On the Witt ring of a Grassmann variety

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Abstract. A $\{+,-\} \times \operatorname{Pic}(X)/2$ - indexed system of maps, with values in proper subquotients of Grothendieck group $K_0(X)$ detects nontrivial bilinear bundles over a variety X, and other objects, considered in connection with bilinear spaces, e. g. Ranicki formations. For any line bundle L on a variety X, the $\mathcal{H}om_{\mathcal{O}_X}(-,L)$ is exact (co)functor on vector bundles and induces an involution $\hat{}^{\,L}$ on the Grothendieck group $K_0(X)$. Tate cohomology groups of the group $\{\mathrm{id}, \hat{}^{\,L}\}$ with values in $K_0(X)$, denoted here $E^+(X,L)$, $E^-(X,L)$, can be effectively computed in many cases. Moreover, the map $e^0 : W(X) \to E^+(X,1)$ is an epimorphism in many cases. For example, if L is trivial then any symmetric or skew - symmetric bilinear space produces a class in Tate cohomology $\hat{H}^1\left(\{\mathrm{id}, \hat{}^{\,L}\}, K_0(X)\right)$. Moreover, Witt equivalent spaces produce the same class.

Functoriality with respect to inverse image map and nice covariant properties yield effective computation of Herbrand index for a Grassmann variety X. This yields a lower bound for order of $E^+(X,L)$. Moreover, for $X = Gr(n,2)$ each element of $E^+(X,1)$ is a value of e^0.

Introduction

The main conjecture on Witt ring of a Grassmann variety is:

Conjecture. *Let $G = Gr(n,k)$ be the Grassmann variety of k - planes in n - dimensional vector space F^n over a field F, $char(F) \neq 2$. Denote $f : G \to \operatorname{Spec} F$ the structure map, $p = [n/2]$ the integral part of $n/2$, $q = [k/2]$ the integral part of $k/2$. If $k \leq n-k$, then*

$$|W(G)/f^*W(F)| \geq 2^{\binom{p}{q}-1}$$

This means that there are many symmetric bilinear forms on a Grassmann variety, which are not extended (or induced) from the field of definition, even up to Witt equivalence. In the case of $k=1$, $G = \mathbb{P}^{n-1}$, Arason proved that $W(G) = f^*W(F) \cong W(F)$ ([**1**, Satz]).

Till now we are able to prove the conjecture for $k=2$ (theorem 9.1 and corollary 9.2 below). To do this, in Section 1 we recall briefly basic notions of the theory of symmetric bilinear forms over a scheme X, including some special properties of bundles of endomorphisms. Probably each non extended Witt equivalence class contains a bundle of endomorphisms with trace of product as a symmetric bilinear

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form. Section 2 contains definition and basic properties of invariant e^0 of symmetric and skew symmetric bilinear forms over schemes. More facts together with complete proofs may be found in [**12**]. The idea is to consider Tate cohomology of two element group with values in $K_0(X)$, where nonzero element $\hat{}$ of the group acts on K_0 as taking dual. Tate cohomology groups are denoted here $E^+(X)$, $E^-(X)$. (Ch. Giffen pointed out to author that the natural action is given by $[\mathcal{A}]\hat{} = -[\hat{\mathcal{A}}]$, so $E^+(G)$ is $\hat{H}^1(\{1,\hat{}\}, K_0(X))$ while $E^-(G)$ is $\hat{H}^0(\{1,\hat{}\}, K_0(X)))$. The forgetful functor induces a homomorphism (an invariant) $e^0 : W(G) \oplus W^-(G) \to E^+(G)$, which is the main tool here. In next two sections we generalise these basic concepts to the case of L-valued symmetric and skew symmetric bilinear forms over schemes for arbitrary line bundle L. In section 5 all four E - groups of a projective space are determined. These are needed as a basis of inductive considerations in following sections. A description of the behaviour of e^0 under regular closed embeddings is given in theorem 6.5 in section 6. Section 7 contains (mainly known) facts on geometry and K - theory of Grassmann varieties. Two decompositions of a Grassmann variety into closed subvariety, which is itself a grassmannian, and open subset, which is an affine bundle over a grassmannian is crucial for estimation of order of E - groups of Grassmann varieties in section 8 (theorem 8.1) , and in the proof of surjectivity of the map $e^0 : W(G) \to E^+(G)$ in section 9 (theorem 9.1).

Throughout $\otimes$ without any subscript means the tensor product of $\mathcal{O}_X$ - modules over $\mathcal{O}_X$, where X is the variety under consideration.

1. Bilinear forms and Witt rings

For completeness we recall here basic definitions for bilinear forms over schemes. Denote $\mathcal{P}\hat{}$ the dual bundle of a bundle $\mathcal{P}$, and $\hat{\varphi} : \mathcal{Q}\hat{} \to \mathcal{P}\hat{}$ the dual morphism for a morphism $\varphi : \mathcal{P} \to \mathcal{Q}$.

Definition 1.1. A *symmetric bilinear form* is a pair $(\mathcal{P}, \beta)$, where $\mathcal{P}$ is a coherent locally free module (a vector bundle) and $\beta : \mathcal{P} \to \mathcal{P}\hat{}$ is an isomorphism such that $\hat{\beta} = \beta$; a *skew symmetric bilinear form* is an analogous pair $(\mathcal{P}, \beta)$ such that the isomorphism β is skew symmetric: $\hat{\beta} = -\beta$.

We omit obvious natural definitions of isomorphisms, direct sums and tensor products of (skew) symmetric bilinear forms.

One may explain the term "bilinear" occurring in above notions by observation that locally, on sections or stalks such an isomorphism β defines "true" bilinear form $(p, p') \longmapsto \beta(p)(p')$. Natural identification $\varepsilon : \mathcal{P} \to \mathcal{P}\hat{}\hat{}$ yields equality's $\hat{\beta}(p)(p') = \hat{\beta}(\varepsilon(p))(p') = (\varepsilon(p) \circ \beta)(p') = \beta(p')(p)$, which explains the term "(skew) symmetric". We simply represent a "true" bilinear form $\mathcal{P} \times \mathcal{P} \to \mathcal{O}_X$ by its adjoint $\mathcal{P} \to \mathcal{P}\hat{}$.

Example 1.2. For arbitrary vector bundle $\mathcal{P}$ consider the bundle $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{P}) \cong \mathcal{P} \otimes \mathcal{P}\hat{}$. Trace of product is a canonical nonsingular symmetric bilinear form on each bundle of the form $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{P})$, since it is symmetric nonsingular on fibers of the bundle and hence it is symmetric nonsingular on stalks of associated locally free sheaf. Note that canonical isomorphism $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{P}) \cong \mathcal{P} \otimes \mathcal{P}\hat{}$ identifies trace of product with the product of evaluation maps: locally

$$tr\big((\alpha \otimes \phi) \circ (\beta \otimes \psi)\big) = \psi(\alpha) \cdot \phi(\beta) \ .$$

The adjoint of trace of product is the switch map $\sigma : \mathcal{P} \otimes \mathcal{P}\hat{} \to \mathcal{P}\hat{} \otimes \mathcal{P} = (\mathcal{P} \otimes \mathcal{P}\hat{})\hat{}$.

For proper definition of orthogonality note first that the dual $\hat{\iota} : \mathcal{P}\hat{} \to \mathcal{V}\hat{}$ for inclusion $\iota : \mathcal{V} \to \mathcal{P}$ of a subbundle is restriction to $\mathcal{V}$.

Definition 1.3. Let $\iota : \mathcal{V} \longrightarrow \mathcal{P}$ be an admissible embedding of vector bundles (i.e. Coker ι is a bundle), where $(\mathcal{P}, \beta)$ is a (skew) symmetric bilinear form. The *orthogonal complement* $\mathcal{V}^\perp$ is the kernel of the map $\hat{\iota} \circ \beta$, so that the sequence

$$0 \longrightarrow \mathcal{V}^\perp \xrightarrow{j} \mathcal{P} \xrightarrow{\hat{\iota}\circ\beta} \mathcal{V}\hat{} \longrightarrow 0$$

is exact.

Definition 1.4. A subbundle $\mathcal{V}$ of a (skew) symmetric bilinear form $(\mathcal{P}, \beta)$ is *totally isotropic* iff $\mathcal{V} \subset \mathcal{V}^\perp$. A (skew) symmetric bilinear form is *metabolic*, iff it possesses a totally isotropic subbundle which coincides with its orthogonal complement, so that there exists exact sequence

$$0 \to \mathcal{V} \xrightarrow{\iota} \mathcal{P} \xrightarrow{\hat{\iota}\circ\beta} \mathcal{V}\hat{} \to 0$$

A *hyperbolic form*

$$(\mathcal{V} \oplus \mathcal{V}\hat{}, \begin{bmatrix} 0 & \mathrm{id}_{\mathcal{V}} \\ \pm\mathrm{id}_{\mathcal{V}\hat{}} & 0 \end{bmatrix})$$

provides an example of a metabolic form. For arbitrary form $(\mathcal{P}, \beta)$ the form $(\mathcal{P}, \beta) \oplus (\mathcal{P}, -\beta) = (\mathcal{P} \oplus \mathcal{P}, [\beta, -\beta])$ is metabolic, since the diagonal subbundle is totally isotropic.

Definition 1.5. Two (skew) symmetric bilinear forms $(\mathcal{P}, \beta)$ and $(\mathcal{Q}, \gamma)$ are *Witt equivalent* iff there exist metabolic forms $(\mathcal{M}, \mu)$ and $(\mathcal{N}, \nu)$ such that

$$(\mathcal{P}, \beta) \oplus (\mathcal{M}, \mu) \cong (\mathcal{Q}, \gamma) \oplus (\mathcal{N}, \nu).$$

Example 1.6. For a totally isotropic subbundle $\mathcal{V}$ of a (skew) symmetric bilinear form $(\mathcal{P}, \beta)$ the map β induces a (skew) symmetric isomorphism $\tilde{\beta} : \mathcal{V}^\perp/\mathcal{V} \to (\mathcal{V}^\perp/\mathcal{V})\hat{}$ such that $\hat{\kappa} \circ \tilde{\beta} \circ \kappa = \hat{j} \circ \beta \circ j$, where j is the embedding $j : \mathcal{V}^\perp \to \mathcal{P}$ and κ is the canonical map $\kappa : \mathcal{V}^\perp \to \mathcal{V}^\perp/\mathcal{V}$. The forms $(\mathcal{P}, \beta)$ and $(\mathcal{V}^\perp/\mathcal{V}, \tilde{\beta})$ are Witt equivalent: $(\mathcal{V}^\perp/\mathcal{V}, \tilde{\beta}) \oplus (\mathcal{V}^\perp/\mathcal{V}, -\tilde{\beta})$ is plainly metabolic; $(\mathcal{P}, \beta) \oplus (\mathcal{V}^\perp/\mathcal{V}, -\tilde{\beta})$ is metabolic too, since the image of the map $\begin{bmatrix} j \\ \kappa \end{bmatrix} : \mathcal{V}^\perp \to \mathcal{P} \oplus \mathcal{V}^\perp/\mathcal{V}$ coincides with its orthogonal complement. Obvious isomorphism

$$(\mathcal{P}, \beta) \oplus \big((\mathcal{V}^\perp/\mathcal{V}, \tilde{\beta}) \oplus (\mathcal{V}^\perp/\mathcal{V}, -\tilde{\beta})\big) \cong (\mathcal{V}^\perp/\mathcal{V}, \tilde{\beta}) \oplus \big((\mathcal{P}, \beta) \oplus (\mathcal{V}^\perp/\mathcal{V}, -\tilde{\beta})\big)$$

shows that $(\mathcal{P}, \beta)$ and $(\mathcal{V}^\perp/\mathcal{V}, \tilde{\beta})$ are Witt equivalent.

Here is another result of this type:

Proposition 1.1. *For an exact sequence*

$$0 \longrightarrow \mathcal{A} \xrightarrow{a} \mathcal{B} \xrightarrow{b} \mathcal{C} \longrightarrow 0$$

of vector bundles the forms $(\mathcal{B} \otimes \mathcal{B}\hat{}, \sigma)$ *and* $(\mathcal{A} \otimes \mathcal{A}\hat{}, \sigma) \oplus (\mathcal{C} \otimes \mathcal{C}\hat{}, \sigma)$ *are Witt equivalent.*

Proof. Tensoring the short exact sequence

$$0 \longrightarrow \mathcal{A} \xrightarrow{a} \mathcal{B} \xrightarrow{b} \mathcal{C} \longrightarrow 0$$

of vector bundles with its dual exact sequence

$$0 \longrightarrow \mathcal{C}\hat{} \xrightarrow{\hat{b}} \mathcal{B}\hat{} \xrightarrow{\hat{a}} \mathcal{A}\hat{} \longrightarrow 0$$

yields commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{C}\otimes\mathcal{C}\hat{} & \xrightarrow{1\otimes\hat{b}} & \mathcal{C}\otimes\mathcal{B}\hat{} & \xrightarrow{1\otimes\hat{a}} & \mathcal{C}\otimes\mathcal{A}\hat{} & \longrightarrow & 0 \\
 & & {\scriptstyle b\otimes 1}\uparrow & & {\scriptstyle b\otimes 1}\uparrow & & {\scriptstyle b\otimes 1}\uparrow & & \\
0 & \longrightarrow & \mathcal{B}\otimes\mathcal{C}\hat{} & \xrightarrow{1\otimes\hat{b}} & \mathcal{B}\otimes\mathcal{B}\hat{} & \xrightarrow{1\otimes\hat{b}} & \mathcal{B}\otimes\mathcal{A}\hat{} & \longrightarrow & 0 \\
 & & {\scriptstyle a\otimes 1}\uparrow & & {\scriptstyle a\otimes 1}\uparrow & & {\scriptstyle a\otimes 1}\uparrow & & \\
0 & \longrightarrow & \mathcal{A}\otimes\mathcal{C}\hat{} & \xrightarrow{1\otimes\hat{b}} & \mathcal{A}\otimes\mathcal{B}\hat{} & \xrightarrow{1\otimes\hat{a}} & \mathcal{A}\otimes\mathcal{A}\hat{} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

which, in turn, yields that total complex of this bicomplex is an exact sequence

$$0 \longrightarrow \mathcal{A}\otimes\mathcal{C}\hat{} \xrightarrow{\begin{bmatrix} -a\otimes 1 \\ 1\otimes\hat{b}\end{bmatrix}} \mathcal{B}\otimes\mathcal{C}\hat{} \oplus \mathcal{A}\otimes\mathcal{B}\hat{} \xrightarrow{\begin{bmatrix} -b\otimes 1 & 0 \\ 1\otimes\hat{b} & a\otimes 1 \\ 0 & -1\otimes\hat{a}\end{bmatrix}} \mathcal{C}\otimes\mathcal{C}\hat{}\oplus\mathcal{B}\otimes\mathcal{B}\hat{}\oplus\mathcal{A}\otimes\mathcal{A}\hat{}$$
$$\xrightarrow{\begin{bmatrix} 1\otimes\hat{b} & b\otimes 1 & 0 \\ 0 & 1\otimes\hat{a} & a\otimes 1\end{bmatrix}} \mathcal{C}\otimes\mathcal{B}\hat{}\oplus\mathcal{B}\otimes\mathcal{A}\hat{} \xrightarrow{[\,-1\otimes\hat{a}\;\; b\otimes 1\,]} \mathcal{C}\otimes\mathcal{A}\hat{} \longrightarrow 0.$$

Exactness of the right half of this sequence yields, via dualisation, that induced map

$$\operatorname{Coker}(\begin{bmatrix} -a\otimes 1 \\ 1\otimes\hat{b}\end{bmatrix}) \longrightarrow \mathcal{C}\otimes\mathcal{C}\hat{}\oplus\mathcal{B}\otimes\mathcal{B}\hat{}\oplus\mathcal{A}\otimes\mathcal{A}\hat{}$$

is an admissible monomorphism. Moreover, the sequence

$$\mathcal{B}\otimes\mathcal{C}\hat{}\oplus\mathcal{A}\otimes\mathcal{B}\hat{} \xrightarrow{\begin{bmatrix} -b\otimes 1 & 0 \\ 1\otimes\hat{b} & a\otimes 1 \\ 0 & -1\otimes\hat{a}\end{bmatrix}} \mathcal{C}\otimes\mathcal{C}\hat{}\oplus\mathcal{B}\otimes\mathcal{B}\hat{}\oplus\mathcal{A}\otimes\mathcal{A}\hat{}$$
$$\xrightarrow{\begin{bmatrix} \hat{b}\otimes 1 & 1\otimes b & 0 \\ 0 & \hat{a}\otimes 1 & -1\otimes a\end{bmatrix}\cdot\begin{bmatrix} -\sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & -\sigma\end{bmatrix}} \mathcal{B}\hat{}\otimes\mathcal{C}\oplus\mathcal{A}\hat{}\otimes\mathcal{B}.$$

is exact. Obviously,

$$\begin{bmatrix} \hat{b}\otimes 1 & 1\otimes b & 0 \\ 0 & \hat{a}\otimes 1 & -1\otimes a\end{bmatrix} = \begin{bmatrix} -b\otimes 1 & 0 \\ 1\otimes\hat{b} & a\otimes 1 \\ 0 & -1\otimes\hat{a}\end{bmatrix}^{\frown}.$$

Hence the image of $\mathcal{B}\otimes\mathcal{C}\hat{}\oplus\mathcal{A}\otimes\mathcal{B}\hat{}$ in $\mathcal{C}\otimes\mathcal{C}\hat{}\oplus\mathcal{B}\otimes\mathcal{B}\hat{}\oplus\mathcal{A}\otimes\mathcal{A}\hat{}$ is a subbundle, which coincides with its orthogonal complement. Thus $\Phi = (\mathcal{C}\otimes\mathcal{C}\hat{},\sigma)\oplus(\mathcal{C}\otimes\mathcal{C}\hat{},-\sigma)\oplus(\mathcal{B}\otimes\mathcal{B}\hat{},\sigma)\oplus(\mathcal{A}\otimes\mathcal{A}\hat{},-\sigma)\oplus(\mathcal{A}\otimes\mathcal{A}\hat{},\sigma)$ is Witt equivalent to $(\mathcal{C}\otimes\mathcal{C}\hat{},\sigma)\oplus(\mathcal{A}\otimes\mathcal{A}\hat{},\sigma)$. On the other hand $(\mathcal{C}\otimes\mathcal{C}\hat{},\sigma)\oplus(\mathcal{C}\otimes\mathcal{C}\hat{},-\sigma)$ and $(\mathcal{A}\otimes\mathcal{A}\hat{},-\sigma)\oplus(\mathcal{A}\otimes\mathcal{A}\hat{},\sigma)$ are metabolic, so Φ is Witt equivalent to $(\mathcal{B}\otimes\mathcal{B}\hat{},\sigma)$. □

DEFINITION 1.7. The *Witt ring $W(X)$ of symmetric bilinear forms* over a scheme X consists of classes of Witt equivalence of symmetric bilinear forms. The addition is induced by direct sum; the opposite to $[(\mathcal{P},\beta)]$ is $[(\mathcal{P},-\beta)]$; the multiplication is induced by the tensor product; the unit element is $[(\mathcal{O}_X,\mu)]$, where $\mu:\mathcal{O}_X\to\mathcal{O}_X\hat{}$ corresponds to the multiplication $\mathcal{O}_X\otimes\mathcal{O}_X\to\mathcal{O}_X$ in the sheaf of rings $\mathcal{O}_X$. The *Witt module $W^-(X)$ of skew symmetric bilinear forms* over a scheme X consists of classes of Witt equivalence of skew symmetric bilinear forms over X with the addition induced by direct sum and the $W(X)$ action induced by the tensor product.

In fact the tensor product converts $W(X)\oplus W^-(X)$ into mod 2 graded ring.

COROLLARY 1.8. *For arbitrary scheme X the correspondence*

$$\mathcal{M}\longmapsto\big(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{M}),\sigma\big)$$

where σ is the adjoint of trace of product, induces a group homomorphism

$$K_0(X)\longrightarrow W(X).$$

This follows immediately from proposition 1.1.

2. Invariant e^0

For a scheme X consider the dualization functor $\hat{}$ on the category of coherent locally free $\mathcal{O}_X$ - modules. The functor $\hat{}$ is exact, so it induces a group homomorphism $\hat{}:K_*(X)\to K_*(X)$. This homomorphism is an involutive ring authomorphism of $K_*(X)$, since the functor $\hat{}$ preserves tensor products. Analogously $\hat{}$ commutes with λ - operations. In fact, $\hat{}$ is the Adams operation ψ^{-1}. The authomorphism $\hat{}$ commutes with inverse image functor homomorphism.

DEFINITION 2.1. For endomorphisms $1\pm\hat{}$ of the Grothendieck group $K_0(X)$ put

$$E^+(X)=\mathrm{Ker}(1-\hat{})/\mathrm{Im}(1+\hat{})\quad,\quad E^-(X)=\mathrm{Ker}(1+\hat{})/\mathrm{Im}(1-\hat{}).$$

Plainly $E^+(X)$ is a ring, and $E^-(X)$ is an $E^+(X)$ - module. Additive groups $E^+(X)$ and $E^+(X)$ are elementary 2 - groups. Both E^+ and E^- are contravariant functors since the dualization functor $\hat{}$ commutes with inverse image functors. It will be convenient to regard superscript $\pm$ in $E^\pm(X)$ as short for ±1:

$$E^{(-1)^k}(X)=\begin{cases}E^+(X) & \text{for even } k,\\ E^-(X) & \text{for odd } k.\end{cases}\tag{2.1}$$

DEFINITION 2.2. The forgetful functor induces a ring homomorphism

$$e^0:W(X)\oplus W^-(X)\longrightarrow E^+(X)$$
$$e^0(\mathcal{P},\beta)=[\mathcal{P}]\big(\mathrm{mod}\ \mathrm{Im}(1+\hat{})\big).$$

The above described invariant e^0 was used in [**12**] for proof of existence of non - extended Witt classes on quadrics, which includes the case of Grassmann variety $Gr(4,2)$ of 2 - spaces in 4 - space (see theorem 5.2 below).

3. L-valued bilinear spaces

We shall generalise the above usual framework to get proper context for investigation of covariant behaviour of $E^+(X)$ and $E^-(X)$. The generalisation consists in taking the functor $\mathcal{H}om_{\mathcal{O}_X}(-, L)$ for a line bundle L instead of the functor $\hat{}$.

For a line bundle L consider exact contravariant functor

$$\hat{}^{\,L} : \mathcal{P} \longmapsto L \otimes \mathcal{P}\hat{}, \varphi\hat{}^{\,L} = \mathrm{id}_L \otimes \hat{\varphi}$$

on the exact category of vector bundles on a scheme X. We shall identify $L \otimes L\hat{} \otimes \mathcal{P}$ with $\mathcal{P}$ for any bundle $\mathcal{P}$.

Definition 3.1. A *symmetric bilinear L - valued form* is a pair $(\mathcal{P}, \beta)$, where $\mathcal{P}$ is a coherent locally free module (a vector bundle) and $\beta : \mathcal{P} \to \mathcal{P}\,\hat{}^{\,L}$ is an isomorphism such that $\beta = \beta\,\hat{}^{\,L}$; a *skew symmetric bilinear L - valued form* is an analogous pair $(\mathcal{P}, \beta)$ such that the isomorphism β is skew symmetric: $\beta\,\hat{}^{\,L} = -\beta$.

Definition 3.2. Let $\iota : V \longrightarrow \mathcal{P}$ be an admissible embedding of vector bundles (i.e. Cokerι is a bundle), where $(\mathcal{P}, \beta)$ is a (skew) symmetric bilinear L-valued form. The *orthogonal complement* $V^\perp$ of subbundle V is the kernel of the map $\iota\,\hat{}^{\,L} \circ \beta$, so that the sequence

$$0 \longrightarrow V^\perp \xrightarrow{\ j\ } \mathcal{P} \xrightarrow{\iota\hat{}^{L}\circ\beta} V\,\hat{}^{\,L} \longrightarrow 0$$

is exact.

Definition 3.3. A subbundle V of a (skew) symmetric bilinear L - valued form $(\mathcal{P}, \beta)$ is *totally isotropic* iff $V \subset V^\perp$. A (skew) symmetric bilinear L - valued form is *metabolic*, iff it possesses a totally isotropic subbundle which coincides with its orthogonal complement, so that there exists exact sequence

$$0 \longrightarrow V \xrightarrow{\iota} \mathcal{P} \xrightarrow{\iota\hat{}^{L}\circ\beta} V\,\hat{}^{\,L} \longrightarrow 0.$$

Definition 3.4. Two (skew) symmetric bilinear L - valued forms $(\mathcal{P}, \beta)$ and $(\mathcal{Q}, \gamma)$ are *Witt equivalent* iff there exist metabolic L - valued forms $(\mathcal{M}, \mu)$ and $(\mathcal{N}, \nu)$ such that $(\mathcal{P}, \beta) \oplus (\mathcal{M}, \mu) \cong (\mathcal{Q}, \gamma) \oplus (\mathcal{N}, \nu)$.

Definition 3.5. The *Witt group $W(X, L)$ of symmetric bilinear L - valued forms* over a scheme X consists of classes of Witt equivalence of symmetric bilinear L - valued forms. The addition is induced by direct sum; the opposite to $[(\mathcal{P}, \beta)]$ is $[(\mathcal{P}, -\beta)]$. The *Witt group $W^-(X, L)$ of skew symmetric bilinear L - valued forms* over a scheme X consists of classes of Witt equivalence of skew symmetric bilinear L - valued forms over X with the addition induced by direct sum.

Note that for any line bundle K map $(\mathcal{P}, \beta) \longmapsto (K \otimes \mathcal{P}, \mathrm{id}_K \otimes \beta)$ defines a natural isomorphism between groups $W(X, L)$ and $W(X, L \otimes K^{\otimes 2})$, so there is a correspondence between types of Witt groups and elements of the factor group $\mathrm{Pic}(X)/2\mathrm{Pic}(X)$. Moreover, the tensor product defines a pairing

$$W(X, L) \times W(X, K) \longrightarrow W(X, L \otimes K).$$

4. E-groups and invariant e^0_L

Any line bundle L defines an involution $\hat{}^L$ on K - groups induced by the exact functor

$$\hat{}^L : \mathcal{P} \longmapsto L \otimes \mathcal{P}\hat{},$$

$$\varphi\,\hat{}^L = \mathrm{id}_L \otimes \hat{\varphi}.$$

We are interested in Tate cohomology of two-element group $\{1, \hat{}^L\}$ with values in $K_0(X)$. Denote $C(X,L)$ the complete resolution

$$C(X,L): \qquad \cdots \longrightarrow K_0(X) \xrightarrow{1+\hat{}^L} K_0(X) \xrightarrow{1-\hat{}^L} K_0(X) \xrightarrow{1+\hat{}^L} \tag{4.1}$$
$$K_0(X) \xrightarrow{1-\hat{}^L} K_0(X) \xrightarrow{1+\hat{}^L} K_0(X) \xrightarrow{1-\hat{}^L} K_0(X) \xrightarrow{1+\hat{}^L} \cdots .$$

Definition 4.1.

$$E^+(X,L) = \mathrm{Ker}(1-\hat{}^L)/\mathrm{Im}(1+\hat{}^L) \quad , \quad E^-(X,L) = \mathrm{Ker}(1+\hat{}^L)/\mathrm{Im}(1-\hat{}^L).$$

We will refer to E-groups meaning collection of $E^+(X,L)$ and $E^-(X,L)$ for all line bundles L. However, types of E-groups of a scheme X correspond to elements of the factor group $\mathrm{Pic}(X)/2\mathrm{Pic}(X)$. It will be convenient to regard the superscript $\pm$ as short for ± 1 (cf. 2.1).

Proposition 4.1. *For arbitrary line bundle K there are isomorphisms*

$$E^+(X, L\otimes K^{\otimes 2}) \cong E^+(X,L) \quad ; \quad E^-(X, L\otimes K^{\otimes 2}) \cong E^-(X,L).$$

Proof. Tensoring with K induces isomorphism of complexes $C(X,L) \xrightarrow{[K]\cdot} C(X, L\otimes K^{\otimes 2})$:

$$\begin{array}{ccccccccccc}
\cdots \xrightarrow{1-\alpha} & K_0(X) & \xrightarrow{1+\alpha} & K_0(X) & \xrightarrow{1-\alpha} & K_0(X) & \xrightarrow{1+\alpha} & K_0(X) & \xrightarrow{1-\alpha} \cdots \\
 & \uparrow \cdot[K] & & \uparrow \cdot[K] & & \uparrow \cdot[K] & & \uparrow \cdot[K] & \\
\cdots \xrightarrow{1-\beta} & K_0(X) & \xrightarrow{1+\beta} & K_0(X) & \xrightarrow{1-\beta} & K_0(X) & \xrightarrow{1+\beta} & K_0(X) & \xrightarrow{1-\beta} \cdots
\end{array}$$

where $\alpha(\mathcal{P}) = L \otimes K^{\otimes 2} \otimes \mathcal{P}\hat{}$ and $\beta(\mathcal{P}) = L \otimes \mathcal{P}\hat{}$. □

Definition 4.2. The forgetful functor induces a group homomorphism

$$e^0_L : \; W(X,L) \oplus W^-(X,L) \longrightarrow E^+(X,L)$$
$$e^0_L(\mathcal{P},\beta) = [\mathcal{P}] \left(\mathrm{mod}\ \mathrm{Im}(1+\hat{}^L)\right).$$

As an example we prove homotopy property of E - groups.

Proposition 4.2 (homotopy property). *If $f: X \to Y$ is a flat morphism of regular noetherian separated schemes whose fibres are affine spaces, then $E^+(Y,L) \cong E^+(X, f^*L)$ and $E^-(Y,L) \cong E^-(X, f^*L)$.*

Proof. By the homotopy property of K - groups the map f^* induced by the inverse image functor f^* provides an isomorphism of complexes

$$\begin{array}{ccccccccccc}
\cdots \xrightarrow{1-\alpha} & K_0(X) & \xrightarrow{1+\alpha} & K_0(X) & \xrightarrow{1-\alpha} & K_0(X) & \xrightarrow{1+\alpha} & K_0(X) & \xrightarrow{1-\alpha} \cdots \\
 & \uparrow f^* & & \uparrow f^* & & \uparrow f^* & & \uparrow f^* & \\
\cdots \xrightarrow{1-\beta} & K_0(X) & \xrightarrow{1+\beta} & K_0(X) & \xrightarrow{1-\beta} & K_0(X) & \xrightarrow{1+\beta} & K_0(X) & \xrightarrow{1-\beta} \cdots
\end{array}$$

where $\alpha = \hat{}^{f^*L}$ and $\beta = \hat{}^{L}$. $\square$

5. E-groups of a projective space

Let the scheme X be a projective space over a field F. Since $\mathrm{Pic}(X)$ is the infinite cyclic group generated by the class of the tautological bundle $[\mathcal{O}_X(-1)]$, there are two types of E - groups, namely the first type - $E^+(X)$ and $E^-(X)$ and the second type - $E^+\big(X, \mathcal{O}_X(-1)\big)$ and $E^-\big(X, \mathcal{O}_X(-1)\big)$.

The following notation is convenient:

$$1 = [\mathcal{O}_X] \qquad \text{the unit element of } K_0(X),$$
$$H = 1 - [\mathcal{O}_X(-1)] \qquad \text{the class of hyperplane section in } K_0(X).$$

It is known that $K_0(X) \cong \mathbb{Z}[T]/\ (T^{N+1})$, where T corresponds to the class H. Let us summarise here several obvious properties of the class of hyperplane section:

LEMMA 5.1. *If $X = \mathbb{P}^N$ and $H = 1 - [\mathcal{O}_X(-1)]$, then*

i) $H^{N+1} = 0$;

ii) $[\mathcal{O}_X(1)] = [\mathcal{O}_X(-1)]\hat{} = (1-H)^{-1} = \sum_{i=0}^{N} H^i$ *in* $K_0(X)$;

iii) $H\hat{} = -H/(1-H) = -\sum_{i=1}^{N} H^i$;

iv) $(H^m)\hat{} = (-H/(1-H))^m = (-1)^m H^m \sum_{i=0}^{N-m} \binom{m+i-1}{i} H^i$;

v) $(H^N)\hat{} = (-1)^N H^N$;

vi) $H^N \cdot [\mathcal{O}_X(-1)] = H^N$;

vii) $[i_*\mathcal{O}_{\mathcal{P}}] = H^N$ *for a rational point* $i : \mathcal{P} \to \mathbb{P}^N$.

PROPOSITION 5.1. *If $X = \mathbb{P}^N$, then*

$$E^+(X) = \mathbb{Z}/2\mathbb{Z}\,[\mathcal{O}_X] \qquad \textit{and} \qquad E^-(X) = \begin{cases} 0 & \textit{for even } N \\ \mathbb{Z}/2\mathbb{Z}\left[H^N\right] & \textit{for odd } N \end{cases};$$

$$E^+\big(X, \mathcal{O}_X(-1)\big) = \begin{cases} \mathbb{Z}/2\mathbb{Z}\left[H^N\right] & \textit{for even } N \\ 0 & \textit{for odd } N \end{cases} \qquad \textit{and} \qquad E^-\big(X, \mathcal{O}_X(-1)\big) = 0.$$

PROOF. Put $L = \mathcal{O}_X(-1)$ for brevity. If $N = 2m$, then with respect to the base $[\mathcal{O}_X(-m)], \ldots, [\mathcal{O}_X(-1)], [\mathcal{O}_X], [\mathcal{O}_X(1)], \ldots, [\mathcal{O}_X(m)]$ of the free abelian group $K_0(X)$ the involutions $\hat{}$ and $\hat{}^{L}$ have matrices

$$\begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & \cdots & 1 & 0 \\ \vdots & & \vdots & \iddots & \vdots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & 0 \\ \vdots & \iddots & \vdots & & \vdots & \vdots \\ 1 & \cdots & 0 & \cdots & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \binom{N+1}{N} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & -\binom{N+1}{N-1} \\ \vdots & \vdots & & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & (-1)^{m+1}\binom{N+1}{m+1} \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & (-1)^{m}\binom{N+1}{m} \\ \vdots & \vdots & \iddots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \binom{N+1}{2} \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & -\binom{N+1}{1} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

respectively, since the identity $[\mathcal{O}_X(m)] \cdot H^{N+1} = 0$ provides expression for

$$[\mathcal{O}_X(m)]\ \hat{}^{L} = [\mathcal{O}_X(-m-1)]\,.$$

Direct computation of eigenvectors for eigenvalue 1 shows that both E^+ - groups have order 2 and nontrivial element is the coset of 1 and the coset of H^N respectively. Analogous inspection of eigenvectors for eigenvalue -1 shows that both E^- - groups are trivial.

If $N = 2m+1$, then with respect to the base

$$[\mathcal{O}_X(-m)]\,, \ldots, [\mathcal{O}_X(-1)]\,, [\mathcal{O}_X]\,, [\mathcal{O}_X(1)]\,, \ldots, [\mathcal{O}_X(m)]\,, [\mathcal{O}_X(m+1)]$$

of the free abelian group $K_0(X)$ the involution $\hat{}$ has matrix

$$\begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & 1 & \binom{N+1}{1} \\ 0 & \cdots & 0 & \cdots & 1 & 0 & -\binom{N+1}{2} \\ \vdots & & \vdots & \iddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & 0 & (-1)^{m}\binom{N+1}{m+1} \\ \vdots & \iddots & \vdots & & \vdots & \vdots & \vdots \\ 1 & \cdots & 0 & \cdots & 0 & 0 & \binom{N+1}{N} \\ 0 & \cdots & 0 & \cdots & 0 & 0 & -1 \end{bmatrix}$$

since the identity $[\mathcal{O}_X(m+1)] \cdot H^{N+1} = 0$ provides expression for $[\mathcal{O}_X(m+1)]\hat{} = [\mathcal{O}_X(-m-1)]$. It is easy to see that $E^+(X) = \mathbb{Z}/2\mathbb{Z}\,[\mathcal{O}_X]$ and $E^-(X) = \mathbb{Z}/2\mathbb{Z}\,\left[H^N\right]$. The most convenient base of $K_0(X)$ for odd $N = 2m+1$ and $\hat{}^{L}$ is

$$[\mathcal{O}_X(-m-1)]\,, [\mathcal{O}_X(-m)]\,, \ldots, [\mathcal{O}_X(-1)]\,, [\mathcal{O}_X]\,, [\mathcal{O}_X(1)]\,, \ldots, [\mathcal{O}_X(m)]\,.$$

Inspection of eigenvectors of the matrix

$$\begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 1 \\ \vdots & & \vdots & \vdots & \iddots & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \iddots & \vdots & \vdots & & \vdots \\ 1 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}$$

of $\hat{\ }^{L}$ with respect to this base shows that $E^+(X,L) = E^-(X,L) = 0$. ☐

More sophisticated technique was used to prove the following main result of [**12**]:

THEOREM 5.2. *i) Let X be a projective quadric hypersurface defined by equation*

$$z^2 + \sum_{i=0}^{m} x_i y_i = 0$$

in $\mathbb{P}^{2m+2} = \mathrm{Proj}F[x_0, y_0, \ldots, x_m, y_m, z]$, where F is a field of characteristic different from two. Denote L_m the class in $K_0(X)$ of maximal linear subspace of X. Then

$$E^+(X) = \mathbb{Z}/2\mathbb{Z}\,[\mathcal{O}_X] \text{ and } E^-(X) = \begin{cases} \mathbb{Z}/2\mathbb{Z}\,[L_m] & \text{for even } m \\ \mathbb{Z}/2\mathbb{Z}\,[H^m] & \text{for odd } m \end{cases};$$

ii) Let X be a projective quadric hypersurface defined by equation

$$\sum_{i=0}^{m} x_i y_i = 0$$

in $\mathbb{P}^{2m+1} = \mathrm{Proj}F[x_0, y_0, \ldots, x_m, y_m]$, where F is a field of characteristic different to two. Denote L_0 the class in $K_0(X)$ of rational point and L'_m, L''_m classes of two (non equivalent) maximal linear subspaces of X. Then

$$E^+(X) = \mathbb{Z}/2\mathbb{Z}\,[\mathcal{O}_X] \oplus \mathbb{Z}/2\mathbb{Z}\,[L_0] \quad \text{and}$$

$$E^-(X) = \begin{cases} 0 & \text{for even } m \\ \mathbb{Z}/2\mathbb{Z}\,[L'_m] \oplus \mathbb{Z}/2\mathbb{Z}\,[L''_m] & \text{for odd } m \end{cases};$$

moreover, the map $e^0 : W(X) \to E^+(X)$ is surjective for $m > 1$.

PROOF. [**12**], Theorem 5.7 and Theorem 6.2 . ☐

6. Covariant properties

Let $i : Y \to X$ be a regular closed embedding of codimension r. Denote $\Theta_{Y/X}$ both the line bundle $\bigwedge^r \mathcal{N}_{Y/X}$ and its class in $K_0(Y)$. The following theorem is a variant of the theorem 16.6 of [**8**] , and the proof is almost a copy of Manin's proof. This in turn is based on Berthelot computations ([**3**]).

THEOREM 6.1. *Let $i : Y \to X$ be a regular closed embedding of regular schemes, $\mathcal{N}_{Y/X}$ - the normal bundle of Y in X, $r = \mathrm{rank}\mathcal{N}_{Y/X} = \mathrm{codim}_X Y$, $\Theta_{Y/X} = \left[\bigwedge^r \mathcal{N}_{Y/X}\right] \in K_0(Y)$. Then for arbitrary $x \in K_0(Y)$ the following equality holds:*

$$\big(i_*(x)\big)\hat{} = (-1)^r \cdot i_*(x\hat{} \cdot \Theta_{Y/X}).$$

PROOF. Consider pairs (i, x) consisting of a regular closed embedding $i : Y \to X$ and an element $x \in K_0(Y)$. A pair (i, x) is *good* if the asserted equality

$$\big(i_*(x)\big)\hat{} = (-1)^r \cdot i_*(x\hat{} \cdot \Theta_{Y/X})$$

holds. We need to show that every pair (i, x) is good. To do this we shall produce an amount of good pairs. Let $\mathcal{I}$ be the sheaf of ideals of Y in X, $X' = \mathrm{Proj}(\bigoplus_{n=0}^{\infty} \mathcal{I}^N)$ - the blowing up X along Y, $Y' = \mathrm{Proj}(\bigoplus_{n=0}^{\infty} \mathcal{I}^N)/\mathcal{I}^{N+1}) \cong \mathbb{P}_Y(\mathcal{N}_{Y/X}\hat{})$ - the exclusive divisor, and

$$\begin{array}{ccc} Y' & \xrightarrow{\ j\ } & X' \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y & \xrightarrow{\ i\ } & X \end{array}$$

- blowing-up diagram. Then the projection $g : Y' \to Y$ is isomorphic to the canonical projection

$$\mathbb{P}_Y(\mathcal{N}_{Y/X}\hat{}) \to Y,$$

Y' is defined in X' by sheaf of ideals $\mathcal{J} = \mathcal{O}_{X'}(1)$, $\mathcal{N}_{Y'/X'} = \mathcal{O}_{Y'}(-1)$, $\Theta_{Y'/'_X} = [\mathcal{O}_{Y'}(-1)]$. Denote $\mathcal{F} = \Omega^1_{Y'/Y}(-1)$, so there is exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow g^*(\mathcal{N}_{Y/X}\hat{}) \longrightarrow \mathcal{O}_{Y'}(1) \longrightarrow 0.$$

LEMMA 6.2.

$$\mathcal{F} = \mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_Y)$$

(compare [**3**], Lemme 3.2).

PROOF. The exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \to 0$$

tensored with $\mathcal{O}_{X'}$ yields exactness of the sequence

$$0 \longrightarrow \mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_Y) \longrightarrow \mathcal{I} \otimes \mathcal{O}_{X'} \longrightarrow \mathcal{O}_{X'} \longrightarrow \mathcal{O}_{Y'} \longrightarrow 0$$

since $\mathrm{Tor}_k^{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_Y) = 0$ for $k > 0$. Thus the image of $\mathcal{I} \otimes \mathcal{O}_{X'}$ in $\mathcal{O}_{X'}$ coincides with $\mathcal{J} = \mathrm{Ker}(\mathcal{O}_{X'} \to \mathcal{O}_{Y'})$ and the sequence

$$0 \longrightarrow \mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_Y) \longrightarrow \mathcal{I} \otimes \mathcal{O}_{X'} \longrightarrow \mathcal{J} \longrightarrow 0$$

is exact. But $\mathcal{J} = \mathcal{O}_{X'}(1)$ is locally free, so this exact sequence splits locally, and therefore remains exact after tensoring with $\mathcal{O}_{Y'} = \mathcal{O}_{X'}/\mathcal{J}$.
Moreover, $\mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_Y)$ vanishes outside Y', so it remains unchanged after tensoring with $\mathcal{O}_{Y'}$. Next,

$$(\mathcal{I} \otimes \mathcal{O}_{X'}) \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}/\mathcal{J} = \mathcal{I} \otimes \mathcal{O}_{X'}/\mathcal{J} = \mathcal{I}/\mathcal{I}^2 \otimes \mathcal{O}_{Y'} = g^*(\mathcal{N}_{Y/X}\hat{}),$$

and

$$\mathcal{J} \otimes_{\mathcal{O}_{X'}} (\mathcal{O}_{X'}/\mathcal{J}) = \mathcal{J}/\mathcal{J}^2 = \mathcal{O}_{Y'}(1).$$

Thus substitution yields exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_Y) \longrightarrow g^*(\mathcal{N}_{Y/X}\hat{\ }) \longrightarrow \mathcal{O}_{Y'}(1) \longrightarrow 0.$$

Therefore $\mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_Y) = \mathcal{F}$. □

Denote, as usual, $\lambda_t(\mathcal{E}) = \sum_{i=0}^{\infty} \left[\bigwedge^i \mathcal{E}\right] t^i \in (K_0(X))\,[t]$.

Lemma 6.3.

$$f^* i_*(y) = j_*\big(g^*(y)\lambda_{-1}(\mathcal{F})\big)$$

for arbitrary $y \in K_0(Y)$.

([**3**] , Proposition 3.4).

Proof. If $y = [\mathcal{E}]$, where $\mathcal{E}$ is a locally free sheaf, then

$$\begin{aligned}
f^* i_*([\mathcal{E}]) &= \sum_{k=0}^{\infty} (-1)^k \left[\mathrm{T}or_k^{\mathrm{O}_X}(\mathcal{O}_{X'}, \mathcal{E})\right] \\
&= \sum_{k=0}^{\infty} (-1)^k \left[\mathrm{T}or_k^{\mathrm{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_Y) \otimes_{\mathcal{O}_{Y'}} g^*(\mathcal{E})\right] \\
&= \sum_{k=0}^{\infty} (-1)^k \left[\bigwedge^k \mathcal{F}\right] \cdot g^*\,[\mathcal{E}] \\
&= \lambda_{-1}(\mathcal{F}) \cdot g^*\,[\mathcal{E}]\,.
\end{aligned}$$

□

The first observation on good pairs is
a) (i, x) *is good iff* $\big(j, g^*(x)\lambda_{-1}(\mathcal{F})\big)$ *is good.*
In fact, $f_* f^* = \mathrm{id}$, so f^* is a monomorphism. The equality

$$\big(i_*(x)\big)\hat{\ } = (-1)^r i_*(x\hat{\ } \cdot \Theta_{Y/X})$$

is equivalent to

$$f^*\big(i_*(x)\big)\hat{\ } = \big(f^* i_*(x)\big)\hat{\ } = (-1)^r f^* i_*(x\hat{\ } \cdot \Theta_{Y/X})$$

which - in force of lemma 6.3 - is equivalent to

$$\begin{aligned}
\Big(j_*\big(g^*(x)\lambda_{-1}(\mathcal{F})\big)\Big)\hat{\ } &= (-1)^r j_*\big(g^*(x)\hat{\ } \cdot \Theta_{Y/X})\lambda_{-1}(\mathcal{F})\big) \\
&= (-1)^r j_*\big(g^*(x\hat{\ })g^*(\Theta_{Y/X})\lambda_{-1}(\mathcal{F})\big).
\end{aligned}$$

Moreover, equalities

$$\begin{aligned}
g^*(\Theta_{Y/X}) = g^*(\left[\bigwedge^r \mathcal{N}_{Y/X}\right]) &= \left[\bigwedge^r g^*(\mathcal{N}_{Y/X})\right] \\
= \left[(\bigwedge^r g^*(\mathcal{N}_{Y/X})\hat{\ })\hat{\ }\right] &= \left[(\bigwedge^{r-1} \mathcal{F} \otimes \mathcal{O}_Y(1))\hat{\ }\right] = \left[\bigwedge^{r-1} \mathcal{F}\hat{\ } \cdot \Theta_{Y'/X'}\right]
\end{aligned}$$

and

$$\left[\bigwedge^k \mathcal{F}\hat{\ }\right] = \left[\bigwedge^{r-1-k} \mathcal{F} \otimes \bigwedge^{r-1} \mathcal{F}\hat{\ }\right]$$

hold, hence the equality under consideration is equivalent to

$$\begin{aligned}\Big(j_*\big(g^*(x)\lambda_{-1}(\mathcal{F})\big)\Big)\hat{} &= (-1)^r j_*\big(g^*(x\hat{})g^*(\Theta_{Y/X})\lambda_{-1}(\mathcal{F})\big)\\ &= (-1)^r j_*\big(g^*(x\hat{})(-1)^{r-1}\lambda_{-1}(\mathcal{F})\hat{}\cdot\Theta_{Y'/X'}\big)\\ &= -j_*\Big(g^*\big(x\lambda_{-1}(\mathcal{F})\big)\hat{}\cdot\Theta_{Y'/X'}\Big).\end{aligned}$$

Thus (i,x) is good iff $\big(j, g^*(x)\lambda_{-1}(\mathcal{F})\big)$ is good.

Next note that

b) if $y \in \operatorname{Im} j^*$ then (j,y) is good.

If, say, $y = j^*(z)$ then the projection formula yields

$$\begin{aligned}j_*(y\hat{}\cdot\Theta_{Y'/X'}) &= j_*\big(j^*(z\hat{})\cdot\Theta_{Y'/X'}\big) = z\hat{}j^*\left[\mathcal{O}_{Y'}(-1)\right]\big)\\ &= z\hat{}\cdot[\mathcal{O}_{X'}(-1)]\left([\mathcal{O}_{X'}]-[\mathcal{O}_{X'}(1)]\right) = z\hat{}\cdot\left([\mathcal{O}_{X'}(-1)]-[\mathcal{O}_{X'}]\right).\end{aligned}$$

On the other hand

$$\begin{aligned}-\big(j_*(y)\big)\hat{} &= -\big(j_*j^*(z)\big)\hat{} = -(z\cdot j_*\left[\mathcal{O}_{Y'}(1)\right])\hat{} = -z\hat{}\big([\mathcal{O}_{X'}]-[\mathcal{O}_{X'}(1)]\big)\hat{}\\ &= -z\hat{}([\mathcal{O}_{X'}]-[\mathcal{O}_{X'}(-1)]) = z\hat{}\cdot([\mathcal{O}_{X'}(-1)]-[\mathcal{O}_{X'}]).\end{aligned}$$

Thus $\big(j_*(y)\big)\hat{} = -j_*(y\hat{}\cdot\Theta_{Y'/X'})$, so (j,y) is good.

c) if there exists $z \in K_0(Y')$ such that $y = z\big(1-[\mathcal{O}_{Y'}(1)]\big)$, then (j,y) is good.

In fact, $Y' = \mathbb{P}_Y(\mathcal{N}_{Y/X}\hat{})$, $K_0(Y') \cong (K_0(Y))[T]/(\lambda_{-T}(\mathcal{N}_{Y/X}\hat{}))$ for a variable T corresponding to the class of $\mathcal{O}_{Y'}(-1)$, $j^*j_*(z) = z(1-[\mathcal{O}_{Y'}(1)]) = y$, so $y \in \operatorname{Im} j^*$ and b) shows that (j,y) is good.

d) if $\lambda_{-1}(\mathcal{F}) = z\big(1-[\mathcal{O}_{Y'}(1)]\big)$ then each pair (i,x) is good.

By c) pair $\big(j, g^*(x)\lambda_{-1}(\mathcal{F})\big)$ is good, thus (i,x) is good by a).

Now we change X. Put $Z = \mathbb{P}^2 \times X = \operatorname{Proj}\mathcal{O}_X\left[t_0,t_1,t_2\right]$. Embed X into Z as subscheme $t_0 = t_1 = 0$ by the map $s : X \longrightarrow Z$. There is exact sequence

$$0 \longrightarrow \mathcal{N}_{Y/X} \longrightarrow \mathcal{N}_{Y/Z} \longrightarrow i^*\mathcal{N}_{X/Z} \longrightarrow 0.$$

Moreover, the bundle $\mathcal{N}_{X/Z}$ is trivial of rank 2:

$$\mathcal{N}_{X/Z} = \mathcal{O}_X^2.$$

Thus the equality $[\mathcal{N}_Y/Z] = \big[\mathcal{N}_{Y/X}\big] + 2$ holds in $K_0(Y)$. Note also that

$$\Theta_{Y/Z} = \left[\bigwedge^{r+2}\mathcal{N}_{Y/Z}\right] = \left[\bigwedge^{r}\mathcal{N}_{Y/X}\right] = \Theta_{Y/X} \text{ and } \Theta_{X/Z} = \left[\bigwedge^{2}\mathcal{N}_{X/Z}\right] = 1.$$

Consider new blowing up diagram:

$$\begin{array}{ccc} Y'' & \xrightarrow{j''} & X'' \\ {\scriptstyle g''}\big\downarrow & & \big\downarrow{\scriptstyle f''} \\ Y & \xrightarrow{s\circ i} & Z \end{array}$$

e) $(s \circ i, x)$ is good for every $x \in K_0(Y)$.

Define the sheaf $\mathcal{F}''$ as the kernel in the exact sequence

$$0 \longrightarrow \mathcal{F}'' \longrightarrow g''^*(\mathcal{N}_{Y/Z}\hat{}) \longrightarrow \mathcal{O}_{Y''}(1) \longrightarrow 0.$$

The map λ_{-1} is a homomorphism, so

$$\begin{aligned}(\lambda_{-1}\mathcal{F}'')\cdot\big(\lambda_{-1}\mathcal{O}_{Y''}(1)\big) &= \lambda_{-1}g''^{*}(\mathcal{N}_{Y/Z}\hat{\ }) = g''^{*}\lambda_{-1}(\mathcal{N}_{Y/Z}\hat{\ })\\ &= g''^{*}\big(\lambda_{-1}(\mathcal{N}_{Y/X}\hat{\ })\cdot\lambda_{-1}(i^{*}{\mathcal{O}_X}^{2})\big)\\ &= g''^{*}\big(\lambda_{-1}(\mathcal{N}_{Y/X}\hat{\ })\big)\cdot g''^{*}\big(\lambda_{-1}({\mathcal{O}_Y}^{2})\big)\end{aligned}$$

and $\lambda_{-1}({\mathcal{O}_Y}^{2})=0$. Thus

$$\big(\lambda_{-1}(\mathcal{F}'')\big)\cdot\Big(\lambda_{-1}\big(\mathcal{O}_{Y''}(1)\big)\Big) = \big(\lambda_{-1}(\mathcal{F}'')\big)\cdot\big(1-[\mathcal{O}_{Y''}(1)]\big)=0.$$

Since

$$Y''={\mathbb{P}^2}_{Y'} \quad\text{and}\quad K_0(Y'')\cong\big(K_0(Y')\big)[T]/(1-T^3)$$

where the variable T corresponds to the class of $\mathcal{O}_{Y''}(1)$, the equality

$$\big(\lambda_{-1}(\mathcal{F}'')\big)\cdot\big(1-[\mathcal{O}_{Y''}(1)]\big)=0$$

yields

$$\lambda_{-1}(\mathcal{F}'')\in\big(1-[\mathcal{O}_{Y''}(1)]\big)\cdot K_0(Y'').$$

Therefore the pair $(s\circ i,x)$ is good by d).

Denote $p:{\mathbb{P}^2}_X\to X$ the structure map.

f) each pair (s,x) is good.

In fact, for arbitrary bundle $\mathcal{E}$ on X one has

$$[s_*(\mathcal{E})]=[p^*(\mathcal{E})]\cdot\big(1-[\mathcal{O}_Z(-1)]\big)^2.$$

It follows that

$$\begin{aligned}(s_*[\mathcal{E}])\hat{\ } &= \Big([p^*(\mathcal{E})]\cdot\big(1-[\mathcal{O}Z(-1)]\big)^2\Big)\hat{\ } = [p^*(\mathcal{E}\hat{\ })]\cdot\big(1-[\mathcal{O}Z(1)]\big)^2\\ &= [p^*(\mathcal{E}\hat{\ })]\cdot[\mathcal{O}_Z(2)]\cdot\big(1-[\mathcal{O}_Z(-1)]\big)^2 = [p^*(\mathcal{E}\hat{\ })]\cdot\big(1-[\mathcal{O}_Z(-1)]\big)^2\\ &= s_*[\mathcal{E}\hat{\ }]\end{aligned}$$

since $[\mathcal{O}_Z(2)]\cdot\big(1-[\mathcal{O}_Z(-1)]\big)^2=\big(1-[\mathcal{O}_Z(-1)]\big)^2$ in $K_0(\mathbb{P}^2_X)$ as was stated in lemma 5.1 vi).

g) (i,x) is good for every $X\in K_0(Y)$.

To see this note first that

$$\big(s_*i_*(x)\big)\hat{\ }=(-1)^{r+2}s_*i_*(x\hat{\ }\Theta_{Y/Z})=(-1)^r s_*i_*(x\hat{\ }\Theta_{Y/X})$$

since $(s\circ i,x)$ is good by e). Next, by f)

$$s_*\big(i_*(x)\hat{\ }\big)=\big(s_*i_*(x)\big)\hat{\ }=(-1)^r s_*i_*(x\hat{\ }\Theta_{Y/X})=s_*\big((-1)^r i_*(x\hat{\ }\Theta_{Y/X})\big).$$

To finish the proof it is enough to note that $p\circ s=\mathrm{id}_X$, so $p_*\circ s_*=\mathrm{id}$. Therefore

$$i_*(x)\hat{\ }=(-1)^r\cdot i_*(x\hat{\ }\Theta_{Y/X})$$

and the theorem is proved. □

Note that under assumptions of theorem 6.1 if in addition X and Y are varieties over a field, then $\Theta_{Y/X}$ can be expressed in terms of canonical bundles:

LEMMA 6.4. *If $i:Y\to X$ is a closed embedding of codimension r of regular varieties over a field, then*

$$\omega_Y=i^*(\omega_X)\otimes_{\mathcal{O}_Y}\bigwedge^r\mathcal{N}_{Y/X}.$$

PROOF. [**5**], Proposition 8.20. □

Theorem 6.5 (Excision for E-groups). *Let $i : Y \to X$ be a regular closed embedding of regular schemes, U - the open complement to Y in X, $\mathcal{N}_{Y/X}$ - the normal bundle of Y in X, $r = rank \mathcal{N}_{Y/X} = codim_X Y$, $\Theta_{Y/X} = \left[\bigwedge^r \mathcal{N}_{Y/X}\right] \in K_0(Y)$. If $i_* : K_0(Y) \to K_0(X)$ is injective, then for arbitrary line bundle L on X, the following hexagon is exact:*

$$
\begin{array}{ccccc}
 & & E^+(X,L) & & \\
 & \nearrow & & \searrow & \\
E^{(-1)^r}(Y, \Theta_{Y/X} \otimes_{\mathcal{O}_Y} i^*L) & & & & E^+(U, j^*L) \\
\uparrow & & & & \downarrow \\
E^-(U, j^*L) & & & & E^{(-1)^{r+1}}(Y, \Theta_{Y/X} \otimes_{\mathcal{O}_Y} i^*L) \\
 & \nwarrow & & \swarrow & \\
 & & E^-(X,L) & &
\end{array}
$$

Proof. The formula

$$(i_*(x))\hat{} = (-1)^r \cdot i_*(x\hat{} \cdot \Theta_{Y/X})$$

of theorem 6.1 yields exact sequence of 2-periodical complexes

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \xrightarrow{1+\hat{}\,j^*L} & K_0(U) & \xrightarrow{1-\hat{}\,j^*L} & K_0(U) & \xrightarrow{1+\hat{}\,j^*L} & K_0(U) & \xrightarrow{1-\hat{}\,j^*L} & \cdots \\
 & & j^* \uparrow & & j^* \uparrow & & j^* \uparrow & & \\
\cdots & \xrightarrow{1+\hat{}} & K_0(X) & \xrightarrow{1-\hat{}} & K_0(X) & \xrightarrow{1+\hat{}} & K_0(X) & \xrightarrow{1-\hat{}} & \cdots \\
 & & i_* \uparrow & & i_* \uparrow & & i_* \uparrow & & \\
\cdots & \xrightarrow{1+(-1)^r\alpha} & K_0(Y) & \xrightarrow{1-(-1)^r\alpha} & K_0(Y) & \xrightarrow{1+(-1)^r\alpha} & K_0(Y) & \xrightarrow{1-(-1)^r\alpha} & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where α stands for $\hat{}\,\Theta_{Y/X} \otimes_{\mathcal{O}_Y} i^*(L)$. Associated long exact homology sequence has period six, hence exactness of the hexagon. □

Lemma 6.4 allows more "invariant" form for exact hexagon of excision:

Corollary 6.6 (Excision for E-groups). *Let $i : Y \to X$ be a closed embedding of regular varieties over a field, U - the open complement to Y in X, $r = codim_X Y$. If $i^* : K_0(Y) \to K_0(X)$ is injective, then for arbitrary line bundle L on X, the* *following hexagon is exact:*

$$
\begin{array}{ccccc}
 & & E^{+}(X, \omega_X \otimes L) & & \\
 & \nearrow & & \searrow & \\
E^{(-1)^r}(Y, \omega_Y \otimes_{\mathcal{O}_Y} i^*L) & & & & E^{+}(U, \omega_U \otimes_{\mathcal{O}_U} j^*L) \\
\uparrow & & & & \downarrow \\
E^{-}(U, \omega_U \otimes_{\mathcal{O}_U} j^*L) & & & & E^{(-1)^{r+1}}(Y, \omega_Y \otimes_{\mathcal{O}_Y} i^*L) \\
 & \nwarrow & & \swarrow & \\
 & & E^{-}(X, \omega_X \otimes L) & &
\end{array}
$$

Proof. Substitution of $\omega_X \otimes L$ in place of L, $\omega_U \otimes_{\mathcal{O}_U} j^*L$ in place of j^*L and $\omega_Y \otimes_{\mathcal{O}_Y} i^*L$ in place of i^*L into commutative diagram from the proof of theorem 6.5 yields commutative diagram, since for arbitrary $X \in K_0(Y)$ the following equalities hold:

$$
\begin{aligned}
\big(i^*(x)\big)\hat{}\cdot [\omega_X \otimes L] &= (-1)^r \cdot i_*(x\hat{}\cdot \Theta_{Y/X}) \cdot [\omega_X \otimes L] \\
&= (-1)^r \cdot i_*(X\hat{}\cdot \Theta_{Y/X} \cdot i^*[\omega_X] \cdot i^*[L]) \\
&= (-1)^r \cdot i_*(x\hat{}\cdot \left[\bigwedge^r \mathcal{N}_{Y/X}\right] \cdot i^*[\omega_X] \cdot i^*[L]) \\
&= (-1)^r \cdot i_*(x\hat{}\cdot \left[\bigwedge^r \mathcal{N}_{Y/X} \otimes i^*\omega_X\right] \cdot i^*[L]) \\
&= (-1)^r \cdot i_*(x\hat{}\cdot [\omega_Y] \cdot i^*[L]).
\end{aligned}
$$

□

7. Geometry of Grassmann varieties

Let F be a field, V a vector space over F of dimension n. Denote by G the Grassmann variety $Gr(V,k)$ of k-dimensional subspaces in V. Moreover, denote $\mathcal{S}$ the universal (or tautological) subbundle of G and $\mathcal{Q}$ the universal factor bundle.

Theorem 7.1. *For $G = Gr(V,k)$ there is an isomorphism of sheaves*

$$\Omega^1_{G/F} \xrightarrow{\sim} \mathcal{Q}\hat{} \otimes \mathcal{S}.$$

Proof. See [**9**], Problem 5.B. □

Corollary 7.2.

The canonical bundle ω_G of a Grassmann variety $G = Gr(V,k)$ is isomorphic to $(\bigwedge^r \mathcal{S})^{\otimes n}$.

Proof. Exactness of the universal sequence $0 \to \mathcal{S} \to \mathcal{O}_G \otimes_F V \to \mathcal{Q} \to 0$ yields that line bundles $\bigwedge^k \mathcal{S}$ and $\bigwedge^{n-k} \mathcal{Q}$ are dual to each other since

$$\bigwedge^k \mathcal{S} \otimes \bigwedge^{n-k} \mathcal{Q} \cong \bigwedge^n (\mathcal{O}_G \otimes_F V) \cong \mathcal{O}_G.$$

Therefore

$$\bigwedge^{k(n-k)} \Omega^1_{G/F} \cong \bigwedge^{k(n-k)} (\mathcal{Q}\hat{} \otimes \mathcal{S}) \cong (\bigwedge^{n-k} \mathcal{Q}\hat{})^{\otimes k} \otimes (\bigwedge^k \mathcal{S})^{\otimes (n-k)} \cong (\bigwedge^k \mathcal{S})^{\otimes n}.$$

□

Remark 7.3. Denote $pl : Gr(n,k) \to \mathbb{P}(\bigwedge^k V)$ the Plücker embedding. By the very definition , $pl^*\mathcal{O}(-1) = \bigwedge^k \mathcal{S}$ and $\omega_G = pl^*\mathcal{O}(-n)$.

The following geometric construction is well known (see [**4**], Example 14.6.2 for details and further references). Fix a decomposition

$$V = W \oplus M$$

into direct sum of codimension 1 subspace W and dimension 1 subspace M of the vector space V . There is induced decomposition of the exterior power:

$$\bigwedge^k V = \bigwedge^k W \oplus \big((\bigwedge^{k-1} V) \wedge M\big).$$

There is natural embedding of Grassmann varieties:

$$i' : Gr(W,k) \to Gr(V,k).$$

In the Plücker coordinates the closed subvariety $Gr(W,k)$ is an intersection of $Gr(V,k)$ with a linear subspace $\mathbb{P}(\bigwedge^k W)$ of $\mathbb{P}(\bigwedge^k V)$. The map $U \longmapsto U \oplus M$ defines closed embedding

$$i'' : Gr(W,k-1) \to Gr(V,k).$$

In the Plücker coordinates the closed subvariety $Gr(W,k-1)$ is an intersection of $Gr(V,k)$ with the linear subspace $\mathbb{P}\big((\bigwedge^{k-1} V) \wedge M\big)$.

For brevity sake denote $G = Gr(V,k)$, $H' = Gr(W,k)$, $H'' = i''\big(Gr(W,k-1)\big)$.

The formula $U \mapsto U \cap W$ defines a morphism $p'' : G \setminus H' \to Gr(W,k-1)$, whose fibres are affine spaces corresponding to the vector spaces $W/U \cap W$. The homotopy property yields thus that the sequence

$$K_0(H') \xrightarrow{i'_*} K_0(G) \xrightarrow{(p''^*)^{-1} \circ j'^*} K_0\big(Gr(W,k-1)\big) \longrightarrow 0$$

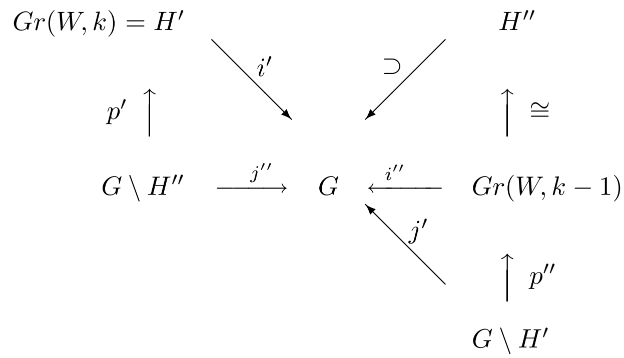
where $j' : G \setminus H' \to G$ is open embedding, is exact. Moreover, i'' is a section for p''.

The open complement $G \setminus H''$ to H'' consists of subspaces which do not contain M. The projection $U \longmapsto (U+M)/M$ of $V = W \oplus M$ onto the first direct summand defines a morphism $p' : G \setminus H'' \to Gr(W,k)$, whose fibres are affine spaces corresponding to the vector spaces $\mathrm{Hom}_F\big((U+M)/M, M\big)$. The homotopy property yields thus that the sequence

$$K_0(H'') \xrightarrow{i''_*} K_0(G) \xrightarrow{(p'^*)^{-1} \circ j''^*} K_0\big(Gr(W,k)\big) \longrightarrow 0$$

is exact. In fact, both above sequences of K-groups are split exact, since all the groups are free abelian groups and $\mathrm{rank}\big(K_0(G)\big) = \binom{n}{k}$ is equal to sum of ranks of $K_0(H') = K_0\big(Gr(n-1,k)\big)$ and $K_0(H'') \cong K_0\big(Gr(n-1,k-1)\big)$.

All the maps needed in sequel fit into following diagram:



Here p'' maps a point $P \in G \setminus H'$ onto $P \cap W \in Gr(W, k-1)$ and p' maps a point $P \in G \setminus H''$ onto $(P+M)/M \in Gr(V/M, k)$. Consequently, denote by $\mathcal{S}_G$, $\mathcal{S}_{H'}$ and $\mathcal{S}_{H''}$ the tautological bundles of G, H' and H'', respectively, and by $\mathcal{Q}_G$, $\mathcal{Q}_{H'}$ and $\mathcal{Q}_{H''}$ the canonical factor bundles of G, H' and H''.

Remark 7.4. Natural isomorphism $G(V,k) \cong G(V\hat{}, n-k)$ replaces H' and H'', $\mathcal{S}$ and $\mathcal{Q}\hat{}$, $\mathcal{S}_{H'}$ and $\mathcal{S}_{H''}$, etc. with each other.

Lemma 7.5. *i) the following homomomorphisms of K-groups are equal:*

$$(p'^*)^{-1} \circ j''^* = i'^* \quad , \quad (p''^*)^{-1} \circ j'^* = i''^*;$$

ii)

$$\mathcal{N}_{H'/G} = \mathcal{S}_{H'}\hat{} \qquad , \quad \mathcal{N}_{H''/G} = \mathcal{Q}_{H''};$$

iii)

$$\Theta_{H'/G} = \left[\bigwedge^{k} \mathcal{S}_{H'} \right]\hat{} \quad , \quad \Theta_{H''/G} = \left[\bigwedge^{k-1} \mathcal{S}_{H''} \right]\hat{} \ .$$

Proof. H': It is easy to check on fibres that the restriction $i'^*(\mathcal{S}_G)$ to H' of the tautological bundle for G coincides with the tautological bundle $\mathcal{S}_{H'}$ for H'. Analogously $j''^*(\mathcal{S}_G) = p'^*(\mathcal{S}_{H'})$. Commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & 0 & \longrightarrow & M \otimes_F \mathcal{O}_{H'} & \xrightarrow{\text{id}} & M \otimes_F \mathcal{O}_{H'} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & i'^*\mathcal{S}_G & \longrightarrow & V \otimes_F \mathcal{O}_{H'} & \longrightarrow & i'^*\mathcal{Q} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{S}_{H'} & \longrightarrow & W \otimes_F \mathcal{O}_{H'} & \longrightarrow & \mathcal{Q}_{H'} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

yields equality $i'^*[\mathcal{Q}] = [\mathcal{Q}_{H'}]+1$. In the same way one can see that $p(^*([\mathcal{Q}_{H'}]+1) = j''^*([\mathcal{Q}])$. The assertion $(p'^*)^{-1} \circ j''^* = i'^*$ follows from the fact, that $[\mathcal{S}]$,$[\mathcal{Q}]$ generate $K_0(G)$ as a λ-ring ([**2**], Théorème 4.6, [**7**], theorem 3.12, see also theorem 7.6 below).

Denoting as usual $\mathcal{T}_G = \Omega^1_{G/F}\hat{}$, $\mathcal{T}_{H'} = \Omega^1_{H'/F}\hat{}$ one may deduce assertion ii) from exactness of the sequence

$$0 \longrightarrow \mathcal{T}_{H'} \longrightarrow i'^*\mathcal{T}_G \longrightarrow \mathcal{N}_{H'/G} \longrightarrow 0$$

([**5**], II§8) and the equalities $\Omega^1_{G/F} = \mathcal{Q}\hat{} \otimes \mathcal{S}$, $\Omega^1_{H'/F} = \mathcal{Q}_{H'}\hat{} \otimes \mathcal{S}_{H'}$ of theorem 7.1 above. By Lemma 6.4 and Corollary 7.2

$$\big(\bigwedge^k \mathcal{S}_{H'}\big)^{\otimes(n-1)} = i^*\Big(\big(\bigwedge^k \mathcal{S}_G\big)^{\otimes n}\Big) \otimes \Theta_{H'/G} = \big(\bigwedge^k \mathcal{S}_{H'}\big)^{\otimes n} \otimes \Theta_{H'/G}.$$

which proves the assertion iii).

H'': We shall identify H'' with $Gr(W, k-1)$. The restriction $i''^*(\mathcal{Q}_G)$ of the universal factor bundle for G coincides with universal factor bundle $\mathcal{Q}''_H$. Thus the restriction $i''^*(\mathcal{S}_G)$ to H'' of the tautological bundle for G is isomorphic to the direct sum $\mathcal{S}_{H''} \oplus (M \otimes_F \mathcal{O}_{H''})$ of tautological bundle $\mathcal{S}_{H''}$ for H'' and a trivial line bundle, hence i) and ii). Moreover

$$i''^*\big(\bigwedge^k \mathcal{S}_G\big) \cong \bigwedge^k(\mathcal{S}_{H''} \oplus M \otimes_F \mathcal{O}_{H''}) \cong M \otimes_F \bigwedge^{k-1} \mathcal{S}_{H''}$$

and, again by Lemma 6.4 and Corollary 7.2

$$\big(\bigwedge^{k-1} \mathcal{S}_{H''}\big)^{\otimes(n-1)} \cong i''^*\Big(\big(\bigwedge^k \mathcal{S}_G\big)^{\otimes n}\Big) \otimes \Theta_{H''/G} \cong \big(\bigwedge^{k-1} S_{H''}\big)^{\otimes n} \otimes M^{\otimes n} \otimes \Theta_{H''/G}.$$

□

Description of the K-theory of a Grassmann variety $Gr(V, k)$ is closely related with classification of irreducible representations of the general linear group GL. Namely, for each non increasing sequence $\mathbf{a} = (a_1, a_2, \ldots, a_k)$ of integers let $b(i) = a_i - a_{i+1}$ for $i = 1, 2, \ldots, k-1$ and $b(k) = a_k$. Next, let

$$\Sigma^{\mathbf{a}}\mathcal{S} = \mathcal{S}^{\otimes b(1)} \otimes \big(\bigwedge^2 \mathcal{S}\big)^{\otimes b(2)} \otimes \cdots \otimes \big(\bigwedge^k \mathcal{S}\big)^{\otimes b(k)}$$

where as usual $\mathcal{S}^{\otimes(-r)} = \mathcal{S}^{\hat{}\otimes r}$.

Theorem 7.6. *If $G = Gr(V, k)$ is the Grassmann variety of k-planes in the n-dimensional vector space V over a field F, then*
a) $K_*(G) \cong K_*(F) \otimes_{\mathbb{Z}} K_0(G)$;
b) the ring $K_0(G)$ is isomorphic to the factor ring $\mathbb{Z}[x_1, ..., x_k, y_1, ..., y_{n-k}]/I$, where I is an ideal generated by

$$x_1 + y_1 - \dim(V),$$
$$x_2 + x_1y_1 + y_2 - \binom{\dim(V)}{2},$$
$$\vdots$$
$$\sum_{i+j=r} x_iy_j - \binom{\dim(V)}{r},$$
$$\vdots$$
$$x_ky_k - 1;$$

under this isomorphism x_i *corresponds to* $\left[\bigwedge^i \mathcal{S}\right]$, *and* y_j *corresponds to* $\left[\bigwedge^j \mathcal{Q}\right]$;
c) $K_0(G)$ *is a free abelian groups with base consisting of* $\Sigma^{\mathbf{a}}\mathcal{S}$ *for all non increasing sequences* $\mathbf{a} = (a_1, a_2, \ldots, a_k)$ *of integers with* $n-k \geq a_1$ *and* $a_k \geq 0$*; in particular* $\operatorname{rank}(K_0(G)) = \binom{n}{k}$;
d) if $\mathbf{a} = (a_1, a_2, \ldots, a_k)$, *then* $(\Sigma^{\mathbf{a}}\mathcal{S})\hat{} \cong \Sigma^{\mathbf{a}}\mathcal{S}\hat{} \cong \Sigma^{\mathbf{a}^*}\mathcal{S}$, *where*

$$\mathbf{a}^* = (-a_k, -a_{k-1}, \ldots, -a_1).$$

Proof. a), c) [**10**], Theorem 5.13, [**6**], Theorem 3.4. b) [**7**], Ch. IV, Theorem 3.12, [**2**], Théorème 4.6. d) If $\mathcal{M}$ is a vector bundle of rank s, then $\bigwedge^r \mathcal{M}\hat{}$ is canonically isomorphic to $\bigwedge^{s-r}\mathcal{M} \otimes \bigwedge^s \mathcal{M}\hat{}$, hence c). □

Proposition 7.1. *There are exact sequences of vector bundles*

$$0 \to \bigwedge^k \mathcal{S}_G \to \cdots \to \bigwedge^2 \mathcal{S}_G \to \mathcal{S}_G \to \mathcal{O}_G \to i'_*\mathcal{O}_{H'} \to 0$$
$$0 \to \bigwedge^{n-k} \mathcal{Q}_G\hat{} \to \cdots \to \bigwedge^2 \mathcal{Q}_G\hat{} \to \mathcal{Q}_G\hat{} \to \mathcal{O}_G \to i''_*\mathcal{O}_{H''} \to 0.$$

Consequently, $i'_*\left[\Sigma^{\mathbf{a}}\mathcal{S}_{H'}\right] = \lambda_{-1}(\mathcal{S}_G)\cdot\left[\Sigma^{\mathbf{a}}\mathcal{S}_G\right]$, $i''_*\left[\Sigma^{\mathbf{b}}\mathcal{Q}_{H''}\right] = \lambda_{-1}(\mathcal{Q}_G\hat{})\cdot\left[\Sigma^{\mathbf{b}}\mathcal{Q}_G\right]$ *for arbitrary non-increasing sequences* $\mathbf{a} = (a_1, ..., a_k), \mathbf{b} = (b_1, ..., b_{n-k})$ *of integers.*

Proof. It is known that $H^0(G, \mathcal{S}\hat{}) = V\hat{}$ ([**6**], Lemma 3.2 a)). Any linear function on V with kernel W produces a global section of $\mathcal{S}\hat{}$ and a map $\mathcal{O}_G \to \mathcal{S}_G\hat{}$. Dualization yields homomorphism $\mathcal{S}_G \to \mathcal{O}_G$ which vanishes exacly at points in H', i.e. subspaces contained in W. This homomorphism produces a differential in exterior algebra of $\mathcal{S}_G$ and resulting Koszul complex provides asserted resolution for $i'_*\mathcal{O}_{H'}$. Analogously $H^0(G, \mathcal{Q}_G) = V$. Any vector from M provides a global section of $\mathcal{Q}_G$ and a map $\mathcal{O}_G \to \mathcal{Q}_G$. Dual map $\mathcal{Q}_G\hat{} \to \mathcal{O}_G$ vanishes exactly at points in H'', i. e. subspaces containing M. The second claim follows easily from projection formula, since $i'^*\Sigma^{\mathbf{a}}\mathcal{S}_G = \Sigma^{\mathbf{a}}\mathcal{S}_{H'}$, $i''^*\Sigma^{\mathbf{b}}\mathcal{S}_G = \Sigma^{\mathbf{b}}\mathcal{S}_{H''}$. □

8. Recurrence for Grassmann varieties

Theorem 8.1. *Let* L *be a line bundle on the Grassmann variety* $G = Gr(n,k)$ *and denote by* p *the integral part of* $\frac{n}{2}$ *and by* q *the integral part of* $\frac{k}{2}$. *Then all E-groups* $E^{\pm}(G, L)$ *are finite and the equality*

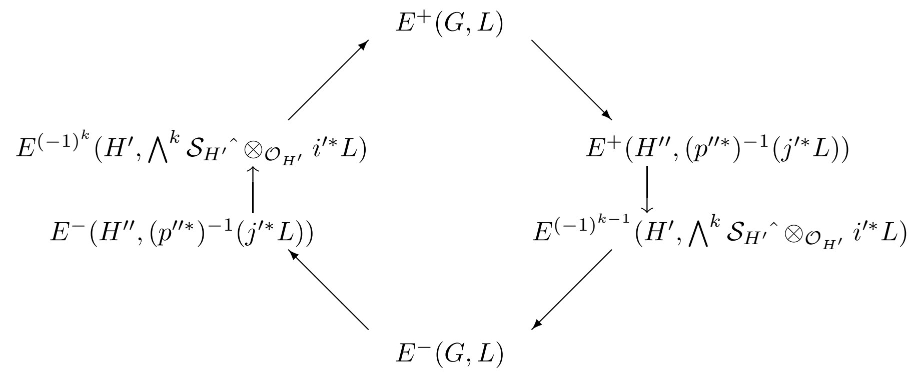
$$\frac{|E^+(G,L|}{|E^-(G,L)|} = 2^{a(n,k)}$$

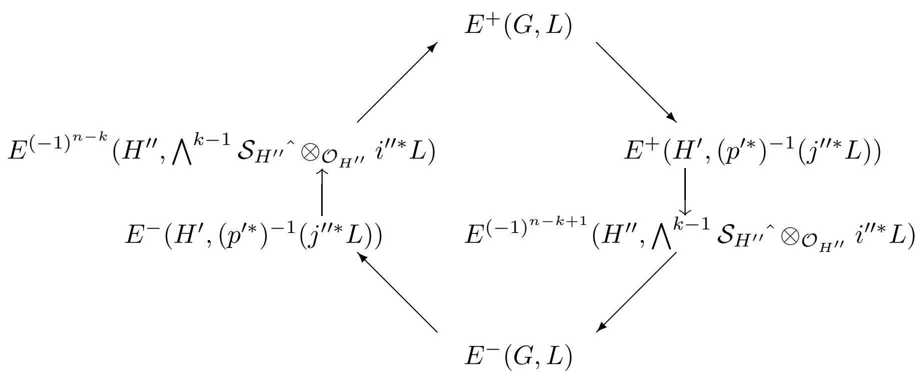
holds, where

$$a(n,k) = \begin{cases} \binom{p}{q} & \textit{iff } k \textit{ is even or } n-k \textit{ is even} \\ 0 & \textit{iff } k \textit{ is odd and } n-k \textit{ is odd} \end{cases}$$

Proof. Excision theorem 6.5 applied to codimension k regular embedding $i' : Gr(W,k) \to G$ and to codimension $n-k$ regular embedding $i'' : Gr(W, k-1) \to G$

with homotopy property (proposition 4.2) yields two exact hexagons:





where H'' is identified with $Gr(W,k-1)$. The Picard group $\mathrm{Pic}(G)$ is an infinite cyclic group, so by the proposition 4.1 there are two cases to treat: the case of trivial bundle $L=\mathcal{O}_G$, and the case of nontrivial, e.g. $L=\bigwedge^k \mathcal{S}_G$. Therefore following four equalities hold:

$$\frac{|E^+(G,\mathbf{1})|}{|E^-(G,\mathbf{1})|}\cdot\frac{\left|E^{(-1)^{k-1}}(H',\bigwedge^k \mathcal{S}_{H'})\right|}{\left|E^{(-1)^k}(H',\bigwedge^k \mathcal{S}_{H'})\right|}\cdot\frac{|E^-(H'',\mathbf{1}'')|}{|E^+(H'',\mathbf{1}'')|}=1$$

$$\frac{\left|E^+(G,\bigwedge^k \mathcal{S}_G)\right|}{\left|E^-(G,\bigwedge^k \mathcal{S}_G)\right|}\cdot\frac{\left|E^{(-1)^{k-1}}(H',\mathbf{1}')\right|}{\left|E^((-1)^k(H',\mathbf{1}')\right|}\cdot\frac{\left|E^-(H'',\bigwedge^{k-1} \mathcal{S}_{H''})\right|}{\left|E^+(H'',\bigwedge^{k-1} \mathcal{S}_{H''})\right|}=1$$

$$\frac{|E^+(G,\mathbf{1})|}{|E^-(G,\mathbf{1})|}\cdot\frac{\left|E^{(-1)^{n-k+1}}(H'',\bigwedge^{k-1} \mathcal{S}_{H''})\right|}{\left|E^{(-1)^{n-k}}(H'',\bigwedge^{k-1} \mathcal{S}_{H''})\right|}\cdot\frac{|E^-(H',\mathbf{1}')|}{|E^+(H',\mathbf{1}')|}=1$$

$$\frac{\left|E^+(G,\bigwedge^k \mathcal{S}_G)\right|}{\left|E^-(G,\bigwedge^k \mathcal{S}_G)\right|}\cdot\frac{\left|E^{(-1)^{n-k+1}}(H'',\mathbf{1}'')\right|}{\left|E^{(-1)^{n-k}}(H'',\mathbf{1}'')\right|}\cdot\frac{\left|E^-(H',\bigwedge^k \mathcal{S}_{H'})\right|}{\left|E^+(H',\bigwedge^k \mathcal{S}_{H'})\right|}=1$$

where $\mathbf{1} = \mathcal{O}_G$, $\mathbf{1}' = \mathcal{O}_{H'}$, $\mathbf{1}'' = \mathcal{O}_{H''}$. Consider logarithms of these quotients with logarithmic base 2 :

$$\frac{|E^+(G,\mathbf{1})|}{|E^-(G,\mathbf{1})|} = 2^{a(n,k)} \qquad , \qquad \frac{\left|E^+(G,\bigwedge^k \mathcal{S}_G)\right|}{\left|E^-(G,\bigwedge^k \mathcal{S}_G)\right|} = 2^{b(n,k)}$$

for all $n > 1$ and all k, $0 < k < n$. The four equations above may be rewritten as follows:

$$\begin{aligned} a(n,k) &= a(n-1,k-1) + (-1)^k \cdot b(n-1,k) \\ &= a(n-1,k) + (-1)^{n-k} \cdot b(n-1,k-1), \\ b(n,k) &= b(n-1,k) + (-1)^{n-k} \cdot a(n-1,k-1) \\ &= b(n-1,k-1) + (-1)^k \cdot a(n-1,k). \end{aligned}$$

$Gr(n,1) \cong Gr(n,n-1) \cong \mathbb{P}^{n-1}$, so proposition 5.1 provides initial values:

$$a(n,1) = b(n,1) = a(n,n-1) = b(n,n-1) = \begin{cases} 0 & \text{for even } n \\ 1 & \text{for odd } n \end{cases} .$$

By induction $a(n,k) = b(n,k)$ in general, the finitness assertion follows and the formula:

$$a(n,k) = \begin{cases} \binom{p}{q} & \text{for } n = 2p,\ k = 2q \text{ and for } n = 2p+1,\ k = 2q \\ & \qquad \text{and for } n = 2p+1,\ k = 2q+1 \\ 0 & \text{for } n = 2p,\ k = 2q+1 \end{cases}$$

holds, hence the theorem. □

Corollary 8.2. *If $n-k$ is even or k is even, then*

$$\left|E^+(Gr(n,k)\right| \geq 2^{\binom{p}{q}}$$

where p is the integral part of $\frac{n}{2}$ and q is the integral part of $\frac{k}{2}$.

Proposition 8.1. *i) If $k = 2q+1$, then maps*

$$i'^* \circ i'_* : E^{\pm}(H', \bigwedge^k \mathcal{S}_{H'}\hat{} \otimes i'^* L) \to E^{\mp}(H', i'^* L),$$

$$i'_* \circ i'^* : E^{\pm}(G, \bigwedge^k \mathcal{S}_G\hat{} \otimes L) \to E^{\mp}(G, L),$$

are trivial.
ii) If $k = 2q$, then maps

$$i'^* \circ i'_* : E^{\pm}(H', \bigwedge^k \mathcal{S}_{H'}\hat{} \otimes i'^* L) \to E^{\pm}(H', i'^* L),$$

$$i'_* \circ i'^* : E^{\pm}(G, \bigwedge^k \mathcal{S}_G\hat{} \otimes L) \to E^{\pm}(G, L),$$

are multiplication by $(-1)^{q+1}\,[\bigwedge^q \mathcal{S}_{H'}]$ $((-1)^{q+1}\,[\bigwedge^q \mathcal{S}_G])$.

Proof. The map $i'_* \circ i'^*$ is - by the projection formula - multiplication by $i'_*[\mathcal{O}_{H'}] = \lambda_{-1}(\mathcal{S}_G)$ (proposition 7.1). The map $i'^* \circ i'_*$ is induced by multiplication by $\lambda_{-1}(\mathcal{S}_{H'})$. Let $\mathcal{S}$ means either $\mathcal{S}_G$ or $\mathcal{S}_{H'}$ Denote

$$p_0 = 0 \ , \ p_i = \sum_{j=0}^{i} (-1)^j \left[\bigwedge^j \mathcal{S} \right] \text{ for } i = 1, 2, \ldots, k.$$

Obviously $p_k = \lambda_{-1}(\mathcal{S})$. Canonical isomorphism $\bigwedge^j \mathcal{S}\hat{} \cong \bigwedge^{k-j} \mathcal{S} \otimes \bigwedge^k \mathcal{S}\hat{}$ yields equality

$$p_i\hat{} = (-1)^k \left[\bigwedge^k \mathcal{S} \right]\hat{} (p_k - p_{k-i-1})$$

for $i = 1, 2, \ldots, k$. Thus, for $k = 2q + 1$, $p_q\hat{} = -\left[\bigwedge^k \mathcal{S}\right]\hat{}(p_k - p_q)$, hence $p_k = p_q - \left[\bigwedge^k \mathcal{S}\right]\hat{}p_q\hat{}$. If $\alpha = \pm\alpha\hat{}[L]\left[\bigwedge^k \mathcal{S}\right]\hat{}$, then

$$\alpha p_k = \alpha \cdot \left(p_q - \left[\bigwedge^k \mathcal{S} \right]\hat{} p_q\hat{} \right) = \alpha p_q \mp \alpha\hat{} p_q\hat{}[L] = 0.$$

In the case $k = 2q$ the equality $p_q\hat{} - (-1)^q [\bigwedge^q \mathcal{S}] = p_{q-1}\hat{} = (-1)^k \left[\bigwedge^k \mathcal{S}\right]\hat{}(p_k - p_q)$ yields

$$p_k + (-1)^q \left[\bigwedge^q \mathcal{S} \right] = p_q + \left[\bigwedge^k \mathcal{S} \right] p_q\hat{} = 0.$$

□

9. Symmetric bilinear forms on Grassmann varieties

It is easy to construct a family of "obvious" symmetric bilinear forms: if $k = 2q$ or $k = 2q+1$ then for arbitrary sequence $a_1 \geq a_2 \geq \cdots \geq a_q$ of nonnegative integers the "palindrom" bundle

$$\Sigma^{(a_1,a_2,\ldots,a_q,-a_q,\ldots,-a_2,-a_1)}\mathcal{S} \text{ for even } k \text{ or}$$

$$\Sigma^{(a_1,a_2,\ldots,a_q,0,-a_q,\ldots,-a_2,-a_1)}\mathcal{S} \text{ for odd } k$$

is self-dual. Moreover, such a bundle is a product of a bundle and its dual, for example:

$$\Sigma^{(a_1,a_2,\ldots,a_q,-a_q,\ldots,-a_2,-a_1)}\mathcal{S} = \Sigma^{(a_1,a_2,\ldots,a_q,0,\ldots,0,0)}\mathcal{S} \otimes \left(\Sigma^{(a_1,a_2,\ldots,a_q,0,\ldots,0,0)}\mathcal{S} \right)\hat{}.$$

So a "palindrom" bundle is a bundle of endomorphisms:

$$\Sigma^{(a_1,a_2,\ldots,a_q,-a_q,\ldots,-a_2,-a_1)}\mathcal{S} = \mathcal{E}nd_{\mathcal{O}_G} \left(\Sigma^{(a_1,a_2,\ldots,a_q,0,\ldots,0,0)}\mathcal{S} \right).$$

For $n = 2p$ or $n = 2p + 1$ there is $\binom{p}{q}$palindrom bundles with $0 < a_1 \leq p - q$. In the case of even $k = 2q$ it is easy to construct an analogous family of "obvious" bundles $\mathcal{A}$ satysfying

$$\mathcal{A} = \mathcal{A}\hat{} \otimes \bigwedge^k \mathcal{S}$$

as

$$\mathcal{A} = \Sigma^{(a_1+1,a_2+1,\ldots,a_q+1,-a_q,\ldots,-a_2,-a_1)}\mathcal{S}$$

(note that $\bigwedge^k \mathcal{S} = \Sigma^{(1,1,\dots,1,1)}\mathcal{S}$). Each such a bundle is a product of a palindrom bundle by $\bigwedge^q \mathcal{S}$.

CONJECTURE. *Classes of above listed "obvious" bundles are independet elements of* $E^+(Gr(n,k), L)$.

THEOREM 9.1. *For* $G = Gr(n,2)$, $n > 2$, *the map*

$$W(G) \xrightarrow{e^0} E^+(G)$$

is surjective.

PROOF. The canonical isomorphism $Gr(n,k) \cong Gr(n,n-k)$ induces natural identification of Grotendieck rings, Witt rings and E-groups. Consider $G = Gr(n,n-2)$ instead of $Gr(n,2)$. Thus $H' = Gr(n-1,n-2) \cong \mathbb{P}^{n-2}$, $H'' = Gr(n-1,n-3)$. Denote $f : K_0(G) \to E^+(G)$ the composition of the endomorphism bundle map $K_0(G) \to W(G)$ of the corrolary 1.8 and the map $e^0 : W(G) \to E^+(G)$. Denote also f', f'' analogously composed maps for H', H''. We will show that, in certain cases, the map f is surjective. It will follows that e^0 is also surjective. Consider following diagrams with exact rows:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & E^+(H', \bigwedge^{n-2}\mathcal{S}\,\hat{}\,) & \xrightarrow{i'_*} & E^+(G) & \xrightarrow{i''^*} & E^+(H'') & \longrightarrow \cdots & \\
& g'\Big\uparrow & & f\Big\uparrow & & f''\Big\uparrow & & \text{for even } n, \\
0 \longrightarrow & K_0(H') & \xrightarrow{i'_*} & K_0(G) & \xrightarrow{i''^*} & K_0(H'') & \longrightarrow 0 &
\end{array}$$

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & E^-(H', \bigwedge^{n-2}\mathcal{S}\,\hat{}\,) & \xrightarrow{i'_*} & E^+(G) & \xrightarrow{i''^*} & E^+(H'') & \longrightarrow \cdots & \\
& g'\Big\uparrow & & f\Big\uparrow & & f''\Big\uparrow & & \text{for odd } n. \\
0 \longrightarrow & K_0(H') & \xrightarrow{i'_*} & K_0(G) & \xrightarrow{i''^*} & K_0(H'') & \longrightarrow 0 &
\end{array}$$

where g' is to be determined. Right-hand squares are plainly commutative.

In the case of odd n, the group $E^-(H', \bigwedge^{n-2}\mathcal{S}'\,\hat{}\,) \cong E^-\big(\mathbb{P}^{n-2}, \mathcal{O}(-1)\big)$ is trivial (proposition 5.1), and g' is the zero map. Thus if f'' is surjective, then f is surjective for odd n.

In the case of even $n = 2p$, the group $E^+(H', \bigwedge^{n-2}\mathcal{S}'\,\hat{}\,) \cong E^+\big(\mathbb{P}^{n-2}, \mathcal{O}(-1)\big)$ is a two element group, generated by $\left[H^{n-2}\right]$ (proposition 5.1), where $H = 1 - [\mathcal{O}(-1)] = 1 - \left[\bigwedge^{n-2}\mathcal{S}'\right]$. H^{n-2} is a class of rational point by lemma 5.1 v). According to proposition 7.1, $\lambda_{-1}(\mathcal{S}')$ is a class of rational point $Gr(n-2,n-2)$, so $H^{n-2} = \lambda_{-1}(\mathcal{S}')$ in $K_0(H')$. In fact there are several relevant equalities in $K_0(H')$, for example:

$$\lambda_{-1}(\mathcal{S}')\lambda_{-1}(\mathcal{Q}') = \lambda_{-1}([\mathcal{S}'] + [\mathcal{Q}']) = \lambda_{-1}(n-1) = (1-1)^{n-1} = 0;$$
$$\lambda_{-1}(\mathcal{S}') = \lambda_{-1}(\mathcal{S}') \cdot [\mathcal{Q}'], \text{ since } \lambda_{-1}(\mathcal{Q}') = 1 - [\mathcal{Q}']$$
$$\lambda_{-1}(\mathcal{S}')\,\hat{}\, = \lambda_{-1}(\mathcal{S}') \cdot [\mathcal{Q}']\,\hat{}\, = \lambda_{-1}(\mathcal{S}').$$

Thus $E^+(H', \bigwedge^{n-2}\mathcal{S}')$ is generated by $[\lambda_{-1}(\mathcal{S}')]$. Define $g' : K_0(H') \to E^+(H', \bigwedge^{n-2}\mathcal{S}')$ by the formula

$$g'(A) = [A \cdot A\,\hat{}\, \cdot \lambda_{-1}(\mathcal{S}')] \,.$$

The map g' is surjective, since $[\lambda_{-1}(\mathcal{S}')] = g'(1)$. Moreover, the square

$$\begin{array}{ccc} E^+(H', \bigwedge^{n-2}\mathcal{S}') & \xrightarrow{i'_*} & E^+(G) \\ {\scriptstyle g'}\uparrow & & \uparrow{\scriptstyle f} \\ K_0(H') & \xrightarrow{i'_*} & K_0(G) \end{array}$$

is commutative: arbitrary A in $K_0(H')$ may be represented as $A = i'^*(B)$ for some B in $K_0(G)$. Then

$$\begin{aligned} i'_* \circ g'(A) &= i'_* \circ g' \circ i'^*(B) = i'_*\left[i'^*(B) \cdot i'_*(B)^{\wedge} \cdot \lambda_{-1}(\mathcal{S}')\right] \\ &= i'_*\left[i'^*(B) \cdot i'^*(B)^{\wedge} \cdot \lambda_{-1}(\mathcal{S}'^{\wedge})\right] = i'_* \circ i'^*\left[(B \cdot B^{\wedge} \cdot \lambda_{-1}(\mathcal{S}^{\wedge})\right] \\ &\qquad = \left[B \cdot B^{\wedge} \cdot \lambda_{-1}\mathcal{S}^{\wedge}\right] \cdot \left[\lambda_{-1}(\mathcal{S})\right] = \left[B \cdot B^{\wedge} \cdot \lambda_{-1}(\mathcal{S}^{\wedge}) \cdot \lambda_{-1}(\mathcal{S})\right], \end{aligned}$$

by proposition 8.1 ii), and

$$f \circ i'_* \circ i'^*(B) = f\big(B \cdot \lambda_{-1}(\mathcal{S})\big) = \left[B \cdot \lambda_{-1}(\mathcal{S}) \cdot B^{\wedge} \cdot \lambda_{-1}(\mathcal{S}^{\wedge})\right].$$

Therefore f is surjective if f'' is. Finally the surjectivity of f follows by the induction. □

Corollary 9.2. *If $G = Gr(n,2)$ and $f : G \to \mathrm{Spec}F$ is the structure map, $n = 2p$ or $n = 2p+1$, then*

$$|W(G)/f^*W(F)| \geq 2^{p-1}.$$

Proof. There is a commutative diagram of abelian groups

$$\begin{array}{ccc} W(G) & \xrightarrow{e^0} & E^+(G) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ W(F) & \xrightarrow{e^0} & \mathbb{Z}/2\mathbb{Z} \end{array}$$

with surjective horizontal arrows (theorem 9.1) and injective right f^*. Moreover, $|E^+(G)| \geq 2^p$ by theorem 8.1 □

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