Chapter 7
Duality

We begin this chapter with some general results on duality in a tensor category, followed by some general results on duality in certain triangulated categories. We then give our main application, showing that the full sub-category $\mathcal{DM}(S)^{\rm pr}$ of $\mathcal{DM}(S)$, gotten by taking the pseudo-abelian hull of the sub-category generated by the motives of projective S-schemes in $\mathbf{Sm}_S$, admits a duality involution. In particular, if $S = \mathrm{Spec}(k)$, and if one has resolution of singularities for k-varieties, then the category $\mathcal{DM}(k)$ has a duality involution, making $\mathcal{DM}(k)$ a rigid triangulated tensor category.

We then give some applications of the duality involution: in (7.4.4)-(7.4.6) we show that, in case the base scheme S is Spec of a field of characteristic zero, the motivic category $\mathcal{DM}(S)$ can be constructed from the "naive" version $\mathcal{A}_{mot}(\mathbf{Sm}_S)^0$, i.e., we may replace all the homotopy identities in the construction of the motivic DG tensor category $\mathcal{A}_{mot}(\mathbf{Sm}_S)$ with strict identities. Combining this with (3.2.6), we arrive at a construction of $\mathbf{D}^b_{mot}(S)$ as a localization of the homotopy category of the usual category of complexes in the additive category $\mathcal{A}_{mot}(\mathbf{Sm}_S)^{*0}$. The tensor structure is induced by the product in the category $\mathcal{L}(\mathbf{Sm}_S)$, similar to the classical Grothendieck construction.

In §7.5, we show how the morphisms in $\mathcal{DM}(S)^{\rm pr}$ can be realized as correspondences; for $S = \mathrm{Spec}(k)$, this allows us to embed the category of R-Chow motives into $\mathcal{DM}(k)_R$. In §7.6, we define the "motive with compact supports", $\mathbb{Z}_X^{c/S}$, as well as motivic homology, motivic compactly supported cohomology and motivic Borel-Moore homology, and verify the standard properties of these theories. In §7.7, we define relative motives; for a smooth S-scheme X which admits a compactification $\bar{X}$ which is smooth over S and has a normal crossing complement $\bar{X}\setminus X = D_1 \cup \ldots \cup D_n$, we identify the motive of X with compact supports as the motive of $\bar{X}$ relative to $D_1, \ldots, D_n$.

In §7.8, we identify the Borel-Moore motive as the motive with supports in a closed subset, and use this to extend the Borel-Moore motive to "smoothly decomposable" S-schemes (7.8.2)(i). Similarly, we extend the definition of the motive with compact supports to S-schemes which admit a "compactifiable closed embedding" into a smooth quasi-projective S scheme (7.8.1)(ii). We show that the resulting motivic cohomology with compact supports/motivic Borel-Moore homology have the usual properties of classical cohomology with compact supports/Borel-Moore homology.

For $S = \mathrm{Spec}(k)$, where k is a field, every quasi-projective k-scheme is smoothly decomposable; if resolution of singularities holds for quasi-projective k-schemes, then every quasi-projective k-scheme admits a compactifiable embedding, so our construction gives a Borel-Moore motive, and a motive with with compact supports for arbitrary quasi-projective k-schemes, as well as the resulting motivic Borel-Moore homology and motivic cohomology with compact supports.

Finally, in §7.9, we define the Tate motivic category, and discuss some of its basic properties.

We assume in this Chapter that the base scheme S is quasi-projective over a Noetherian ring A.

7.1. Duality in tensor categories

We recall in this section some basic facts about duality in tensor categories. This material appears somewhat different form in [D] and in [S]; we give the treatment here mainly to fix notation and to keep our presentation self-contained.

(7.1.1)

Let $\mathcal{A}$ be an tensor category, X and X' objects of $\mathcal{A}$, $\iota\colon 1 \to X \otimes X'$ a morphism. For objects A and B of $\mathcal{A}$, we have the homomorphisms

$$\begin{aligned} \iota'(A,B)&\colon \mathrm{Hom}_{\mathcal{A}}(X' \otimes A, B) \to \mathrm{Hom}_{\mathcal{A}}(A, X \otimes B) \\ \iota''(A,B)&\colon \mathrm{Hom}_{\mathcal{A}}(A \otimes X, B) \to \mathrm{Hom}_{\mathcal{A}}(A, B \otimes X'), \end{aligned} \tag{7.1.1.1}$$

where $\iota'(A,B)(f)$ is the composition

$$A \cong 1 \otimes A \overset{\iota\otimes\mathrm{id}_f}{\longrightarrow} X \otimes X' \otimes A \overset{\mathrm{id}_X\otimes f}{\longrightarrow} X \otimes B,$$

and $\iota''(A,B)(g)$ is the composition

$$A \cong A \otimes 1 \overset{\mathrm{id}_A\otimes\iota}{\longrightarrow} A \otimes X \otimes X' \overset{g\otimes\mathrm{id}_X}{\longrightarrow} B \otimes X'.$$

In case $A = 1$, or $B = 1$, we will often make the identifications

$$X \otimes 1 \cong 1 \otimes X \cong X; \qquad X' \otimes 1 \cong 1 \otimes X' \cong X'$$

giving the maps

$$\begin{aligned} \iota'(1,B)&\colon \mathrm{Hom}_{\mathcal{A}}(X', B) \to \mathrm{Hom}_{\mathcal{A}}(1, X \otimes B) \\ \iota'(A,1)&\colon \mathrm{Hom}_{\mathcal{A}}(X' \otimes A, 1) \to \mathrm{Hom}_{\mathcal{A}}(A, X) \\ \iota'(1,1)&\colon \mathrm{Hom}_{\mathcal{A}}(X', 1) \to \mathrm{Hom}_{\mathcal{A}}(1, X), \end{aligned}$$

and similarly for ι''.

Clearly, the maps (7.1.1.1) define natural transformations

$$\begin{aligned} \iota'&\colon \mathrm{Hom}_{\mathcal{A}}(X' \otimes ?_1, ?_2) \to \mathrm{Hom}_{\mathcal{A}}(?_1, X \otimes ?_2) \\ \iota''&\colon \mathrm{Hom}_{\mathcal{A}}(?_1 \otimes X, ?_2) \to \mathrm{Hom}_{\mathcal{A}}(?_1, ?_2 \otimes X'), \end{aligned} \tag{7.1.1.2}$$

of the functors

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{A}}(X' \otimes ?_1, ?_2), \mathrm{Hom}_{\mathcal{A}}(?_1, X \otimes ?_2)\colon \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \to \mathcal{A}, \\ &\mathrm{Hom}_{\mathcal{A}}(?_1 \otimes X, ?_2), \mathrm{Hom}_{\mathcal{A}}(?_1, ?_2 \otimes X')\colon \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \to \mathcal{A}. \end{aligned}$$

(7.1.2) DEFINITION

Let X be an object of $\mathcal{A}$. A *dual* to X is a pair (X^D, ι_X), with X^D an object of $\mathcal{A}$, and $\iota_X: 1 \to X \otimes X^D$ a morphism, such that the natural transformations (7.1.1.2) are isomorphisms. □

Clearly, the relation of duality is symmetric: if (X^D, ι_X) is a dual to X, then (X, ι_{X^D}) is a dual to X^D, where $\iota_{X^D} = \tau_{X,X^D} \circ \iota_X$.

(7.1.3) LEMMA

Let X be an object of $\mathcal{A}$, (X^D, ι_X) and (X^{*D}, ι_X^*) two duals to X. Then there is a unique morphism

$$f: X^{*D} \to X^D$$

such that $(\mathrm{id}_X \otimes f)(\iota_X^*) = \iota_X$. In addition, f is an isomorphism.

Proof. We have the isomorphisms

$$\iota'_X := \iota'_X(1, X^D): \mathrm{Hom}_{\mathcal{A}}(X^D, X^D) \to \mathrm{Hom}_{\mathcal{A}}(1, X \otimes X^D)$$
$$\iota^{*\prime}_X := \iota^{*\prime}_X(1, X^D): \mathrm{Hom}_{\mathcal{A}}(X^{*D}, X^D) \to \mathrm{Hom}_{\mathcal{A}}(1, X \otimes X^D).$$

Letting $f = (\iota^{*\prime}_X)^{-1}(\iota_X)$ gives the desired morphism $f: X^{*D} \to X^D$.

If $g: X^{*D} \to X^D$ satisfies $(\mathrm{id}_X \otimes g)(\iota_X^*) = \iota_X$, then

$$\iota^{*\prime}_X(g) = \iota_X = \iota^{*\prime}_X(f);$$

since $\iota^{*\prime}_X$ is an isomorphism, we have $g = f$, hence f is unique.

By symmetry, there is an $h: X^D \to X^{*D}$ such that $(\mathrm{id}_X \otimes h)(\iota_X) = \iota_X^*$, hence

$$(\mathrm{id}_X \otimes h \circ f)(\iota_X^*) = \iota_X^*; \quad (\mathrm{id}_X \otimes f \circ h)(\iota_X) = \iota_X.$$

By the uniqueness just proven, we have $h = f^{-1}$, hence f is an isomorphism. □

(7.1.4)

By (7.1.3), we may speak of *the* dual (X^D, ι_X) to X.

If X and Y are objects of $\mathcal{A}$, with duals (X^D, ι_X), (Y^D, ι_Y), we have the isomorphism

$$(-)^D: \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(Y^D, X^D) \tag{7.1.4.1}$$

given as the composition

$$\mathrm{Hom}_{\mathcal{A}}(X, Y) \xrightarrow{\iota''_X} \mathrm{Hom}_{\mathcal{A}}(1, Y \otimes X^D) \xrightarrow{(\iota'_Y)^{-1}} \mathrm{Hom}_{\mathcal{A}}(Y^D, X^D).$$

(7.1.5) LEMMA

i) If X, Y and Z are objects of $\mathcal{A}$, with duals (X^D, ι_X), (Y^D, ι_Y) and (Z^D, ι_Z), and $f\colon Y \to Z$, $g\colon X \to Y$ are morphisms, then

$$(f \circ g)^D = g^D \circ f^D.$$

The dual of the identity map id_X is id_{X^D}.

ii) Let (X^{*D}, ι_X^*), (Y^{*D}, ι_Y^*) be another choice for the duals of X and Y, $F\colon X^{*D} \to X^D$, $G\colon Y^{*D} \to Y^D$ the canonical isomorphisms. Let

$$^{*D}\colon \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(Y^{*D}, X^{*D})$$

be the isomorphism (7.1.4.1), taken with respect to the duals (X^{*D}, ι_X^*) and (Y^{*D}, ι_Y^*). Then, for $f\colon X \to Y$, we have

$$F \circ f^{*D} = f^D \circ G.$$

iii) Let $f\colon X \to Y$ be a map in $\mathcal{A}$. Take $(X, \tau_{X,X^D} \circ \iota_X)$, $(Y, \tau_{Y,Y^D} \circ \iota_Y)$ for duals to X^D and Y^D. Then

$$(f^D)^D = f.$$

iv) Let $f\colon X \to Y$ be a morphism in $\mathcal{A}$, and take duals as in (iii). Let A and B be in $\mathcal{A}$. Then the diagrams

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(X^D \otimes A, B) & \xrightarrow{\iota'_X(A,B)} & \mathrm{Hom}_{\mathcal{A}}(A, X \otimes B) \\ {\scriptstyle (f^D \otimes \mathrm{id}_A)^*}\downarrow & & \downarrow{\scriptstyle (f \otimes \mathrm{id}_B)_*} \\ \mathrm{Hom}_{\mathcal{A}}(Y^D \otimes A, B) & \xrightarrow[\iota'_Y(A,B)]{} & \mathrm{Hom}_{\mathcal{A}}(A, Y \otimes B) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(A \otimes Y, B) & \xrightarrow{\iota''_Y(A,B)} & \mathrm{Hom}_{\mathcal{A}}(A, B \otimes Y^D) \\ {\scriptstyle (\mathrm{id}_A \otimes f)^*}\downarrow & & \downarrow{\scriptstyle (\mathrm{id}_B \otimes f^D)^*} \\ \mathrm{Hom}_{\mathcal{A}}(A \otimes X, B) & \xrightarrow[\iota''_X(A,B)]{} & \mathrm{Hom}_{\mathcal{A}}(A, B \otimes X^D) \end{array}$$

commute.

Proof. All four assertions follow easily from the definitions; we leave the details to the reader. □

(7.1.6) THEOREM

Let $\mathcal{A}$ be a tensor category. Suppose each object X of $\mathcal{A}$ has a dual (X^D, ι_X).

i) sending X to X^D, $f\colon X \to Y$ to $f^D\colon Y^D \to X^D$ defines a functor (of additive categories)

$$(-)^D\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{A}.$$

ii) the functor $(-)^D$ is independent, up to natural isomorphism, of the choice of duals.

iii) suppose we have $1^D = 1$, *with* $\iota_1 : 1 \to 1 \otimes 1$ *the inverse to the multiplication* $\mu : 1 \otimes 1 \to 1$. *Then the functor* $(-)^D$ *is a pseudo tensor functor, i.e.:*

a) for each pair of objects X, Y *of* $\mathcal{A}$, *there is an isomorphism*

$$\rho_{X,Y} : (X \otimes Y)^D \to X^D \otimes Y^D$$

such that

$$(\rho_{X,Y} \otimes \mathrm{id}_Z) \circ \rho_{X \otimes Y, Z} = (\mathrm{id}_X \otimes \rho_{Y,Z}) \circ \rho_{X, Y \otimes Z}.$$

b) the maps $\rho_{X,Y}$ *intertwine the symmetry isomorphisms* $\tau^D_{X,Y}$ *and* τ_{X^D,Y^D}, *and the maps* $\rho_{1,X}$ *(resp.* $\rho_{X,1}$*) intertwine the multiplication isomorphisms* $\mu^D_{X,l}$ *(resp.* $\mu^D_{X,r}$*) and* $\mu_{X^D,l}$ *(resp.* $\mu_{X^D,r}$*).*

iv) there is a natural isomorphism

$$\mathrm{id}_{\mathcal{A}} \to ((-)^D)^D.$$

Proof. (i) follows directly from (7.1.5)(i); (ii) follows from (7.1.3) and (7.1.5)(ii). (iv) follows from (7.1.3) and (7.1.5)(iii). For (iii), we note that

$$(X^D \otimes Y^D, \iota^*_{X \otimes Y} = \tau \circ (\iota_X \otimes \iota_Y))$$

is a dual to $X \otimes Y$, where (X^D, ι_X) is a dual to X, (Y^D, ι_Y) is a dual to Y,

$$\tau : X \otimes X^D \otimes Y \otimes Y^D \to X \otimes Y \otimes X^D \otimes Y^D$$

is the permutation isomorphism. Indeed, for objects A and B of $\mathcal{A}$, the maps

$$\iota'_Y(A \otimes X^D, B) : \mathrm{Hom}_{\mathcal{A}}(A \otimes X^D \otimes Y^D, B) \to \mathrm{Hom}_{\mathcal{A}}(A \otimes X^D, B \otimes Y)$$
$$\iota'_X(A, B \otimes Y) : \mathrm{Hom}_{\mathcal{A}}(A \otimes X^D, B \otimes Y) \to \mathrm{Hom}_{\mathcal{A}}(A, B \otimes Y \otimes X)$$

are isomorphisms. This implies that $\iota^{*\prime}_{X \otimes Y}(A, B)$ is an isomorphism. Similarly, $\iota^{*\prime\prime}_{X \otimes Y}(A, B)$ is an isomorphism. Via (7.1.3), we have the canonical isomorphism

$$\rho_{X,Y} : (X \otimes Y)^D \to X^D \otimes Y^D.$$

The relation of (iii) follows from the uniqueness portion of (7.1.3). □

(7.1.7) REMARK

If $\mathcal{A}$ is a *graded* tensor category, and if $X \in \mathcal{A}$ has a dual (X^D, ι), then $(X^D[-1], \iota')$ is a dual to $X[1]$, where

$$\iota' : 1 \to X[1] \otimes X^D[-1]$$

is the image of ι under composition with the canonical isomorphism

$$X \otimes X^D \to X[1] \otimes X^D[-1].$$

We call this choice of dual on $X[1]$ the *canonical* dual.

If we have a morphism $f: X \to Y$, where X and Y have duals (X^D, ι_X), (Y^D, ι_Y), and if we make the canonical choice of dual for $X[1]$ and $Y[1]$, we have the identity

$$f[1]^D = f^D[-1].$$

Thus, if $\mathcal{A}$ is a graded tensor category such that each object has a translated which has a dual, then each object of $\mathcal{A}$ has a dual, and, we may assume that the dual of $X[1]$ is the canonical dual $(X^D[-1], \iota')$ for each X in $\mathcal{A}$. In this case, (7.1.6) extends to a graded version, in which the duality functor of (i) is a graded functor. □

(7.1.8) The trace map

We now give a criterion for a given morphism $\iota: 1 \to X \otimes X^D$ to give a dual (X^D, ι) to X. We prove a somewhat more general statement, for later use.

Let $\iota: 1 \to X \otimes X^D$ be a morphism in $\mathcal{A}$, $F: \mathcal{A} \to \mathcal{B}$ a tensor functor, and let A and B be objects of $\mathcal{B}$. Define the map

$$\iota'_F(A, B): \mathrm{Hom}_{\mathcal{B}}(F(X^D) \otimes A, B) \to \mathrm{Hom}_{\mathcal{B}}(A, F(X) \otimes B) \tag{7.1.8.1}$$

by setting $\iota'_F(A, B)(f)$ equal to the composition

$$\begin{aligned} A \cong 1 \otimes A &\xrightarrow{F(\iota)\otimes \mathrm{id}_A} F(X \otimes X^D) \otimes A = F(X) \otimes F(X^D) \otimes A \\ &\xrightarrow{\mathrm{id}_{F(X)}\otimes f} F(X) \otimes B. \end{aligned}$$

Define the map

$$\iota''_F(A, B): \mathrm{Hom}_{\mathcal{B}}(A \otimes F(X), B) \to \mathrm{Hom}_{\mathcal{B}}(A, B \otimes F(X^D)) \tag{7.1.8.2}$$

similarly by setting $\iota''_F(A, B)(f)$ equal to the composition

$$\begin{aligned} A \cong A \otimes 1 &\xrightarrow{\mathrm{id}_A \otimes F(\iota)} A \otimes F(X \otimes X^D) = A \otimes F(X) \otimes F(X^D) \\ &\xrightarrow{f \otimes \mathrm{id}_{F(X^D)}} B \otimes F(X^D). \end{aligned}$$

If F is the identity functor on $\mathcal{A}$, these maps are just the maps ι' and ι'' defined in (7.1.1.1).

(7.1.9) Proposition

Suppose there is a map

$$\epsilon : X^D \otimes X \to 1$$

in $\mathcal{A}$ such that the compositions

$$\begin{aligned} &X \cong 1 \otimes X \xrightarrow{\iota\otimes \mathrm{id}_X} X \otimes X^D \otimes X \xrightarrow{\mathrm{id}_X \otimes \epsilon} X \otimes 1 \cong X \\ &X^D \cong X^D \otimes 1 \xrightarrow{\mathrm{id}_{X^D}\otimes \iota} X^D \otimes X \otimes X^D \xrightarrow{\epsilon \otimes \mathrm{id}_{X^D}} 1 \otimes X^D \cong X^D \end{aligned} \tag{7.1.9.1}$$

are the respective identity maps. Then the maps (7.1.8.1) and (7.1.8.2) are isomorphisms for all A and B in $\mathcal{B}$. In particular, (X^D, ι) is a dual to X.

Proof. Fix A and B in $\mathcal{B}$, and write ι'_F for $\iota'_F(A,B)$. Define the map

$$\sigma'_F : \mathrm{Hom}_{\mathcal{A}}(A, F(X) \otimes B) \to \mathrm{Hom}_{\mathcal{A}}(F(X^D) \otimes A, B)$$

by sending $g: A \to F(X) \otimes B$ to the composition

$$\begin{aligned} F(X^D) \otimes A &\xrightarrow{\mathrm{id}_{F(X^D)} \otimes g} F(X^D) \otimes F(X) \otimes B = F(X^D \otimes X) \otimes B \\ &\xrightarrow{F(\epsilon_X)\otimes \mathrm{id}_B} 1 \otimes B \cong B. \end{aligned}$$

Let $f: F(X^D) \otimes A \to B$ be a morphism in $\mathcal{B}$. Then $\sigma'_F(\iota'_F(f))$ is given by the composition

$$\begin{aligned} F(X^D) \otimes A \cong F(X^D) \otimes 1 \otimes A &\xrightarrow{\mathrm{id}_{F(X^D)} \otimes F(\iota) \otimes \mathrm{id}_A} F(X^D) \otimes F(X \otimes X^D) \otimes A \\ &= F(X^D \otimes X) \otimes F(X^D) \otimes A \xrightarrow{\mathrm{id} \otimes f} F(X^D \otimes X) \otimes B \\ &\xrightarrow{F(\epsilon) \otimes \mathrm{id}_B} B \otimes 1 \cong B. \end{aligned}$$

We may commute $\mathrm{id} \otimes f$ and $F(\epsilon) \otimes \mathrm{id}$, hence $\sigma'_F(\iota'_F(f))$ is equal to the composition

$$\begin{aligned} F(X^D) \otimes A \cong F(X^D) \otimes 1 \otimes A &= F(X^D \otimes 1) \otimes A \\ &\xrightarrow{F(\mathrm{id}_{X^D} \otimes \iota) \otimes \mathrm{id}_A} F(X^D \otimes X \otimes X^D) \otimes A \\ &\xrightarrow{F(\epsilon \otimes \mathrm{id}_{X^D}) \otimes \mathrm{id}_A} F(1 \otimes X^D) \otimes A \\ &= 1 \otimes (F(X^D) \otimes A) \xrightarrow{\mathrm{id} \otimes f} 1 \otimes B \cong B \end{aligned}$$

By assumption, the composition

$$F(\epsilon \otimes \mathrm{id}_{X^D}) \circ F(\mathrm{id}_{X^D} \otimes \iota)$$

is the canonical isomorphism

$$F(X^D \otimes 1) \to F(1 \otimes X^D).$$

Since 1 is the unit in $\mathcal{A}$, we have

$$\sigma'_F(\iota'_F(f)) = f.$$

Now let $g\colon A \to F(X) \otimes B$ be a map in $\mathcal{B}$. Then $\iota'_F(\sigma_F(g))$ is given by the composition

$$\begin{aligned} A \cong 1 \otimes A &\xrightarrow{F(\iota)\otimes \mathrm{id}_A} F(X \otimes X^D) \otimes A \\ &\xrightarrow{\mathrm{id}\otimes g} F(X \otimes X^D) \otimes F(X) \otimes B = F(X) \otimes F(X^D \otimes X) \otimes B \\ &\xrightarrow{\mathrm{id}\otimes F(\epsilon)\otimes \mathrm{id}} F(X) \otimes 1 \otimes B \cong F(X) \otimes B. \end{aligned}$$

We may commute $\mathrm{id} \otimes g$ and $F(\iota) \otimes \mathrm{id}_A$,and rewrite this as the composition

$$\begin{aligned} A \cong 1 \otimes A &\xrightarrow{\mathrm{id}\otimes g} 1 \otimes F(X) \otimes B = F(1 \otimes X) \otimes B \\ &\xrightarrow{F(\iota\otimes \mathrm{id}_X)\otimes \mathrm{id}_B} F(X \otimes X^D \otimes X) \otimes B \xrightarrow{F(\mathrm{id}_X\otimes\epsilon)\otimes \mathrm{id}_B} F(X \otimes 1) \otimes B \\ &= F(X) \otimes 1 \otimes B \cong F(X) \otimes B. \end{aligned}$$

By assumption, the composition

$$F(\mathrm{id}_X \otimes \epsilon) \circ F(\iota \otimes \mathrm{id}_X)$$

is the canonical isomorphism

$$F(1 \otimes X) \to F(X \otimes 1).$$

Since 1 is the unit in $\mathcal{A}$, we have

$$\iota'_F(\sigma'_F(g)) = g.$$

Thus σ'_F is the inverse to ι'_F, hence $\iota'_F = \iota'_F(A, B)$ is an isomorphism. The proof that $\iota''_F(A, B)$ is an isomorphism is essentially the same. □

There is a converse to (7.1.9), namely,

(7.1.10) PROPOSITION

Suppose (X^D, ι) is a dual to X. Then there is a unique map

$$\epsilon\colon X^D \otimes X \to 1$$

such that the compositions (7.1.9.1) are the respective identity maps.

Proof. If such an ϵ exists, then the first composition in (7.1.9.1) is $\iota'(X,X)(\epsilon)$, after making the canonical identification of $X \otimes 1$ with X. Similarly, the second composition in (7.1.9.1) is $\iota''(X^D, X^D)(\epsilon)$, after making a similar identification. Since $\iota'(X,X)$ is an isomorphism, ϵ is unique.

To show existence, we use the canonical structure for the dual of $X \otimes X^D$, i.e.,

$$(X \otimes X^D)^D = X^D \otimes X,$$

with map

$$\iota_{X \otimes X^D} \colon 1 \to X \otimes X^D \otimes X^D \otimes X$$

being the composition

$$\begin{aligned} 1 \cong 1 \otimes 1 &\xrightarrow{\iota \otimes \iota} X \otimes X^D \otimes X \otimes X^D \xrightarrow{\tau_{3,4}} X \otimes X^D \otimes X^D \otimes X \\ &\xrightarrow{\tau_{2,3}} X \otimes X^D \otimes X^D \otimes X. \end{aligned} \tag{1}$$

Taking the dual of ι gives the map

$$\iota^D \colon X^D \otimes X \to 1;$$

we claim that $\epsilon = \iota^D$ is the desired map. It suffices to show that $\iota'(X,X)(\iota^D)$ is the identity on X and that $\iota''(X^D, X^D)(\iota^D)$ is the identity on X^D, after making the identifications as above.

We use the canonical dual for 1: $1^D = 1$ with $\iota_1 \colon 1 \to 1 \otimes 1$ the inverse of the multiplication. By the definition of duality, ι^D is characterized by the fact that the composition

$$\begin{aligned} 1 &\xrightarrow{\iota_{X \otimes X^D}} X \otimes X^D \otimes X^D \otimes X \\ &\xrightarrow{\mathrm{id} \otimes \iota^D} X \otimes X^D \otimes 1 \cong X \otimes X^D \end{aligned}$$

is the map ι. By definition, the map $\iota'(X,X)(\iota^D) \colon X \to X$ is the composition

$$X \cong 1 \otimes X \xrightarrow{\iota \otimes \mathrm{id}_X} X \otimes X^D \otimes X \xrightarrow{\mathrm{id}_X \otimes \iota^D} X \otimes 1 \cong X.$$

From this, it follows that the map $\iota''(1,X)(\iota'(X,X)(\iota^D))$ is the composition

$$\begin{aligned} 1 &\xrightarrow{\iota} X \otimes X^D \cong 1 \otimes X \otimes X^D \xrightarrow{\iota \otimes \mathrm{id}_X \otimes \mathrm{id}_{X^D}} X \otimes X^D \otimes X \otimes X^D \\ &\xrightarrow{\mathrm{id}_X \otimes \iota^D \otimes \mathrm{id}_{X^D}} X \otimes 1 \otimes X^D \cong X \otimes X^D. \end{aligned} \tag{2}$$

Using the above characterization of ι^D, together with the definition (1) of $\iota_{X \otimes X^D}$, it follows easily from (2) that

$$\iota''(1,X)(\iota'(X,X)(\iota^D)) = \iota.$$

Since $\iota''(1,X)(\mathrm{id}_X)=\iota$ as well, and since $\iota''(1,X)$ is an isomorphism, we have

$$\iota'(X,X)(\iota^D)=\mathrm{id}_X.$$

The identity

$$\iota''(X^D,X^D)(\iota^D)=\mathrm{id}_{X^D}$$

is verified similarly, completing the proof. $\square$

The dual maps $\epsilon=\iota^D\colon X^D\otimes X\to 1$ give a description of the composition law in $\mathcal{A}$, as follows:

(7.1.11) PROPOSITION

Let X and Y be in $\mathcal{A}$, with respective duals (X^D,ι_X), (Y^D,ι_Y). Let $\epsilon_Y\colon Y^D\otimes Y\to 1$ be the map given by (7.1.10). Suppose we have maps

$$f\colon X\to Y;\quad g\colon Y\to Z$$

in $\mathcal{A}$. Then $\iota''_X(1,Z)(g\circ f)\colon 1\to Z\otimes X^D$ is the composition

$$\begin{aligned}1\cong 1\otimes 1&\xrightarrow{\iota''_Y(1,Z)(g)\otimes\iota''_X(1,Y)(f)}Z\otimes Y^D\otimes Y\otimes X^D\\&\xrightarrow{\mathrm{id}_Z\otimes\epsilon_Y\otimes\mathrm{id}_{X^D}}Z\otimes 1\otimes X^D\cong Z\otimes X^D.\end{aligned}\tag{1}$$

Proof. We may expand the composition (1) as

$$\begin{aligned}1\cong 1\otimes 1&\xrightarrow{\mathrm{id}_1\otimes\iota_X}1\otimes X\otimes X^D\xrightarrow{\iota_Y\otimes\mathrm{id}_{X\otimes X^D}}Y\otimes Y^D\otimes X\otimes X^D\\&\xrightarrow{\mathrm{id}_{Y\otimes Y^D}\otimes f\otimes\mathrm{id}_{X^D}}Y\otimes Y^D\otimes Y\otimes X^D\\&\xrightarrow{g\otimes\mathrm{id}_{Y^D}\otimes\mathrm{id}_Y\otimes\mathrm{id}_{X^D}}Z\otimes Y^D\otimes Y\otimes X^D\\&\xrightarrow{\mathrm{id}_Z\otimes\epsilon_Y\otimes\mathrm{id}_{X^D}}Z\otimes 1\otimes X^D\cong Z\otimes X^D.\end{aligned}$$

We may then commute $g\otimes\mathrm{id}_{Y^D}\otimes\mathrm{id}_Y\otimes\mathrm{id}_{X^D}$ with $\mathrm{id}_Z\otimes\epsilon_Y\otimes\mathrm{id}_{X^D}$, and $\iota_Y\otimes\mathrm{id}_{X\otimes X^D}$ with $\mathrm{id}_{Y\otimes Y^D}\otimes f\otimes\mathrm{id}_{X^D}$ to give the composition

$$\begin{aligned}1\cong 1\otimes 1&\xrightarrow{\mathrm{id}_1\otimes\iota_X}1\otimes X\otimes X^D\xrightarrow{\mathrm{id}_1\otimes f\otimes\mathrm{id}_{X^D}}1\otimes Y\otimes X^D\\&\xrightarrow{\iota_Y\otimes\mathrm{id}_Y\otimes\mathrm{id}_{X^D}}Y\otimes Y^D\otimes Y\otimes X^D\xrightarrow{\mathrm{id}_Y\otimes\epsilon_Y\otimes\mathrm{id}_{X^D}}Y\otimes 1\otimes X^D\\&\cong Y\otimes X^D\xrightarrow{g\otimes\mathrm{id}_{X^D}}Z\otimes X^D.\end{aligned}$$

Since $(\mathrm{id}_Y\otimes\epsilon_Y)\circ(\iota_Y\otimes\mathrm{id}_Y)$ is the canonical identification $1\otimes Y\cong Y\otimes 1$, we may rewrite this composition as

$$
\begin{aligned}
1 \cong & 1 \otimes 1 \xrightarrow{\mathrm{id}_1 \otimes \iota_X} 1 \otimes X \otimes X^D \xrightarrow{\mathrm{id}_1 \otimes f \otimes \mathrm{id}_{X^D}} 1 \otimes Y \otimes X^D \\
& \cong Y \otimes X^D \xrightarrow{g \otimes \mathrm{id}_{X^D}} Z \otimes X^D.
\end{aligned}
$$

Eliminating the superfluous 1's gives the composition

$$
1 \xrightarrow{\iota_X} X \otimes X^D \xrightarrow{f \otimes \mathrm{id}_{X^D}} Y \otimes X^D \xrightarrow{g \otimes \mathrm{id}_{X^D}} Z \otimes X^D; \tag{1}
$$

since

$$
(g \otimes \mathrm{id}_{X^D}) \circ (f \otimes \mathrm{id}_{X^D}) = (g \circ f) \otimes \mathrm{id}_{X^D},
$$

the composition (1) is $\iota''_X(1, Z)(g \circ f)$, completing the proof. □

7.2. Duality in triangulated tensor categories

We show how the existence of duals for generating objects in certain triangulated tensor categories gives rise to an exact duality on the full category.

(7.2.1)

Let $\mathcal{A}$ be a DG tensor category without unit. We may then form the category of complexes, $\mathbf{C}^b(\mathcal{A})$ and the homotopy category $\mathbf{K}^b(\mathcal{A})$. The tensor product on $\mathcal{A}$ induces the structure of a DG tensor category without unit on $\mathbf{C}^b(\mathcal{A})$, and the structure of a triangulated tensor category without unit on $\mathbf{K}^b(\mathcal{A})$. We may form a localization $\mathcal{D}$ of $\mathbf{K}^b(\mathcal{A})$ with respect to a thick tensor subcategory; $\mathcal{D}$ is then a triangulated tensor category without unit.

If we have two distinguished triangles in $\mathcal{D}$:

$$
\begin{gathered}
X_1 \to Y_1 \to Z_1 \to X_1[1] \\
Z_2 \to Y_2 \to X_2 \to Z_2[1],
\end{gathered}
$$

we may then form the commutative square

$$
\begin{array}{ccccccc}
X_1 \otimes Z_2 & \to & Y_1 \otimes Z_2 & \to & Z_1 \otimes Z_2 & \to & X_1[1] \otimes Z_2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X_1 \otimes Y_2 & \to & Y_1 \otimes Y_2 & \to & Z_1 \otimes Y_2 & \to & X_1[1] \otimes Y_2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X_1 \otimes X_2 & \to & Y_1 \otimes X_2 & \to & Z_1 \otimes X_2 & \to & X_1[1] \otimes X_2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X_1 \otimes Z_2[1] & \to & Y_1 \otimes Z_2[1] & \to & Z_1 \otimes Z_2[1] & \to & X_1[1] \otimes Z_2[1]
\end{array} \tag{7.2.1.1}
$$

Identifying $X_1[1] \otimes Z_2$ with $X_1 \otimes Z_2[1]$ by the canonical isomorphism, we let

$$Z_1 \otimes Z_2 \oplus X_1 \otimes X_2 \to X_1 \otimes Z_2[1]$$

be the difference of the two maps in (7.2.1.1), and form the distinguished triangle

$$K \xrightarrow{\alpha} Z_1 \otimes Z_2 \oplus X_1 \otimes X_2 \to X_1 \otimes Z_2[1] \to K[1] \tag{7.2.1.2}$$

in $\mathcal{D}$. In addition, the sum of the two maps $Y_1 \otimes Y_2 \to Z_1 \otimes Y_2$, $Y_1 \otimes Y_2 \to Y_1 \otimes X_2$ gives the map

$$Y_1 \otimes Y_2 \to Z_1 \otimes Y_2 \oplus Y_1 \otimes X_2, \tag{7.2.1.3}$$

and the direct sum of the maps $Z_1 \otimes Z_2 \to Z_1 \otimes Y_2$, $X_1 \otimes X_2 \to Y_1 \otimes X_2$ gives the map

$$Z_1 \otimes Z_2 \oplus X_1 \otimes X_2 \to Z_1 \otimes Y_2 \oplus Y_1 \otimes X_2. \tag{7.2.1.4}$$

Putting the maps (7.2.1.3) and (7.2.1.4) together gives the diagram

$$\begin{array}{ccc} & & Z_1 \otimes Z_2 \oplus X_1 \otimes X_2 \\ & & \downarrow \\ Y_1 \otimes Y_2 & \to & Z_1 \otimes Y_2 \oplus Y_1 \otimes X_2 \end{array} \tag{7.2.1.5}$$

(7.2.2) LEMMA

There is a morphism $\beta\colon K \to Y_1 \otimes Y_2$ so that the diagram (7.2.1.5) fills in to a commutative diagram

$$\begin{array}{ccccccc} K & \xrightarrow{\alpha} & Z_1 \otimes Z_2 \oplus X_1 \otimes X_2 & \longrightarrow & X_1 \otimes Z_2[1] & \longrightarrow & K[1] \\ {\scriptstyle\beta}\downarrow & & \downarrow & & & & \\ Y_1 \otimes Y_2 & \longrightarrow & Z_1 \otimes Y_2 \oplus Y_1 \otimes X_2, & & & & \end{array} \tag{7.2.2.1}$$

with the top row the distinguished triangle (7.2.1.2).

Proof. Let X and Y be objects of $\mathcal{D}$, and $f\colon X \to Y$ a morphism in $\mathcal{D}$. As $\mathcal{D}$ is a localization of $\mathbf{K}^b(\mathcal{A})$, the morphism f can be factored as a composition

$$X \xrightarrow{i} Z \xrightarrow{j^{-1}} Y$$

with i and j morphisms in $\mathbf{K}^b(\mathcal{A})$, and j invertible in $\mathcal{D}$.

Thus, it suffices to prove the lemma in the case of standard distinguished triangles from $\mathbf{C}^b(\mathcal{A})$:

$$\begin{gathered} Z_1[-1] \xrightarrow{f_1} X_1 \xrightarrow{g_1} Y_1 = \mathrm{Cone}(f_1) \xrightarrow{h_1} Z_1 \\ Z_2 \xrightarrow{f_2} Y_2 \xrightarrow{g_2} X_2 = \mathrm{Cone}(f_2) \xrightarrow{h_2} Z_2[1] \end{gathered} \tag{1}$$

We may then take K to be given by

$$K := \operatorname{Cone}(f_1[1] \otimes \mathrm{id}_{Z_2} - \mathrm{id}_{X_1} \otimes h_2 \colon Z_1 \otimes Z_2 \oplus X_1 \otimes X_2 \to X_1 \otimes Z_2[1])[-1], \tag{2}$$

and the sequence (7.2.1.2) to be the standard Cone sequence. Let

$$\alpha \colon K \to Z_1 \otimes Z_2 \oplus X_1 \otimes X_2$$

be the canonical map.

The morphisms

$$\mathrm{id}_{X_1} \otimes g_2 \colon X_1 \otimes Y_2 \to X_1 \otimes X_2$$
$$h_1 \otimes \mathrm{id}_{Z_2} \colon Y_1 \otimes Z_2 \to Z_1 \otimes Z_2$$

satisfy

$$(\mathrm{id}_{X_1} \otimes h_2) \circ (\mathrm{id}_{X_1} \otimes g_2) = 0$$
$$(f_1[1] \otimes \mathrm{id}_{Z_2}) \circ (h_1 \otimes \mathrm{id}_{Z_2}) = 0$$

hence the direct sum $h_1 \otimes \mathrm{id}_{Z_2} \oplus \mathrm{id}_{X_1} \otimes g_2$ extends canonically to the map of complexes

$$\gamma \oplus \gamma' \colon Y_1 \otimes Z_2 \oplus X_1 \otimes Y_2 \to K.$$

Let

$$\gamma'' \colon X_1 \otimes Z_2 \to Y_1 \otimes Z_2 \oplus X_1 \otimes Y_2$$

be the map $(g_1 \otimes \mathrm{id}_{Z_2}, -\mathrm{id}_{X_1} \otimes f_2)$. By (2), the sequence

$$X_1 \otimes Z_2 \xrightarrow{\gamma''} Y_1 \otimes Z_2 \oplus X_1 \otimes Y_2 \xrightarrow{\gamma \oplus \gamma'} K \tag{3}$$

identifies K with $\operatorname{Cone}(\gamma'')$.

Let

$$\beta' \colon Y_1 \otimes Z_2 \oplus X_1 \otimes Y_2 \to Y_1 \otimes Y_2$$

be the map

$$\beta' = \mathrm{id}_{Y_1} \otimes f_2 + g_1 \otimes \mathrm{id}_{Y_2}.$$

As

$$\beta' \circ \gamma'' = 0,$$

the identification (3) of K with $\operatorname{Cone}(\gamma'')$ gives the canonical extension of β' to the map

$$\beta \colon K \to Y_1 \otimes Y_2.$$

By construction we have

$$(\mathrm{id}_{Z_1} \otimes f_2 \oplus g_1 \otimes \mathrm{id}_{X_2}) \circ \alpha \circ (\gamma \oplus \gamma') = (h_1 \otimes \mathrm{id}_{Y_2}, \mathrm{id}_{Y_1} \otimes g_2) \circ \beta'$$

which gives the desired commutativity

$$(\mathrm{id}_{Z_1} \otimes f_2 \oplus g_1 \otimes \mathrm{id}_{X_2}) \circ \alpha = (h_1 \otimes \mathrm{id}_{Y_2}, \mathrm{id}_{Y_1} \otimes g_2) \circ \beta.$$

□

We suppose for the remainder of this section that the category $\mathcal{D}$ is a triangulated tensor category with unit 1.

(7.2.3) LEMMA

Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

be a distinguished triangle in $\mathcal{D}$. Suppose that X has a dual (X^D, ι_X), and Z has a dual (Z^D, ι_Z). Then Y has a dual (Y^D, ι_Y). In addition, the sequence

$$X^D[-1] \xrightarrow{h^D} Z^D \xrightarrow{g^D} Y^D \xrightarrow{f^D} X^D \tag{7.2.3.1}$$

is a distinguished triangle in $\mathcal{D}$.

Proof. We use the canonical dual $X[1]^D = X^D[-1]$, with $\iota_{X[1]}$ the image of ι_X under the canonical isomorphism $X \otimes X^D \to X[1] \otimes X^D[-1]$. The map $h: Z \to X[1]$ gives the dual map $h^D: X^D[-1] \to Z^D$; we define Y^D and the maps $i: Z^D \to Y^D$, $j: Y^D \to X^D$ by requiring that the sequence

$$X^D[-1] \xrightarrow{h^D} Z^D \xrightarrow{i} Y^D \xrightarrow{j} X^D \tag{1}$$

be a distinguished triangle in $\mathcal{D}$. Similarly, let

$$p: X \otimes X^D \oplus Z \otimes Z^D \to X \otimes Z^D[1]$$

be the map $\mathrm{id} \otimes h^D[1] - h \otimes \mathrm{id}$, and let

$$K \xrightarrow{q} X \otimes X^D \oplus Z \otimes Z^D \xrightarrow{p} X \otimes Z^D[1] \to K[1]$$

be the extension of p to a distinguished triangle in $\mathcal{D}$.

We identify $Z \otimes Z^D$ with $Z[-1] \otimes Z^D[1] = Z[-1] \otimes Z[-1]^D$ by the canonical isomorphism and let $\iota_{Z[-1]}: 1 \to Z[-1] \otimes Z[-1]^D$ be the map corresponding to ι_Z. Then, by definition, we have

$$(h \otimes 1) \circ \iota_Z = (h[-1] \otimes 1) \circ \iota_{Z[-1]} = \iota''_{Z[-1]}(h[-1]),$$

$$(h^D[1] \otimes 1) \circ \iota_X = (h[-1]^D \otimes 1) \circ \iota_X = \iota'_X(h[-1]^D),$$

$$h[-1]^D = (\iota'_X)^{-1}(\iota''_{Z[-1]}(h[-1])).$$

Thus, we have

$$p \circ (\iota_X, \iota_Z) = 0,$$

hence there is a map

$$\iota_K : 1 \to K \tag{2}$$

with

$$q \circ \iota_K = (\iota_X, \iota_Z). \tag{3}$$

From (7.2.2) we have the commutative diagram (7.2.2.1):

$$\begin{array}{ccc} K & \xrightarrow{\beta} & Y \otimes Y^D \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle (\mathrm{id}\otimes j, g\otimes \mathrm{id})} \\ X \otimes X^D \oplus Z \otimes Z^D & \underset{f\otimes\mathrm{id}\oplus\mathrm{id}\otimes i}{\longrightarrow} & Y \otimes X^D \oplus Z \otimes Y^D; \end{array} \tag{4}$$

let $\iota_Y = \beta \circ \iota_K$. Then we have

$$\begin{aligned} \iota'_Y(j) &= (\mathrm{id} \otimes j) \circ \iota_Y \\ &= (f \otimes \mathrm{id}) \circ \iota_X \\ &= \iota''_X(f), \\ \iota''_Y(g) &= (g \otimes \mathrm{id}) \circ \iota_Y \\ &= (\mathrm{id} \otimes i) \circ \iota_Z \\ &= \iota'_Z(i). \end{aligned} \tag{5}$$

Let A and B be objects of $\mathcal{D}$, and consider the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(X^D \otimes A, B) & \xrightarrow{\iota''_X} & \mathrm{Hom}_{\mathcal{D}}(A, X \otimes B) \\ {\scriptstyle (j\otimes\mathrm{id})^*}\downarrow & & \downarrow{\scriptstyle (f\otimes\mathrm{id})_*} \\ \mathrm{Hom}_{\mathcal{D}}(Y^D \otimes A, B) & \underset{\iota''_Y}{\longrightarrow} & \mathrm{Hom}_{\mathcal{D}}(A, Y \otimes B) \end{array} \tag{6}$$

For a map $\alpha: X^D \otimes A \to B$, the map $(f \otimes \mathrm{id})_*(\iota''_X(\alpha))$ is the composition

$$A \cong 1 \otimes A \xrightarrow{\iota_X \otimes \mathrm{id}_A} X \otimes X^D \otimes A \xrightarrow{\mathrm{id}_X \otimes \alpha} X \otimes B \xrightarrow{f \otimes \mathrm{id}_B} Y \otimes B.$$

We may commute the last two maps in this composition, giving the identity

$$(f \otimes \mathrm{id})_*(\iota''_X(\alpha)) = (\mathrm{id}_Y \otimes \alpha) \circ ([(f \otimes \mathrm{id}_{X^D}) \circ \iota_X] \otimes \mathrm{id}_A).$$

(we ignore the identification of A and $1 \otimes A$). By (5), this gives the identity

$$(f \otimes \mathrm{id})_*(\iota''_X(\alpha)) = (\mathrm{id}_Y \otimes \alpha) \circ ([\mathrm{id}_Y \otimes j \circ \iota_Y] \otimes \mathrm{id}_A),$$

which shows that the diagram (6) commutes.

One shows that the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(A\otimes Y,B) & \xrightarrow{\iota'_Y} & \mathrm{Hom}_{\mathcal{D}}(A,B\otimes Y^D) \\ {\scriptstyle(\mathrm{id}\otimes f)^*}\downarrow & & \downarrow{\scriptstyle(\mathrm{id}\otimes j)_*} \\ \mathrm{Hom}_{\mathcal{D}}(A\otimes X,B) & \xrightarrow[\iota'_X]{} & \mathrm{Hom}_{\mathcal{D}}(A,B\otimes X^D) \end{array} \tag{7}$$

commutes, using a similar argument.

Using the second identity of (5), the same argument gives the commutativity of the diagrams

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(Y^D\otimes A,B) & \xrightarrow{\iota''_Y} & \mathrm{Hom}_{\mathcal{D}}(A,Y\otimes B) \\ {\scriptstyle(i\otimes\mathrm{id})^*}\downarrow & & \downarrow{\scriptstyle(g\otimes\mathrm{id})_*} \\ \mathrm{Hom}_{\mathcal{D}}(Z^D\otimes A,B) & \xrightarrow[\iota''_Z]{} & \mathrm{Hom}_{\mathcal{D}}(A,Z\otimes B) \end{array} \tag{8}$$

and

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(A\otimes Z,B) & \xrightarrow{\iota'_Z} & \mathrm{Hom}_{\mathcal{D}}(A,B\otimes Z^D) \\ {\scriptstyle(\mathrm{id}\otimes g)^*}\downarrow & & \downarrow{\scriptstyle(\mathrm{id}\otimes i)_*} \\ \mathrm{Hom}_{\mathcal{D}}(A\otimes Y,B) & \xrightarrow[\iota'_Y]{} & \mathrm{Hom}_{\mathcal{D}}(A,B\otimes Y^D). \end{array} \tag{9}$$

Since we are using the canonical duals of (7.1.7), the diagrams (6)-(9) remain commutative after applying a shift. Thus, the commutativity of diagrams (6)-(9), together with (7.1.5)(iv), gives the commutativity of the diagrams

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(A\otimes X[1],B) & \xrightarrow{\iota'_{X[1]}} & \mathrm{Hom}_{\mathcal{D}}(A,B\otimes X^D[-1]) \\ {\scriptstyle(\mathrm{id}\otimes h)^*}\downarrow & & \downarrow{\scriptstyle(\mathrm{id}\otimes h^D)_*} \\ \mathrm{Hom}_{\mathcal{D}}(A\otimes Z,B) & \xrightarrow{\iota'_Z} & \mathrm{Hom}_{\mathcal{D}}(A,B\otimes Z^D) \\ {\scriptstyle(\mathrm{id}\otimes g)^*}\downarrow & & \downarrow{\scriptstyle(\mathrm{id}\otimes i)_*} \\ \mathrm{Hom}_{\mathcal{D}}(A\otimes Y,B) & \xrightarrow{\iota'_Y} & \mathrm{Hom}_{\mathcal{D}}(A,B\otimes Y^D) \\ {\scriptstyle(\mathrm{id}\otimes f)^*}\downarrow & & \downarrow{\scriptstyle(\mathrm{id}\otimes j)_*} \\ \mathrm{Hom}_{\mathcal{D}}(A\otimes X,B) & \xrightarrow{\iota'_X} & \mathrm{Hom}_{\mathcal{D}}(A,B\otimes X^D) \\ {\scriptstyle(\mathrm{id}\otimes h[-1])^*}\downarrow & & \downarrow{\scriptstyle(\mathrm{id}\otimes h^D[1])_*} \\ \mathrm{Hom}_{\mathcal{D}}(A\otimes Z[-1],B) & \xrightarrow{\iota'_{Z[-1]}} & \mathrm{Hom}_{\mathcal{D}}(A,B\otimes Z^D[1]) \end{array} \tag{10}$$

and

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}}(Z^D[1]\otimes A,B) & \xrightarrow{\iota''_{Z[-1]}} & \mathrm{Hom}_{\mathcal{D}}(A,Z[-1]\otimes B)\\
{\scriptstyle (h^D[1]\otimes \mathrm{id})^*}\downarrow & & \downarrow{\scriptstyle (h[-1]\otimes \mathrm{id})_*}\\
\mathrm{Hom}_{\mathcal{D}}(X^D\otimes A,B) & \xrightarrow{\iota''_{X}} & \mathrm{Hom}_{\mathcal{D}}(A,X\otimes B)\\
{\scriptstyle (j\otimes \mathrm{id})^*}\downarrow & & \downarrow{\scriptstyle (f\otimes \mathrm{id})_*}\\
\mathrm{Hom}_{\mathcal{D}}(Y^D\otimes A,B) & \xrightarrow{\iota''_{Y}} & \mathrm{Hom}_{\mathcal{D}}(A,Y\otimes B)\\
{\scriptstyle (i\otimes \mathrm{id})^*}\downarrow & & \downarrow{\scriptstyle (g\otimes \mathrm{id})_*}\\
\mathrm{Hom}_{\mathcal{D}}(Z^D\otimes A,B) & \xrightarrow{\iota''_{Z}} & \mathrm{Hom}_{\mathcal{D}}(A,Z\otimes B)\\
{\scriptstyle (h^D\otimes \mathrm{id})^*}\downarrow & & \downarrow{\scriptstyle (h\otimes \mathrm{id})_*}\\
\mathrm{Hom}_{\mathcal{D}}(X^D[-1]\otimes A,B) & \xrightarrow{\iota''_{X[1]}} & \mathrm{Hom}_{\mathcal{D}}(A,X[1]\otimes B)
\end{array} \tag{11}$$

As the columns of (10) and (11) are Hom sequences arising from distinguished triangles, they are exact; the five lemma then implies that the maps ι'_Y and ι''_Y are isomorphisms. Thus, (Y^D,ι_Y) is a dual to Y.

Finally, the identities (2) show that $g^D = i$ and $f^D = j$; the distinguished triangle (1) is thus the desired triangle (7.2.3.1). □

(7.2.4) REMARK

Consider the diagram ((7.2.3), (4)). Suppose we have chosen duals (X^D,ι) to X and (Z^D,ι_Z) to Z; this gives the object Y^D by the sequence ((7.2.3), (1)). From (7.2.2) we have the diagram (7.2.2.1)

$$\begin{array}{ccc}
K & \xrightarrow{\beta} & Y\otimes Y^D\\
{\scriptstyle q}\downarrow & & \downarrow\\
X\otimes X^D\oplus Z\otimes Z^D & \longrightarrow & Y\otimes Y^D\oplus Z\otimes Y^D.
\end{array}$$

It follows from the proof of (7.2.3) that, if we have a map as in ((7.2.3), (2))

$$\iota\colon 1\to K$$

in $\mathcal{D}$, satisfying the identity ((7.2.3), (3))

$$q\circ\iota = (\iota_X,\iota_Z)$$

then $(Y^D,\beta\circ\iota)$ is a dual to Y. □

(7.2.5) THEOREM

Suppose $\mathcal{D}$ is generated (as a triangulated category) by a set of objects $\mathcal{S}$ such that each $X\in\mathcal{S}$ has a dual (X^D,ι_X). Then

i) every object Y of $\mathcal{D}$ has a dual (Y^D,ι_Y).

ii) if we assume that the choice of dual for $X[1]$ is the canonical one (see (7.1.7)) for each X in $\mathcal{D}$, then, sending X to its dual X^D and a morphism $f: X \to Y$ to the dual morphism $f^D: Y^D \to X^D$ defines an exact functor

$$(-)^D: \mathcal{D}^{\mathrm{op}} \to \mathcal{D}$$

The functor $(-)^D$ is a pseudo tensor functor (see (7.1.6)(iii)).

Proof. Part (i) follows directly from (7.2.3). From (i), (7.1.6) and (7.1.7), sending X to its dual X^D and f to its dual f^D defines a graded functor

$$(-)^D: \mathcal{D}^{\mathrm{op}} \to \mathcal{D}$$

which is a pseudo tensor functor. Thus, we need only show that $(-)^D$ is exact.

Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

be a distinguished triangle in $\mathcal{D}$. By (7.2.3), there is a choice of dual (Y^{*D}, ι_{*Y}) for Y such that the sequence

$$X^D[-1] \xrightarrow{h^D} Z^D \xrightarrow{g^{*D}} Y^{*D} \xrightarrow{f^{*D}} X^D$$

is a distinguished triangle, where g^{*D} and f^{*D} are the maps defined with respect to the choice (Y^{*D}, ι_{*Y}) of dual for Y. Let $F: Y^{*D} \to Y^D$ be the canonical isomorphism given by (7.1.3). By (7.1.5), we have

$$F \circ g^{*D} = g^D; \qquad f^{*D} = f^D \circ F.$$

Thus, we have a commutative diagram

$$\begin{array}{ccccccc} X^D[-1] & \xrightarrow{h^D} & Z^D & \xrightarrow{g^{*D}} & Y^{*D} & \xrightarrow{f^{*D}} & X^D \\ \| & & \| & & {\scriptstyle F}\downarrow & & \| \\ X^D[-1] & \xrightarrow[h^D]{} & Z^D & \xrightarrow[g^D]{} & Y^D & \xrightarrow[f^D]{} & X^D \end{array}$$

As F is an isomorphism, the sequence

$$X^D[-1] \xrightarrow{h^D} Z^D \xrightarrow{g^D} Y^D \xrightarrow{f^D} X^D$$

is a distinguished triangle, completing the proof. □

7.3. The diagonal and co-diagonal

We examine the diagonal morphism for a smooth projective S-scheme.

(7.3.1)

Let $p_X\colon X \to S$ be a smooth projective S-scheme of dimension d over S. We have the diagonal

$$\delta_X\colon X \to X \times_S X,$$

giving the maps

$$\delta_X^*\colon \mathbb{Z}_{X\times_S X}(d)[2d] \to \mathbb{Z}_X(d)[2d]$$

and

$$\delta_*^X\colon \mathbb{Z}_X \to \mathbb{Z}_{X\times_S X}(d)[2d].$$

We also have the external products

$$\begin{aligned}\boxtimes_{X,X}\colon \mathbb{Z}_X(a) \otimes \mathbb{Z}_X(d-a)[2d] \to \mathbb{Z}_{X\times_S X}(d)[2d]\\ \boxtimes_{X,X}\colon \mathbb{Z}_X(d-a)[2d] \otimes \mathbb{Z}_X(a) \to \mathbb{Z}_{X\times_S X}(d)[2d]\end{aligned}$$

We define the maps in $\mathcal{D}$

$$\begin{aligned}\iota_X\colon \mathbb{Z}_S = 1 \to \mathbb{Z}_X(a) \otimes \mathbb{Z}_X(d-a)[2d]\\ \epsilon_X\colon \mathbb{Z}_X(d-a)[2d] \otimes \mathbb{Z}_X(a) \to 1\end{aligned} \tag{7.3.1.1}$$

by

$$\begin{aligned}\iota_X &= \boxtimes_{X,X}^{-1} \circ \delta_*^X \circ p_X^*\\ \epsilon_X &= p_{X*} \circ \delta_X^* \circ \boxtimes_{X,X}\end{aligned}$$

We may form the composition

$$\begin{aligned}\mathbb{Z}_X(a) \cong& 1 \otimes \mathbb{Z}_X(a) \xrightarrow{\iota_X\otimes \mathrm{id}} \mathbb{Z}_X(a) \otimes \mathbb{Z}_X(d-a)[2d] \otimes \mathbb{Z}_X(a)\\ &\xrightarrow{\mathrm{id}\otimes\epsilon_X} \mathbb{Z}_X(a) \otimes 1 \to \mathbb{Z}_X(a)\end{aligned} \tag{7.3.1.2}$$

and the composition

$$\begin{aligned}\mathbb{Z}_X(d-a)[2d] &\cong \mathbb{Z}_X(d-a)[2d] \otimes 1\\ &\xrightarrow{\mathrm{id}\otimes\iota_X} \mathbb{Z}_X(d-a)[2d] \otimes \mathbb{Z}_X(a) \otimes \mathbb{Z}_X(d-a)[2d]\\ &\xrightarrow{\epsilon_X\otimes\mathrm{id}} 1 \otimes \mathbb{Z}_X(d-a)[2d] \to \mathbb{Z}_X(d-a)[2d]\end{aligned} \tag{7.3.1.3}$$

(7.3.2) Lemma

The compositions (7.3.1.2) and (7.3.1.3) are the identity.

Proof. Let

$$\delta_X^{12}: X \times X \to X \times X \times X$$
$$\delta_X^{23}: X \times X \to X \times X \times X$$

be the maps

$$\delta_X^{12} = \delta_X \times \mathrm{id}_X$$
$$\delta_X^{23} = \mathrm{id}_X \times \delta_X$$

We recall from (3.2.8) that the multiplication isomorphism

$$\mu_l: 1 \otimes \mathbb{Z}_X(a) \to \mathbb{Z}_X(a)$$

is the external product

$$\boxtimes_{S,X}: \mathbb{Z}_S \otimes \mathbb{Z}_X(a) \to \mathbb{Z}_{S \times_S X}(a) \overset{p_2^*}{\cong} \mathbb{Z}_X(a).$$

We have the isomorphism

$$\boxtimes_{X,X,X}: \mathbb{Z}_X(a) \otimes \mathbb{Z}_X(d-a)[2d] \otimes \mathbb{Z}_X(a) \to \mathbb{Z}_{X \times X \times X}(d+a)[2d];$$

applying (6.4.9), we find that (7.3.1.2) is equal to the composition

$$\begin{aligned} \mathbb{Z}_X(a) \xrightarrow{p_2^*} \mathbb{Z}_{X \times X}(a) \xrightarrow{\delta_*^{12}} \mathbb{Z}_{X \times X \times X}(d+a)[2d] \\ \xrightarrow{\delta_X^{23*}} \mathbb{Z}_{X \times X}(d+a)[2d] \xrightarrow{p_{1*}} \mathbb{Z}_X(a). \end{aligned} \tag{1}$$

We have the cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{\delta_X} & X \times_S X \\ {\scriptstyle \delta_X}\downarrow & & \downarrow{\scriptstyle \delta_X^{23}} \\ X \times_S X & \underset{\delta_X^{12}}{\longrightarrow} & X \times_S X \times_S X. \end{array}$$

By (6.4.8), we have

$$\delta_X^{23*} \circ \delta_*^{12} = \delta_{X*} \circ \delta_X^*.$$

Thus (7.3.1.2) is equal to the composition

$$\begin{aligned} \mathbb{Z}_X(a) \xrightarrow{p_2^*} \mathbb{Z}_{X \times X}(a) \xrightarrow{\delta_X^*} \mathbb{Z}_X(a) \\ \xrightarrow{\delta_{X*}} \mathbb{Z}_{X \times X}(d+a)[2d] \xrightarrow{p_{1*}} \mathbb{Z}_X(a). \end{aligned} \tag{2}$$

Since

$$p_2 \circ \delta_X = \mathrm{id}_X,$$

we have

$$\delta_X^* \circ p_2^* = \mathrm{id}.$$

Since

$$p_1 \circ \delta_X = \mathrm{id}_X,$$

it follows from (6.4.6) that

$$p_{1*} \circ \delta_{X*} = \mathrm{id}.$$

Thus, the composition (2) is the identity, completing the proof that (7.3.1.2) is the identity. The proof that (7.3.1.3) is the identity is similar, and is left to the reader. □

7.4. *The duality involution*

We describe the duality structure on the motivic category. We also show that the category $\mathcal{DM}(S)$ may be constructed (up to equivalence) using the tensor category $\mathcal{A}_{mot}(\mathbf{Sm}_S)^0$ (3.1.6) rather than the DG tensor category $\mathcal{A}_{mot}(\mathbf{Sm}_S)$, in case $S = \mathrm{Spec}(k)$, with k a field of characteristic zero.

(7.4.1) THE DUAL FOR PROJECTIVE X

We let $\mathcal{DM}(S)^{\mathrm{pr}}$ denote the smallest full triangulated subcategory of $\mathcal{DM}(S)$ containing the objects $\mathbb{Z}_X(a)$, with X in $\mathbf{Sm}_S$ projective over S, and closed under taking summands. Since $\mathbb{Z}_X(a) \otimes \mathbb{Z}_Y(b)$ is isomorphic to $\mathbb{Z}_{X\times_S Y}(a+b)$, $\mathcal{DM}(S)^{\mathrm{pr}}$ is an triangulated tensor subcategory of $\mathcal{DM}(S)$.

For $X \in \mathbf{Sm}_S$, projective over S, we set

$$\mathbb{Z}_X(a)[b]^D := \mathbb{Z}_X(d-a)[2d-b]. \tag{7.4.1.1}$$

We have the morphism (7.3.1.1)

$$\iota_X : 1 \to \mathbb{Z}_X(a)[b] \otimes \mathbb{Z}_X(a)[b]^D;$$

by (7.3.2) and (7.1.9), $(\mathbb{Z}_X(a)^D, \iota_X)$ is a dual to $\mathbb{Z}_X(a)$. Thus, for X and Y in $\mathbf{Sm}_S$ and projective over S, we have the isomorphism

$$(-)^D : \mathrm{Hom}_{\mathcal{DM}}(\mathbb{Z}_X(a)[b], \mathbb{Z}_Y(a')[b']) \to \mathrm{Hom}_{\mathcal{DM}}(\mathbb{Z}_Y(a')[b']^D, \mathbb{Z}_X(a)[b]^D). \tag{7.4.1.2}$$

(7.4.2) THEOREM

The operation $(-)^D$ defined for projective X by (7.4.1.1) and (7.4.1.2):

$$\mathbb{Z}_X(a)[b] \mapsto \mathbb{Z}_X(a)[b]^D,$$
$$(f : \mathbb{Z}_X(a)[b] \to \mathbb{Z}_Y(a')[b']) \mapsto (f^D : \mathbb{Z}_Y(a')[b']^D \to \mathbb{Z}_X(a)[b]^D)$$

extends to an exact pseudo tensor functor (see (7.1.6)(iii))

$$(-)^D: (\mathcal{DM}(S)^{\mathrm{pr}})^{\mathrm{op}} \to \mathcal{DM}(S)^{\mathrm{pr}};$$

defining an exact duality on $\mathcal{DM}(S)^{\mathrm{pr}}$*, i.e., for* A*,* B *and* C *in* $\mathcal{DM}(S)^{\mathrm{pr}}$*, there are natural isomorphisms*

$$\mathrm{Hom}_{\mathcal{DM}(S)}(A \otimes B^D, C) \to \mathrm{Hom}_{\mathcal{DM}(S)}(A, C \otimes B)$$

$$\mathrm{Hom}_{\mathcal{DM}(S)}(A \otimes B, C) \to \mathrm{Hom}_{\mathcal{DM}(S)}(A, C \otimes B^D)$$

which are exact in the variables A*,* B *and* C*. In addition, there is a natural isomorphism*

$$\mathrm{id} \to ((-)^D)^D.$$

Proof. If we let $\mathbf{D}^b_{mot}(\mathbf{Sm}_S)^{\mathrm{pr}}$ be the full triangulated subcategory of $\mathbf{D}^b_{mot}(\mathbf{Sm}_S)$ generated by the objects $\mathbb{Z}_X(a)$ for X in $\mathbf{Sm}_S$ projective over S, then the extension of the operation $(-)^D$ to an exact, pseudo tensor functor

$$(-)^D: (\mathbf{D}^b_{mot}(\mathbf{Sm}_S)^{\mathrm{pr}})^{\mathrm{op}} \to \mathbf{D}^b_{mot}(\mathbf{Sm}_S)^{\mathrm{pr}},$$

with a natural isomorphism

$$\mathrm{id} \to ((-)^D)^D,$$

follows directly from (7.2.5).

We have the functor $_\#$ (see II, §2.5) on the category of triangulated tensor categories, where $\mathcal{A}_\#$ is the pseudo-abelian hull of a category $\mathcal{A}$. Applying (7.2.5) again, it suffices to show that, if $\mathcal{A}$ is a tensor category having a duality involution $(-)^D$, if A is an object of $\mathcal{A}$, and if B is the summand of A in $\mathcal{A}_\#$ corresponding to an idempotent endomorphism $p: A \to A$, then B has a dual (B^D, ι_B) in $\mathcal{A}_\#$.

To see this, the idempotent endomorphism $p: A \to A$ gives rise to the endomorphism

$$p^D: A^D \to A^D.$$

Since $(-)^D$ is a functor on $\mathcal{A}$, p^D is an idempotent endomorphism of A^D. Let B^D be the summand of A^D in $\mathcal{A}_\#$ corresponding to p^D. The idempotent endomorphism $p \otimes p^D$ then defines the summand $B \otimes B^D$ of $A \otimes A^D$. We let $\iota_B: 1 \to B \otimes B^D$ be the map gotten by projecting $\iota_A: 1 \to A \otimes A^D$ onto the summand $B \otimes B^D$:

$$\iota_B = (p \otimes p^D) \circ \iota_A \in (p \otimes p^D) \circ \mathrm{Hom}_{\mathcal{A}}(1, A \otimes A^D) := \mathrm{Hom}_{\mathcal{A}_\#}(1, B \otimes B^D).$$

It is then an elementary exercise to show that (B^D, ι_B) is a dual to B, which completes the proof of the theorem. □

(7.4.3) THEOREM

Let $F: \mathcal{DM}(S)_R \to \mathcal{A}$ be an exact R-tensor functor. Suppose the map

$$F(1,\Gamma): \mathrm{Hom}_{\mathcal{DM}(S)}(1,\Gamma) \to \mathrm{Hom}_{\mathcal{A}}(1, F(\Gamma))$$

is an isomorphism for each Γ in $\mathcal{DM}(S)_R$. Then, for each Δ in $\mathcal{DM}(S)_R^{\mathrm{pr}}$, and each Γ in $\mathcal{DM}(S)_R$, the map

$$F(\Delta,\Gamma): \mathrm{Hom}_{\mathcal{DM}(S)_R}(\Delta,\Gamma) \to \mathrm{Hom}_{\mathcal{A}}(F(\Delta), F(\Gamma))$$

is an isomorphism, hence the restriction of F to $\mathcal{DM}(S)_R^{\mathrm{pr}}$ is a fully faithful embedding. In particular, if $\mathcal{DM}(S)_R^{\mathrm{pr}} = \mathcal{DM}(S)_R$, then F is a fully faithful embedding.

Proof. We give the proof for $R = \mathbb{Z}$. Each object of $\mathcal{DM}(S)^{\mathrm{pr}}$ is a summand of an iterated cone of objects of the form $\mathbb{Z}_X(a)[b]$, with X smooth and projective over S. Since F is exact, it suffices to show that $F(\Delta,\Gamma)$ is an isomorphism for $\Delta = \mathbb{Z}_X(a)[b]$.

This follows from the hypothesis on F, and (7.4.2). □

(7.4.4)

We recall the tensor category $\mathcal{A}_{mot}(\mathbf{Sm}_S)^0$ (3.1.6), and the DG tensor functor (3.1.6.2)

$$H_{mot}: \mathcal{A}_{mot}(\mathbf{Sm}_S) \to \mathcal{A}_{mot}(\mathbf{Sm}_S)^0.$$

The category $\mathcal{A}_{mot}(\mathbf{Sm}_S)^0$ and functor H_{mot} are characterized by the identity

$$\mathrm{Hom}_{\mathcal{A}_{mot}(\mathbf{Sm}_S)^0}(\mathbb{Z}_X, \mathbb{Z}_Y(a)) = H^{2a}(\mathrm{Hom}_{\mathcal{A}_{mot}(\mathbf{Sm}_S)}(\mathbb{Z}_X, \mathbb{Z}_Y(a)).$$

From (3.2.11), we have the triangulated tensor category $\mathbf{K}^b_{mot}(\mathbf{Sm}_S)^0$:

$$\mathbf{K}^b_{mot}(\mathbf{Sm}_S)^0 = \mathbf{K}^b(\mathcal{A}_{mot}(\mathbf{Sm}_S)^0),$$

and the exact tensor functor

$$\mathbf{K}^b(H_{mot}): \mathbf{K}^b_{mot}(\mathbf{Sm}_S) \to \mathbf{K}^b_{mot}(\mathbf{Sm}_S)^0.$$

The category $\mathbf{D}^b_{mot}(\mathbf{Sm}_S)^0$ is gotten from $\mathbf{K}^b_{mot}(\mathbf{Sm}_S)^0$ by inverting the morphisms of (2.1.3), and the category $\mathcal{DM}(S)^0$ is gotten from $\mathbf{D}^b_{mot}(\mathbf{Sm}_S)^0$ by forming the pseudo-abelian hull. This gives the exact tensor functors

$$\mathbf{D}^b(H_{mot}): \mathbf{D}^b_{mot}(\mathbf{Sm}_S) \to \mathbf{D}^b_{mot}(\mathbf{Sm}_S)^0$$

$$\mathcal{DM}(H_{mot}): \mathcal{DM}(S) \to \mathcal{DM}(S)^0$$

(7.4.5) LEMMA

Let X be in $\mathbf{Sm}_S$. Suppose there is an open immersion

$$j: X \to \bar{X}$$

with $\bar{X}$ smooth and projective over S, such that

i) the complement $Z := \bar{X}\setminus X$ is a union of smooth projective irreducible S-schemes:

$$Z = \cup_{i=1}^{N} Z_i.$$

ii) For each collection of indices $i_1, \ldots, i_s$, the closed subset $Z_{i_1} \cap \ldots \cap Z_{i_s}$ of $\bar{X}$ is smooth over S.

Then X is in $\mathbf{Sm}_S^{\mathrm{pr}}$.

Proof. Let $U = \bar{X}\setminus Z_1$, $Z_U = Z \cap U$. We have the distinguished triangles (2.2.9.2) in $\mathcal{DM}(S)$:

$$\mathbb{Z}_{\bar{X},Z}(a) \to \mathbb{Z}_{\bar{X}}(a) \to \mathbb{Z}_X(a) \to \mathbb{Z}_{\bar{X},Z}(a)[1],$$
$$\mathbb{Z}_{\bar{X},Z_1}(a) \to \mathbb{Z}_{\bar{X},Z}(a) \to \mathbb{Z}_{U,Z_U}(a) \to \mathbb{Z}_{\bar{X},Z_1}(a)[1],$$
$$\mathbb{Z}_{\bar{X},Z_1}(a) \to \mathbb{Z}_{\bar{X}}(a) \to \mathbb{Z}_U(a) \to \mathbb{Z}_{\bar{X},Z_1}(a)[1].$$

Since Z_1 is smooth, say of codimension d, we have the isomorphism (6.1.1.2)

$$\mathbb{Z}_{\bar{X},Z_1}(a) \cong \mathbb{Z}_{Z_1}(a-d)[-2d],$$

hence $\mathbb{Z}_U(a)$ is in $\mathcal{DM}(S)^{\mathrm{pr}}$. Similarly, $\mathbb{Z}_{Z_i \cap U}(b)$ is in $\mathcal{DM}(S)^{\mathrm{pr}}$ for each $i = 2, \ldots, N$; by induction, this implies $\mathbb{Z}_{U,Z_U}(a)$ is in $\mathcal{DM}(S)^{\mathrm{pr}}$. Thus $\mathbb{Z}_{\bar{X},Z}(a)$ is in $\mathcal{DM}(S)^{\mathrm{pr}}$, hence $\mathbb{Z}_X(a)$ is in $\mathcal{DM}(S)^{\mathrm{pr}}$. □

(7.4.6) THEOREM

Suppose $S = \mathrm{Spec}(k)$ for a field k, and that, if $\mathrm{char}(k) > 0$, the coefficient ring R is $\mathbb{Q}$. Suppose further that, for each X in $\mathbf{Sm}_k$, there is an open immersion

$$j: X \to \bar{X}$$

with $\bar{X}$ smooth and projective over k, such that

i) the complement $Z := \bar{X}\setminus X$ is a union of smooth projective irreducible k-schemes:

$$Z = \cup_{i=1}^{N} Z_i.$$

ii) For each collection of indices $i_1, \ldots, i_s$, the closed subset $Z_{i_1} \cap \ldots \cap Z_{i_s}$ of $\bar{X}$ is smooth over k.

Then the functors $\mathbf{D}^b(H_{mot})$ and $\mathcal{DM}(H_{mot})$ are equivalences.

Proof. It suffices to show that $\mathbf{D}^b(H_{mot})$ is an equivalence of categories (after tensoring with R). By (7.4.5), $\mathbf{D}^b_{mot}(\mathbf{Sm}_k)^{\mathrm{pr}} = \mathbf{D}^b_{mot}(\mathbf{Sm}_k)$. Since $\mathbf{D}^b_{mot}(\mathbf{Sm}_k)^0$ is generated by the objects in the image of $\mathbf{D}^b_{mot}(\mathbf{Sm}_k)$, it suffices to show that $\mathbf{D}^b(H_{mot})$ is fully faithful, after tensoring with R.

By (4.6.6) and (4.4.9), the functor $\mathbf{D}^b(H_{mot})$ gives an isomorphism

$$\mathrm{Hom}_{\mathbf{D}^b_{mot}(\mathbf{Sm}_k)}(1, \mathbb{Z}_X(a)[b]) \otimes R \to \mathrm{Hom}_{\mathbf{D}^b_{mot}(\mathbf{Sm}_k)^0}(1, \mathbb{Z}_X(a)[b]) \otimes R$$

for each X in $\mathbf{Sm}_k$. Since $\mathbf{D}^b_{mot}(\mathbf{Sm}_k)$ is generated as a triangulated category by the objects $\mathbb{Z}_X(a)$, it follows that $\mathbf{D}^b(H_{mot})$ gives an isomorphism

$$\mathrm{Hom}_{\mathbf{D}^b_{mot}(\mathbf{Sm}_k)}(1, \Gamma) \otimes R \to \mathrm{Hom}_{\mathbf{D}^b_{mot}(\mathbf{Sm}_k)^0}(1, \Gamma) \otimes R$$

for each Γ in $\mathbf{D}^b_{mot}(\mathbf{Sm}_k)$. Since $\mathbf{D}^b_{mot}(\mathbf{Sm}_k)^{\mathrm{pr}} = \mathbf{D}^b_{mot}(\mathbf{Sm}_k)$, it follows from (7.4.3) that $\mathbf{D}^b(H_{mot})$ is fully faithful after tensoring with R, completing the proof. □

7.5. *Correspondences*

Via the duality isomorphism of §7.4, we may interpret maps in $\mathcal{DM}(S)$ between motives of projective varieties X and Y as classes in the motivic cohomology of the product. In this section, we show how the category of Chow motives over a field k motives generalizes to our setting.

(7.5.1) LEMMA

Let X be a projective S-scheme in $\mathbf{Sm}_S$, of dimension d over S, let Y be in $\mathbf{Sm}_S$, and let $f\colon \mathbb{Z}_X \to \mathbb{Z}_Y(a)[b]$ be a morphism in $\mathcal{DM}(S)$.

i) Let

$$\zeta\colon 1 \to \mathbb{Z}_{Y\times_S X}(a+d)[b+2d]$$

be the map $\boxtimes_{Y,X} \circ \iota''_X(f)$. Then f is equal to the composition

$$\mathbb{Z}_X \xrightarrow{p_2^*} \mathbb{Z}_{Y\times_S X} \xrightarrow{\cup_{Y\times_S X}\zeta} \mathbb{Z}_{Y\times_S X}(a+d)[b+2d] \xrightarrow{p_{1*}} \mathbb{Z}_Y(a)[b].$$

ii) If, in addition, Y is projective over S of dimension e, and

$$\zeta^D\colon 1 \to \mathbb{Z}_{X\times_S Y}(a+d)[b+2d]$$

is the map $\boxtimes_{X,Y} \circ \iota''_Y(f^D)$, then

$$\zeta^D = t^*_{X,Y}(\zeta),$$

where $t_{X,Y}\colon X\times_S Y\to Y\times_S X$ is the exchange of factors. Thus, f^D is equal to the composition

$$\mathbb{Z}_Y(e-a)[2e-b]\xrightarrow{p_1^*}\mathbb{Z}_{Y\times_S X}(e-a)[2e-b] \xrightarrow{\cup_{Y\times_S X}\zeta}\mathbb{Z}_{Y\times_S X}(d+e)[2d+2e]\xrightarrow{p_{2*}}\mathbb{Z}_X(d)[2d].$$

Proof. We first prove (i). If we denote the composition in (i) by g, it suffices to show that $\iota''_X(f)=\iota''_X(g)$. The map $\iota''_X(g)$ is the composition

$$1\xrightarrow{\iota_X}\mathbb{Z}_X\otimes\mathbb{Z}_X(d)[2d]\xrightarrow{p_2^*\otimes\mathrm{id}}\mathbb{Z}_{Y\times_S X}\otimes\mathbb{Z}_X(d)[2d] \xrightarrow{\cup_{Y\times_S X}\zeta\otimes\mathrm{id}}\mathbb{Z}_{Y\times_S X}(a+d)[b+2d]\otimes\mathbb{Z}_X(d)[2d]\xrightarrow{p_{1*}\otimes\mathrm{id}}\mathbb{Z}_Y(a)[b]\otimes\mathbb{Z}_X(d)[2d]$$

We may rewrite this as

$$\boxtimes_{Y,X}\circ\iota''_X(g)=p_{13*}\circ\cup_{Y\times_S X\times_S X}(p_{12}^*\zeta)\circ p_{23}^*\circ\delta_{X*}\circ p_X^*.$$

We have the cartesian diagram

$$\begin{array}{ccc} Y\times_S X & \xrightarrow{\mathrm{id}_Y\times\delta_X} & Y\times_S X\times_S X \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_{23}} \\ X & \xrightarrow[\delta_X]{} & X\times_S X; \end{array}$$

applying (6.4.8), we have the identity

$$p_{23}^*\circ\delta_{X*}=(\mathrm{id}_Y\times\delta_X)_*\circ p_2^*.$$

Using this and the projection formula (6.4.7), we may rewrite $\boxtimes_{Y,X}\circ\iota''_X(g)$ as

$$\begin{aligned}\boxtimes_{Y,X}\circ\iota''_X(g)&=p_{13*}\circ(\mathrm{id}_Y\times\delta_X)_*\circ\cup_{Y\times_S X}(\zeta)\circ p_{Y\times_S X}^*\\&=\mathrm{id}_{Y\times_S X*}\circ\zeta\\&=\zeta.\end{aligned}$$

Thus $\iota''_X(g)=\boxtimes_{Y,X}^{-1}\circ\zeta=\iota''_X(f)$, completing the proof of (i).

For (ii), we note the $\iota'_Y(f^D)=\iota''_X(f)$, by definition of f^D. On the other hand, it follows from the symmetry of the diagonal map that the maps

$$\iota_{\mathbb{Z}_Y(a)[b]}\colon 1\to\mathbb{Z}_Y(a)[b]\otimes\mathbb{Z}_Y(e-a)[2e-b],$$
$$\iota_{\mathbb{Z}_Y(e-a)[2e-b]}\colon 1\to\mathbb{Z}_Y(e-a)[2e-b]\otimes\mathbb{Z}_Y(a)[b]$$

are related by

$$t_{Y,Y}^*\circ\boxtimes_{Y,Y}\circ\iota_{\mathbb{Z}_Y(a)[b]}=\boxtimes_{Y,Y}\circ\iota_{\mathbb{Z}_Y(e-a)[2e-b]}.$$

From this, it follows that

$$\boxtimes_{X,Y} \circ \iota''_Y(f^D) = t^*_{X,Y} \circ \boxtimes_{Y,X} \circ \iota'_Y(f^D),$$

i.e.,

$$\zeta^D = t^*_{X,Y} \circ \zeta,$$

proving (ii). □

(7.5.2) Lemma

Suppose X, Y and Z are in $\mathbf{Sm}_S$, with X and Y projective over S, of relative dimensions d and e, respectively. Let

$$f: \mathbb{Z}_X \to \mathbb{Z}_Y(a)[b],\ \ g: \mathbb{Z}_Y(a)[b] \to \mathbb{Z}_Z(a'+a)[b'+b]$$

be morphisms in $\mathcal{DM}(S)$, and let

$$\begin{aligned}
&\zeta_f: 1 \to \mathbb{Z}_{Y\times_S X}(a+d)[b+2d],\\
&\zeta_g: 1 \to \mathbb{Z}_{Z\times_S Y}(a'+e)[b'+2e],\\
&\zeta_{g\circ f}: 1 \to \mathbb{Z}_{Z\times_S X}(a+a'+d)[b+b'+2d]
\end{aligned}$$

be the respective morphisms

$$\boxtimes_{Y,X} \circ \iota''_X(f), \quad \boxtimes_{Z,Y} \circ \iota''_Y(g), \quad \boxtimes_{Z,X} \circ \iota''_X(g\circ f).$$

Let

$$\begin{aligned}
&p_{12}: Z\times_S Y\times_S X \to Z\times_S Y,\\
&p_{13}: Z\times_S Y\times_S X \to Z\times_S X,\\
&p_{23}: Z\times_S Y\times_S X \to Y\times_S X
\end{aligned}$$

be the projections. Then

$$\zeta_{g\circ f} = p_{13*}(p^*_{12}(\zeta_g) \cup p^*_{23}(\zeta_f)).$$

Proof. By (7.1.11), and the uniqueness (7.1.10) of the co-diagonal

$$\mathbb{Z}_Y(e-a)[2e-b] \otimes \mathbb{Z}_Y(a)[b] \to 1,$$

$\zeta_{g\circ f}$ is the composition

$$\begin{aligned}1 \cong\, & 1\otimes 1\\ &\xrightarrow{\iota''_Y(g)\otimes\iota''_X(f)} \mathbb{Z}_Z(a+a')[b+b']\otimes\mathbb{Z}_Y(e-a)[2e-b]\otimes\mathbb{Z}_Y(a)[b]\otimes\mathbb{Z}_X(d)[2d]\\ &\xrightarrow{\mathrm{id}\otimes\epsilon_Y\otimes\mathrm{id}} \mathbb{Z}_Z(a+a')[b+b']\otimes 1\otimes\mathbb{Z}_X(d)[2d]\\ &\cong \mathbb{Z}_Z(a+a')[b+b']\otimes\mathbb{Z}_X(d)[2d]\\ &\xrightarrow{\boxtimes_{Z,X}} \mathbb{Z}_{Z\times_S X}(a+a'+d)[b+b'+2d].\end{aligned}$$

Since

$$\epsilon_Y = p_{Y*}\circ\delta_Y^*\circ\boxtimes_{Y,Y}$$

we may rewrite this as the composition

$$\begin{aligned}1 \cong\, & 1\otimes 1\\ &\xrightarrow{\iota_Y\otimes\iota_X} \mathbb{Z}_Y(a)[b]\otimes\mathbb{Z}_Y(e-a)[2e-b]\otimes\mathbb{Z}_X\otimes\mathbb{Z}_X(d)[2d]\\ &\xrightarrow{g\otimes\mathrm{id}\otimes f\otimes\mathrm{id}} \mathbb{Z}_Z(a+a')[b+b']\otimes\mathbb{Z}_Y(e-a)[2e-b]\otimes\mathbb{Z}_Y(a)[b]\otimes\mathbb{Z}_X(d)[2d]\\ &\xrightarrow{\mathrm{id}\otimes(\delta_Y^*\circ\boxtimes_{Y,Y})\otimes\mathrm{id}} \mathbb{Z}_Z(a+a')[b+b']\otimes\mathbb{Z}_Y(e)[2e]\otimes\mathbb{Z}_X(d)[2d]\\ &\xrightarrow{\mathrm{id}\otimes p_{Y*}\otimes\mathrm{id}} \mathbb{Z}_Z(a+a')[b+b']\otimes 1\otimes\mathbb{Z}_X(d)[2d]\\ &\cong \mathbb{Z}_Z(a+a')[b+b']\otimes\mathbb{Z}_X(d)[2d]\\ &\xrightarrow{\boxtimes_{Z,X}} \mathbb{Z}_{Z\times_S X}(a+a'+d)[b+b'+2d].\end{aligned}$$

Using the definition of ζ_f and ζ_g, and (6.4.9), this in turn may be rewritten as the composition

$$\begin{aligned}1 \cong\, & 1\otimes 1\\ &\xrightarrow{\boxtimes_{Z\times_S Y,Y\times_S X}\circ(\zeta_g\otimes\zeta_f)} \mathbb{Z}_{Z\times_S Y\times_S Y\times_S X}(a+a'+e+d)[b+b'+2e+2d]\\ &\xrightarrow{(\mathrm{id}_Z\times\delta_Y\times\mathrm{id}_X)^*} \mathbb{Z}_{Z\times_S Y\times_S X}(a+a'+e+d)[b+b'+2e+2d]\\ &\xrightarrow{p_{13*}} \mathbb{Z}_{Z\times_S X}(a+a'+d)[b+b'+2d].\end{aligned}\tag{1}$$

Since the composition

$$\begin{aligned}1 \cong\, & 1\otimes 1\\ &\xrightarrow{\boxtimes_{Z\times_S Y,Y\times_S X}\circ(\zeta_g\otimes\zeta_f)} \mathbb{Z}_{Z\times_S Y\times_S Y\times_S X}(a+a'+e+d)[b+b'+2e+2d]\end{aligned}$$

is the same as the cup product $p_{12}^*(\zeta_g) \cup p_{34}^*(\zeta_f)$, the composition (1) is the same as the composition

$$\begin{aligned} 1 &\xrightarrow{p_{12}^*(\zeta_g)\cup p_{23}^*(\zeta_f)} \mathbb{Z}_{Z\times_S Y\times_S X}(a+a'+e+d)[b+b'+2e+2d] \\ &\xrightarrow{p_{13*}} \mathbb{Z}_{Z\times_S X}(a+a'+d)[b+b'+2d], \end{aligned}$$

completing the proof. □

(7.5.3) LEMMA

Let X be a projective S-scheme in $\mathbf{Sm}_S$, of dimension d over S

i) Let Y be in $\mathbf{Sm}_S$, and let $f\colon Y \to X$ be a morphism in $\mathbf{Sm}_S$, giving the morphism $f^*\colon \mathbb{Z}_X \to \mathbb{Z}_Y$. Let $\zeta_{f^*} = \boxtimes_{Y,X} \circ \iota''_X(f^*)$, and let $\Gamma_f \subset Y \times_S X$ be the graph of f. Then

$$\zeta_{f^*} = \mathrm{cl}^d_{Y\times_S X}(\Gamma_f).$$

ii) Let Y be in $\mathbf{Sm}_S$, and let $f\colon X \to Y$ be a morphism in $\mathbf{Sm}_S$ of relative codimension a, giving the morphism $f_*\colon \mathbb{Z}_X \to \mathbb{Z}_Y(a)[2a]$. Let $\zeta_{f_*} = \boxtimes_{Y,X} \circ \iota''_X(f_*)$, and let $\Gamma^t_f \subset Y \times_S X$ be the transpose of the graph of f:

$$\Gamma^t_f = t_{X,Y}(\Gamma_f).$$

Then

$$\zeta_{f_*} = \mathrm{cl}^{a+d}_{Y\times_S X}(\Gamma^t_f).$$

iii) Let $Z \in \mathcal{Z}^q(Y \times_S X/S)$ be a codimension q cycle on $Y \times_S X$. Let

$$\gamma_Z\colon \mathbb{Z}_X \to \mathbb{Z}_Y(q-d)[2q-2d]$$

be the composition

$$\mathbb{Z}_X \xrightarrow{p_2^*} \mathbb{Z}_{X\times_S Y} \xrightarrow{\cup \mathrm{cl}^q(Z)} \mathbb{Z}_{X\times_S Y}(q)[2q] \xrightarrow{p_{1*}} \mathbb{Z}_Y(q-d)[2q-2d],$$

and let $\zeta_{\gamma_Z} = \boxtimes_{Y,X} \circ \iota''_X(\gamma_Z)$. Then

$$\zeta_{\gamma_Z} = \mathrm{cl}^q_{Y\times_S X}(Z).$$

iv) Let Y be in $\mathbf{Sm}_S$, and let $Z \in \mathcal{Z}^q(Y/S)$ be a codimension q cycle on Y, giving the map

$$\mathrm{cl}^q_Y(Z)\colon 1 \to \mathbb{Z}_Y(q)[2q].$$

Let $\zeta_Z = \boxtimes_{Y,S} \circ \iota''_S(\mathrm{cl}^q_Y(Z))$. Then

$$\zeta_Z = \mathrm{cl}^q_Y(Z).$$

v) let $Z \in \mathcal{Z}^q(X/S)$ be a codimension q cycle on X, giving the map

$$\cup\mathrm{cl}^q_X(Z)\colon \mathbb{Z}_X \to \mathbb{Z}_X(q)[2q].$$

Let $\zeta_{\cup Z} = \boxtimes_{X,X} \circ \iota''_X(\cup\mathrm{cl}^q(Z))$. Then

$$\zeta_{\cup Z} = \mathrm{cl}^{d+q}_{X\times_S X}(\delta_{X*}(Z)).$$

Proof. For (i), we use the relation

$$\begin{aligned} \boxtimes_{Y,X} \circ \iota_X &= \delta_{X*} \circ p_X^* \\ &= \delta_{X*} \circ p_X^* \circ \mathrm{cl}^0_S(|S|) && (3.3.4) \\ &= \delta_{X*} \circ \mathrm{cl}^0_X(|X|) && (3.3.3) \\ &= \mathrm{cl}^d_{X\times_S X}(\delta_{X*}(|X|)) && (6.2.3) \\ &= \mathrm{cl}^d_{X\times_S X}(\Delta_X). \end{aligned}$$

From this it follows that

$$\begin{aligned} \boxtimes_{Y,X} \circ \iota''_X(f^*) &= (f \times \mathrm{id}_X)^*(\mathrm{cl}^d_{X\times_S X}(\Delta_X)) \\ &= \mathrm{cl}^d_{Y\times_S X}(\Gamma_f) && (3.3.3), \end{aligned}$$

proving (i).

For (ii), we have

$$\begin{aligned} \boxtimes_{Y,X} \circ \iota''_X(f_*) &= (f \times \mathrm{id}_X)_* \circ \delta_{X*} \circ p_X^* \\ &= (f \times \mathrm{id}_X)_* \circ \delta_{X*} \circ p_X^* \circ \mathrm{cl}^0_S(|S|) && (3.3.4) \\ &= (f, \mathrm{id}_X)_* \circ \mathrm{cl}^0_X(|X|) && (6.4.6) \\ &= \mathrm{cl}^d_{Y\times_S X}(\Gamma^t_f) && (6.2.3). \end{aligned}$$

The assertion (iii) follows directly from (7.5.1). As the cycle class map $\mathrm{cl}^q_Y(Z)$ may be factored as the composition

$$1 = \mathbb{Z}_S \xrightarrow{p_2^*} \mathbb{Z}_{Y\times_S S} \xrightarrow{\cup\mathrm{cl}^q(Z)} \mathbb{Z}_{Y\times_S S}(q)[2q] \xrightarrow{p_{1*}} \mathbb{Z}_Y(q)[2q],$$

(iv) follows from (iii). For (v), we have

$$\begin{aligned} p_{1*} \circ (p_2^*(-) \cup [\delta_{X*} \circ \mathrm{cl}^q)_X(Z)]) &= p_{1*} \circ \delta_{X*}(\delta_X^* \circ p_2^*(-) \cup \mathrm{cl}^q_X(Z)) && (6.4.7) \\ &= \cup\mathrm{cl}^q_X(Z) && (6.4.6). \end{aligned}$$

By (6.2.3),

$$\delta_{X*} \circ \mathrm{cl}^q_Y(Z) = \mathrm{cl}^{d+q}_{X\times_S X}(\delta_{X*}(Z));$$

this and (iii) proves (v).

(7.5.4) Remark

Let X and Y be projective S-schemes in $\mathbf{Sm}_S$.

i) Let $f\colon X \to Y$ a morphism. Suppose X has dimension d over S and Y has dimension e over S. Then the morphism

$$f^*\colon \mathbb{Z}_Y(a) \to \mathbb{Z}_X(a)$$

has dual the morphism

$$f_*\colon \mathbb{Z}_X(d-a)[2d] \to \mathbb{Z}_Y(e-a)[2e].$$

The morphism

$$f_*\colon \mathbb{Z}_X(a) \to \mathbb{Z}_Y(e-d+a)[2e-2d]$$

has dual

$$f^*\colon \mathbb{Z}_Y(d-a)[2d] \to \mathbb{Z}_X(d-a)[2d].$$

ii) Let $Z \in \mathcal{Z}^q(X/S)$ be a cycle. Then the morphism

$$\cup_X \mathrm{cl}^q_X(Z)\colon \mathbb{Z}_X(a) \to \mathbb{Z}_X(a+q)[2q]$$

has dual the morphism

$$\cup_X \mathrm{cl}^q_X(Z)\colon \mathbb{Z}_X(d-a-q)[2d-2q] \to \mathbb{Z}_X(d-a)[2d].$$

The morphism

$$\mathrm{cl}^q_X(Z)\colon 1 \to \mathbb{Z}_X(q)[2q]$$

has dual the composition

$$\mathbb{Z}_X(d-q)[2d-2q] \xrightarrow{\cup_X \mathrm{cl}^q_X(Z)} \mathbb{Z}_X(d)[2d] \xrightarrow{p_{X*}} 1.$$

Indeed, the computations of the dual of f^*, f_* and $\cup_X \mathrm{cl}^q_X(Z)$ follow easily from (7.5.1) and (7.5.3). For the dual of $\mathrm{cl}^q_X(Z)$ we have

$$\mathrm{cl}^q_X(Z) = (\cup \mathrm{cl}^q_X(Z)) \circ p_X^*,$$

hence

$$(\mathrm{cl}^q_X(Z))^D = (p_X^*)^D \circ (\cup \mathrm{cl}^q_X(Z))^D = p_{X*} \circ (\cup \mathrm{cl}^q_X(Z)).$$

□

(7.5.5) CHOW MOTIVES

We recall the construction of the category of *graded Chow motives* with R-coefficients Mot_R: let k be a field. The category Pre-Mot_R has objects $\mathcal{M}(X)(a)$, where X is a smooth projective k-scheme, and a an integer. The morphisms are given by

$$\mathrm{Hom}_{\text{Pre-Mot}_R}(\mathcal{M}(X)(a), \mathcal{M}(Y)(b)) = \mathrm{CH}^{d_X+b-a}(Y \times X) \otimes_{\mathbb{Z}} R$$

if X has dimension d_X over k. The composition law is given by

$$Z \circ W = p_{ZX*}(p^*_{ZY}(Z) \cup p^*_{YX}(W)),$$

for $W \in \mathrm{CH}^{d_X+n}(Y \times X)$ and $Z \in \mathrm{CH}^{d_Y+m}(Z \times Y)$. Pre-Mot_R is an R-tensor category, with direct sum being disjoint union, and tensor product induced by the product over k. The duality involution on Pre-Mot_R is given by the interchange of factors in $Y \times X$. The category Mot_R is the R-tensor category $\text{Pre-Mot}_{R\#}$ gotten from Pre-Mot_R by taking the pseudo-abelian hull.

Let R be a localization of $\mathbb{Z}$. It follows immediately from (7.5.1), (7.5.2) and (7.5.3) that sending $\mathcal{M}(X)(a)$ to $R_X(a)[2a]$ and

$$Z \in \mathrm{Hom}_{\text{Pre-Mot}_R}(\mathcal{M}(X)(a), \mathcal{M}(Y)(b)) = \mathrm{CH}^{d_X+b-a}(Y \times X) \otimes_{\mathbb{Z}} R$$

to

$$(\boxtimes_{Y,X} \circ \iota''_X)^{-1}(\mathrm{cl}^{d_X+b-a}_{Y\times_S X}(Z))\colon \mathbb{Z}_X(a)[2a] \to \mathbb{Z}_Y(b)[2b]$$

extends to an R-tensor functor

$$\mathrm{Mot}_R \to \mathcal{DM}(S)_R$$

compatible with the respective duality involutions. From (4.6.6), and (7.4.3), this functor is a fully faithful embedding in case $\mathrm{char}(k) = 0$, or if $R = \mathbb{Q}$.

7.6. *Homology and compactly supported cohomology*

We use the results of §7.4-5 to define and relate motivic homology, Borel-Moore homology, and compactly supported cohomology.

(7.6.1)

Let $\mathbf{Sm}^{\mathrm{pr}}_S$ be the full subcategory of $\mathbf{Sm}_S$ with objects being those X in $\mathbf{Sm}_S$ for which $\mathbb{Z}_X$ is in $\mathcal{DM}(S)^{\mathrm{pr}}$. In particular, if $X \in \mathbf{Sm}_S$ is projective over S, then X is in $\mathbf{Sm}^{\mathrm{pr}}_S$; more generally, if U is an open subset of a smooth projective S-scheme $X \in \mathbf{Sm}_S$, and if we can write the complement of U as

$$X - U = \cup_{i=1}^n D_i$$

such that the closed subsets $D_{i_1} \cap \ldots \cap D_{i_s}$ are smooth over S for each collection of indices $i_1, \ldots, i_s$, then, by (7.4.5), U is in $\mathbf{Sm}^{\mathrm{pr}}_S$.

(7.6.2) DEFINITION

i) Let X be in $\mathbf{Sm}_S$. The motivic homology of X, $H_p(X,\mathbb{Z}(q))$, is defined by

$$H_p(X,\mathbb{Z}(q)) = \mathrm{Hom}_{\mathcal{DM}(S)}(\mathbb{Z}_X(q)[p],1).$$

ii) Let X be in $\mathbf{Sm}_S^{\mathrm{pr}}$; suppose X is of dimension d over S. We define the object $\mathbb{Z}_X^{c/S}$ of $\mathcal{DM}(S)$ by

$$\mathbb{Z}_X^{c/S} = \mathbb{Z}_X^D(-d)[-2d].$$

We extend these notions to arbitrary $X \in \mathbf{Sm}_S^{\mathrm{pr}}$ by taking direct sums over the connected components of X. The compactly supported cohomology of X, $H^p_{c/S}(X,\mathbb{Z}(q))$, is defined by

$$H^p_{c/S}(X,\mathbb{Z}(q)) = \mathrm{Hom}_{\mathcal{DM}(S)}(1,\mathbb{Z}_X^{c/S}(q)[p]).$$

iii) Let X be in $\mathbf{Sm}_S^{\mathrm{pr}}$. The Borel-Moore homology of X, $H_p^{\mathrm{B.M.}}(X,\mathbb{Z}(q))$, is defined by

$$H_p^{\mathrm{B.M.}}(X,\mathbb{Z}(q)) = \mathrm{Hom}_{\mathcal{DM}(S)}(\mathbb{Z}_X^{c/S}(q)[p],1).$$

□

We recall from (6.4.11) the Borel-Moore motive $\mathbb{Z}_X^{\mathrm{B.M.}}$ of $X \in \mathbf{Sm}_S$. The identity

$$\mathbb{Z}_X^{c/S} = (\mathbb{Z}_X^{\mathrm{B.M.}})^D$$

for $X \in \mathbf{Sm}_S^{\mathrm{pr}}$ follows immediately from the definition (7.6.2). Applying the duality involution gives the natural isomorphisms

$$H_p^{\mathrm{B.M.}}(X,\mathbb{Z}(q)) \cong \mathrm{Hom}_{\mathcal{DM}(S)}(\mathbb{Z}_S,\mathbb{Z}_X^{\mathrm{B.M.}}(-q)[-p]),$$
$$H^p_{c/S}(X,\mathbb{Z}(q)) \cong \mathrm{Hom}_{\mathcal{DM}(S)}(\mathbb{Z}_X^{\mathrm{B.M.}}(-q)[-p],\mathbb{Z}_S).$$

(7.6.3) MOTIVIC HOMOLOGY

Sending X to $H_p(X,\mathbb{Z}(q))$ extends to a functor

$$H_p(-,\mathbb{Z}(q))\colon \mathbf{Sm}_S \to \mathbf{Ab};$$

for a map $f\colon X \to Y$ in $\mathbf{Sm}_S$, we denote the map $H_p(f,\mathbb{Z}(q))$ by f_*. The homotopy and Künneth isomorphisms, and the Mayer-Vietoris and Gysin distinguished triangles, yield the corresponding properties of the motivic homology groups: The map

$$p_*\colon H_p(X\times_S \mathbb{A}^1,\mathbb{Z}(q)) \to H_p(X,\mathbb{Z}(q))$$

is an isomorphism, there are external products in motivic homology, and there are natural long exact Mayer-Vietoris sequences for $X = U \cup V$, U and V Zariski opens in X:

$$\begin{aligned}\ldots \to & H_p(U \cup V, \mathbb{Z}(q)) \to H_p(U, \mathbb{Z}(q)) \oplus H_p(V, \mathbb{Z}(q)) \to H_p(X, \mathbb{Z}(q)) \\ & \to H_{p-1}(U \cup V, \mathbb{Z}(q)) \to \ldots .\end{aligned}$$

One has functorial pull-back for projective morphisms $f: Y \to X$ of relative dimension d:

$$f^*: H_p(X, \mathbb{Z}(q)) \to H_{p+2d}(Y, \mathbb{Z}(q+d)).$$

and a long exact Gysin sequence for $i: Z \to X$ a closed embedding in $\mathbf{Sm}_S$ of codimension d, with complement $j: U \to X$:

$$\begin{aligned}\ldots \to H_p(U, \mathbb{Z}(q)) \to & H_p(X, \mathbb{Z}(q)) \to H_{p-2d}(Z, \mathbb{Z}(q-d)) \\ & \to H_{p-1}(X, \mathbb{Z}(q)) \to \ldots .\end{aligned}$$

These properties all follow immediately from the analogous properties of the objects $\mathbb{Z}_X(q)$, etc., in the category $\mathcal{DM}(S)$.

We have the map

$$\begin{aligned}\mathrm{id}_X \cup_X (-): \mathrm{Hom}_{\mathcal{DM}(S)}(\mathbb{Z}_S, \mathbb{Z}_X(q')[p']) &\to \mathrm{Hom}_{\mathcal{DM}(S)}(\mathbb{Z}_X, \mathbb{Z}_X(q')[p']) \\ &\cong \mathrm{Hom}_{\mathcal{DM}(S)}(\mathbb{Z}_X(q-q')[p-p'], \mathbb{Z}_X(q)[p])\end{aligned}$$

defined as the composition

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{DM}(S)}(\mathbb{Z}_S, \mathbb{Z}_X(q')[p']) \\ &\qquad \xrightarrow{\mathrm{id}_X \otimes (-)} \mathrm{Hom}_{\mathcal{DM}(S)}(\mathbb{Z}_X \otimes \mathbb{Z}_S, \mathbb{Z}_X \otimes \mathbb{Z}_X(q')[p']) \\ &\qquad \xrightarrow{\boxtimes_{X,X} \circ (-) \circ \boxtimes_{X,S}^{-1}} \mathrm{Hom}_{\mathcal{DM}(S)}(\mathbb{Z}_X, \mathbb{Z}_{X \times_S X}(q')[p']) \\ &\qquad \xrightarrow{\Delta_X^* \circ (-)} \mathrm{Hom}_{\mathcal{DM}(S)}(\mathbb{Z}_X, \mathbb{Z}_X(q')[p']) \\ &\qquad \cong \mathrm{Hom}_{\mathcal{DM}(S)}(\mathbb{Z}_X(q-q')[p-p'], \mathbb{Z}_X(q)[p]).\end{aligned}$$

Combining $\mathrm{id}_X \cup_X (-)$ with the operation of composition

$$\begin{aligned}\mathrm{Hom}_{\mathcal{DM}(S)}(\mathbb{Z}_X(q-q')[p-p'], \mathbb{Z}_X(q)[p]) &\otimes \mathrm{Hom}_{\mathcal{DM}(S)}(\mathbb{Z}_X(q)[p], \mathbb{Z}_S) \\ \to \mathrm{Hom}_{\mathcal{DM}(S)}(\mathbb{Z}_X(q-q')[p-p'], \mathbb{Z}_S)\end{aligned}$$

gives us the cap product pairing

$$\cap^X: H^{p'}(X, \mathbb{Z}(q')) \otimes H_p(X, \mathbb{Z}(q)) \to H_{p-p'}(X, \mathbb{Z}(q-q')).$$

If we have purity for motivic cohomology (e.g., $S = \mathrm{Spec}(k)$ for k a field of characteristic zero, see (6.2.5)), we have the canonical identification

$$H_0(S, \mathbb{Z}_S(0)) \cong \mathbb{Z};$$

we may then follow $\cap^X$ with the push-forward by the structure morphism p_X to give the pairing

$$H^p(X, \mathbb{Z}(q)) \otimes H_p(X, \mathbb{Z}(q)) \to \mathbb{Z}.$$

Let $f: Y \to X$ be a morphism in $\mathbf{Sm}_S$; one easily verifies the identity

$$f^* \circ (\mathrm{id}_X \cup \alpha) = (\mathrm{id}_Y \cup f^*(\alpha)) \circ f^*$$

for $\alpha: \mathbb{Z}_S \to \mathbb{Z}_X(p')[q']$. This, together with the associativity of composition, immediately implies the projection formula:

$$f_*(f^*(\alpha) \cap^Y \beta) = \alpha \cap^X f_*(\beta)$$

for elements $\alpha \in H^{p'}(X, \mathbb{Z}(q'))$, $\beta \in H_p(Y, \mathbb{Z}(q))$.

(7.6.4) COHOMOLOGY WITH COMPACT SUPPORTS, AND BOREL-MOORE HOMOLOGY

For a morphism $f: X \to Y$ in $\mathbf{SM}_S^{\mathrm{pr}}$ of relative dimension d_f, we let

$$f_!: \mathbb{Z}_X^{c/S} \to \mathbb{Z}_Y^{c/S}(-d_f)[-2d_f] \tag{7.6.4.1}$$

denote the (shifted and twisted) dual of the pull-back map

$$f^*: \mathbb{Z}_Y \to \mathbb{Z}_X$$

If f is projective, the shifted and twisted dual of the push-forward f_* defines the morphism

$$f^!: \mathbb{Z}_Y^{c/S} \to \mathbb{Z}_X^{c/S} \tag{7.6.4.2}$$

The maps $f_!$ are functorial after shift and twist:

$$(f \circ g)_! = (f_!)(-d_g)[-2d_g] \circ g_!.$$

Sending X to $\mathbb{Z}_X^{c/S}$ and f to $f^!$ defines the functor

$$\mathbb{Z}^{c/S}: \mathbf{Sm}_{S,\mathrm{proj}}^{\mathrm{op}} \to \mathcal{DM}(S).$$

The maps $f^!$ and $f_!$ induce maps on $H^p_{c/S}(X, \mathbb{Z}(q))$ and $H_p^{\mathrm{B.M.}}(X, \mathbb{Z}(q))$ in the obvious way: for $f: X \to Y$ of relative dimension d, we have

$$f_!: H^p_{c/S}(X, \mathbb{Z}(q)) \to H^{p-2d}_{c/S}(Y, \mathbb{Z}(q-d));$$

the groups $H_p^{\mathrm{B.M.}}(X, \mathbb{Z}(q))$ have a similar contravariant functoriality

$$f_!^*: H_p^{\mathrm{B.M.}}(Y, \mathbb{Z}(q)) \to H_{p+2d}^{\mathrm{B.M.}}(X, \mathbb{Z}(q+d)).$$

We have the homotopy property, Mayer-Vietoris sequences for Zariski open covers and reverse functoriality for projective morphisms:

$$f^!: H^p_{c/S}(Y, \mathbb{Z}(q)) \to H^p_{c/S}(X, \mathbb{Z}(q))$$
$$f^{!*}: H_p^{\mathrm{B.M.}}(X, \mathbb{Z}(q)) \to H_p^{\mathrm{B.M.}}(Y, \mathbb{Z}(q)).$$

Taking the dual of the Gysin distinguished triangle (6.1.2.2) gives the Gysin distinguished triangle for motives with compact support

$$\mathbb{Z}_U^{c/S} \xrightarrow{j_!} \mathbb{Z}_X^{c/S} \xrightarrow{i^!} \mathbb{Z}_Z^{c/S} to \mathbb{Z}_U^{c/S}[1]$$

if any two of X, U and Z are in $\mathbf{Sm}_S^{\mathrm{pr}}$. This gives rise to the standard Gysin sequences for Borel-Moore homology and cohomology with compact supports.

Taking the inverse of the dual of the Künneth isomorphism gives the Künneth isomorphism

$$\boxtimes_{X,Y}^{c/S}: \mathbb{Z}_X^{c/S} \otimes \mathbb{Z}_Y^{c/S} \to \mathbb{Z}_{X\times Y}^{c/S}$$

This gives external products in compactly supported cohomology

$$\cup_{X,Y}^{c/S}: H^p_{c/S}(X, \mathbb{Z}(q)) \otimes H^{p'}_{c/S}(Y, \mathbb{Z}(q')) \to H^{p+p'}_{c/S}(X \times_S Y, \mathbb{Z}(q+q'))$$

If $X = Y$, we may then pull-back by $\Delta_X^!$, giving cup product in compactly supported cohomology:

$$\cup_X^{c/S}: H^p_{c/S}(X, \mathbb{Z}(q)) \otimes H^{p'}_{c/S}(X, \mathbb{Z}(q')) \to H^{p+p'}_{c/S}(X, \mathbb{Z}(q+q')),$$
$$\cup_X^{c/S} = \Delta_X^! \circ \cup_{X,X}^{c/S}.$$

This makes

$$H^*_{c/S}(X, \mathbb{Z}(*)) := \oplus_{p,q} H^p_{c/S}(X, \mathbb{Z}(q))$$

into a bi-graded ring (without unit).

Composition defines as above the cap product

$$\cap_X^c: H^{p'}_{c/S}(X, \mathbb{Z}(q')) \otimes H_p^{\mathrm{B.M.}}(X, \mathbb{Z}(q)) \to H_{p-p'}^{\mathrm{B.M.}}(X, \mathbb{Z}(q-q'))$$

satisfying

$$f_{!*}(f_!(\alpha) \cap_Y^c \beta) = \alpha \cap_X^c f_!^*(\beta)$$

for $\alpha \in H^{p'}_{c/S}(X, \mathbb{Z}(q'))$, $\beta \in H^{\mathrm{B.M.}}_p(Y, \mathbb{Z}(q))$, and $f\colon X \to Y$ a morphism in $\mathbf{Sm}_S$. This gives

$$H^{\mathrm{B.M.}}_*(X, \mathbb{Z}(*)) := \oplus_{p,q} H^{\mathrm{B.M.}}_p(X, \mathbb{Z}(q))$$

the structure of a bi-graded module over $H^*_{c/S}(X, \mathbb{Z}(*))$.

(7.6.5) Poincaré duality

For X smooth and projective of dimension d over S, the identity (7.4.1.1)

$$\mathbb{Z}_X^D = \mathbb{Z}_X(d)[2d],$$

the identification (7.5.4)

$$f^{*D} = f_*; \quad f_*^D = f^*$$

for a morphism $f\colon X \to Y$ of smooth projective S-schemes, and the fact that duality is an exact involution, gives the functorial isomorphisms

$$\begin{aligned} &H_p(X, \mathbb{Z}(q)) \cong H^{2d-p}(X, \mathbb{Z}(2d-q)) \\ &H^p_{c/S}(X, \mathbb{Z}(q)) \cong H^p(X, \mathbb{Z}(q)) \\ &H^{\mathrm{B.M.}}_p(X, \mathbb{Z}(q)) \cong H_p(X, \mathbb{Z}(q)) \cong H^{2d-p}(X, \mathbb{Z}(2d-q)). \end{aligned}$$

Via these isomorphisms, the cap products defined above are identified with the cup product in motivic cohomology. □

7.7. *Relative homology and cohomology*

We identify the compactly supported cohomology with the cohomology of a projective compactification over S, relative to a "normal crossing complement at infinity" in case such exists; we identify the Borel-Moore homology with a similarly defined relative homology group. In particular, if $S = \mathrm{Spec}(k)$ for a field k, and if one has resolution of singularities for quasi-projective k schemes, then one has this interpretation of compactly supported motivic cohomology, and Borel-Moore homology, for all quasi-projective k-schemes. We begin with the construction of relative motivic cohomology.

(7.7.1) n-cubes

Let $<n>$ be opposite of the category associated to the partially ordered set of subsets of the finite set $\{1, \ldots, n\}$, i.e., an object of $<n>$ is a subset I of $\{1, \ldots, n\}$, and there is a morphism $J \to I$ if and only if $J \supset I$. The category $<n>$ is usually called the *n-cube*. For a category $\mathcal{C}$ we have the category $\mathcal{C}(<n>)$, the category of *n-cubes in* $\mathcal{C}$, being the category of functors

$$X\colon <n> \to \mathcal{C}.$$

(7.7.2) LIFTING n-CUBES TO $\mathcal{L}(\mathcal{V})$

Let

$$X_*: \langle n\rangle \to \mathbf{Sm}_S,$$
$$I \mapsto X_I,$$

be a functor and let $(X_\emptyset, f_\emptyset: X' \to X_\emptyset)$ be a lifting of $X_\emptyset$ to an object of $\mathcal{L}(\mathbf{Sm}_S)$. For each $I \subset \{1, \ldots, n\}$, form the Cartesian diagram

$$\begin{array}{ccc} X'_I := X' \times_{X_\emptyset} X_I & \xrightarrow{p_1} & X' \\ {\scriptstyle f_I := p_2}\downarrow & & \downarrow{\scriptstyle f_\emptyset} \\ X_I & \xrightarrow[X_{I\supset\emptyset}]{} & X_\emptyset. \end{array}$$

The maps $X_{J\supset I}$ induce the maps

$$X'_{J\supset I}: X'_J \to X'_I$$

defining the n-cube

$$X'_*: : \langle n\rangle \to \mathbf{Sch}_S;$$

the maps f_I give the map of n-cubes

$$f_*: X'_* \to X_*.$$

Supposing that the X'_I are smooth over S for all I, we define the lifting of X_* to a functor

$$(X_*, f^X_*): \langle n\rangle \to \mathcal{L}(\mathbf{Sm}_S) \tag{7.7.2.1}$$

by setting

$$X'_{*\supset I} := \coprod_{J\subset I} X'_J,$$

$$f^X_I := \cup_{J\subset I} X_{J\supset I} \circ f_J,$$

$$f^X_I: X'_{*\supset I} \to X_I.$$

(compare with (4.1.6.1)). We then apply the functor

$$\mathbb{Z}(0): \mathcal{L}(\mathbf{Sm}_S)^{\mathrm{op}} \to \mathcal{A}_{mot}(\mathbf{Sm}_S)$$

to (7.7.2.1), and take the associated complex with term $\oplus_{I,|I|=s}\mathbb{Z}_{X_I}(0)_{f^X_I}$ in degree s, and differential

$$\partial^s: \oplus_{I,|I|=s}\mathbb{Z}_{X_I}(0)_{g^X_I} \to \oplus_{I,|I|=s+1}\mathbb{Z}_{X_I}(0)_{f^X_I}$$

given by setting

$$\partial^s_{I,i}: \mathbb{Z}_{X_I}(0)_{f^X_I} \to \mathbb{Z}_{X_{I\cup\{i\}}}(0)_{f^X_{I\cup\{i\}}}$$

$$\partial^s_{I,i} = \begin{cases} X^*_{I\cup\{i\}\supset I} & \text{for } i \notin I \\ 0 & \text{for } i \in I \end{cases}$$

$$\partial^s = \sum_{I,|I|=s} \sum_{i=1}^n (-1)^i \partial^s_{I,i}.$$

We denote the resulting object of $\mathbf{C}^b_{mot}(\mathbf{Sm}_S)$ by $\mathbb{Z}_{X_*}(0)_{f_\emptyset}$.

If we take $f_\emptyset = \mathrm{id}_{X_\emptyset}$, then sending X^* to $\mathbb{Z}_{X^*}(0) := \mathbb{Z}_{X^*}(0)_{\mathrm{id}_{X_\emptyset}}$ gives the functor

$$\mathbb{Z}(0): \mathcal{L}(\mathbf{Sm}_S)(<n>) \to \mathbf{C}^b_{mot}(\mathbf{Sm}_S) \tag{7.7.2.2}$$

extending the functor (4.1.7.1)

$$\mathbb{Z}(0): \mathcal{L}(\mathbf{Sm}_S) \to \mathcal{A}^b_{mot}(\mathbf{Sm}_S).$$

(7.7.3) n-cubes and Cones

The main utility of the n-cube follows from the elementary remark that the category of n-cubes in a category $\mathcal{C}$ is equivalent to the category of maps of $n-1$-cubes in $\mathcal{C}$ by associating to a map of $n-1$-cubes

$$f_*: X^-_* \to X^+_*$$

the n-cube $X(f_*)_*$ with

$$X(f_*)_I = \begin{cases} X^-_I & \text{if } n \notin I, \\ X^+_{I\setminus\{n\}} & \text{if } n \in I, \end{cases}$$

$$X(f_*)_{J\supset I} = \begin{cases} X^-_{J\supset I} & \text{if } n \notin J, \\ X^+_{J\setminus\{n\}\supset I\setminus\{n\}} & \text{if } n \in I, \end{cases}$$

$$X(f_*)_{I\cup\{n\}\supset I} = f_I \text{ for } I \subset \{1, \ldots, n-1\}.$$

(this unique determines $X(f_*)_*$). If we have an n-cube X_* in $\mathcal{L}(\mathbf{Sm}_S)$, which we may then write as

$$X_* = X(f_*)_*$$

for the uniquely determined map f_* of $n-1$-cubes in $\mathcal{L}(\mathbf{Sm}_S)$, we have the identity

$$\mathbb{Z}_{X_*}(0) = \mathrm{Cone}(\mathbb{Z}_{f_*}(0): \mathbb{Z}_{X^-_*}(0) \to \mathbb{Z}_{X^+_*}(0))[-1]. \tag{7.7.3.1}$$

Thus, each n-cube in $\mathcal{L}(\mathbf{Sm}_S)$ gives rise to a sequence of linked distinguished triangles in $\mathbf{K}^b_{mot}(\mathbf{Sm}_S)$

(7.7.4) Relative motives

Suppose we have a smooth S-scheme X, with subschemes

$$D_1, \ldots, D_n \subset X$$

such that, for each index $I = (1 \leq i_1 < \ldots < i_s \leq n)$, the subscheme D_I,

$$D_I := D_{i_1} \cap \ldots \cap D_{i_s}$$

is smooth over S. We say in this case that $D_1, \ldots, D_n$ *intersect transversely*. Suppose we have a lifting $(X, f\colon X' \to X)$ of X to $\mathcal{L}(\mathcal{V})$ such that the pull-backs

$$f_I := p_2\colon X' \times_X D_I \to D_I$$

are in $\mathbf{Sm}_S$. We let

$$(X; D_1, \ldots, D_n)_*\colon <n> \to \mathbf{Sm}_S$$

be the n-cube in $\mathbf{Sm}_S$ with

$$(X; D_1, \ldots, D_n)_I = D_I;$$

for $J \subset I$, we let

$$(X; D_1, \ldots, D_n)_{J \subset I}\colon D_J \to D_I$$

be the inclusion. Applying the construction described above gives the lifting of the n-cube $(X; D_1, \ldots, D_n)^*$ to the n-cube

$$((X; D_1, \ldots, D_n)_*, f_*^X)\colon <n> \to \mathcal{L}(\mathbf{Sm}_S)$$

which in turn gives us the object $\mathbb{Z}_{(X;D_1,\ldots,D_n)}(0)_f$ of $\mathbf{C}^b_{mot}(\mathbf{Sm}_S)$; the identification (7.7.3.1) of $\mathbb{Z}_{X_*}(0)_f$ as a Cone gives us the distinguished triangle in $\mathbf{K}^b_{mot}(\mathbf{Sm})$:

$$\begin{aligned}\mathbb{Z}_{(X;D_1,\ldots,D_n)}(0)_f \to \mathbb{Z}_{(X;D_1,\ldots,D_{n-1})}(0)_f \to \mathbb{Z}_{(D_n;D_{1,n},\ldots,D_{n-1,n})}(0)_{f_n} \\ \xrightarrow{i_n} \mathbb{Z}_{(X;D_1,\ldots,D_n)}(0)_f[1]\end{aligned} \tag{7.7.4.1}$$

We call the object $\mathbb{Z}_{(X;D_1,\ldots,D_n)}(0)$ of $\mathcal{DM}(S)$ the *motive of X, relative to $D_1, \ldots, D_n$*. We define the *relative motivic cohomology* $H^p((X; D_1, \ldots, D_n), \mathbb{Z}(q))$ by

$$H^p((X; D_1, \ldots, D_n), \mathbb{Z}(q)) = \mathrm{Hom}_{\mathcal{DM}(S)}(1, \mathbb{Z}_{(X;D_1,\ldots,D_n)}(q)[p]).$$

(7.7.5) Relative motives and duality

Let X be smooth and projective over S. Let $D_1, \ldots, D_n$ be closed subschemes of X which intersect transversely. Let $U = X \setminus \cup_{i=1}^n D_i$. We call the collection $(X; D_1, \ldots, D_n)$ a *good compactification of U over S*. If U admits a good compactification, then, by (7.4.5), U is in $\mathbf{Sm}_S^{\mathrm{pr}}$.

Let X be in $\mathbf{Sm}_S$, $E, D_1, \ldots, D_n$ closed subschemes of X which intersect transversely. Let $U = X\setminus(D_1 \cup \ldots \cup D_n)$, and let

$$\delta_U : U \to X \times_S U$$
$$\delta_U(u) = (u, u),$$

be the diagonal inclusion. Consider the object

$$\mathbb{Z}_{(X\times_S U; D_1\times_S U, \ldots, D_n\times_S U)}(0)_{\delta_U}$$

of $\mathbf{C}^b_{mot}(\mathbf{Sm}_S)$. By (3.2.4), we have identification

$$\mathrm{Hom}_{\mathbf{K}^b_{mot}(\mathbf{Sm}_S)}(\mathfrak{e}\otimes 1, \mathbb{Z}_{(X\times_S U; D_1\times_S U, \ldots, D_n\times_S U)}(q)_{\delta_U}[2q])$$
$$\cong \cap_{j=1}^n \ker(i_{D_j^*} : \mathcal{Z}^q(X\times_S U)_{\delta_U} \to \mathcal{Z}^q(D_i \times_S U))$$

In particular, the diagonal $\Delta_U := \delta_{U*}(|U|)$ gives the map (see (1.2.10))

$$[\Delta_U]^S : \mathfrak{e}\otimes 1 \to \mathbb{Z}_{(X\times_S U; D_1\times_S U, \ldots, D_n\times_S U)}(d_U)_{\delta_U}[2d_U];$$

and the cycle class map in $\mathbf{D}^b_{mot}(\mathbf{Sm}_S)$:

$$\mathrm{cl}(\Delta_U) : 1 \to \mathbb{Z}_{(X\times_S U; D_1\times_S U, \ldots, D_n\times_S U)}(d_U)_{\delta_U}[2d_U];$$

We let

$$\iota_U : 1 \to \mathbb{Z}_{(X; D_1, \ldots, D_n)} \otimes \mathbb{Z}_U(d_U)[2d_U] \tag{7.7.5.1}$$

be the map in $\mathbf{D}^b_{mot}(\mathbf{Sm}_S)$ defined by composing $\mathrm{cl}(\Delta_U)$ with the inverse of the isomorphism

$$\mathbb{Z}_{(X; D_1, \ldots, D_n)} \otimes \mathbb{Z}_U(d_U)[2d_U] \xrightarrow{\boxtimes_{*,U}} \mathbb{Z}_{(X\times_S U; D_1\times_S U, \ldots, D_n\times_S U)}(d_U)[2d_U]$$
$$\xrightarrow{\rho^{-1}_{\delta_U}} \mathbb{Z}_{(X\times_S U; D_1\times_S U, \ldots, D_n\times_S U)}(d_U)_{\delta_U}[2d_U]$$

Suppose we have a closed subschemes $E, D_1, \ldots, D_n$ of X, with transverse intersection; we suppose that E has codimension $d_{E:X}$ in X. For $I = (i_1, \ldots, i_s)$, let $D_I = \cap_j D_{i_j}$, and let $i_{E,I} : E\cap D_I \to D_I$ be the inclusion. The collection of maps $i^*_{E,I}$ defines the morphism

$$i^*_E : \mathbb{Z}_{(X; D_1, \ldots, D_n)} \to \mathbb{Z}_{(E; E\cap D_1, \ldots, E\cap D_n)}. \tag{7.7.5.2}$$

(7.7.6) LEMMA

Let X be in $\mathbf{Sm}_S$, $E, D_1, \ldots, D_n$ closed subschemes of X which intersect transversely. Let $U = X\setminus(D_1 \cup \ldots \cup D_n)$, and let $E_U = E\cap U$. Then

i) the pair $(\mathbb{Z}_{(X; D_1, \ldots, D_n)}, \iota_U)$ (7.7.5.1) is the dual of $\mathbb{Z}_U(d_U)[2d_U]$.

ii) The map

$$i_{E_U*}\colon \mathbb{Z}_{E_U}(-d_{E:X})[-2d_{E:X}] \to \mathbb{Z}_U(0)$$

is the dual of the map (7.7.5.2).

Proof. We prove (i) and (ii) together by induction on n, the case $n = 0$ for (i) being the definition (7.4.1.1) of the dual of $\mathbb{Z}_X(d_X)[2d_X]$ for X smooth and projective over S, and for (ii) being (7.5.4). We may suppose that X is equi-dimensional over S and each D_i has pure codimension d_i on X.

Let $V = X\setminus(D_1 \cup \ldots \cup D_{n-1})$, and $D_V = D_n \cap V$. Let

$$d = d_X = d_V = d_U, \quad d' = d_{D_V},$$

and let

$$j\colon U \to V, \quad i\colon D_V \to V$$

be the inclusions. We have the (shifted) Gysin distinguished triangle (6.1.2.2)

$$\mathbb{Z}_V(d)[2d] \xrightarrow{j^*} \mathbb{Z}_U(d)[2d] \longrightarrow \mathbb{Z}_{D_V}(d')[2d'+1] \xrightarrow{i_*} \mathbb{Z}_V(d)[2d+1] \tag{1}$$

Applying our induction hypothesis, the dual of the map i_* is the map

$$i^*\colon \mathbb{Z}_{(X;D_1,\ldots,D_{n-1})}[-1] \to \mathbb{Z}_{(D_n;D_{1,n},\ldots,D_{n-1,n})}[-1],$$

which fits into the distinguished triangle

$$\begin{aligned}\mathbb{Z}_{(X;D_1,\ldots,D_{n-1})}[-1] \xrightarrow{i^*} \mathbb{Z}_{(D_n;D_{1,n},\ldots,D_{n-1,n})}[-1] \xrightarrow{i_n} \mathbb{Z}_{(X;D_1,\ldots,D_n)}\\ \xrightarrow{j_n} \mathbb{Z}_{(X;D_1,\ldots,D_{n-1})}.\end{aligned} \tag{2}$$

Letting $\mathbb{Z}_{V,D_V}$ denote as usual the motive of V with supports on D_V:

$$\mathbb{Z}_{V,D_V} := \mathrm{Cone}(j^*\colon \mathbb{Z}_V \to \mathbb{Z}_U),$$

the distinguished triangle (1) is by definition isomorphic to the distinguished triangle

$$\begin{aligned}\mathbb{Z}_V(d)[2d] \xrightarrow{j^*} \mathbb{Z}_U(d)[2d] \xrightarrow{i_1} \mathbb{Z}_{V,D_V}(d)[2d+1]\\ \xrightarrow{j_0} \mathbb{Z}_V(d)[2d+1]\end{aligned} \tag{3}$$

In addition, the sequences (2) and (3) are Cone sequences in $\mathbf{C}^b_{mot}(\mathbf{Sm}_S)$, hence define distinguished triangles in $\mathbf{K}^b_{mot}(\mathbf{Sm}_S)$.

We may then form the 4×4 diagram by tensoring (2) with (3), using the tensor product $\times$ in the category $\mathbf{C}^b(\mathbf{Sm}_S)^*$. Let

$$X_1 = \mathbb{Z}_{(D_n;D_{1,n},\ldots,D_{n-1,n})}[-1],\ Y_1 = \mathbb{Z}_{(X;D_1,\ldots,D_n)},\ Z_1 = \mathbb{Z}_{(X;D_1,\ldots,D_{n-1})}$$

$$X_2 = \mathbb{Z}_{V,D_V}(d)[2d+1],\ Y_2 = \mathbb{Z}_U(d)[2d],\ Z_2 = \mathbb{Z}_V(d)[2d].$$

and let

$$K = \mathrm{Cone}(\mathrm{id} \times j_0 - i^*[1] \times \mathrm{id}\colon X_1 \times X_2 \oplus Z_1 \times Z_2 \to X_1 \times Z_2[1])[-1].$$

As in §7.2, (see (7.2.1.5) and (7.2.2)) we have the maps in $\mathbf{C}^b_{mot}(\mathbf{Sm}_S)$:

$$\begin{array}{ccc} K & \xrightarrow{\beta} & Y_1 \times Y_2 \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle (j_n \times \mathrm{id}, \mathrm{id} \times i_1)} \\ X_1 \times X_2 \oplus Z_1 \times Z_2 & \xrightarrow{i_n \times \mathrm{id} \oplus \mathrm{id} \times j^*} & Y_1 \times X_2 \oplus Z_1 \times Y_2 \\ {\scriptstyle \mathrm{id} \times j_0 - i^* \times \mathrm{id}}\downarrow & & \\ X_1 \times Z_2[1] & & \end{array} \tag{4}$$

where the left-hand column is the Cone sequence. The map β is given as follows: The identity

$$\mathbb{Z}_{V,D_V}(d)[2d+1] = \mathrm{Cone}(j^*\colon \mathbb{Z}_V(d)[2d] \to \mathbb{Z}_U(d)[2d])$$

gives the map in $\mathbf{C}^b_{mot}(\mathbf{Sm}_S)$

$$\gamma\colon X_1 \times Y_2 \to K$$

with $q \circ \gamma = \mathrm{id} \times i_1$. Similarly, the identity

$$\mathbb{Z}_{(X;D_1,\ldots,D_n)} = \mathrm{Cone}(i^*\colon \mathbb{Z}_{(X;D_1,\ldots,D_{n-1})} \to \mathbb{Z}_{(D_n;D_{1,n},\ldots,D_{n-1,n})})[-1]$$

gives the map

$$\gamma'\colon Y_1 \times Z_2 \to K$$

with $\alpha \circ \gamma' = j_n \times \mathrm{id}$. Letting

$$\gamma''\colon X_1 \times Z_2 \to X_1 \times Y_2 \oplus Y_1 \times Z_2$$

be the map

$$\gamma'' = (\mathrm{id} \times j^*, -i_n \times \mathrm{id})$$

we have the Cone sequence in $\mathbf{C}^b_{mot}(\mathbf{Sm}_S)$:

$$X_1 \times Z_2 \xrightarrow{\gamma''} X_1 \times Y_2 \oplus Y_1 \times Z_2 \xrightarrow{\gamma + \gamma'} K \to X_1 \times Z_2[1]. \tag{5}$$

Let

$$\beta'\colon X_1 \times Y_2 \oplus Y_1 \times Z_2 \to Y_1 \times Y_2$$

be the map

$$\beta' = i_n \times \mathrm{id} + \mathrm{id} \times j^* \tag{6}$$

Then $\beta' \circ \gamma'' = 0$, and β is the map determined by β'.

Now let Δ_V be the diagonal in $X \times V$, and Δ_{D_V} the diagonal in $D_n \times D_V$. Let $\mathcal{K}$ be the localization of $\mathbf{K}^b_{mot}(\mathbf{Sm}_S)$ with respect to the maps of (2.1.3)(e), i.e, we invert the maps

$$\mathrm{id}_Y^*: \mathbb{Z}_Y(d)_f \to \mathbb{Z}_Y(d)$$

The cycles Δ_V and Δ_{D_V} determine the maps in $\mathcal{K}$ (see (1.2.10))

$$[\Delta_V]^S: \mathfrak{e} \otimes 1 \to Z_1 \times Z_2$$
$$[\Delta_{D_V}]^S: \mathfrak{e} \otimes 1 \to X_1 \times \mathbb{Z}_{D_V}(d')[2d']$$
$$[i_*(\Delta_{D_V})]^S: \mathfrak{e} \otimes 1 \to X_1 \times \mathbb{Z}_V(d)[2d]$$
$$[(\mathrm{id} \times i)_*(\Delta_{D_V})]^S_{D_V}: \mathfrak{e} \otimes 1 \to X_1 \times \mathbb{Z}_{V,D_V}(d)[2d]$$

By (6.2.3), we have

$$(\mathrm{id} \times i)_* \circ [\Delta_{D_V}]^S = [(\mathrm{id} \times i)_*(\Delta_{D_V})]^S_{D_V} \tag{7}$$

in $\mathbf{D}^b_{mot}(\mathbf{Sm}_S)$. Let

$$\mathrm{cl}_{D_V}((\mathrm{id} \times i)_*(\Delta_{D_V})): 1 \to X_1 \times X_2$$

be the cycle class map with supports corresponding to $[(\mathrm{id} \times i)_*(\Delta_{D_V})]^S_{D_V}$, and let

$$\boxtimes: X_1 \otimes X_2 \to X_1 \times X_2$$

be the external product.

By our induction hypothesis, the cycle class maps in $\mathbf{D}^b_{mot}(\mathbf{Sm}_S)$

$$\mathrm{cl}(\Delta_V): 1 \to Z_1 \times Z_2$$
$$\mathrm{cl}(\Delta_{D_V}): 1 \to X_1 \times \mathbb{Z}_{D_V}(d')[2d'],$$

composed with inverse of the respective external products

$$\boxtimes: Z_1 \otimes Z_2 \to Z_1 \times Z_2$$
$$\boxtimes: X_1 \otimes \mathbb{Z}_{D_V}(d')[2d'] \to X_1 \times \mathbb{Z}_{D_V}(d')[2d'],$$

gives the duals

$$(Z_1, \boxtimes^{-1} \circ \mathrm{cl}(\Delta_V)), \quad (X_1, \boxtimes^{-1} \circ \mathrm{cl}(\Delta_{D_V}))$$

to Z_2 and $\mathbb{Z}_{D_V}(d')[2d']$. By (7),

$$(X_1, \boxtimes^{-1} \circ \mathrm{cl}_{D_V}((\mathrm{id} \times i)_*(\Delta_{D_V})))$$

is the dual to X_2.

The matrix

$$\begin{pmatrix} [(\mathrm{id} \times i)_*(\Delta_{D_V})]^S_{D_V} \\ 0, [\Delta_V] \end{pmatrix}$$

determines the map

$$\delta\colon \mathfrak{e}\otimes 1 \to \mathrm{Cone}(\gamma'') \cong K.$$

in $\mathcal{K}$. By the functoriality of the cycle maps, it follows easily that

$$q\circ\delta = (\mathrm{cl}(\Delta_V), \mathrm{cl}_{D_V}((\mathrm{id}\times i)_*(\Delta_{D_V}))).$$

By the remark (7.2.4), this implies that $(Y_1, \boxtimes^{-1}\beta\circ\delta\circ\nu_1^{-1})$ is the dual of Y_2, where

$$\boxtimes\colon Y_1\otimes Y_2 \to Y_1\times Y_2$$

is the external product and

$$\nu_1\colon \mathfrak{e}\otimes 1\to 1$$

the isomorphism (2.2.4.1).

Since

$$\beta\circ\delta = \mathrm{cl}(\Delta_U),$$

part (i) is proven.

Part (ii) follows from (i), the identity of cycles on $E\times U$

$$(\mathrm{id}_E\times i_{E_U})_*(\Delta_{E_U}) = (i_E\times \mathrm{id}_U)^*(\Delta_U)$$

and (6.2.3). □

(7.7.7) Proposition

Let $(X; D_1,\ldots,D_n)$ be a good compactification of U over S. Then there is a canonical isomorphism

$$\mathbb{Z}_{(X;D_1,\ldots,D_n)} \to \mathbb{Z}_U^{c/S}.$$

In particular, if $(X; D_1,\ldots,D_n)$ and $(X'; D'_1,\ldots,D'_m)$ are two good compactifications of U over S, then there is a canonical isomorphism

$$\mathbb{Z}_{(X;D_1,\ldots,D_n)} \to \mathbb{Z}_{(X';D'_1,\ldots,D'_m)}$$

Proof. This follows from (7.7.6), (7.1.3) and the definition of $\mathbb{Z}_U^{c/S}$ (7.6.2). □

(7.7.8) Borel-Moore homology as relative homology

We define the relative motivic homology $H_p((X; D_1,\ldots,D_n),\mathbb{Z}(q))$ by

$$H_p((X; D_1,\ldots,D_n),\mathbb{Z}(q)) = \mathrm{Hom}_{\mathcal{DM}(S)}(\mathbb{Z}_{(X;D_1,\ldots,D_n)}(q)[p], \mathbb{Z}_S); \tag{7.7.8.1}$$

this is compatible with our earlier definition (7.7.4) of relative motivic cohomology as

$$H^p((X; D_1,\ldots,D_n),\mathbb{Z}(q)) = \mathrm{Hom}_{\mathcal{DM}(S)}(\mathbb{Z}_S, \mathbb{Z}_{(X;D_1,\ldots,D_n)}(q)[p]).$$

Via (7.7.6) and (7.7.7), we may identify the Borel-Moore homology, respectively the compactly supported cohomology, of an S-scheme U which admits a good compactification $(X; D_1, \ldots, D_n)$ with relative motivic (co)homology:

$$\begin{aligned} H_p^{\mathrm{B.M.}}(U, \mathbb{Z}(q)) &\cong H_p((X; D_1, \ldots, D_n), \mathbb{Z}(q)) \\ H^p_{c/S}(U, \mathbb{Z}(q)) &\cong H^p((X; D_1, \ldots, D_n), \mathbb{Z}(q)). \end{aligned} \tag{7.7.8.2}$$

7.8. *Homology and cohomology of singular schemes*

We now relate the Borel-Moore homology to homology with supports in a "smoothly decomposable" closed subscheme (7.8.1)(i). This enables us to extend the definition of the Borel-Moore motive and Borel-Moore homology to such S-schemes. We also consider the extension of the motive with compact support to certain S-schemes which are not smooth over S: those which are smoothly decomposable and admit a "compactifiable" closed embedding into a smooth S-scheme (7.8.2)(ii). For such S-schemes, we define the motive with compact supports and the resulting motivic cohomology with compact supports.

(7.8.1)

We recall from (2.1.2.1), for W a closed subset of a smooth, quasi-projective S-scheme X with complement $j: U \to X$, we have the "motive of X with supports in W", $\mathbb{Z}_{X,W}$ defined by

$$\mathbb{Z}_{X,W} = \mathrm{Cone}(j^*: \mathbb{Z}_X \to \mathbb{Z}_U)[-1].$$

If X and U are in $\mathbf{Sm}_S^{\mathrm{pr}}$, we may take the dual of j^*; taking the appropriate shift and twist gives the map (7.6.4.1)

$$j_!: \mathbb{Z}_U^{c/S} \to \mathbb{Z}_X^{c/S}$$

(7.8.2) DEFINITION

i) Let W be a reduced quasi-projective S-scheme. A sequence of closed subsets of W:

$$\emptyset = W_0 \subset W_1 \subset \ldots \subset W_{n-1} \subset W_n = W$$

is an *S-smooth stratification of W* if $W_{i+1} \backslash W_i$ is smooth over S for each $i = 0, \ldots, n-1$, and W_i is equi-dimensional over S for each $i = 1, \ldots, n$. We call W *smoothly decomposable over S* if W has an S-smooth stratification.

ii) Let W be a smoothly decomposable S-scheme, $i: W \to X$ a closed embedding of W into a smooth S-scheme X, with complement U. We call the embedding i *compactifiable* if X and U are in $\mathbf{Sm}_S^{\mathrm{pr}}$. □

(7.8.3) Lemma

Suppose W is a smoothly decomposable S-scheme. If we have closed embeddings of W into smooth, quasi-projective S-schemes

$$i: W \to X; \qquad i': W \to X'$$

with

$$\dim_S(X) = d_X; \qquad \dim_S(X') = d_{X'},$$

then there is a canonical isomorphism

$$\phi_{W,i,i'}: \mathbb{Z}_{X,i(W)}(d_X)[2d_X] \to \mathbb{Z}_{X',i'(W)}(d_{X'})[2d_{X'}]$$

Proof. Suppose X is a locally closed subscheme of a projective space $\mathbb{P}^N_S$ over S, so that X is a closed subscheme of some open subscheme U of $\mathbb{P}^N_S$ via $i_X: X \to U$. We have the Gysin isomorphism

$$\mathbb{Z}_{X,i(W)}(d_X)[2d_X] \xrightarrow{i_{X*}} \mathbb{Z}_{U,(i_X \circ i)(W)}(N)[2N]$$

and similarly for X'; thus, we may assume that X and X' are open subschemes of projective spaces:

$$j_X: X \to \mathbb{P}^N_S; \qquad j_{X'}: X' \to \mathbb{P}^M_S.$$

Let

$$i'': W \to X \times_S X'$$

be the diagonal embedding. Then $(\mathrm{id}_X \times j_{X'})(i''(W))$ is a closed codimension $e+M$ subset W'' of $X \times \mathbb{P}^M_S$; by the excision axiom (2.1.3)(b), the morphism

$$(\mathrm{id}_X \times j_{X'})^*: \mathbb{Z}_{X\times\mathbb{P}^M_S,W''} \to \mathbb{Z}_{X\times X',W''}$$

is an isomorphism. Let

$$f: W'' \to i(W) \times \mathbb{P}^M_S$$

be the inclusion, inducing the morphism

$$f_*: \mathbb{Z}_{X\times\mathbb{P}^M_S,W''} \to \mathbb{Z}_{X\times\mathbb{P}^M_S,i(W)\times\mathbb{P}^M_S}.$$

Define the morphism in $\mathcal{DM}(S)$

$$p_{1*} = p^i_{1*}: \mathbb{Z}_{X\times X',W''}(N+M)[2(N+M)] \to \mathbb{Z}_{X,i(W)}(N)[2N] \tag{1}$$

as the composition

$$\begin{aligned}
&\mathbb{Z}_{X\times X',W''}(N+M)[2(N+M)]\\
&\qquad\xrightarrow{((\mathrm{id}_X\times j_{X'})^*)^{-1}}\mathbb{Z}_{X\times\mathbb{P}^M_S,W''}(N+M)[2(N+M)]\\
&\qquad\xrightarrow{f_*}\mathbb{Z}_{X\times\mathbb{P}^M_S,i(W)\times\mathbb{P}^M_S}(N+M)[2(N+M)]\\
&\qquad\xrightarrow{p_{1*}}\mathbb{Z}_{X,i(W)}(N)[2N].
\end{aligned}$$

We first show that the morphism (1) is an isomorphism.

Suppose at first that W is a smooth S-scheme. Then we have the Gysin isomorphisms (6.1.1.2)

$$\begin{aligned}
&i_*\colon\mathbb{Z}_W\to\mathbb{Z}_{X,i(W)}(N)[2N]\\
&i''_*\colon\mathbb{Z}_W\to\mathbb{Z}_{X\times X',W''}(N+M)[2(N+M)]
\end{aligned}$$

It follows from (6.4.6) that

$$p_{1*}\circ i''_*=i_*,$$

hence p_{1*} is an isomorphism. In general, we have the smoothly decomposable closed subset W_{n-1} of W. Let $W''_{n-1}=i''(W_{n-1})$; we may assume by induction that the map

$$p^{n-1}_{1*}\colon\mathbb{Z}_{X\times X',W''_{n-1}}(N+M)[2(N+M)]\to\mathbb{Z}_{X,i(W_{n-1})}(N)[2N],$$

defined similarly to p_{1*}, is an isomorphism. Let $U=X\setminus i(W_{n-1})$, $U'=X'\setminus i'(W_{n-1})$. We have the commutative diagram, where the columns are the localization sequences for motives with support (2.2.9.2),

$$\begin{array}{ccc}
\mathbb{Z}_{X\times X',W''_{n-1}}(e+M)[2(N+M)] & \xrightarrow{p^{n-1}_{1*}} & \mathbb{Z}_{X,i(W_{n-1})}(N)[2N]\\
\downarrow & & \downarrow\\
\mathbb{Z}_{X\times X',W''}(N)[2(N+M)] & \xrightarrow{p_{1*}} & \mathbb{Z}_{X,i(W)}(N)[2N]\\
\downarrow & & \downarrow\\
\mathbb{Z}_{U\times U',W''\setminus W''_{n-1}}(N)[2(N+M)] & \xrightarrow[p_{1*}]{} & \mathbb{Z}_{U,i(W\setminus W_{n-1})}(N)[2N];
\end{array}$$

the commutativity follows from the naturality of the excision isomorphism, and the naturality of the projective bundle formula (which is used to define the push-forward for a projection, see (6.3.1.1)). As the columns are distinguished triangles, and the top and bottom horizontal maps are isomorphisms, the middle horizontal map is an isomorphism as well.

Define the map $p^{i'}_{2*}$ by reversing the roles of X and X'; we let $\phi_{W,i,i'}$ be the composition $p^{i'}_{2*}\circ(p^i_{1*})^{-1}$. The functoriality of projective push-forward (6.4.6) gives the identity

$$p^i_{2*}=p^i_{1*}\circ t^*_{X',X}\tag{2}$$

where $t_{X',X}: X' \times_S X \to X \times_S X'$ is the exchange of factors. The identity (2) implies the identity

$$\phi_{W,i',i''} \circ \phi_{W,i,i'} = \phi_{W,i,i''}$$

which proves that the isomorphism $\phi_{W,i,i'}$ is canonical. □

(7.8.4) Lemma

Let W be a smoothly decomposable S-scheme, and let $i: W \to X$ and $i': W \to X'$ be compactifiable closed embedding with complements

$$j: U \to X; \qquad j': U' \to X'.$$

Then there is a canonical isomorphism

$$\psi_{W,i,i'}: \mathrm{Cone}(j_!: \mathbb{Z}_U^{c/S} \to \mathbb{Z}_X^{c/S}) \to \mathrm{Cone}(j'_!: \mathbb{Z}_{U'}^{c/S} \to \mathbb{Z}_{X'}^{c/S})$$

Proof. We may assume that X and X' are equi-dimensional over S; one then dualizes the isomorphism $\phi_{W,i,i'}$. □

(7.8.5) Definition

i) Let W be a smoothly decomposable S-scheme. Let $i: W \to X$ be a closed embedding of W into a smooth quasi-projective S-scheme X of dimension N over S. We define the Borel-Moore motive of W, $\mathbb{Z}_W^{\mathrm{B.M.}}$, by

$$\mathbb{Z}_W^{\mathrm{B.M.}} = \mathbb{Z}_{X,i(W)}(N)[2N].$$

ii) Let $i: W \to X$ be a compactifiable embedding of a smoothly decomposable S-scheme with complement $j: U \to X$. Then we define the motive of W with compact supports, $\mathbb{Z}_W^{c/S}$, as

$$\mathbb{Z}_W^{c/S} = \mathrm{Cone}(j_!: \mathbb{Z}_U^{c/S} \to \mathbb{Z}_X^{c/S}).$$

By (7.8.3) and (7.8.4), these notions are well-defined, independent of the choice of (compactifiable) embedding. □

(7.8.6) Remark

i) If W is already smooth over S, we may take i to be the identity map. From this, one sees that $\mathbb{Z}_W^{\mathrm{B.M.}}$ agrees with the definition of $\mathbb{Z}_W^{\mathrm{B.M.}}$ given in (6.4.11). Similarly, for W in $\mathbf{Sm}_S^{\mathrm{pr}}$, the above definition of $\mathbb{Z}_W^{c/S}$ agrees with that of (7.6.2)(ii).

ii) Since $(j^*)^D = j_!$ (7.6.4.1), and since duality is an exact involution (7.2.5), we have the canonical isomorphism

$$(\mathbb{Z}_W^{\mathrm{B.M.}})^D \cong \mathbb{Z}_W^{c/S}$$

for all smoothly decomposable W which admit a compactifiable closed embedding, in particular, for all W in $\mathbf{Sm}_S^{\mathrm{pr}}$. □

(7.8.7) PULL-BACK AND PUSH-FORWARD FOR THE BOREL-MOORE MOTIVES

Let $h: W \to W'$ be a projective S-morphism of smoothly decomposable S-schemes, and take closed embeddings

$$i: W \to X; \qquad i': W' \to X';$$

we may assume that X and X' are open subschemes of projective spaces $\mathbb{P}^N_S$, $\mathbb{P}^M_S$. Let Γ_h be the graph of h, and

$$i_h: \Gamma_h \to X \times_S X'$$

the embedding induced by i and i'. Suppose W has dimension d_W over S, and W' has dimension $d_{W'}$ over S. Letting $q: W \to \Gamma_h$ be the inverse of the projection $p_1: \Gamma_h \to X$, we have the isomorphism

$$\phi_{W,i,i_h \circ q}: \mathbb{Z}_{X,i(W)}(N)[2N] \to \mathbb{Z}_{X \times_S X', i_h(\Gamma_h)}(N+M)[2(N+M)].$$

Since h is proper, $i_h(\Gamma_h)$ is closed in $\mathbb{P}^N_S \times_S X'$; define the morphism

$$p_{2*}: \mathbb{Z}_{X \times X', i_h(\Gamma_h)}(N+M)[2(N+M)] \to \mathbb{Z}_{X', i'(W')}(M)[2M]$$

as the composition

$$\begin{aligned}
&\mathbb{Z}_{X \times X', i_f(\Gamma_f)}(N - d_W + M)[2(N+M)] \\
&\qquad \xrightarrow{((j_X \times \mathrm{id}_{X'})^*)^{-1}} \mathbb{Z}_{\mathbb{P}^N_S \times X', i_f(\Gamma_f)}(N - d_W + M)[2(N+M)] \\
&\qquad \xrightarrow{g_*} \mathbb{Z}_{\mathbb{P}^N_S \times X', i\mathbb{P}^N_S \times i'(W')}(N+M)[2(N+M)] \\
&\qquad \xrightarrow{p_{2*}} \mathbb{Z}_{X, i'(W')}(M)[2M].
\end{aligned}$$

We then define

$$h_*: \mathbb{Z}^{\mathrm{B.M.}}_W \to \mathbb{Z}^{\mathrm{B.M.}}_{W'} \tag{7.8.7.1}$$

as the composition $p_{2*} \circ \phi_{W,i,i_h \circ q}$.

Similarly, let $g: W \to W'$ be an open immersion with complement F. Take a closed embedding

$$i': W' \to X'$$

with X' of dimension N over S, and let

$$i: W \to X := X' \backslash i'(F)$$

be the restriction of i'. Let $j: X \to X'$ be the inclusion. Then j gives the map

$$j^*: \mathbb{Z}_{X', i'(W')} \to \mathbb{Z}_{X, i(W)};$$

we let

$$g^*: \mathbb{Z}^{\mathrm{B.M.}}_{W'} \to \mathbb{Z}^{\mathrm{B.M.}}_W \tag{7.8.7.2}$$

be the shift and twist of j^*

(7.8.8) LEMMA

The maps (7.8.7.1) and (7.8.7.2) are independent of the choice of embeddings i and i'. In addition, we have the functorialities

$$(h \circ h')_* = h_* \circ h'_*; \qquad (g \circ g')^* = g'^* \circ g^*$$

when defined. Furthermore, if we have a cartesian diagram

$$\begin{array}{ccc} W' \times_W W'' & \xrightarrow{\bar{g}} & W' \\ {\scriptstyle \bar{h}}\downarrow & & \downarrow{\scriptstyle h} \\ W'' & \xrightarrow[g]{} & W \end{array}$$

with h projective and g an open immersion, then

$$\bar{h}_* \circ \bar{g}^* = g^* \circ h_*.$$

Proof. The independence of (7.8.7.1) and (7.8.7.2) on the choice of embeddings, as well as the functorialities, follows from (7.8.4), and the functoriality of projective push-forward (6.4.6). The compatibility of pull-back and push-forward for a cartesian square follows from the functoriality of push-forward. □

(7.8.9)

Assuming that W and W' admit compactifiable closed embeddings, we may take duals of the maps (7.8.7.1) and (7.8.7.2); applying (7.8.6)(ii), we get the morphisms

$$h^!: \mathbb{Z}^{c/S}_{W'} \to \mathbb{Z}^{c/S}_{W}$$

$$g_!: \mathbb{Z}^{c/S}_{W} \to \mathbb{Z}^{c/S}_{W'}$$

satisfying the dual of the relations given in (7.8.8).

(7.8.10)

We recall (*cf.* (6.4.11)) that $\mathbf{Sm}_{S,\mathrm{proj}}$ is the subcategory of $\mathbf{Sm}_S$ with the same objects, and with morphisms being the projective morphisms. We have the functor (6.4.12)

$$\mathbb{Z}^{\mathrm{B.M.}}: \mathbf{Sm}_{S,\mathrm{proj}} \to \mathcal{DM}(S) \tag{7.8.10.1}$$

sending X of dimension d_X over S to $\mathbb{Z}_X(d_X)[2d_X] := \mathbb{Z}^{\mathrm{B.M.}}_X$. Let $\mathbf{SDS}_S$ be the full subcategory of $\mathbf{Sch}_S$ with objects the smoothly decomposable S-schemes, and let $\mathbf{SDS}_{S\mathrm{proj}}$ the sub-category of $\mathbf{SDS}_S$ with the same objects, and with morphisms being the projective morphisms. We let $\mathbf{Sm}^{\mathrm{pr}}_{S\mathrm{proj}}$ be the full subcategory of $\mathbf{Sm}_{S\mathrm{proj}}$ with the same objects as $\mathbf{Sm}^{\mathrm{pr}}_S$, and let $\mathbf{SDS}^{\mathrm{pr}}_{S\mathrm{proj}}$ be the full subcategory of $\mathbf{SDS}_{S\mathrm{proj}}$ with objects those W which admit a compactifiable closed embedding. Taking the dual of the functor (7.8.10.1) gives the functor

$$\mathbb{Z}^{c/S} : (\mathbf{Sm}^{\mathrm{pr}}_{S\mathrm{proj}})^{\mathrm{op}} \to \mathcal{DM}(S). \tag{7.8.10.2}$$

Putting the two lemmas (7.8.4) and (7.8.8) together proves

(7.8.11) THEOREM

The functors (7.8.10.1) and (7.8.10.2) extends to functors

$$\mathbb{Z}^{\mathrm{B.M.}} : \mathbf{SDS}_{S\mathrm{proj}} \to \mathcal{DM}(S)$$
$$\mathbb{Z}^{c/S} : (\mathbf{SDS}^{\mathrm{pr}}_{S\mathrm{proj}})^{\mathrm{op}} \to \mathcal{DM}(S)$$

□

(7.8.12) DEFINITION

i) Let W be a smoothly decomposable S-scheme. The *motivic Borel-Moore homology of* W is defined by

$$H^{\mathrm{B.M.}}_p(W, \mathbb{Z}(q)) = \mathrm{Hom}_{\mathcal{DM}(S)}(1, \mathbb{Z}^{\mathrm{B.M.}}_W(-q)[-p]).$$

ii) Let W be a smoothly decomposable S-scheme which has a compactifiable closed embedding $i: W \to X$ into a smooth quasi-projective S-scheme X. The *motivic cohomology of* W *with compact supports* is defined by

$$H^p_{c/S}(W, \mathbb{Z}(q)) = \mathrm{Hom}_{\mathcal{DM}(S)}(1, \mathbb{Z}^{c/S}_W(q)[p])$$

□

Since we have the duality isomorphism

$$\mathrm{Hom}_{\mathcal{DM}(S)}(1, \mathbb{Z}^{\mathrm{B.M.}}_X(-q)[-p]) \cong \mathrm{Hom}_{\mathcal{DM}(S)}(\mathbb{Z}^{c/S}_X(q)[p], 1)$$

for X in $\mathbf{Sm}^{\mathrm{pr}}$, the definition (7.8.12) of Borel-Moore homology and compactly supported cohomology extends that given in (7.6.2). It follows from (7.8.8) that the Borel-Moore homology is covariantly functorial for projective maps, and contravariantly functorial for open immersions; in addition, the pull-back and push-forward are compatible in cartesian squares. The dual statements follows for the compactly supported cohomology via the duality involution, using (7.8.6)(ii).

The cup and cap products for Borel-Moore homology and compactly supported cohomology defined in (7.6.4) extend in the obvious way to the singular case whenever all the groups are defined; one applies Mayer-Vietoris and the Künneth formula to give canonical isomorphisms

$$\mathbb{Z}^{\mathrm{B.M.}}_W \otimes \mathbb{Z}^{\mathrm{B.M.}}_{W'} \to \mathbb{Z}^{\mathrm{B.M.}}_{W \times_S W'},$$

and then takes the inverse of the dual to give canonical isomorphisms

$$\mathbb{Z}_W^{c/S} \otimes \mathbb{Z}_{W'}^{c/S} \to \mathbb{Z}_{W\times_S W'}^{c/S};$$

the remainder of the construction of cup and cap product then proceeds formally the same way as the smooth case. The various properties: functoriality, projection formula, etc. described in (7.6.4) also extend without trouble. In particular, there is a functorial bi-graded ring structure on the compactly supported cohomology, and Borel-Moore homology is a bi-graded module for the compactly supported cohomology ring.

(7.8.13) Motives over a field

Suppose we take $S = \mathrm{Spec}(k)$, where k is a perfect field. Then all reduced quasi-projective k-schemes are smoothly decomposable, hence the Borel-Moore motive, and Borel-Moore homology are defined for all reduced quasi-projective k-schemes. If, in addition, resolution of singularities holds for reduced quasi-projective k-schemes, then, by (7.4.5), all reduced quasi-projective k-schemes admit a compactifiable closed embedding into a smooth quasi-projective k-scheme. Thus the compactly supported cohomology is defined for all reduced quasi-projective k-schemes.

If we restrict to reduced projective k-schemes, this gives an interesting extension of motivic cohomology of smooth projective k-schemes to the singular case. For example, if W is the union of smooth closed subschemes $D_1, \ldots, D_n$ of a smooth projective k-variety X, which intersect transversely on X, then the compactly supported cohomology of W is given via a Mayer-Vietoris type spectral sequence with E_1-term being the motivic cohomology of the irreducible components of the intersections $D_{i_0} \cap \ldots \cap D_{i_a}$:

$$E_1^{a,b} = \bigoplus_{1\le i_0<\ldots<i_b\le n} H^a(D_{i_0} \cap \ldots \cap D_{i_b}, \mathbb{Z}(q)) \Longrightarrow H^{a+b}_{c/S}(W, \mathbb{Z}(q)).$$

One can extend this computation to the general case: if X is smooth and projective, W a closed subset with complement U, then, by (7.7.7) and (7.7.8.2), the compactly supported cohomology of U is the relative motivic cohomology

$$H^p_{c/S}(U, \mathbb{Z}(q)) = H^p((\tilde{X}; D_1, \ldots, D_m), \mathbb{Z}(q))$$

where $\tilde{X}$ is a smooth compactification of U with normal crossing divisor $D_1 + \ldots + D_m$ at infinity. Applying the definition of $\mathbb{Z}_W^{c/S}$ as the Cone

$$\mathbb{Z}_W^{c/S} = \mathrm{Cone}(j_!\colon \mathbb{Z}_U^{c/S} \to \mathbb{Z}_X^{c/S})$$

gives a spectral sequence converging to $H^p_{c/S}(W, \mathbb{Z}(q))$ with E_1-terms involving the cohomology of X, $\tilde{X}$ and the irreducible components of the intersections $D_{i_0} \cap \ldots \cap D_{i_a}$:

$$E_1^{a,b} \Longrightarrow H_{c/S}^{a+b}(W, \mathbb{Z}(q));$$

$$E_1^{a,b} = \begin{cases} \bigoplus\limits_{1 \le i_0 < \ldots < i_b \le n} H^a(D_{i_0} \cap \ldots \cap D_{i_b}, \mathbb{Z}(q)) & \text{for } b > 0, \\ H^a(X, \mathbb{Z}(q)) \oplus \oplus_{1 \le i \le n} H^a(D_i, \mathbb{Z}(q)) & \text{for } b = 0, \\ H^a(\tilde{X}, \mathbb{Z}(q)) & \text{for } b = -1. \end{cases}$$

7.9. The triangulated Tate motivic category

We give the definition of the triangulated Tate motivic category $\mathcal{DTM}(S)_R$ as a subcategory of $\mathcal{DM}(S)_R$. If the base scheme S is Spec(k) for k a field, we show that $\mathcal{DTM}(S)_R$ is equivalent to respective subcategory of the categories $\mathbf{D}^b_{mot}(\mathbf{Sm}_k)_R$ and $\mathbf{D}^b_{mot}(\mathbf{Sm}_k)^0_R$. In addition, the duality involution on $\mathcal{DM}(\mathbf{Sm}_S)_R$ restricts to a duality involution on $\mathcal{DTM}(S)_R$.

(7.9.1) DEFINITION

Let S be a reduced scheme and R a localization of $\mathbb{Z}$. The *triangulated Tate motivic category $\mathcal{DTM}(S)_R$* is the strictly full triangulated tensor subcategory of $\mathcal{DM}(S)_R$ generated by the objects $R_S(q)$, $q = 0, \pm 1$. □

(7.9.2) LEMMA

The category $\mathcal{DTM}(S)_R$ is equal to the strictly full triangulated subcategory of $\mathcal{DM}(S)_R$ generated by the objects $\mathbb{Z}_S(q)$, $q \in \mathbb{Z}$.

Proof. This follows immediately from the exactness of the tensor product operation in $\mathcal{DM}(S)$, and the Künneth isomorphism (2.1.3)(c)

$$\mathbb{Z}_S(a) \otimes \mathbb{Z}_S(b) \cong \mathbb{Z}_S(a+b).$$

□

(7.9.3) PROPOSITION

The duality involution (7.4.2)

$$(-)^D : \mathcal{DM}(S)_R^{\mathrm{op}} \to \mathcal{DM}(S)_R$$

restricts to an involution

$$(-)^D : \mathcal{DTM}(S)_R^{\mathrm{op}} \to \mathcal{DTM}(S)_R$$

Proof. We have $\mathbb{Z}_S^D = 1^D = 1 = \mathbb{Z}_S$, hence $\mathbb{Z}_S(q)^D = \mathbb{Z}_S(-q)$ for each integer q. Since the involution $(-)^D$ is exact (7.4.2), this implies that the strictly full triangulated subcategory of $\mathcal{DM}(S)^{\mathrm{op}}$ is mapped into its opposite by $(-)^D$. Applying (7.9.2) completes the proof. □

(7.9.4)

We recall the tensor category $\mathcal{A}_{mot}(\mathcal{V})^0$ (3.1.6), and the DG tensor functor (3.1.6.2)

$$H_{mot}\colon \mathcal{A}_{mot}(\mathcal{V}) \to \mathcal{A}_{mot}(\mathcal{V})^0.$$

We have as well the triangulated tensor categories $\mathbf{D}^b_{mot}(\mathcal{V})^0$ and $\mathcal{DM}(\mathcal{V})^0_R$ formed from $\mathcal{A}_{mot}(\mathcal{V})^0_R$ in a way paralleling the construction of $\mathbf{D}^b_{mot}(\mathcal{V})$ and $\mathcal{DM}(\mathcal{V})$ from $\mathcal{A}_{mot}(\mathcal{V})$ (see (3.2.11) and (7.4.4)). In particular, we have the commutative diagram

$$\begin{array}{ccccc} \mathcal{A}_{mot}(\mathcal{V}) & \to & \mathbf{D}^b_{mot}(\mathcal{V}) & \to & \mathcal{DM}(\mathcal{V}) \\ {\scriptstyle H_{mot}}\downarrow & & {\scriptstyle \mathbf{D}^b_{mot}(H_{mot})}\downarrow & & \downarrow{\scriptstyle \mathcal{DM}(H_{mot})} \\ \mathcal{A}_{mot}(\mathcal{V})^0 & \to & \mathbf{D}^b_{mot}(\mathcal{V})^0_R & \to & \mathcal{DM}(\mathcal{V})^0_R \end{array}$$

If we take $\mathcal{V} = \mathbf{Sm}_k$ for k a field, then, if, e.g., we have resolution of singularities for k-varieties, it follows from (7.4.6) that the functors $\mathbf{D}^b_{mot}(H_{mot})$ and $\mathcal{DM}(H_{mot})$ are equivalences (we need to assume $R = \mathbb{Q}$ in case $\mathrm{char}(k) > 0$).

Let $\mathcal{DT}(S)_R$ be the full triangulated tensor subcategory of $\mathbf{D}^b_{mot}(\mathbf{Sm}_S) \otimes R$ generated by the objects $\mathbb{Z}_S(q)$, $q = 0, \pm 1$, and let $\mathcal{DT}(S)^0_R$ be the full triangulated tensor subcategory of $\mathbf{D}^b_{mot}(\mathbf{Sm}_S)^0 \otimes R$ generated by the objects $\mathbb{Z}_S(q)$, $q = 0, \pm 1$.

(7.9.5) THEOREM

The functors

$$\mathbf{D}^b_{mot}(\mathbf{Sm}_S) \otimes R \to \mathcal{DM}(S)_R$$

$$\mathbf{D}^b_{mot}(\mathbf{Sm}_S)^0 \otimes R \to \mathcal{DM}(S)^0_R$$

induce equivalences

$$\mathcal{DT}^b(S) \to \mathcal{DTM}(S)$$

$$\mathcal{DT}^b(S) \to \mathcal{DTM}(S)$$

Under the hypothesis of (7.4.6), the functors $\mathbf{D}^b_{mot}(H_{mot})$ and $\mathcal{DM}(H_{mot})$ induce equivalences

$$\mathcal{DT}(S)_R \to \mathcal{DT}(S)^0_R,$$

$$\mathcal{DTM}(S)_R \to \mathcal{DTM}(S)^0_R.$$

Proof. As the objects $\mathbb{Z}_S(q)$ generating $\mathcal{DTM}(S)$ are in $\mathcal{DT}(S)$, and as the functor

$$\mathbf{D}^b_{mot}(\mathbf{Sm}_S) \to \mathcal{DM}(S)$$

is a fully faithful embedding, the categories $\mathcal{DT}(S)$ and $\mathcal{DTM}(S)$ are equivalent. The second pair of equivalences follows from (7.4.6). □

(7.9.6) FUNCTORIALITY

We recall from §2.3 that the formation of the category $\mathcal{DM}(S)$ is functorial in S. If $p: T \to S$ is a map of reduced schemes, the functor

$$\mathcal{DM}(p^*): \mathcal{DM}(S)_R \to \mathcal{DM}(T)_R$$

induces the functor

$$\mathcal{DTM}(p^*): \mathcal{DTM}(S) \to \mathcal{DTM}(T)$$

This determines the functor

$$\begin{aligned} &\mathcal{DTM}(-)_R: \mathbf{Sch} \to \mathbf{TT}_R \\ &S \mapsto \mathcal{DTM}(S_{\mathrm{red}})_R \end{aligned}$$

from the category of schemes to the category of triangulated rigid R-tensor categories.

Chapter 8
Realization of the motivic category

In this chapter, we describe a mapping property satisfied by the category $\mathcal{DM}(\mathcal{V})$. The main theorem of this chapter, (8.3.1), gives a criterion for a cohomology theory defined by a complex of sheaves F on a Grothendieck site to define the "F-realization" of $\mathcal{DM}(\mathcal{V})$. One should view this more as a prototype than a final result; many interesting cohomology theories have been defined in a somewhat more general setting than the one described above, but it seems difficult to give an all-encompassing result covering all the known cases. We will consider various important examples of such cohomology theories in the §8.4, where we give the realizations corresponding to singular cohomology, étale cohomology, Hodge (Deligne) cohomology, and Jannsen's "motivic" cohomology built from compatible realizations.

8.1. Geometric cohomology theories

We give an axiomatic description of some cohomology theories which admit realization functors.

(8.1.1)

Let $\mathcal{C}$ be a full subcategory of $\mathbf{Sch}_S$, closed under finite fiber products, arbitrary coproducts, and taking open and closed subsets. Following Bloch-Ogus [B-O] and Gillet [G], a graded cohomology theory $\Gamma(*)$ on $\mathcal{C}$ is a graded complex of sheaves $\Gamma^*(*)$ on the big Zariski site $\mathcal{C}_{Zar}$ of $\mathcal{C}$, together with a pairing in the derived category of graded complexes of sheaves of R-modules on $\mathcal{C}_{Zar}$:

$$\Gamma^*(*) \otimes^L \Gamma^*(*) \to \Gamma^*(*)$$

which is associative with unit and graded-commutative, and satisfies certain additional axioms. We give here a slightly different version of this notion.

We refer the reader to (II, Chapter 6) for the notions related to rigidifications.

(8.1.2)

We begin with a rigid Grothendieck topology $(\mathfrak{S}, \mathfrak{r}\mathfrak{S})$ (see II, (6.2.3) and (6.10.2)) subcategory $\mathcal{C}$ of $\mathbf{Sch}_S$ containing $\mathcal{V}$ (or a sub-category $\mathcal{C}$ of $\mathbf{An}_{S_{\mathrm{an}}}$ containing the image $\mathcal{V}_{\mathrm{an}}$ in $\mathbf{An}_{S_{\mathrm{an}}}$); we assume that $\mathcal{C}$ satisfies the conditions of (II, (5.1.1)).

For X in $\mathcal{V}$, $W \subset X$ a closed subset, we let $\mathcal{Z}^q_W(X/S)$ denote subgroup of $\mathcal{Z}^q(X/S)$ consisting of cycles with support in W. We let $p_X\colon X \to S$ denote the structure morphism.

Consider the category $\mathrm{Sh}^{\mathfrak{S}}_{\mathcal{C}}(\mathcal{A})$ of sheaves on $\mathcal{C}$ with values in an abelian tensor category $\mathcal{A}$. We suppose that filtered direct limits are representable in $\mathcal{A}$, that $\mathcal{A}$ has enough injectives, and that each object of $\mathcal{C}$ has finite cohomological dimension in the topology $\mathfrak{S}$. For an object X of $\mathcal{C}$, we denote the category of $\mathcal{A}$-valued sheaves on X by $\mathrm{Sh}^{\mathfrak{S}}_X(\mathcal{A})$.

We say a sheaf $\mathcal{F} \in \mathrm{Sh}^{\mathfrak{S}}_{\mathcal{C}}(\mathcal{A})$ is *flat* if for each U in $\mathcal{C}$, the functor $-\otimes \mathcal{F}(U): \mathcal{A} \to \mathcal{A}$ an exact functor.

For X in $\mathcal{C}$ and W a closed subset of X, we have the functors

$$p_{X*}: \mathrm{Sh}^{\mathfrak{S}}_X(\mathcal{A}) \to \mathrm{Sh}^{\mathfrak{S}}_S(\mathcal{A})$$
$$p^W_{X*}: \mathrm{Sh}^{\mathfrak{S}}_X(\mathcal{A}) \to \mathrm{Sh}^{\mathfrak{S}}_S(\mathcal{A})$$

where p^W_{X*} is the functor "sections with support in W". This gives the derived functors

$$Rp_{X*}: \mathrm{Sh}^{\mathfrak{S}}_X(\mathcal{A}) \to \mathbf{D}^+(\mathrm{Sh}^{\mathfrak{S}}_S(\mathcal{A}))$$
$$Rp^W_{X*}: \mathrm{Sh}^{\mathfrak{S}}_X(\mathcal{A}) \to \mathbf{D}^+(\mathrm{Sh}^{\mathfrak{S}}_S(\mathcal{A}))$$

and the natural transformation

$$Ri^!_W: Rp^W_{X*} \to Rp_{X*}. \tag{8.1.2.1}$$

(8.1.3) Definition

Let $\mathcal{F} = \oplus_{q=0}^{\infty}\mathcal{F}(q) \in \mathbf{C}^+(\mathrm{Sh}^{\mathfrak{S}}_{\mathcal{C}}(\mathcal{A}))$ be a graded complex of flat sheaves, having a graded product

$$\mathcal{F} \otimes \mathcal{F} \to \mathcal{F} \tag{8.1.3.1}$$

We say that F defines a *geometric cohomology theory on* $\mathcal{V}$ if

(i) The product (8.1.3.1) is associative and graded-commutative.

(ii) (homotopy) Let $p: X \to Y$ be the inclusion of a closed codimension one subscheme. Let $T \subset Y$ be a reduced closed subscheme, and let $W = p^{-1}(T)$. Suppose that the inclusion $p: X \to Y$ is a map in $\mathcal{V}$. Suppose further that $T \cong \mathbb{A}^1_W$, and that, via this isomorphism, $p: W \to T$ is the inclusion of $W \times 0$ into $\mathbb{A}^1_W$. Then the map

$$p^*: Rp^T_{Y*}\mathcal{F}_Y \to Rp^W_{X*}\mathcal{F}_X$$

is an isomorphism in $\mathbf{D}^+(\mathrm{Sh}^{\mathfrak{S}}_S(\mathcal{A}))$.

(iii) (cycle classes) Let X be in $\mathcal{V}$, and $W \subset X$ a closed subset such that W is the support of an effective cycle in $\mathcal{Z}^q(X/S)$. Then there is a homomorphism

$$\mathrm{cl}^q_{X,W}: \mathcal{Z}^q_W(X/S) \to \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Sh}^{\mathfrak{S}}_S(\mathcal{A}))}(1_S, Rp^W_{X*}\mathcal{F}_X(q)[2q]).$$

The maps $\mathrm{cl}^q_{X,W}$ are functorial in the following sense:

a) If $f\colon Y \to X$ is a map in $\mathcal{V}$, and if $f^{-1}(W)$ is contained in the support W' of some effective cycle in $\mathcal{Z}^q(Y/S)$, then the diagram

$$\begin{array}{ccc} \mathcal{Z}^q_W(X/S) & \xrightarrow{f^*} & \mathcal{Z}^q_{W'}(Y/S) \\ {\scriptstyle \mathrm{cl}^q_{X,W}}\downarrow & & \downarrow{\scriptstyle \mathrm{cl}^q_{Y,W'}} \\ \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Sh}^{\mathfrak{S}}_S(\mathcal{A}))}(Rp^W_{X*}\mathcal{F}_X(q)[2q]) & \xrightarrow[f^*]{} & \mathrm{Hom}_{\mathbf{D}^+(\mathrm{Sh}^{\mathfrak{S}}_S(\mathcal{A}))}(1_S, Rp^{W'}_{Y*}\mathcal{F}_Y(q)[2q]) \end{array}$$

commutes (by our assumption on W and f, the cycle $f^*(Z)$ is defined for all $Z \in \mathcal{Z}^q_W(X/S)$).

b) If $T \subset Y$ is the support of an effective cycle in $\mathcal{Z}^{q'}(Y/S)$, then

$$\mathrm{cl}^q_{X,W}(Z) \boxtimes \mathrm{cl}^{q'}_{Y,T}(Z') = \mathrm{cl}^{q+q'}_{X\times_S Y, W\times_S T}(Z \times_{/S} Z')$$

for all $Z \in \mathcal{Z}^q_W(X)$, $Z' \in \mathcal{Z}^{q'}_T(Y)$.

(iv) (purity) Let X be in $\mathcal{V}$, and $W \subset X$ a closed subset which is the support of an effective cycle in $\mathcal{Z}^q(X/S)$. Then

$$\mathrm{Hom}_{\mathbf{D}^+(\mathrm{Sh}^{\mathfrak{S}}_S(\mathcal{A}))}(1_S, Rp^W_{X*}\mathcal{F}_X(q)[2q-p]) = 0$$

for $p > 0$.

(v) (Künneth formula) For all X, Y in $\mathcal{V}$, the external products

$$\boxtimes^{q_1,q_2}_{\mathcal{F}}(X,Y)\colon Rp_{X*}\mathcal{F}(q_1) \otimes Rp_{Y*}\mathcal{F}_Y(q_2) \to Rp_{X\times_S Y*}\mathcal{F}_{X\times_S Y}(q_1+q_2)$$

induced by the product (8.1.3.1) are isomorphisms in $\mathbf{D}^+(\mathrm{Sh}^{\mathfrak{S}}_S(\mathcal{A}))$.

(vi) (Gysin morphism) Let $p\colon P \to X$ be a smooth morphism in $\mathcal{V}$ of relative dimension d, with section $s\colon X \to P$. Let

$$\mathrm{cyc}^d_{s(X),P}(s(X))\colon 1_S \to R^{s(X)}_{P*}\mathcal{F}_P(d)[2d]$$

be the map $\mathrm{cl}^d_{s(X),P}(s(X))$. Then the composition

$$Rp_{X*}\mathcal{F}_X(q) \xrightarrow{p^*} Rp_{P*}\mathcal{F}_P(q) \xrightarrow{(-)\cup \mathrm{cyc}^d_{s(X),P}(s(X))} Rp^{s(X)}_{P*}\mathcal{F}_P(q+d)[2d]$$

is an isomorphism.

(vii) (unit) The cycle class map associated to the fundamental class on S:

$$\mathrm{cl}^0(|S|)\colon 1_S \to \mathcal{F}_S(0)$$

is a quasi-isomorphism.

(viii) the unit $1 \in \mathcal{A}$ admits a finite projective resolution

$$P_M \to \ldots \to P_0 \to 1$$

in $\mathcal{A}$ such

a) $P_*^{\otimes n} \to 1^{\otimes n}$ is a projective resolution of $1^{\otimes n}$ for each $n \geq 1$

b) $- \otimes P_m$ is an exact functor on $\mathcal{A}$ for all $0 \leq m \leq M$.

c) the complex

$$\mathrm{Hom}_{\mathbf{C}^b(\mathcal{A})}(P_*^{\otimes n}, P_*^{\otimes m})$$

is 2-torsion free for all $m, n \geq 1$.

□

(8.1.4) REMARK

Suppose we have a *twisted duality theory* $\Gamma(*)$ in the sense of [B-O] or [G]. Then, for $p: X \to Y$ the inclusion of a closed codimension d subscheme, with X and Y smooth, we have the Poincaré duality isomorphism

$$H^p(X, \Gamma(q)) \to H_X^{p+2d}(Y, \Gamma(q+d)).$$

This implies part (vi) above as a special case, and reduces part (ii) to the usual form of the homotopy axiom:

(ii)′ Let X be in $\mathcal{V}$, and let $p: \mathbb{A}^1_X \to X$ be the projection. Then the map

$$p^*: Rp_{X*}\mathcal{F}_X \to Rp_{\mathbb{A}^1_X*}\mathcal{F}_{\mathbb{A}^1_X}$$

is an isomorphism.

If the base is a perfect field, the purity condition (iv) reduces to $H^p(X, \Gamma(q)) = 0$ for $p < 0$, and (iii) is implied by requiring that $H^0(X, \Gamma(0))$ is the free $H^0(S, \Gamma(0))$-module on the fundamental class of X, together with the projection formula. In particular, for $S = \mathrm{Spec}(k)$, k a perfect field, a twisted duality theory $\Gamma(*)$ gives rise to a geometric cohomology theory if $\Gamma(*)$ (with its product) is given as

$$\Gamma(*) = R\alpha_*(\mathcal{F}(*))$$

where

$$\alpha: \mathcal{C}_{\mathfrak{S}} \to \mathcal{C}_{\mathrm{Zar}}$$

is a map of a rigid Grothendieck topology $(\mathfrak{S}, \mathfrak{r}\mathfrak{S})$ to the Zariski topology on $\mathcal{C}$, and $\mathcal{F}(*)$ is a graded complex of sheaves on $\mathcal{C}$ for $\mathfrak{S}$ with an associative and graded-commutative product.

□

8.2. Cohomology with supports

We show how to define canonical co-chain complexes for "cohomology with support in codimension q".

(8.2.1)

We recall from (II, (6.7.1)), the category $\mathcal{C}_2$ of pairs (X,W), with X in $\mathcal{C}$ and W a closed subset of X. We have the category of rigid hypercovers over $\mathcal{C}_2$ (II, (6.7.2)),

$$\pi_{\mathcal{C}_2}\colon \mathrm{HCov}_{\mathfrak{r}\mathfrak{S}}(\mathcal{C}_2) \to \mathcal{C}_2, \tag{8.2.1.1}$$

which is a fibered symmetric semi-monoidal category over $\mathcal{C}_2$ (see II, (4.3.4) and (6.7.2)). We denote the fiber of $\pi_{\mathcal{C}_2}$ over (X,W) by $\mathrm{HCov}_{\mathfrak{r}\mathfrak{S}}(X,W)$.

We have the category (II, §4.1) of formal pointed objects of $\mathcal{C}$, $\mathcal{C}^+$: for $X\in\mathcal{C}$, we have $X^+ : X\coprod *$, with base-point $*$. We extend the site $\mathcal{C}_{\mathfrak{S}}$ to the site $\mathcal{C}^+_{\mathfrak{S}}$, as explained in (II, (6.9.1)), giving the category of sheaves on X^+, $\mathrm{Sh}^{\mathfrak{S}}_X(\mathcal{A})$, and the category of presheaves on X $\mathrm{PreSh}^{\mathfrak{S}}_X(\mathcal{A})$.

An $\mathcal{A}$-valued sheaf on $\mathcal{C}_{\mathfrak{S}}$ (or $X_{\mathfrak{S}}$) extends to an $\mathcal{A}$-valued sheaf on $\mathcal{C}^+_{\mathfrak{S}}$ (or $X^+_{\mathfrak{S}}$) with value 0 on $*$, as explained in (II, (6.9.1)).

We have the category of pointed simplicial objects (II, (6.7.2.2) and (6.8.1.3))

$$\mathcal{C}^+(\Delta^{\mathrm{op}})$$

the functors

$$\begin{aligned} p^+_{\mathrm{HCov2}}&\colon \mathrm{HCov}_{\mathfrak{r}\mathfrak{S}}(\mathcal{C}_2)\to\mathcal{C}^+(\Delta^{\mathrm{op}})\\ p^!_{\mathrm{HCov2}}&\colon \mathrm{HCov}_{\mathfrak{r}\mathfrak{S}}(\mathcal{C}_2)\to\mathcal{C}^+(\Delta^{\mathrm{op}})\end{aligned} \tag{8.2.1.2}$$

and natural transformation (II, (6.8.1.4))

$$i^!\colon p^+_{\mathrm{HCov2}} \to p^!_{\mathrm{HCov2}}. \tag{8.2.1.3}$$

The functors (8.2.1.2) are lax symmetric semi-monoidal functors, and (8.2.1.3) is a natural transformation of lax symmetric semi-monoidal functors (see II, (1.1.1)(iii), (6.5.4), (6.7.2.5) and (6.8.3)); we omit from the notation the natural transformations which are part of the data of a lax symmetric semi-monoidal functor.

Let (X,f,q) be in $\mathcal{L}(\mathcal{V})\times\mathbb{Z}$, and let $(X,f)^{(q)}$ denote the set of closed subsets $W\subset X$ such that W is the support of an effective cycle in $\mathcal{Z}^q(X)_f$.

Recall the symmetric monoidal category $\mathcal{L}(\mathcal{V})^*$ (1.1.7), with the same objects as $\mathcal{L}(\mathcal{V})\times\mathbb{Z}$, and faithful symmetric monoidal functor (1.1.7.1)

$$\mathcal{L}(\mathcal{V})^{\mathrm{op}}\times\mathbb{Z}\to\mathcal{L}(\mathcal{V})^*.$$

We have as well the faithful functor

$$\begin{aligned} i\colon (\mathcal{L}(\mathcal{V})^*)^{\mathrm{op}} &\to \mathcal{V} \\ i(X,f,q) &= X. \end{aligned} \tag{8.2.1.4}$$

Via i we identitify

$$\mathrm{Hom}_{\mathcal{L}(\mathcal{V})^*}((Y,f',q),(X,f,q))$$

with a subset of $\mathrm{Hom}_{\mathcal{V}}(X,Y)$; there are no morphisms from (Y,f',q') to (X,f,q) if $q \neq q'$.

(8.2.2) Lemma

Let $g\colon (Y,f',q) \to (X,f,q)$ be a map in $\mathcal{L}(\mathcal{V})^*$. Then for each $W \in (X,f)^{(q)}$, $g^{-1}(W)$ is in $(Y,f')^{(q)}$.

Proof. Suppose W is the support of an effective cycle $Z \in \mathcal{Z}^q(X)_f$. By (1.1.6), $g^*(Z)$ is defined and is in $\mathcal{Z}^q(Y)_{f'}$. Since Z is effective, and the map g is a map of smooth S-schemes, $g^*(Z)$ is effective, and $g^{-1}(W)$ is the support of $g^*(Z)$, hence $g^{-1}(W)$ is in $(Y,f')^{(q)}$. □

(8.2.3)

Set

$$\mathcal{W} = (\mathcal{L}(\mathcal{V})^*)^{\mathrm{op}},$$

with functor

$$i\colon \mathcal{W} \to \mathcal{C} \tag{8.2.3.1}$$

being the projection (8.2.1.4).

Let $\mathcal{C}_2^*$ be the full subcategory of $\mathcal{C}_2 \times_{\mathcal{C}} \mathcal{W}$ with objects being pairs $((X,W),(X,f,q))$ such that W is in $(X,f)^{(q)}$; $\mathcal{C}_2^*$ is clearly a symmetric semi-monoidal subcategory of the fiber product category $\mathcal{C}_2 \times_{\mathcal{C}} \mathcal{W}$. The projections thus define symmetric semi-monoidal functors

$$\begin{aligned} p_1\colon \mathcal{C}_2^* &\to \mathcal{C}_2 \\ p_2\colon \mathcal{C}_2^* &\to \mathcal{W}. \end{aligned}$$

If we have a morphism

$$g\colon (Y,f',q) \to (X,f,q)$$

in $\mathcal{W}$, and $(X,W) \in (X,f)^{(q)}$, we define

$$g^*((X,W)) = (Y, g^{-1}(W));$$

by (8.2.2), g^* defines a functor

$$g^*\colon p_2^{-1}((X,f,q)) \to p_2^{-1}((Y,f',q)).$$

We have the map

$$(g,g)\colon (g^*((X,W)),(Y,f',q)) \to ((X,W),(X,f,q))$$

over g; this gives $\mathcal{C}_2^*$ the structure of a fibered symmetric semi-monoidal category over $\mathcal{W}$ (with trivial comparison isomorphisms).

Let

$$\pi_{\mathcal{C}_2^*}\colon \mathrm{HCov}_{\mathfrak{r}\mathfrak{S}}(\mathcal{W}) \to \mathcal{C}_2^*$$

be the pull-back of the functor (8.2.1.1) by the functor p_1,and let

$$\pi_{\mathcal{W}}\colon \mathrm{HCov}_{\mathfrak{r}\mathfrak{S}}(\mathcal{W}) \to \mathcal{W} \tag{8.2.3.2}$$

be the composition of $\pi_{\mathcal{C}_2^*}$ with the projection p_2. We let $\mathrm{HCov}_{\mathfrak{r}\mathfrak{S}}((X,f,q))$ denote the fiber of $\pi_{\mathcal{W}}$ over (X,f,q).

The functors (8.2.1.2) and natural transformation (8.2.1.3) give the functors

$$\begin{aligned} p^+_{\mathrm{HCov}}&\colon \mathrm{HCov}_{\mathfrak{r}\mathfrak{S}}(\mathcal{W}) \to \mathcal{C}^+(\Delta^{\mathrm{op}}) \\ p^!_{\mathrm{HCov}}&\colon \mathrm{HCov}_{\mathfrak{r}\mathfrak{S}}(\mathcal{W}) \to \mathcal{C}^+(\Delta^{\mathrm{op}}) \end{aligned} \tag{8.2.3.3}$$

and natural transformation

$$i^!\colon p^+_{\mathrm{HCov}} \to p^!_{\mathrm{HCov}}. \tag{8.2.3.4}$$

(8.2.4) Lemma

i) The structure of a fibered symmetric semi-monoidal category over $\mathcal{C}_2$ on (8.2.1.1) induces the structure of a fibered symmetric semi-monoidal category over $\mathcal{W}$ on (8.2.3.2).

ii) The lax symmetric semi-monoidal functors (8.2.1.2) gives (8.2.3.3) the structure of lax symmetric semi-monoidal functors; the natural transformation (8.2.3.4) is a natural transformation of lax symmetric semi-monoidal functors.

iii) For each (X,f,q) in $\mathcal{W}$, the fiber $\mathrm{HCov}_{\mathfrak{r}\mathfrak{S}}((X,f,q))$ over (X,f,q) is left-directed.

iv) Let (X,f,q) be in $\mathcal{W}$, take $\mathcal{U}$ in $\mathrm{HCov}_{\mathfrak{r}\mathfrak{S}}((X,f,q))$, and let

$$\mathrm{HCov}_{\mathfrak{r}\mathfrak{S}}((X,f,q))_{\mathcal{U}}$$

denote the full subcategory of $\mathrm{HCov}_{\mathfrak{r}\mathfrak{S}}((X,f,q))$ with objects being those $\mathcal{V}$ which admit a map $\mathcal{V}\to\mathcal{U}$ in $\mathrm{HCov}_{\mathfrak{r}\mathfrak{S}}((X,f,q))$. Then $\mathrm{HCov}_{\mathfrak{r}\mathfrak{S}}((X,f,q))_{\mathcal{U}}$ a left final subcategory of $\mathrm{HCov}_{\mathfrak{r}\mathfrak{S}}((X,f,q))$.

Proof. The assertions (i) and (ii) are obvious consequences of the analogous properties for $\mathrm{HCov}_{\mathfrak{r}\mathfrak{S}}(\mathcal{C}_2)$ (II, (6.5.4) and (6.7.2)), together with the fact that

$$p_1\colon \mathcal{C}_2^* \to \mathcal{C}_2$$

is a symmetric semi-monoidal functor, and

$$p_2 \colon \mathcal{C}_2^* \to \mathcal{W}$$

is a fibered symmetric semi-monoidal functor.

For (iii) and (iv), we note that the fiber of p_2 over (X, f, q) is isomorphic to the lattice of closed subsets W of X with $W \in (X, f)^{(q)}$, ordered under reverse inclusion. As $(X, f)^{(q)}$ is closed under finite union, the fiber $p_2^{-1}(X, f, q)$ is a left-directed category, and each object is left final. The assertions (iii) and (iv) follows from this, the fact that p_2 is fibered, and the analogous statements for the categories $\mathrm{HCov}_{\mathfrak{r}\mathfrak{S}}((X, W))$ (II, *loc. cit.*).
□

If U_* is in $\mathcal{C}^+(\Delta^{\mathrm{op}})$, we let $\mathcal{F}(n)(U_*)^*$ denote the total complex associated to the cosimplicial object $\mathcal{F}(n)(U_*)$ of $\mathbf{C}^+(\mathcal{A})$.

(8.2.5) DEFINITION

i) For $(X, f, q) \in \mathcal{W}$, we define $\check{\mathcal{F}}^{(q)}(X)(n)_f$ as the direct limit:

$$\check{\mathcal{F}}^{(q)}(X)(n)_f = \varinjlim_{\mathcal{U} \in \mathrm{HCov}_{\mathfrak{r}\mathfrak{S}}((X,f,q))} \mathcal{F}(n)(p^!_{\mathrm{HCov}}(\mathcal{U}))^*.$$

We define $\check{\mathcal{F}}(X)(n)$ as the direct limit:

$$\check{\mathcal{F}}(X)(n) = \varinjlim_{\mathcal{U} \in \mathrm{HCov}_{\mathfrak{r}\mathfrak{S}}((X,f,q))} \mathcal{F}(n)(p^+_{\mathrm{HCov}}(\mathcal{U}))^*.$$

ii) We define $\check{\mathcal{F}}_X^{(q)}(n)_f$ as the sheaf on S associated to the presheaf

$$(U \to S) \mapsto \check{\mathcal{F}}^{(q)}(U \times_S X)(n)_f.$$

We define $\check{\mathcal{F}}_X(n)$ as the sheaf on S associated to the presheaf

$$(U \to S) \mapsto \check{\mathcal{F}}(U \times_S X)(n).$$

We define $Rp^{(q)}_{X*}\mathcal{F}(X)(n)_f$ as the direct limit:

$$Rp^{(q)}_{X*}\mathcal{F}(X)(n)_f = \varinjlim_{W \in (X,f)^{(q)}} Rp^W_{X*}\mathcal{F}(X)(n)_f.$$

□

Sending (X, f, q) to $\check{\mathcal{F}}_X^{(q)}(n)_f$ or $\check{\mathcal{F}}_X(n)$ defines the functors

$$\begin{aligned} &\check{\mathcal{F}}_-^{(-)}(n)_-: \mathcal{W} \to \mathbf{C}^+(\mathrm{Sh}_S^{\mathfrak{S}}(\mathcal{A})) \\ &\check{\mathcal{F}}_-(n): \mathcal{W} \to \mathbf{C}^+(\mathrm{Sh}_S^{\mathfrak{S}}(\mathcal{A})) \end{aligned} \tag{8.2.5.1}$$

The natural transformation (8.2.3.4) defines the natural transformation

$$\check{i}^!: \check{\mathcal{F}}_-^{(-)}(n)_- \to \check{\mathcal{F}}_-(n). \tag{8.2.5.2}$$

Similarly, sending (X, f, q) to $Rp_{X*}^{(q)}\mathcal{F}_X(n)_f$ or $Rp_{X*}\mathcal{F}_X(n)$ defines functors

$$\begin{aligned} Rp_{X*}^{(-)}\mathcal{F}_-(n)_-: \mathcal{W} \to \mathbf{D}^+(\mathrm{Sh}_S^{\mathfrak{S}}(\mathcal{A})) \\ Rp_*\mathcal{F}_-(n): \mathcal{W} \to \mathbf{D}^+(\mathrm{Sh}_S^{\mathfrak{S}}(\mathcal{A})) \end{aligned}$$

The natural transformation (8.1.2.1) defines the natural transformation

$$Ri^!\mathcal{F}_{(-)}: Rp_{X*}^{(-)}\mathcal{F}_-(n)_- \to Rp_*\mathcal{F}_-(n).$$

(8.2.6) LEMMA

i) The complex of sheaves $\check{\mathcal{F}}_X(n)$ depends only on X, not on the object (X, f, q) of $\mathcal{W}$ over X.

ii) For $(X, f, q) \in \mathcal{W}$, complexes $\check{\mathcal{F}}_X^{(q)}(n)_f$ and $\check{\mathcal{F}}_X(n)$ are complexes of acyclic sheaves on S.

iii) For $\mathcal{G} \in \mathbf{C}^+(\mathrm{Sh}_S^{\mathfrak{S}}(\mathcal{A}))$, we let $R\mathcal{G}$ denote the image of $\mathcal{G}$ in $\mathbf{D}^+(\mathrm{Sh}_S^{\mathfrak{S}}(\mathcal{A}))$. There are canonical isomorphisms of functors

$$\begin{aligned} &R\check{\mathcal{F}}_-^{(-)}(n)_- \to Rp_*^{(-)}\mathcal{F}_-(n)_- \\ &R\check{\mathcal{F}}_-(n) \to Rp_*\mathcal{F}_-(n). \end{aligned}$$

In addition, the diagram

$$\begin{array}{ccc} R\check{\mathcal{F}}_-^{(-)}(n)_- & \to & Rp_*^{(-)}\mathcal{F}_-(n)_- \\ {\scriptstyle R\check{i}^!}\downarrow & & \downarrow{\scriptstyle Ri^!\mathcal{F}_{(-)}} \\ R\check{\mathcal{F}}_-(n) & \to & Rp_*\mathcal{F}_-(n) \end{array}$$

commutes.

Proof. For the first assertion, we note that the functor p_{HCov2}^+ (8.2.1.2) is the pull-back of the functor (II, (6.4.5)(ii))

$$p_{\mathrm{HCov}}: \mathrm{HCov}_{\mathfrak{r}\mathfrak{S}}(\mathcal{C}) \to \mathcal{C}(\Delta^{\mathrm{op}})$$

via
$$p_1: \mathcal{C}_2 \to \mathcal{C},$$
followed by the functor
$$+: \mathcal{C}(\Delta^{\mathrm{op}}) \to \mathcal{C}^+(\Delta^{\mathrm{op}}).$$
Thus, we may rewrite the limit defining $\check{\mathcal{F}}(X)(n)$ as
$$\check{\mathcal{F}}(X)(n) = \varinjlim_{\mathcal{U} \in \mathrm{HCov}_{\mathfrak{r}\mathfrak{S}}(X)} \mathcal{F}(n)(p_{\mathrm{HCov}}(\mathcal{U}))^*. \tag{1}$$

This proves (i).

For (ii), we may rewrite the limits in (8.2.5) as
$$\varinjlim_{(X,W)\in(X,f)^{(q)}} [\varinjlim_{\mathcal{U}\in \mathrm{HCov}_{\mathfrak{r}\mathfrak{S}}((X,W))} (?)].$$

As the ordered set $(X, f)^{(q)}$ is left-directed, taking
$$\varinjlim_{(X,W)\in(X,f)^{(q)}}$$
is an exact functor. The second assertion then follows from (1), and (II, (6.10.8), (6.10.9)). □

8.3. The construction of the realization functor

We now give the construction of the realization functor
$$\mathrm{Re}_{\mathcal{F}}: \mathcal{DM}(\mathcal{V}) \to \mathbf{D}^+(\mathrm{Sh}_S^{\mathfrak{S}}(\mathcal{A})).$$
associated to a geometric cohomology theory $\mathcal{F}$ on $\mathcal{V}$. Except for one point, the construction would be an essentially straightforward step-by-step extension of the functor
$$\check{\mathcal{F}}_{(-)}(-): \mathcal{V}^{\mathrm{op}} \times \mathbb{Z} \to \mathbf{C}^+(\mathbf{C}^+(\mathrm{Sh}_S^{\mathfrak{S}}(\mathcal{A})))$$
$$(X, q) \mapsto \check{\mathcal{F}}_X(q),$$
to the DG tensor category $\mathcal{A}_{mot}(\mathcal{V})$, and from there, a direct extension to the category of complexes $\mathbf{C}^b_{mot}(\mathcal{V})$, the homotopy category $\mathbf{K}^b_{mot}(\mathcal{V})$, and the localization $\mathbf{D}^b_{mot}(\mathcal{V})$. One then applies (II, (2.5.5)), to give the extension to $\mathcal{DM}(\mathcal{V})$. The problem is that in general one cannot define the external product on the complexes $\check{\mathcal{F}}_X$ to be associative and graded-commutative (on the level of complexes), but only graded-commutative up to homotopy (although still associative). We must then replace the DG category $\mathcal{A}_{mot}(\mathcal{V})$ with an up-to-homotopy commutative model $\mathcal{A}^{\natural}_{mot}(\mathcal{V})$, and use (II, (2.3.2)), to get back

to the homotopy category $\mathbf{K}^b_{mot}(\mathcal{V})$. The extension to $\mathcal{DM}(\mathcal{V})$ then proceeds as outlined above. We now proceed to give the details of this construction.

(8.3.1) THEOREM

Let $\mathcal{F} = \oplus_{q=0}^{\infty}\mathcal{F}(q) \in \mathbf{C}^+(\mathrm{Sh}^{\mathfrak{S}}_{\mathcal{C}}(\mathcal{A}))$ be a graded complex of flat sheaves on the site $\mathcal{C}_{\mathfrak{S}}$, with values in an abelian tensor category $\mathcal{A}$, having an associative, graded-commutative product (8.1.3.1). Suppose $\mathcal{F}$ defines a geometric cohomology theory on $\mathcal{V}$ (8.1.3). Then the functor

$$\check{\mathcal{F}}_{(-)}(-): \mathcal{V}^{\mathrm{op}} \times \mathbb{Z} \to \mathbf{C}^+(\mathrm{Sh}^{\mathfrak{S}}_S(\mathcal{A}))$$
$$(X, q) \mapsto \check{\mathcal{F}}_X(q),$$

extends to an exact tensor functor

$$\mathrm{Re}_{\mathcal{F}}: \mathcal{DM}(\mathcal{V}) \to \mathbf{D}^+(\mathrm{Sh}^{\mathfrak{S}}_S(\mathcal{A})).$$

The functor $\mathrm{Re}_{\mathcal{F}}$ is natural in the geometric cohomology theory $\mathcal{F}$ and in the category $\mathcal{V}$; in addition, the functor $\mathrm{Re}_{\mathcal{F}}$ is independent of the choice of rigidification $\mathfrak{r}\mathfrak{S}$ of the topology $\mathfrak{S}$, up to canonical isomorphism.

The proof proceeds in a series of steps:

(8.3.2)

- *Step 1. The extension to $\mathcal{A}_1(\mathcal{V})$:*

The functors (8.2.5.1) define the functors

$$\begin{aligned} \check{\mathcal{F}}^*&: \mathcal{L}(\mathcal{V})^* \to \mathbf{C}^+(\mathrm{Sh}_{\mathcal{A}}(S)) \\ \check{\mathcal{F}}&: \mathcal{L}(\mathcal{V})^* \to \mathbf{C}^+(\mathrm{Sh}_{\mathcal{A}}(S)) \end{aligned} \tag{8.3.2.1}$$

by

$$\check{\mathcal{F}}^*(X, f, q) = \check{\mathcal{F}}^{(q)}_X(q)_f$$
$$\check{\mathcal{F}}(X, f, q) = \check{\mathcal{F}}_X(q).$$

The natural transformation (8.2.5.2) gives the natural transformation

$$\check{i}^! \mathcal{F}: \check{\mathcal{F}}^* \to \check{\mathcal{F}}. \tag{8.3.2.2}$$

Using the additive structure on $\mathbf{C}^+(\mathrm{Sh}^{\mathfrak{S}}_S(\mathcal{A}))$, and the fact that sheaves transform disjoint unions to direct sums, the functors (8.3.2.1) and natural transformation (8.3.2.2) extend to functors

$$\begin{aligned} \check{\mathcal{F}}^*_1&: \mathcal{A}_1(\mathcal{V}) \to \mathbf{C}^+(\mathrm{Sh}^{\mathfrak{S}}_S(\mathcal{A})) \\ \check{\mathcal{F}}_1&: \mathcal{A}_1(\mathcal{V}) \to \mathbf{C}^+(\mathrm{Sh}^{\mathfrak{S}}_S(\mathcal{A})) \end{aligned} \tag{8.3.2.3}$$

with

$$\check{\mathcal{F}}_1^*(\mathbb{Z}_X(a)_f) = \check{\mathcal{F}}_X^*(a)_f; \qquad \check{\mathcal{F}}_1(\mathbb{Z}_X(a)_f) = \check{\mathcal{F}}_X(a)_f$$

and natural transformation

$$\check{i}^! \mathcal{F}_1 : \check{\mathcal{F}}_1^* \to \check{\mathcal{F}}_1. \tag{8.3.2.4}$$

(see (1.2.1)).

(8.3.3)

• *Step 2. The category $\mathcal{A}_2^{\mathfrak{h}}(\mathcal{V})$, and the extension to $\mathcal{A}_2^{\mathfrak{h}}(\mathcal{V})$:*

We now apply the constructions of (II, §3.1). Using the notation of (II, §1.6), and referring to (1.2.3), the category $\mathcal{A}_2(\mathcal{V})$ is the universal commutative external product, $\mathcal{A}_1(\mathcal{V})^{\otimes,c}$, on $\mathcal{A}_1(\mathcal{V})$. Using the construction of (II, (3.1.6)), and applying (II, (3.1.8)), we have the DG tensor category $\mathcal{A}_1(\mathcal{V})^{\otimes,\mathfrak{h}}$, and the DG tensor functor

$$\mathfrak{c}: \mathcal{A}_1(\mathcal{V})^{\otimes,\mathfrak{h}} \to \mathcal{A}_1(\mathcal{V})^{\otimes,c},$$

which is the identity on objects, surjective on morphisms and a homotopy equivalence. We have as well the additive functor

$$i^{\mathfrak{h}}: \mathcal{A}_1(\mathcal{V}) \to \mathcal{A}_1(\mathcal{V})^{\otimes,\mathfrak{h}}$$

with $\mathfrak{c} \circ \mathfrak{i}^{\mathfrak{h}}$ the canonical functor (II, (1.6.1))

$$i^c: \mathcal{A}_1(\mathcal{V}) \to \mathcal{A}_1(\mathcal{V})^{\otimes,c}.$$

We set

$$\mathcal{A}_2^{\mathfrak{h}}(\mathcal{V}) := \mathcal{A}_1(\mathcal{V})^{\otimes,\mathfrak{h}}.$$

We now apply the results of (II, §4.4, (4.4.3)) where we take

$$\mathcal{W} = (\mathcal{L}(\mathcal{V}^*)^{\mathrm{op}}$$

and

$$w: \mathcal{W} \to \mathbb{Z}$$

the projection on $\mathbb{Z}$. We take the fibered symmetric semi-monoidal category of (II, (4.4.3))

$$\pi: \mathrm{RSCov}(\mathcal{W}) \to \mathcal{W}$$

to be the functor (8.2.3.2)

$$\pi_{\mathcal{W}}: \mathrm{HCov}_{\mathfrak{r}\mathfrak{S}}(\mathcal{W}) \to \mathcal{W}.$$

We make two choices of functors

$$p: \mathrm{RSCov}(\mathcal{W}) \to \mathcal{C}^+(\Delta^{\mathrm{op}}),$$

namely, the functors (8.2.3.3). By (8.2.4), the conditions required by (II, (4.4.3.1)) are satisfied. In addition, we have

$$\check{\mathcal{F}}(X, f, q) = \begin{cases} \check{\mathcal{F}}_X^{(q)}(q)_f & \text{for } p = p^!_{\mathrm{HCov}}, \\ \check{\mathcal{F}}_X(q) & \text{for } p = p^+_{\mathrm{HCov}}, \end{cases}$$

Let $\mathcal{L}(\mathcal{V})^*_{\mathbb{Z}}$ be the additive category generated by $\mathcal{L}(\mathcal{V})^*$. By (II, (4.4.4)) we have the functors:

$$\begin{aligned} \check{\mathcal{F}}^*\colon (\mathcal{L}(\mathcal{V})^*_{\mathbb{Z}})^{\otimes,\mathfrak{h}} &\to \mathbf{C}^+(\mathcal{A}) \\ \check{\mathcal{F}}\colon (\mathcal{L}(\mathcal{V})^*_{\mathbb{Z}})^{\otimes,\mathfrak{h}} &\to \mathbf{C}^+(\mathcal{A}) \end{aligned} \tag{8.3.3.1}$$

extending the functors (8.3.2.3). As the construction of the functor $\check{\mathfrak{F}}$ is natural in the category RSCov($\mathcal{W}$), the natural transformation (8.3.2.4) extends to the natural transformation

$$\check{\mathcal{F}}^* \to \check{\mathcal{F}}. \tag{8.3.3.2}$$

It follows directly from the definition (II, (3.1.6)) of the functor $^{\otimes,\mathfrak{h}}$ that

$$(\mathcal{A}_1(\mathcal{V}))^{\otimes,\mathfrak{h}} := \mathcal{A}_2^{\mathfrak{h}}(\mathcal{V})$$

is isomorphic to the DG tensor category gotten by imposing the relations of (1.2.1) on $(\mathcal{L}(\mathcal{V}^*_{\mathbb{Z}})^{\otimes,\mathfrak{h}}$. Thus, the functors (8.3.3.1) and the natural transformation (8.3.3.2) extend to the functors

$$\begin{aligned} \check{\mathcal{F}}_2^*\colon \mathcal{A}_2^{\mathfrak{h}}(\mathcal{V}) &\to \mathbf{C}^+(\mathrm{Sh}_S^{\mathfrak{S}}(\mathcal{A})) \\ \check{\mathcal{F}}_2\colon \mathcal{A}_2^{\mathfrak{h}}(\mathcal{V}) &\to \mathbf{C}^+(\mathrm{Sh}_S^{\mathfrak{S}}(\mathcal{A})) \end{aligned} \tag{8.3.3.3}$$

and natural transformation

$$\check{\mathcal{F}}_2^* \to \check{\mathcal{F}}_2. \tag{8.3.3.4}$$

Before proceeding further with our construction, we note the following result; the proof is elementary and is left to the reader:

(8.3.4) LEMMA

Let $F\colon \mathcal{A} \to \mathcal{B}$ be a DG tensor functor of DG tensor categories without unit. Suppose that

i) F is an isomorphism on objects
ii) F is surjective on morphisms
iii) F is a homotopy equivalence.

Let $f\colon F(X) \to F(Y)$ be a map of degree a in $\mathcal{B}$ such that $Df = 0$. Then there is a map $s\colon X \to Y$ of degree a in $\mathcal{A}$ such that $F(s) = f$ and $Ds = 0$. In addition, let $\mathcal{A}[h_s]$ be the DG tensor category without unit gotten by adjoining a morphism

$$h_s\colon X \to Y$$

of degree $a-1$ with $dh_s = s$, let $\mathcal{B}[h_f]$ the DG tensor category without unit defined by adjoining a morphism

$$h_f\colon F(X) \to F(Y)[a-1]$$

of degree $a-1$ with $dh_f = f$, and let

$$F'\colon \mathcal{A}[h_s] \to \mathcal{B}[h_f]$$

be the extension of F with $F'(h_s) = h_f$. Then F' satisfies (i) and (ii). □

(8.3.5)

• *Step 3. The category $\mathcal{A}^{\natural}_{mot}(\mathcal{V})$ and the extension to $\mathcal{A}_{mot}(\mathcal{V})$:*

We now form a sequence of DG tensor categories without unit

$$\begin{array}{ccccccc} \mathcal{A}^{\natural}_2(\mathcal{V}) & \to & \mathcal{A}^{\natural}_3(\mathcal{V}) & \to & \mathcal{A}^{\natural}_4(\mathcal{V}) & \to & \mathcal{A}^{\natural}_5(\mathcal{V}) \\ &&&&&& \cup \\ &&&&&& \mathcal{A}^{\natural}_{mot}(\mathcal{V}) \end{array}$$

analogous to the sequence of DG tensor categories formed in §1.2.

We recall the homotopy unit category $\mathbb{E}$ constructed in (II, §2.4). $\mathbb{E}$ is a DG tensor category without unit, with the generating object $\mathfrak{e}$. $\mathbb{E}$ has no morphisms of positive degree, no morphisms from $\mathfrak{e}^{\otimes m}$ to $\mathfrak{e}^{\otimes n}$ if $n \neq m$, and

$$H^q(\mathrm{Hom}(\mathfrak{e}^{\otimes n}, \mathfrak{e}^{\otimes n})^*) = \begin{cases} \mathbb{Z}\cdot \mathrm{id} & \text{for } q=0, \\ 0 & \text{otherwise.} \end{cases} \tag{8.3.5.1}$$

In addition, the Hom-complex $\mathrm{Hom}(\mathfrak{e}^{\otimes n}, \mathfrak{e}^{\otimes n})^*$ is a complex of free (left or right) $\mathbb{Z}[S_n]$-modules, where $\sigma \in S_n$ acts by (left or right) composition with the symmetry isomorphism τ_σ.

We have the coproduct of DG tensor categories without unit $\mathcal{A}_2(\mathcal{V})[\mathbb{E}]$; the DG tensor category $\mathcal{A}_3(\mathcal{V})$ (1.2.5) is formed from the DG tensor category $\mathcal{A}_2(\mathcal{V})[\mathbb{E}]$ by adjoining morphisms

$$[Z]\colon \mathfrak{e} \to \mathbb{Z}_X(n)_f.$$

of degree $2n$ for each non-zero $Z \in \mathcal{Z}^n(X)_f$. We form the DG tensor category $\mathcal{A}^{\natural}_3(\mathcal{V})$ from the coproduct $\mathcal{A}^{\natural}_2(\mathcal{V})[\mathbb{E}]$ by adjoining morphisms of degree $2n$

$$[Z]^{\natural}\colon \mathfrak{e} \to \mathbb{Z}_X(n)_f$$

with $d[Z]^{\natural} = 0$ for each non-zero $Z \in \mathcal{Z}^n(X)_f$. We extend the functor

$$\mathfrak{c}[\mathrm{id}_{\mathbb{E}}]\colon \mathcal{A}^{\natural}_2(\mathcal{V})[\mathbb{E}] \to \mathcal{A}_2(\mathcal{V})[\mathbb{E}]$$

to

$$\mathfrak{c}_3\colon \mathcal{A}^{\natural}_3(\mathcal{V}) \to \mathcal{A}_3(\mathcal{V}) \tag{8.3.5.2}$$

by setting

$$\mathfrak{c}_3([Z]^\natural) = [Z].$$

By (II, (2.3.4)), the functor (8.3.5.2) is a homotopy equivalence.

For each pair of objects Γ, Δ of $\mathcal{A}_1(\mathcal{V})$, we have the external product

$$\boxtimes_{\Gamma,\Delta}: \Gamma \otimes \Delta \to \Gamma \times \Delta$$

in $\mathcal{A}_2(\mathcal{V})$, and the lifting of $\boxtimes_{\Gamma,\Delta}$ to the external product

$$\boxtimes^\natural_{\Gamma,\Delta}: \Gamma \otimes \Delta \to \Gamma \times \Delta$$

in $\mathcal{A}^\natural_2(\mathcal{V})$ (see (II, (3.1.7.2) and (3.1.8)), where the lifting $\boxtimes^\natural_{**}$ is denoted $\boxtimes^\delta_{**}$). We note that $d\boxtimes^\natural_{\Gamma,\Delta} = 0$

The DG tensor category $\mathcal{A}_4(\mathcal{V})$ is formed from graded tensor category $\mathcal{A}_3(\mathcal{V})$ by selecting certain morphisms f in $\mathcal{A}_3(\mathcal{V})$, and adjoining morphisms h_f with $dh_f = f$ (see (1.2.7)). The morphisms f are all constructed from the morphisms $[Z]$, $\boxtimes_{**}$ and $\otimes$, together with morphisms of the category $\mathcal{A}_1(\mathcal{V})$. Given such an expression for a morphism f, we let $f^\natural$ be the morphism in $\mathcal{A}^\natural_3(\mathcal{V})$ gotten by replacing each occurrence of the morphism $[Z]$ with the morphism $[Z]^\natural$, and replacing $\boxtimes_{**}$ with $\boxtimes^\natural_{**}$. Since

$$d[Z]^\natural = 0,\ d\boxtimes^\natural_{**} = 0,$$

we have

$$df^\natural = 0$$

as well. We then adjoin, for each such f, a morphism $h^\natural_f$ to $\mathcal{A}^\natural_3(\mathcal{V})$ with $dh^\natural_f = f^\natural$, forming the DG tensor category without unit $\mathcal{A}^\natural_4(\mathcal{V})$.

We extend (8.3.5.2) to

$$\mathfrak{c}_4: \mathcal{A}^\natural_4(\mathcal{V}) \to \mathcal{A}_4(\mathcal{V}) \tag{8.3.5.3}$$

by setting $\mathfrak{c}_4(h^\natural_f) = h_f$. By (II, (2.3.4)), $\mathfrak{c}_4$ is a homotopy equivalence; by (8.3.4), $\mathfrak{c}_4$ is the identity on objects and surjective on morphisms.

The category $\mathcal{A}_5(\mathcal{V})$ is formed from $\mathcal{A}_4(\mathcal{V})$ by forming a succession of categories

$$\mathcal{A}_5(\mathcal{V})^{(0)} = \mathcal{A}_4(\mathcal{V}) \subset \ldots \subset \mathcal{A}_5(\mathcal{V})^{(r-1)} \subset \mathcal{A}_5(\mathcal{V})^{(r)} \subset \ldots$$

and letting $\mathcal{A}_5(\mathcal{V})$ be the direct limit. The category $\mathcal{A}_5(\mathcal{V})^{(r)}$ is formed from $\mathcal{A}_5(\mathcal{V})^{(r-1)}$ by adjoining morphisms

$$h_f: \mathfrak{e}^{\otimes k} \to \mathbb{Z}_X(n)_f$$

of degree $2n - r - 1$, with

$$dh_f = f$$

for each non-zero morphism

$$f: \mathfrak{e}^{\otimes k} \to \mathbb{Z}_X(n)_f \tag{8.3.5.4}$$

of degree r in $\mathcal{A}_5(\mathcal{V})^{(r-1)}$ with $df = 0$. This is done successively for $k = 1, 2, \ldots$. Using (8.3.4) and (II, (2.3.4)), we may construct inductively the sequence of DG tensor categories

$$\mathcal{A}_5^{\mathfrak{h}}(\mathcal{V})^{(0)} = \mathcal{A}_4^{\mathfrak{h}}(\mathcal{V}) \subset \ldots \subset \mathcal{A}_5^{\mathfrak{h}}(\mathcal{V})^{(r-1)} \subset \mathcal{A}_5^{\mathfrak{h}}(\mathcal{V})^{(r)} \subset \ldots$$

and DG tensor functors

$$\mathfrak{c}_5^{(r)}: \mathcal{A}_5^{\mathfrak{h}}(\mathcal{V})^{(r)} \to \mathcal{A}_5(\mathcal{V})^{(r)} \tag{8.3.5.5}$$

which are homotopy equivalences, the identity on objects and surjective on morphisms as follows: Assuming we have constructed the sequence up to $r-1$, we may lift each morphism (8.3.5.4) to a morphism

$$f^{\mathfrak{h}}: 1^{\otimes k} \to \mathbb{Z}_X(n)_f$$

in $\mathcal{A}_5^{\mathfrak{h}}(\mathcal{V})^{(r-1)}$ with $df^{\mathfrak{h}} = 0$. We may then adjoin morphisms

$$h_f^{\mathfrak{h}}: 1^{\otimes k} \to \mathbb{Z}_X(n)_f$$

with $dh_f^{\mathfrak{h}} = f^{\mathfrak{h}}$, forming the DG tensor category $\mathcal{A}_5^{\mathfrak{h}}(\mathcal{V})^{(r)}$. The extension of $\mathfrak{c}_5^{(r-1)}$ to $\mathfrak{c}_5^{(r)}$ is defined by

$$\mathfrak{c}_5^{(r)}(h_f^{\mathfrak{h}}) = h_f.$$

Taking the direct limit over r of (8.3.5.5) gives the DG tensor functor

$$\mathfrak{c}_5: \mathcal{A}_5^{\mathfrak{h}}(\mathcal{V}) \to \mathcal{A}_5(\mathcal{V}), \tag{8.3.5.6}$$

which is a homotopy equivalence.

Finally, the category $\mathcal{A}_{mot}(\mathcal{V})$ is defined as the full DG tensor subcategory of $\mathcal{A}_5(\mathcal{V})$ generated by objects of the form $\mathbb{Z}_X(n)_f$ or $\mathbb{Z}_X(n)_f \otimes \mathfrak{e}^{\otimes a}$, $a \geq 1$ (see (1.2.9)). We let $\mathcal{A}_{mot}^{\mathfrak{h}}(\mathcal{V})$ be the full DG tensor subcategory of $\mathcal{A}_5^{\mathfrak{h}}(\mathcal{V})$ generated by objects of the form $\mathbb{Z}_X(n)_f$ or $\mathbb{Z}_X(n)_f \otimes \mathfrak{e}^{\otimes a}$, $a \geq 1$. Since (8.3.5.6) is a homotopy equivalence, the identity on objects and surjective on morphisms, the same is true for the restriction

$$\mathfrak{c}_{mot}: \mathcal{A}_{mot}^{\mathfrak{h}}(\mathcal{V}) \to \mathcal{A}_{mot}(\mathcal{V}). \tag{8.3.5.7}$$

It is now a straightforward matter to extend the functors (8.3.3.3) and the natural transformation (8.3.3.4) to $\mathcal{A}_{mot}^{\mathfrak{h}}(\mathcal{V})$. Indeed, by (8.1.3)(viii), we have a finite projective resolution

$$P_* \to 1$$

of the unit $1 \in \mathcal{A}$, satisfying the conditions (a)-(c). The condition (a) implies that we have the canonical isomorphisms

$$\mathrm{Hom}_{K^b(\mathcal{A})}(P_*^{\otimes n}, P_*^{\otimes m}[p]) \cong \begin{cases} 0 & \text{for } p < 0, \\ \mathrm{Hom}_{\mathcal{A}}(1^{\otimes n}, 1^{\otimes m}) & \text{for } p = 0. \end{cases}$$

This, together with (8.3.5.1), the condition (b) of (8.1.3)(vi), and the universal mapping property of the category $\mathbb{E}$ (II, (2.4.12)) shows that there is a functor of DG tensor categories

$$I\colon \mathbb{E} \to C^b(\mathcal{A})$$

with $I(\mathfrak{e}^{\otimes n}) = P_*^{\otimes n}$. In addition, I is unique up to homotopy.

Let

$$P_{*S}^{\otimes n} \to 1_S^{\otimes n}$$

be the augmented complex of constant sheaves on S corresponding to $P_*^{\otimes n}$. Let

$$\begin{aligned} I_S&\colon \mathbb{E} \to \mathrm{Sh}_{\mathcal{A}}^{\mathfrak{S}}(S) \\ I_S(\mathfrak{e}^{\otimes n}) &= (P_{S*})^{\otimes n} \cong P_{S*}^{\otimes n} \end{aligned}$$

be the sheafification of the functor I; by condition (c) of (8.1.3)(viii), $(P_{S*})^{\otimes n}$ is a complex of flat sheaves on S. Taking the coproduct of I_S with the functors (8.3.3.3) gives the DG tensor functors

$$\begin{aligned} \check{\mathcal{F}}_2^*[I_S]&\colon \mathcal{A}_2^{\natural}(\mathcal{V})[\mathbb{E}] \to \mathbf{C}^+(\mathrm{Sh}_{\mathcal{A}}(S)) \\ \check{\mathcal{F}}_2[I_S]&\colon \mathcal{A}_2^{\natural}(\mathcal{V})[\mathbb{E}] \to \mathbf{C}^+(\mathrm{Sh}_{\mathcal{A}}(S)). \end{aligned} \tag{8.3.5.8}$$

The natural transformation (8.3.3.4) extends similarly to the natural transformation

$$\check{\mathcal{F}}_2^*[I_S] \to \check{\mathcal{F}}_2[I_S]. \tag{8.3.5.9}$$

Since the complexes $P_{S*}^{\otimes n}$ are constant sheaves associated to a projective resolution of $1^{\otimes n} \cong 1$, and since the complexes $\check{\mathcal{F}}_X^n(n)_f$ are complexes of acyclic sheaves on S, it follows from (8.2.6) that we have the isomorphism

$$\mathrm{Hom}_{K^+(\mathrm{Sh}_S^{\mathfrak{S}}(\mathcal{A}))}(P_{S*}^{\otimes n}, \check{\mathcal{F}}_X^n(n)_f) \cong \varinjlim_{W\in(X,f)^{(q)}} \mathrm{Hom}_{D^+(\mathrm{Sh}_S^{\mathfrak{S}}(\mathcal{A}))}(1_S, Rp_{X*}\mathcal{F}_X^n(n)_f) \tag{8.3.5.10}$$

For non-zero Z in $\mathcal{Z}^n(X)_f$, define

$$\check{\mathcal{F}}_3^*([Z]^{\natural})\colon P_{S*} \to \check{\mathcal{F}}_X^n(n)_f[2n]$$

to be a choice of a map representing the map

$$\mathrm{cl}^q_{X,W}(Z\otimes 1)\colon 1_S \to Rp_{X*}\mathcal{F}_X^n(n)_f[2n].$$

in $\mathbf{D}^+(\mathrm{Sh}_S^{\mathfrak{S}}(\mathcal{A}))$ given by (8.1.3)(iii). We define

$$\check{\mathcal{F}}_3([Z]^{\natural})\colon P_{S*} \to \check{\mathcal{F}}_X(n)_f$$

to be the composition of $\check{\mathcal{F}}_3^*([Z]^{\mathfrak{h}})$ with the natural map

$$\check{\mathcal{F}}_X^*(n)_f \to \check{\mathcal{F}}_X(n)_f.$$

This gives the extension of the functors (8.3.5.8) to functors

$$\begin{aligned} \check{\mathcal{F}}_3^*&: \mathcal{A}_3^{\mathfrak{h}}(\mathcal{V}) \to \mathbf{C}^+(\mathrm{Sh}_{\mathcal{A}}(S)) \\ \check{\mathcal{F}}_3&: \mathcal{A}_3^{\mathfrak{h}}(\mathcal{V}) \to \mathbf{C}^+(\mathrm{Sh}_{\mathcal{A}}(S)) \end{aligned} \tag{8.3.5.11}$$

and natural transformation (8.3.5.9) extends to the natural transformation

$$\check{\mathcal{F}}_3^* \to \check{\mathcal{F}}_3. \tag{8.3.5.12}$$

The extension of (8.3.5.11) to the category $\mathcal{A}_4^{\mathfrak{h}}(\mathcal{V})$ is accomplished using the functoriality of the cycle classes in (8.1.3)(iii). For example, let

$$f: (Y, g) \to (X, f)$$

be a morphism in $\mathcal{L}(\mathcal{V})$, giving the map

$$f^*: \mathbb{Z}_X(n)_f \to \mathbb{Z}_Y(n)_g$$

in $\mathcal{A}_1(\mathcal{V})$. Take $Z \in \mathcal{Z}^n(X)_f$. Then we have the map

$$h^{\mathfrak{h}}_{X,Y,[Z],f^*}: \mathfrak{e} \to \mathbb{Z}_Y(n)_g$$

in $\mathcal{A}_4^{\mathfrak{h}}(\mathcal{V})$ with

$$dh^{\mathfrak{h}}_{X,Y,[Z],f^*} = f^* \circ [Z]^{\mathfrak{h}} - [f^*(Z)].$$

The functoriality of the cycle classes gives the relation

$$\check{\mathcal{F}}_3^*(f^* \circ [Z]^{\mathfrak{h}}) - \check{\mathcal{F}}_3^*([f^*(Z)]) = d\beta$$

for some map

$$\beta: P_{S*} \to \check{\mathcal{F}}_3^*(\mathbb{Z}_Y(n)_g)$$

of degree $2n-1$. We then define

$$\check{\mathcal{F}}_4^*(h^{\mathfrak{h}}_{X,Y,[Z],f^*}) = \beta.$$

The definition of $\check{\mathcal{F}}_4^*$ for the other types of maps adjoined to form $\mathcal{A}_4^{\mathfrak{h}}(\mathcal{V})$ is similar; we let $\check{\mathcal{F}}_4$ be the composition of $\check{\mathcal{F}}_4^*$ with the natural transformation (8.3.5.12).

The extension to $\mathcal{A}_5^{\mathfrak{h}}(\mathcal{V})$ is accomplished in a similar manner, relying on the purity hypothesis (8.1.3)(iv) for the cohomology theory $\mathcal{F}$. Restricting to the subcategory $\mathcal{A}_{mot}^{\mathfrak{h}}(\mathcal{V})$ gives the desired functor

$$\check{\mathcal{F}}_{mot}^{\mathfrak{h}}\colon \mathcal{A}_{mot}^{\mathfrak{h}}(\mathcal{V}) \to \mathbf{C}^+(\mathrm{Sh}_S^{\mathfrak{S}}(\mathcal{A})). \tag{8.3.5.13}$$

(8.3.6)

• *Step 4. The extension to $\mathcal{DM}(\mathcal{V})$:*

Applying the functor $\mathbf{C}^b$ (II, (2.1.2)) to the DG tensor functor (8.3.5.13) gives the functor

$$\mathbf{C}^b(\check{\mathcal{F}}_{mot}^{\mathfrak{h}})\colon \mathbf{C}^b(\mathcal{A}_{mot}^{\mathfrak{h}}(\mathcal{V})) \to \mathbf{C}^b(\mathbf{C}^+(\mathrm{Sh}_S^{\mathfrak{S}}(\mathcal{A})));$$

composing with the equivalence (see II, (2.1.4))

$$\mathrm{Tot}\colon \mathbf{C}^b(\mathbf{C}^+(\mathrm{Sh}_{\mathcal{A}}^{\mathfrak{S}}(S))) \to \mathbf{C}^+(\mathrm{Sh}_S^{\mathfrak{S}}(\mathcal{A}))$$

gives the DG tensor functor

$$\tilde{\mathbf{C}}^b(\check{\mathcal{F}}_{mot}^{\mathfrak{h}})\colon \mathbf{C}^b(\mathcal{A}_{mot}^{\mathfrak{h}}(\mathcal{V})) \to \mathbf{C}^+(\mathrm{Sh}_S^{\mathfrak{S}}(\mathcal{A})).$$

Passing to the homotopy category gives the exact tensor functor

$$\tilde{\mathbf{K}}^b(\check{\mathcal{F}}_{mot}^{\mathfrak{h}})\colon \mathbf{K}^b(\mathcal{A}_{mot}^{\mathfrak{h}}(\mathcal{V})) \to \mathbf{K}^+(\mathrm{Sh}_S^{\mathfrak{S}}(\mathcal{A})). \tag{8.3.6.1}$$

By (II, (2.3.2)), the functor

$$\mathbf{K}^b(\mathfrak{c}_{mot})\colon \mathbf{K}^b(\mathcal{A}_{mot}^{\mathfrak{h}}(\mathcal{V})) \to \mathbf{K}^b(\mathcal{A}_{mot}(\mathcal{V})) = \mathbf{K}_{mot}^b(\mathcal{V})$$

is an equivalence of exact tensor categories; the functor (8.3.6.1) thus gives the exact tensor functor

$$\mathbf{K}^b(\check{\mathcal{F}}_{mot})\colon \mathbf{K}_{mot}^b(\mathcal{V}) \to \mathbf{K}^+(\mathrm{Sh}_S^{\mathfrak{S}}(\mathcal{A})). \tag{8.3.6.2}$$

Let X be in $\mathcal{V}$, and suppose X is a union of two open subschemes in $\mathcal{V}$:

$$X = U \cup V.$$

Let

$$j\colon \check{\mathcal{F}}_U(n) \oplus \check{\mathcal{F}}_V(n) \to \check{\mathcal{F}}_{U\cap V}(n)$$

be the difference of the two restriction maps. Since the the image of $\check{\mathcal{F}}_X(n)$ in the derived category is isomorphic to $Rp_{X*}\mathcal{F}_X(n)$, the natural map

$$\check{\mathcal{F}}_X(n) \to \mathrm{Cone}(j)[-1]$$

is a quasi-isomorphism. This, together with (8.1.3)(ii), (iv)-(vii), implies that the composition of (8.3.6.2) with the canonical map

$$\mathbf{K}^+(\mathrm{Sh}_S^{\mathfrak{S}}(\mathcal{A})) \to \mathbf{D}^+(\mathrm{Sh}_S^{\mathfrak{S}}(\mathcal{A})).$$

factors through the localization $\mathbf{D}^b_{mot}(\mathcal{V})$ of $\mathbf{K}^b_{mot}(\mathcal{V})$ (see (2.1.3)), giving the functor

$$\mathbf{D}^b(\check{\mathcal{F}}_{mot})\colon \mathbf{D}^b_{mot}(\mathcal{V}) \to \mathbf{D}^+(\mathrm{Sh}_S^{\mathfrak{S}}(\mathcal{A})). \tag{8.3.6.3}$$

Finally, applying the argument of (II, (2.5.4)), the functor (8.3.6.3) extends canonically to the category $\mathcal{DM}(\mathcal{V})$, giving the desired realization functor

$$\mathrm{Re}_{\mathcal{F}}\colon \mathcal{DM}(\mathcal{V}) \to \mathbf{D}^+(\mathrm{Sh}_S^{\mathfrak{S}}(\mathcal{A})). \tag{8.3.6.4}$$

This completes the proof of (8.3.1), except to show that the functor (8.3.6.4) is independent of the choice of rigidification, and the other choices made along the way.

(8.3.7)

We first note that the extension from $\mathcal{A}_2^{\mathfrak{h}}$ to $\mathcal{A}_{mot}^{\mathfrak{h}}$ is independent of the various choices of the maps in Step 3 up to a homotopy of the resulting functors; this follows directly from the purity hypothesis (8.1.3)(iv), and the fact that the category $\mathcal{A}_{mot}^{\mathfrak{h}}$ is freely generated from $\mathcal{A}_2^{\mathfrak{h}}$ by the adjoined maps.

Now suppose we have made two choices of rigidifications; this results in two categories $\mathrm{HCov}_{\mathfrak{r}\mathfrak{S}^1}(\mathcal{C}_2)$ and $\mathrm{HCov}_{\mathfrak{r}\mathfrak{S}^2}(\mathcal{C}_2)$, and two functors

$$p^1\colon \mathrm{HCov}_{\mathfrak{r}\mathfrak{S}^1}(\mathcal{C}_2) \to \mathrm{HCov}_{\mathfrak{S}}(\mathcal{C}_2),$$
$$p^2\colon \mathrm{HCov}_{\mathfrak{r}\mathfrak{S}^2}(\mathcal{C}_2) \to \mathrm{HCov}_{\mathfrak{S}}(\mathcal{C}_2),$$

We may form the fiber product category

$$\mathrm{HCov}_{\mathfrak{r}\mathfrak{S}^{12}}(\mathcal{C}_2) := \mathrm{HCov}_{\mathfrak{r}\mathfrak{S}^1}(\mathcal{C}_2) \times_{\mathcal{C}_2} \mathrm{HCov}_{\mathfrak{r}\mathfrak{S}^2}(\mathcal{C}_2) \to \mathcal{W}.$$

We define the functor

$$p^{12}\colon \mathrm{HCov}_{\mathfrak{r}\mathfrak{S}^{12}}(\mathcal{C}_2) \to \mathrm{HCov}_{\mathfrak{S}}(\mathcal{C}_2)$$

by sending a pair (V^1_*, V^2) over (X, U) to the fiber product

$$\mathrm{diag} V^1_* \times_{(X,U)_*} V^2_* \to (X, U)_*;$$

it follows easily from the compatibility of coskeleton and fiber products that $\mathrm{diag} V^1_* \times_{(X,U)_*} V^2_*$ is indeed a hypercover of (X, U). The projections give natural maps of hypercovers

$$\begin{array}{ccc} & \mathrm{diag} V^1_* \times_{(X,U)_*} V^2_* & \\ \swarrow & & \searrow \\ V^1_* & & V^2_*. \end{array} \tag{8.3.7.1}$$

We may then replace the parameter categories $\mathrm{HCov}_{\mathfrak{r}\mathfrak{S}^2}(\mathcal{C}_2)$ and $\mathrm{HCov}_{\mathfrak{r}\mathfrak{S}^1}(\mathcal{C}_2)$ with the category $\mathrm{HCov}_{\mathfrak{r}\mathfrak{S}^{12}}(\mathcal{C}_2)$ when forming the extension to $\mathcal{A}_2^{\mathfrak{h}}$ in Step 2; as the construction of (II, (4.4.4)) is natural in the category RSCov and functor p, the diagram (8.3.7.1) determines the desired natural isomorphism of functors to the derived category. This completes the proof of (8.3.1). □

8.4. The Betti and étale realizations

We define the Betti and étale realizations of $\mathcal{DM}(S)_R$.

(8.4.1) RIGIDIFICATIONS

We begin by recalling Friedlander's construction [F] of a rigidification of the classical and étale topologies. We first give the étale version.

Let S be a quasi-projective scheme over a Noetherian ring. For each point s of S, fix an algebraic closure $\overline{k(s)}$ of the residue field $k(s)$ of s. If $X \to S$ is a quasi-projective S-scheme, a *geometric point* of X is an equivalence class of maps over S

$$x\colon \mathrm{Spec}(\overline{k(s)}) \to X$$

where x is equivalent to x' if x and x' differ by an automorphism of $\mathrm{Spec}(\overline{k(s)})$ over X. We let X_{geom} denote the set of geometric points of X.

A *pointed étale map*

$$f\colon (U,u) \to (X,x)$$

is an étale map $f\colon U \to X$, with geometric points u of U, x of X such that $f(u) = x$.

We let $\mathcal{C}$ be the full subcategory of $\mathbf{Sch}_S$ with objects being arbitrary disjoint unions of quasi-projective S-schemes. We extend the above notions to $\mathcal{C}$ in the obvious way.

Let $\mathfrak{r}\mathcal{C}$ be the category with objects pointed connected quasi-projective S-schemes (X,x) and maps being pointed maps in $\mathbf{Sch}_S$; forgetting the point x defines the functor

$$q\colon \mathfrak{r}\mathcal{C} \to \mathcal{C}.$$

Given a diagram

$$\begin{array}{ccc} & & (Y,y) \\ & & \downarrow \\ (Z,z) & \to & (X,x) \end{array}$$

in $\mathfrak{r}\mathcal{C}$, let

$$(Y \times_X Z)^{(y,z)}$$

be the connected component of $Y \times_X Z$ containing the point (y,z). Define the rigid product $(Y,y) \times^{\mathfrak{r}}_{(X,x)} (Z,z)$ by

$$(Y,y) \times^{\mathfrak{r}}_{(X,x)} (Z,z) := ((Y \times_X Z)^{(y,z)}, (y,z)).$$

This gives the commutative diagram

$$\begin{array}{ccc} (Y,y)\times^{\mathfrak{r}}_{(X,x)}(Z,z) & \to & (Y,y) \\ \downarrow & & \downarrow \\ (Z,z) & \to & (X,x) \end{array}$$

and it is easy to see that this defines a fiber product in $\mathfrak{r}\mathcal{C}$. The inclusion map

$$\Theta_{(Y,y),(Z,z),(X,x)}\colon (Y\times_X Z)^{(y,z)} \to Y\times_X Z$$

gives the functor q the structure of a lax functor of categories with fiber products (cf. II, (6.1.3)).

For $(X,x)\in \mathfrak{r}\mathcal{C}$, we set $\mathfrak{r}\text{ét}((X,x))$ equal to the full subcategory of $\mathfrak{r}\mathcal{C}/(X,x)$ with objects the pointed étale maps

$$(U,u)\to (X,x).$$

The fact that $(\mathfrak{r}\mathcal{C}, q, \Theta, \mathfrak{r}\text{ét}(-))$ defines a rigidification $\mathfrak{r}\text{ét}$ of $\mathcal{C}_{\text{ét}}$ (II, (6.2.3)) follows from two elementary properties:

1) Each étale cover has a refinement which admits a rigidification

2) If $f\colon U\to X$ is an étale map, and $h\colon (U,u)\to (U,u)$ is a pointed map over X, then h is the identity.

One also easily verifies that the additional properties of (II, (6.10.3)) hold, so that $(\mathfrak{r}\text{ét}, \text{ét})$ defines a rigid Grothendieck topology on $\mathcal{C}$.

One defines similarly a rigidification of the classical topology $\text{an}_{\mathbb{C}}$ for analytic spaces by defining the category $\mathfrak{r}\mathcal{C}_{\text{an}_{\mathbb{C}}}$ as the category of connected pointed analytic spaces (X,x) where X has only finitely many irreducible components, and the category $\mathfrak{r}\text{an}_{\mathbb{C}}((X,x)$ as the category of pointed open immersions

$$(U,u)\to (X,x).$$

The pointed fiber products are defined as in the étale case by taking the connected component of the usual fiber product containing the product of base-points. As above, this gives us the rigid Grothendieck topology $(\mathfrak{r}\text{an}_{\mathbb{C}}, \text{an}_{\mathbb{C}})$ on the category $\mathcal{C}_{\text{an}_{\mathbb{C}}}$ of disjoint unions of connected analytic spaces with finitely may irreducible components.

We may form the real version of the analytic rigid topology by using analytic spaces with a real structure throughout; this defines the rigid Grothendieck topology $(\mathfrak{r}\text{an}_{\mathbb{R}}, \text{an}_{\mathbb{R}})$ on the category $\mathcal{C}_{\text{an}_{\mathbb{R}}}$ of disjoint unions of connected analytic spaces with real structure.

(8.4.2) The Betti realization

For the Betti realization (either over $\mathbb{C}$ or over $\mathbb{R}$), we may take $\mathcal{V}$ to be any subcategory of the category of smooth, quasi-project $\mathbb{C}$-schemes or $\mathbb{R}$-schemes for which $\mathcal{DM}(\mathcal{V})$ is defined. We take the cohomology theory defined by the graded sheaf

$$\mathbb{Z}(*) = \oplus_{q=0}^{\infty}(2\pi i)^q\mathbb{Z}.$$

The well known properties of singular cohomology show that $\mathbb{Z}(*)$ defines a geometric cohomology theory; we then apply (8.3.1) to give the $\mathbb{C}$-Betti realization functor

$$Re_{\mathfrak{B}_{\mathbb{C}}}: \mathcal{DM}(\mathcal{V}) \to \mathbf{D}^+(\mathbf{Ab}).$$

For the $\mathbb{R}$-Betti realization, we work in the category $\mathrm{Sh}^{\mathrm{an}_{\mathbb{R}}}(\mathbf{Ab})$ of sheaves of abelian groups with a $\mathbb{Z}/2$-action over the action of complex conjugation F_∞; over a point, this is the category $\mathbf{Ab}_{F_\infty}$, i.e., the category of modules over the group ring $\mathbb{Z}[\mathbb{Z}/2]$. Applying (8.3.1) gives the $\mathbb{R}$-Betti realization functor

$$Re_{\mathfrak{B}_{\mathbb{R}}}: \mathcal{DM}(\mathcal{V}) \to \mathbf{D}^+(\mathbf{Ab}_{F_\infty}),$$

for $\mathcal{V}$ a category of $\mathbb{R}$-schemes.

Using the same method, we may form the Betti realization over an arbitrary base scheme S over $\mathbb{C}$ or over $\mathbb{R}$. This gives the realization functors

$$Re_{\mathfrak{B}_{S,\mathbb{C}}}: \mathcal{DM}(\mathcal{V}) \to \mathbf{D}^+(\mathrm{Sh}_S^{\mathrm{an}_{\mathbb{C}}}(\mathbf{Ab})),$$

and

$$Re_{\mathfrak{B}_{S,\mathbb{R}}}: \mathcal{DM}(\mathcal{V}) \to \mathbf{D}^+(\mathrm{Sh}_S^{\mathrm{an}_{\mathbb{R}}}(\mathbf{Ab})).$$

(8.4.3) THE ÉTALE REALIZATION

For the étale realization, let X be an S-scheme . We consider the triangulated tensor category

$$\mathbf{D}^+ \varprojlim \mathrm{Sh}_X^{\text{ét}}(\mathbb{Z}_l)$$

constructed from inverse systems of of complexes of étale sheaves of $\mathbb{Z}/l^n$-modules ($n = 1, 2, \ldots$), as defined by Ekedahl in [Ek]; we have as well the category

$$\mathbf{D}^+ \varprojlim \mathrm{Sh}^{\text{ét}}(\mathbb{Z}_l)/S$$

constructed in a similar manner from inverse systems of complexes on the big étale site over S. Fix a prime l different from the residue characteristics of the base scheme S. We have the objects $\mathbb{Z}_{l,X}(q)$ defined as the inverse system of étale sheaves

$$\mathbb{Z}/l(q) \leftarrow \mathbb{Z}/l^2(q) \leftarrow \ldots \leftarrow \mathbb{Z}/l^\nu(q) \leftarrow \ldots$$

on X; sending X to $\mathbb{Z}_{l,X}(q)$; this gives us the objects $\mathbb{Z}_l(q)$ of $\mathbf{D}^+ \lim_{\leftarrow} \mathrm{Sh}^{\text{ét}}(\mathbb{Z}_l)/S$ with value $\mathbb{Z}_{l,X}(q)$ on X. The continuous l-adic cohomology is then defined as the cohomology theory associated to the graded object

$$\mathbb{Z}_{l,\text{ét}}(*) := \oplus_{q=0}^\infty \mathbb{Z}_l(q)$$

of $\mathbf{D}^+\lim\limits_{\leftarrow}\mathrm{Sh}^{\text{ét}}(\mathbb{Z}_l)/S$. Although (8.3.1) does not directly apply to this situation, the same proof, replacing the category of sheaves over S with the category of inverse systems of complexes of sheaves over S, yields the analogous result. The work of Jannsen [J] verifies that $\mathbb{Z}_{l,\text{ét}}(*)$ satisfies the axioms of a geometric cohomology theory, except that of the unit 1 having a finite projective resolution. The construction of the realization functor goes through without change until the middle of Step 3, where we use the finite projective resolution of 1 to give the isomorphism (8.3.5.10). We alter the construction at this point as follows: the unit 1 is the limit object $\mathbb{Z}_{l,S}$. Suppose we have an inverse system of complexes of acyclic étale sheaves $\mathbb{Z}/l^n$-modules on S

$$\ldots \to C^*_{n+1} \to C^*_n \to\to C^*_{n-1} \ldots \tag{8.4.3.1}$$

such that each map on global sections

$$\Gamma(S, C^m_n) \to \Gamma(S, C^m_{n-1})$$

is surjective. Suppose in addition that the C^*_n are all bounded below. Then the analog of (8.3.5.10) holds, i.e., we have the isomorphism

$$\mathrm{Hom}_{\mathbf{K}^+}(\mathbb{Z}_{l,S}, C^*_*) \cong \mathrm{Hom}_{\mathbf{D}^+}(\mathbb{Z}_{l,S}, C^*_*) \tag{8.4.3.2}$$

where C^*_* denotes the limit object determined by the sequence (8.4.3.1), $\mathbf{K}^+$ is the homotopy category of inverse systems of complexes of sheaves on S, and $\mathbf{D}^+$ is the category $\mathbf{D}^+\lim_{\leftarrow}\mathrm{Sh}^{\text{ét}}_S$. It is easy to verify that the complexes of sheaves $\check{\mathcal{F}}^{(q)}_X(q)_f$ (8.2.5), considered as a sequence (8.4.3.1), have the necessary surjectivity property. We then use the isomorphism (8.4.3.2) instead of (8.3.5.10), and the construction goes through without further change. This gives us the l-adic realization functor

$$Re_{\text{ét},l}\colon \mathcal{DM}(\mathcal{V}) \to \mathbf{D}^+\lim\limits_{\leftarrow}\mathrm{Sh}^{\text{ét}}_S(\mathbb{Z}_l).$$

(8.4.4) The mod-n realization

To form the mod-n realization, first take the product of the l-adic realization functors for all l dividing n. If Γ is an object of $\mathbf{C}^b_{mot}(\mathcal{V})$, define $\Gamma \otimes \mathbb{Z}/n$ to be the object

$$\mathrm{Cone}(\Gamma \xrightarrow{\times n} \Gamma).$$

Let $\mathcal{DM}(\mathcal{V}; \mathbb{Z}/n)$ be the full subcategory of $\mathcal{DM}(\mathcal{V})$ generated by the objects $\Gamma \otimes \mathbb{Z}/n$. Restricting the product of the l-adic realizations to $\mathcal{DM}(\mathcal{V}; \mathbb{Z}/n)$ defines the mod n-realization

$$Re_{\text{ét},\mathbb{Z}/n}\colon \mathcal{DM}(\mathcal{V}; \mathbb{Z}/n) \to \mathbf{D}^+\lim\limits_{\leftarrow}\mathrm{Sh}^{\text{ét}}_S(\mathbb{Z}_l);$$

using the quasi-isomorphism

$$\mu_n^{\otimes q} \to \mathrm{Cone}(\prod_{l|n} \mathbb{Z}_l(q) \xrightarrow{\times n} \prod_{l|n} \mathbb{Z}_l(q)),$$

we may form an equivalent realization in the usual category of étale sheaves

$$Re_{\text{ét},\mathbb{Z}/n}\colon \mathcal{DM}(\mathcal{V};\mathbb{Z}/n) \to \mathbf{D}^+(\mathrm{Sh}_S^{\text{ét}}(\mathbb{Z}/n));$$

(8.4.5) THE $\mathbb{Q}_l$ REALIZATION

We tensor the l-adic realization with $\mathbb{Q}$, and use the argument of (II, (2.5.4)) to give the $\mathbb{Q}_l$ realization

$$Re_{\text{ét},l}\colon \mathcal{DM}(\mathcal{V})_{\mathbb{Q}} \to \mathbf{D}^+ \lim_{\leftarrow} \mathrm{Sh}_S^{\text{ét}}(\mathbb{Z}_l) \otimes \mathbb{Q}.$$

8.5. The absolute Hodge realization

We construct the Hodge realization via a modification of Beilinson's category of Hodge complexes.

(8.5.1) ABSOLUTE HODGE COMPLEXES

Let $\mathcal{H}_R$ denote the category of R-mixed Hodge structures. We begin by recalling Beilinson's construction of the category of absolute Hodge complexes. This consists of a subcategory of the category of diagrams

$$\mathcal{F} = \begin{array}{ccccccc} & \mathcal{F}'_{\mathbb{Q}} & & & (\mathcal{F}'_{\mathbb{C}}, W'_{\mathbb{C}\bullet}) & & \\ \nearrow & & \nwarrow & \nearrow & & \nwarrow & \\ \mathcal{F}_R & & & (\mathcal{F}_{\mathbb{Q}}, W_{\mathbb{Q}\bullet}) & & & (\mathcal{F}_{\mathbb{C}}, W_{\mathbb{C}\bullet}, F^\bullet) \end{array}$$

Here, $\mathcal{F}_R$, $\mathcal{F}'_{\mathbb{Q}}$, $\mathcal{F}_{\mathbb{Q}}$, $\mathcal{F}_{\mathbb{C}}$ and $\mathcal{F}'_{\mathbb{C}}$ are complexes of R-modules, $R\otimes\mathbb{Q}$-modules, $R\otimes\mathbb{Q}$-modules, $\mathbb{C}$-vector spaces and $\mathbb{C}$-vector spaces, resp., $W_\bullet$, $W'_\bullet$ denotes an increasing filtration, and $F^\bullet$ denotes a decreasing filtration. The arrows in the diagram denote the following maps:

The arrow $\mathcal{F}_R \to \mathcal{F}'_{\mathbb{Q}}$ is a quasi-isomorphism $\mathcal{F}_R \otimes \mathbb{Q} \to \mathcal{F}'_{\mathbb{Q}}$,

The arrow $(\mathcal{F}_{\mathbb{Q}}, W_{\mathbb{Q}\bullet}) \to \mathcal{F}'_{\mathbb{Q}}$ is a quasi-isomorphism $\mathcal{F}_{\mathbb{Q}} \to \mathcal{F}'_{\mathbb{Q}}$,

The arrow $(\mathcal{F}_{\mathbb{Q}}, W_{\mathbb{Q}\bullet}) \to (\mathcal{F}'_{\mathbb{C}}, W'_{\mathbb{C}\bullet})$ is a filtered quasi-isomorphism

$$(\mathcal{F}_{\mathbb{Q}} \otimes \mathbb{C}, W_{\mathbb{Q}\bullet} \otimes \mathbb{C}) \to (\mathcal{F}'_{\mathbb{C}}, W'_{\mathbb{C}\bullet}),$$

The arrow $(\mathcal{F}_{\mathbb{C}}, W_{\mathbb{C}\bullet}, F^\bullet) \to (\mathcal{F}'_{\mathbb{C}}, W'_{\mathbb{C}\bullet})$ is a filtered quasi-isomorphism

$$(\mathcal{F}_{\mathbb{C}}, W_{\mathbb{C}\bullet}) \to (\mathcal{F}'_{\mathbb{C}}, W'_{\mathbb{C}\bullet}).$$

The category $C^*_{\mathcal{H}}$ ($* =$ a boundedness condition) of mixed Hodge complexes are those diagrams as above for which the following conditions are satisfied (see [AHC] Definition 3.2):

(i) $H^\bullet(\mathcal{F}_R)$ are finitely generated R-modules.

(ii) For a in $\mathbb{Z}$, consider the filtered complex $(Gr^{W_\mathbb{C}}_a \mathcal{F}_\mathbb{C}, Gr^{W_\mathbb{C}}_a F^\bullet)$. The differential of this complex is strictly compatible with the filtration.

(iii) This filtration, together with the isomorphism

$$H^\bullet(Gr^{W_\mathbb{Q}}_a \mathcal{F}_\mathbb{Q}) \otimes \mathbb{C} \to H^\bullet(Gr^{W_\mathbb{C}}_a \mathcal{F}_\mathbb{C})$$

that comes from the diagram, defines on $H^\bullet(Gr^{W_\mathbb{Q}}_a \mathcal{F}_\mathbb{Q})$ a pure $R \otimes \mathbb{Q}$-Hodge structure of weight a. □

In particular, taking the cohomology of all the above complexes defines a mixed Hodge structure on $H^\bullet(\mathcal{F}_R)$; let $\underline{H}^\bullet(\mathcal{F})$ denote the resulting mixed Hodge structure.

The category $C^*_{\mathcal{H}}$ is closed under taking cones, and thus the homotopy category $K^*_{\mathcal{H}}$ is a triangulated category; the functor $\underline{H}^\bullet(-)$ defines a cohomological functor from $K^*_{\mathcal{H}}$ to the derived category $D^*(\mathcal{H})$. Localizing $K^*_{\mathcal{H}}$ with respect to $\underline{H}^\bullet(-)$ gives the category $D^*_{\mathcal{H}}$; Beilinson shows ([AHC], Theorem 3.4) that the resulting functor

$$D^b_{\mathcal{H}} \to D^b(\mathcal{H})$$

is an equivalence of categories.

(8.5.2) Multiplication

For a diagram $\mathcal{F}$ in $C^*_{\mathcal{H}}$, we have the resulting diagram of R-modules

$$\mathcal{D}_{\mathcal{H}}(\mathcal{F}) := \begin{array}{ccccccc} & \mathcal{F}'_\mathbb{Q} & & & W'_{\mathbb{C}0}(\mathcal{F}'_\mathbb{C}) & & \\ \nearrow & & \nwarrow & \nearrow & & \nwarrow & \\ \mathcal{F}_R & & W_{\mathbb{Q}0}(\mathcal{F}_\mathbb{Q}) & & & & W_{\mathbb{C}0} \cap F^0(\mathcal{F}_\mathbb{C}). \end{array}$$

For an arbitrary diagram

$$\mathcal{D} \;=\; \begin{array}{ccccccccc} & B_1 & & & B_2 & & & \dots \\ f_1 \nearrow & & \nwarrow g_1 \quad f_2 \nearrow & & & \nwarrow g_2 \quad f_3 \nearrow & & \\ A_1 & & & A_2 & & & A_3 & \end{array} \qquad (8.5.2.1)$$

let

$$\tilde{\Gamma}^0(\mathcal{D}) = \oplus_i A_i, \quad \tilde{\Gamma}^1(\mathcal{D}) = \oplus_i B_i, \quad \phi_1 = \Sigma_i f_i, \quad \phi_2 = \Sigma_i g_i,$$

and set

$$\tilde{\Gamma}(\mathcal{D}) = \mathrm{Cone}(\phi_1 - \phi_2 \colon \tilde{\Gamma}^0(\mathcal{D}) \to \tilde{\Gamma}^1(\mathcal{D}))[-1].$$

Set

$$\tilde{\Gamma}_{\mathcal{H}}(\mathcal{F}) := \tilde{\Gamma}(\mathcal{D}_{\mathcal{H}}(\mathcal{F}))$$

as a functor from $C^*_{\mathcal{H}}$ to $\mathbf{D}^*(\mathbf{Mod}_R)$; we may also consider $\tilde{\Gamma}_{\mathcal{H}}$ as a functor from $\mathcal{H}$ to $\mathbf{D}^*(\mathbf{Mod}_R)$ via the embedding of $\mathcal{H}$ in $C^*_{\mathcal{H}}$ as complexes concentrated in degree zero. We have as well the functor

$$\Gamma_{\mathcal{H}}: C^*_{\mathcal{H}} \to \mathbf{D}^*(\mathbf{Mod}_R) \tag{8.5.2.2}$$

defined as

$$\Gamma_{\mathcal{H}}(\mathcal{F}) = \mathrm{Hom}_{C^*_{\mathcal{H}}}(R(0), \mathcal{F}) = \ker(\phi_1 - \phi_2: \tilde{\Gamma}_0 \to \tilde{\Gamma}_1).$$

The natural map $\Gamma_{\mathcal{H}} \to \tilde{\Gamma}_{\mathcal{H}}$ defines the natural transformation of functors from $D^*_{\mathcal{H}}$ to $\mathbf{D}^*(\mathbf{Mod}_R)$, $R\Gamma_{\mathcal{H}} \to \tilde{\Gamma}_{\mathcal{H}}$, which is, by ([AHC], Lemma 3.6), an isomorphism.

Beilinson describes multiplication in diagrams of the form (8.5.2.1) as follows: Let $\mathcal{D}$, $\mathcal{D}'$ be diagrams of this form. The tensor product $\mathcal{D} \otimes \mathcal{D}'$ is the diagram

$$\mathcal{D} \otimes \mathcal{D}' \;=$$

$$\begin{array}{ccccccc} & & B_1 \otimes B_1' & & & & B_2 \otimes B_2' \ldots \\ & f_1 \otimes f_1' \nearrow & & \nwarrow g_1 \otimes g_1' & & f_2 \otimes f_2' \nearrow & & \nwarrow g_2 \otimes g_2' \ldots \\ A_1 \otimes A_1' & & & & A_2 \otimes A_2' & & & & A_3 \otimes A_3' \quad \ldots \end{array}$$

For $\alpha \in R$, Beilinson defines the map

$$*_\alpha: \tilde{\Gamma}(\mathcal{D}) \otimes \tilde{\Gamma}(\mathcal{D}') \to \tilde{\Gamma}(\mathcal{D} \otimes \mathcal{D}')$$

by the formulas

$$\begin{aligned} a *_\alpha a' &= a \otimes a'; \quad ; a *_\alpha b' = (-1)^{\deg a}(\alpha\phi_1(a) + (1-\alpha)\phi_2(a)) \otimes b' \\ b *_\alpha b' &= 0; \quad b *_\alpha a' = b \otimes ((1-\alpha)\phi_1(a') + \alpha\phi_2(a')). \end{aligned}$$

By ([AHC], Lemma 1.11), all the multiplications $*_\alpha$ are homotopic, the multiplications $*_0$ and $*_1$ are associative, and under the canonical isomorphism $\mathcal{D} \otimes \mathcal{D}' \to \mathcal{D}' \otimes \mathcal{D}$, the multiplication $*_\alpha$ transforms to $*_{1-\alpha}$. Thus, we have an associative, graded-commutative multiplication

$$*: \tilde{\Gamma}(\mathcal{D}) \otimes \tilde{\Gamma}(\mathcal{D}') \to \tilde{\Gamma}(\mathcal{D} \otimes \mathcal{D}')$$

in the homotopy category $K^*(\mathbf{Mod}_R)$. Given a map $\mathcal{F} \otimes \mathcal{F}' \to \mathcal{F}''$, the above multiplication gives rise to the multiplication

$$*: \tilde{\Gamma}_{\mathcal{H}}(\mathcal{F}) \otimes \tilde{\Gamma}_{\mathcal{H}}(\mathcal{F}') \to \tilde{\Gamma}_{\mathcal{H}}(\mathcal{F}'').$$

(8.5.3) ENLARGED DIAGRAMS

We consider a slight modification of the above constructions. Form the category $C^*_{\mathcal{H}'}$ as the category of diagrams

$$\mathcal{F} = \begin{array}{ccccccc} & \mathcal{F}'_{\mathbb{Q}} & & (\mathcal{F}'_{\mathbb{C}}, W'_{\mathbb{C}\bullet}) & & (\mathcal{F}''_{\mathbb{C}}, W''_{\mathbb{C}\bullet}) & \\ \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \\ \mathcal{F}_R & & (\mathcal{F}_{\mathbb{Q}}, W_{\mathbb{Q}\bullet}) & & (\mathcal{F}_{\mathbb{C}}, W_{\mathbb{C}\bullet}) & & (\mathcal{F}_{\mathbb{C}}, W_{\mathbb{C}\bullet}, F^\bullet) \end{array} \tag{8.5.3.1}$$

where the arrows are as above quasi-isomorphisms of the appropriate objects in the appropriate category. We map $C^*_{\mathcal{H}'}$ to $C^*_{\mathcal{H}}$ by replacing the portion

$$\begin{array}{ccccc} (\mathcal{F}'_{\mathbb{C}}, W'_{\mathbb{C}\bullet}) & & & & (\mathcal{F}''_{\mathbb{C}}, W''_{\mathbb{C}\bullet}) \\ & f \nwarrow & & \nearrow g & \\ & & (\mathcal{F}_{\mathbb{C}}, W_{\mathbb{C}\bullet}) & & \end{array}$$

of the diagram (8.5.3.1) with

$$\overline{(\mathcal{F}'_{\mathbb{C}}, W'_{\mathbb{C}\bullet})} := \mathrm{Cone}((f,g)\colon \mathcal{F}_{\mathbb{C}} \to \mathcal{F}'_{\mathbb{C}} \oplus \mathcal{F}''_{\mathbb{C}}$$

forming the diagram

$$\overline{\mathcal{F}} := \quad \begin{array}{ccccccc} & & \mathcal{F}'_{\mathbb{Q}} & & & \overline{(\mathcal{F}'_{\mathbb{C}}, W'_{\mathbb{C}\bullet})} & \\ & \nearrow & & \nwarrow & \nearrow & & \nwarrow \\ \mathcal{F}_R & & & (\mathcal{F}_{\mathbb{Q}}, W_{\mathbb{Q}\bullet}) & & & (\mathcal{F}_{\mathbb{C}}, W_{\mathbb{C}\bullet}, F^\bullet) \end{array}$$

We map $C^*_{\mathcal{H}}$ to $C^*_{\mathcal{H}'}$ by adding two identity maps, forming the diagram

$$\begin{array}{ccccccccc} & & \mathcal{F}'_{\mathbb{Q}} & & (\mathcal{F}'_{\mathbb{C}}, W'_{\mathbb{C}\bullet}) & & & (\mathcal{F}'_{\mathbb{C}}, W'_{\mathbb{C}\bullet}) & \\ & \nearrow & & \nwarrow \ \ \nearrow & & \nwarrow \mathrm{id} & \mathrm{id} \nearrow & & \nwarrow \\ \mathcal{F}_R & & (\mathcal{F}_{\mathbb{Q}}, W_{\mathbb{Q}\bullet}) & & & (\mathcal{F}'_{\mathbb{C}}, W'_{\mathbb{C}\bullet}) & & & (\mathcal{F}_{\mathbb{C}}, W_{\mathbb{C}\bullet}, F^\bullet) \end{array}$$

These maps give an equivalence of the homotopy categories $K^*_{\mathcal{H}} \to K^*_{\mathcal{H}'}$; this equivalence respects the functor $\tilde{\Gamma}$ and the multiplication $*$.

(8.5.4) CANONICAL ČECH COMPLEXES

Let $\mathcal{F}$ be a sheaf of A-modules on a complex manifold X, let U be an open subset of X (for the classical topology), and let $\mathcal{U}$ be an open cover of U. Let $\check{\mathcal{F}}(\mathcal{U})$ denote the complex of the Čech co-chains of $\mathcal{F}$ for the cover $\mathcal{U}$. We let $\check{\mathcal{F}}(U)$ denote the limit of the complexes $\check{\mathcal{F}}(\mathcal{U})$, as $\mathcal{U}$ runs over rigid open covers of U.

Let $j\colon V \to U$ be an open subset of U for the classical topology. If $\mathcal{U}$ is a rigid open cover of U

$$\mathcal{U} = \prod_{x \in U} x \in U_x \subset U$$

we have the rigid pull-back to V:

$$j^{*_{\mathrm{r}}}\mathcal{U} := \prod_{x \in V} x \in U_x \cap V \subset V$$

Including $U_x \cap V$ into U_x defines the canonical map of open covers over the map j

$$\tilde{j}\colon j^{*_{\mathrm{r}}}\mathcal{U} \to \mathcal{U}.$$

The Čech complex $\check{\mathcal{F}}(\mathcal{U})$ has degree n term given by

$$\check{\mathcal{F}}(\mathcal{U})^n = \prod_{(x_0,\ldots,x_n)\in X^n} \mathcal{F}(U_{x_0}\cap\ldots\cap U_{x_n}).$$

Projecting on the set of n-tuples in V, and taking the restriction defines the map

$$j_{\mathcal{U}}^{*,n}: \check{\mathcal{F}}(\mathcal{U})^n \to \check{\mathcal{F}}(j^{*\mathfrak{r}}\mathcal{U})^n;$$

giving the map of complexes

$$j_{\mathcal{U}}^{*}: \check{\mathcal{F}}(\mathcal{U}) \to \check{\mathcal{F}}(j^{*\mathfrak{r}}\mathcal{U}).$$

Passing to the limit over rigid covers $\mathcal{U}$ gives the map of A-modules

$$j^{*,n}: \check{\mathcal{F}}^n(U) \to \check{\mathcal{F}}^n(V).$$

and the map of complexes

$$j^{*}: \check{\mathcal{F}}(U) \to \check{\mathcal{F}}(V).$$

One easily verifies that this makes $\check{\mathcal{F}}^n$ into a presheaf of A-modules on X, and $\check{\mathcal{F}}$ a complex of presheaves; it is also easy to see from the direct description of $\check{\mathcal{F}}^n$ given above that $\check{\mathcal{F}}^n$ is a flasque sheaf on X.

As is well known, the Čech cohomology of sheaf on X is canonically isomorphic to the sheaf cohomology, hence the augmented complex of sheaves

$$\mathcal{F} \to \check{\mathcal{F}}$$

is a flasque resolution of $\mathcal{F}$. In particular, the image of the complex $\check{\mathcal{F}}$ in the derived category is canonically isomorphic to $R\Gamma(X,\mathcal{F})$.

(8.5.5) Alternating complexes

For an open cover $\mathcal{U}$ of X, the symmetric group S_{n+1} acts on the degree n component of $\mathcal{F}(\mathcal{U})$ by

$$f^{\sigma}_{\alpha_0,\ldots,\alpha_n} = f_{\alpha_{\sigma 0},\ldots,\alpha_{\sigma n}};$$

we let $\mathrm{Alt}\check{\mathcal{F}}(\mathcal{U})$ denote the sub-complex consisting of alternating co-chains (one checks that $\mathrm{Alt}\check{\mathcal{F}}(\mathcal{U})$ is indeed a sub-complex, not merely a sub-module, of $\check{\mathcal{F}}(\mathcal{U})$). Similarly, if X is a smooth $\mathbb{C}$-scheme, and $\mathcal{F}$ a sheaf on X_{an} we let $\mathrm{Alt}\check{\mathcal{F}}$ denote the sub-complex of $\check{\mathcal{F}}$ consisting of alternating co-chains. We extend these notions to complexes of sheaves $F^\bullet$ by taking the total complex of the double complex $\mathrm{Alt}\check{\mathcal{F}}^\bullet$.

If $\mathcal{F}$ is a complex of sheaves on X, we have the canonical augmentation

$$i: \Gamma(X,F) \to \mathrm{Alt}\check{\mathcal{F}}(X),$$

inducing the map (in the derived category $\mathbf{D}^+(\mathbf{Mod}_A)$)

$$Ri\colon \mathrm{Alt}\check{\mathcal{F}}(X) \to R\Gamma(X, \mathcal{F}).$$

The construction of $\check{\mathcal{F}}$ and $\mathrm{Alt}\check{\mathcal{F}}$ are functorial in the complex $\mathcal{F}$, hence extend to the analogous constructions for a complex $\mathcal{F}$ on the analytic site $\mathcal{C}_{an}$, for a category of complex manifolds $\mathcal{C}$. In particular, we have the functor $\mathrm{Alt}\check{\mathcal{F}}$ from $\mathcal{C}^{\mathrm{op}}$ to $\mathbf{C}^+(\mathbf{Mod}_A)$, and the natural transformation of functors from $\mathcal{C}^{\mathrm{op}}$ to $\mathbf{D}^+(\mathbf{Mod}_A)$

$$Ri\colon \mathrm{Alt}\check{\mathcal{F}} \to R\Gamma(-, \mathcal{F}).$$

(8.5.6) Lemma

Let A be a commutative $\mathbb{Q}$-algebra. Let F be a complex of sheaves of flat A-modules on the site $\mathcal{C}_{an}$ associated to the classical topology on a category $\mathcal{C}$ of complex manifolds. Then the natural map

$$Ri\colon \mathrm{Alt}\check{\mathcal{F}} \to R\Gamma(-, \mathcal{F})$$

is an isomorphism. Suppose $\mathcal{C}$ is closed under products, and that we have an associative, graded commutative exterior product

$$\boxtimes\colon F \otimes F \to F.$$

Then there is an associative, graded commutative product

$$\mathrm{Alt}\check{\boxtimes}\colon \mathrm{Alt}\check{F} \otimes \mathrm{Alt}\check{F} \to \mathrm{Alt}\check{F},$$

compatible, via Ri, with $\boxtimes$.

Proof. Since $A \supset \mathbb{Q}$, we have the alternating projection

$$\mathrm{Alt}\colon \check{F} \to \mathrm{Alt}\check{F},$$

splitting the inclusion, so the cohomology of $\mathrm{Alt}\check{F}$ is a summand of that of $\check{F}$; in particular, $H^p(\mathrm{Alt}\check{F}) = 0$ for F a flasque sheaf on X and $p > 0$. As $H^0(\check{F}) = H^0(\mathrm{Alt}\check{F})$, using (8.5.4), this implies that $\mathrm{Alt}\check{F} \to R\Gamma(X, F)$ is an isomorphism, proving the first assertion.

For the assertion on products, define $\mathrm{Alt}\check{\boxtimes}$ to be the restriction of $\mathrm{Alt}\circ\check{\boxtimes}$ to $\mathrm{Alt}\check{F}\otimes\mathrm{Alt}\check{F}$. A direct computation shows that $\mathrm{Alt}\check{\boxtimes}$ is associative and graded-commutative, completing the proof. □

(8.5.7) Compactifications

Let X be a smooth quasi-projective $\mathbb{C}$-scheme; a *compactification* of X is a birational inclusion $j\colon X \to \overline{X}$ of X as an open subscheme of a smooth projective $\mathbb{C}$-scheme $\overline{X}$, such that the complement $D := \overline{X} \backslash X$ is a normal crossing divisor. Form the category $\mathcal{D}^*(X, \overline{X})$ ($* = b,+,-$ or $\emptyset$ is a boundedness condition) of diagrams

$$\begin{array}{ccccccc} & \mathcal{F}'_{\mathbb{Q}} & & (\mathcal{F}_{\mathbb{C}}, W_{\mathbb{C}\bullet}) & & (\mathcal{F}''_{\mathbb{C}}, W''_{\mathbb{C}\bullet}) & \\ \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \\ \mathcal{F}_R & & (\mathcal{F}_{\mathbb{Q}}, W_{\mathbb{Q}\bullet}) & & (\mathcal{F}'_{\mathbb{C}}, W'_{\mathbb{C}\bullet}) & & (\mathcal{G}_{\mathbb{C}}, W_{\mathbb{C}\bullet}, F^\bullet) \end{array}$$

Here $\mathcal{F}_R$ is a complex of sheaves of R-modules on X, $\mathcal{F}'_{\mathbb{Q}}$ is a complex of sheaves of $R \otimes \mathbb{Q}$-vector spaces on $\overline{X}$, $(\mathcal{F}_{\mathbb{Q}}, W_{\mathbb{Q}\bullet})$ is a filtered complex of $R \otimes \mathbb{Q}$-vector spaces on $\overline{X}$, $(\mathcal{F}_{\mathbb{C}}, W_{\mathbb{C}\bullet})$, $(\mathcal{F}'_{\mathbb{C}}, W'_{\mathbb{C}\bullet})$ and $(\mathcal{F}''_{\mathbb{C}}, W''_{\mathbb{C}\bullet})$ are filtered complexes of $\mathbb{C}$-vector spaces on $\overline{X}$, and $(\mathcal{G}_{\mathbb{C}}, W_{\mathbb{C}\bullet}, F^\bullet)$ is a bi-filtered complex of $\mathbb{C}$-vector spaces on $\overline{X}$. The arrows in the diagram are as follows:

The arrow $\mathcal{F}_R \to \mathcal{F}'_{\mathbb{Q}}$ is a quasi-isomorphism $\mathcal{F}_R \otimes \mathbb{Q} \to j^*(\mathcal{F}'_{\mathbb{Q}})$.

The arrow $(\mathcal{F}_{\mathbb{Q}}, W_{\mathbb{Q}\bullet}) \to \mathcal{F}'_{\mathbb{Q}}$ is a quasi-isomorphism $\mathcal{F}_{\mathbb{Q}} \to \mathcal{F}'_{\mathbb{Q}}$

The arrow $(\mathcal{F}_{\mathbb{Q}}, W_{\mathbb{Q}\bullet}) \to (\mathcal{F}_{\mathbb{C}}, W_{\mathbb{C}\bullet})$ is a filtered quasi-isomorphism

$$(\mathcal{F}_{\mathbb{Q}} \otimes \mathbb{C}, W_{\mathbb{Q}\bullet} \otimes \mathbb{C}) \to (\mathcal{F}_{\mathbb{C}}, W_{\mathbb{C}\bullet}).$$

The arrows $(\mathcal{F}'_{\mathbb{C}}, W'_{\mathbb{C}\bullet}) \to (\mathcal{F}_{\mathbb{C}}, W_{\mathbb{C}\bullet})$ and $(\mathcal{F}'_{\mathbb{C}}, W'_{\mathbb{C}\bullet}) \to (\mathcal{F}''_{\mathbb{C}}, W''_{\mathbb{C}\bullet})$ are filtered quasi-isomorphisms.

The arrow $(\mathcal{G}_{\mathbb{C}}, W_{\mathbb{C}\bullet}, F^\bullet) \to (\mathcal{F}''_{\mathbb{C}}, W''_{\mathbb{C}\bullet})$ is a filtered quasi-isomorphism

$$(\mathcal{G}_{\mathbb{C}}, W_{\mathbb{C}\bullet}) \to (\mathcal{F}''_{\mathbb{C}}, W''_{\mathbb{C}\bullet}).$$

Maps in $\mathcal{D}^*(X, \overline{X})$ are component-wise; $\mathcal{D}^*(X, \overline{X})$ forms a DG tensor category, with the tensor product given component-wise, and the Cone functors on each component defines the Cone functor for $\mathcal{D}^*(X, \overline{X})$. We have the full DG tensor subcategory $\mathcal{D}^*_{\mathcal{H}}(X, \overline{X})$ consisting of diagrams for which the diagram

$$\begin{array}{ccccccc} & R\Gamma(\overline{X}, \mathcal{F}'_{\mathbb{Q}}) & & R\Gamma(\overline{X}, (\mathcal{F}_{\mathbb{C}}, W_{\mathbb{C}\bullet})) & & R\Gamma(\overline{X}, (\mathcal{F}''_{\mathbb{C}}, W''_{\mathbb{C}\bullet})) & \\ \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \\ R\Gamma(X, \mathcal{F}_R) & & R\Gamma(\overline{X}, (\mathcal{F}_{\mathbb{Q}}, W_{\mathbb{Q}\bullet})) & & R\Gamma(\overline{X}, (\mathcal{F}'_{\mathbb{C}}, W'_{\mathbb{C}\bullet})) & & R\Gamma(\overline{X}, (\mathcal{G}_{\mathbb{C}}, W_{\mathbb{C}\bullet}, F^\bullet)) \end{array}$$

is in $D^*_{\mathcal{H}'}$.

(8.5.8) THE MULTIPLICATIVE DIAGRAM ASSOCIATED TO A COMPACTIFICATION

Let $j\colon X \to \overline{X}$ be a compactification. For a complex of sheaves $\mathcal{F}^\bullet$ on $\overline{X}$, we let $(\mathcal{F}^\bullet, \tau_\leq)$ denote the *canonical filtration*, i.e.

$$(\tau_{\leq p}\mathcal{F})^q = \begin{cases} \mathcal{F}^q; & \text{if } q < p \\ \ker(d^q\colon \mathcal{F}^q \to \mathcal{F}^{q+1}); & \text{if } q = p \\ 0; & \text{if } q > p. \end{cases}$$

Let $\Omega^\bullet_X$ denote the analytic DeRham complex, and let $\Omega^\bullet_{\overline{X}}(\log D)$ be the complex of forms with log poles. Let $(\Omega^\bullet_{\overline{X}}(\log D), W_\bullet)$ denote the filtration by order of pole, and $(\Omega^\bullet_{\overline{X}}(\log D), F^\bullet)$ the stupid filtration. We have the following maps

$$\begin{aligned}
\mathbb{Z}_X \otimes \mathbb{Q} &\to j^*(j_*(\mathrm{Alt}\check{\mathbb{Z}}_X \otimes \mathbb{Q}))\\
(j_*(\mathrm{Alt}\check{\mathbb{Z}}_X \otimes \mathbb{Q}) \otimes \mathbb{C}, \tau_\leq \otimes \mathbb{C}) &\to (j_*(\mathrm{Alt}\check{\Omega}^\bullet_X), \tau_\leq)\\
(\Omega^\bullet_{\overline{X}}(\log D), \tau_\leq) &\to (j_*(\mathrm{Alt}\check{\Omega}^\bullet_X), \tau_\leq)\\
(\Omega^\bullet_{\overline{X}}(\log D), \tau_\leq) &\to (\Omega^\bullet_{\overline{X}}(\log D), W_\bullet).
\end{aligned}$$

The first is the quasi-isomorphism of complexes of sheaves on X, $\mathbb{Z}_X \otimes \mathbb{Q} \to \mathrm{Alt}\check{\mathbb{Z}}_X \otimes \mathbb{Q}$, the second is the filtered quasi-isomorphism of complexes of sheaves on $\overline{X}$ induced by the quasi-isomorphism of complexes of sheaves on X: $\mathbb{C}_X \to \Omega_X$, and the isomorphism $\mathbb{Z}_X \otimes \mathbb{Q} \otimes \mathbb{C} \to \mathbb{C}_X$. The third is the filtered quasi-isomorphism of sheaves on $\overline{X}$ induced by the quasi-isomorphism $\Omega^\bullet_{\overline{X}}(\log D) \to j_*(\Omega^\bullet_X)$ and the last line is the filtered quasi-isomorphism induced by the identity map on $\Omega^\bullet_{\overline{X}}(\log D)$ (the canonical filtration is finer than the weight filtration). This gives us the diagram

$$D[X, \overline{X}] :=$$

$$\begin{array}{ccccccc}
 & j_*(\mathrm{Alt}\check{\mathbb{Z}}_X \otimes \mathbb{Q}) & & (j_*(\mathrm{Alt}\check{\Omega}^\bullet_X), \tau_\leq) & & (\Omega^\bullet_{\overline{X}}(\log D), W_\bullet) & \\
\nearrow & & \nwarrow \quad \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \\
\mathbb{Z}_X & & (j_*(\mathrm{Alt}\check{\mathbb{Z}}_X \otimes \mathbb{Q}), \tau_\leq) & & (\Omega^\bullet_{\overline{X}}(\log D), \tau_\leq) & & (\Omega^\bullet_{\overline{X}}(\log D), W_\bullet, F^\bullet)
\end{array}$$

in $\mathcal{D}^+(X, \overline{X})$, Beilinson shows that this diagram is in fact in $\mathcal{D}^+_{\mathcal{H}'}(X, \overline{X})$.

(8.5.9) THE TATE OBJECT

For an abelian group A, let $W(q)_{A\bullet}$ be the filtration on A given by

$$W(q)_{A,p} = \begin{cases} A; & \text{if } p \geq q \\ 0; & \text{if } p < q. \end{cases}$$

We let $F(q)^\bullet$ be the filtration on $\mathbb{C}$ given by

$$F(q)^p = \begin{cases} \mathbb{C}; & \text{if } p \leq q \\ 0; & \text{if } p > q. \end{cases}$$

We have the "R-Tate mixed Hodge structure"

$$R(q) :=$$

$$\begin{array}{ccccccc}
 & (2\pi i)^q R \otimes \mathbb{Q} & & (\mathbb{C}, W(q)_{\mathbb{C}\bullet}) & & (\mathbb{C}, W(q)_{\mathbb{C}\bullet}) & \\
\nearrow & & \nwarrow \quad \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \\
(2\pi i)^q R & & ((2\pi i)^q R \otimes \mathbb{Q}, W(q)_{(2\pi i)^q R\otimes\mathbb{Q}\bullet}) & & (\mathbb{C}, W(q)_{\mathbb{C}\bullet}) & & (\mathbb{C}, W(q)_{\mathbb{C}\bullet}, F(q)^\bullet)
\end{array}$$

The object $R(0)$ is the unit for the tensor structure on $C^{+}_{\mathcal{H}'}$. Define $R^{\mathrm{Hdg}}_{X}(q)$ to be the object $R(q) \otimes Rp_{X*}D[X, \bar{X}]$ of $D^{+}_{\mathcal{H}'}$.

(8.5.10) Theorem

Sending (X, q) to $R^{\mathrm{Hdg}}_{X}(q)$ extends to a functor

$$Re_{\mathrm{Hdg}} \colon \mathcal{DM}(\mathcal{V}) \to D^{+}_{\mathcal{H}'}.$$

Proof. The proof is essentially the same as the proof of (8.3.1) with $\mathcal{H}$ playing the role of the abelian tensor category $\mathcal{A}$; there are three main differences:

i) we replace the categories $\mathbf{C}^{+}(\mathcal{H})$, $\mathbf{K}^{+}(\mathcal{H})$ and $\mathbf{D}^{+}(\mathcal{H})$ with the more convenient categories $C^{+}_{\mathcal{H}'}$, $K^{+}_{\mathcal{H}'}$ and $D^{+}_{\mathcal{H}'}$.

ii) after forming the limit of the complexes formed from the diagram $D[X, \overline{X}]$ by using rigid hypercovers with respect to the analytic rigidification, we then form the direct limit with respect to compactifications.

iii) We also need to address the lack of projective resolutions in the category $\mathcal{H}$, as we use a projective resolution of the unit object $1 \in \mathcal{A}$ in the proof of (8.3.1).

We give the construction for $R = \mathbb{Z}$ for simplicity.

For this last point, consider the "enlarged" mixed Hodge complex P_* which in degree 0 is

$$\begin{array}{ccccccccccc}
 & & \mathbb{Q} \oplus \mathbb{Q} & & & (\mathbb{C} \oplus \mathbb{C}, W(0)_{\mathbb{C}^\bullet} \oplus W(0)_{\mathbb{C}^\bullet}) & & & (\mathbb{C} \oplus \mathbb{C}, W(0)_{\mathbb{C}^\bullet} \oplus W(0)_{\mathbb{C}^\bullet}) & & \\
 & \nearrow & & \nwarrow & \nearrow & & \nwarrow & \nearrow & & \nwarrow & \\
\mathbb{Z} & & & & (\mathbb{Q}, W(0)_{\mathbb{Q}^\bullet}) & & (\mathbb{C}, W(0)_{\mathbb{C}^\bullet}) & & & & (\mathbb{C}, W(0)_{\mathbb{C}^\bullet}, F(0)^\bullet)
\end{array}$$

and in degree -1 is

$$\begin{array}{ccccccccccc}
 & & \mathbb{Q} & & & (\mathbb{C}, W(0)_{\mathbb{C}^\bullet}) & & (\mathbb{C}, W(0)_{\mathbb{C}^\bullet}) & & \\
 & \nearrow & & \nwarrow & \nearrow & & \nwarrow \; \nearrow & & \nwarrow & \\
0 & & & & 0 & & 0 & & & 0
\end{array}$$

The maps in the degree 0 portion are the obvious ones, with the left-hand side from the lower row going into the left-hand summand, and the right-hand side from the lower row going into the right-hand summand. The differential is the diagonal map. We map P_0 to $\mathbb{Z}(0)$ by the identity on the lower row and the difference map on the upper row. This determines the map of extended mixed Hodge complexes

$$P_* \to \mathbb{Z}(0) \tag{1}$$

which is an isomorphism in $D^{+}_{\mathcal{H}'}$. It is easy to see that the map (1) gives an isomorphism of functors

$$\mathrm{Hom}_{K^{+}_{\mathcal{H}'}}(P_*^{\otimes n}, -) \to \mathrm{Hom}_{D^{+}_{\mathcal{H}'}}(\mathbb{Z}(0), -)$$

for all $n \geq 0$. Using P_* instead of a projective resolution of $1 \in \mathcal{A}$ and using the categories $C^+_{\mathcal{H}'}$, $K^+_{\mathcal{H}'}$ and $D^+_{\mathcal{H}'}$ instead of $\mathbf{C}^+(\mathcal{H})$, $\mathbf{K}^+(\mathcal{H})$ and $\mathbf{D}^+(\mathcal{H})$, the proof of (8.3.1) goes through word for word. □

We may compose the functor Re_{Hdg} with the derived functor

$$R\mathrm{Hdg}\colon D^+_{\mathcal{H}'} \to \mathbf{D}^+(\mathbf{Mod}_R)$$

of the absolute Hodge cohomology functor $\underline{H}^0$ to give the absolute Hodge realization

$$Re_{\mathrm{AHdg}}\colon \mathcal{DM} \to \mathbf{D}^+(\mathbf{Mod}_R).$$

For $R = \mathbb{Z}$, the resulting cohomology groups $H^p(Re_{\mathrm{AHdg}}(\mathbb{Z}_X(q)))$ are the absolute Hodge cohomology groups $H^p_{\mathrm{AH}}(X, \mathbb{Z}(q))$; when X is projective, these agree with the Deligne cohomology groups $H^p_{\mathcal{D}}(X, \mathbb{Z}(q))$.

8.6. The motivic realization

Let k be a field of characteristic zero, and let $\mathcal{V}_k$ denote the category of smooth, quasi-projective varieties over k. We conclude this section with an extension of the above construction of the Hodge realization to give a realization of $\mathcal{DM}(\mathcal{V}_k)_{\mathbb{Z}}$ into Jannsen's category of mixed absolute Hodge complexes up to quasi-isomorphism. This is constructed similarly to Beilinson's category $D^+_{\mathcal{H}}$; the essential difference is the addition of l-adic data to the Betti-Hodge data encoded in $D^b_{\mathcal{H}}$. We will give here a brief resumé of the construction, somewhat modified as in the previous section; for details see ([J2], §6, especially pages 97-104).

(8.6.1) DEFINITION

Let G_k denote the Galois group of $\overline{k}$ over k. A *polarizable mixed absolute Hodge complex* (MAH-complex) over k is a diagram $\mathcal{D}$ of the following form:

$$\mathcal{D} :=$$

$$\begin{array}{cccccccc}
\prod_l K'_{\mathbb{Q},l} & & \prod_{l,\sigma:k\to\mathbb{C}} (K_{\mathbb{Q},l,\sigma}, W) & & \prod_{\sigma:k\to\mathbb{C}} (K'_{\mathbb{C},\sigma}, W) & & \prod_{\sigma:k\to\mathbb{C}} (K''_{\mathbb{C},\sigma}, W) & \\
f_1 \nearrow & \nwarrow g_1 \quad f_2 \nearrow & & \nwarrow g_2 \quad f_3 \nearrow & & \nwarrow g_3 \quad f_4 \nearrow & & \nwarrow g_4 \\
\prod_l K_l & \prod_l (K_{\mathbb{Q},l}, W) & & \prod_{\sigma:k\to\mathbb{C}} (K_{\mathbb{Q},\sigma}, W) & & \prod_{\sigma:k\to\mathbb{C}} (K_{\mathbb{C},\sigma}, W) & & (K_k, W, F) \\
\mathrm{id} \downarrow & & & \mathrm{id} \downarrow & & & & \\
\prod_l K_l & \prod_{\sigma:k\to\mathbb{C}} K_\sigma & & \prod_{\sigma:k\to\mathbb{C}} (K_{\mathbb{Q},\sigma}, W) & & & & \\
f'_1 \searrow & \swarrow g'_1 \quad f'_2 \searrow & & \swarrow g'_2 & & & & \\
\prod_{l,\sigma:k\to\mathbb{C}} K'_{l,\sigma} & & \prod_{\sigma:k\to\mathbb{C}} K'_{\mathbb{Q},\sigma} & & & & &
\end{array}$$

where $\sigma\colon k \to \mathbb{C}$ denotes an embedding, l is a prime number, and

(i) for each l, K_l, $K_{\mathbb{Q},l}$ and $K'_{\mathbb{Q},l}$ are bounded below complexes of continuous G_k-modules. K_l is a complex of $\mathbb{Z}_l$-modules such that the homology groups are finitely generated $\mathbb{Z}_l$-modules, $K_{\mathbb{Q},l}$ and $K'_{\mathbb{Q},l}$ are complexes of $\mathbb{Q}_l$-vector spaces such that the homology groups are finite dimensional $\mathbb{Q}_l$-vector spaces; W is a decreasing filtration on $K_{\mathbb{Q},l}$.

(ii) for each l and each $\sigma\colon k \to \mathbb{C}$, $(K_{\mathbb{Q},l,\sigma}, W)$ is a bounded below, filtered complex of $\mathbb{Q}_l$-vector spaces with finite dimensional homology, and $K'_{l,\sigma}$ is a bounded below complex of $\mathbb{Z}_l$-modules with homology finitely generated over $\mathbb{Z}_l$.

(iii) for each $\sigma\colon k \to \mathbb{C}$, $K_{\mathbb{Q},\sigma}$, $K'_{\mathbb{Q},\sigma}$ (resp. $K_{\mathbb{C},\sigma}$, $K'_{\mathbb{C},\sigma}$, $K''_{\mathbb{C},\sigma}$) are bounded below, complexes of $\mathbb{Q}$-vector spaces (resp. $\mathbb{C}$-vector spaces) with finite dimensional homology. For each

$\sigma: k \to \mathbb{C}$, $K_{\mathbb{Q},\sigma}$, K_σ is a bounded below complex of abelian groups, with finitely generated homology. W denotes a decreasing filtration on the various complexes.

(iv) K_k is a complex of k-vector spaces, bounded below, with finite dimensional homology, an increasing filtration W, and a decreasing filtration F.

(v) $f_1 = \prod_l f_{1,l}$, $g_1 = \prod_l g_{1,l}$, where $f_{1,l}: K_l \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \to K'_{\mathbb{Q},l}$ and $g_{1,l}: K_{\mathbb{Q},l} \to K'_{\mathbb{Q},l}$ are quasi-isomorphisms.

(vi) $f'_1 = \prod_l f'_{1,l,\sigma}$, $f_2 = \prod_{l,\sigma} f_{2,l,\sigma}$, $g'_1 = \prod_{l,\sigma} g'_{1,l,\sigma}$ where for each $\sigma: k \to \mathbb{C}$, $f_{1,l,\sigma}$ is a family of quasi-isomorphisms

$$f_{1,l,\overline{\sigma}}: K_l \to K_{l,\sigma}$$

indexed by the set of extensions $\overline{\sigma}: \overline{k} \to \mathbb{C}$ of σ, with $f_{1,l,\overline{\sigma}\rho}$ homotopic to $f_{1,l,\overline{\sigma}}$ for each $\rho \in G_k$, $f_{2,l,\sigma}$ is a family of filtered quasi-isomorphisms

$$f_{2,l,\overline{\sigma}}: (K_{\mathbb{Q},l}, W) \to (K_{\mathbb{Q},l,\sigma}, W)$$

indexed by the set of extensions $\overline{\sigma}: \overline{k} \to \mathbb{C}$ of σ, with $f_{2,l,\overline{\sigma}\rho}$ homotopic to $f_{2,l,\overline{\sigma}}$ for each $\rho \in G_k$, and $g'_{1,l,\sigma}$ is a family of quasi-isomorphisms

$$g'_{1,l,\overline{\sigma}}: K_\sigma \otimes \mathbb{Z}_l \to K'_{l,\sigma}$$

indexed by the set of extensions $\overline{\sigma}: \overline{k} \to \mathbb{C}$ of σ, with $g'_{1,l,\overline{\sigma}\rho}$ homotopic to $g'_{1,l,\overline{\sigma}}$ for each $\rho \in G_k$.

(vii) $g_2 = \prod_{l,\sigma} g_{2,l,\sigma}$, where, for each (l, σ),

$$g_{2,l,\sigma}: (K_{\mathbb{Q},\sigma}, W) \otimes_{\mathbb{Q}} \mathbb{Q}_l \to (K_{\mathbb{Q},l,\sigma}, W)$$

is a filtered quasi-isomorphism.

(viii) $f'_2 = \prod_\sigma f'_{2,\sigma}$, $g'_2 = \prod_\sigma g'_{2,\sigma}$, $f_3 = \prod_\sigma f_{3,\sigma}$ $g_3 = \prod_\sigma g_{3,\sigma}$, $f_4 = \prod_\sigma f_{4,\sigma}$, $g_4 = \prod_\sigma g_{4,\sigma}$, where, for each $\sigma: k \to \mathbb{C}$,

$$\begin{aligned} f_{3,\sigma}: (K_{\mathbb{Q},\sigma}, W) \otimes_{\mathbb{Q}} \mathbb{C} &\to (K'_{\mathbb{C},\sigma}, W), \\ g_{3,\sigma}: (K_{\mathbb{C},\sigma}, W) &\to (K'_{\mathbb{C},\sigma}, W), \\ f_{4,\sigma}: (K_{\mathbb{C},\sigma}, W) &\to (K''_{\mathbb{C},\sigma}, W), \\ g_{4,\sigma}: (K_k, W) \otimes_{k,\sigma} \mathbb{C} &\to (K'_{\mathbb{C},\sigma}, W) \end{aligned}$$

are filtered quasi-isomorphisms, and

$$\begin{aligned} f'_{2,\sigma}: K_\sigma \otimes \mathbb{Q} &\to K'_{\mathbb{Q},\sigma} \\ g'_{2,\sigma}: K_{\mathbb{Q},\sigma} &\to K'_{\mathbb{Q},\sigma} \end{aligned}$$

are quasi-isomorphisms.

(ix) For each $\sigma\colon k\to\mathbb{C}$, the diagram

$$\begin{array}{ccccccc}
 & K'_{\mathbb{Q},\sigma} & & (K'_{\mathbb{C},\sigma},W) & & (K''_{\mathbb{C},\sigma},W) & \\
f'_{2,\sigma}\nearrow & & \nwarrow g'_{2,\sigma}\quad f_{3,\sigma}\nearrow & & \nwarrow g_{3,\sigma}\quad f_{4,\sigma}\nearrow & & \nwarrow g_{4,\sigma}\\
K_\sigma & & (K_{\mathbb{Q},\sigma},W) & & (K_{\mathbb{C},\sigma},W) & & (K_k,W,F)\otimes_{k,\sigma}\mathbb{C}
\end{array}$$

is in $C^b_{\mathcal{H}',\mathbb{Q}}$, and defines on $H^i(Gr^W_m K_\sigma)$ a pure, polarizable $\mathbb{Z}$-Hodge structure of weight $m+i$.

(x) Let $\underline{H}$ denote the collection of graded cohomologies $gr^W_m H^i$ arising from the diagram $\mathcal{D}$. Then there are bilinear forms

$$\underline{H}_? \otimes \underline{H}_? \to \mathbb{Q}_?(-m)$$

for each component, which are compatible under the various comparison isomorphisms, and which give a polarization of the real Hodge structure given by the diagram in (ix). □

(8.6.2)

Let $C^b_{\mathcal{MAH},k}$ denote the category of polarizable MAH-complexes over k; a homomorphism $\mathcal{D}_1\to\mathcal{D}_2$ consists of a collection of maps on each component, in the appropriate category of complexes, such that each of the resulting squares commutes. We have the Tate object $\mathbb{Z}_{\mathcal{MAH},k}(q)$ in $C^+_{\mathcal{MAH},k}$, defined as the diagram:

$$\begin{array}{ccccccccc}
 & \prod_l \mathbb{Q}_l(q) & & \prod_{l,\sigma:k\to\mathbb{C}} (\mathbb{Q}_l(q),W(q)) & & \prod_{\sigma:k\to\mathbb{C}} (\mathbb{C},W(q)) & & \prod_{\sigma:k\to\mathbb{C}} (\mathbb{C},W(q)) & \\
f_1\nearrow & & \nwarrow g_1\quad f_2\nearrow & & \nwarrow g_2\quad f_3\nearrow & & \nwarrow \mathrm{id}\quad \mathrm{id}\nearrow & & \nwarrow g_4\\
\prod_l \mathbb{Z}_l(q) & & \prod_l (\mathbb{Q}_l(q),W(q)) & & \prod_{\sigma:k\to\mathbb{C}} (\mathbb{Q}(q),W(q)) & & \prod_{\sigma:k\to\mathbb{C}} (\mathbb{C},W(q)) & & (k,W(q),F(q))\\
\mathrm{id}\downarrow & & & & \mathrm{id}\downarrow & & & & \\
\prod_l \mathbb{Z}_l(q) & & \prod_{\sigma:k\to\mathbb{C}} \mathbb{Z}(q) & & \prod_{\sigma:k\to\mathbb{C}} (\mathbb{Q}(q),W(q)) & & & & \\
f'_1\searrow & & \swarrow g'_1\quad f'_2\searrow & & \swarrow g'_2 & & & & \\
 & \prod_l \mathbb{Z}_l(q) & & \prod_{\sigma:k\to\mathbb{C}} \mathbb{Q}(q) & & & & &
\end{array}$$

where $\mathbb{Z}_l(q)$ and $\mathbb{Q}_l(q)$ are the standard Tate Galois modules, $\mathbb{Z}(q)=(2\pi i)^q\mathbb{Z}$, $\mathbb{Q}(q)=(2\pi i)^q\mathbb{Q}$, and $W(q)$, $F(q)$ are the Tate filtrations (8.5.9). The maps are the obvious ones.

(8.6.3)

$C^b_{\mathcal{MAH},k}$ is a DG-tensor category ($\otimes$ given component-wise) with a Cone functor; the unit is the Tate object $\mathbb{Z}_{\mathcal{MAH},k}(0)$. This gives the homotopy category $K^b_{\mathcal{MAH},k}$ the structure of a triangulated tensor category. Taking the cohomology of each component of a diagram $\mathcal{D}$ in $C^b_{\mathcal{MAH},k}$ gives an object $\underline{H}^p$ in the category of diagrams equivalent to Jannsen's category of polarizable mixed realizations $\underline{MR}^p_k$; the functor $\underline{H}^0$ extends to a cohomological functor from $K^b_{\mathcal{MAH},k}$ to $\underline{MR}_k$.

A map f in $K^b_{\mathcal{MAH},k}$ is called a quasi-isomorphism if f induces a quasi-isomorphism (not necessarily filtered) in each component. Localizing $K^b_{\mathcal{MAH},k}$ with respect to quasi-isomorphisms defines the triangulated tensor category $D^b_{\mathcal{MAH},k}$; the cohomological functors $\underline{H}^p$ extend to $D^b_{\mathcal{MAH},k}$.

Beilinson's arguments in [AHC] extend to show that the natural map

$$\mathbf{D}^b(\underline{MR}^p_k) \to D^b_{\mathcal{MAH},k}$$

is an equivalence.

We now proceed to define the motivic realization

$$Re_{mot}\colon \mathcal{DM}_{mot}(\mathcal{V}_k) \to D^b_{\mathcal{MAH},k}.$$

(8.6.4)

Let ét_k denote the big étale site over k; $\mathfrak{r}\text{ét}_k$ the rigidification described in in the beginning of this section. We use a slightly different site for the classical topology than the usual one. For an embedding $\sigma\colon k \to \mathbb{C}$ and for $X \in \mathcal{V}$, let X_σ denote the complex manifold associated to the $\mathbb{C}$-scheme $X \times_{k,\sigma} \mathbb{C}$. Let an_k denote the site where an open cover of $X \in \mathcal{V}_k$ is a collection of maps $f_\sigma\colon U_\sigma \to X_\sigma$, where f_σ is surjective, and f_σ is locally (on U_σ) a homeomorphism. We rigidify the site an_k as in the beginning of this section by taking the product of the analytic rigidifications for each σ.

We have the map of sites

$$\alpha\colon \text{ét}_k \to \text{an}_k$$

given by sending $U \to X$ to $\prod_\sigma U_\sigma \to X_\sigma$; this extends to a map of rigidified sites in the obvious way. Similarly, letting Zar_k denote the big Zariski site over k, we have the map of sites

$$\beta\colon \text{Zar}_k \to \text{an}_k;$$

rigidifying Zar_k as above by taking pointed open sets, this map of sites extends to a map of rigidified sites.

(8.6.5)

Let X be in $\mathcal{V}_k$. There is a compactification $X \to \overline{X}$ defined over k, and the category of compactifications of X over k forms a directed filtering category. For a compactification $X \to \overline{X}$, with normal crossing divisor D at infinity, we form the diagram of complexes of sheaves

$$\mathcal{D}[X,\overline{X}] :=$$

$$\begin{array}{ccccccccc}
 & \prod_{l} \mathcal{F}'_{\mathbb{Q},l} & & \prod_{l,\sigma:k\to\mathbb{C}} (\mathcal{F}_{\mathbb{Q},l,\sigma},W) & & \prod_{\sigma:k\to\mathbb{C}} (\mathcal{F}'_{\mathbb{C},\sigma},W) & & \prod_{\sigma:k\to\mathbb{C}} (\mathcal{F}''_{\mathbb{C},\sigma},W) & \\
f_1 \nearrow & & \nwarrow g_1 \quad f_2 \nearrow & & \nwarrow g_2 \quad f_3 \nearrow & & \nwarrow \mathrm{id} \quad \mathrm{id} \nearrow & & \nwarrow g_4 \\
\prod_{l} \mathcal{F}_l & & \prod_{l} (\mathcal{F}_{\mathbb{Q},l},W) & & \prod_{\sigma:k\to\mathbb{C}} (\mathcal{F}_{\mathbb{Q},\sigma},W) & & \prod_{\sigma:k\to\mathbb{C}} (\mathcal{F}_{\mathbb{C},\sigma},W) & & (\mathcal{F}_k,W,F) \\
\mathrm{id}\downarrow & & & & \mathrm{id}\downarrow & & & & \\
\prod_{l} \mathcal{F}_l & & \prod_{\sigma:k\to\mathbb{C}} \mathcal{F}_\sigma & & \prod_{\sigma:k\to\mathbb{C}} (\mathcal{F}_{\mathbb{Q},\sigma},W) & & & & \\
f'_1 \searrow & & \swarrow g'_1 \quad f'_2 \searrow & & \swarrow g'_2 & & & & \\
 & \prod_{l,\sigma:k\to\mathbb{C}} \mathcal{F}'_{l,\sigma} & & \prod_{\sigma:k\to\mathbb{C}} \mathcal{F}'_{\mathbb{Q},\sigma} & & & & &
\end{array}$$

where:

$$\mathcal{F}_l = \mathbb{Z}^{\text{ét}}_{l,X}, \quad \mathcal{F}'_{\mathbb{Q},l} = \check{\mathrm{Alt}}\mathbb{Q}^{\text{ét}}_{l,X}, \quad (\mathcal{F}_{\mathbb{Q},l},W) = (j_*\check{\mathrm{Alt}}\mathbb{Q}^{\text{ét}}_{l,X},\tau_{\le}) \tag{1}$$

$$\begin{gathered}\mathcal{F}_\sigma = \mathbb{Z}^{\text{an}}_{X^\sigma}, \quad \mathcal{F}'_{l,\sigma} = \mathbb{Z}^{\text{an}}_{l,X^\sigma}, \quad \mathcal{F}'_{\mathbb{Q},\sigma} = \check{\mathrm{Alt}}\mathbb{Q}^{\text{an}}_{l,X^\sigma}, \\ (\mathcal{F}_{\mathbb{Q},l,\sigma},W) = (j_*\check{\mathrm{Alt}}\mathbb{Q}^{\text{an}}_{l,X^\sigma},\tau_{\le}), \quad (\mathcal{F}_{\mathbb{Q},\sigma},W) = (j_*\check{\mathrm{Alt}}\mathbb{Q}^{\text{an}}_{X^\sigma},\tau_{\le})\end{gathered} \tag{2}$$

$$\begin{gathered}(\mathcal{F}'_{\mathbb{C},\sigma},W) = (j_*\check{\mathrm{Alt}}\Omega^{\text{an}}_{X^\sigma},\tau_{\le}), \quad (\mathcal{F}_{\mathbb{C},\sigma},W) = (\Omega^{\text{an}}_{\overline{X}^\sigma}(\log(D)),\tau_{\le}), \\ (\mathcal{F}''_{\mathbb{C},\sigma},W) = (\Omega^{\text{an}}_{\overline{X}^\sigma}(\log(D)),W^{\text{an}}_\bullet)\end{gathered} \tag{3}$$

$$(\mathcal{F}_k,W,F) = (\Omega^{\text{Zar}}_{\overline{X}/k}(\log(D)),W^{\text{Zar}}_\bullet,F^\bullet). \tag{4}$$

Here, the superscripts "ét", "an", and "Zar" denote the relevant topologies, "$\tau_{\le}$" is the canonical filtration, $\Omega^{\text{an}}_{X^\sigma}$, (resp. $\Omega^{\text{an}}_{\overline{X}^\sigma}(\log(D))$) is the complex of sheaves of holomorphic forms (resp. holomorphic forms with log poles), $\Omega^{\text{Zar}}_{\overline{X}/k}(\log(D))$ is the complex of sheaves of algebraic forms with log poles. $W^{\text{an}}_\bullet$ and $W^{\text{Zar}}_\bullet$ are the filtrations by order of pole, and $F^\bullet$ is the stupid filtration. The maps are the change of topology morphisms:

$$\mathbb{Z}^{\text{ét}}_{l,X} \to \alpha^*\mathbb{Z}^{\text{an}}_{l,X^\sigma}; \quad (j_*\check{\mathrm{Alt}}\mathbb{Q}^{\text{ét}}_{l,X},\tau_{\le}) \to \alpha^*(j_*\check{\mathrm{Alt}}\mathbb{Q}^{\text{an}}_{l,X^\sigma},\tau_{\le}),$$

the change of coefficient rings:

$$\begin{gathered}\mathbb{Z}^{\text{ét}}_{l,X}\otimes\mathbb{Q}_l \to j_*\check{\mathrm{Alt}}\mathbb{Q}^{\text{ét}}_{l,X}; \quad \mathbb{Z}^{\text{an}}_{X^\sigma}\otimes\mathbb{Q} \to \check{\mathrm{Alt}}\mathbb{Q}^{\text{an}}_{X^\sigma} \\ \mathbb{Z}^{\text{an}}_{X^\sigma}\otimes\mathbb{Z}_l \to \mathbb{Z}^{\text{an}}_{l,X^\sigma}; \quad (j_*\check{\mathrm{Alt}}\mathbb{Q}^{\text{an}}_{X^\sigma},\tau_{\le})\otimes\mathbb{Q}_l \to (j_*\check{\mathrm{Alt}}\mathbb{Q}^{\text{an}}_{l,X^\sigma},\tau_{\le}),\end{gathered}$$

forgetting the filtration:

$$j^*(j_*\check{\mathrm{Alt}}\mathbb{Q}^{\mathrm{an}}_{l,X^\sigma}, \tau_\leq) \to \check{\mathrm{Alt}}\mathbb{Q}^{\mathrm{an}}_{l,X^\sigma}$$

$$j^*(j_*\check{\mathrm{Alt}}\mathbb{Q}^{\mathrm{an}}_{X^\sigma}, \tau_\leq) \to \check{\mathrm{Alt}}\mathbb{Q}^{\mathrm{an}}_{X^\sigma};$$

the arrow $(\mathcal{F}_k, W, F) \to (\mathcal{F}''_{\mathbb{C},\sigma}, W)$ is the product over $\sigma: k \to \mathbb{C}$ of the change of topology maps

$$(\Omega^{\mathrm{Zar}}_{\overline{X}/k}(\log(D)), W^{\mathrm{Zar}}_\bullet) \to \beta^*(\Omega^{\mathrm{an}}_{\overline{X}^\sigma}(\log(D)), W^{\mathrm{an}}_\bullet).$$

Let $\mathbb{Z}_{\mathcal{MAH},k,X}(q)$ be the diagram $\mathbb{Z}_{\mathcal{MAH},k}(q) \otimes Rp_{X*}\mathcal{D}[X, \bar{X}]$. The arguments used to construct the Hodge realization give

(8.6.6) THEOREM

Sending (X, q) in $\mathcal{V}_k \times \mathbb{Z}$ to $\mathbb{Z}_{\mathcal{MAH},k,X}(q)$ extends to an exact tensor functor

$$Re_{\mathcal{MAH},k}: \mathcal{DM}(\mathcal{V}_k))_{\mathbb{Z}} \to D^+_{\mathcal{MAH},k}.$$

□