$H_\star(GL_n S(\star, \star))[1] \simeq H_\star(GL_n, \bar{S}(\star, \star))$$

□

Theorem 3.5. *For any $q \leq n$ the complex*

$$0 \leftarrow S(n-q, q) \xleftarrow{\partial} \dots \xleftarrow{\partial} S(n-1, q) \xleftarrow{\partial} S(n, q)$$

is acyclic up to dimension $n-1$.

Proof. Consider some fixed ordered set $V = \{v_1, \dots, v_q\}$ of linearly independent vectors. Let $C_V(p)$ be the submodule of $S(p, q)$ generated by matrices

$$(v_{\sigma^{-1}(1)}, v_{\sigma^{-1}(2)}, \dots, v_{\sigma^{-1}(q)}),$$

where $v_j \in V$ and $\sigma \in \Sigma_q$. Since the face operators ∂_j change only the order of columns in a matrix, we can rewrite the complex $S(\star, q)$ in the form

$$S(\star, q) \simeq \coprod_{\substack{V=\{v_1, v_2, \dots, v_q\} \\ v_i \text{ linearly independent}}} C_V(\star).$$

All summands on the right-hand side are isomorphic to each other. It's sufficient to verify that the complex $C_V(\star)$ is acyclic for some set V . Let $T^q(p)$ be a set of all ordered subsets of $\{1, \dots, q\}$ having cardinality p . Consider the complex

$$D_\star^q = \left(\mathbb{A}[T^q(q)] \xrightarrow{d} \mathbb{A}[T^q(q-1)] \xrightarrow{d} \dots \xrightarrow{d} \mathbb{A}[T^q(1)] \xrightarrow{d} \mathbb{A} \right),$$

where $d(\{x_1, \dots, x_p\}) = \sum_{i=1}^p (-1)^i \{x_1, \dots, \hat{x}_i, \dots, x_p\}$. Obviously, $C_V(\star)[n-q] \simeq D_\star^q$, where $q = \#(V)$. Let us prove that $H_p D_\star^q = 0$ for $p < q$. We will make an induction on q . The case $q = 1$ is trivial. Assume that $H_p D_\star^q = 0$ for $p < q \leq k-1$. Consider the following filtration on the complex $D_\star^k$.

Set $F_l D_m^k$ to be a free $\mathbb{A}$ -module generated by all elements $\{x_1, \dots, x_m\} \in T^k(m)$ satisfying a condition: if $1 \in \{x_1, \dots, x_m\}$ then $1 = x_i$ with $1 \leq i \leq l$, in particular, F_0 is generated by all subsets which don't contain 1.

The term $F_{l/l-1} D_m^k$ of factor filtration is generated by sets

$$\{x_1, \dots, x_{l-1}, 1, x_{l+1}, \dots, x_m\}.$$

In the factor complex $F_{l/l-1} D^k$ operators d_i are trivial for $i \leq l$, so we have a canonical decomposition

$$F_{l/l-1} D_\star^k[l] = \coprod_{T^{k-1}(l-1)} D_\star^{k-l}.$$

Using the induction hypophesis, we get $H_p F_{l/l-1} D^k = 0$ for $p < k$. Consider a spectral sequence of the filtered complex converging to the homology of $D^k/F_0 D^k$. We have $E_{lm}^1 = H_{l+m}(F_{l/l-1} D^k)$. But as just was proved it is 0 for $l+m < k$. Consider now a long exact sequence:

$$\dots \rightarrow H_{p+1}(D^k/F_0 D^k) \rightarrow H_p(F_0 D^k) \xrightarrow{\alpha} H_p(D^k) \rightarrow H_p(D^k/F_0 D^k) \rightarrow \dots$$

One checks easily that the inclusion $F_0 D^k \hookrightarrow D^k$ is homotopic to 0 and hence the above map α is trivial. Thus, $H_p(D^k) \hookrightarrow H_p(D^k/F_0 D^k)$ for $p < k$ and since $H_p(D^k/F_0 D^k) = 0$, we conclude that $H_p(D^k) = 0$. This completes the proof of the theorem. $\square$

Lemma 3.6. *Let $x \in S(n, q)$, $0 \leq q \leq n$. If $\partial x = 0$ then $dx = 0$.*

Proof. There is a homomorphism $\alpha : S(n-1, q) \rightarrow S(n, q-1)$

$$\alpha((v_1, v_2, \dots, v_{q-1} | v_q)) = (v_1, v_2, \dots, v_{q-1})$$

satisfying the formula: $\alpha \partial = d$. $\square$

Proposition 3.7.

$$H_\star(GL_n, \bar{S}) = \coprod_{0 \leq j \leq n} H_\star(GL_n, \text{Ker}(S(n, j) \rightarrow S(n-1, j)))[-n-j]$$

Proof. Denote $\text{Ker}(S(n, j) \xrightarrow{\partial} S(n-1, j))$ by $K(j)$. There is a natural map between the complex $0 \leftarrow K(0) \xleftarrow{d} K(1) \xleftarrow{d} \dots \xleftarrow{d} K(n)$ and $\bar{S}[n, 0]$ induced by ∂ . Theorem 3.5 shows this map is a quasiisomorphism. Using lemma 3.6 we see that the differential of the complex $K(\star)$ is trivial. Because of that, $H_\star(GL_n, \bar{S}) = \coprod_{0 \leq j \leq n} H_\star(GL_n, K(j))[-n-j]$. $\square$

Lemma 3.8. *For any $0 \leq p, q \leq n$ and $p + q \geq n$*

$$H_*(GL_n, S(p, q)) \simeq H_*(GL_{n-q})$$

Proof. Since $q \leq n$, the group GL_n acts transitively on the canonical basis of $S(p, q)$. The Shapiro's lemma tells that the homology under consideration equals to the homology of the stabiliser of an arbitrary basis element. But the stabiliser of the typical element $(e_{n-q+1}, e_{n-q+2}, \dots, e_n)$ has a form

$$\text{Stab} = \begin{pmatrix} GL_{n-q} & 0 & 0 \\ \star & I_{p+q-n} & 0 \\ \star & \star & \hat{\Sigma}_{n-p} \end{pmatrix},$$

where $\hat{\Sigma}_{n-p} \subset GL_{n-p}$ is a subgroup of permutation matrices. Using theorem 1.11[5] and the fact that the group $\hat{\Sigma}_{n-p}$ doesn't have homology except H_0 we obtain: $H_*(\text{Stab}) \simeq H_*(GL_{n-q})$. $\square$

Lemma 3.9. *Let $0 \leq p, q \leq n$ and $p + q \geq n$. Then the map $\partial_* : H_*(GL_{n-q}) \simeq H_*(GL_n, S(p, q)) \rightarrow H_*(GL_n, S(p-1, q)) \simeq H_*(GL_{n-q})$ is trivial if $p + q - n$ is even and coincides with the identity map otherwise.*

Proof. The map ∂_* can be rewritten as follows: $\partial_* = \sum (-1)^i \partial_{i*}$, where ∂_{i*} is the map between homology groups induced by the i -th face operator. It's sufficient to prove that ∂_{i*} is the identity map. Consider the category of pairs (group, module). The isomorphism given in the previous lemma is induced by the morphism (i, u) , where i is the canonical imbedding and $u(1) = (e_{n-q+1}, \dots, e_n)_{\Sigma_{n-p}}$. The map ∂_i acts on the reper $(e_{n-q+1}, \dots, e_n)$ as some permutation matrix α . This matrix commutes with $i(GL_{n-q})$. Therefore, we have $(\alpha i \alpha^{-1}, \alpha u) = (i, \alpha u)$. But inner automorphisms act trivially in homology so we get an equality $(i, u)_* = (i, \alpha u)_*$ which completes the proof. $\square$

Proposition 3.10. *For any $0 \leq j \leq n$ we have*

$$H_k(GL_n, K(j)) \simeq \begin{cases} 0, & j \text{ is odd} \\ H_k(GL_{n-j}), & \text{otherwise.} \end{cases}$$

Proof. Let's consider two hyperhomology spectral sequences of GL_n with coefficients in $S(\star, j)$. The first one has the form

$$E_{lk}^2 = E_{lk}^\infty = \begin{cases} H_k(GL_n, K(j)), & l = n \\ 0, & \text{otherwise.} \end{cases}$$

The second one has the first term $\tilde{E}_{pr}^1 = H_p(GL_n, S(r, j)) \simeq H_p(GL_{n-j})$ and the differential d^1 has the form (Lemma 3.9)

$$d^1 = \begin{cases} 0, & \text{if } r + j - n \text{ is even} \\ \text{id}, & \text{otherwise.} \end{cases}$$

Therefore,

$$\tilde{E}_{pr}^\infty = \tilde{E}_{pr}^2 = \begin{cases} H_p(GL_n, S(n, j)) \simeq H_p(GL_{n-j}), & \text{if } j \text{ is even and } r = n \\ 0, & \text{otherwise.} \end{cases}$$

□

Proof of theorem 1.4. Consider the following chain of isomorphisms

$$\coprod_{\substack{0 \leq i \leq n \\ i \text{ is even}}} H_{k-i}(GL_{n-i}) \xrightarrow{(\text{prop. 3.10})} \coprod_{0 \leq j \leq n} H_{k-j}(GL_n, K(j)) \xrightarrow{(\text{prop. 3.7})}$$

$$H_{n+k}(GL_n, \bar{S}(\star, \star)) \xrightarrow{(\text{prop. 3.4})} H_{n+k+1}(GL_n, S(\star, \star)) \xrightarrow{(\text{prop. 2.5})} H_{k+1}(G^n(\star, \star))$$

□

Now we consider the behavior of truncated bi-Grassmannian complexes for different values of n . (Up to the end of the paper any object related to the complex G^n will be supplied by superscript n .)

Theorem 3.11. *For any $n \geq 0$ the following diagram commutes.*

$$\begin{array}{ccc} \coprod_{0 \leq i \leq n/2} H_{k-2i}(GL_{n-2i}) & \xrightarrow{\sim} & H_{k+1}(G^n(\star, \star)) \\ \downarrow & & \downarrow \\ \coprod_{0 \leq i \leq (n+1)/2} H_{k-2i}(GL_{n+1-2i}) & \xrightarrow{\sim} & H_{k+1}(G^{n+1}(\star, \star)) \end{array}$$

Here the right vertical arrow is induced by the embedding of bi-Grassmannian complexes and the left one by the natural embeddings $GL_{n-2i} \hookrightarrow GL_{n+1-2i}$.

Proof. The Shapiro's lemma implies that the map induced by the inclusion

$$\lambda : K^n(n-k) \rightarrow K^{n+1}(n-k) \quad (\lambda(A) = \begin{pmatrix} A \\ 0 \end{pmatrix}) \text{ makes the diagram}$$

$$\begin{array}{ccc} H_i(GL_k) & \longrightarrow & H_i(GL_n, K^n(n-k)) \\ \downarrow & & \downarrow \lambda \\ H_i(GL_{k+1}) & \longrightarrow & H_i(GL_{n+1}, K^{n+1}(n-k)) \end{array}$$

commutative. The map λ can be extended to the map of bicomplexes $\bar{S}^n \xrightarrow{\lambda} \bar{S}^{n+1}[1, 0]$. Consider another map of these bicomplexes $\bar{S}^n \xrightarrow{\mu} \bar{S}^{n+1}[0, 1]$ given as follows. For $A \in S^n(p, q)$ set

$$\mu(A) = \begin{pmatrix} A & 0 \\ & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix} \in S^{n+1}(p, q+1).$$

One can easily see that the map $s : \bar{S}^n \rightarrow \bar{S}^{n+1}[1, 1]$

$$A \mapsto \left(\begin{array}{cccc|ccc} a_{11} & \cdots & a_{1,p+q-n} & 0 & a_{1,p+q-n+1} & \cdots & a_{1,q} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,p+q-n} & 0 & a_{n,p+q-n+1} & \cdots & a_{n,q} \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{array} \right) \in S^{n+1}(p+1, q+1)$$

gives a chain homotopy between λ and μ .

The map μ can be extended to the map of resolutions $S^n \xrightarrow{\mu} S^{n+1}[0, 1]$ and finally we have a commutative diagram of bicomplexes:

$$\begin{array}{ccc} GL_n S^n[0, n] & \longrightarrow & G^n \\ \mu \downarrow & & \downarrow \\ GL_{n+1} S^{n+1}[0, n+1] & \longrightarrow & G^{n+1} \end{array}$$

whose rows are quasiisomorphisms. (See prop. 2.5.) $\square$

Theorem 3.11 shows that we may really pass to direct limits on n and deduce corollary 1.5 from theorem 1.4.

As an immediate application of theorem 3.11 we can also obtain the following case of the Stabilisation theorem for the linear groups (see [3]). Let's assume that $(n+1)!$ is invertible in the coefficient ring $\mathbb{A}$. Consider the natural embedding $G^n \hookrightarrow G^{n+1}$. This map gives us an isomorphism in homology groups up to degree $n-1$

$$H_k(G^n, \mathbb{A}) \simeq H_k(G^{n+1}, \mathbb{A}) \quad (k < n)$$

and an epimorphism

$$H_n(G^n, \mathbb{A}) \twoheadrightarrow H_n(G^{n+1}, \mathbb{A})$$

Corollary 3.12. *The map $H_k(GL_n, \mathbb{A}) \rightarrow H_k(GL_{n+1}, \mathbb{A})$ induced by the natural embedding $GL_n \hookrightarrow GL_{n+1}$ is an isomorphism if $k < n$ and epimorphism if $k = n$, provided that $(n+1)!$ is invertible in the coefficient ring.*

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