

Chapter 4

Co-chain operations

We now apply the constructions of Chapter 3 to give co-chain operations on certain complexes $\check{\mathfrak{F}}$ constructed from an associative, graded-commutative complex of presheaves \mathfrak{F} . In applications, the complexes $\check{\mathfrak{F}}$ are constructed as limits of the complexes formed from \mathfrak{F} by taking “rigid hypercovers” following the method of Friedlander; in this section we give a formal version of the construction and will use a more concrete form when we give our main applications in I, Chapter 8. The co-chain operations fit together to give a functor

$$\check{\mathfrak{F}}: (\mathcal{W}_{\mathbb{Z}}^{\text{op}})^{\otimes, \mathfrak{h}} \longrightarrow \mathbf{C}(\mathcal{A}),$$

whose construction is the main object of this chapter.

To fix ideas, we work in a subcategory of \mathbf{Sch}_S ; the same construction will define the functor $\check{\mathfrak{F}}$ for a subcategory of analytic spaces, or analytic spaces with a real structure.

4.1. Adjoining a disjoint base-point

We construct the category of objects of \mathcal{C} together with a “disjoint base-point”.

(4.1.1)

Let \mathcal{C} be a full subcategory of \mathbf{Sch}_S , closed under disjoint union. We extend the category \mathcal{C} to the category of “pointed objects in \mathcal{C} ”, \mathcal{C}^+ by defining \mathcal{C}^+ to have the same objects as \mathcal{C} (we denote the object of \mathcal{C}^+ corresponding to X in \mathcal{C} by X^+), and with

$$\text{Hom}_{\mathcal{C}^+}(X^+, Y^+) = \text{Hom}_{\mathcal{C}}(X, Y \amalg S).$$

Composition is given by defining the composition

$$X^+ \xrightarrow{f} Y^+ \xrightarrow{g} Z^+$$

as the map $X^+ \rightarrow Z^+$ corresponding to

$$(g \cup \text{id}_S) \circ f.$$

The identity id_{X^+} is the canonical map

$$i_{X, X \amalg S}: X \longrightarrow X \amalg S.$$

We denote the object \emptyset^+ by $*$; the canonical map $\emptyset \rightarrow X \amalg S$ defines the map

$$i_{X^+}: * \longrightarrow X^+$$

while the structure morphism $X \rightarrow S \cong \emptyset \amalg S$ defines the map

$$p_{X^+}: X^+ \longrightarrow *.$$

This makes $*$ into an initial and final object in \mathcal{C}^+ .

(4.1.2)

For a morphism $f: X \rightarrow Y$ in \mathcal{C} , let

$$f^+: X^+ \longrightarrow Y^+$$

be the morphism given as the composition

$$f^+ = i_{Y,Y} \amalg_S \circ f.$$

Sending X to X^+ and f to f^+ defines a faithful functor

$$+: \mathcal{C} \longrightarrow \mathcal{C}^+. \quad (4.1.2.1)$$

If $Y = X^+$, we write Y^- for X ; if $g: Y \rightarrow Y'$ is the map $f^+: X^+ \rightarrow X'^+$, we write $g^-: Y^- \rightarrow Y'^-$ for $f: X \rightarrow X'$.

(4.1.3) PRODUCTS AND COPRODUCTS

Let

$$\{i_\alpha: X_\alpha \longrightarrow X \mid \alpha \in A\}$$

be a coproduct of the X_α in \mathcal{C} . Then the collection of maps

$$\{i_\alpha^+: X_\alpha^+ \longrightarrow X^+ \mid \alpha \in A\}$$

is a coproduct of the X_α^+ in \mathcal{C}^+ . We write the operation of coproduct as \vee ; in particular, we have

$$X^+ \vee Y^+ = (X \amalg Y)^+.$$

If we have maps $f: Z \rightarrow X \amalg S$, $g: W \rightarrow Y \amalg S$ in \mathcal{C} , defining maps $f: Z^+ \rightarrow X^+$ and $g: W^+ \rightarrow Y^+$ in \mathcal{C}^+ , we let

$$f \vee g: Z^+ \vee W^+ \longrightarrow X^+ \vee Y^+$$

be the map

$$f \cup_{p_S} \amalg_S g: Z \amalg W \longrightarrow (X \amalg S) \cup_{p_S} \amalg_S (Y \amalg S) \cong X \amalg Y \amalg S$$

where $p_S \coprod_S: S \coprod S \rightarrow S$ is the structure morphism. The functor $+$ then respects the coproducts.

To define the the smash product \wedge_{Z^+} , we identify objects over Z^+ in \mathcal{C}^+ with objects over $Z \coprod S$ in \mathcal{C} , and define

$$X^+ \wedge_{Z^+} Y^+ := (X \times_Z \coprod_S Y)^+.$$

We have as well the absolute smash product

$$X^+ \wedge Y^+ := X^+ \wedge_* Y^+.$$

The smash product \wedge gives \mathcal{C}^+ and $\mathcal{C}^+/\mathcal{C}^+$ the structure of symmetric monoidal categories, and makes $+$ a symmetric monoidal functor.

(4.1.4) PRESHEAVES

Let \mathcal{G} be a presheaf on \mathcal{C} , with values in an additive category \mathcal{A} . Suppose that \mathcal{G} transforms finite disjoint unions into direct sums (e.g., if \mathcal{G} is a sheaf for the Zariski topology). We extend \mathcal{G} to a presheaf (also denoted \mathcal{G}) on \mathcal{C}^+ by setting

$$\mathcal{G}(X^+) = \mathcal{G}(X) \tag{4.1.4.1}$$

If we have a morphism

$$f: Y \rightarrow X \coprod S$$

in \mathcal{C} , defining the morphism

$$f: Y^+ \rightarrow X^+$$

in \mathcal{C}^+ , we define

$$f^*: \mathcal{G}(X^+) \rightarrow \mathcal{G}(Y^+)$$

as the composition

$$\mathcal{G}(X^+) = \mathcal{G}(X) \xrightarrow{i_{X^*}} \mathcal{G}(X \coprod S) \xrightarrow{f^*} \mathcal{G}(Y) = \mathcal{G}(Y^+).$$

Here

$$i_{X^*}: \mathcal{G}(X) \rightarrow \mathcal{G}(X \coprod S) \cong \mathcal{G}(X) \oplus \mathcal{G}(S)$$

is the canonical map given by the assumption that the presheaf transform coproducts in \mathcal{C} to direct sums in \mathcal{A} .

4.2. Complexes associated to multi-simplicial objects

(4.2.1)

Let \mathcal{A} be a tensor category, and let \mathfrak{F} be a graded complex of presheaves on \mathcal{C} , with values in \mathcal{A} :

$$\mathfrak{F} \in \mathbf{C}(\text{PreSh}_{\mathcal{C}}(\mathcal{A})); \quad \mathfrak{F} = \oplus_q \dots \longrightarrow \mathfrak{F}^n(q) \longrightarrow \mathfrak{F}^{n+1}(q) \longrightarrow \dots \quad (4.2.1.1)$$

We suppose that we have associative and bi-graded products (graded-commutative in n and commutative in q).

$$\boxtimes: \mathfrak{F} \otimes \mathfrak{F} \longrightarrow \mathfrak{F}; \quad (4.2.1.2)$$

i.e., for X and Y in \mathcal{C} , we have the map of complexes

$$\boxtimes_{X,Y}^{q,q'}: \mathfrak{F}(q)(X) \otimes \mathfrak{F}(q')(Y) \longrightarrow \mathfrak{F}(q+q')(X \times_S Y),$$

functorially in X and Y , which is associative and graded-commutative in the obvious sense (and is commutative with respect to the q -grading). We also assume that \mathfrak{F} transforms finite disjoint unions to direct sums.

In particular, we have the extension of the complex of presheaves \mathfrak{F} to a graded complex of presheaves on \mathcal{C}^+ ; the product (4.2.1.2) extends to maps

$$\boxtimes_{X^+,Y^+}^{q,q'}: \mathfrak{F}(q)(X^+) \otimes \mathfrak{F}(q')(Y^+) \longrightarrow \mathfrak{F}(q+q')(X^+ \wedge Y^+), \quad (4.2.1.3)$$

functorially in X^+ and Y^+ , by taking

$$\boxtimes_{X^+,Y^+}^{q,q'} = \boxtimes_{X,Y}^{q,q'}: \mathfrak{F}(q)(X) \otimes \mathfrak{F}(q')(Y) \longrightarrow \mathfrak{F}(q+q')(X \times_S Y).$$

That $\boxtimes_{X^+,Y^+}^{q,q'}$ is indeed functorial in X^+ and Y^+ follows from the identity $\mathfrak{F}(\ast) = 0$. The product (4.2.1.3) remains associative and graded-commutative.

(4.2.2) THE FUNCTOR $\mathfrak{F}(q)(-)^*$

Let $\mathfrak{C} = (\mathcal{C}^+)^{\text{op}}$; \mathfrak{C} is a symmetric semi-monoidal category with product \wedge . We have the symmetric monoidal category $(\mathbb{Z}, +)$; we may apply the construction of (3.2.2), giving the DG tensor category $[\mathfrak{C} \times \mathbb{Z}]_{\infty}^{\otimes}$. We note that a cosimplicial object of \mathbb{Z} is constant, hence we may write the objects of $[\mathfrak{C} \times \mathbb{Z}]_{\infty}^{\otimes}$ as direct sums of sequences

$$((U_*^1, u_1), \dots, (U_*^n, u_n))$$

with

$$U_*^j \in \mathfrak{C}(\Delta); \quad u_j \in \mathbb{Z}.$$

Let U_* be in

$$\mathfrak{C}(\Delta) = [\mathcal{C}^+(\Delta^{\text{op}})]^{\text{op}}.$$

Composing U_* with $\mathfrak{F}(q)$ gives the functor

$$\mathfrak{F}(q) \circ U_*: \Delta \longrightarrow \mathbf{C}(\mathcal{A});$$

we may then take the associated total complex, giving the element

$$\mathfrak{F}(q)(U_*)^* \in \mathbf{C}(\mathcal{A}).$$

Sending U_* to $\mathfrak{F}(q)(U_*)^*$ thus defines a functor

$$\mathfrak{F}(q)(-)^*: \mathfrak{C}(\Delta) \longrightarrow \mathbf{C}(\mathcal{A}). \tag{4.2.2.1}$$

(4.2.3) The functor \mathfrak{F}^∞

We now extend the functors (4.2.2.1) to a DG tensor functor

$$\mathfrak{F}^\infty: [\mathfrak{C} \times \mathbb{Z}]_\infty^\otimes \longrightarrow \mathbf{C}(\mathcal{A}) \tag{4.2.3.1}$$

with

$$\mathfrak{F}^\infty((U_*^1, u_1), \dots, (U_*^n, u_n)) := \mathfrak{F}(u_1)(U_*^1)^* \otimes \dots \otimes \mathfrak{F}(u_n)(U_*^n)^*.$$

To define \mathfrak{F}^∞ on morphisms, take $F: n \rightarrow m$ in Ω_0 , take $U_*^1, \dots, U_*^n, V_*^1, \dots, V_*^m$ in $\mathfrak{C}(\Delta)$, and take non-negative integers $p_1, \dots, p_n, q_1, \dots, q_m$. We write

$$[p_*] = ([p_1], \dots, [p_n]); \quad [q_*] = ([q_1], \dots, [q_m]),$$

and take

$$h_{p_1, \dots, p_n}^{q_1, \dots, q_m} \in \text{Hom}_{[\mathfrak{C} \times \mathbb{Z}]_\infty}(((U_*^1, u_1), \dots, (U_*^n, u_n)), ((V_*^1, v_1), \dots, (V_*^m, v_m)))_F([p_*], [q_*]).$$

We may write

$$\begin{aligned} & \Pi_{\mathfrak{C} \times \mathbb{Z}}(F)((U_*^1, u_1), \dots, (U_*^n, u_n)) \\ &= ((\Pi_{\mathfrak{C}}(F)(U_*^1, \dots, U_*^n)_1, \Pi_{\mathbb{Z}}(F)(u_1, \dots, u_n)_1), \\ & \quad \dots, (\Pi_{\mathfrak{C}}(F)(U_*^1, \dots, U_*^n)_m, \Pi_{\mathbb{Z}}(F)(u_1, \dots, u_n)_m)), \end{aligned}$$

and write $h_{p_1, \dots, p_n}^{q_1, \dots, q_m}$ as an m -tuple

$$\begin{aligned} h_{p_1, \dots, p_n}^{q_1, \dots, q_m} &= \prod_{j=1}^m h_{p_1, \dots, p_n, j}^{q_1, \dots, q_m} \\ h_{p_1, \dots, p_n, j}^{q_1, \dots, q_m} &: \Pi_{\mathfrak{C}}(F)(U_*^1([p_1]), \dots, U_*^n([p_n]))_j \longrightarrow V_*^j([q_j]); \end{aligned}$$

where we have the condition

$$\Pi_{\mathbb{Z}}(F)(u_1, \dots, u_n)_j = v_j; \quad j = 1, \dots, m.$$

The map $h_{p_1, \dots, p_n}^{q_1, \dots, q_m}$ thus gives the map

$$\begin{aligned} & \mathfrak{F}(v_j)(\Pi_{\mathbf{C}}(F)(U_*^1([p_1]), \dots, U_*^n([p_n]))_j) \\ & \xrightarrow{\mathfrak{F}(h_{p_1, \dots, p_n}^{q_1, \dots, q_m})} \mathfrak{F}(v_j)(V_*^j([q_j])) \end{aligned} \quad (4.2.3.2)$$

for each j . As in (3.1.2.3), we have

$$\begin{aligned} \Pi_{\mathbf{C}}(F)(U_*^1([p_1]), \dots, U_*^n([p_n]))_j &= U_*^{i_1^j}([p_{i_1^j}]) \wedge \dots \wedge U_*^{i_{s_j}^j}([p_{i_{s_j}^j}]) \\ \Pi_{\mathbb{Z}}(F)(u_1, \dots, u_n)_j &= u_{i_1^j} + \dots + u_{i_{s_j}^j} = v_j. \end{aligned}$$

for suitable integers i_k^j (depending only on F).

The product structure \boxtimes on \mathfrak{F} gives us the map

$$\begin{aligned} & \mathfrak{F}(u_{i_1^j})(U_*^{i_1^j}([p_{i_1^j}])) \otimes \dots \otimes \mathfrak{F}(u_{i_{s_j}^j})(U_*^{i_{s_j}^j}([p_{i_{s_j}^j}])) \\ & \xrightarrow{\Pi(F)([p_1], \dots, [p_n])_j^*} \mathfrak{F}(v_j)(\Pi_{\mathbf{C}}(F)(U_*^1([p_1]), \dots, U_*^n([p_n]))_j). \end{aligned} \quad (4.2.3.3)$$

Composing (4.2.3.2) with (4.2.3.3), taking the tensor product over j , and composing with the canonical permutation isomorphism

$$\begin{aligned} & \mathfrak{F}(u_1)((U_*^1([p_1])) \otimes \dots \otimes \mathfrak{F}(u_n)((U_*^n([p_n]))) \\ & \xrightarrow{\tau} \otimes_{j=1}^m \mathfrak{F}(u_{i_1^j})(U_*^{i_1^j}([p_{i_1^j}])) \otimes \dots \otimes \mathfrak{F}(u_{i_{s_j}^j})(U_*^{i_{s_j}^j}([p_{i_{s_j}^j}])), \end{aligned}$$

gives the map

$$\begin{aligned}
 & \mathfrak{F}((U_*^1([p_1]), u_1), \dots, (U_*^n([p_n]), u_n)) \\
 & \quad \xrightarrow{\mathfrak{F}(h_{p_1, \dots, p_n}^{q_1, \dots, q_m})} \mathfrak{F}((V_*^1([q_1]), v_1), \dots, (V_*^m([q_m]), v_m)); \\
 & \mathfrak{F}((U_*^1([p_1]), u_1), \dots, (U_*^n([p_n]), u_n)) := \bigotimes_{i=1}^n \mathfrak{F}(u_i)(U_*^i([p_i])), \\
 & \mathfrak{F}((V_*^1([q_1]), v_1), \dots, (V_*^m([q_m]), v_m)) := \bigotimes_{j=1}^m \mathfrak{F}(v_j)(V_*^j([q_j])) \\
 & \mathfrak{F}(h_{p_1, \dots, p_n}^{q_1, \dots, q_m}) := \left(\bigotimes_{j=1}^m \mathfrak{F}(h_{p_1, \dots, p_n, j}^{q_1, \dots, q_m}) \right) \circ \Pi_{\mathfrak{C}}(F)([p_1], \dots, [p_n])_*^* \circ \tau.
 \end{aligned} \tag{4.2.3.4}$$

If we now have an element

$$\begin{aligned}
 h & \in \text{Hom}_{[\mathfrak{C} \times \mathbb{Z}]_{\infty}^{\otimes}}(((U_*^1, u_1), \dots, (U_*^n, u_n)), ((V_*^1, v_1), \dots, (V_*^m, v_m)))_F^d \\
 h & = \prod_{\substack{p_1, \dots, p_n; q_1, \dots, q_m \\ \sum_j q_j - \sum_i p_i = d}} h_{p_1, \dots, p_n}^{q_1, \dots, q_m},
 \end{aligned}$$

the collection of maps (4.2.3.4) thus defines the map

$$\mathfrak{F}^{\infty}(h): \mathfrak{F}^{\infty}((U_*^1, u_1), \dots, (U_*^n, u_n)) \longrightarrow \mathfrak{F}^{\infty}((V_*^1, v_1), \dots, (V_*^m, v_m))[d]. \tag{4.2.3.5}$$

Let $\eta: F \rightarrow \eta \cdot F$ be a 2-morphism in Ω ; the identity

$$\mathfrak{F}^{\infty}(h) = \mathfrak{F}^{\infty}(\eta \cdot h)$$

follows from the definitions and the graded-commutativity of \mathfrak{F} . Thus the map (4.2.3.5) depends only on the image of h in

$$\text{Hom}_{[\mathfrak{C} \times \mathbb{Z}]_{\infty}^{\otimes}}(((U_*^1, u_1), \dots, (U_*^n, u_n)), ((V_*^1, v_1), \dots, (V_*^m, v_m)))^d.$$

This completes the definition of the functor (4.2.3.1); the verification that \mathfrak{F}^{∞} gives a DG tensor functor is straightforward, and is left to the reader. Clearly, the functor (4.2.3.1) is natural in \mathfrak{F} and \mathfrak{C} .

(4.2.4) REMARK

If each $\mathfrak{F}(q)$ is bounded below:

$$\mathfrak{F}(q) \in \mathbf{C}^+(\text{PreSh}_{\mathfrak{C}}(\mathcal{A})); \quad \mathfrak{F}(q) = \mathfrak{F}^{N_q}(q) \longrightarrow \dots \longrightarrow \mathfrak{F}^n(q) \longrightarrow \mathfrak{F}^{n+1}(q) \longrightarrow \dots$$

then the functor (4.2.3.1) takes values in $\mathbf{C}^+(\mathcal{A})$. \square

4.3. Fibered and cofibered categories

We recall the definition of fibered and cofibered categories, and define the notion of a fibered symmetric semi-monoidal category.

(4.3.1) DEFINITION

We recall that a *fibered category* \mathfrak{W} over a category \mathcal{W} consists of a functor

$$\pi: \mathfrak{W} \longrightarrow \mathcal{W}$$

together with “pull-back functors”

$$f^*: \mathfrak{W}(X) := \pi^{-1}(X) \longrightarrow \mathfrak{W}(Y) := \pi^{-1}(Y),$$

a natural transformation

$$\phi_f: f^* \longrightarrow i_{\mathfrak{W}(X)},$$

for each morphism $f: Y \rightarrow X$ in \mathcal{W} , where $i_{\mathfrak{W}(Y)}$ is the inclusion of $\mathfrak{W}(Y)$ in \mathfrak{W} , and natural isomorphisms

$$\theta_{f,g}: g^* \circ f^* \longrightarrow (f \circ g)^*$$

for each pair of composable morphisms f, g in \mathcal{W} , such that, for each pair of composable morphisms f, g in \mathcal{W} ,

i) the diagram

$$\begin{array}{ccc} h^* \circ (g^* \circ f^*) & \xrightarrow{h^* \circ \theta_{f,g}} & h^* \circ (f \circ g)^* \\ & & \searrow \theta_{f \circ g, h} \\ & & (f \circ g \circ h)^* \\ & & \nearrow \theta_{f, g \circ h} \\ \parallel & & \\ (h^* \circ g^*) \circ f^* & \xrightarrow{\theta_{g,h} \circ f^*} & (g \circ h)^* \circ f^* \end{array}$$

commutes

ii) We have

$$\phi_{f \circ g} \circ \theta_{f,g} = \phi_f \circ (\phi_g \circ f^*).$$

iii) For a morphism $f: Y \rightarrow X$ in \mathcal{W} , and for $W \in \mathfrak{W}(Y)$, $W' \in \mathfrak{W}(X)$, we let

$$\mathrm{Hom}_{\mathfrak{W}}(W, W')/f$$

be the subset of $\mathrm{Hom}_{\mathfrak{W}}(W, W')$ of morphisms F with $\pi(F) = f$. Then the functor

$$\circ\phi_f: \mathrm{Hom}_{\mathfrak{W}(Y)}(W, f^*W') \longrightarrow \mathrm{Hom}_{\mathfrak{W}}(W, W')/f$$

is an isomorphism.

We have as well the dual notion, that of a *cofibered* category.

(4.3.2) PULL-BACKS

Let $\mathfrak{W} \rightarrow \mathcal{W}$ be a fibered category over \mathcal{W} , and let

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ u \downarrow & & \downarrow v \\ Y' & \xrightarrow{f'} & X' \end{array}$$

be a commutative diagram in \mathcal{W} . Let W be in $\mathfrak{W}(Y)$, W' in $\mathfrak{W}(X)$ We may send a map

$$g: W \longrightarrow f^* W'$$

in $\mathfrak{W}(Y)$ to the composition

$$\begin{aligned} u^*(W) &\xrightarrow{u^*(g)} u^*(f^*(W')) \xrightarrow{\theta_{f,u}(W')} (f \circ u)^*(W') \\ &= (v \circ f')^*(W') \xrightarrow{\theta_{v,f'}(W')^{-1}} f'^*(v^*(W')) \end{aligned}$$

Composing with the isomorphisms $\circ\phi_f$ and $\circ\phi_{f'}$ gives the pull-back maps

$$(u, v)^*: \text{Hom}_{\mathfrak{W}}(W, W')/f \longrightarrow \text{Hom}_{\mathfrak{W}}(u^*(W), v^*(W'))/f', \quad (4.3.2.1)$$

which is functorial under composition of such commutative squares.

(4.3.3) THE CATEGORY OF FIBERED CATEGORIES

A functor of fibered categories

$$F: \mathfrak{W}_1 \longrightarrow \mathfrak{W}_2$$

over a functor $p_0: \mathcal{W}_1 \longrightarrow \mathcal{W}_2$ is a pair (p, Θ_*) consisting of a functor

$$p: \mathfrak{W}_1 \longrightarrow \mathfrak{W}_2$$

over p_0 , and a natural transformation Θ_f for each morphism f in \mathcal{W}_1 ,

$$\Theta_f: p \circ f^{*1} \longrightarrow p_0(f)^{*2} \circ p$$

such that

- i) For morphism $f: Y \rightarrow X$ in \mathcal{W}_1 , and each W in $\mathfrak{W}_1(X)$, we have

$$p(\phi_f^1(W)) = \phi_{p_0(f)}^2(p(W)) \circ \Theta_f(W).$$

ii) the diagram

$$\begin{array}{ccc}
 p_0(g)^{*2} \circ p_0(f)^{*2} \circ p & \xleftarrow{p_0(g)^{*2} \circ \Theta_f} & p_0(g)^{*2} \circ p \circ f^{*1} \\
 \theta_{f,g}^2 \downarrow & & \searrow \Theta_g \circ f^{*1} \\
 & & p \circ g^{*1} \circ f^{*1} \\
 p_0(f \circ g)^{*2} \circ p & \xleftarrow{\Theta_{f \circ g}} & p \circ (f \circ g)^{*1} \\
 & & \swarrow p \circ \theta_{f,g}^1
 \end{array}$$

commutes for each pair of composable morphisms f, g in \mathcal{W} .

A natural transformation

$$\rho: (p, \Theta_*) \longrightarrow (q, \Xi_*)$$

over a natural transformation

$$\rho_0: p_0 \longrightarrow q_0$$

is given by assigning to each $W \in \mathfrak{W}_1(X)$ a morphism

$$\rho(W): p(W) \longrightarrow q(W),$$

over $\rho_0(X)$, such that, for each morphism $f: Y \rightarrow X$ in \mathcal{W} and each morphism

$$F: W' \longrightarrow f^{*1}W$$

in $\mathfrak{W}_1(Y)$, the diagram

$$\begin{array}{ccc}
 p(W') & \xrightarrow{\rho(W')} & q(W') \\
 p(F) \downarrow & & \downarrow q(F) \\
 p(f^{*1}(W)) & \xrightarrow{\rho(f^{*1}(W))} & q(f^{*1}(W)) \\
 \Theta_f \downarrow & & \downarrow \Xi_f \\
 p_0(f)^{*2}(p(W)) & \xrightarrow{(p_0(f), q_0(f))^{*2}(\rho(W))} & q_0(f)^{*2}(q(W))
 \end{array}$$

commutes, where the pull-back $(p_0(f), q_0(f))^{*2}$ is defined with respect to the commutative diagram

$$\begin{array}{ccc}
 p_0(X) & \xrightarrow{\rho_0(X)} & q_0(X) \\
 p_0(f) \downarrow & & \downarrow q_0(f) \\
 p_0(Y) & \xrightarrow{\rho_0(Y)} & q_0(Y)
 \end{array}$$

(4.3.4) FIBERED SYMMETRIC MONOIDAL CATEGORIES

The symmetric monoidal structure in **cat** (1.1.2) induces a symmetric monoidal structure on the category of small fibered categories; via the method of §1.2, this in turn gives the notion of a (strictly associative) fibered symmetric semi-monoidal category over a symmetric semi-monoidal category. Explicitly, this is given by a fibered category $\mathfrak{W} \rightarrow$

\mathcal{W} over a symmetric monoidal category $(\mathcal{W}, \times_{\mathcal{W}}, t_{\mathcal{W}})$, together with the structure of a symmetric semi-monoidal category on \mathfrak{W} , $(\mathfrak{W}, \times_{\mathfrak{W}}, t_{\mathfrak{W}})$, such that

$$\times_{\mathfrak{W}}: \mathfrak{W} \times \mathfrak{W} \longrightarrow \mathfrak{W}$$

is a functor of fibered categories over the functor

$$\times_{\mathcal{W}}: \mathcal{W} \times \mathcal{W} \longrightarrow \mathcal{W}$$

and with $t_{\mathfrak{W}}$ a natural transformation over $t_{\mathcal{W}}$. The associativity isomorphisms are handled similarly if we do not assume strict associativity.

4.4. Canonical co-chains and co-chain operations

We now show how a multiplicative complex of presheaves on \mathcal{C} , with values in a tensor category \mathcal{A} , together with a choice of “canonical co-chains” parametrized by a category \mathcal{W} , gives a representation of $(\mathcal{W}^{\text{op}})^{\otimes, \flat}$ into the category of complexes in \mathcal{A} . We assume that filtered direct limits are representable in \mathcal{A} , and that the tensor product commutes with filtered direct limits.

(4.4.1)

A category \mathcal{B} is called *left-directed* if

- i) for each pair of objects x, y of \mathcal{B} , there is an object z of \mathcal{B} , and morphisms $z \rightarrow x$, $z \rightarrow y$.
- ii) for each pair of objects x, y of \mathcal{B} , there is at most one morphism $x \rightarrow y$.

A functor

$$F: \mathcal{B} \longrightarrow \mathcal{B}'$$

is called *left-final* if, for each morphism

$$f': a' \longrightarrow b'$$

in \mathcal{B}' , there is a morphism

$$f: a \longrightarrow b$$

in \mathcal{B} and a commutative diagram in \mathcal{B}'

$$\begin{array}{ccc} F(a) & \xrightarrow{F(f)} & F(b) \\ \downarrow & & \downarrow \\ a' & \xrightarrow{f'} & b' \end{array}$$

in \mathcal{B}' .

(4.4.2)

As in earlier sections of this chapter, we let \mathfrak{C} denote the category $(\mathcal{C}^+)^{\text{op}}$. The category of pointed simplicial objects $\mathcal{C}^+(\Delta^{\text{op}})$ is a symmetric semi-monoidal category with product $\wedge^{\mathfrak{s}}$:

$$U_* \wedge^{\mathfrak{s}} V_* = \text{diag}[U_* \wedge V_*]_*.$$

(4.4.3)

Let (\mathcal{W}, w) be a symmetric semi-monoidal category, with product \times , together with symmetric semi-monoidal functor

$$w: \mathcal{W} \longrightarrow \mathbb{Z}.$$

We let $\mathcal{W}_{\mathbb{Z}}$ denote the additive category generated by \mathcal{W} ; the symmetric semi-monoidal structure on \mathcal{W} makes $\mathcal{W}_{\mathbb{Z}}$ a tensor category without unit.

We suppose that we have a symmetric semi-monoidal category $\text{RSCov}(\mathcal{W})$, with product $\times^{\mathfrak{ts}}$, a symmetric semi-monoidal functor

$$\pi: \text{RSCov}(\mathcal{W}) \longrightarrow \mathcal{W},$$

together with a functor

$$p: \text{RSCov}(\mathcal{W}) \longrightarrow \mathcal{C}^+(\Delta^{\text{op}}),$$

and a natural transformation

$$\chi: p \circ \times^{\mathfrak{ts}} \longrightarrow \wedge^{\mathfrak{s}} \circ p$$

such that

(4.4.3.1)

i) the category $\text{RSCov}(\mathcal{W})$ is fibered over \mathcal{W} via

$$(\pi, \Theta): \text{RSCov}(\mathcal{W}) \longrightarrow \mathcal{W},$$

and the symmetric semi-monoidal structure on $\text{RSCov}(\mathcal{W})$ extends to make $\text{RSCov}(\mathcal{W})$ a fibered symmetric semi-monoidal category over \mathcal{W} .

ii) The pair (p, χ) defines a lax functor (see (1.1.1)(iii)) of symmetric semi-monoidal categories .

iii) For each X in \mathcal{W} , the fiber of $\text{RSCov}(\mathcal{W})$ over X , $\text{RSCov}(X)$, is left directed.

iv) Given an object U_* in $\text{RSCov}(X)$, let $\text{RSCov}(X)_{U_*}$ denote the full subcategory of $\text{RSCov}(X)$ of objects V_* which have a morphism $V_* \rightarrow U_*$. Then $\text{RSCov}(X)_{U_*}$ is left final in $\text{RSCov}(X)$.

We denote the pull-back in $\text{RSCov}(\mathcal{W})$ associated to a morphism $f: Y \rightarrow X$ in \mathcal{W} by $f^{*\mathfrak{ts}}$.

□

The category $\text{RSCov}(\mathcal{W})$ will arise when we apply Friedlander's technique of "rigid hypercovers" to give canonically defined complexes \mathfrak{F} representing the derived object $R\Gamma\mathfrak{F}$ of a complex \mathfrak{F} of sheaves on a Grothendieck site.

(4.4.4) THE FUNCTOR $\check{\mathfrak{F}}$

Let \mathfrak{F} be a graded complex of presheaves on \mathcal{C} , as in (4.2.1.1), with associative, bi-graded product (4.2.1.2); we assume as in (4.2.1) that \mathfrak{F} transforms finite disjoint unions to direct sums. We have the functor (4.2.3.1); we now restrict to the sub-categories $\text{RSCov}(X)$, and take limits to eliminate the dependence on the simplicial objects U_* . This will define a DG tensor functor

$$\check{\mathfrak{F}}: (\mathcal{W}_{\mathbb{Z}}^{\text{op}})^{\otimes, \mathfrak{h}} \longrightarrow \mathbf{C}(\mathcal{A}). \tag{4.4.4.1}$$

(see (4.4.3) and (3.1.6)).

For X in \mathcal{W} , define $\check{\mathfrak{F}}(X)$ as the (filtered) direct limit

$$\check{\mathfrak{F}}(X) = \varinjlim_{U_* \in \text{RSCov}(X)^{\text{op}}} \mathfrak{F}(w(X))(p(U_*))^*. \tag{4.4.4.2}$$

We extend the definition of $\check{\mathfrak{F}}$ to objects of the additive category $\mathcal{W}_{\mathbb{Z}}$ generated by \mathcal{W} by taking direct sums, and to objects of $(\mathcal{W}_{\mathbb{Z}}^{\text{op}})^{\otimes, \mathfrak{h}}$ by setting

$$\check{\mathfrak{F}}(X_1, \dots, X_n) = \check{\mathfrak{F}}(X_1) \otimes \dots \otimes \check{\mathfrak{F}}(X_n).$$

As tensor products commute with filtered direct limits in \mathcal{A} by assumption, we have the natural isomorphism

$$\check{\mathfrak{F}}(X_1, \dots, X_n) \cong \varinjlim_{(U_*^1, \dots, U_*^n)} \mathfrak{F}^{\infty}((p(U_*^1), w(X_1)), \dots, (p(U_*^n), w(X_n)))$$

for X_1, \dots, X_n in \mathcal{W} , where the limit is over

$$(U_*^1, \dots, U_*^n) \in (\text{RSCov}(X_1) \times \dots \times \text{RSCov}(X_n))^{\text{op}}.$$

To define \check{F} on morphisms, proceed as follows: Let $F: n \rightarrow m$ be a map in Ω_0 , let

$$g = (g_1, \dots, g_m): Y_* \longrightarrow \Pi_{\mathcal{W}}(F)(X_*) \tag{4.4.4.3}$$

be a map in \mathcal{W}^m , and let

$$h = \prod_{\sum_j q_j - \sum_j p_j = d} \prod_{i=1}^n h_{p_1, \dots, p_n, i}^{q_1, \dots, q_m}$$

$$h_{p_1, \dots, p_n, i}^{q_1, \dots, q_m}: [p_i] \longrightarrow [q_{F(i)}]$$

be a collection of maps in Δ , giving the map

$$g \otimes h: X_* \longrightarrow Y_*[d]$$

in $(\mathcal{W}_{\mathbb{Z}}^{\text{op}})^{\otimes, \flat}$. We write

$$\begin{aligned} X_* &:= (X_1, \dots, X_n), & Y_* &:= (Y_1, \dots, Y_m), \\ U_*^* &:= (U_*^1, \dots, U_*^n), & V_*^* &:= (V_*^1, \dots, V_*^m), \\ (p(U_*^*), w(X_*)) &:= ((p(U_*^1), w(X_1)), \dots, (p(U_*^n), w(X_n))), \\ (p(V_*^*), w(Y_*)) &:= ((p(V_*^1), w(Y_1)), \dots, (p(V_*^m), w(Y_m))), \end{aligned}$$

etc. We may write

$$\begin{aligned} \Pi_{\mathcal{W}}(F)(X_*) &= (\Pi_{\mathcal{W}}(F)(X_*)_1, \dots, \Pi_{\mathcal{W}}(F)(X_*)_m), \\ \Pi_{\mathcal{W}}(F)(X_*)_j &= X_{i_1^j} \times \dots \times X_{i_{s_j}^j}, \end{aligned}$$

where the sequences $(i_1^j, \dots, i_{s_j}^j)$ are determined by the map F , and satisfy

$$F^{-1}(j) = \{i_1^j, \dots, i_{s_j}^j\}.$$

Take $U_*^i \in \text{RSCov}(X_i)$ for $i = 1, \dots, n$. Then

$$\begin{aligned} \Pi_{\mathcal{C}^+}(F)(p(U_*^*)) &= (\Pi_{\mathcal{C}^+}(F)(p(U_*^*))_1, \dots, \Pi_{\mathcal{C}^+}(F)(p(U_*^*))_m), \\ \Pi_{\mathcal{C}^+}(F)(p(U_*^*))_j &= p(U_*^{i_1^j}) \wedge \dots \wedge p(U_*^{i_{s_j}^j}). \end{aligned}$$

In addition, we have

$$\begin{aligned} \Pi_{\mathbb{Z}}(F)(w(X_*))_j &= w(X_{i_1^j}) + \dots + w(X_{i_{s_j}^j}) \\ &= w(X_{i_1^j} \times \dots \times X_{i_{s_j}^j}) \\ &= w(\Pi_{\mathcal{W}}(F)(X_*)_j) \end{aligned} \tag{4.4.4.4}$$

Fix p_1, \dots, p_n and q_1, \dots, q_m , with $\sum_j q_j - \sum_j p_j = d$, and write

$$\begin{aligned} p_* &= p_1, \dots, p_n, & [p_*] &= ([p_1], \dots, [p_n]), \\ q_* &= q_1, \dots, q_m, & [q_*] &= ([q_1], \dots, [q_m]), \end{aligned}$$

Applying the functor

$$p(U_*^{i_k^j}): \Delta^{\text{op}} \longrightarrow \mathcal{C}^+$$

to $h_{p_*, i_k^j}^{q*}$ for $k = 1, \dots, s^j$ defines the maps

$$\begin{aligned} p(U_*^{i_k^j})(h_{p_*, i_k^j}^{q*}) : p(U_*^{i_k^j})([p_{i_k^j}]) &\longrightarrow p(U_*^{i_k^j})([q_j]); \\ k = 1, \dots, s^j. \end{aligned} \quad (4.4.4.5)$$

in \mathcal{C}^+ . We let

$$\begin{aligned} p(U_*^{i_1^j}([q_j])) \wedge \dots \wedge p(U_*^{i_{s^j}^j}([q_j])) \\ \xrightarrow{p(U_*)(h_{p_*, i_k^j}^{q*})} p(U_*^{i_1^j}([p_{i_1^j}])) \wedge \dots \wedge p(U_*^{i_{s^j}^j}([p_{i_{s^j}^j}])) \end{aligned} \quad (4.4.4.6)$$

be the product over k of the maps (4.4.4.5).

Let $p(U_*)^{j\wedge}$ be the simplicial object

$$p(U_*)^{j\wedge} := p(U_*^{i_1^j}) \wedge^{\mathfrak{s}} \dots \wedge^{\mathfrak{s}} p(U_*^{i_{s^j}^j}),$$

i.e., $p(U_*)^{j\wedge}$ has q -simplices

$$p(U_*)^{j\wedge}([q]) = p(U_*^{i_1^j})([q]) \wedge \dots \wedge p(U_*^{i_{s^j}^j})([q]).$$

Similarly, let $U_*^{j \times \mathfrak{rs}}$ be the object of $\text{RSCov}(\Pi_{\mathcal{W}}(F)(X_1, \dots, X_n)_j)$,

$$U_*^{j \times \mathfrak{rs}} := U_*^{i_1^j} \times^{\mathfrak{rs}} \dots \times^{\mathfrak{rs}} U_*^{i_{s^j}^j}$$

By our assumption (4.4.3.1)(ii), we have the natural map

$$\chi_j : p(U_*^{j \times \mathfrak{rs}}) \longrightarrow p(U_*)^{j\wedge}. \quad (4.4.4.7)$$

We have the maps (4.4.4.3)

$$g_j : Y_j \longrightarrow \Pi_{\mathcal{W}}(F)(X_1, \dots, X_n)_j; \quad j = 1, \dots, m;$$

in \mathcal{W} ; as \mathbb{Z} has only the identity morphisms, we have by (4.4.4.4)

$$w(Y_j) = w(\Pi_{\mathcal{W}}(F)(X_*)_j) = \Pi_{\mathbb{Z}}(F)(w(X_*)_j). \quad (4.4.4.8)$$

We may form the pull-backs

$$g_j^{*\mathfrak{rs}}(U_*^{j \times \mathfrak{rs}}) \in \text{RSCov}(Y_j)$$

for each j .

Take V_*^j in $\text{RSCov}(Y_j)_{g_j^{*\text{rs}}}(U_*^{j \times \text{rs}})$ (4.4.3.1)(iv) for $j = 1, \dots, m$. Since each $\text{RSCov}(Y)$ is left directed by (4.4.3.1)(iii), there are unique maps in $\text{RSCov}(Y_j)$

$$\phi_j: V_*^j \longrightarrow g_j^{*\text{rs}}(U_*^{j \times \text{rs}}); \quad j = 1, \dots, m.$$

giving the maps

$$\Phi_j: V_*^j \longrightarrow U_*^{j \times \text{rs}}$$

over g_j . Applying p gives the maps in \mathcal{C}^+

$$p(\Phi_j): p(V_*^j) \longrightarrow p(U_*^{j \times \text{rs}}). \quad (4.4.4.9)$$

We may then take the composition of the maps (4.4.4.6), (4.4.4.7) and (4.4.4.9):

$$\begin{aligned} p(V_*^j)([q_j]) &\xrightarrow{p(\Phi_j)([q_j])} p(U_*^{j \times \text{rs}})([q_j]) \\ &\xrightarrow{\chi_j([q_j])} p(U_*)^{j \wedge}([q_j]) \\ &= p(U_*^{i_1^j})([q_j]) \wedge \dots \wedge p(U_*^{i_{s_j}^j})([q_j]) \\ &\xrightarrow{p(U_*)^{(h_{p_*, j}^{q_*})}} p(U_*^{i_1^j})([p_{i_1^j}]) \wedge \dots \wedge p(U_*^{i_{s_j}^j})([p_{i_{s_j}^j}]), \end{aligned}$$

which we denote by

$$(U_*^*, V_*^j)(g \otimes h)_{p_*}^{q_*}. \quad (4.4.4.10)$$

The opposite map to the product of the maps (4.4.4.10)

$$(U_*^*, V_*^*) (g \otimes h)_{p_*}^{q_*} := \left[\prod_{j=1}^n (U_*^*, V_*^j)(g \otimes h)_{p_*}^{q_*} \right]^{\text{op}}$$

defines by (4.4.4.8) an element

$$(U_*^*, V_*^*) (g \otimes h)_{p_*}^{q_*} \quad (4.4.4.11)$$

of

$$\text{Hom}_{[\mathcal{C} \times \mathbb{Z}]_\infty} ((p(U_*^*), w(X_*)), (p(V_*^*), w(Y_*)))([p_*]; [q_*])_F.$$

Taking the product over all $p_* = p_1, \dots, p_n$ and $q_* = q_1, \dots, q_m$ of the maps (4.4.4.11) defines the element

$$(U_*^*, V_*^*) (g \otimes h) \quad (4.4.4.12)$$

of

$$\text{Hom}_{[\mathcal{C} \times \mathbb{Z}]_\infty^\otimes} ((p(U_*^*), w(X_*)), (p(V_*^*), w(Y_*))).$$

We may then apply the functor \mathfrak{F}^∞ to (4.4.4.12), giving the morphism

$$\mathfrak{F}^\infty((p(U_*^*), w(X_*))) \xrightarrow{\mathfrak{F}^\infty((U_*^*, V_*^*)(g \otimes h))} \mathfrak{F}^\infty((p(V_*^*), w(Y_*))). \tag{4.4.4.13}$$

in $\mathbf{C}(\mathcal{A})$. By our assumptions (4.4.3.1)(iii), (iv), the natural map

$$\varinjlim_{V_*^j \in \text{RSCov}(Y_j) \xrightarrow{g_j^* \tau^s} (U_*^j \times \tau^s)} \mathfrak{F}^\infty((p(V_*^*), w(Y_*))) \longrightarrow \check{\mathfrak{F}}(Y_1, \dots, Y_m)$$

is an isomorphism. As the map $\mathfrak{F}^\infty((U_*^*, V_*^*)(g \otimes h))$ is clearly natural in U_*^* and V_*^* , we may take the limit in (4.4.4.13), giving

$$\begin{aligned} \check{\mathfrak{F}}(g \otimes h) &: \check{\mathfrak{F}}(X_1, \dots, X_n) \longrightarrow \check{\mathfrak{F}}(Y_1, \dots, Y_m) \\ \check{\mathfrak{F}}(g \otimes h) &:= \varinjlim_{U_*^i \in \text{RSCov}(X_i)} \varinjlim_{V_*^j \in \text{RSCov}(Y_j) \xrightarrow{g_j^* \tau^s} (U_*^j \times \tau^s)} \mathfrak{F}^\infty((U_*^*, V_*^*)(g \otimes h)). \end{aligned} \tag{4.4.4.14}$$

It follows easily from the functoriality of \mathfrak{F}^∞ that the maps (4.4.4.14) do indeed define a functor, natural in $\check{\mathfrak{F}}$, \mathcal{W} and $\text{RSCov}(\mathcal{W})$:

$$\check{\mathfrak{F}} = \check{\mathfrak{F}}_{\text{RSCov}(\mathcal{W})}: (\mathcal{W}_{\mathbb{Z}}^{\text{op}})^{\otimes, \mathfrak{h}} \longrightarrow \mathbf{C}(\mathcal{A}). \tag{4.4.4.15}$$

The verification that (4.4.4.15) is a DG tensor functor follows from the fact that \mathfrak{F}^∞ is a DG tensor functor, together with the assumptions of (4.4.3.1); this verification is straightforward, albeit tedious, and is left to the reader.

(4.4.5) REMARK

If each functor complex $\check{\mathfrak{F}}(q)$ is *bounded below*:

$$\check{\mathfrak{F}}(q) \in \mathbf{C}^+(\text{PreSh}_{\mathcal{A}}(\mathcal{C})); \quad \check{\mathfrak{F}}(q) = \check{\mathfrak{F}}^{N_q}(q) \longrightarrow \dots \longrightarrow \check{\mathfrak{F}}^n(q) \longrightarrow \check{\mathfrak{F}}^{n+1}(q) \longrightarrow \dots ,$$

then it follows from (4.2.4) and the construction of this section that the functor (4.4.4.15) takes values in $\mathbf{C}^+(\mathcal{A})$. □

Chapter 5

Grothendieck topologies and hypercovers

In this chapter, we review the construction of the coskeleton functors, and the notion of a hypercover from [SGA-4].

5.1. Simplicial objects and the coskeleton

(5.1.1) GROTHENDIECK TOPOLOGIES

Let \mathfrak{S} be a Grothendieck topology on a full subcategory \mathcal{C} of \mathbf{Sch}_S ; we assume that \mathcal{C} is a symmetric monoidal sub-category of \mathbf{Sch}_S (with product \times_S), and is closed under finite fiber products, arbitrary disjoint union, and taking open subschemes and closed subschemes. Finally, we assume that the topology \mathfrak{S} is finer than the Zariski topology.

We have the category \mathcal{C}/\mathcal{C} of morphisms in \mathcal{C} , which is a category fibered and cofibered over \mathcal{C} via the functor

$$\begin{aligned} \text{rng}: \mathcal{C}/\mathcal{C} &\longrightarrow \mathcal{C} \\ \text{rng}(X \longrightarrow Y) &= Y. \end{aligned}$$

(see §4.3 for details on fibered categories). We also have the functor

$$\begin{aligned} \text{dom}: \mathcal{C}/\mathcal{C} &\longrightarrow \mathcal{C} \\ \text{dom}(X \longrightarrow Y) &= X. \end{aligned}$$

The topology \mathfrak{S} has as data the full fibered subcategory $\mathfrak{S}(\mathcal{C})$ of \mathcal{C}/\mathcal{C} , and the full fibered subcategory $\text{Cov}_{\mathfrak{S}}(\mathcal{C})$ of $\mathfrak{S}(\mathcal{C})$. The fiber $\mathfrak{S}(X)$ of $\mathfrak{S}(\mathcal{C})$ over $X \in \mathcal{C}$ is the category of “opens of X ” $U \rightarrow X$ for the topology \mathfrak{S} , and the fiber $\text{Cov}_{\mathfrak{S}}(X)$ of $\text{Cov}_{\mathfrak{S}}(\mathcal{C})$ is the category of covers $U \rightarrow X$ of X for the topology \mathfrak{S} . These categories satisfy various axioms; we refer the reader to ([SGA-4], II) for a complete list.

(5.1.2) SIMPLICIAL OBJECTS OF \mathcal{C}

For an integer $n \geq 0$, we have the full subcategory $\Delta^{\leq n\text{op}}$ of Δ^{op} with objects $[0], \dots, [n]$, and the inclusion functor

$$i_n: \Delta^{\leq n\text{op}} \longrightarrow \Delta^{\text{op}} \tag{5.1.2.1}$$

We have the category $\mathcal{C}(\Delta^{\text{op}})$ of simplicial objects of \mathcal{C} and the category $\mathcal{C}(\Delta^{\leq n\text{op}})$ of n -truncated simplicial objects of \mathcal{C} . For $X \in \mathcal{C}$, we have the category of objects over X , \mathcal{C}/X , the category of simplicial objects in \mathcal{C}/X , $\mathcal{C}(\Delta^{\text{op}})/X$, and the category of n -truncated simplicial objects in \mathcal{C}/X , $\mathcal{C}(\Delta^{\leq n\text{op}})/X$. An object of $\mathcal{C}(\Delta^{\text{op}})/X$ or $\mathcal{C}(\Delta^{\leq n\text{op}})/X$ is called

augmented over X . The category $\mathcal{C}(\Delta^{\text{op}})/\mathcal{C}$ is the category of pairs (X_*, X) with X in \mathcal{C} and $X_* \rightarrow X$ augmented over X ; a morphism

$$(f_*, f): (X_* \rightarrow X) \rightarrow (Y_* \rightarrow Y)$$

is a morphism $f_*: X_* \rightarrow Y_*$ in $\mathcal{C}(\Delta^{\text{op}})$ such that $f_n: X_n \rightarrow Y_n$ is a map over f for each n .

Given objects

$$Y_* \rightarrow X, \quad Z_* \rightarrow X$$

in $\mathcal{C}(\Delta^{\text{op}})/X$, we may form the fiber product (as functors)

$$(Y_* \times_X Z_*)_*; \quad (Y_* \times_X Z_*)_n = Y_n \times_X Z_n;$$

this gives the functor

$$\text{rng}: \mathcal{C}(\Delta^{\text{op}})/\mathcal{C} \rightarrow \mathcal{C}$$

the structure of a fibered category over \mathcal{C} ; the truncated version

$$\text{rng}: \mathcal{C}(\Delta^{\leq n \text{op}})/\mathcal{C} \rightarrow \mathcal{C}$$

is defined similarly.

The functor (5.1.2.1) induces the restriction functor

$$i_n^*: \mathcal{C}(\Delta^{\text{op}}) \rightarrow \mathcal{C}(\Delta^{\leq n \text{op}})$$

the fibered version

$$i_{n,\mathcal{C}}^*: \mathcal{C}(\Delta^{\text{op}})/\mathcal{C} \rightarrow \mathcal{C}(\Delta^{\leq n \text{op}})/\mathcal{C}$$

and the fiber over X

$$i_{n,X}^*: \mathcal{C}(\Delta^{\text{op}})/X \rightarrow \mathcal{C}(\Delta^{\leq n \text{op}})/X. \quad (5.1.2.2)$$

We often denote the simplicial object $i_{n,X}^* Y_*$ by $Y_*^{\leq n}$.

(5.1.3) THE COSKELETON

For integers $m, n \geq 0$, we let $\Delta^{\leq n}/[m]$ be the full subcategory of the category of morphisms to $[m]$ in Δ , $\Delta/[m]$, with objects

$$[k] \rightarrow [m]$$

where $0 \leq k \leq n$. Take $Y_* \in \mathcal{C}(\Delta^{\leq n \text{op}})/X$, giving the functor

$$Y_* \circ \text{dom}: (\Delta^{\leq n}/[m])^{\text{op}} \rightarrow \mathcal{C}/X,$$

$$[k] \rightarrow [m] \mapsto Y_k.$$

Let

$$i_{n,*}^X(Y_*) \in \mathcal{C}(\Delta^{\text{op}})/X$$

be given by

$$i_{n,*}^X(Y_*)_m = \lim_{(\Delta^{\leq n}/[m])^{\text{op}}} Y_* \circ \text{dom};$$

for a morphism $h: [m'] \rightarrow [m]$ in Δ , the functor

$$h_*: (\Delta^{\leq n}/[m'])^{\text{op}} \longrightarrow (\Delta^{\leq n}/[m])^{\text{op}}$$

defines the morphism over X

$$i_{n,*}^X(Y_*)(h): i_{n,*}^X(Y_*)_m \longrightarrow i_{n,*}^X(Y_*)_{m'}.$$

This defines the functor

$$i_{n,*}^X: \mathcal{C}(\Delta^{\leq n \text{op}})/X \longrightarrow \mathcal{C}(\Delta^{\text{op}})/X \quad (5.1.3.1)$$

which is right adjoint to the functor (5.1.2.2); we let

$$\text{cosk}_n^X: \mathcal{C}(\Delta^{\text{op}})/X \longrightarrow \mathcal{C}(\Delta^{\text{op}})/X \quad (5.1.3.2)$$

be the composition $i_{n,*}^X \circ i_{n,X}^*$.

For $X_* \rightarrow X$, the identity on $X_*^{\leq n}$ thus gives the canonical map

$$\nu_*^{X_*,n}: X_* \longrightarrow \text{cosk}_n^X(X_*)_* \quad (5.1.3.3)$$

In addition, as the category $\Delta^{\leq n}/[m]$ has the initial object $\text{id}_{[m]}$ in case $m \leq n$, the map $\nu_m^{X_*,n}$ is an isomorphism for $m \leq n$.

(5.1.4)

We now extend the functors $i_{n,X}^*$ and cosk_n^X to functors

$$\begin{aligned} i_{n,\mathcal{C}}^*: \mathcal{C}(\Delta^{\text{op}})/\mathcal{C} &\longrightarrow \mathcal{C}(\Delta^{\leq n \text{op}})/\mathcal{C} \\ \text{cosk}_n^{\mathcal{C}}: \mathcal{C}(\Delta^{\text{op}})/\mathcal{C} &\longrightarrow \mathcal{C}(\Delta^{\text{op}})/\mathcal{C}. \end{aligned} \quad (5.1.4.1)$$

The definition of $i_{n,\mathcal{C}}^*$ is obvious; for $\text{cosk}_n^{\mathcal{C}}$, we note that a morphism

$$(g_*, f): (Y_* \longrightarrow X) \longrightarrow (Y'_* \longrightarrow X')$$

in $\mathcal{C}(\Delta^{\leq n \text{op}})$ over $f: X \rightarrow X'$ gives the natural transformation

$$g_*: Y_* \circ \text{dom} \longrightarrow Y'_* \circ \text{dom}$$

This then defines the morphism

$$i_{n,*}^f(g_*): i_{n,*}^X(Y_*) \longrightarrow i_{n,*}^{X'}(Y'_*),$$

which defines the functor

$$i_{n*}^{\mathcal{C}}: \mathcal{C}(\Delta^{\leq n \text{op}})/\mathcal{C}(\Delta^{\text{op}}) \longrightarrow \mathcal{C}(\Delta^{\text{op}})/\mathcal{C}(\Delta^{\text{op}}) \quad (5.1.4.2)$$

right adjoint to $i_{n,\mathcal{C}}^*$. We set

$$\text{cosk}_n^{\mathcal{C}} = i_{n*}^{\mathcal{C}} \circ i_{n,\mathcal{C}}^*. \quad (5.1.4.3)$$

We often write $\text{cosk}_n^f(f_*)$ or $i_{n*}^f(f_*)$ for $\text{cosk}_n^{\mathcal{C}}(f_*)$ or $i_{n*}^{\mathcal{C}}(f_*)$ applied to a morphism f_* over f .

(5.1.5) REMARKS

i) Let

$$f: Y \longrightarrow X$$

be a map in \mathcal{C} . We have the functor

$$f_*: \mathcal{C}(\Delta^{\text{op}})/Y \longrightarrow \mathcal{C}(\Delta^{\text{op}})/X$$

gotten by composing the augmentation over Y with f . It follows from the adjoint property

$$\text{Hom}_{\mathcal{C}(\Delta^{\text{op}})/Y}(U_*, f^*V_*) \cong \text{Hom}_{\mathcal{C}(\Delta^{\text{op}})/X}(f_*U_*, V_*)$$

for $U_* \rightarrow Y$ in $\mathcal{C}(\Delta^{\text{op}})/Y$, $V_* \rightarrow X$ in $\mathcal{C}(\Delta^{\text{op}})/X$, that the projection

$$p_1: f^*V_* = V_* \times_X Y \longrightarrow V_*$$

induces the isomorphism

$$\text{cosk}_n^Y \circ f^*(-) \longrightarrow f^*(\text{cosk}_n^X(-)).$$

ii) For augmented simplicial schemes

$$U_* \longrightarrow X; \quad V_* \longrightarrow X,$$

we may form the diagonal simplicial scheme

$$\begin{aligned} \text{diag}[U_* \times_X V_*]_* &\longrightarrow X \\ \text{diag}[U_* \times_X V_*]_n &= U_n \times_X V_n, \end{aligned}$$

which we denote by $U_* \times_X^5 V_*$. The operation \times_X^5 gives products in $\mathcal{C}(\Delta^{\text{op}})/X$. Arguing as in (i), we see that the natural transformation

$$\text{d.c.}_{n,X}: \text{cosk}_n^X \circ [- \times_X^5 -] \longrightarrow \text{cosk}_n^X(-) \times_X^5 \text{cosk}_n^X(-)$$

induced by the projections

$$p_1: U_* \times_X^5 V_* \longrightarrow U_*; \quad p_2: U_* \times_X^5 V_* \longrightarrow V_*$$

is an isomorphism.

iii) Let $U_* \longrightarrow X$ and $V_* \longrightarrow Y$ be in $\mathcal{C}(\Delta^{\text{op}})/\mathcal{C}$. We may form the bi-simplicial object $U_* \times_S V_*$, with

$$(U_* \times_S V_*)_{nm} = U_n \times_S V_m,$$

and the diagonal simplicial object

$$\begin{aligned} \text{diag}[U_* \times_S V_*]_* &, \\ \text{diag}[U_* \times_S V_*]_k &= U_k \times_S V_k. \end{aligned}$$

We write $U_* \times^5 V_*$ for $\text{diag}[U_* \times_S V_*]_*$. Arguing as in (i), we see that the natural transformation

$$\text{d.c.}_{n,X,Y}: \text{cosk}_n^X(-) \times^5 \text{cosk}_n^Y(-) \longrightarrow \text{cosk}_n^{X \times_S Y} \circ [- \times^5 -]$$

induced by the projections is an isomorphism. The operation \times^5 defines a product in $\mathcal{C}(\Delta^{\text{op}})/\mathcal{C}$, hence gives $\mathcal{C}(\Delta^{\text{op}})/\mathcal{C}$ the structure of a symmetric semi-monoidal category, and gives the functor

$$\text{rng}: \mathcal{C}(\Delta^{\text{op}})/\mathcal{C} \longrightarrow \mathcal{C}$$

the structure of fibered functor of symmetric semi-monoidal categories.

iv) For U_* in $\mathcal{C}(\Delta^{\text{op}})/X$, we write $c_n^X(U_*)$ for $\text{cosk}_{n-1}^X(U_*)_n$; for V_* in $\mathcal{C}(\Delta^{\text{op}})/Y$ and a map $f_*: V_* \rightarrow U_*$ over a map $f: Y \rightarrow X$, we write

$$c_n^f(f_*): c_n^Y(V_*) \longrightarrow c_n^X(U_*)$$

for the map

$$\text{cosk}_{n-1}^f(f_*)_n: \text{cosk}_{n-1}^Y(V_*)_n \longrightarrow \text{cosk}_{n-1}^X(U_*)_n.$$

We write

$$\nu_n^{U_*}: U_n \longrightarrow c_n^X(U_*)$$

for the map $\nu_n^{U_*, n-1}$ (5.1.3.3). We set $c_0^X(U_*) = X$, $c_0^f(f_*) = f$ and let

$$\nu_0^{U_*}: U_0 \longrightarrow X$$

be the structure morphism. □

5.2. Hypercovers

(5.2.1) DEFINITION

i) Let X be in $\mathcal{C}(\Delta^{\text{op}})$. A *hypercov*er of X for the topology \mathfrak{S} is an augmented simplicial object $V_* \rightarrow X$ such that for each $n = 0, 1, 2, \dots$, the natural map (5.1.5)(iv)

$$\nu_n^{V_*}: V_n \longrightarrow c_n^X(V_*)$$

is in $\text{Cov}_{\mathfrak{S}}(c_n^X(V_*))$.

ii) The category $\text{HCov}_{\mathfrak{S}}(\mathcal{C})$ of hypercovers is the full sub-category of $\mathcal{C}(\Delta^{\text{op}})/\mathcal{C}$ with fiber over X having objects the hypercovers of X . For X in \mathcal{C} , we denote the fiber of $\text{HCov}_{\mathfrak{S}}(\mathcal{C})$ over X by $\text{HCov}_{\mathfrak{S}}(X)$. \square

(5.2.2) REMARK

i) It follows from (5.1.5)(i) that, if $U_* \rightarrow X$ is a hypercover of X , and $f: Y \rightarrow X$ is a simplicial map, then $f^*U_* \rightarrow X$ is a hypercover of X . This gives $\text{HCov}_{\mathfrak{S}}(\mathcal{C})$ the structure of a fibered subcategory of $\mathcal{C}(\Delta^{\text{op}})/\mathcal{C}$. If $U_* \rightarrow X$ and $V_* \rightarrow X$ are hypercovers of X , then from (5.1.5)(ii), the fiber product $U_* \times_X^s V_*$ is a hypercover of X , giving products in $\text{HCov}_{\mathfrak{S}}(X)$; if $U_* \rightarrow X$ is a hypercover of X and $V_* \rightarrow Y$ is a hypercover of Y , then from (5.1.5)(iii), the product $U_* \times^s V_*$ is a hypercover of $X \times_S Y$. This makes $\text{HCov}_{\mathfrak{S}}(\mathcal{C})$ a symmetric semi-monoidal subcategory of $\mathcal{C}(\Delta^{\text{op}})/\mathcal{C}$.

ii) Let $U_* \rightarrow X$ be a hypercover of X for the topology \mathfrak{S} . Then each map

$$U_n \longrightarrow X$$

is in $\text{Cov}_{\mathfrak{S}}(X)$. Indeed, $U_0 \rightarrow X$ is in $\text{Cov}_{\mathfrak{S}}(X)$ by definition; if we assume by induction that $U_0 \rightarrow X, \dots, U_{n-1} \rightarrow X$ are in $\text{Cov}_{\mathfrak{S}}(X)$, then, as $\text{Cov}_{\mathfrak{S}}(X)$ is closed under finite inverse limits, it follows that

$$c_n^X(U_*) \longrightarrow X$$

is in $\text{Cov}_{\mathfrak{S}}(X)$. As $U_n \rightarrow c_n^X(U_*)$ is in $\text{Cov}_{\mathfrak{S}}(X)$ by definition, it follows that $U_n \rightarrow X$ is in $\text{Cov}_{\mathfrak{S}}(X)$. \square

(5.2.3)

For a category \mathcal{A} , and for $X \in \mathcal{C}$, we let $\text{PreSh}_X^{\mathfrak{S}}(\mathcal{A})$ denote the category of presheaves on X for the topology \mathfrak{S} , with values in \mathcal{A} , and $\text{Sh}_X^{\mathfrak{S}}(\mathcal{A})$ the category of sheaves. Let U_* be in

$$\mathfrak{S}(X)(\Delta^{\text{op}}) \subset \mathcal{C}(\Delta^{\text{op}})/X$$

If \mathcal{F} is in $\text{PreSh}_X^{\mathfrak{S}}(\mathcal{A})$, then we may apply \mathcal{F} to U_* , forming the augmented cosimplicial object

$$\mathcal{F}(X) \longrightarrow \mathcal{F}(U_*) \in \mathcal{F}(X) \setminus \mathcal{A}(\Delta).$$

If \mathcal{A} is an additive category, we may form the associated augmented complex

$$\mathcal{F}(X) \longrightarrow \mathcal{F}(U_*)^* \in \mathcal{F}(X) \setminus \mathbf{C}^+(\mathcal{A})$$

with

$$\mathcal{F}(U_*)^n = \mathcal{F}(U_n)$$

and differential the usual alternating sum.

(5.2.4) LEMMA

Let $U_* \rightarrow X$ be in $\mathrm{HCov}_{\mathfrak{S}}(X)$, and let \mathcal{I} be a sheaf of abelian groups on \mathfrak{S}_X , with values in \mathcal{A} . Then the augmentation induces an isomorphism

$$\mathcal{I}(X) \longrightarrow H^0(\mathcal{I}(U_*)^*).$$

If \mathcal{I} is an injective sheaf, then

$$H^p(\mathcal{I}(U_*)^*) = 0$$

for all $p > 0$.

Proof. This follows from ([SGA-4], V, Théorème 7.3.2(3)). □

Chapter 6

Rigid topologies and rigid hypercovers

In this chapter we give an axiomatic treatment of Friedlander's construction of a rigid Grothendieck topology and rigid hypercovers. This will lead to the construction of categories satisfying the properties required of $\text{RSCov}(\mathcal{W})$ in (4.4.3.1). In addition, the choice of the proper functor

$$p: \text{RSCov}(\mathcal{W}) \longrightarrow \mathcal{C}^+(\Delta^{\text{op}})/\mathcal{W}$$

in (4.4.3.1) will give functorial models for the derived functor $R\Gamma_X \mathcal{F}$ of a sheaf \mathcal{F} , as well as the derived functor of the functor of sections with support in a closed subset $R\Gamma_X^W \mathcal{F}$.

6.1. Some categorical preliminaries

(6.1.1)

A functor

$$p: \mathcal{A} \longrightarrow \mathcal{B}$$

is called *left dense* if, for each object b of \mathcal{B} , there is an object a of \mathcal{A} and a morphism

$$p(a) \longrightarrow b$$

in \mathcal{B} .

(6.1.2) FUNCTORS OF CATEGORIES WITH FIBER PRODUCTS

For a category \mathcal{A} , we have the category of maps in \mathcal{A} , \mathcal{A}/\mathcal{A} and the category of commutative squares in \mathcal{A} , $\text{Sq}\mathcal{A}$ with objects commutative squares

$$A_{**} := \begin{array}{ccc} A_{11} & \longrightarrow & A_{01} \\ \downarrow & & \downarrow \\ A_{10} & \longrightarrow & A_{00} \end{array}$$

and maps $f_{**}: A_{**} \rightarrow B_{**}$ being collections $f_{ij}: A_{ij} \rightarrow B_{ij}$ such that the obvious diagram is commutative. Sending A_{**} to A_{00} gives the functor

$$\pi_{00}: \text{Sq}\mathcal{A} \longrightarrow \mathcal{A}.$$

Deleting A_{11} gives the functor

$$\pi^{11}: \text{Sq}\mathcal{A} \longrightarrow \mathcal{A}/\mathcal{A} \times_{\mathcal{A}} \mathcal{A}/\mathcal{A}.$$

Sending \mathcal{A} to $\text{Sq}\mathcal{A}$ gives a functor Sq from \mathbf{cat} to \mathbf{cat} , and π_{00} is a natural transformation from Sq to id .

If \mathcal{A} has fiber products, sending a pair $(Y \rightarrow X, Z \rightarrow X)$ to the diagram

$$\begin{array}{ccc} Y \times_X Z & \xrightarrow{p_1} & Y \\ p_2 \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

gives the functor

$$- \times_{-} -: \mathcal{A}/\mathcal{A} \times_{\mathcal{A}} \mathcal{A}/\mathcal{A} \longrightarrow \text{Sq}\mathcal{A}.$$

over the identity on $\mathcal{A}/\mathcal{A} \times_{\mathcal{A}} \mathcal{A}/\mathcal{A}$

(6.1.3) LAX FUNCTORS

Let \mathcal{A} and \mathcal{B} be categories with fiber products $\times^{\mathcal{A}}$ and $\times^{\mathcal{B}}$. A *lax functor of categories with fiber products* is a pair (p, Θ) , where

$$p: \mathcal{A} \longrightarrow \mathcal{B}$$

is a functor, and Θ is a natural transformation

$$\Theta: \text{Sq}(p) \circ (- \times_{-}^{\mathcal{A}} -) \longrightarrow (- \times_{-}^{\mathcal{B}} -) \circ (p/p \times_p p/p)$$

over $p/p \times_p p/p$ (i.e. $\Theta_{ij} = \text{id}$ for $(i, j) \neq (1, 1)$) such that Θ is associative: given $Y \rightarrow X$, $Z \rightarrow X$, $Z \rightarrow X'$ and $W \rightarrow X'$, we have $Y \times_X Z \rightarrow X'$ and $Z \times_{X'} W \rightarrow X$. Then

$$\begin{aligned} & \alpha_{p(Y), p(Z), p(W)}^{\mathcal{B}} \circ [\Theta_{11}((Y \rightarrow X), (Z \rightarrow X)) \times \text{id}_{p(W)}] \circ \Theta_{11}((Y \times_X Z \rightarrow X'), (W \rightarrow X')) \\ &= [\text{id}_{p(Y)} \times \Theta_{11}((Z \rightarrow X'), (W \rightarrow X'))] \circ \Theta_{11}((Y \rightarrow X), (Z \times_{X'} W \rightarrow X)) \circ p(\alpha_{Y, Z, W}^{\mathcal{A}}), \end{aligned}$$

where $\alpha^{\mathcal{A}}$ and $\alpha^{\mathcal{B}}$ are the respective associativity isomorphisms. We require in addition that p respects the canonical symmetry isomorphisms:

$$\begin{aligned} p(\tau_{Y \rightarrow X, Z \rightarrow X}: Y \times_X Z \rightarrow Z \times_X Y) &= \tau_{p(Y) \rightarrow p(X), p(Z) \rightarrow p(X)} \\ \tau_{p(Y) \rightarrow p(X), p(Z) \rightarrow p(X)}: p(Y) \times_{p(X)} p(Z) &\rightarrow p(Z) \times_{p(X)} p(Y). \end{aligned}$$

We define a lax functor of categories with products similarly by taking X to be the final object in \mathcal{A} .

6.2. Rigidifications

(6.2.1)

For X in \mathcal{C} , let $C(X)$ denote the set of connected components of X (for the Zariski topology); for $X_j \in C(X)$, we let

$$i_{X_j}: X_j \longrightarrow X$$

denote the canonical map. We let \mathcal{C}_{con} denote the full subcategory of \mathcal{C} with objects the connected schemes in \mathcal{C} ; we assume that, if X is in \mathcal{C} , and X_0 is a connected component of X , then X_0 is in \mathcal{C} .

(6.2.2)

If we have a functor

$$F: \mathcal{A} \longrightarrow \mathcal{B}$$

such that, for each object a of \mathcal{A} , the fiber $F^{-1}(a)$ is a small category, and if \mathcal{S} is a set of objects of \mathcal{A} , we let $F^{-1}(\mathcal{S})$ denote the set of objects b of \mathcal{B} with $F(b) \in \mathcal{S}$.

Suppose we have a category $F: \mathcal{B} \rightarrow \mathcal{C}_{\text{con}}$ which is cofibered over \mathcal{C}_{con} , and such that the fiber $F^{-1}(X)$ is a small category with only identity morphisms for all X in \mathcal{C}_{con} . For each map $j: Y \rightarrow X$ in \mathcal{C}_{con} , we have the corresponding map of sets

$$j_*: F^{-1}(Y) \longrightarrow F^{-1}(X).$$

We say that F respects disjoint unions if, for each finite decomposition

$$X = \coprod_{j=1}^N Y_j$$

of $X \in \mathcal{C}_{\text{con}}$ as a disjoint union of connected locally closed subsets

$$j_i: Y_j \longrightarrow X,$$

the map

$$\cup_{i=1}^N j_{i*}: \coprod_{j=1}^N F^{-1}(Y_j) \longrightarrow F^{-1}(X).$$

is an isomorphism of sets.

(6.2.3) DEFINITION

Let \mathfrak{S} be a Grothendieck topology on \mathcal{C} , as in (5.1.1). A rigidification $\mathfrak{r}\mathfrak{S}$ of \mathfrak{S} consists of

i) a category $\mathfrak{r}\mathcal{C}$ with fiber products $\times^{\mathfrak{r}}$, and a faithful lax functor of categories with fiber products

$$(q, \Theta): \mathfrak{r}\mathcal{C} \longrightarrow \mathcal{C}$$

ii) a full fibered sub-category $\mathfrak{r}\mathfrak{S}(\mathfrak{r}\mathcal{C})$ of $\mathfrak{r}\mathcal{C}/\mathfrak{r}\mathcal{C}$ with fiber $\mathfrak{r}\mathfrak{S}(X')$ over X'

such that

- a) For X' in $\mathfrak{r}\mathcal{C}$, $q(X')$ is in \mathcal{C}_{con} , and the category $\mathfrak{r}\mathcal{C}$ is cofibered over \mathcal{C}_{con} .
- b) For each X' in $\mathfrak{r}\mathcal{C}$, the sub-category $\mathfrak{r}\mathfrak{S}(X')$ of $\mathfrak{r}\mathcal{C}/X'$ is closed under $\times_{X'}^{\mathfrak{r}}$.
- c) For each X in \mathcal{C}_{con} , the category $q^{-1}(X)$ is a small category with only identity morphisms. The functor q is compatible with disjoint unions. We denote the category

$$\prod_{X' \in q^{-1}(C(X))} \mathfrak{r}\mathfrak{S}(X').$$

by $\text{Cov}_{\mathfrak{r}\mathfrak{S}}(X)$.

- d) For each X and Y in \mathcal{C} , the structure morphisms

$$p_X: X \longrightarrow S; \quad p_Y: Y \longrightarrow S$$

and projections

$$p_1: X \times_S Y \longrightarrow X; \quad p_2: X \times_S Y \longrightarrow Y,$$

define the commutative diagram of sets

$$\begin{array}{ccc} q^{-1}(C(X \times_S Y)) & \xrightarrow{p_{1*}} & q^{-1}(C(X)) \\ p_{2*} \downarrow & & \downarrow \\ q^{-1}(C(Y)) & \xrightarrow{p_{Y*}} & q^{-1}(C(S)). \end{array}$$

Then the induced map

$$q^{-1}(C(X \times_S Y)) \longrightarrow q^{-1}(C(X)) \times_{q^{-1}(C(S))} q^{-1}(C(Y))$$

is an isomorphism.

- e) A morphism

$$j: U \longrightarrow X'$$

in $\mathfrak{r}\mathcal{C}$ is in $\mathfrak{r}\mathfrak{S}(X')$ if and only if

$$q(j): q(U) \longrightarrow q(X')$$

is in $\mathfrak{S}(q(X'))$.

- f) Sending an object

$$\prod_{X' \in q^{-1}(C(X))} j'_{X'}: U'_{X'} \longrightarrow X'$$

of $\text{Cov}_{\mathfrak{r}\mathfrak{S}}(X)$ to

$$\cup_{X' \in q^{-1}(C(X))} i_{q(X')} \circ q(j'_{X'}): \coprod_{X' \in q^{-1}(C(X))} q(U'_{X'}) \rightarrow X$$

sends $\text{Cov}_{\mathfrak{r}\mathfrak{S}}(X)$ into $\text{Cov}_{\mathfrak{S}}(X)$; we denote the resulting functor by

$$q_X: \text{Cov}_{\mathfrak{r}\mathfrak{S}}(X) \longrightarrow \text{Cov}_{\mathfrak{S}}(X).$$

g) for each X' in $\mathfrak{r}\mathcal{C}$, the category $\mathfrak{r}\mathfrak{S}(X')$ is left-directed.

h) For each X in \mathcal{C} , the functor

$$q_X: \text{Cov}_{\mathfrak{r}\mathfrak{S}}(X) \longrightarrow \text{Cov}_{\mathfrak{S}}(X)$$

is left dense. □

(6.2.4) REMARKS

i) It follows from (b) that $\mathfrak{r}\mathcal{C}$ is strictly cofibered over \mathcal{C}_{con} , i.e., we have the identity of co-base change functors

$$(f \circ g)_* = f_* \circ g_*$$

for composable morphisms f and g in \mathcal{C}_{con} . Similarly, as $\mathfrak{r}\mathcal{C}$ is cofibered over \mathcal{C}_{con} , (b) implies that the functor q is faithful. Thus, for each morphism $f: Y \rightarrow X$ in \mathcal{C}_{con} , and each $Y' \in q^{-1}(Y)$, we have the canonical lifting of f to

$$f_{Y'}: Y' \longrightarrow f_*(Y').$$

The map $f_{Y'}$ is the unique lifting of f with domain Y' , hence

$$f_{g_*(Z')} \circ g_{Z'} = (f \circ g)_{Z'}$$

for morphisms $f: Y \rightarrow X$, $g: Z \rightarrow Y$ in \mathcal{C}_{con} , and $Z' \in q^{-1}(Z)$. If we have a connected scheme W_0 in \mathcal{C} , a map

$$i_{W_0}: W_0 \longrightarrow W$$

and a map $f: W \rightarrow Y$ in \mathcal{C} , there is a unique connected component

$$i_{Y_0}: Y_0 \longrightarrow Y$$

and a map

$$f_0: W_0 \longrightarrow Y_0$$

making the diagram

$$\begin{array}{ccc} W_0 & \xrightarrow{i_{W_0}} & W \\ f_0 \downarrow & & \downarrow f \\ Y_0 & \xrightarrow{i_{Y_0}} & Y \end{array}$$

commute. If W'_0 is in $q^{-1}(W_0)$, we define $f_*(W'_0)$ to be $f_{0*}(W_0)$. This defines the push-forward

$$f_*: q^{-1}(C(W)) \longrightarrow q^{-1}(C(Y)) \quad (1)$$

for each map $f: W \rightarrow Y$ in \mathcal{C} . For X in \mathcal{C} , we let $\mathfrak{r}(X)$ denote the set $q^{-1}(C(X))$, and we write the morphism (1) as

$$\mathfrak{r}(f): \mathfrak{r}(W) \longrightarrow \mathfrak{r}(Y).$$

This defines the functor

$$\mathfrak{r}: \mathcal{C} \longrightarrow \mathbf{Sets}.$$

ii) Suppose we have a cartesian diagram in $\mathfrak{r}\mathcal{C}$:

$$\begin{array}{ccc} X' \times_{Z'}^{\mathfrak{r}} Y' & \xrightarrow{p_2^{\mathfrak{r}}} & Y' \\ p_1^{\mathfrak{r}} \downarrow & & \downarrow f \\ X' & \xrightarrow{g} & Z' \end{array}$$

giving the diagram in \mathcal{C} :

$$\begin{array}{ccc} q(X' \times_{Z'}^{\mathfrak{r}} Y') & & \\ \searrow \Theta & & \\ q(X') \times_{q(Z')} q(Y') & \xrightarrow{p_2} & q(Y') \\ p_1 \downarrow & & \downarrow q(f) \\ q(X') & \xrightarrow{q(g)} & q(Z'). \end{array}$$

with

$$q(p_1^{\mathfrak{r}}) = p_1 \circ \Theta; \quad q(p_2^{\mathfrak{r}}) = p_2 \circ \Theta.$$

Then, by the uniqueness mentioned in (i), we have

$$\begin{aligned} X' &= (p_1 \circ \Theta)_*(X' \times_{Z'}^{\mathfrak{r}} Y') \\ Y' &= (p_2 \circ \Theta)_*(X' \times_{Z'}^{\mathfrak{r}} Y') \end{aligned}$$

and

$$p_i^{\mathfrak{r}} = (p_i \circ \Theta)_{X' \times_{Z'}^{\mathfrak{r}} Y'}; \quad i = 1, 2.$$

iii) Let X and Y be in \mathcal{C} . For a pair

$$(X', Y') \in \mathfrak{r}(X) \times_{\mathfrak{r}(S)} \mathfrak{r}(Y)$$

we may form the fiber product

$$X' \times_{S'}^{\mathfrak{r}} Y'; \quad S' = p_{X*}(X') = p_{Y*}(Y'),$$

where p_X and p_Y are the structure morphisms. The natural transformation Θ sends $q(X' \times_{S'}^{\mathfrak{r}} Y')$ into a connected component of $X \times_S Y$; applying the co-base change, we have the canonical morphism

$$\theta_{X',Y'}: X' \times_{S'}^{\mathfrak{r}} Y' \longrightarrow W' := \Theta_*(X' \times_{S'}^{\mathfrak{r}} Y').$$

This determines the map

$$\times^{\mathfrak{r}}: \mathfrak{r}(X) \times_{\mathfrak{r}(S)} \mathfrak{r}(Y) \longrightarrow \mathfrak{r}(X \times_S Y).$$

By (d) and (i) above, $\times^{\mathfrak{r}}$ is an isomorphism.

iv) From (g), it follows that $\text{Cov}_{\mathfrak{r}\mathfrak{S}}(X)$ is left-directed for each X in \mathcal{C}

v) It follows from (e) and the faithfulness of q that: For each $j: U' \rightarrow X'$ in $\mathfrak{r}\mathfrak{S}(X')$, the functor

$$j \circ (-): \mathfrak{r}\mathfrak{S}(U') \longrightarrow \mathfrak{r}\mathcal{C}/j$$

sends $\mathfrak{r}\mathfrak{S}(U')$ isomorphically onto $\mathfrak{r}\mathfrak{S}(X')/j$. Let U and X be in \mathcal{C}_{con} , and let $j: U \rightarrow X$ be a map in $\mathfrak{S}(X)$. Then, for each $U' \in q^{-1}(U)$, the map

$$j_*: \mathfrak{r}\mathcal{C}/U' \longrightarrow \mathfrak{r}\mathcal{C}/j_*U'$$

sends $\mathfrak{r}\mathfrak{S}(U')$ into $\mathfrak{r}\mathfrak{S}(j_*U')$.

vi) Let U be in $\text{Cov}_{\mathfrak{r}\mathfrak{S}}(X)$. We write the morphism $q_X(U)$ as

$$q_U: q^{\perp} \text{dom}(U) \longrightarrow X.$$

vii) As a guide to the subsequent material, we give an example of a rigidification: Let $\mathfrak{S} = \text{Zar}$ be the Zariski topology on the category \mathcal{C} of disjoint unions of schemes of finite type over a field k . Fixing an algebraic closure \bar{k} of k , we have, for each $X \in \mathcal{C}$, the set of \bar{k} -points $X(\bar{k})$. Let $\mathfrak{r}\mathcal{C}$ be the category of pairs (X, x) where X in \mathcal{C} is connected, and x is in $X(\bar{k})$; a morphism

$$f: (X, x) \longrightarrow (Y, y)$$

is a morphism $f: X \rightarrow Y$ in \mathcal{C} sending x to y . Given a diagram

$$\begin{array}{ccc} & (Y, y) & \\ & \downarrow & \\ (Z, z) & \longrightarrow & (X, x) \end{array}$$

in $\mathfrak{r}\mathcal{C}$, define $(Y \times_X Z)^{(y,z)}$ to be the connected component of $Y \times_X Z$ containing (y, z) . We then set

$$(Y, y) \times_{(X,x)}^{\mathfrak{r}} (Z, z) = ((Y \times_X Z)^{(y,z)}, (y, z)).$$

and let

$$\Theta_{(X,x),(Y,y),(Z,z)}: (Y \times_X Z)^{(y,z)} \longrightarrow Y \times_X Z$$

be the inclusion.

We have the functor

$$\begin{aligned} q: \mathfrak{r}\mathcal{C} &\longrightarrow \mathcal{C} \\ (X, x) &\mapsto X. \end{aligned}$$

For $(X, x) \in \mathfrak{r}\mathcal{C}$, let $\mathfrak{r}\text{Zar}((X, x))$ be the full subcategory of $\mathfrak{r}\mathcal{C}/(X, x)$ consisting of

$$j: (U, u) \longrightarrow (X, x)$$

with $j: U \rightarrow X$ an open immersion. It is then an easy exercise to check that $(\mathfrak{r}\mathcal{C}, q, \Theta, \mathfrak{r}\text{Zar})$ defines a rigidification of Zar. In addition, the functor \mathfrak{r} of (i) is given by

$$\mathfrak{r}(X) = X(\bar{k}).$$

□

6.3. Functorial properties of rigidifications

(6.3.1) RIGID PULL-BACK

Let $f: Y' \rightarrow X'$ be a map in $\mathfrak{r}\mathcal{C}$; denote the pull-back map

$$(U \rightarrow X') \mapsto (U \times_{X'}^{\mathfrak{r}} Y' \rightarrow Y')$$

by

$$f^{*\mathfrak{r}}: \mathfrak{r}\mathcal{C}/X' \longrightarrow \mathfrak{r}\mathcal{C}/Y'.$$

Let $f: Y \rightarrow X$ be a morphism in \mathcal{C} . Define the rigid pull-back map

$$f^{*\mathfrak{r}}: \text{Cov}_{\mathfrak{r}\mathfrak{S}}(X) \longrightarrow \text{Cov}_{\mathfrak{r}\mathfrak{S}}(Y) \tag{6.3.1.1}$$

by

$$f^{*\mathfrak{r}}\left(\prod_{x \in \mathfrak{r}(X)} U_x \rightarrow x\right) = \prod_{y \in \mathfrak{r}(Y)} f_y^{*\mathfrak{r}}(U_{f_*y} \rightarrow f_*y).$$

Using these pull-back maps, we define the fibered category $\text{Cov}_{\mathfrak{r}\mathfrak{S}}(\mathcal{C})$ over \mathcal{C} , with fiber $\text{Cov}_{\mathfrak{r}\mathfrak{S}}(X)$ over X ; the functors q_X then give a functor of fibered categories over \mathcal{C}

$$q_{\mathcal{C}}: \text{Cov}_{\mathfrak{r}\mathfrak{S}}(\mathcal{C}) \longrightarrow \text{Cov}_{\mathfrak{S}}(\mathcal{C}).$$

By the uniqueness of the liftings $f_{Y'}$ discussed above, the functor $q_{\mathcal{C}}$ is faithful.

(6.3.2) RIGID FIBER PRODUCTS

For X in \mathcal{C} , we have the rigid product

$$\times_X^r: \text{Cov}_{\mathfrak{r}\mathfrak{S}}(X) \times \text{Cov}_{\mathfrak{r}\mathfrak{S}}(X) \longrightarrow \text{Cov}_{\mathfrak{r}\mathfrak{S}}(X)$$

defined by

$$\left(\prod_{x \in \mathfrak{r}(X)} U_x \rightarrow x \right) \times_X^r \left(\prod_{x \in \mathfrak{r}(X)} V_x \rightarrow x \right) = \prod_{x \in \mathfrak{r}(X)} U_x \times_x^r V_x \rightarrow x.$$

The natural transformation Θ gives the natural maps

$$\begin{aligned} \Theta_X: q_X \left(\left(\prod_{x \in \mathfrak{r}(X)} U_x \rightarrow x \right) \times_X^r \left(\prod_{x \in \mathfrak{r}(X)} V_x \rightarrow x \right) \right) \\ \longrightarrow q_X \left(\prod_{x \in \mathfrak{r}(X)} U_x \rightarrow x \right) \times_X q_X \left(\prod_{x \in \mathfrak{r}(X)} V_x \rightarrow x \right), \end{aligned}$$

which gives q_X the structure of a lax symmetric semi-monoidal functor (see (1.1.1)(iii)).

More generally, let $f: Y \rightarrow X$ and $g: Z \rightarrow X$ be maps in \mathcal{C} . By (6.2.3)(d), we have the isomorphism

$$\mathfrak{r}(Y \times_X Z) \longrightarrow \mathfrak{r}(Y) \times_{\mathfrak{r}(X)} \mathfrak{r}(Z).$$

Take

$$\begin{aligned} U &:= \prod_{y \in \mathfrak{r}(Y)} U_y \rightarrow y \in \text{Cov}_{\mathfrak{r}}(Y) \\ V &:= \prod_{z \in \mathfrak{r}(Z)} V_z \rightarrow z \in \text{Cov}_{\mathfrak{r}}(Z). \end{aligned}$$

Define $U \times_X V \in \text{Cov}_{\mathfrak{r}}(Y \times_X Z)$ by

$$U \times_X V = \prod_{(y,z) \in \mathfrak{r}(Y) \times_{\mathfrak{r}(X)} \mathfrak{r}(Z)} U_y \times_{f_* y}^r V_z \rightarrow y \times_{f_* y}^r z$$

In particular, for Y and Z in \mathcal{C} , we have the product

$$\times^r: \text{Cov}_{\mathfrak{r}\mathfrak{S}}(Y) \times \text{Cov}_{\mathfrak{r}\mathfrak{S}}(Z) \longrightarrow \text{Cov}_{\mathfrak{r}\mathfrak{S}}(Y \times_S Z).$$

6.4. Rigid hypercovers

(6.4.1) ADJOINING COPRODUCTS

For a category \mathcal{A} , we let $\mathcal{A}^{\perp\perp}$ be the category with objects consisting of pairs (S, f) , with S a set and $f: S \rightarrow \text{Obj}\mathcal{A}$ a map, and morphisms $(S, f) \rightarrow (T, g)$ being a pair (F, G) with $F: S \rightarrow T$ a map of sets, and G a collection of maps

$$G_s: f(s) \longrightarrow g(F(s)); \quad s \in S.$$

Composition of

$$(S, f) \xrightarrow{(F_1, G_1)} (T, g) \xrightarrow{(F_2, G_2)} (U, h)$$

is given by

$$(F_2, G_2) \circ (F_1, G_1) = (F_2 \circ F_1, G_3),$$

where

$$G_{3s}: f(s) \longrightarrow h(F_2(F_1(s)))$$

is the composition

$$f(s) \xrightarrow{G_{1s}} g(F_1(s)) \xrightarrow{G_{2F_1(s)}} h(F_2(F_1(s))).$$

We write an object of $\mathcal{A}^{\perp\perp}$ as $\coprod_{s \in S} f(s)$.

(6.4.2)

Let X be in \mathcal{C} . For $U \in \text{Cov}_{\mathfrak{r}\mathfrak{S}}(X)$, write

$$U = \prod_{x \in \mathfrak{r}(X)} j_x: U_x \rightarrow x$$

We have the commutative diagram

$$\begin{array}{ccc} q(U_x) & \xrightarrow{i_{U_x}} & \coprod(U) \\ q(j_x) \downarrow & & \downarrow p_U \\ q(x) & \xrightarrow{i_x} & X. \end{array}$$

We define the object $\text{dom}U$ of $\mathfrak{r}\mathcal{C}^{\perp\perp}$ by

$$\text{dom}U := \prod_{x \in \mathfrak{r}(X)} U_x.$$

Sending an object $\coprod_{s \in S} f(s)$ of $\mathfrak{r}\mathcal{C}^{\perp\perp}$ to $\prod_{s \in S} q(f(s))$ defines the functor

$$q^{\perp\perp}: \mathfrak{r}\mathcal{C}^{\perp\perp} \longrightarrow \mathcal{C}.$$

This agrees with the notation of (6.2.4)(v).

(6.4.3) DEGENERATE SIMPLICIAL OBJECTS

We have the subcategory Δ_{deg} of Δ , with the same objects, where the morphisms $f: [m] \rightarrow [n]$ are the surjective maps in Δ . A functor

$$F_*: \Delta_{\text{deg}}^{\text{op}} \longrightarrow \mathcal{A}$$

is called a *degenerate simplicial object of \mathcal{A}* ; the notions of an n -truncated degenerate simplicial object, and a map of degenerate simplicial objects are the obvious ones.

(6.4.4) DEFINITION

i) Let X be in \mathcal{C} . A *rigid pre-hypercover of X* is a triple (U_*, \tilde{U}_*, i_*) consisting of $U_* \rightarrow X$ in $\mathcal{C}(\Delta^{\text{op}})/X$, a collection of objects

$$\tilde{U}_n \in \text{Cov}_{\mathfrak{r}\mathfrak{S}}(c_n^X(U_*)); \quad n = 0, 1, \dots,$$

a degenerate simplicial object $\text{dom} \tilde{U}_*$ in $\mathfrak{r}\mathcal{C}^{\perp\perp}$ with n -simplices $\text{dom} \tilde{U}_n$ and maps

$$i_n: q^{\perp\perp} \text{dom}(\tilde{U}_n) \longrightarrow U_n$$

such that

a) We have the structure map (6.2.4)(vi)

$$q_{\tilde{U}_n}: q^{\perp\perp} \text{dom}(\tilde{U}_n) \longrightarrow c_n^X(U_*)$$

and the canonical morphism (5.1.5)(iv)

$$\nu_n^{U_*}: U_n \longrightarrow c_n^X(U_*).$$

Then i_n is a map over $c_n^X(U_*)$.

b) The maps i_n define a map of degenerate simplicial objects

$$i_*: q^{\perp\perp} \text{dom}(\tilde{U}_*) \longrightarrow U_*$$

over X .

If the map i_* is an isomorphism, we call (U_*, \tilde{U}_*, i_*) a *rigid hypercover of X_** .

ii) Let $f: Y \rightarrow X$ be in \mathcal{C} , let (U_*, \tilde{U}_*, i_*) be a rigid pre-hypercover of X , and let (V_*, \tilde{V}_*, j_*) be a rigid pre-hypercover of Y . A map

$$(U_*, \tilde{U}_*, i_*) \longrightarrow (V_*, \tilde{V}_*, j_*)$$

of rigid pre-hypercovers over f is a collection of maps

$$F_*: U_* \longrightarrow V_*; \quad \tilde{F}_n: \tilde{U}_n \longrightarrow \tilde{V}_n; \quad n = 0, 1, \dots,$$

such that F_* is a map in $\mathcal{C}(\Delta^{\text{op}})$ over f , \tilde{F}_n is a map over

$$c_n^f(F_*): c_n^Y(V_*) \longrightarrow c_n^X(U_*),$$

the maps

$$\text{dom}\tilde{F}_n: \text{dom}\tilde{U}_n \longrightarrow \text{dom}\tilde{V}_n$$

define a map of degenerate simplicial objects of $\mathfrak{r}\mathcal{C}^{\perp\perp}$, and

$$j_n \circ \perp\perp(\tilde{F}_n) = F_n \circ i_n; \quad n = 0, 1, \dots$$

This defines the category of rigid pre-hypercovers $\text{PHCov}_{\mathfrak{r}\mathfrak{S}}(\mathcal{C})$ as a category over \mathcal{C} ; we denote the fiber over X by $\text{PHCov}_{\mathfrak{r}\mathfrak{S}}(X)$. We let $\text{HCov}_{\mathfrak{r}\mathfrak{S}}(\mathcal{C})$ denote the full subcategory of $\text{PHCov}_{\mathfrak{r}\mathfrak{S}}(\mathcal{C})$ with objects the rigid hypercovers, and write $\text{HCov}_{\mathfrak{r}\mathfrak{S}}(X)$ for the fiber of $\text{HCov}_{\mathfrak{r}\mathfrak{S}}(\mathcal{C})$ over X . \square

We define the n -truncated versions of rigid pre-hypercovers and rigid hypercovers by requiring the conditions (a), (b) up to degree n ; maps are defined similarly.

(6.4.5) REMARK

i) Since the functor q is faithful, so is the functor

$$q^{\perp\perp}: \mathfrak{r}\mathcal{C}^{\perp\perp} \longrightarrow \mathcal{C}$$

In particular, if (U_*, \tilde{U}_*, i_*) is a rigid hypercover of X , then the degenerate simplicial structure on \tilde{U}_* is determined by the simplicial object U_* and the isomorphism i_* , hence is unique.

ii) The projection on the first factor defines the functor over \mathcal{C}

$$p_{\text{PHCov}}: \text{PHCov}_{\mathfrak{r}\mathfrak{S}}(\mathcal{C}) \longrightarrow \mathcal{C}(\Delta^{\text{op}})/\mathcal{C}.$$

iii) By (5.2.1), (6.2.3)(f) and (6.4.4), the restriction of p_{PHCov} to $\text{HCov}_{\mathfrak{r}\mathfrak{S}}(\mathcal{C})$ defines the functor over \mathcal{C}

$$p_{\text{HCov}}: \text{HCov}_{\mathfrak{r}\mathfrak{S}}(\mathcal{C}) \longrightarrow \text{HCov}_{\mathfrak{S}}(\mathcal{C}).$$

iv) Since the category $\text{Cov}_{\mathfrak{r}\mathfrak{S}}(Z)$ is left-directed for each Z in \mathcal{C} , and is fibered over \mathcal{C} , for objects (U_*, \tilde{U}_*, i_*) and (V_*, \tilde{V}_*, j_*) of $\text{HCov}_{\mathfrak{r}\mathfrak{S}}(X)$, there is at most one map

$$(F_*, \tilde{F}_*): \mathcal{U} \longrightarrow \mathcal{V}.$$

in $\text{HCov}_{\mathfrak{r}\mathfrak{S}}(X)$. Indeed, as the maps i_n and j_n are isomorphisms, the maps F_n are determined by the maps \tilde{F}_n . There is at most one map

$$\tilde{F}_0: \tilde{U}_0 \longrightarrow \tilde{V}_0$$

over X , hence (F_0, \tilde{F}_0) is unique. If we assume there is a unique n -truncated map $(F_*, \tilde{F}_*)^{\leq n}$, then, as

$$\tilde{F}_{n+1}: \tilde{U}_n \longrightarrow \tilde{V}_n$$

is a map over $c_n^f(F_n)$, there is at most one choice for \tilde{F}_{n+1} . □

(6.4.6) LEMMA

Let \mathcal{U} be a rigid pre-hypercover of X . Then there is a rigid hypercover $i_{H^*}^X \mathcal{U}$ of X_* , and a map

$$\omega_{\mathcal{U}}: i_{H^*}^X \mathcal{U} \longrightarrow \mathcal{U}$$

over id_{X^*} , which defines a right adjoint

$$i_{H^*}^X: \text{PHCov}_{\mathfrak{r}\mathfrak{S}}(X_*) \longrightarrow \text{HCov}_{\mathfrak{r}\mathfrak{S}}(X_*)$$

to the inclusion

$$i_H^X: \text{HCov}_{\mathfrak{r}\mathfrak{S}}(X_*) \longrightarrow \text{PHCov}_{\mathfrak{r}\mathfrak{S}}(X_*),$$

i.e., the map $\omega_{\mathcal{U}}$ is universal for maps over the identity of rigid hypercovers to \mathcal{U} .

Proof. Write

$$\mathcal{U} = (U_*, \tilde{U}_*, i_*); \quad i_{H^*}^X \mathcal{U} = (V_*, \tilde{V}_*, j_*), \quad \omega_{\mathcal{U}} = (F_*, \tilde{F}_*).$$

We define (V_*, \tilde{V}_*, j_*) and the map (F_*, \tilde{F}_*) inductively by degree. We set

$$\begin{aligned} \tilde{V}_0 &= \tilde{U}_0, & V_0 &= q^{\perp} \text{dom}(\tilde{V}_0), \\ \tilde{F}_0 &= \text{id}_{\tilde{U}_0}, & j_0 &= \text{id}_{V_0}, & F_0 &= i_0. \end{aligned}$$

A 0-truncated hypercover of X is just an object \tilde{W}_0 of $\text{Cov}_{\mathfrak{r}\mathfrak{S}}(X)$, together with an isomorphism

$$k_0: q^{\perp} \text{dom}(\tilde{W}_0) \longrightarrow W_0$$

over X . It is easily seen that each map of rigid 0-truncated pre-hypercovers

$$(W_0, \tilde{W}_0, k_0) \longrightarrow (U_*, \tilde{U}_*, i_*)^{\leq 0}$$

factors uniquely through

$$(F_0, \tilde{F}_0): (V_0, \tilde{V}_0, j_0) \longrightarrow (U_*, \tilde{U}_*, i_*)^{\leq 0}$$

Now suppose we have constructed an $n - 1$ -truncated rigid hypercover of X ,

$$(V_*, \tilde{V}_*, j_*)^{\leq n-1},$$

together with a map over id_X

$$(F_*, \tilde{F}_*)^{\leq n-1}: (V_*, \tilde{V}_*, j_*)^{\leq n-1} \longrightarrow (U_*, \tilde{U}_*, i_*)^{\leq n-1}.$$

which is universal for maps over id_X

$$\mathcal{W} \longrightarrow (U_*, \tilde{U}_*, i_*)^{\leq n-1}$$

of $n - 1$ -truncated rigid hypercovers of X . The map F_* gives the maps

$$c_l^X(F_*): c_l^X(V_*) \longrightarrow c_l^X(U_*); \quad l = 0, \dots, n-1,$$

we assume in addition that

$$\begin{aligned} \tilde{V}_l &= c_l^X(F_*)^{*\tau}(\tilde{U}_l), \\ V_l &= q^\perp \text{dom}(\tilde{V}_l) \\ j_l &= \text{id}_{V_l}, \end{aligned} \tag{1}$$

and

$$\tilde{F}_l: \tilde{V}_l \longrightarrow \tilde{U}_l \tag{2}$$

is the canonical map over $c_l^X(F_*)$. From (1), we have

$$F_l = i_l \circ q^\perp \text{dom}(\tilde{F}_l).$$

The map F_* induces the map

$$i_{n-1*}^X(F_*)_*: i_{n-1*}^X(V_*) \longrightarrow \text{cosk}_{n-1}^X(U_*)_*.$$

We define \tilde{V}_n by

$$\tilde{V}_n = i_{n-1*}^X(F_*)_n^{*\tau}(\tilde{U}_n),$$

and set

$$\begin{aligned} V_n &= q^\perp \text{dom}(\tilde{V}_n) \\ j_n &= \text{id}_{V_n}, \\ F_n &= i_n \circ q^\perp \text{dom}(\tilde{F}_n), \end{aligned}$$

where

$$\tilde{F}_n: \tilde{V}_n \longrightarrow \tilde{U}_n$$

is the canonical map over $\text{cosk}_{n-1}^X(F_*)_n$.

Let $\sigma: [n] \rightarrow [k]$ be a surjective map in Δ^{op} , with $k < n$, and take

$$x_k \in \mathfrak{t}(c_k^X(V_*)).$$

This gives the factor

$$V_{x_k} \rightarrow x_k$$

in \tilde{V}_k . Let

$$y_k = (F_k)_*(x_k) \in \mathfrak{r}(c_k^X(U_*)).$$

We have the factor

$$U_{y_k} \rightarrow y_k$$

in \tilde{U}_k , and the map

$$\tilde{F}'_{x_k} : (V_{x_k} \rightarrow x_k) \longrightarrow (U_{y_k} \rightarrow y_k)$$

over $(F_k, c_k^X(F_*))$. The map

$$\text{dom}\tilde{U}(\sigma) : \text{dom}\tilde{U}_k \longrightarrow \text{dom}\tilde{U}_n$$

determines the element $y_n := y_n(y_k, \sigma)$ of $\mathfrak{r}(c_n^X(U_*))$, the corresponding factor

$$U_{y_n} \rightarrow y_n$$

in \tilde{U}_n , and the map

$$\tilde{U}(\sigma)_{y_k} : U_{y_k} \longrightarrow U_{y_n}$$

We have the map

$$i_{n-1*}^X(V_*)(\sigma) : V_k \longrightarrow i_{n-1*}^X(V_*)_n;$$

we let $x_n := x_n(x_k, \sigma)$ be the element of $\mathfrak{r}(i_{n-1*}^X(V_*)_n)$ determined by

$$x_n = i_{n-1*}^X(V_*)(\sigma)_*(V_k) \tag{4}$$

and let

$$V_n \rightarrow x_n$$

be the component of \tilde{V}_n corresponding to x_n . By conditions (a) and (b) of (6.2.3), we have

$$\begin{aligned} y_n &= q_{\tilde{U}_n*}(U_{y_n}) \\ &= \nu_{n*}^{U_*}(i_{n*}U_{y_n}) \\ &= \nu_{n*}^{U_*}(i_{n*}(q^{\perp} \text{dom}(\tilde{U}(\sigma))_*(U_{y_k}))) \\ &= \nu_{n*}^{U_*}(U(\sigma)_*(i_{k*}(U_{y_k}))) \\ &= \text{cosk}_{k-1}^X(U_*)(\sigma)_*(i_{k*}U_{y_k}). \end{aligned} \tag{5}$$

By the commutativity of the diagram

$$\begin{array}{ccc} V_k & \xrightarrow{i_{n-1*}^X(V_*)(\sigma)} & i_{n-1*}^X(V_*)_n \\ F_k \downarrow & & \downarrow i_{n-1*}^X(F_*)_n \\ U_k & \xrightarrow{\text{cosk}_{n-1}^X(U_*)(\sigma)} & c_n^X(U_*) \end{array}$$

the conditions of (6.4.4)(ii), and the identities (4) and (5), we have

$$\begin{aligned}
y_n &= \text{cosk}_{n-1}^X(U_*)(\sigma)_*(i_{k*}U_{y_k}) \\
&= \text{cosk}_{n-1}^X(U_*)(\sigma)_*(i_{k*}F_{k*}V_{x_k}) \\
&= i_{n-1*}^X(F_*)_{n*}(i_{n-1*}^X(V_*)(\sigma)V_{x_k}) \\
&= i_{n-1*}^X(F_*)_{n*}(x_n).
\end{aligned}$$

and the map $i_{n-1*}^X(F_*)_n$ lifts to the map

$$h_n: x_n \longrightarrow y_n.$$

Similarly, the map $i_{n-1*}^X(V_*)(\sigma)$ lifts to the map

$$g_{V_k}: V_{x_k} \rightarrow x_n$$

giving the commutative diagram

$$\begin{array}{ccc}
V_{x_k} & \xrightarrow{g_{V_k}} & x_n \\
\tilde{U}(\sigma)' \circ \tilde{F}'_k \downarrow & & \downarrow h_n \\
U_{y_n} & \longrightarrow & y_n
\end{array} \tag{6}$$

We have the identity

$$V_{x_n} = U_{y_n} \times_{y_n}^{\mathfrak{r}} x_n;$$

thus, the diagram (6) defines the map

$$\text{dom}\tilde{V}(\sigma)_{x_k}: V_{x_k} \longrightarrow V_{x_n(x_k, \sigma)};$$

the collection of these maps over $x_k \in \mathfrak{r}(c_k^X(V_*))$ thus determines a morphism

$$\text{dom}\tilde{V}(\sigma): \text{dom}\tilde{V}_k \longrightarrow \text{dom}\tilde{V}_n$$

in $\mathfrak{r}\mathcal{C}^{\perp\perp}$.

It is then easy to check that the diagrams

$$\begin{array}{ccc}
q^{\perp\perp} \text{dom}(\tilde{V}_k) & \xrightarrow{q^{\perp\perp} \text{dom}\tilde{V}(\sigma)} & q^{\perp\perp} \text{dom}(\tilde{V}_n) \\
\parallel & & \downarrow \nu_n^{V_*} \\
q^{\perp\perp} \text{dom}(\tilde{V}_k) & \xrightarrow{i_{n-1*}^X(V_*)(\sigma)} & i_{n-1*}^X(V_*)_n
\end{array}$$

and

$$\begin{array}{ccc}
\text{dom}(\tilde{V}_k) & \xrightarrow{\text{dom}(\tilde{V}(\sigma))} & \text{dom}(\tilde{V}_n) \\
\text{dom}(\tilde{F}_k) \downarrow & & \downarrow \text{dom}(\tilde{F}_n) \\
\text{dom}(\tilde{U}_k) & \xrightarrow{\text{dom}(\tilde{V}(\sigma))} & \text{dom}(\tilde{U}_n)
\end{array}$$

commute.

Now let $\delta: [k] \rightarrow [n]$ be an arbitrary map in Δ , with $k < n$. We have the map

$$i_{n-1*}^X(V_*)(\delta): i_{n-1*}^X(V_*)_n \longrightarrow V_k;$$

set

$$V_*(\delta): V_n \longrightarrow V_k$$

to be the composition of $i_{n-1*}^X(V_*)(\delta)$ with the structure morphism

$$V_n = q^{\perp\perp} \text{dom}(\tilde{V}_n) \longrightarrow i_{n-1*}^X(V_*)_n. \tag{7}$$

For an arbitrary map

$$\delta: [n] \longrightarrow [k]; \delta \neq \text{id}_{[n]},$$

in Δ , we have the unique factorization of δ as

$$[n] \xrightarrow{\sigma} [m] \xrightarrow{\eta} [k]$$

with σ surjective and η injective, hence $m < n$. We then define

$$V(\delta): V_k \longrightarrow V_n$$

by

$$V(\delta) = [q^{\perp\perp} \text{dom} \tilde{V}(\sigma)] \circ V(\eta),$$

which is well-defined by (1).

The necessary simplicial identities for the maps $V(\delta)$ follow from the simplicial identities satisfied by $i_{n-1*}^X(V_*)$, the fact that the maps $V(\delta)$ for $\delta: [k] \rightarrow [n]$, $k < n$, and $\text{dom}(\tilde{V})(\sigma)$ factor through the structure morphism (7), and the degenerate simplicial identities satisfied by $\text{dom}(\tilde{V})(\sigma)$. This gives $V_*^{\leq n}$ the structure of a simplicial object over X_* ; the map $F_n: V_n \rightarrow U_n$ extends $F_*^{\leq n-1}$ to the simplicial map

$$F_*^{\leq n}: V_*^{\leq n} \longrightarrow U_*^{\leq n}.$$

The triple $(V_*, \tilde{V}_*, j_*)^{\leq n}$ satisfies the conditions for a rigid hypercover of X_* by construction, and the pair $(F_*, \tilde{F}_*)^{\leq n}$ gives a map of rigid pre-hypercovers over the identity. The additional inductive assumptions (1) and (2) are satisfied by construction. We now check that the map $(F_*, \tilde{F}_*)^{\leq n}$ is universal. Let $(W_*, \tilde{W}_*, k_*)^{\leq n}$ be an n -truncated rigid hypercover of X_* , and let

$$(G_*, \tilde{G}_*)^{\leq n}: (W_*, \tilde{W}_*, k_*)^{\leq n} \longrightarrow (U_*, \tilde{U}_*, i_*)^{\leq n}$$

a morphism of n -truncated rigid pre-hypercovers. By assumption, the truncation

$$(G_*, \tilde{G}_*)^{\leq n-1}: (W_*, \tilde{W}_*, k_*)^{\leq n-1} \longrightarrow (U_*, \tilde{U}_*, i_*)^{\leq n-1}$$

factors uniquely through $(F_*, \tilde{F}_*)^{\leq n-1}$ by a morphism $(H_*, \tilde{H}_*)^{\leq n-1}$; this gives the commutative diagram

$$\begin{array}{ccc} c_n^X(W_*) & \xrightarrow{c_n^X(H_*)} & c_n^X(V_*) \\ \parallel & & \downarrow c_n^X(F_*) \\ c_n^X(W_*) & \xrightarrow{c_n^X(G_*)} & c_n^X(U_*). \end{array}$$

and \tilde{H}_n is a map over $c_n^X(H_*)$. Thus, we have the canonical map

$$r: \tilde{W}_n \longrightarrow c_n^X(H_*)^{*\tau}(\tilde{U}_n);$$

this gives the factorization of \tilde{G}_n as

$$\tilde{W}_n \xrightarrow{\tilde{H}_n} \tilde{V}_n \xrightarrow{r''} \tilde{U}_n \tag{8}$$

Since q is faithful and cofibered, the map \tilde{H}_n is the unique map over $c_n^X(H_*)$ fitting into such a factorization. We have the commutative diagram

$$\begin{array}{ccc} q^{\perp\perp} \text{dom}(\tilde{W}_n) & \xrightarrow{q^{\perp\perp} \text{dom}(\tilde{H}_n)} & q^{\perp\perp} \text{dom}(\tilde{U}_n) \\ k_n \downarrow & & \downarrow i_n \\ W_n & \xrightarrow{H_n} & U_n; \end{array}$$

as k_n is an isomorphism, we may define the map

$$H_n: W_n \longrightarrow V_n$$

by

$$H_n = q^{\perp\perp} \text{dom}(\tilde{H}_n) \circ k_n^{-1}.$$

It is then an easy matter to check that the pair (H_*, \tilde{H}_*) gives the desired map

$$(H_*, \tilde{H}_*): (W_*, \tilde{W}_*, k_*)^{\leq n} \longrightarrow (V_*, \tilde{V}_*, j_*)^{\leq n}.$$

The uniqueness of the map (H_*, \tilde{H}_*) follows from the uniqueness of \tilde{H}_n in the factorization (8).

We may therefore continue the induction, completing the proof. □

6.5. Structure of the category of rigid hypercovers

We now give several constructions based on (6.4.6).

(6.5.1) PULL-BACK OF RIGID HYPERCOVERS

Let $f: Y \rightarrow X$ be in \mathcal{C} , and let (U_*, \tilde{U}_*, i_*) be a rigid hypercover of X . We define the pull-back of (U_*, \tilde{U}_*, i_*) , $f^{*\mathfrak{r}\mathfrak{s}}(U_*, \tilde{U}_*, i_*)$ as follows: We have the simplicial pull-back $f^*U_* = U_* \times_X Y \rightarrow Y$; the map

$$p_{1*}: f^*U_* \longrightarrow U_*$$

over f induces the maps

$$g^k: \text{cosk}_k^Y(f^*U_*)_* \longrightarrow \text{cosk}_k^X(U_*)_*$$

Thus, the pull-back

$$\tilde{V}_k := (g_k^{k-1})^{*\mathfrak{r}}(\tilde{U}_k)$$

is in $\text{Cov}_{\mathfrak{r}\mathfrak{S}}(c_k^Y(f^*U_*))$. Since we have the isomorphisms (5.1.5)(i)

$$\phi^k: \text{cosk}_k^Y(f^*U_*)_* \longrightarrow f^* \text{cosk}_k^X(U_*)_*,$$

the natural transformation Θ induces the map over $c_k^Y(f^*U_*)$

$$j_k: q^{\perp} \text{dom}(\tilde{V}_k) \longrightarrow f^*U_k.$$

Let

$$\tilde{p}_{1k}: \tilde{V}_k \longrightarrow \tilde{U}_k$$

be the canonical map; we then have

$$i_k \circ q^{\perp} \text{dom}(\tilde{p}_{1k}) = p_{1k} \circ j_k.$$

The degenerate simplicial structure on $\text{dom}(\tilde{U}_*)$ induces a degenerate simplicial structure on $\text{dom}(\tilde{V}_*)$ via pull-back: Let w_k be in $\mathfrak{r}(c_k^Y(f^*U_*))$, and let

$$V_{w_k} \longrightarrow w_k$$

be the corresponding factor of \tilde{V}_k . Let $\sigma: [n] \rightarrow [k]$ be a surjection in Δ . As in the proof of (6.4.6), we have

$$w_n \in \mathfrak{r}(c_n^Y(f^*U_*))$$

$$t_n \in \mathfrak{r}(c_n^X(U_*))$$

$$t_k \in \mathfrak{r}(c_k^X(U_*))$$

determined by

$$\begin{aligned} w_n &= \text{cosk}_{n-1}^Y(f^*U_*)(\sigma)_*(j_{k*}V_{w_k}) \\ t_n &= (g_n^{n-1})_*w_n \\ t_k &= (g_k^{k-1})_*w_k. \end{aligned}$$

We have the canonical map

$$w(\sigma): V_{w_k} \longrightarrow w_n$$

over $\text{cosk}_{n-1}^Y(f^*U_*)(\sigma)$. The corresponding factors of \tilde{V}_n , \tilde{U}_n and \tilde{U}_k satisfy

$$\begin{aligned} \text{dom}(\tilde{U})(\sigma)_{t_k}: U_{t_k} &\longrightarrow U_{t_n} \\ V_{w_k} &= U_{t_k} \times_{t_k} w_k \\ V_{w_n} &= U_{t_n} \times_{t_n} w_n. \end{aligned}$$

Thus, the maps

$$\begin{aligned} \text{dom}(\tilde{U})(\sigma) \circ p_1: U_{y_k} \times_{t_k} w_k &\longrightarrow U_{t_n} \\ w(\sigma) \circ p_1: U_{y_k} \times_{t_k} w_k &\longrightarrow w_n \end{aligned}$$

determine the map

$$\text{dom}(\tilde{V})(\sigma)_{w_k}: V_{w_k} \longrightarrow V_{w_n}.$$

It is then easy to check that this defines a degenerate simplicial object with n -simplices $\text{dom}\tilde{V}_n$, and makes the triple

$$(f^*U_*, \tilde{V}_*, j_*)$$

a rigid pre-hypercover of Y_* . The pair (p_{1*}, \tilde{p}_{1*}) gives a map of rigid pre-hypercovers over f . We then define $f^{*\text{rs}}(U_*, \tilde{U}_*, i_*)$ by

$$f^{*\text{rs}}(U_*, \tilde{U}_*, i_*) = i_{H*}^Y(f^*U_*, \tilde{V}_*, j_*), \tag{6.5.1.1}$$

and let

$$p_1^{\text{rs}}: f^{*\text{rs}}(U_*, \tilde{U}_*, i_*) \longrightarrow (U_*, \tilde{U}_*, i_*) \tag{6.5.1.2}$$

be the composition of (p_{1*}, \tilde{p}_{1*}) with the canonical map

$$\omega_{(f^*U_*, \tilde{V}_*, j_*)}: f^{*\text{rs}}(U_*, \tilde{U}_*, i_*) \longrightarrow (f^*U_*, \tilde{V}_*, j_*).$$

given by (6.4.6). In particular, we have the factorization of the map of simplicial objects

$$p_{\text{HCov}}(p_1^{\text{rs}}): f^{*\text{rs}}(U_*) \longrightarrow U_*$$

as

$$f^{*\text{rs}}(U_*) \xrightarrow{\omega_{(f^*U_*, \tilde{V}_*, j_*)}^1} f^*U_* \xrightarrow{p_{1*}} U_*.$$

For $\mathcal{U} = (U_*, \tilde{U}_*, i_*)$, we write

$$\omega_{f, \mathcal{U}}^{*rs}: p_{\text{HCov}}(f^{*rs}\mathcal{U}) \longrightarrow f^*(p_{\text{HCov}}(\mathcal{U})) \tag{6.5.1.4}$$

for the map $\omega_{(f^*U_*, \tilde{V}_*, j_*)}^1$.

The pull-back maps f^{*rs} together with the projections p_1^{rs} gives $\text{HCov}_{\mathfrak{rS}}(\mathcal{C})$ the structure of a fibered category over \mathcal{C} : one constructs, following the inductive construction of f^{*rs} , i_{H^*} and ω , a morphism

$$(f \circ g)^{*rs}\mathcal{U} \longrightarrow g^{*rs}(f^{*rs}\mathcal{U})$$

starting with the canonical map

$$(f \circ g)^{*r}(\tilde{U}_0) \longrightarrow g^{*r}(f^{*r}(\tilde{U}_0)).$$

As $\text{HCov}_{\mathfrak{rS}}(Z)$ is left-directed for each $Z \in \mathcal{C}$, this morphism is unique, and thus defines a natural transformation of the desired form. Using the left-directedness again, it follows that the necessary coherence diagram commutes.

The pair

$$(p_{\text{HCov}}, \omega_{*,*}^{*rs}): \text{HCov}_{\mathfrak{rS}}(\mathcal{C}) \longrightarrow \text{HCov}_{\mathfrak{S}}(\mathcal{C}) \tag{6.5.1.3}$$

defines a functor of fibered categories over \mathcal{C} .

(6.5.2) FIBER-PRODUCT OF RIGID HYPERCOVERS

Let X be in \mathcal{C} , and take (U_*, \tilde{U}_*, i_*) (V_*, \tilde{V}_*, j_*) in $\text{HCov}_{\mathfrak{rS}}(X)$. We have the simplicial fiber product

$$U_* \times_X^5 V_*$$

and the isomorphism over X

$$\phi^j: \text{cosk}_j^X(U_* \times_X^5 V_*) \longrightarrow \text{cosk}_j^X(U_*) \times_X^5 \text{cosk}_j^X(V_*).$$

Taking the rigid fiber products gives

$$\tilde{U}_n \times_X^{\mathfrak{r}} \tilde{V}_n \in \text{Cov}_{\mathfrak{r}}(c_n^X(U_*) \times_X c_n^X(V_*)).$$

We may then take the pull-back by ϕ_n^{n-1} , giving

$$\tilde{W}_n := \phi_n^{n-1*}(\tilde{U}_n \times_X^{\mathfrak{r}} \tilde{V}_n) \in \text{Cov}_{\mathfrak{r}}(c_n^X(U_* \times_X^{\mathfrak{r}} V_*)).$$

The degenerate simplicial structure on $\text{dom}(\tilde{U}_*)$ and $\text{dom}(\tilde{V}_*)$ gives, as in (6.5.1), a canonical degenerate simplicial structure for $\text{dom}(\tilde{W}_n)$; the canonical maps

$$k_n: q^{\perp} \text{dom}(\tilde{W}_n) \longrightarrow U_n \times_X V_n$$

then form a map of degenerate simplicial objects over $c_n^X(U_* \times_X V_*)$. This forms the rigid pre-hypercover $(U_* \times_X V_*, \tilde{W}_*, k_*)$ of X , with morphisms

$$\begin{aligned}\pi_1: (U_* \times_X V_*, \tilde{W}_*, k_*) &\longrightarrow (U_*, \tilde{U}_*, i_*) \\ \pi_2: (U_* \times_X V_*, \tilde{W}_*, k_*) &\longrightarrow (V_*, \tilde{V}_*, j_*)\end{aligned}\tag{6.5.2.1}$$

over the identity.

We then define $(U_*, \tilde{U}_*, i_*) \times_X^{\text{rs}} (V_*, \tilde{V}_*, j_*)$ by

$$(U_*, \tilde{U}_*, i_*) \times_X^{\text{rs}} (V_*, \tilde{V}_*, j_*) := i_{H_*}^X (U_* \times_X V_*, \tilde{W}_*, k_*);$$

the canonical map given by (6.4.6),

$$\omega_{(U_* \times_X V_*, \tilde{W}_*, k_*)}: (U_*, \tilde{U}_*, i_*) \times_X^{\text{rs}} (V_*, \tilde{V}_*, j_*) \longrightarrow (U_* \times_X V_*, \tilde{W}_*, k_*)\tag{6.5.2.2}$$

together with the projections (6.5.2.1) give the ‘‘rigid projections’’

$$\begin{aligned}p_1^{\text{rs}}: (U_*, \tilde{U}_*, i_*) \times_X^{\text{rs}} (V_*, \tilde{V}_*, j_*) &\longrightarrow (U_*, \tilde{U}_*, i_*) \\ p_2^{\text{rs}}: (U_*, \tilde{U}_*, i_*) \times_X^{\text{rs}} (V_*, \tilde{V}_*, j_*) &\longrightarrow (V_*, \tilde{V}_*, j_*).\end{aligned}\tag{6.5.2.3}$$

It follows from the universal property of the maps (6.5.2.2), together with the fact that rigid product \times_X^{r} and simplicial product \times_X^{s} are fiber products, that \times_X^{rs} , together with the projections (6.5.2.3) forms a product for the category $\text{HCov}_{\text{r}\mathfrak{S}}(X)$.

The natural transformation

$$\omega_{\mathcal{U}, \mathcal{V}}^{\times_X^{\text{rs}}}: p_{\text{HCov}}(\mathcal{U} \times_X^{\text{rs}} \mathcal{V}) \longrightarrow p_{\text{HCov}}(\mathcal{U}) \times_X^{\text{s}} p_{\text{HCov}}(\mathcal{V})\tag{6.5.2.4}$$

determined by the maps (6.5.2.2) gives the functor

$$p_{\text{HCov}}(X): \text{HCov}_{\text{r}\mathfrak{S}}(X) \longrightarrow \text{HCov}_{\mathfrak{S}}(X)$$

the structure of a lax functor of categories with products.

(6.5.3) PRODUCTS OF RIGID HYPERCOVERS

We define a product

$$\times^{\text{rs}}: \text{HCov}_{\text{r}\mathfrak{S}}(X) \times \text{HCov}_{\text{r}\mathfrak{S}}(Y) \longrightarrow \text{HCov}_{\text{r}\mathfrak{S}}(X \times_S Y)\tag{6.5.3.1}$$

by a similar procedure as in (6.5.2), where we replace \times_X^{r} , \times_X^{s} and \times_X with \times^{r} , \times^{s} and \times_S . This gives $\text{HCov}_{\text{r}\mathfrak{S}}(\mathcal{C})$ the structure of a fibered symmetric semi-monoidal category over \mathcal{C} . As above, we construct along with the product \times^{rs} the natural transformation

$$\omega^{\text{rs}}: p_{\text{HCov}} \circ [(-) \times^{\text{rs}} (-)] \longrightarrow \times^{\text{s}} \circ [p_{\text{HCov}}(-) \times p_{\text{HCov}}(-)].\tag{6.5.3.2}$$

which gives

$$(p_{\mathrm{HCov}}, \omega^{\mathrm{ts}}): \mathrm{HCov}_{\tau\mathfrak{S}}(\mathcal{C}) \longrightarrow \mathrm{HCov}_{\mathfrak{S}}(\mathcal{C})$$

the structure of a lax symmetric semi-monoidal functor (1.1.1)(iii).

We collect the results of this section in

(6.5.4) THEOREM

i) *The category*

$$\mathrm{HCov}_{\tau\mathfrak{S}}(\mathcal{C}) \longrightarrow \mathcal{C}$$

is a fibered symmetric semi-monoidal category over \mathcal{C} , with pull-back (6.5.1.1), and product (6.5.3.1).

ii) *The functor*

$$p_{\mathrm{HCov}}: \mathrm{HCov}_{\tau\mathfrak{S}}(\mathcal{C}) \longrightarrow \mathrm{HCov}_{\mathfrak{S}}(\mathcal{C})$$

$$(U_*, \tilde{U}_*, i_*) \mapsto U_*$$

together with the natural transformation (6.5.3.2) forms a lax symmetric semi-monoidal functor

$$(p_{\mathrm{HCov}}, \omega^{\mathrm{ts}}): \mathrm{HCov}_{\tau\mathfrak{S}}(\mathcal{C}) \longrightarrow \mathrm{HCov}_{\mathfrak{S}}(\mathcal{C}).$$

over \mathcal{C} .

iii) *For each $X \in \mathcal{C}$, the category $\mathrm{HCov}_{\tau\mathfrak{S}}(X)$ is left-directed.*

iv) *Let X be in \mathcal{C} , and let \mathcal{U} be in $\mathrm{HCov}_{\tau\mathfrak{S}}(X)$. Let $\mathrm{HCov}_{\tau\mathfrak{S}}(X)_{\mathcal{U}}$ denote the full subcategory of $\mathrm{HCov}_{\tau\mathfrak{S}}(X)$ with objects \mathcal{V} admitting a map to \mathcal{U} . Then $\mathrm{HCov}_{\tau\mathfrak{S}}(X)_{\mathcal{U}}$ is left-final in $\mathrm{HCov}_{\tau\mathfrak{S}}(X)$.*

Proof. The assertions (i) and (ii) are proved in (6.5.1) and (6.5.3). The assertion (iii) follows from (6.4.5)(iv) and the existence of products (6.5.2) in $\mathrm{HCov}_{\tau\mathfrak{S}}(X)$; similarly, (iv) follows from (iii) and the existence of products in $\mathrm{HCov}_{\tau\mathfrak{S}}(X)$. \square

6.6. Cohomological properties of rigid hypercovers

(6.6.1) LEMMA

Let X be in \mathcal{C} , $\tilde{U} \in \text{Cov}_{\mathfrak{r}\mathfrak{S}}(X)$, and let

$$g: W \longrightarrow q^{\perp\perp} \text{dom}(\tilde{U})$$

be in $\text{Cov}_{\mathfrak{S}}(q^{\perp\perp} \text{dom}(\tilde{U}))$. Then there is a map

$$\rho: \tilde{V} \longrightarrow \tilde{U}$$

in $\text{Cov}_{\mathfrak{r}\mathfrak{S}}(X)$ such that $q^{\perp\perp} \text{dom} \rho$ factors through g in \mathcal{C}/X .

Proof. Let $Y = q^{\perp\perp} \text{dom}(\tilde{U})$. By (6.2.3)(i), we may assume that $W \rightarrow q^{\perp\perp} \text{dom}(\tilde{U})$ is in the image of

$$\begin{aligned} q_Y: \text{Cov}_{\mathfrak{r}\mathfrak{S}}(Y) &\longrightarrow \text{Cov}_{\mathfrak{S}}(Y); \\ (W \longrightarrow q^{\perp\perp} \text{dom}(\tilde{U})) &= q_{\tilde{W}}: q^{\perp\perp} \text{dom}(\tilde{W}) \longrightarrow Y. \end{aligned}$$

Write \tilde{U} as

$$\tilde{U} = \prod_{x \in \mathfrak{r}(X)} U_x \longrightarrow x.$$

so

$$Y = \prod_{x \in \mathfrak{r}(X)} q(U_x).$$

We may then write \tilde{W} as

$$\tilde{W} = \prod_{x \in \mathfrak{r}(X)} \prod_{Y_x \in \mathfrak{r}(q(U_x))} W_{Y_x} \longrightarrow Y_x.$$

In particular, we have the factor

$$W_{U_x} \longrightarrow U_x; \tag{1}$$

by (6.2.4)(v), the composition

$$W_{U_x} \longrightarrow U_x \rightarrow x$$

is in $\mathfrak{r}\mathfrak{S}(x)$, hence the product

$$\tilde{V} := \prod_{x \in \mathfrak{r}(X)} W_{U_x} \rightarrow x$$

is in $\text{Cov}_{\mathfrak{r}\mathfrak{S}}(X)$. The maps (1) define the map

$$\rho: \tilde{V} \longrightarrow \tilde{U}$$

in $\text{Cov}_{\mathfrak{r}\mathfrak{S}}(X)$; the inclusions

$$q(W_{U_x}) \longrightarrow q^{\perp\perp} \text{dom}(\tilde{W}) = W$$

define the map

$$f: q^{\perp\perp} \text{dom}(\tilde{V}) \longrightarrow q^{\perp\perp} \text{dom}(\tilde{W}).$$

over X . Clearly we have

$$q^{\perp\perp} \text{dom}(\rho) = g \circ f,$$

completing the proof. \square

(6.6.2) PROPOSITION

Let (U_*, \tilde{U}_*, i_*) be a rigid hypercover of X , and suppose we have a cover $W \rightarrow U_n$, $W \in \text{Cov}_{\mathfrak{S}}(U_n)$. Then there is a map of rigid hypercovers

$$(V_*, \tilde{V}_*, j_*) \longrightarrow (U_*, \tilde{U}_*, i_*)$$

with $V_n \rightarrow U_n$ factoring through $W \rightarrow U_n$.

Proof. We first note that the composition

$$W \longrightarrow U_n \longrightarrow c_n^X(U_*)$$

is in $\text{Cov}_{\mathfrak{S}}(c_n^X(U_*))$, since

$$q^{\perp\perp} \text{dom}(\tilde{U}_n) \longrightarrow c_n^X(U_*)$$

is in $\text{Cov}_{\mathfrak{S}}(c_n^X(U_*))$, and U_n is isomorphic to $q^{\perp\perp} \text{dom}(\tilde{U}_n)$ over $\text{Cov}_{\mathfrak{S}}(c_n^X(U_*))$. By (6.6.1), there is a map

$$\tilde{f}: \tilde{W} \longrightarrow \tilde{U}_n$$

in $\text{Cov}_{\mathfrak{r}\mathfrak{S}}(c_n^X(U_*))$ and a commutative diagram over $c_n^X(U_*)$

$$\begin{array}{ccc} q^{\perp\perp} \text{dom}(\tilde{W}) & \xrightarrow{q^{\perp\perp} \text{dom}(\tilde{f})} & q^{\perp\perp} \text{dom}(\tilde{U}_n) \\ \downarrow & & \downarrow i_n \\ W & \longrightarrow & U_n. \end{array}$$

We now define objects \tilde{W}_k of $\text{Cov}_{\mathfrak{r}\mathfrak{S}}(\text{cosk}_{k-1}^X(U_*)_k)$ for $k = 0, \dots, n$, together with a functor

$$\text{dom} \tilde{W}_*^{\leq n}: \Delta_{\text{deg}}^{\leq n \text{op}} \longrightarrow \mathfrak{r}\mathcal{C}^{\perp\perp}$$

satisfying

$$\text{dom} \tilde{W}_k^{\leq n} = \text{dom}(\tilde{W}_k),$$

and a map of truncated degenerate simplicial objects

$$\tilde{f}_*^{\leq n}: \text{dom} \tilde{W}_*^{\leq n} \longrightarrow \text{dom} \tilde{U}_*^{\leq n}.$$

over X_* . We proceed by downwards induction on k ; we begin by setting

$$\tilde{W}_n = \tilde{W}$$

and \tilde{f}_n the map induced by \tilde{f} .

Let $\Delta_{\text{deg}}^{k+1 \leq * \leq n \text{op}}$ denote the full subcategory of $\Delta_{\text{deg}}^{\leq n \text{op}}$ with objects $[k+1], \dots, [n]$, and let $\text{dom} \tilde{U}_*^{k+1 \leq * \leq n}$ be the restriction of $\text{dom} \tilde{U}_*$ to $\Delta_{\text{deg}}^{k+1 \leq * \leq n \text{op}}$. Suppose we have defined $\tilde{W}_{k+1}, \dots, \tilde{W}_n$, together with a functor

$$\text{dom} \tilde{W}_*^{k+1 \leq * \leq n}: \Delta_{\text{deg}}^{k+1 \leq * \leq n \text{op}} \longrightarrow \mathfrak{rC}^{\perp\perp},$$

and a map

$$\tilde{f}_*^{k+1 \leq * \leq n}: \text{dom} \tilde{W}_*^{k+1 \leq * \leq n} \longrightarrow \text{dom} \tilde{U}_*^{k+1 \leq * \leq n}.$$

For x_j in $\mathfrak{r}(c_j^X(U_*))$, we let

$$U_{x_j} \longrightarrow x_j$$

be the factor in \tilde{U}_j corresponding to x_j and let

$$W_{x_j} \longrightarrow x_j$$

be the factor in \tilde{W}_j corresponding to x_j . Let $\sigma: [j] \rightarrow [k]$ be a surjection in Δ . For each

$$x_k \in \mathfrak{r}(c_k^X(U_*)),$$

the map $\text{dom} \tilde{U}(\sigma)$ determines a corresponding $x_j(x_k, \sigma)$ in $\mathfrak{r}(c_j^X(U_*))$ and a map over $X(\sigma)$

$$\text{dom} \tilde{U}(x_k, \sigma): U_{x_k} \longrightarrow U_{x_j(x_k, \sigma)}$$

The natural transformation $\tilde{f}_*^{k+1 \leq * \leq n}$ gives the map

$$\tilde{f}_j(x_j(x_k, \sigma)): W_{x_j(x_k, \sigma)} \longrightarrow U_{x_j(x_k, \sigma)}$$

over X_j ; we let W_{x_k} be the fiber product

$$(U_{x_k} \times_{U_{x_{j_1}(x_k, \sigma_1)}}^{\mathfrak{r}} W_{x_{j_1}(x_k, \sigma_1)}) \times_{U_{x_k}}^{\mathfrak{r}} \dots \times_{U_{x_k}}^{\mathfrak{r}} (U_{x_k} \times_{U_{x_{j_t}(x_k, \sigma_t)}}^{\mathfrak{r}} W_{x_{j_t}(x_k, \sigma_t)}),$$

where the $\sigma_l: [j_l] \rightarrow [k]$ run over the distinct surjective maps $[j] \rightarrow [k]$ in Δ with $k+1 \leq j \leq n$. We then have the canonical map

$$\tilde{f}_k(x_k): W_{x_k} \longrightarrow U_{x_k}.$$

Since $\mathfrak{rS}(x_k)$ is closed under pull-back and fiber products, W_{x_k} is in $\mathfrak{rS}(x_k)$.

For each surjective $\sigma: [j] \rightarrow [k]$, the projection on the corresponding factor $W_{x_j(x_k, \sigma)}$ of W_{x_k} , defines the map

$$\text{dom} \tilde{W}(x_k, \sigma): W_{x_k} \longrightarrow W_{x_j(x_k, \sigma)}.$$

Let

$$\tilde{W}_k = \prod_{x_k \in \text{t}(c_k^X(U_*))} W_{x_k} \longrightarrow x_k.$$

This clearly extends the degenerate simplicial structure $\text{dom} \tilde{W}_*$ and map $\tilde{f}_*^{k+1 \leq * \leq n}$ to $\Delta_{\text{deg}}^{k \leq * \leq n \text{op}}$, and the induction goes through.

We now extend $\tilde{W}_*^{\leq n}$ to \tilde{W}_* by setting

$$\tilde{W}_m = \tilde{U}_m$$

for $m > n$; the degenerate simplicial structure on \tilde{U}_* , together with the map $\tilde{f}_*^{\leq n}$ give $\text{dom} \tilde{W}_*$ a unique degenerate simplicial structure with the property that the extension of $\tilde{f}_*^{\leq n}$ to

$$\tilde{f}_*: \tilde{W}_* \longrightarrow \tilde{U}_*$$

with

$$\tilde{f}_m = \text{id}_{\tilde{U}_m}$$

defines a map of degenerate simplicial objects.

Taking $j_m = i_m \circ q^{\perp\perp} \text{dom}(\tilde{f}_m)$, we have defined a rigid pre-hypercover (U_*, \tilde{W}_*, j_*) of X_* and a map of rigid pre-hypercovers

$$(\text{id}_{U_*}, \tilde{f}_*): (U_*, \tilde{W}_*, j_*) \longrightarrow (U_*, \tilde{U}_*, j_*)$$

with the property that

$$i_n \circ q^{\perp\perp} \text{dom}(\tilde{f}_n): \perp\perp(\tilde{W}_n) \longrightarrow U_n$$

factors through the given cover $W \rightarrow U_n$. We then take

$$(V_*, \tilde{V}_*, k_*) := i_{H_*}^X(U_*, \tilde{W}_*, j_*),$$

with the canonical map

$$\omega_{(U_*, \tilde{W}_*, j_*)}: (V_*, \tilde{V}_*, k_*) \longrightarrow (U_*, \tilde{W}_*, j_*),$$

and let

$$(G_*, \tilde{G}_*): (V_*, \tilde{V}_*, k_*) \longrightarrow (U_*, \tilde{U}_*, i_*).$$

be the composition $(\text{id}_{U_*}, \tilde{f}_*) \circ \omega_{(U_*, \tilde{W}_*, j_*)}$. The map

$$G_n: V_n \longrightarrow U_n$$

thus factors as

$$V_n \xrightarrow{k_n^{-1}} q^\perp \text{dom}(\tilde{V}_n) \longrightarrow q^\perp \text{dom}(\tilde{W}_n) \longrightarrow U_n$$

hence G_n factors through $W \rightarrow U_n$, completing the proof. □

(6.6.3)

Let \mathcal{A} be an abelian category, and take X in \mathcal{C} . Each $U_* \in \text{HCov}_\mathfrak{S}(X)$ determines a δ -functor (with values in \mathcal{A})

$$H^*(U_*, -): \text{PreSh}_\mathfrak{A}^\mathfrak{S}(X) \longrightarrow \mathcal{A}$$

by sending \mathcal{F} to the cohomology of the complex $\mathcal{F}(U_*)^*$ associated to the cosimplicial object $\mathcal{F}(U_*)$ of \mathcal{A} :

$$H^*(U_*, \mathcal{F}) = H^*(\mathcal{F}(U_*)^*).$$

We set

$$\underline{H}_{\mathfrak{r}\mathfrak{S}}^p(X, \mathcal{F}) := \lim_{\substack{\longrightarrow \\ \mathcal{U} \in \text{HCov}_{\mathfrak{r}\mathfrak{S}}(X)}} H^p(p_{\text{HCov}}(\mathcal{U}), \mathcal{F}).$$

(6.6.4) THEOREM

Let X be in \mathcal{C} and let \mathcal{F} be a presheaf on X .

i) $\underline{H}_{\mathfrak{r}\mathfrak{S}}^p(X, -)$ defines a δ -functor on $\text{PreSh}_\mathfrak{A}^\mathfrak{S}(X)$.

ii) Let $\tilde{\mathcal{F}}$ denote the \mathfrak{S} -sheaf associated to \mathcal{F} . There is a natural isomorphism of δ -functors

$$\underline{H}_{\mathfrak{r}\mathfrak{S}}^p(X, \mathcal{F}) \longrightarrow H_{\mathfrak{S}}^p(X, \tilde{\mathcal{F}})$$

Proof. The first assertion follows (6.5.4)(iii), and the fact that direct limits over left-directed categories are exact functors.

We note that (ii) is equivalent to saying that the canonical map in the derived category $D^+(\mathcal{A})$

$$\lim_{\substack{\longrightarrow \\ \mathcal{U} \in \text{HCov}_{\mathfrak{r}\mathfrak{S}}(X)}} \mathcal{F}(p_{\text{HCov}}(\mathcal{U}))^* \longrightarrow R\Gamma(X, \tilde{\mathcal{F}})$$

is an isomorphism. For this, it suffices to show that the map in $D^+(\mathbf{Ab})$

$$\lim_{\substack{\longrightarrow \\ \mathcal{U} \in \text{HCov}_{\mathfrak{r}\mathfrak{S}}(X)}} \text{Hom}_X(\mathcal{G}, \mathcal{F}(p_{\text{HCov}}(\mathcal{U}))^*) \longrightarrow R\Gamma(X, \text{Hom}_X(\mathcal{G}, \tilde{\mathcal{F}}))$$

is an isomorphism for all \mathcal{A} -values sheaves \mathcal{G} on X , where Hom_X denotes sheaf Hom. This reduces us to the case $\mathcal{A} = \mathbf{Ab}$.

If \mathcal{F} is an injective sheaf of abelian groups, it follows from (5.2.4) and (6.4.5)(iii) that the limit defining $\underline{H}_{\mathfrak{r}\mathfrak{S}}^p(X, \mathcal{F})$ is constant, and we have

$$\underline{H}_{\mathfrak{r}\mathfrak{S}}^p(X, \mathcal{F}) = \begin{cases} 0 & \text{for } p > 0 \\ \mathcal{F}(X) & \text{for } p = 0. \end{cases}$$

Let \mathcal{F} be an abelian presheaf with the property that each section of \mathcal{F} on an open $U \rightarrow X$ vanishes on some cover $W \rightarrow U$: we call \mathcal{F} a locally zero presheaf. It follows immediately from (6.6.2) that

$$\underline{H}_{\mathfrak{r}\mathfrak{G}}^p(X, \mathcal{F}) = 0$$

if \mathcal{F} is locally zero. If we have a short exact sequence of sheaves:

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0,$$

we may factor this sequence into exact sequences of abelian presheaves:

$$\begin{aligned} 0 &\longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{F}' \longrightarrow 0, \\ 0 &\longrightarrow \ker \longrightarrow \mathcal{F}/\mathcal{F}' \longrightarrow \mathcal{F}'' \longrightarrow \text{cok} \longrightarrow 0, \end{aligned}$$

with \mathcal{F}/\mathcal{F}' , \ker and cok being locally zero. As $\underline{H}_{\mathfrak{r}\mathfrak{G}}^p(X, -)$ is a δ -functor on presheaves, the vanishing of $\underline{H}_{\mathfrak{r}\mathfrak{G}}^p(X, -)$ on presheaves which are locally zero then implies that $\underline{H}_{\mathfrak{r}\mathfrak{G}}^p(X, -)$ is a δ -functor on the category of abelian sheaves. It follows easily from the definition of a rigid hypercover and from (6.6.2) that

$$\underline{H}_{\mathfrak{r}\mathfrak{G}}^0(X, \mathcal{F}) = H^0(X, \tilde{\mathcal{F}}).$$

We thus have the canonical natural transformation of δ -functors on abelian sheaves

$$\underline{H}_{\mathfrak{r}\mathfrak{G}}^p(X, -) \longrightarrow H_{\mathfrak{G}}^p(X, -)$$

by the computation of $\underline{H}_{\mathfrak{r}\mathfrak{G}}^p(X, -)$ for injective abelian sheaves, this map is an isomorphism. \square

6.7. The category of closed subsets

(6.7.1)

We define the category of closed subsets in \mathcal{C} , \mathcal{C}_2 , as the category with objects (X, W) , with W a closed subset of X , where a morphism

$$f: (X, W) \longrightarrow (X', W')$$

is a morphism $f: X \rightarrow X'$ in \mathcal{C} such that $W' \subset f(W)$. We have the functor

$$p_1: \mathcal{C}_2 \longrightarrow \mathcal{C}.$$

We define the product of pairs (X, W) and (X', W') by

$$(X, W) \times_S^2 (X', W') = (X \times_S X', (W \times_S W')_{\text{red}}).$$

This makes \mathcal{C}_2 into a symmetric semi-monoidal category, and p_1 is then a symmetric semi-monoidal functor.

We define the symmetric semi-monoidal fibered category

$$\mathcal{C}(\Delta^{\text{op}})/\mathcal{C}_2 \longrightarrow \mathcal{C}_2$$

over \mathcal{C}_2 as the pull-back of $\mathcal{C}(\Delta^{\text{op}})/\mathcal{C}$ via p_1 ; we denote the pull-back over a morphism

$$f: (X, W) \longrightarrow (X', W')$$

by f^* , and the symmetric product by $\times^{\mathfrak{s}}$.

(6.7.2)

We have (6.5.4) the fibered symmetric semi-monoidal category

$$\text{rng}: \text{HCov}_{\tau\mathfrak{S}}(\mathcal{C}) \longrightarrow \mathcal{C}$$

over \mathcal{C} ; we define the fibered symmetric semi-monoidal category

$$\text{rng}: \text{HCov}_{\tau\mathfrak{S}}(\mathcal{C}_2) \longrightarrow \mathcal{C}_2$$

as the pull-back of $\text{HCov}_{\tau\mathfrak{S}}(\mathcal{C})$ via p_1 . We denote the pull-back associated to f by $f^{*\tau\mathfrak{s}}$, and the product by $\times^{\tau\mathfrak{s}}$. The lax symmetric semi-monoidal functor (6.5.4)

$$(p_{\text{HCov}}, \omega^{\tau\mathfrak{s}}): \text{HCov}_{\tau\mathfrak{S}}(\mathcal{C}) \longrightarrow \mathcal{C}(\Delta^{\text{op}})/\mathcal{C}$$

defines by pull-back the lax symmetric semi-monoidal functor

$$(p_{\text{HCov}_2}, \omega_2^{\text{rs}}): \text{HCov}_{\text{rS}}(\mathcal{C}_2) \longrightarrow \mathcal{C}(\Delta^{\text{op}})/\mathcal{C}_2. \quad (6.7.2.1)$$

We have the functor

$$p_1^+ : \mathcal{C}_2 \longrightarrow \mathcal{C}^+;$$

we define the category

$$\mathcal{C}^+(\Delta^{\text{op}})/\mathcal{C}_2 \longrightarrow \mathcal{C}_2$$

as the pull-back of the fibered category

$$\mathcal{C}^+(\Delta^{\text{op}})/\mathcal{C}^+ \longrightarrow \mathcal{C}^+$$

via p_1^+ . Composing the functor (6.7.2.1) with

$$+ : \mathcal{C}(\Delta^{\text{op}})/\mathcal{C}_2 \longrightarrow \mathcal{C}^+(\Delta^{\text{op}})/\mathcal{C}_2$$

gives the lax symmetric semi-monoidal functor

$$p_{\text{HCov}_2}^+ : \text{HCov}_{\text{rS}}(\mathcal{C}_2) \longrightarrow \mathcal{C}^+(\Delta^{\text{op}})/\mathcal{C}_2. \quad (6.7.2.2)$$

6.8. Cones and rigid hypercovers with support

(6.8.1) THE FUNCTOR $p_{\text{HCov}_2}^!$

Let $f: Y_* \rightarrow X_*$ be a map in $\mathcal{C}(\Delta^{\text{op}})$. We let $\text{Cone}(f)_*$ be the object of $\mathcal{C}^+(\Delta^{\text{op}})$ defined by taking the quotient of $(X_* \times \Delta[0] \coprod U_* \times \Delta[1])^+$ by the maps

$$\begin{aligned} f_n \times \text{id} : U_n \times 1^n &\longrightarrow X_n \times 1^n \\ p_{U_n} \times \text{id} : U_n \times 0^n &\longrightarrow * \times 0^n. \end{aligned}$$

We have the natural map in $\mathcal{C}^+(\Delta^{\text{op}})$

$$\pi_{X_*} : X_*^+ \longrightarrow \text{Cone}(f_*). \quad (6.8.1.1)$$

Let (X_*, \tilde{X}_*, i_*) be a rigid hypercover of $X \in \mathcal{C}$, and let $i_W: W \rightarrow X$ be a closed subset with complement $j: U \rightarrow X$. We have the pull-back $j^{*\text{rs}}(X_*, \tilde{X}_*, i_*)$, which is a rigid hypercover of U , and the canonical map (6.5.1.2)

$$p_1^{\text{rs}}(j) : j^{*\text{rs}}(X_*, \tilde{X}_*, i_*) \longrightarrow (X_*, \tilde{X}_*, i_*)$$

over j ; we set

$$p_{(X,W)}^!(X_*, \tilde{X}_*, i_*) := \text{Cone}(p_1^{\text{rs}}(j)).$$

This defines the functor

$$p_{\mathrm{HCov}2}^! : \mathrm{HCov}_{\mathfrak{r}\mathfrak{G}}(\mathcal{C}_2) \longrightarrow \mathcal{C}^+(\Delta^{\mathrm{op}}); \quad (6.8.1.2)$$

the maps (6.8.1.1) give the natural transformation

$$i^! : p_{\mathrm{HCov}2}^+ \longrightarrow p_{\mathrm{HCov}2}^!. \quad (6.8.1.3)$$

(6.8.2) THE NATURAL TRANSFORMATION Ω

Now let $i_W : W \rightarrow X$, $i_T : T \rightarrow Y$ be inclusions of closed subsets, giving the inclusion

$$i_{W \times_S T} : W \times_S T \longrightarrow X \times_S Y.$$

Let (X_*, \tilde{X}_*, i_*) and (Y_*, \tilde{Y}_*, j_*) be rigid hypercovers of X and Y , giving the rigid hypercover $(X_*, \tilde{X}_*, i_*) \times^{\mathfrak{r}\mathfrak{s}} (Y_*, \tilde{Y}_*, j_*)$ of $X \times_S Y$. We now define a map

$$\begin{aligned} p_{(X,W) \times_S^2 (Y,T)}^! [(X_*, \tilde{X}_*, i_*) \times^{\mathfrak{r}\mathfrak{s}} (Y_*, \tilde{Y}_*, j_*)] \\ \xrightarrow{\Omega_{(X,W),(Y,T)}} p_{(X,W)}^! (X_*, \tilde{X}_*, i_*) \times^{\mathfrak{s}} p_{(Y,T)}^! (Y_*, \tilde{Y}_*, j_*). \end{aligned} \quad (6.8.2.1)$$

Write

$$\begin{aligned} (X_*, \tilde{X}_*, i_*) \times^{\mathfrak{r}\mathfrak{s}} (Y_*, \tilde{Y}_*, j_*) &= (Z_*, \tilde{Z}_*, k_*) \\ X \times_S Y &= Z \\ Z \setminus W \times_S T &= U_Z; \quad X \setminus W = U_X; \quad Y \setminus T = U_Y. \end{aligned}$$

We write the open inclusions as

$$j_X : U_X \longrightarrow X; \quad j_Y : U_Y \longrightarrow Y; \quad j_Z : U_Z \longrightarrow Z.$$

We have the map of rigid pre-hypercovers

$$(F_*, \tilde{F}_*) : (Z_*, \tilde{Z}_*, k_*) \longrightarrow (X_* \times^{\mathfrak{s}} Y_*, \tilde{X}_* \times^{\mathfrak{r}} \tilde{Y}_*, i_* \times^{\mathfrak{r}} j_*)$$

giving the maps

$$c_n(F_*) : c_n^Z(Z_*) \longrightarrow c_n^X(X_*) \times_S c_n^Y(Y_*),$$

the objects

$$\tilde{X}_n \times^{\mathfrak{r}} \tilde{Y}_n \in \mathrm{Cov}_{\mathfrak{r}\mathfrak{G}}(c_n^X(X_*) \times_S c_n^Y(Y_*)),$$

and the identity

$$\tilde{Z}_n = c_n(F_*)^{*\mathfrak{r}}(\tilde{X}_n \times^{\mathfrak{r}} \tilde{Y}_n).$$

Write

$$\tilde{Z}_n = \prod_{a \in \mathfrak{r}(c_n^Z(Z_*))} Z_a \longrightarrow a, \quad \tilde{X}_n = \prod_{b \in \mathfrak{r}(c_n^X(X_*))} X_b \longrightarrow b, \quad \tilde{Y}_n = \prod_{c \in \mathfrak{r}(c_n^Y(Y_*))} Y_c \longrightarrow c.$$

Then

$$\begin{aligned} & p_{(X,W) \times_S^2(Y,T)}^!(Z_*, \tilde{Z}_*, k_*)_n \\ &= \left[\prod_{a \in \mathfrak{r}(c_n^Z(Z_*))} q(Z_a) \times 1^n \prod_{i=1}^{n-1} \prod_{a' \in \mathfrak{r}(c_n^Z(Z_*) \times_Z U_Z)} q(j_Z^{*\mathfrak{r}} Z_{j_Z^* a'}) \times 0^i 1^{n-i} \right]^+ \end{aligned}$$

Similarly,

$$\begin{aligned} & p_{(X,W)}^!(X_*, \tilde{X}_*, i_*) \times^s p_{(Y,T)}^!(Y_*, \tilde{Y}_*, j_*)_n = \\ & \left[\prod_{(b,c) \in \mathfrak{r}(c_n^X(X_*)) \times \mathfrak{r}(c_n^Y(Y_*))} q(X_b) \times_S q(Y_c) \times 1_1^n 1_2^n \right. \\ & \prod_{j=1}^{n-1} \prod_{(b,c') \in \mathfrak{r}(c_n^X(X_*)) \times \mathfrak{r}(c_n^Y(Y_*) \times_Y U_Y)} q(X_b) \times_S q(j_Y^{*\mathfrak{r}} Y_{j_Y^* c'}) \times 1_1^n 0_2^j 1_2^{n-j} \\ & \prod_{i=1}^{n-1} \prod_{(b',c) \in \mathfrak{r}(c_n^X(X_*) \times_X U_X) \times \mathfrak{r}(c_n^Y(Y_*))} q(j_X^{*\mathfrak{r}} X_{j_X^* b'}) \times_S q(Y_c) \times 0_1^i 1_1^{n-i} 1_2^n \\ & \left. \prod_{i,j=1}^{n-1} \prod_{(b',c') \in \mathfrak{r}(c_n^X(X_*) \times_X U_X) \times \mathfrak{r}(c_n^Y(Y_*) \times_Y U_Y)} q(j_X^{*\mathfrak{r}} X_{j_X^* b'}) \times_S q(j_Y^{*\mathfrak{r}} Y_{j_Y^* c'}) \times 0_1^i 1_1^{n-i} 0_2^j 1_2^{n-j} \right]^+ \end{aligned}$$

We define the map Ω on the components of $p_{(X,W) \times_S^2(Y,T)}^!(Z_*, \tilde{Z}_*, k_*)_n$ by dividing into four separate cases.

Case1: For a component of the form

$$Z := q(Z_a) \times 1^n; \quad a \in \mathfrak{r}(c_n^Z(Z_*)),$$

let

$$(b, c) = c_n(F_*)_*(a),$$

giving the identity

$$Z_a = (X_b \times_s^{\mathfrak{r}} Y_c) \times_{b \times_s^{\mathfrak{r}} c} a; \quad s = p_{c_n^Z(Z_*)_*}(a).$$

We have the projection

$$p_1: Z_a \longrightarrow X_b \times_s^{\mathfrak{r}} Y_c;$$

then may form the composition

$$q(Z_a) \text{maprov} q(p_1) q(X_b \times_s^{\mathfrak{r}} Y_c) \xrightarrow{\Theta} q(X_b) \times_S q(Y_c),$$

and send $q(Z_a) \times 1^n$ to $q(X_b) \times_S q(Y_c) \times 1_1^n 1_2^n$ via this map.

We write U_Z as a disjoint union of locally closed subsets:

$$U_Z = W \times U_Y \coprod U_X \times T \coprod U_X \times U_Y;$$

we write the inclusions of the various components as

$$j_1: U_Z^1 := W \times U_Y \longrightarrow U_Z$$

$$j_2: U_Z^2 := U_X \times T \longrightarrow U_Z$$

$$j_3: U_Z^3 := U_X \times U_Y \longrightarrow U_Z$$

This decomposition induces a similar decomposition of each U_Z -scheme $Q \rightarrow U_Z$ as

$$Q = Q^1 \coprod Q^2 \coprod Q^3,$$

with inclusions

$$j_1: Q^1 \longrightarrow Q, \quad j_2: Q^2 \longrightarrow Q, \quad j_3: Q^3 \longrightarrow Q.$$

By (6.2.3)(c), this decomposition gives the decomposition of $\mathfrak{r}(c_n^Z(Z_*) \times_Z U_Z)$ as

$$\mathfrak{r}(c_n^Z(Z_*) \times_Z U_Z) = j_{1*}(\mathfrak{r}(c_n^Z(Z_*)^1)) \coprod j_{2*}(\mathfrak{r}(c_n^Z(Z_*)^2)) \coprod j_{3*}(\mathfrak{r}(c_n^Z(Z_*)^3)).$$

Similarly, the decompositions

$$X = W \coprod U_X; \quad Y = T \coprod U_Y := Y^1 \coprod Y^2$$

with inclusions

$$j_1^X: X^1 := W \longrightarrow X; \quad j_2^X: X^2 := U_X \longrightarrow X$$

$$j_1^Y: Y^1 := T \longrightarrow Y; \quad j_2^Y: Y^2 := U_Y \longrightarrow Y$$

give decompositions

$$\mathfrak{r}(c_n^X(X_*)) = j_{1*}^X(\mathfrak{r}(c_n^X(X_*)^1)) \coprod j_{2*}^X(\mathfrak{r}(c_n^X(X_*)^2))$$

$$\mathfrak{r}(c_n^Y(Y_*)) = j_{1*}^Y(\mathfrak{r}(c_n^Y(Y_*)^1)) \coprod j_{2*}^Y(\mathfrak{r}(c_n^Y(Y_*)^2))$$

For a component of the form

$$q(j_Z^{\mathfrak{r}} Z_{j_Z^{\mathfrak{r}} a'}) \times 0^i 1^{n-i}; \quad a' \in \mathfrak{r}(c_n^Z(Z_*) \times_Z U_Z),$$

we have three remaining cases:

Case 2: $a' = j_{1*}a_1$ for some $a_1 \in r((c_n^Z(Z_*) \times_Z U)^1)$.

This is the same as requiring that

$$c_n(F_*)_*(j_{Z*}a') = (b, j_{2*}^Y c'), \text{ with } b \in j_{1*}^X(\mathfrak{r}(c_n^X(X_*)^1), c' \in \mathfrak{r}(c_n^Y(Y_*)^2).$$

We have the map

$$p_1: j_Z^{*\mathfrak{r}} Z_{j_{Z*}a'} \longrightarrow X_b \times_s^{\mathfrak{r}} j_Y^{*\mathfrak{r}} Y'_{j_{Y*}c'}$$

defined as in Case 1; this gives the composition

$$q(j_Z^{*\mathfrak{r}} Z_{j_{Z*}a'}) \xrightarrow{q(p_1)} q(X_b \times_s^{\mathfrak{r}} j_Y^{*\mathfrak{r}} Y'_{j_{Y*}c'}) \longrightarrow q(X_b) \times_S q(j_Y^{*\mathfrak{r}} q(Y'_{j_{Y*}c'})).$$

We then map the component $q(j_Z^{*\mathfrak{r}} Z_{j_{Z*}a'}) \times 0^i 1^{n-i}$ to the component

$$q(X_b) \times_S q(j_Y^{*\mathfrak{r}} q(Y'_{j_{Y*}c'})) \times 1_1^n 0_2^i 1_2^{n-i}$$

via the above map.

Case 3: $a' = j_{2*}a_2$ for some $a_2 \in \mathfrak{r}(c_n^Z(Z_*) \times_Z U^2)$.

This is the same as Case 2, reversing the roles of X and Y .

Case 4: $a' = j_{3*}a_3$ for some $a_3 \in \mathfrak{r}(c_n^Z(Z_*) \times_Z U^3)$.

This is the same as requiring that

$$c_n(F_*)_*(j_{Z*}a') = (j_{2*}^X b', j_{2*}^Y c'), \text{ with } b' \in \mathfrak{r}(c_n^X(X_*)^2), c' \in \mathfrak{r}(c_n^Y(Y_*)^2).$$

We have the map

$$p_1: j_Z^{*\mathfrak{r}} Z_{j_{Z*}a'} \longrightarrow j_X^{*\mathfrak{r}} X_{j_{X*}b'} \times_s^{\mathfrak{r}} j_Y^{*\mathfrak{r}} Y'_{j_{Y*}c'}$$

defined as in Case 2; this gives the composition

$$q(j_Z^{*\mathfrak{r}} Z_{j_{Z*}a'}) \xrightarrow{q(p_1)} q(j_X^{*\mathfrak{r}} X_{j_{X*}b'} \times_s^{\mathfrak{r}} j_Y^{*\mathfrak{r}} Y'_{j_{Y*}c'}) \longrightarrow q(j_X^{*\mathfrak{r}} X_{j_{X*}b'}) \times_S q(j_Y^{*\mathfrak{r}} q(Y'_{j_{Y*}c'})).$$

We then map the component $q(j_Z^{*\mathfrak{r}} Z_{j_{Z*}a'}) \times 0^i 1^{n-i}$ to the component

$$q(j_X^{*\mathfrak{r}} X_{j_{X*}b'}) \times_S q(j_Y^{*\mathfrak{r}} q(Y'_{j_{Y*}c'})) \times 0_1^i 1_1^{n-i} 0_2^i 1_2^{n-i}.$$

It is an elementary matter to check that this does indeed define a map of simplicial objects, as desired. In addition, we have by construction, the maps

$$\begin{aligned} & \Omega_{(X,W),(Y,T)} \circ \pi_{(X_*, \tilde{X}_*, i_*) \times^{\mathfrak{r}5} (Y_*, \tilde{Y}_*, j_*)}: \\ & (\pi_{(X_*, \tilde{X}_*, i_*)} \wedge^5 \pi_{(Y_*, \tilde{Y}_*, j_*)}) \circ \omega_{(X_*, \tilde{X}_*, i_*)}^{\mathfrak{r}5}, (Y_*, \tilde{Y}_*, j_*)}: \\ & (X_*, \tilde{X}_*, i_*) \times^{\mathfrak{r}5} (Y_*, \tilde{Y}_*, j_*) \longrightarrow p_{(X,W)}^! (X_*, \tilde{X}_*, i_*) \wedge^5 p_{(Y,T)}^! (Y_*, \tilde{Y}_*, j_*) \end{aligned} \tag{6.8.2.2}$$

agree.

(6.8.3) PROPOSITION

i) The maps (6.8.2.1) define a natural transformation

$$\Omega: p_{\mathrm{HCov}2}^! \circ [(-) \times^{\mathrm{rs}} (-)] \longrightarrow p_{\mathrm{HCov}2}^!(-) \wedge^5 p_{\mathrm{HCov}2}^!(-),$$

and the pair $(p_{\mathrm{HCov}2}^!, \Omega)$ defines a lax symmetric semi-monoidal functor

$$(p_{\mathrm{HCov}2}^!, \Omega) \longrightarrow \mathcal{C}^+(\Delta^{\mathrm{op}})/\mathcal{C}_2$$

over \mathcal{C}_2 .

ii) The natural transformation (6.8.1.3) defines a natural transformation of lax symmetric semi-monoidal functors (see (6.5.4)(iii))

$$i^!: (p_{\mathrm{HCov}2}, \omega_2^{\mathrm{rs}})^+ \longrightarrow (p_{\mathrm{HCov}2}^!, \Omega)$$

Proof. The maps (6.8.2.1) define a natural transformation of the desired form; by construction, the maps (6.8.2.1) are symmetric in (X, W) and (Y, T) , hence commute with the appropriate symmetry isomorphisms. The associative property can be verified by a direct, although tedious, computation, which we leave to the reader. This proves (i). The assertion (ii) follows from the identity (6.8.2.2). \square

6.9. Cohomological properties of rigid hypercovers with support

(6.9.1)

Suppose we have a Grothendieck topology \mathfrak{S} on \mathcal{C} : for $X \in \mathcal{C}$, we denote the category of opens of X by $\mathfrak{S}(X)$, and the category of covers of X by $\mathrm{Cov}_{\mathfrak{S}}(X)$. We assume that \mathfrak{S} is finer than the Zariski topology. We extend the site $\mathcal{C}_{\mathfrak{S}}$ to the site $\mathcal{C}_{\mathfrak{S}}^+$ on \mathcal{C}^+ by setting

$$\begin{aligned} \mathfrak{S}(X^+) &= +(\mathfrak{S}(X)), \\ \mathrm{Cov}_{\mathfrak{S}}(X^+) &= +(\mathrm{Cov}_{\mathfrak{S}}(X)), \end{aligned} \tag{6.9.1.1}$$

where $+$ is the functor (4.1.2.1). This defines the sub-categories $\mathfrak{S}(\mathcal{C}^+)$ and $\mathrm{Cov}(\mathcal{C}^+)$ of $\mathcal{C}^+/\mathcal{C}^+$, forming the topology \mathfrak{S}^+ on \mathcal{C}^+ .

By our definition of the topology \mathfrak{S}^+ , we see that the extension of a sheaf on \mathcal{C} to a presheaf on \mathcal{C}^+ is automatically a sheaf on \mathcal{C}^+ , and that the operation of extending a sheaf is exact.

(6.9.2) LEMMA

Let (X, W) be in \mathcal{C}_2 , and let $j: U \rightarrow X$ be the complement of W . Then the pull-back map

$$j^{*\tau^s}: \mathrm{HCov}_{\tau^s}(X) \longrightarrow \mathrm{HCov}_{\tau^s}(U)$$

is left final.

Proof. Since the category $\mathrm{HCov}_{\tau^s}(U)$ is left-directed by (6.5.4), it suffices to show that, given a rigid hypercover \mathcal{U} of U , there is a rigid hypercover \mathcal{X} of X and a map of rigid hypercovers

$$j^{*\tau^s}: \mathcal{X} \longrightarrow \mathcal{U}.$$

Write

$$\mathcal{U} = (U_*, \tilde{U}_*, i_*).$$

Define the rigid pre-hypercover $j_*\mathcal{U}$ inductively as follows: The decomposition

$$X = W \coprod U := X_W \coprod X_U$$

defines a decomposition of $f: Z \rightarrow X$, for each X -scheme Z , as

$$Z = f^{-1}(W) \coprod f^{-1}(U) := Z_W \coprod Z_U, \quad (1)$$

with inclusions

$$j_Z^W: Z_W \rightarrow Z; \quad j_Z^U: Z_U \rightarrow Z.$$

By (6.2.3)(c), the decomposition (1) gives the decomposition of $\tau(Z)$ as

$$\tau(Z) = j_{Z_*}^W(\tau(Z_W)) \coprod j_{Z_*}^U(\tau(Z_U)).$$

If we have an object \tilde{V}_U in $\mathrm{Cov}_{\tau^s}(Z_U)$ given as the product

$$\tilde{V}_U = \prod_{z_U \in \tau(Z_U)} V_{z_U} \rightarrow z_U,$$

we define $j_{Z_*}^U(\tilde{V}_U)$ in $\mathrm{Cov}_{\tau^s}(Z)$ by

$$j_{Z_*}^U(\tilde{V}_U) = \left[\prod_{z_U \in \tau(Z_U)} V_{z_U} \rightarrow z_U \rightarrow j_{Z_*}^U z_U \right] \prod_{z_W \in \tau(Z_W)} \prod \mathrm{id}_{j_{Z_*}^W z_W}.$$

Note that we have

$$j_Z^{U*\tau}(j_{Z_*}^U(\tilde{V}_U)) = \tilde{V}_U, \quad (2)$$

hence we have defined a functor $j_{Z_*}^U$, natural in Z , splitting the pull-back $j_Z^{U*\tau}$ (6.3.1.1). We have the canonical map

$$j_Z^U(\tilde{V}_U): \tilde{V}_U \longrightarrow j_{Z_*}^U(\tilde{V}_U) \quad (3)$$

over j_Z^U , gotten from the identity (2).

Writing $j_*\mathcal{U}$ as (X_*, \tilde{X}_*, j_*) , we set

$$\tilde{X}_0 = j_{X^*}^U(\tilde{U}_0), \quad X_0 = q^\perp \text{dom}(\tilde{X}_0), \quad j_0 = \text{id}.$$

From the definition (6.5.1) of $j^{*\text{rs}}$, together with the identity (2), we have

$$\begin{aligned} j^{*\text{rs}}(X_0, \tilde{X}_0, j_0) &= (q^\perp \text{dom}(j^{*\text{r}} \tilde{X}_0), j^{*\text{r}} \tilde{X}_0, \text{id}) \\ &= (q^\perp \text{dom}(\tilde{U}_0), \tilde{U}_0, \text{id}). \end{aligned}$$

We then have the map of 0-truncated pre-hypercovers over id_U

$$(i_0, \text{id}): j^{*\text{rs}}(X_0, \tilde{X}_0, j_0) \longrightarrow (U_0, \tilde{U}_0, i_0);$$

this defines the 0-truncation $j_*\mathcal{U}^{\leq 0}$, with morphism

$$(F_*, \tilde{F}_*)^{\leq 0}: j_*\mathcal{U}^{\leq 0} \longrightarrow \mathcal{U}^{\leq 0}$$

over id_U .

Suppose we have defined the n -truncation $j_*\mathcal{U}^{\leq n}$, with morphism

$$(F_*, \tilde{F}_*)^{\leq n}: j_*\mathcal{U}^{\leq n} = (X_*, \tilde{X}_*, j_*)^{\leq n} \longrightarrow \mathcal{U}$$

over j , satisfying

$$\begin{aligned} \tilde{X}_k &= j_{c_k^X(X_*)}^U c_k(F_*)^{*\text{rs}}(\tilde{U}_k) \\ X_k &= q^\perp \text{dom}(\tilde{X}_k) \\ j_k &= \text{id} \\ F_k &= q^\perp \text{dom}(\tilde{F}_k) \end{aligned} \tag{4}$$

and that \tilde{F}_k is the map given by composing the map (3) with the projection

$$c_k(F_*)^{*\text{r}}(\tilde{U}_k) \longrightarrow \tilde{U}_k.$$

As in (6.5.1), there is a canonical n -truncated degenerate simplicial structure on $\text{dom}(\tilde{F}_*)^{\leq n}$ so that the above data defines a map of n -truncated pre-hypercovers. We then set

$$\tilde{X}_{n+1} = j_{c_k^X(X_*)}^U c_{n+1}(F_*)^{*\text{r}}(\tilde{U}_{n+1})$$

and fill-in the rest of the data to satisfy (4); the degenerate simplicial structure on \tilde{U}_* then defines an extension of degenerate simplicial structure to $(\tilde{X}_*)^{\leq n+1}$, and the induction goes

through. This gives us the map of pre-hypercovers over j :

$$\phi: j_*\mathcal{U} \longrightarrow \mathcal{U}$$

We have the hypercover of X

$$i_{H^*}^X(j_*\mathcal{U})$$

and the map of pre-hypercovers of X

$$\omega(j_*\mathcal{U}): i_{H^*}^X(j_*\mathcal{U}) \longrightarrow j_*\mathcal{U}.$$

given by (6.4.6). The composition $\phi \circ \omega(j_*\mathcal{U})$ defines the map of hypercovers of U

$$\psi: j^{*\text{rs}}(i_{H^*}^X(j_*\mathcal{U})) \longrightarrow \mathcal{U},$$

completing the proof. □

(6.9.3)

Let \mathcal{A} be an abelian category. For (X, W) in \mathcal{C}_2 , we have the functor

$$\Gamma^W(X, -): \text{PreSh}_X^{\mathfrak{S}}(\mathcal{A}) \longrightarrow \mathcal{A}$$

which associates to a presheaf \mathcal{F} the sections of \mathcal{F} on X with support in W . We have the derived functors

$$H_W^*(X, -) = R^*\Gamma^W(X, -)$$

of the restriction of $\Gamma^W(X, -)$ to $\text{Sh}_X^{\mathfrak{S}}(\mathcal{A})$.

For a presheaf \mathcal{F} with values in an abelian category \mathcal{A} , define

$$\underline{H}_{\text{r}\mathfrak{S}}^p W(X, \mathcal{F}) = \varinjlim_{\mathcal{X} \in \text{HCov}_{\text{r}\mathfrak{S}}((X, W))} H^p(\mathcal{F}(p_{\text{HCov}_2}^! \mathcal{X})^*).$$

(6.9.4) PROPOSITION

- i) The objects $\underline{H}_{\text{r}\mathfrak{S}}^* W(X, \mathcal{F})$ define a δ -functor on the category of presheaves.
- ii) Let $\tilde{\mathcal{F}}$ be the sheaf associated to \mathcal{F} . Then the canonical map of δ -functors

$$\underline{H}_{\text{r}\mathfrak{S}}^p W(X, \mathcal{F}) \longrightarrow H_W^p(X, \tilde{\mathcal{F}}),$$

is an isomorphism.

- iii) The natural transformation (6.8.1.3) gives a natural transformation of δ -functors

$$i^!(X, W): \underline{H}_{\text{r}\mathfrak{S}}^p W(X, \mathcal{F}) \longrightarrow \underline{H}^p \text{r}\mathfrak{S}(X, \mathcal{F})$$

6.10. The canonical rigid resolution

We conclude with a construction of a canonical acyclic resolution of a sheaf \mathcal{F} on a scheme X , given via the rigid hypercovers of the opens on X . This construction appears, in slightly different form and for the case of the classical topology, in the appendix of Godement's book [G].

(6.10.1) THE COMPLEX OF SHEAVES $\mathcal{F}_{\mathfrak{r}}^*$

Let \mathcal{F} be a sheaf on $X_{\mathfrak{S}}$, with values in an abelian category \mathcal{A} , and let $Y \rightarrow X$ be an open in $X_{\mathfrak{S}}$. We define the complex $\mathcal{F}_{\mathfrak{r}}(Y)^*$ by

$$\mathcal{F}_{\mathfrak{r}}(Y)^* = \varinjlim_{\mathcal{U} \in \mathrm{HCov}_{\mathfrak{r}\mathfrak{S}}(Y)} \mathcal{F}(p_{\mathrm{HCov}}(\mathcal{U}))^*. \quad (6.10.1.1)$$

The pull-back morphisms

$$f^{*\mathfrak{r}\mathfrak{S}}: \mathrm{HCov}_{\mathfrak{r}\mathfrak{S}}(Y) \longrightarrow \mathrm{HCov}_{\mathfrak{r}\mathfrak{S}}(Y')$$

associated to a map $f: Y' \rightarrow Y$ over X , together with the canonical maps over f

$$f^{*\mathfrak{r}\mathfrak{S}}\mathcal{U} \longrightarrow \mathcal{U}$$

define the morphisms

$$\mathcal{F}_{\mathfrak{r}}(f)^*: \mathcal{F}_{\mathfrak{r}}(Y)^* \longrightarrow \mathcal{F}_{\mathfrak{r}}(Y')^*,$$

and defines the complex of presheaves $\mathcal{F}_{\mathfrak{r}}^*$, together with the canonical augmentation (as presheaves)

$$\mathcal{F} \longrightarrow \mathcal{F}_{\mathfrak{r}}^*. \quad (6.10.1.2)$$

(6.10.2)

In order to study $\mathcal{F}_{\mathfrak{r}}^*$ effectively, we need to add some conditions to the rigidification $\mathfrak{r}\mathfrak{S}$ and the topology \mathfrak{S} .

We have the functor of (6.2.4)(i)

$$\mathfrak{r}: \mathcal{C} \longrightarrow \mathbf{Sets}; \quad (6.10.2.1)$$

by (6.2.3), the functor (6.10.2.1) preserves finite limits, and sends disjoint unions to disjoint unions.

(6.10.3) DEFINITION

i) The rigidification $\mathfrak{r}\mathfrak{S}$ is said to be *covering* if the functor (6.10.2.1) sends covers

$$j: U \longrightarrow X \in \mathrm{Cov}_{\mathfrak{S}}(X)$$

to surjections

$$j_*: \mathfrak{r}(U) \longrightarrow \mathfrak{r}(X).$$

ii) The rigidification is called *stable* if, the following condition holds: Let

$$U = \prod_{x \in \mathfrak{r}(X)} U_x \longrightarrow x$$

be in $\text{Cov}_{\mathfrak{r}\mathfrak{S}}(X)$, and take $p: Y \rightarrow X$ in $\text{Cov}_{\mathfrak{S}}(X)$ such that for each $y \in \mathfrak{r}(Y)$ with $p_*(y) = x \in \mathfrak{r}(X)$, the open

$$q(U_x) \longrightarrow X$$

factors through the connected component $q(y)$ of Y . Then the map

$$\Theta_U: q_Y(p^*\mathfrak{r}U) \longrightarrow p^*q_X(U)$$

defined by the natural transformation Θ , (6.2.3)(i), is an isomorphism.

iii) An object $f: Y \rightarrow X$ of $\text{Cov}_{\mathfrak{S}}(X)$ is called *pseudo-finite* if the map

$$f_*: \mathfrak{r}(Y) \longrightarrow \mathfrak{r}(X)$$

has finite fibers. The rigidification $\mathfrak{r}\mathfrak{S}$ is called *pseudo-finite* if each $f: Y \rightarrow X$ in $\text{Cov}_{\mathfrak{S}}(X)$ has a pseudo-finite refinement, and each $f: Y \rightarrow X$ in $\text{Cov}_{\mathfrak{S}}(X)$ is a direct limit of pseudo-finite covers.

iv) A topology \mathfrak{S} on \mathcal{C} , together with covering, stable, pseudo-finite rigidification $\mathfrak{r}\mathfrak{S}$, is called a *rigid Grothendieck topology* on \mathcal{C} .

□

We have the site $\mathbf{Sets}_{\text{surj}}$, where every map is open, and the covers are surjections.

(6.10.4) LEMMA

Let $\mathfrak{r}\mathfrak{S}$ be a covering rigidification. Then the functor

$$\mathfrak{r}: \mathcal{C} \longrightarrow \mathbf{Sets}$$

defines a morphism of sites

$$\mathfrak{r}: \mathcal{C}_{\mathfrak{S}} \longrightarrow \mathbf{Sets}_{\text{surj}}.$$

In particular, if U_* is a hypercover of X in \mathcal{C} for the topology \mathfrak{S} , then $\mathfrak{r}(U_*)$ is a hypercover of $\mathfrak{r}(X)$ for the topology surj .

Proof. This follows from the fact that \mathfrak{r} preserves finite inverse limits, sends opens to opens and covers to covers. □

Applying the functor \mathfrak{r} to a hypercover U_* of Y thus gives the hypercover

$$\mathfrak{r}(U_*) \longrightarrow \mathfrak{r}(Y)$$

of $\mathfrak{r}(Y)$ for the topology surj ; in particular, we have the truncated $m - 1$ -coskeleton

$$\mathfrak{r}(c_m^Y(U_*)) \begin{array}{c} \xrightarrow{\quad} \\ \xleftrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} \mathfrak{r}(U_{m-1}) \begin{array}{c} \xrightarrow{\quad} \\ \xleftrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} \cdots \begin{array}{c} \xrightarrow{\quad} \\ \xleftrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} \mathfrak{r}(U_0) \rightarrow \mathfrak{r}(Y)$$

As \mathfrak{r} is compatible with inverse limits, we have the canonical isomorphism

$$\text{cosk}_{m-1}^{\mathfrak{r}(Y)}(\mathfrak{r}(U_*)) \cong \mathfrak{r}(\text{cosk}_{m-1}^Y(U_*)) \tag{6.10.4.1}$$

(6.10.5) NON-DEGENERATE SIMPLICES AND THE NON-DEGENERATE SUBCOMPLEX

Let

$$S_*: \Delta^{\text{op}} \longrightarrow \mathbf{Sets}$$

be a simplicial set, and let \mathcal{A} be an object of the abelian category \mathcal{A} ; we assume that products (indexed by sets) exist in \mathcal{A} . Form the cosimplicial object of \mathcal{A} :

$$\prod_{S_*} A: \Delta \longrightarrow \mathcal{A}$$

with n cosimplices $\prod_{S_n} A$, and with the maps

$$\prod_{S_*} A(g): \prod_{S_n} A \longrightarrow \prod_{S_m} A$$

for $g: n \rightarrow m$ in Δ being the projection induced by

$$S(g): S_m \longrightarrow S_n.$$

We have the complex

$$\prod_{S_*} A^*$$

associated to $\prod_{S_*} A$.

For a simplicial set S_* , we let

$$S_m^{\text{deg}} \subset S_m$$

denote the union

$$\cup_{\sigma} S(\sigma)(S_{m-1})$$

as σ runs over surjective maps $[m] \rightarrow [m - 1]$ in Δ , and let $S_m^{\text{non-deg}}$ be the complement:

$$S_m = S_m^{\text{non-deg}} \coprod S_m^{\text{deg}}.$$

Similarly, if A^* is a cosimplicial object of an abelian category \mathcal{A} , we have the sub-object

$$A_{\text{non-deg}}^m := \bigcap_{\sigma: [m] \rightarrow [m-1]} \ker[A(\sigma): A^m \rightarrow A^{m-1}]$$

of A^m ; we define A_{deg}^m as the quotient

$$0 \rightarrow A_{\text{non-deg}}^m \rightarrow A^m \rightarrow A_{\text{deg}}^m \rightarrow 0.$$

(6.10.6) LEMMA

Suppose the homology $H_p(S_*, \mathbb{Z})$ of S_* is zero for $p > 0$, and \mathbb{Z} for $p = 0$. Then

$$H^p\left(\prod_{S_*} A^*\right) = \begin{cases} 0 & \text{for } p > 0 \\ A & \text{for } p = 0. \end{cases}$$

Proof. We may assume that $\mathcal{A} = \mathbf{Ab}$; then we have the isomorphism of cosimplicial abelian groups

$$\prod_{S_*} A \cong \text{Hom}(\mathbb{Z}(S_*), A).$$

The result then follows from the universal coefficient theorem. □

(6.10.7) LEMMA

Let

$$\mathcal{U}^{\leq n-1} := (U_*, \tilde{U}_*, i_*)^{\leq n-1}$$

be an $n - 1$ -truncated rigid hypercover of $Y \in \mathcal{C}$, with respect to a rigid Grothendieck topology $(\mathfrak{S}, \mathfrak{r}\mathfrak{S})$. Suppose we have, for each $u_n \in \mathfrak{r}(c_n^Y(U_*)^{\text{non-deg}})$, an object

$$U_{u_n} \rightarrow u_n \in \mathfrak{r}\mathfrak{S}(u_n)$$

Then there is a rigid hypercover of Y ,

$$\mathcal{U} := (U_*, \tilde{U}_*, i_*),$$

extending $\mathcal{U}^{\leq n-1}$, with the factor in \tilde{U}_n corresponding to u_n equal to the given $U_{u_n} \rightarrow u_n$.

Proof. We first prove the following

SUBLEMMA

Let $\sigma: [m] \rightarrow [m-1]$ be a surjective map in Δ , with $m < n$. Take $u_{m-1} \in \mathfrak{r}(c_m^Y(U_*))$, and let

$$U_{u_{m-1}} \longrightarrow u_{m-1}$$

be the corresponding factor in \tilde{U}_{m-1} . We have the map

$$\text{dom}\tilde{U}(\sigma): \text{dom}\tilde{U}_{m-1} \longrightarrow \text{dom}\tilde{U}_m;$$

let

$$U_{u_m} \longrightarrow u_m$$

be the factor of \tilde{U}_m such giving the factor

$$\text{dom}\tilde{U}(\sigma)_{u_{m-1}}: U_{u_{m-1}} \longrightarrow U_{u_m} \tag{1}$$

of $\text{dom}\tilde{U}(\sigma)$. Then the map (1) is an isomorphism.

Proof of sublemma. Indeed, there is a map

$$\delta: [m-1] \longrightarrow [m]$$

in Δ with

$$\sigma \circ \delta = \text{id}_{[m-1]}.$$

Thus, the map

$$g_{m-1} := i_{m-1}^{-1} \circ U(\delta) \circ U(\sigma) \circ i_{m-1}: q^{\perp\perp} \text{dom}\tilde{U}_{m-1} \longrightarrow q^{\perp\perp} \text{dom}\tilde{U}_{m-1}$$

restricts to the identity on the connected component $q(U_{u_{m-1}})$. Since we have the lifting (1) of $q^{\perp\perp} \text{dom}\tilde{U}(\sigma)$, we have

$$q^{\perp\perp} \text{dom}\tilde{U}(\sigma)_*(U_{u_{m-1}}) = U_{u_m};$$

since

$$\text{id} = g_{m-1*} = q^{\perp\perp} \text{dom}\tilde{U}(\delta)_* \circ q^{\perp\perp} \text{dom}\tilde{U}(\sigma)_*,$$

we must have

$$q^{\perp\perp} \text{dom}\tilde{U}(\delta)_*(U_{u_m}) = U_{u_{m-1}}.$$

We may thus lift $q^{\perp\perp} \text{dom}\tilde{U}(\delta)$ to a map

$$f_m: U_{u_m} \longrightarrow U_{u_{m-1}}$$

with

$$f_m \circ \tilde{U}(\sigma) = \text{id}_{U_{u_{m-1}}}.$$

On the other hand, the maps $\tilde{U}(\sigma)$ and f_m are liftings of maps over id_X , hence there is a $y \in \mathfrak{r}(Y)$ such that $\tilde{U}(\sigma)$ and f_m are maps over

$$U_{u_m} \longrightarrow y; \quad U_{u_{m-1}} \longrightarrow y. \tag{2}$$

As U_* is a hypercover of Y , applying q to the maps in (2) defines opens

$$q(U_{u_m}) \longrightarrow Y; \quad q(U_{u_{m-1}}) \longrightarrow Y,$$

hence, by (6.2.3)(e), the maps (2) define objects of $\mathfrak{r}\mathfrak{S}(y)$. As the category $\mathfrak{r}\mathfrak{S}(y)$ is left-directed, we must have

$$\tilde{U}(\sigma) \circ f_m = \text{id}_{U_{u_m}},$$

hence $\tilde{U}(\sigma)$ is an isomorphism, as claimed. □

We now continue with the proof of the lemma. It suffices to show that we may extend the $n - 1$ -truncated rigid hypercover to a n -truncated rigid hypercover with the given factors of \tilde{U}_n over $\mathfrak{r}(c_n^Y(U_*))^{\text{non-deg}}$. Let u_{n-1} be in $\mathfrak{r}(c_{n-1}^Y(U_*))$, and let

$$U_{n-1}^{u_{n-1}} \longrightarrow u_{n-1}$$

the corresponding factor of \tilde{U}_{n-1} . Let $\sigma: [n] \rightarrow [n - 1]$ be a surjective map in Δ , giving the map

$$h := \text{cosk}_{n-1}^Y(\sigma): U_{n-1} \longrightarrow c_n^Y(U_*).$$

Set

$$u_n = h_*(U_{n-1}^{u_{n-1}}) \in \mathfrak{r}(c_n^Y(U_*))^{\text{deg}},$$

giving the map

$$j: U_{n-1}^{u_{n-1}} \longrightarrow u_n \tag{3}$$

in $\mathfrak{r}\mathcal{C}$. As $q(u_n)$ is a connected component of $c_n^Y(U_*)$, and

$$c_n^Y(U_*) \longrightarrow Y$$

is an open of Y , the map

$$q(j): q(U_{n-1}^{u_{n-1}}) \longrightarrow q(u_n)$$

is an open of $q(u_n)$. Thus, by (6.2.3)(e), (3) defines an object of $\mathfrak{r}\mathfrak{S}(u_n)$, which we write as

$$U_{u_n} \longrightarrow u_n.$$

Since we have the canonical isomorphism (6.10.4.1)

$$\mathfrak{r}(c_n^Y(U_*)) \cong c_n^{\mathfrak{r}(Y)}(\mathfrak{r}(U_*)),$$

we have the identity

$$\mathfrak{r}(c_n^Y(U_*)) = \lim_{\substack{\longrightarrow \\ \sigma: [r] \rightarrow [k]}} \mathfrak{r}(U_k).$$

From this, we see that u_n and σ determine $U_{n-1}^{u_{n-1}}$; from the sublemma, it follows that $U_{n-1}^{u_{n-1}}$ is determined by u_n , up to canonical isomorphism.

Similarly, if

$$u_n = \mathfrak{r}(\text{cosk}_{n-1}^Y(\sigma'))(U_k^{u_k})$$

for some map $\sigma': [n] \rightarrow [k]$ with $k < n$, then there is a surjective $\sigma: [n] \rightarrow [n-1]$ and a map $\sigma'': [n-1] \rightarrow [k]$ such that

$$\sigma'' \circ \sigma = \sigma'.$$

We then have the factor

$$U_{n-1}^{u_{n-1}} \longrightarrow u_{n-1}$$

of \tilde{U}_{n-1} with

$$\mathfrak{r}(U_*)(\sigma'')(U_k^{u_k}) = U_{n-1}^{u_{n-1}},$$

giving the map

$$\tilde{U}(\sigma''): U_k^{u_k} \longrightarrow U_{n-1}^{u_{n-1}} = U_{u_n}. \tag{4}$$

It then follows from the sublemma as above that the map $\tilde{U}(\sigma'')$ is independent of the choice of σ'' ; we may then write (4) as

$$\tilde{U}(\sigma'): U_k^{u_k} \longrightarrow U_{u_n}.$$

If we then set

$$\tilde{U}_n = \coprod_{u_n \in \mathfrak{r}(c_n(U_*))} U_{u_n} \rightarrow u_n$$

with the degenerate portion being determined as above, we have the extension of the degenerate $n-1$ -truncated simplicial object $(\text{dom} \tilde{U}_*)^{\leq n-1}$ to an n -truncated degenerate simplicial object $(\text{dom} \tilde{U}_*)^{\leq n}$.

It follows from the sublemma that the direct limit

$$\lim_{\substack{\longrightarrow \\ \sigma: [k] \rightarrow [m] \in \Delta/[m]}} U_k$$

is isomorphic to the disjoint union

$$\coprod_{u_n \in \mathfrak{r}(c_n(U_*))_{\text{deg}}} q(U_{u_n}).$$

From this it follows that the canonical map

$$i_n: q^{\perp} \text{dom} \tilde{U}_n \longrightarrow c_n^Y(U_*)$$

extends the map

$$i_*^{\leq n-1}: (q^{\perp} \text{dom} \tilde{U}_*)^{\leq n-1} \longrightarrow U_*^{\leq n-1}$$

as maps of degenerate truncated simplicial objects. Set

$$U_n = q^{\perp\perp} \text{dom} \tilde{U}_n; i_n = \text{id};$$

the structure morphisms for \tilde{U}_n determine the map

$$q_{\tilde{U}_n} : U_n \longrightarrow c_n^Y(U_*).$$

For a map

$$\delta: [n - 1] \longrightarrow [n]$$

in Δ , we let

$$U(\delta): U_n \longrightarrow U_{n-1}$$

be the composition

$$U_n \xrightarrow{q_{\tilde{U}_n}} c_n^Y(U_*) \xrightarrow{\text{cosk}_{n-1}^Y(\delta)} U_{n-1}.$$

This completes the construction. □

(6.10.8) THEOREM

Let \mathcal{F} be a sheaf on $X \in \mathcal{C}$, and suppose we have a rigidification $\mathfrak{r}\mathfrak{S}$ of \mathfrak{S} which gives a rigid Grothendieck topology $(\mathfrak{S}, \mathfrak{r}\mathfrak{S})$, and let $f: X \rightarrow X'$ be a map in \mathcal{C} . Then

- i) The presheaves $\mathcal{F}_{\mathfrak{r}}^n$ (6.10.1.2) are sheaves on X .
- ii) the map (6.10.1.2) defines an acyclic resolution of \mathcal{F} .
- iii) the sheaves $f_* \mathcal{F}_{\mathfrak{r}}^n$ are acyclic sheaves on X' .

Proof. We note that (ii) follows from (i), (iii) and (6.6.4).

Let Y_* be a hypercover of $Y \in \mathcal{C}$; we call Y pseudo-finite if each of the maps

$$Y_k \longrightarrow Y$$

are pseudo-finite covers. By (6.10.3)(iii), we may compute the cohomology of a sheaf \mathcal{G} on Y by computing the cohomology of the complexes $\mathcal{G}(Y_*)$, as Y_* runs over pseudo-finite hypercovers of Y . In order to prove (i) and (iii), it thus suffices to show that

$$H^p(Y_*, \mathcal{F}_{\mathfrak{r}}^n) = \begin{cases} 0 & \text{for } p > 0 \\ \mathcal{F}_{\mathfrak{r}}^n(Y) & \text{for } p = 0. \end{cases} \tag{1}$$

for each open $Y \rightarrow X$ of X , and each pseudo-finite hypercover Y_* of Y .

Let then $Y_* \rightarrow Y$ be a pseudo-finite hypercover of an open $Y \rightarrow X$ of X . Fix an integer $N > 0$; it suffices to prove (1) for all $p \leq N$. Consider a rigid hypercover $\mathcal{U} = (U_*, \tilde{U}_*, i_*)$ of Y . To compute $\mathcal{F}_{\mathfrak{r}}^m$ for $m \leq n$, we may assume that the rigid covers

$$q^{\perp\perp} \text{dom} \tilde{U}_m \longrightarrow c_m(U_*)$$

factor through each projection

$$j_{k,m}: Y_k \times_Y c_m(U_*) \longrightarrow c_m(U_*),$$

for all $k \leq N$ and $m \leq n$, in the fashion given by (6.10.3)(ii). Thus, as the rigidification $\mathfrak{r}\mathfrak{S}$ is stable, we have the canonical isomorphisms

$$p_{\text{HCov}}(j_k^{*\mathfrak{r}\mathfrak{S}}U)^{\leq n} \cong j_k^*(U_*)^{\leq n} \tag{2}$$

For a Y -scheme $f: Z \rightarrow Y$ in \mathcal{C} , and for $y \in \mathfrak{r}(Y)$, we let

$$\mathfrak{r}(Z)_y = f_*^{-1}(y) \subset \mathfrak{r}(Z).$$

This gives the decomposition of the hypercover $\mathfrak{r}(U_*)$ of $\mathfrak{r}(Y)$ as a disjoint union

$$\mathfrak{r}(U_*) = \coprod_{y \in \mathfrak{r}(Y)} \mathfrak{r}(U_*)_y$$

with each $\mathfrak{r}(U_*)_y$ a hypercover of y ; by ([SGA-4], Théorème 7.3.2(3)) we have

$$H_p(\mathbb{Z}(\mathfrak{r}(U_*)_y)) = \begin{cases} 0 & \text{for } p > 0 \\ \mathbb{Z} & \text{for } p = 0. \end{cases} \tag{3}$$

For $u_m \in \mathfrak{r}(c_m(U_*))$, we have the corresponding factor

$$\tilde{U}_m^{u_m} \longrightarrow u_m$$

of \tilde{U}_m , and the connected component

$$U_m^{u_m} \longrightarrow c_m(U_*) = i_m(q(\tilde{U}_m^{u_m} \longrightarrow u_m))$$

of U_m .

Fix $y \in \mathfrak{r}(Y)$, and $u \in \mathfrak{r}(c_n(U_*))_y^{\text{non-deg}}$. Let

$$\mathcal{F}_u = \varinjlim_{(U \rightarrow u) \in \mathfrak{r}\mathfrak{S}(u)} \mathcal{F}(\text{dom}(U)).$$

As the subcategory of $U \rightarrow u$ which factor through Y_k is left final in $\mathfrak{r}\mathfrak{S}(u)$, we have the isomorphism

$$\mathcal{F}_u \cong \varinjlim_{(U \rightarrow v) \in \mathfrak{r}\mathfrak{S}(v)} \mathcal{F}(\text{dom}(U)) \tag{4}$$

for each $v \in \mathfrak{r}(Y_k)_y$.

We have the decompositions

$$\begin{aligned}\mathfrak{r}(c_m^Y(U_*)) &= \mathfrak{r}(c_m(U_*))^{\text{non-deg}} \coprod \mathfrak{r}(c_m(U_*))^{\text{deg}} \\ \mathfrak{r}(c_m^Y(U_*))_y &= \mathfrak{r}(c_m(U_*))_y^{\text{non-deg}} \coprod \mathfrak{r}(c_m(U_*))_y^{\text{deg}} \\ &= (\mathfrak{r}(c_m(U_*))^{\text{non-deg}})_y \coprod (\mathfrak{r}(c_m(U_*))^{\text{deg}})_y\end{aligned}$$

For each $y \in \mathfrak{r}(Y)$, let

$$\mathcal{F}(j_k^{*\tau s} \mathcal{U}^{\leq n-1})_{y \text{ non-deg}}^n := \prod_{u_n \in \mathfrak{r}(Y_k \times_Y c_n(U_*))_y^{\text{non-deg}}} \mathcal{F}_{u_n}$$

and let

$$\mathcal{F}(j_k^{*\tau s} \mathcal{U}^{\leq n-1})_{\text{non-deg}}^n := \prod_{y \in \mathfrak{r}(Y)} \mathcal{F}(j_k^{*\tau s} \mathcal{U}^{\leq n-1})_{y \text{ non-deg}}^n.$$

We have as well

$$\mathcal{F}(\mathcal{U}^{\leq n-1})_{y \text{ non-deg}}^n := \prod_{u_n \in \mathfrak{r}(c_n(U_*))_y^{\text{non-deg}}} \mathcal{F}_{u_n}$$

and let

$$\mathcal{F}(\mathcal{U}^{\leq n-1})_{\text{non-deg}}^n := \prod_{y \in \mathfrak{r}(Y)} \mathcal{F}(\mathcal{U}^{\leq n-1})_{y \text{ non-deg}}^n.$$

This forms the N -truncated cosimplicial object of \mathcal{A}

$$\mathcal{F}(j_*^{*\tau s} \mathcal{U}^{\leq n-1})_{\text{non-deg}}^n$$

with k -simplices $\mathcal{F}(j_k^{*\tau s} \mathcal{U}^{\leq n-1})_{\text{non-deg}}^n$, augmented over $\mathcal{F}(\mathcal{U}^{\leq n-1})_{\text{non-deg}}^n$.

By (4), we have the isomorphism

$$\mathcal{F}(j_k^{*\tau s} \mathcal{U}^{\leq n-1})_{y \text{ non-deg}}^n \cong \prod_{\mathfrak{r}(Y_k)_y} \mathcal{F}(\mathcal{U}^{\leq n-1})_{y \text{ non-deg}}^n.$$

Applying (3) and (6.10.6), we have

$$H^p(\mathcal{F}(j_*^{*\tau s} \mathcal{U}^{\leq n-1})_{\text{non-deg}}^n) = 0$$

for $0 < p < N$, and the augmentation induces an isomorphism

$$H^0(\mathcal{F}(j_*^{*\tau s} \mathcal{U}^{\leq n-1})_{\text{non-deg}}^n) \cong \mathcal{F}(\mathcal{U}^{\leq n-1})_{\text{non-deg}}^n.$$

Let $\mathcal{F}(Y_k)_{\text{non-deg}}^n$ be the direct limit

$$\varinjlim_{\mathcal{U}^{\leq n-1}} \mathcal{F}(j_k^* \tau_{\mathfrak{s}} \mathcal{U}^{\leq n-1})_{\text{non-deg}}^n.$$

where the limit is over $n - 1$ -truncated rigid hypercovers $\mathcal{U}^{\leq n-1}$ of Y . This forms the augmented N -truncated cosimplicial object $\mathcal{F}(Y_*)_{\text{non-deg}}^n$. As filtered direct limits are exact, we have

$$H^p(\mathcal{F}(Y_*)_{\text{non-deg}}^n) = 0 \tag{5}$$

for $0 < p < N$, and the augmentation induces an isomorphism

$$H^0(\mathcal{F}(Y_*)_{\text{non-deg}}^n) \cong \mathcal{F}(Y)_{\text{non-deg}}^n. \tag{6}$$

On the other hand, by (6.10.7), we may take the limit

$$\varinjlim \mathcal{F}(p_{\text{HCov}}(\mathcal{U}))^*$$

defining \mathcal{F}_{τ}^n by first fixing the $n - 1$ st truncation $\mathcal{U}^{\leq n-1}$, taking the limit as above in the non-degenerate part of $c_n(U_*)$, and then taking the limit in the $n - 1$ st truncation $\mathcal{U}^{\leq n-1}$. Thus, we have the isomorphisms

$$\begin{aligned} \mathcal{F}_{\tau}^n(Y)_{\text{non-deg}} &\cong \mathcal{F}(Y)_{\text{non-deg}}^n \\ \mathcal{F}_{\tau}^n(Y_k)_{\text{non-deg}} &\cong \mathcal{F}(Y_k)_{\text{non-deg}}^n \end{aligned}$$

In particular, applying (6), we see that the sub-presheaf $\mathcal{F}_{\tau}^n(-)_{\text{non-deg}}$ of $\mathcal{F}_{\tau}^n(-)$ is an acyclic sheaf.

Finally, we have the finite functorial filtration on $F^* \mathcal{F}_{\tau}^n(-)$ with graded quotient presheaves isomorphic to finite direct sums of the presheaves $\mathcal{F}_{\tau}^m(-)_{\text{non-deg}}$, with $m \leq n$. This completes the proof. \square

(6.10.9) COROLLARY

Let \mathcal{F} be a sheaf on $X \in \mathcal{C}$, and suppose we have a rigidification $\tau \mathfrak{S}$ of \mathfrak{S} which gives a rigid Grothendieck topology $(\mathfrak{S}, \tau \mathfrak{S})$. Let $j: U \rightarrow X$ be the inclusion of a Zariski open subscheme, and let $\mathcal{F}_{\tau, U}^*$ be the complex (see (6.8.1.2))

$$\mathcal{F}_{\tau, U}^* = \varinjlim_{\mathcal{X} \in \text{HCov}(X)} \mathcal{F}(p_{\text{HCov}2}^! \mathcal{X})^*$$

Let $f: X \rightarrow X'$ be a map in \mathcal{C} . Then

i) The presheaves $\mathcal{F}_{\tau, U}^n$ are sheaves on X .

ii) The natural transformation (6.8.1.3) induces the sequence of complexes

$$\mathcal{F}_{\mathfrak{t},U}^* \longrightarrow \mathcal{F}_{\mathfrak{t}}^* \longrightarrow [j^* \mathcal{F}]_{\mathfrak{t}}^* \longrightarrow \mathcal{F}_{\mathfrak{t},U}^*[1]$$

isomorphic to the Cone sequence associated to the natural map

$$j^*: \mathcal{F}_{\mathfrak{t}}^* \longrightarrow [j^* \mathcal{F}]_{\mathfrak{t}}^*.$$

iii) the sheaves $f_* \mathcal{F}_{\mathfrak{t}}^n$ are acyclic sheaves on X' .

Proof. It follows from (6.9.2) that we have the canonical homotopy equivalence

$$\mathcal{F}_{\mathfrak{t},U}^* \longrightarrow \text{Cone}(j^*: \mathcal{F}_{\mathfrak{t}}^* \longrightarrow [j^* \mathcal{F}]_{\mathfrak{t}}^*)$$

The three assertions then follow from (6.10.8). □

References for Part II

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$\mathbb{Z}_{X^*}(q)_{f^*}^{* \leq N}$	67	$c_i(E)$	144
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$\mathbb{Z}_{\mathcal{U}_*^{\text{n.d.}}}(q)_f^{* \leq n}$	68	$H_p(X, \mathcal{GL}/S; \mathbb{Z})$	150
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$q(\tau, h_*, p)$	70	$\mathbb{Z}_X^{\text{B.M.}}$	177
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$p_{\text{HCov}2}^+$	239	$R_X^{\text{Hdg}}(q)$	267
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