THE HOMOLOGY OF SPECIAL LINEAR GROUPS OVER POLYNOMIAL RINGS

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March 24, 1996

ABSTRACT. We study the homology of $SL_n(F[t, t^{-1}])$ by examining the action of the group on a suitable simplicial complex. The E^1 -term of the resulting spectral sequence is computed and the differential, d^1 , is calculated in some special cases to yield information about the low-dimensional homology groups of $SL_n(F[t, t^{-1}])$. In particular, we show that if F is an infinite field, then $H_2(SL_n(F[t, t^{-1}]), \mathbb{Z}) = K_2(F[t, t^{-1}])$ for $n \geq 3$. We also prove an unstable analogue of homotopy invariance in algebraic K -theory; namely, if F is an infinite field, then the natural map $SL_n(F) \rightarrow SL_n(F[t])$ induces an isomorphism on integral homology for all $n \geq 2$.

Since Quillen's definition of the higher algebraic K -groups of a ring [15], much attention has been focused upon studying the (co)homology of linear groups. There have been some successes—Quillen's computation [14] of the mod l cohomology of $GL_n(\mathbb{F}_q)$, Soulé's results [18] on the cohomology of $SL_3(\mathbb{Z})$ —but few explicit calculations have been completed. Most known results concern the stabilization of the homology of linear groups. For example, van der Kallen [11], Charney [7], and others have proved quite general stability theorems for GL_n of a ring. Also, Suslin [19] proved that if F is an infinite field, then the natural map

$$H_i(GL_m(F)) \longrightarrow H_i(GL_n(F))$$

is an isomorphism for $i \leq m$. Other noteworthy results include Borel's computation of the stable cohomology of arithmetic groups [1], [2], the computation of $H^\bullet(SL_n(F), \mathbb{R})$ for F a number field by Borel and Yang [3], and Suslin's isomorphism [20] of $H_3(SL_2(F))$ with the indecomposable part of $K_3(F)$.

This paper is concerned with studying the homology of linear groups defined over the polynomial rings $F[t]$ and $F[t, t^{-1}]$. One motivation for this is an attempt to find unstable analogues of the fundamental theorem of algebraic K -theory [15]: If R is a regular ring, then there are natural isomorphisms

$$(1) \quad K_i(R[t]) \cong K_i(R)$$

Supported by an Alfred P. Sloan Doctoral Dissertation Fellowship

and

$$(2) \quad K_i(R[t, t^{-1}]) \cong K_i(R) \oplus K_{i-1}(R).$$

In this paper, we study the homology of $SL_n(F[t, t^{-1}])$. Before stating our main result, we first establish some notation.

The group $SL_n(F[t, t^{-1}])$ acts on a contractible $(n-1)$ -dimensional building $\mathcal{X}$ with fundamental domain an $(n-1)$ -simplex $\mathcal{C}$. This yields a spectral sequence converging to the homology of $SL_n(F[t, t^{-1}])$ with E^1 -term satisfying

$$(3) \quad E_{p,q}^1 = \bigoplus_{\dim \sigma = p} H_q(\Gamma_\sigma)$$

where Γ_σ denotes the stabilizer of the p -simplex σ in $SL_n(F[t, t^{-1}])$, and σ is contained in $\mathcal{C}$. The vertex stabilizers are isomorphic to $SL_n(F[t])$, and the other stabilizers break up into isomorphism classes in such a way that in each class, there is a group Γ_σ which fits into a split short exact sequence

$$1 \longrightarrow K \longrightarrow \Gamma_\sigma \xrightarrow{t=0} P_\sigma \longrightarrow 1$$

where P_σ is a parabolic subgroup of $SL_n(F)$ and K consists of the matrices in $SL_n(F[t])$ which are congruent to the identity modulo t . Our main result is the following.

Theorem (cf. Theorem 5.1). *If F is an infinite field, then the inclusion $P_\sigma \longrightarrow \Gamma_\sigma$ induces an isomorphism*

$$H_\bullet(P_\sigma, \mathbb{Z}) \longrightarrow H_\bullet(\Gamma_\sigma, \mathbb{Z}).$$

If σ is a vertex, we have $\Gamma_\sigma = SL_n(F[t])$ and $P_\sigma = SL_n(F)$. In this case the theorem reduces to the following unstable analogue of (1).

Theorem (cf. Theorem 3.4). *If F is an infinite field, then for all $n \geq 2$ the inclusion $SL_n(F) \longrightarrow SL_n(F[t])$ induces an isomorphism*

$$H_\bullet(SL_n(F), \mathbb{Z}) \longrightarrow H_\bullet(SL_n(F[t]), \mathbb{Z}).$$

This theorem improves on a result of Soulé [17].

Theorem 5.1 completes the computation of the E^1 -term of the spectral sequence (3). However, the differential d^1 is difficult to calculate in general. In Section 6 we compute the map in a few special cases and obtain information about the low dimensional homology groups of $SL_n(F[t, t^{-1}])$. In particular, we show that if F is an infinite field, then for $n \geq 3$, there is an isomorphism

$$H_2(SL_n(F[t, t^{-1}]), \mathbb{Z}) \cong K_2(F[t, t^{-1}]).$$

The homology of $SL_2(F[t, t^{-1}])$ was studied by the author in [12] using slightly different techniques than those used here. The main result of [12] is the following.

Theorem (cf. [12, Theorem 5.1]). Let F be a number field and denote by r_1 (resp. r_2) the number of real (resp. conjugate pairs of complex) embeddings of F . Then for $k \geq 2r_1 + 3r_2 + 2$ there is a natural isomorphism

$$H_k(SL_2(F[t, t^{-1}]), \mathbb{Q}) \cong H_{k-1}(F^\times, \mathbb{Q}).$$

The results of this paper reprove and generalize the results of [12]. In particular, Theorems 3.1 and 4.3 of [12] hold for infinite fields of arbitrary characteristic, not just fields of characteristic zero.

This paper is organized as follows:

In Section 1 we present the necessary background material on the Bruhat–Tits building $\mathcal{X}$. We also introduce a complex $\mathcal{Y}$ which will be used in subsequent sections.

In Section 2 we study the action of $SL_n(F[t, t^{-1}])$ on $\mathcal{X}$ and examine the structure of the various stabilizers.

In Section 3 we prove Theorem 3.4, the unstable version of (1). Even though this is a special case of Theorem 5.1, we prove it separately for two reasons. First, it is a striking result which deserves to be called a theorem in its own right, and second, the proof sets the stage for the proof of Theorem 5.1.

In Section 4 we find fundamental domains for the actions of the various stabilizers on the complex $\mathcal{Y}$ introduced in Section 1.

In Section 5 we prove Theorem 5.1.

Finally, in Section 6 we compute the d^1 -map in the spectral sequence (3) in some special cases.

Notation. If G is a group acting on a simplicial complex X and if σ is a simplex in X , we denote the stabilizer of σ in G by G_σ . If R is a ring, we denote the group of units by $R^\times$. The set of $n \times n$ matrices over R will be denoted by $M_n(R)$. Unless otherwise stated, F will be an infinite field of arbitrary characteristic.

Acknowledgements. It is a pleasure to thank those who provided valuable insight during the preparation of this paper. In particular, I wish to thank Howard Garland, John Harer, Jun Yang, and Andrei Suslin for many helpful conversations. I am especially indebted to my advisor, Richard Hain, for his guidance and support during my graduate career.

This paper is dedicated to Ellen Wall.

1. PRELIMINARIES ON BUILDINGS

In this section, we summarize the basic facts about the Bruhat–Tits building associated to a vector space over a field with discrete valuation. The building was constructed in [6]; more detailed information may be found there (or see Brown [4, Ch. V]).

Let K be a field with discrete valuation, v . Denote by $\mathcal{O}$ the valuation ring of v ; that is,

$$\mathcal{O} = \{x \in K : v(x) \geq 0\}.$$

Choose a field element π satisfying $v(\pi) = 1$, and denote by k the residue field $\mathcal{O}/\pi\mathcal{O}$. By a *lattice* in K^n , we mean a finitely generated $\mathcal{O}$ -submodule which spans K^n ; such a submodule is free of rank n . Two lattices L, L' are called *equivalent* if there is some nonzero field element x such that $L' = xL$. Denote the equivalence class of the lattice L by $[L]$. If $v_1, \dots, v_n$ are linearly independent elements of K^n , denote the equivalence class of the lattice they span by $[v_1, \dots, v_n]$.

Assign a *type* to a lattice class as follows. If $[v_1, \dots, v_n]$ is a lattice class, we define its type to be the element

$$v(\det(v_1, \dots, v_n))$$

modulo n , where $\det(v_1, \dots, v_n)$ denotes the determinant of the matrix having $v_1, \dots, v_n$ as columns.

Construct a simplicial complex X in the following manner. The vertices of X are equivalence classes of lattices in K^n . A collection of vertices $\Lambda_0, \Lambda_1, \dots, \Lambda_m$ forms an m -simplex if there exist representatives $L_0, L_1, \dots, L_m$ satisfying

$$\pi L_m \subset L_0 \subset L_1 \subset \dots \subset L_m.$$

Since $L_i/\pi L_m$ is a subspace of the n -dimensional k -vector space $L_m/\pi L_m$, the maximal simplices of X have n vertices; that is, $\dim X = n - 1$. Moreover, the complex X is contractible [4, p. 137]. There is an obvious action of $GL_n(K)$ on X . Note that this action is transitive on the vertices of X .

We now find a fundamental domain for the action of $SL_n(K)$ on X . Let $\mathcal{C}$ be the $(n-1)$ -simplex with vertices $[e_1, \dots, e_i, \pi e_{i+1}, \dots, \pi e_n], i = 1, \dots, n$, where $e_1, \dots, e_n$ is the standard basis of K^n . Then we have the following result (see [4, p. 137])

Proposition 1.1. *The $(n-1)$ -simplex $\mathcal{C}$ is a fundamental domain for the action of $SL_n(K)$ on X .*

Proof. Let $\mathcal{C}'$ be an arbitrary $(n-1)$ -simplex with vertices $\Lambda_0, \dots, \Lambda_{n-1}$, with Λ_i of type $n-i$. By the Invariant Factor Theorem, there is a basis $f_1, \dots, f_n$ of K^n such that

$$\Lambda_0 = [f_1, \dots, f_n], \Lambda_1 = [f_1, \pi f_2, \dots, \pi f_n], \dots, \Lambda_{n-1} = [f_1, \dots, \pi f_n],$$

and $\det(f_1, \dots, f_n) = \pi^{nr}u$ for some integer r and $u \in \mathcal{O}^\times$. Replacing f_1 by $\pi^{-r}u^{-1}f_1$, and f_i by $\pi^{-r}f_i$, $i = 2, \dots, n$, we still have

$$\Lambda_0 = [f_1, \dots, f_n], \dots, \Lambda_{n-1} = [f_1, \dots, \pi f_n],$$

but now $\det(f_1, \dots, f_n) = 1$. Let g be the matrix having $f_1, \dots, f_n$ as columns. Then g takes $\mathcal{C}$ to $\mathcal{C}'$. Since the action of $SL_n(K)$ preserves type, it follows that $\mathcal{C}$ is a fundamental domain. $\square$

The stabilizer of $[e_1, \dots, e_n]$ in $SL_n(K)$ is the subgroup $SL_n(\mathcal{O})$. Thus, the stabilizer of $[e_1, \dots, e_i, \pi e_{i+1}, \dots, \pi e_n]$ is

$$g_i SL_n(\mathcal{O}) g_i^{-1},$$

where

$$g_i = \text{diag}(1, 1, \dots, 1, \pi, \dots, \pi),$$

the first π appearing in the $(i + 1)$ st column. The stabilizer of an edge is the intersection of the stabilizers of its vertices; the stabilizer of a 2-simplex is the intersection of the stabilizers of its edges, and so on.

In this paper, we shall be interested in studying various group actions on two Bruhat–Tits buildings associated to two different fields associated to a field F .

Example 1.2. Denote by $\mathcal{L}$ the field of formal Laurent series over F . Define a valuation v on $\mathcal{L}$ by

$$v\left(\sum_{i \geq n_0} a_i t^i\right) = n_0, \quad a_{n_0} \neq 0.$$

Here, we choose $\pi = t$. Observe that the ring $F[t, t^{-1}]$ is dense in $\mathcal{L}$. Denote by $\mathcal{X}$ the Bruhat–Tits building associated to $\mathcal{L}^n$.

Example 1.3. Denote by $F(t)$ the field of fractions of $F[t]$. Define a valuation v_∞ on $F(t)$ by

$$v_\infty(a/b) = \deg b - \deg a, \quad b \neq 0.$$

In this case, we choose $\pi = 1/t$. Denote by $\mathcal{Y}$ the Bruhat–Tits building associated to $F(t)^n$.

Remark. Denote by $\widehat{K}$ the completion of K with respect to the valuation v . Then the Bruhat–Tits buildings of K and $\widehat{K}$ are isomorphic. In particular, the completion $\widehat{F(t)}$ of $F(t)$ is isomorphic to $\mathcal{L}$ via the map $t \mapsto t^{-1}$. It follows that the complexes $\mathcal{X}$ and $\mathcal{Y}$ are isomorphic. Although these complexes are isomorphic, it will be convenient to distinguish them when doing homological computations.

2. THE ACTION OF $SL_n(F[t, t^{-1}])$ ON $\mathcal{X}$

We now investigate the action of the group $SL_n(F[t, t^{-1}])$ on the complex $\mathcal{X}$ of Example 1.2. Since $F[t, t^{-1}]$ is a dense subring of the field $\mathcal{L}$, we have the following result.

Lemma 2.1. *The subgroup $SL_n(F[t, t^{-1}])$ is dense in $SL_n(\mathcal{L})$.*

Proof. The closure of $SL_n(F[t, t^{-1}])$ in $SL_n(\mathcal{L})$ contains the subgroup of elementary matrices over $\mathcal{L}$. Since these matrices generate $SL_n(\mathcal{L})$, the result follows. $\square$

Denote by V the vector space $\mathcal{L}^n$ and let $GL(V)^\circ$ denote the kernel of the homomorphism

$$v \circ \det : GL(V) \longrightarrow \mathbb{Z}.$$

Then we have the following (cf. [16, Thm. 2, p. 78]).

Proposition 2.2. *If G is a subgroup of $GL(V)^\circ$ whose closure contains $SL(V)$, then the $(n - 1)$ -simplex $\mathcal{C}$ (see Proposition 1.1) is a fundamental domain for the action of G on $\mathcal{X}$.*

Proof. We know that $\mathcal{C}$ is a fundamental domain for the action of $SL(V)$ on $\mathcal{X}$. Let $\mathcal{C}'$ be an $(n-1)$ -simplex in $\mathcal{X}$. There is an element s of $SL(V)$ with

$$s\mathcal{C} = \mathcal{C}'.$$

Let U be the subgroup of $GL_n(\mathcal{O})$ consisting of the matrices which are congruent to the identity mod t ; this is an open subgroup of $GL(V)$. By hypothesis, there is an element u of U and an element g of G with $g = su$. Observe that u fixes each vertex of $\mathcal{C}$. Hence, we have the chain of equalities

$$g\mathcal{C} = su\mathcal{C} = s\mathcal{C} = \mathcal{C}',$$

and since G preserves type, it follows that $\mathcal{C}$ is a fundamental domain for the action of G on $\mathcal{X}$. $\square$

The preceding two results imply that the $(n-1)$ -simplex $\mathcal{C}$ is a fundamental domain for the action of $SL_n(F[t, t^{-1}])$ on $\mathcal{X}$.

We now identify the stabilizers in $SL_n(F[t, t^{-1}])$ of the simplices of $\mathcal{C}$. Label the vertices of $\mathcal{C}$ as

$$p_i = [e_1, \dots, e_{i-1}, te_i, \dots, te_n], \quad i = 1, 2, \dots, n.$$

Note that $p_1 = [te_1, \dots, te_n] = [e_1, \dots, e_n]$. Evidently, the stabilizer of p_1 in $SL_n(F[t, t^{-1}])$ is the subgroup

$$SL_n(F[t]) = SL_n(\mathcal{O}) \cap SL_n(F[t, t^{-1}]).$$

Denote by g_i the matrix

$$g_i = \text{diag}(1, \dots, 1, t, \dots, t), \quad i = 2, \dots, n$$

where the first $i-1$ entries are equal to 1. Then the stabilizer of p_i in $SL_n(F[t, t^{-1}])$ is

$$g_i SL_n(F[t]) g_i^{-1}.$$

Denote by $\Gamma_{i_1, \dots, i_k}$ the stabilizer of the $(k-1)$ -simplex having vertices $p_{i_1}, \dots, p_{i_k}$. The group $\Gamma_{i_1, \dots, i_k}$ is the intersection of the stabilizers $\Gamma_{i_1}, \dots, \Gamma_{i_k}$ of the vertices of the simplex. Elements of $\Gamma_{i_1, \dots, i_k}$ have the form

$$\begin{pmatrix} L_1 & V_{12} & V_{13} & \cdots & V_{1,k} & t^{-1}V_{1,k+1} \\ tV_{21} & L_2 & V_{23} & \cdots & V_{2,k} & V_{2,k+1} \\ tV_{31} & tV_{32} & L_3 & \cdots & V_{3,k} & V_{3,k+1} \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & \vdots \\ tV_{k+1,1} & tV_{k+1,2} & tV_{k+1,3} & \cdots & tV_{k+1,k} & L_{k+1} \end{pmatrix}$$

where we have

$$\begin{aligned} L_r &\in \mathbb{M}_{i_r - i_{r-1}}(F[t]), & 1 \leq r \leq k+1 \\ V_{r,s} &\in \mathbb{M}_{i_r - i_{r-1}, i_s - i_{s-1}}(F[t]), & 1 \leq r, s \leq k+1 \end{aligned}$$

(here, we set $i_0 = 1$ and $i_{k+1} = n+1$).

Consider the stabilizers $\Gamma_{1,j_2,\dots,j_k}$. These are subgroups of $\Gamma_1 = SL_n(F[t])$. Elements of the group $\Gamma_{1,j_2,\dots,j_k}$ have the form

$$\begin{pmatrix} L_1 & V_{12} & V_{13} & \cdots & V_{1,k-1} & V_{1,k} \\ tV_{21} & L_2 & V_{23} & \cdots & V_{2,k-1} & V_{2,k} \\ tV_{31} & tV_{32} & L_3 & \cdots & V_{3,k-1} & V_{3,k} \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & \vdots \\ tV_{k,1} & tV_{k,2} & tV_{k,3} & \cdots & tV_{k,k-1} & L_k \end{pmatrix}$$

where we have

$$\begin{aligned} L_r &\in \mathbb{M}_{j_{r+1}-j_r}(F[t]), & 1 \leq r \leq k \\ V_{r,s} &\in \mathbb{M}_{j_{r+1}-j_r, j_{s+1}-j_s}(F[t]), & 1 \leq r, s \leq k \end{aligned}$$

(here, we set $j_1 = 1$ and $j_{k+1} = n + 1$).

These groups are related as follows.

Proposition 2.3. *The group $\Gamma_{i_1,\dots,i_k}$ is conjugate to $\Gamma_{1,(i_2-i_1+1),\dots,(i_k-i_1+1)}$ inside $GL_n(F[t, t^{-1}])$.*

Proof. First conjugate $\Gamma_{i_1,\dots,i_k}$ by the element

$$g = \text{diag}(t, t, \dots, t, 1, \dots, 1)$$

where the first $i_1 - 1$ entries are equal to t . The resulting group has elements of the form

$$\begin{pmatrix} L_1 & tV_{12} & tV_{13} & \cdots & tV_{1,k} & V_{1,k+1} \\ V_{21} & L_2 & V_{23} & \cdots & V_{2,k} & V_{2,k+1} \\ V_{31} & tV_{32} & L_3 & \cdots & V_{3,k} & V_{3,k+1} \\ \vdots & & & \ddots & & \vdots \\ V_{k,1} & tV_{k,2} & tV_{k,3} & \cdots & L_k & V_{k,k+1} \\ V_{k+1,1} & tV_{k+1,2} & tV_{k+1,3} & \cdots & tV_{k+1,k} & L_{k+1} \end{pmatrix}$$

where the L_r and $V_{r,s}$ are as above. Now conjugate by the permutation matrix corresponding to the permutation

$$\begin{array}{ccc}
 1 & \mapsto & n - i_1 + 2 \\
 2 & \mapsto & n - i_1 + 3 \\
 \vdots & & \vdots \\
 i_1 - 1 & \mapsto & n \\
 i_1 & \mapsto & 1 \\
 \vdots & & \vdots \\
 i_2 - 1 & \mapsto & i_2 - i_1 \\
 i_2 & \mapsto & i_2 - i_1 + 1 \\
 i_2 + 1 & \mapsto & i_2 - i_1 + 2 \\
 \vdots & & \vdots \\
 n & \mapsto & n - i_1 + 1.
 \end{array}$$

Note that if τ denotes the n -cycle $(12 \cdots n)$, then this permutation is simply τ^{i_1-1} . The resulting group has the form

$$\begin{pmatrix}
 L_2 & V_{23} & V_{24} & \cdots & \cdots & V_{2,k+1} & V_{21} \\
 tV_{32} & L_3 & V_{34} & \cdots & \cdots & V_{3,k+1} & V_{31} \\
 tV_{42} & tV_{43} & L_4 & \cdots & \cdots & V_{4,k+1} & V_{41} \\
 \vdots & & & \ddots & & & \vdots \\
 tV_{k,2} & tV_{k,3} & tV_{k,4} & \cdots & L_k & V_{k,k+1} & V_{k,1} \\
 tV_{k+1,2} & tV_{k+1,3} & tV_{k+1,4} & \cdots & tV_{k+1,k} & L_{k+1} & V_{k+1,1} \\
 tV_{12} & tV_{13} & tV_{14} & \cdots & tV_{1,k} & V_{1,k+1} & L_1
 \end{pmatrix},$$

which is precisely the group $\Gamma_{1,(i_2-i_1+1),\dots,(i_k-i_1+1)}$. $\square$

If σ is a p -simplex in $\mathcal{C}$, denote by Γ_σ the stabilizer of σ in $SL_n(F[t, t^{-1}])$. Since the complex $\mathcal{X}$ is contractible, we have a spectral sequence converging to the homology of $SL_n(F[t, t^{-1}])$ with E^1 -term

$$(4) \quad E_{p,q}^1 = \bigoplus_{\dim \sigma = p} H_q(\Gamma_\sigma)$$

where σ ranges over the p -simplices of $\mathcal{C}$. By Proposition 2.3, we need only compute the homology of each $\Gamma_{1,j_2,\dots,j_k}$; we do this in Section 5.

In the next section we single out the Γ_i , $i = 1, \dots, n$ and compute their homology.

3. THE VERTEX STABILIZERS—THE HOMOLOGY OF $SL_n(F[t])$

Notation. For G a subgroup of $GL_n(R)$, R a commutative ring with unit, denote by $\overline{G}$ the subgroup $G \cap SL_n(R)$.

Consider the stabilizers $\Gamma_1, \dots, \Gamma_n$ of the vertices of $\mathcal{C}$. Each of these is isomorphic to $SL_n(F[t])$. To compute homology we use the Bruhat–Tits building $\mathcal{Y}$ of Example 1.3. Recall that this is the building associated to the n -dimensional vector space $V = F(t)^n$.

There is an obvious left action of $SL_n(F[t])$ on $\mathcal{Y}$. Let $e_1, \dots, e_n$ be the standard basis of V . Then the subcomplex $\mathcal{T}$ having vertices

$$[e_1 t^{r_1}, e_2 t^{r_2}, \dots, e_{n-1} t^{r_{n-1}}, e_n], \text{ where } r_1 \geq r_2 \geq \dots \geq r_{n-1} \geq 0$$

is a fundamental domain for the action of $SL_n(F[t])$ on $\mathcal{Y}$ [17].

The complex $\mathcal{T}$ is an infinite wedge. Denote by v_0 the vertex $[e_1, \dots, e_n]$ and by v_i the vertex $[e_1 t, e_2 t, \dots, e_i t, e_{i+1}, \dots, e_n]$, $i = 1, 2, \dots, n-1$. For a k element subset $I = \{i_1, \dots, i_k\}$ of $\{1, 2, \dots, n-1\}$, define $E_I^{(k)}$ to be the subcomplex of $\mathcal{T}$ which is the union of all rays with origin v_0 passing through the $(k-1)$ -simplex $\langle v_{i_1}, \dots, v_{i_k} \rangle$. There are $\binom{n-1}{k}$ such $E_I^{(k)}$. Observe that if $I = \{1, 2, \dots, n-1\}$, then $E_I^{(n-1)} = \mathcal{T}$. When we write $E_J^{(l)}$, the superscript l denotes the cardinality of the set J .

Define a filtration $V^\bullet$ of $\mathcal{T}$ by setting $V^{(0)} = v_0$ and

$$(5) \quad V^{(k)} = \bigcup_I E_I^{(k)}, \quad 1 \leq k \leq n-1$$

where I ranges over all k -element subsets of $\{1, 2, \dots, n-1\}$. Note that $V^{(n-1)} = \mathcal{T}$.

Evidently, the stabilizer of v_0 in $SL_n(F[t])$ is the subgroup $SL_n(F)$. For any other vertex $v = [e_1 t^{r_1}, e_2 t^{r_2}, \dots, e_{n-1} t^{r_{n-1}}, e_n]$ in $\mathcal{T}$, let Γ_v denote the stabilizer of v in $SL_n(F[t])$. The subgroup Γ_v is the semidirect product of a reductive group L_v contained in $SL_n(F)$ and a unipotent group U_v contained in $SL_n(F[t])$. If p_{kl} denotes the polynomial in the k th row and l th column of an element of Γ_v , then we have $\deg p_{kl} \leq r_k - r_l$. It follows that the subgroup Γ_v has a block form

$$\Gamma_v = \overline{\begin{pmatrix} L_1 & V_{12} & V_{13} & \cdots & V_{1m} \\ & L_2 & V_{23} & \cdots & V_{2m} \\ & & \ddots & & \vdots \\ & 0 & & L_{m-1} & V_{m-1,m} \\ & & & & L_m \end{pmatrix}}$$

where the L_k and V_{kl} satisfy

$$L_k \in GL_{i_k - i_{k-1}}(F), \quad \text{where } r_{i_{k-1}+1} = r_{i_{k-1}+2} = \dots = r_{i_k}$$

$$V_{kl} \in \mathbb{M}_{i_k - i_{k-1}, i_l - i_{l-1}}(F[t]), \quad \text{where } r_{i_{k-1}+1} = r_{i_{k-1}+2} = \dots = r_{i_k} \\ r_{i_{l-1}+1} = r_{i_{l-1}+2} = \dots = r_{i_l}$$

(we set $i_0 = 0$). Observe that the stabilizers Γ_{v_i} , $i = 1, 2, \dots, n-1$, have the block form of the $n-1$ maximal parabolic subgroups in SL_n . If $I = \{i_1, \dots, i_k\}$ and if v is a vertex in $E_I^{(k)}$ which does not lie in any $E_J^{(k-1)}$, where $J \subset I$, then Γ_v has the block form of the intersection $\Gamma_{v_{i_1}} \cap \dots \cap \Gamma_{v_{i_k}}$. Observe that if v is a vertex of $\mathcal{T}$ not lying in any $E_J^{(n-2)}$, then the r_i are positive and distinct and hence the group Γ_v is upper triangular.

If e is an edge with vertices v, w , then the stabilizer Γ_e is simply the intersection $\Gamma_v \cap \Gamma_w$. Similarly, the stabilizer of a 2-simplex is the intersection of the edge

FIGURE 1. The Fundamental Domain $\mathcal{T}$ for $n = 3$.

stabilizers, and so on. It follows that if $l \leq k$ and if σ is an l -simplex in $E_I^{(k)}$, where $I = \{i_1, \dots, i_k\}$, not lying entirely in any $E_J^{(k-1)}$, where $J \subset I$, then Γ_σ has the block form of the intersection $\Gamma_{v_{i_1}} \cap \dots \cap \Gamma_{v_{i_k}}$.

The case $n = 3$ is shown in Figure 1.

Since the complex $\mathcal{Y}$ is contractible, we have a spectral sequence converging to $H_\bullet(SL_n(F[t]), \mathbb{Z})$ with E^1 -term satisfying

$$(6) \quad E_{p,q}^1 = \bigoplus_{\dim \sigma = p} H_q(\Gamma_\sigma)$$

where σ ranges over the simplices of $\mathcal{T}$.

3.1. The Homology of the Stabilizers. We now compute the homology of the groups Γ_σ . Suppose that A is an F -algebra. Let P be a subgroup of $GL_{n+m}(A)$ having block form

$$P = \begin{pmatrix} L_1 & M \\ 0 & L_2 \end{pmatrix}$$

where $L_1 \subseteq GL_n(A)$, $L_2 \subseteq GL_m(A)$, and M is a vector subspace of $\mathbb{M}_{n,m}(A)$ such that $L_1 M = M = M L_2$. Suppose that each L_i contains the group of diagonal

matrices over F . Denote by L the subgroup of P defined by

$$L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}.$$

A proof of the following is deduced easily from [10, Lemma 9] by observing that the argument used works with F replaced by A . Recall that $\overline{G}$ denotes the intersection $G \cap SL_n(R)$.

Proposition 3.1. *If F is an infinite field, then the inclusion $\overline{L} \longrightarrow \overline{P}$ induces an isomorphism*

$$H_\bullet(\overline{L}, \mathbb{Z}) \longrightarrow H_\bullet(\overline{P}, \mathbb{Z}). \quad \square$$

Corollary 3.2. *Suppose that P is a subgroup of $GL_n(A)$ having block form*

$$\begin{pmatrix} L_1 & V_{12} & V_{13} & \cdots & V_{1m} \\ & L_2 & V_{23} & \cdots & V_{2m} \\ & & \ddots & & \vdots \\ & 0 & & L_{m-1} & V_{m-1,m} \\ & & & & L_m \end{pmatrix}$$

where each $L_i \subseteq GL_{n_i}(A)$ and each V_{ij} is a vector subspace of $\mathbb{M}_{n_i, n_j}(A)$ such that $L_i V_{ij} = V_{ij} = V_{ij} L_j$. Assume that each L_i contains the group of diagonal matrices over F . Denote by L the subgroup

$$L = \begin{pmatrix} L_1 & & 0 \\ & \ddots & \\ 0 & & L_m \end{pmatrix}$$

of P . Then the inclusion $\overline{L} \longrightarrow \overline{P}$ induces an isomorphism

$$H_\bullet(\overline{L}, \mathbb{Z}) \longrightarrow H_\bullet(\overline{P}, \mathbb{Z}).$$

Proof. Consider the sequence of inclusions

$$\begin{aligned} \overline{L} &\rightarrow \overline{\begin{pmatrix} L_1 & 0 & 0 \\ & \ddots & \vdots \\ 0 & L_{m-1} & V_{m-1,m} \\ & 0 & L_m \end{pmatrix}} \rightarrow \overline{\begin{pmatrix} L_1 & 0 & 0 & 0 \\ & \ddots & \vdots & \vdots \\ 0 & L_{m-2} & V_{m-2,m-1} & V_{m-2,m} \\ & 0 & L_{m-1} & V_{m-1,m} \\ 0 & & 0 & L_m \end{pmatrix}} \\ &\dots \rightarrow \overline{\begin{pmatrix} L_1 & 0 & \cdots & \cdots & 0 \\ 0 & L_2 & V_{23} & \cdots & V_{2m} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & L_{m-1} & V_{m-1,m} \\ 0 & 0 & & & L_m \end{pmatrix}} \longrightarrow \overline{P}. \end{aligned}$$

By Proposition 3.1, each of these maps induces a homology isomorphism. It follows that the inclusion $\overline{L} \rightarrow \overline{P}$ induces an isomorphism

$$H_\bullet(\overline{L}, \mathbb{Z}) \longrightarrow H_\bullet(\overline{P}, \mathbb{Z}). \quad \square$$

If σ is a simplex in $\mathcal{T}$, then the subgroup Γ_σ has a block form as in the corollary. We have an extension

$$1 \longrightarrow U_\sigma \longrightarrow \Gamma_\sigma \longrightarrow L_\sigma \longrightarrow 1$$

where U_σ is a unipotent group and L_σ is a reductive subgroup of $SL_n(F)$. The corollary implies that the inclusion $L_\sigma \rightarrow \Gamma_\sigma$ induces an isomorphism

$$H_\bullet(L_\sigma, \mathbb{Z}) \longrightarrow H_\bullet(\Gamma_\sigma, \mathbb{Z}).$$

Let $I = \{i_1, \dots, i_k\}$ be a subset of $\{1, 2, \dots, n-1\}$. If σ is a simplex in

$$E_I^{(k)} - \bigcup_{J \subset I} E_J^{(k-1)},$$

then Γ_σ has the block form of the intersection $\Gamma_{v_{i_1}} \cap \dots \cap \Gamma_{v_{i_k}}$. If τ is another such simplex, then Γ_τ has the same block form. Thus, $L_\sigma = L_\tau$ and it follows that Γ_σ and Γ_τ have the same homology. Moreover, if σ is a face of τ , then the map $\Gamma_\tau \rightarrow \Gamma_\sigma$ induces an isomorphism on homology.

3.2. The Homology of $SL_n(F[t])$. Given a coefficient system $\mathcal{M}$ on a simplicial complex Z (i.e., a covariant functor from the simplices of Z to the category of abelian groups), we may define the chain complex $C_\bullet(Z, \mathcal{M})$ by setting

$$C_p(Z, \mathcal{M}) = \bigoplus_{\dim \sigma = p} \mathcal{M}(\sigma)$$

with boundary map the alternating sum of the maps induced by the face maps in Z .

We shall make use of the following result (compare with [18, Lemma 6]).

Lemma 3.3. *Suppose $F^{(0)} \subset F^{(1)} \subset \dots \subset F^{(k)} = Z$ is a filtration of the simplicial complex Z by subcomplexes such that each $F^{(i)}$ and each component of $F^{(i)} - F^{(i-1)}$ is contractible. Suppose that $\mathcal{M}$ is a coefficient system on Z such that the restriction of $\mathcal{M}$ to each component of $F^{(i)} - F^{(i-1)}$ is constant. Then the inclusion $F^{(0)} \longrightarrow Z$ induces an isomorphism*

$$H_\bullet(F^{(0)}, \mathcal{M}) \longrightarrow H_\bullet(Z, \mathcal{M}).$$

Proof. The filtration of Z induces a filtration of $C_\bullet(Z, \mathcal{M})$. This yields a spectral sequence converging to $H_\bullet(Z, \mathcal{M})$ with E^1 -term having i th column

$$H_\bullet(F^{(i)}, F^{(i-1)}; \mathcal{M}).$$

Consider the relative chain complex $C_\bullet(F^{(i)}, F^{(i-1)}; \mathcal{M})$. By hypothesis, this chain complex is a direct sum of chain complexes with constant coefficients. Since each $F^{(i)}$ is contractible, it follows that

$$H_\bullet(F^{(i)}, F^{(i-1)}; \mathcal{M}) = 0, \quad i \geq 1.$$

Thus, only the 0th column $H_\bullet(F^{(0)}, \mathcal{M})$ is nonzero. This proves the lemma. $\square$

We may now compute $H_\bullet(SL_n(F[t]), \mathbb{Z})$. The argument in the proof below is used implicitly by Soulé in the proof of Theorem 5 of [17].

Theorem 3.4. *If F is an infinite field, then the natural inclusion $SL_n(F) \rightarrow SL_n(F[t])$ induces an isomorphism*

$$H_\bullet(SL_n(F), \mathbb{Z}) \longrightarrow H_\bullet(SL_n(F[t]), \mathbb{Z}).$$

Proof. Recall the spectral sequence (6). The E^1 -term satisfies

$$E_{p,q}^1 = \bigoplus_{\dim \sigma = p} H_q(\Gamma_\sigma) \implies H_{p+q}(SL_n(F[t])).$$

For each $q \geq 0$, define a coefficient system $\mathcal{F}_q$ on $\mathcal{T}$ by

$$\mathcal{F}_q(\sigma) = H_q(\Gamma_\sigma).$$

Then the q th row in the spectral sequence is simply $C_\bullet(\mathcal{T}, \mathcal{F}_q)$ and the d^1 -map is the boundary map in this chain complex.

Recall the filtration $V^\bullet$ of $\mathcal{T}$ (5). For each simplex in

$$E_I^{(k)} - \bigcup_{J \subset I} E_J^{(k-1)},$$

the stabilizers have the same reductive part and hence have the same homology (see the discussion following the proof of Corollary 3.2). It follows that the restriction of $\mathcal{F}_q$ to each component of $V^{(i)} - V^{(i-1)}$ is constant. By Lemma 3.3, the inclusion $v_0 \rightarrow \mathcal{T}$ induces an isomorphism

$$H_\bullet(v_0, \mathcal{F}_q) \longrightarrow H_\bullet(\mathcal{T}, \mathcal{F}_q).$$

Observe that

$$H_p(v_0, \mathcal{F}_q) = \begin{cases} H_q(SL_n(F)) & p = 0 \\ 0 & p > 0. \end{cases}$$

It follows that the E^2 -term of the spectral sequence (6) satisfies

$$E_{p,q}^2 = \begin{cases} H_q(SL_n(F)) & p = 0 \\ 0 & p > 0. \end{cases} \quad \square$$

Remark. Theorem 3.4 may be viewed as an unstable version of Quillen's homotopy invariance in algebraic K -theory [15].

Remark. The $n = 2$ case of Theorem 3.4 was proved for fields of characteristic zero in [12] by considering the Mayer-Vietoris sequence associated to the amalgamated free product decomposition (due to Nagao [13])

$$(7) \quad SL_2(F[t]) \cong SL_2(F) *_{B(F)} B(F[t])$$

where $B(R)$ denotes the upper triangular group over R . Proposition 3.2 of [12] shows that $B(F)$ and $B(F[t])$ are the same homologically. This implies that the Mayer–Vietoris sequence associated to (7) breaks into short exact sequences

$$0 \longrightarrow H_k(B(F)) \longrightarrow H_k(B(F[t])) \oplus H_k(SL_2(F)) \longrightarrow H_k(SL_2(F[t])) \longrightarrow 0,$$

from which it follows that $H_\bullet(SL_2(F), \mathbb{Z}) \cong H_\bullet(SL_2(F[t]), \mathbb{Z})$.

As an immediate consequence of Theorem 3.4 we have the following result.

Corollary 3.5. *The natural inclusion $GL_n(F) \rightarrow GL_n(F[t])$ induces an isomorphism*

$$H_\bullet(GL_n(F), \mathbb{Z}) \longrightarrow H_\bullet(GL_n(F[t]), \mathbb{Z}).$$

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & SL_n(F) & \longrightarrow & GL_n(F) & \longrightarrow & F^\times & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \parallel & & \\ 1 & \longrightarrow & SL_n(F[t]) & \longrightarrow & GL_n(F[t]) & \longrightarrow & F^\times & \longrightarrow & 1. \end{array}$$

This yields a map of spectral sequences which by Theorem 3.4 is an isomorphism at the E^2 -level. $\square$

By applying a theorem of Suslin, we have the following stability result.

Corollary 3.6. *If $n \leq m$, then the natural map*

$$H_i(GL_n(F[t]), \mathbb{Z}) \longrightarrow H_i(GL_m(F[t]), \mathbb{Z})$$

is an isomorphism for $i \leq n$.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} H_i(GL_n(F), \mathbb{Z}) & \longrightarrow & H_i(GL_m(F), \mathbb{Z}) \\ \downarrow & & \downarrow \\ H_i(GL_n(F[t]), \mathbb{Z}) & \longrightarrow & H_i(GL_m(F[t]), \mathbb{Z}). \end{array}$$

By [19, 3.4], the top horizontal map is an isomorphism for $i \leq n$ and by Corollary 3.5, so is each of the two vertical maps. $\square$

4. THE LEVEL t CONGRUENCE SUBGROUP AND A FUNDAMENTAL DOMAIN FOR THE ACTION OF $\Gamma_{1,j_2,\dots,j_k}$ ON $\mathcal{Y}$

Consider the exact sequence

$$1 \longrightarrow K \longrightarrow SL_n(F[t]) \xrightarrow{t=0} SL_n(F) \longrightarrow 1$$

where K consists of those matrices which are congruent to the identity modulo t . In the preceding section we described a fundamental domain, $\mathcal{T}$, for the action of $SL_n(F[t])$ on the complex $\mathcal{Y}$ of Example 1.2. In order to find a fundamental domain for the action of $\Gamma_{1,j_2,\dots,j_k}$ on $\mathcal{Y}$, we proceed in steps. First, we find a fundamental domain for the action of K , then a fundamental domain for the action of $\Gamma_{1,2,\dots,n}$, and finally, a fundamental domain for the action of $\Gamma_{1,j_2,\dots,j_k}$.

Denote by $B_n(F)$ the upper triangular subgroup of $SL_n(F)$ and choose a set S of coset representatives for $SL_n(F)/B_n(F)$. Set

$$\mathcal{T}' = \bigcup_{s \in S} s\mathcal{T}.$$

Proposition 4.1. *The complex $\mathcal{T}'$ is a fundamental domain for the action of K on $\mathcal{Y}$.*

Proof. Let σ be an $(n-1)$ -simplex of $\mathcal{Y}$. There exists some x in $SL_n(F[t])$ and a unique simplex σ_0 of $\mathcal{T}$ such that $\sigma = x\sigma_0$. Write

$$x = ky, \quad k \in K, \quad y \in SL_n(F)$$

and

$$y = su, \quad s \in S, \quad u \in B_n(F).$$

Then

$$\sigma = ksu\sigma_0.$$

Note that u acts trivially on $\mathcal{T}$; i.e., $u\sigma_0 = \sigma_0$. Hence, $\sigma = ksu\sigma_0$, and thus

$$\sigma \equiv s\sigma_0 \pmod{K}.$$

It remains to show that no two vertices of $\mathcal{T}'$ are identified by K .

Suppose $x : s_1\Lambda_1 \rightarrow s_2\Lambda_2$ where the s_i belong to S and x is some element of K . Then

$$s_1s_2^{-1}x : s_1\Lambda_1 \rightarrow s_1\Lambda_2.$$

Now, $s_1s_2^{-1}x$ belongs to $SL_n(F[t])$ and the $s_1\Lambda_i$ are inequivalent modulo $SL_n(F[t])$ (i.e., we could have taken $s_1\mathcal{T}$ as a fundamental domain). Hence, $\Lambda_1 = \Lambda_2$. Denote this common vertex by Λ . Moreover, $s_1s_2^{-1}x$ stabilizes $s_1\Lambda$. Observe that the stabilizer of $s_1\Lambda$ in $SL_n(F[t])$ is

$$s_1(SL_n(F[t]))_\Lambda s_1^{-1}.$$

It follows that

$$s_1s_2^{-1}x = s_1\gamma s_1^{-1}$$

where γ stabilizes Λ . So,

$$(8) \quad x = s_2\gamma s_1^{-1}.$$

We have a split exact sequence

$$1 \rightarrow (K \cap (SL_n(F[t]))_\Lambda) \rightarrow (SL_n(F[t]))_\Lambda \xrightarrow{t=0} P_\Lambda \rightarrow 1$$

where P_Λ is a parabolic subgroup of $SL_n(F)$. Write $\gamma = kv$, where $k \in K$ and $v \in P_\Lambda$. Then

$$\begin{aligned} x &= s_2kvs_1^{-1} \\ &= s_2(vs_1^{-1})(s_1v^{-1})k(vs_1^{-1}). \end{aligned}$$

Since K is a normal subgroup of $SL_n(F[t])$, we have

$$(s_1 v^{-1})k(vs_1^{-1}) \in K.$$

Denote this element by k' . Then we may write

$$x = s_2(vs_1^{-1})k'$$

or

$$(9) \quad x(k')^{-1} = s_2(vs_1^{-1}).$$

Now, the element $x(k')^{-1}$ belongs to K while the element $s_2(vs_1^{-1})$ belongs to $SL_n(F)$. Since the groups K and $SL_n(F)$ intersect in the identity, both sides of equation (9) must equal 1. It follows that

$$s_2 = s_1 v^{-1}.$$

Since v^{-1} stabilizes Λ , we have

$$s_2 \Lambda = (s_1 v^{-1}) \Lambda = s_1 \Lambda.$$

It follows that $\mathcal{T}'$ is a fundamental domain for the action of K on $\mathcal{Y}$. $\square$

Remark. When $n = 2$, Proposition 4.1 allows us to deduce the free product decomposition

$$(10) \quad K = *_{s \in \mathbb{P}^1(F)} s C s^{-1}$$

where

$$C = \left\{ \begin{pmatrix} 1 & tp(t) \\ 0 & 1 \end{pmatrix} : p(t) \in F[t] \right\}$$

(here, the set S of coset representatives of $SL_2(F)/B_2(F)$ may be identified with $\mathbb{P}^1(F)$). For further details see [12, 4.1].

Now consider the stabilizer $\Gamma_{1,2,\dots,n}$ of the simplex $\mathcal{C}$ (see Proposition 1.1). We have a split short exact sequence

$$1 \longrightarrow K \longrightarrow \Gamma_{1,2,\dots,n} \xrightarrow{t=0} B_n(F) \longrightarrow 1.$$

Choose a set of representatives for the permutation group Σ_n in $SL_n(F)$ (e.g., we could take even permutations of the identity matrix along with odd permutations of the matrix $\text{diag}(-1, 1, \dots, 1)$). Denote by $\mathcal{D}_{1,2,\dots,n}$ the subcomplex of $\mathcal{Y}$ defined by

$$\mathcal{D}_{1,2,\dots,n} = \bigcup_{p \in \Sigma_n} p\mathcal{T}.$$

Proposition 4.2. *The subcomplex $\mathcal{D}_{1,2,\dots,n}$ is a fundamental domain for the action of $\Gamma_{1,2,\dots,n}$ on $\mathcal{Y}$.*

Proof. We have a split extension

$$1 \longrightarrow U \longrightarrow B_n(F) \xrightarrow{\pi} T \longrightarrow 1$$

where U is the unipotent radical of $B_n(F)$ and T is the diagonal subgroup. The composition of π with the map

$$\Gamma_{1,2,\dots,n} \xrightarrow{t=0} B_n(F)$$

yields a split extension

$$1 \longrightarrow G \longrightarrow \Gamma_{1,2,\dots,n} \longrightarrow T \longrightarrow 1.$$

Here, the group G consists of matrices of the form

$$\begin{pmatrix} 1 + tp_{11} & p_{12} & \cdots & \cdots & p_{1n} \\ tp_{21} & 1 + tp_{22} & \cdots & \cdots & p_{2n} \\ \vdots & & & & \vdots \\ tp_{n1} & \cdots & \cdots & tp_{n,n-1} & 1 + tp_{nn} \end{pmatrix}$$

where the p_{ij} lie in $F[t]$. We first show that $\mathcal{D}_{1,2,\dots,n}$ is a fundamental domain for the action of G on $\mathcal{Y}$.

Consider the extension

$$1 \longrightarrow K \longrightarrow G \xrightarrow{t=0} U \longrightarrow 1.$$

Suppose that σ is an $(n-1)$ -simplex in $\mathcal{Y}$. Then there exist $k \in K$, $s \in S$, and $\sigma_0 \in \mathcal{T}$ such that

$$\sigma = ks\sigma_0.$$

Recall the *Bruhat decomposition* of $SL_n(F)$ (see *e.g.*, [9, p. 172]):

$$SL_n(F) = \bigcup_{p \in \Sigma_n} UpB$$

(here, $B = B_n(F)$). From this it follows that if s is an element of the set S , then we may write $s = upv$ for some $u \in U$, $p \in \Sigma_n$, and $v \in B_n(F)$. Then we have the chain of equalities

$$\sigma = ks\sigma_0 = kupv\sigma_0 = kup\sigma_0.$$

The last equality follows since $B_n(F)$ acts trivially on $\mathcal{T}$. Now, ku lies in G . Hence,

$$\sigma \equiv p\sigma_0 \pmod{G}.$$

It follows that $\mathcal{D}_{1,2,\dots,n}$ is a fundamental domain for the action of G on $\mathcal{Y}$. Observe that the diagonal subgroup T acts trivially on $\mathcal{D}_{1,2,\dots,n}$.

Lemma 4.3. *Suppose a group H acts on a simplicial complex $\mathcal{Z}$, and that there is a split extension*

$$1 \longrightarrow N \longrightarrow H \longrightarrow Q \longrightarrow 1.$$

Suppose further that the subcomplex $\mathcal{A}$ is a fundamental domain for the action of N on $\mathcal{Z}$ and that Q acts trivially on $\mathcal{A}$. Then $\mathcal{A}$ is a fundamental domain for the action of H on $\mathcal{Z}$.

Proof. It suffices to show that no two vertices of $\mathcal{A}$ are identified by the action of H . Suppose that v_1 and v_2 are vertices of $\mathcal{A}$ and that there is an element h in H with $hv_1 = v_2$. Write $h = nq$, where $n \in N$, and $q \in Q$. Then we have

$$v_2 = hv_1 = nqv_1 = nv_1.$$

Since the vertices of $\mathcal{A}$ are inequivalent modulo N , we must have $v_1 = v_2$. $\square$

The lemma implies that $\mathcal{D}_{1,2,\dots,n}$ is a fundamental domain for the action of $\Gamma_{1,2,\dots,n}$ on $\mathcal{Y}$. This completes the proof of Proposition 4.2. $\square$

Finally, consider the group $\Gamma_{1,j_2,\dots,j_k}$. Note that $\Gamma_{1,j_2,\dots,j_k}$ contains the subgroup H of Σ_n consisting of permutation matrices that are products of the form

$$\sigma_1 \sigma_2 \cdots \sigma_{k-1}$$

where σ_i is a permutation of the set

$$\{j_i, j_i + 1, \dots, j_{i+1} - 1\}$$

(we take $j_1 = 1$). Let N be a set of coset representatives of $H \backslash \Sigma_n$ containing the identity. Define a subcomplex $\mathcal{D}_{1,j_2,\dots,j_k}$ by

$$\mathcal{D}_{1,j_2,\dots,j_k} = \bigcup_{p \in N} p\mathcal{T}.$$

Proposition 4.4. *The complex $\mathcal{D}_{1,j_2,\dots,j_k}$ is a fundamental domain for the action of $\Gamma_{1,j_2,\dots,j_k}$ on $\mathcal{Y}$.*

Proof. Observe that $\Gamma_{1,j_2,\dots,j_k}$ contains the group $\Gamma_{1,2,\dots,n}$. It follows that a fundamental domain for the action of $\Gamma_{1,j_2,\dots,j_k}$ on $\mathcal{Y}$ is no larger than $\mathcal{D}_{1,2,\dots,n}$. If σ is an $(n-1)$ -simplex in $\mathcal{Y}$, then there exist $g \in \Gamma_{1,2,\dots,n}$, $p \in \Sigma_n$, and $\sigma_0 \in \mathcal{T}$ such that

$$\sigma = gp\sigma_0.$$

Write $p = hn$, where $h \in H$ and $n \in N$. Then we have the chain of equalities

$$\sigma = gp\sigma_0 = ghen\sigma_0.$$

Since gh lies in $\Gamma_{1,j_2,\dots,j_k}$, it follows that

$$\sigma \equiv n\sigma_0 \pmod{\Gamma_{1,j_2,\dots,j_k}},$$

and hence, $\mathcal{D}_{1,j_2,\dots,j_k}$ is a fundamental domain for the action of $\Gamma_{1,j_2,\dots,j_k}$ on $\mathcal{Y}$. $\square$

5. THE HOMOLOGY OF $\Gamma_{1,j_2,\dots,j_k}$

We now compute the homology of the various $\Gamma_{1,j_2,\dots,j_k}$. This will complete the computation of the E^1 -term of the spectral sequence (4) since by Proposition 2.3 each $\Gamma_{i_1,\dots,i_k}$ is isomorphic to some $\Gamma_{1,j_2,\dots,j_k}$.

We have a split short exact sequence

$$1 \longrightarrow K \longrightarrow \Gamma_{1,j_2,\dots,j_k} \xrightarrow{t=0} P_{1,j_2,\dots,j_k} \longrightarrow 1$$

where $P_{1,j_2,\dots,j_k}$ is a parabolic subgroup of $SL_n(F)$.

Theorem 5.1. *The natural inclusion $P_{1,j_2,\dots,j_k} \longrightarrow \Gamma_{1,j_2,\dots,j_k}$ induces an isomorphism*

$$H_\bullet(P_{1,j_2,\dots,j_k}, \mathbb{Z}) \longrightarrow H_\bullet(\Gamma_{1,j_2,\dots,j_k}, \mathbb{Z}).$$

Proof. Since the complex $\mathcal{Y}$ is contractible, we obtain a spectral sequence converging to the homology of $\Gamma_{1,j_2,\dots,j_k}$ satisfying

$$(11) \quad E_{p,q}^1 = \bigoplus_{\dim \sigma = p} H_q(G_\sigma)$$

where G_σ is the stabilizer of the p -simplex σ in $\Gamma_{1,j_2,\dots,j_k}$ ($\sigma \subset \mathcal{D}_{1,j_2,\dots,j_k}$).

Recall the filtration $V^\bullet$ of $\mathcal{T}$ (5) defined in Section 3. Define a filtration $W^\bullet$ of $\mathcal{D}_{1,j_2,\dots,j_k}$ by setting

$$W^{(l)} = \bigcup_{p \in N} pV^{(l)}, \quad 0 \leq l \leq n-1.$$

Note that $W^{(0)} = v_0$ and that the group G_{v_0} is precisely $P_{1,j_2,\dots,j_k}$. Define a coefficient system $\mathcal{G}_q$ on $\mathcal{D}_{1,j_2,\dots,j_k}$ by

$$\mathcal{G}_q(\sigma) = H_q(G_\sigma).$$

Then the q th row of the spectral sequence (11) is the chain complex

$$C_\bullet(\mathcal{D}_{1,j_2,\dots,j_k}, \mathcal{G}_q).$$

On each component of $W^{(i)} - W^{(i-1)}$, the coefficient system $\mathcal{G}_q$ is constant (*i.e.*, the stabilizers in the translate $p\mathcal{T}$ are conjugate to the stabilizers in $\mathcal{T}$ and hence have isomorphic homology). So we may apply Lemma 3.3 to deduce that the inclusion $v_0 \longrightarrow \mathcal{D}_{1,j_2,\dots,j_k}$ induces an isomorphism

$$H_\bullet(v_0, \mathcal{G}_q) \longrightarrow H_\bullet(\mathcal{D}_{1,j_2,\dots,j_k}, \mathcal{G}_q).$$

Now the E^2 -term of the spectral sequence (11) satisfies

$$E_{p,q}^2 = \begin{cases} H_q(P_{1,j_2,\dots,j_k}) & p = 0 \\ 0 & p > 0. \end{cases}$$

This completes the proof of Theorem 5.1. $\square$

Remark. Theorem 3.4 is the special case $\Gamma_1 = SL_n(F[t])$ and $P_1 = SL_n(F)$.

Remark. In the case of $\Gamma_{1,2,\dots,n}$ and $P_{1,2,\dots,n} = B_n(F)$, it is not necessary to define the filtration $W^\bullet$ of $\mathcal{D}_{1,2,\dots,n}$ to prove the result. Indeed, Corollary 3.2 implies that each G_σ is homologically equivalent to $B_n(F)$. It follows that the q th row of spectral sequence (11) is the chain complex

$$C_\bullet(\mathcal{D}_{1,2,\dots,n}, H_q(B_n(F))).$$

Since $\mathcal{D}_{1,2,\dots,n}$ is contractible, the homology of the complex vanishes except in dimension zero, where we get $H_q(B_n(F))$.

Remark. When $n = 2$, we only have the group Γ_{12} . In this case, Theorem 5.1 states that

$$H_\bullet(\Gamma_{12}) \cong H_\bullet(B_2(F)).$$

This was proved in [12] for fields of characteristic zero by examining the Lyndon–Hochschild–Serre spectral sequence associated to the extension

$$1 \longrightarrow K \longrightarrow \Gamma_{12} \longrightarrow B_2(F) \longrightarrow 1.$$

The free product decomposition (10) for K allows us to deduce that

$$H_k(K) = \bigoplus_{s \in \mathbb{P}^1(F)} H_k(sCs^{-1}), \quad k \geq 1.$$

Utilizing Shapiro’s Lemma and a standard center kills argument, Proposition 4.4 of [12] shows that

$$H_\bullet(B_2(F), H_k(K)) = 0, \quad k \geq 1.$$

The $n = 2$ case of Theorem 5.1 follows easily. In [12], we used the action of $B_2(F)$ to kill the homology of K rather than finding a fundamental domain for the action of Γ_{12} on $\mathcal{Y}$. This approach works well in that case, but fails for $n \geq 3$ since we no longer have the free product decomposition for K .

6. THE d^1 -MAP

Having completed the computation of the E^1 -term of the spectral sequence (4), we now turn our attention to the differential, d^1 . Unfortunately, the computation of this map is rather difficult as it depends upon computing the maps induced on homology by the various inclusions $P_I \longrightarrow P_J$, where P_I and P_J are parabolic subgroups of $SL_n(F)$. To get a feel for the oddities which may occur, we present the following two results. Recall that for a field F , we denote by $B_2(F)$ the subgroup of $SL_2(F)$ consisting of upper triangular matrices.

Proposition 6.1 (Dupont–Sah [8]). *The natural map*

$$H_2(B_2(\mathbb{C})) \longrightarrow H_2(SL_2(\mathbb{C}))$$

is surjective. $\square$

The following result and its proof were communicated to me by J. Yang.

Proposition 6.2. *If F is a number field, then the natural map*

$$j : H_2(B_2(F), \mathbb{Q}) \longrightarrow H_2(SL_2(F), \mathbb{Q})$$

is trivial. $\square$

Proof. If F is a number field, then the group $K_2(F)$ is torsion. Since the map $H_2(B_2(F), \mathbb{Z}) \rightarrow H_2(SL_2(F), \mathbb{Z})$ factors through the map $H_2(B_2(F), \mathbb{Z}) \rightarrow K_2(F)$, it follows that after tensoring with $\mathbb{Q}$, the map j is trivial. $\square$

In light of these results, it seems to be a difficult question to compute the map

$$H_k(P_I) \longrightarrow H_k(P_J)$$

in general. Still, we are able to compute some special cases. In particular, we shall compute the maps $d_{*,0}^1$ and $d_{*,1}^1$.

6.1. The $q = 0$ case. Since the group $H_0(\Gamma_\sigma) = \mathbb{Z}$ for each simplex σ of $\mathcal{C}$, the $q = 0$ row of the spectral sequence (4) is simply the simplicial chain complex $S_\bullet(\mathcal{C})$. Since the simplex $\mathcal{C}$ is contractible, we have

$$E_{p,0}^2 = \begin{cases} \mathbb{Z} & p = 0 \\ 0 & p > 0. \end{cases}$$

6.2. The $q = 1$ case. Because we can find explicit representatives for elements of the various $H_1(\Gamma_\sigma)$, we are able to compute the map $d_{*,1}^1$. We begin by writing down the map explicitly.

Consider the group $\Gamma_{1,j_2,\dots,j_k}$. By Theorem 5.1, we have

$$H_1(\Gamma_{1,j_2,\dots,j_k}) \cong H_1(P_{1,j_2,\dots,j_k}).$$

By Corollary 3.2, the group $P_{1,j_2,\dots,j_k}$ has the same homology as its reductive part $L_{1,j_2,\dots,j_k}$. The group $L_{1,j_2,\dots,j_k}$ has the form

$$\overline{\begin{pmatrix} B_1 & & 0 \\ & B_2 & \\ & & \ddots \\ 0 & & & B_k \end{pmatrix}}$$

where each $B_i = GL_{j_{i+1}-j_i}(F)$ (see section 2). Now, for each i , $H_1(B_i) = F^\times$ (via the determinant map) and hence by the Künneth formula, $H_1(B_1 \times B_2 \times \cdots \times B_k) = (F^\times)^k$. It follows that

$$H_1(L_{1,j_2,\dots,j_k}) \cong (F^\times)^{k-1},$$

via the map

$$\begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_k \end{pmatrix} \mapsto (\det A_1, \det A_2, \dots, \det A_{k-1}).$$

Since each $\Gamma_{i_1, \dots, i_k}$ is conjugate to some $\Gamma_{1, j_2, \dots, j_k}$, it follows that

$$H_1(\Gamma_{i_1, \dots, i_k}) \cong (F^\times)^{k-1}.$$

Denote the simplex with vertices $i_1, i_2, \dots, i_k$ by $\sigma_{i_1 \dots i_k}$. We now compute the map

$$H_1(\Gamma_{i_1, \dots, i_k}) \longrightarrow H_1(\Gamma_{i_1, \dots, \widehat{i_l}, \dots, i_k})$$

induced by the face map $\sigma_{i_1 \dots i_k} \longrightarrow \sigma_{i_1 \dots \widehat{i_l} \dots i_k}$.

Lemma 6.3. *Let $\sigma_{i_1 \dots i_k}$ be a $(k-1)$ -simplex in $\mathcal{C}$ and suppose that $\sigma_{i_1 \dots \widehat{i_l} \dots i_k}$ is a face of $\sigma_{i_1 \dots i_k}$. Then the map*

$$H_1(\Gamma_{i_1, \dots, i_k}) \longrightarrow H_1(\Gamma_{i_1, \dots, \widehat{i_l}, \dots, i_k})$$

is the map

$$(F^\times)^{k-1} \longrightarrow (F^\times)^{k-2}$$

defined by

$$(\alpha_1, \dots, \alpha_{k-1}) \mapsto \begin{cases} (\alpha_2, \alpha_3, \dots, \alpha_{k-1}) & l = 1 \\ (\alpha_1, \dots, \alpha_{l-1} \alpha_l, \widehat{\alpha_l}, \dots, \alpha_{k-1}) & 2 \leq l \leq k-2 \\ (\alpha_1, \alpha_2, \dots, \alpha_{k-2}) & l = k-1. \end{cases}$$

Proof. To compute the map, we must chase elements around the following diagram:
(for $2 \leq l \leq k-1$)

$$\begin{array}{ccccc} \Gamma_{i_1, \dots, i_k} & \rightarrow & \Gamma_{1, (i_2-i_1+1), \dots, (i_k-i_1+1)} & \rightarrow & L_{1, (i_2-i_1+1), \dots, (i_k-i_1+1)} \\ \downarrow & & & & \\ \Gamma_{i_1, \dots, \widehat{i_l}, \dots, i_k} & \rightarrow & \Gamma_{1, \dots, (\widehat{i_l-i_1+1}), \dots, (i_k-i_1+1)} & \rightarrow & L_{1, \dots, (\widehat{i_l-i_1+1}), \dots, (i_k-i_1+1)} \\ & & & & \\ & & \dots & \rightarrow & (F^\times)^{k-1} \\ & & & & \downarrow \\ & & \dots & \rightarrow & (F^\times)^{k-2} \end{array}$$

Consider first the case $2 \leq l \leq k-2$. Here the first maps are the same in each row.

We follow elements around the diagram. In the first row, we have

$$\begin{aligned}
 & \begin{pmatrix} L_1 & V_{12} & V_{13} & \cdots & V_{1,k} & t^{-1}V_{1,k+1} \\ tV_{21} & L_2 & V_{23} & \cdots & V_{2,k} & V_{2,k+1} \\ tV_{31} & tV_{32} & L_3 & \cdots & V_{3,k} & V_{3,k+1} \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & \vdots \\ tV_{k+1,1} & tV_{k+1,2} & tV_{k+1,3} & \cdots & tV_{k+1,k} & L_{k+1} \end{pmatrix} \\
 & \mapsto \begin{pmatrix} L_2 & V_{23} & V_{24} & \cdots & \cdots & V_{2,k+1} & V_{21} \\ tV_{32} & L_3 & V_{34} & \cdots & \cdots & V_{3,k+1} & V_{31} \\ tV_{42} & tV_{43} & L_4 & \cdots & \cdots & V_{4,k+1} & V_{41} \\ \vdots & & & \ddots & & & \vdots \\ tV_{k,2} & tV_{k,3} & tV_{k,4} & \cdots & L_k & V_{k,k+1} & V_{k,1} \\ tV_{k+1,2} & tV_{k+1,3} & tV_{k+1,4} & \cdots & tV_{k+1,k} & L_{k+1} & V_{k+1,1} \\ tV_{12} & tV_{13} & tV_{14} & \cdots & tV_{1,k} & V_{1,k+1} & L_1 \end{pmatrix} \\
 & \mapsto \begin{pmatrix} L_2 & & & & & & 0 \\ & L_3 & & & & & \\ & & \ddots & & & & \\ & & & L_{l-1} & 0 & & \\ & & & 0 & L_l & & \\ & & & & & \ddots & \\ & & & & & & L_{k+1} & V_{k+1,1} \\ 0 & & & & & & V_{1,k+1} & L_1 \end{pmatrix} \\
 & \mapsto (\det L_2, \det L_3, \dots, \det L_k).
 \end{aligned}$$

In the second row, we have

$$\begin{pmatrix} L_1 & V_{12} & V_{13} & \cdots & V_{1,k} & t^{-1}V_{1,k+1} \\ tV_{21} & L_2 & V_{23} & \cdots & V_{2,k} & V_{2,k+1} \\ tV_{31} & tV_{32} & L_3 & \cdots & V_{3,k} & V_{3,k+1} \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & \vdots \\ tV_{k+1,1} & tV_{k+1,2} & tV_{k+1,3} & \cdots & tV_{k+1,k} & L_{k+1} \end{pmatrix} \mapsto$$

$$\begin{aligned}
& \mapsto \begin{pmatrix} L_2 & V_{23} & V_{24} & \cdots & \cdots & V_{2,k+1} & V_{21} \\ tV_{32} & L_3 & V_{34} & \cdots & \cdots & V_{3,k+1} & V_{31} \\ tV_{42} & tV_{43} & L_4 & \cdots & \cdots & V_{4,k+1} & V_{41} \\ \vdots & & & \ddots & & & \vdots \\ tV_{k,2} & tV_{k,3} & tV_{k,4} & \cdots & L_k & V_{k,k+1} & V_{k,1} \\ tV_{k+1,2} & tV_{k+1,3} & tV_{k+1,4} & \cdots & tV_{k+1,k} & L_{k+1} & V_{k+1,1} \\ tV_{12} & tV_{13} & tV_{14} & \cdots & tV_{1,k} & V_{1,k+1} & L_1 \end{pmatrix} \\
& \mapsto \begin{pmatrix} L_2 & & & & & & 0 \\ & L_3 & & & & & \\ & & \ddots & & & & \\ & & & L_{l-1} & V_{l-1,l} & & \\ & & & 0 & L_l & & \\ & & & & & \ddots & \\ & & & & & & L_{k+1} & V_{k+1,1} \\ 0 & & & & & & V_{1,k+1} & L_1 \end{pmatrix} \\
& \mapsto (\det L_2, \dots, \det \begin{pmatrix} L_{l-1} & V_{l-1,l} \\ 0 & L_l \end{pmatrix}, \dots, \det L_k) \\
& = (\det L_2, \dots, \det L_{l-1} \det L_l, \det L_{l+1}, \dots, \det L_k).
\end{aligned}$$

So we see that the map $(F^\times)^{k-1} \longrightarrow (F^\times)^{k-2}$ is given by

$$(\alpha_1, \dots, \alpha_{k-1}) \mapsto (\alpha_1, \dots, \alpha_{l-1} \alpha_l, \widehat{\alpha_l}, \dots, \alpha_{k-1}).$$

Next, consider the case $l = k - 1$. Here the map in the second row is as follows:

$$\begin{pmatrix} L_1 & V_{12} & V_{13} & \cdots & V_{1,k} & t^{-1}V_{1,k+1} \\ tV_{21} & L_2 & V_{23} & \cdots & V_{2,k} & V_{2,k+1} \\ tV_{31} & tV_{32} & L_3 & \cdots & V_{3,k} & V_{3,k+1} \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & \vdots \\ tV_{k+1,1} & tV_{k+1,2} & tV_{k+1,3} & \cdots & tV_{k+1,k} & L_{k+1} \end{pmatrix} \mapsto$$

$$\begin{aligned}
& \mapsto \begin{pmatrix} L_2 & V_{23} & V_{24} & \cdots & \cdots & V_{2,k+1} & V_{21} \\ tV_{32} & L_3 & V_{34} & \cdots & \cdots & V_{3,k+1} & V_{31} \\ tV_{42} & tV_{43} & L_4 & \cdots & \cdots & V_{4,k+1} & V_{41} \\ \vdots & & & \ddots & & & \vdots \\ tV_{k,2} & tV_{k,3} & tV_{k,4} & \cdots & L_k & V_{k,k+1} & V_{k,1} \\ tV_{k+1,2} & tV_{k+1,3} & tV_{k+1,4} & \cdots & tV_{k+1,k} & L_{k+1} & V_{k+1,1} \\ tV_{12} & tV_{13} & tV_{14} & \cdots & tV_{1,k} & V_{1,k+1} & L_1 \end{pmatrix} \\
& \mapsto \begin{pmatrix} L_2 & & & & & & \\ & L_3 & & & & & \\ & & \ddots & & & & \\ & & & L_{k-1} & & & \\ & & & & \begin{pmatrix} L_k & V_{k,k+1} & V_{k,1} \\ 0 & L_{k+1} & V_{k+1,1} \\ 0 & V_{1,k+1} & L_1 \end{pmatrix} & & \end{pmatrix} \\
& \mapsto (\det L_2, \dots, \det L_{k-1}).
\end{aligned}$$

So, the map $(F^\times)^{k-1} \longrightarrow (F^\times)^{k-2}$ is simply

$$(\alpha_1, \dots, \alpha_{k-1}) \mapsto (\alpha_1, \dots, \alpha_{k-2}).$$

Finally, consider the case $l = 1$. In this case, we are omitting the first vertex i_1 . Thus, we use different conjugation maps in the isomorphisms

$$\Gamma_{i_1, \dots, i_k} \longrightarrow \Gamma_{1, (i_2 - i_1 + 1), \dots, (i_k - i_1 + 1)}$$

and

$$\Gamma_{i_2, \dots, i_k} \longrightarrow \Gamma_{1, (i_3 - i_2 + 1), \dots, (i_k - i_2 + 1)}.$$

Now the second row of the diagram looks like

$$\begin{pmatrix} L_1 & V_{12} & V_{13} & \cdots & V_{1,k} & t^{-1}V_{1,k+1} \\ tV_{21} & L_2 & V_{23} & \cdots & V_{2,k} & V_{2,k+1} \\ tV_{31} & tV_{32} & L_3 & \cdots & V_{3,k} & V_{3,k+1} \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & \vdots \\ tV_{k+1,1} & tV_{k+1,2} & tV_{k+1,3} & \cdots & tV_{k+1,k} & L_{k+1} \end{pmatrix} \mapsto$$

$$\begin{aligned}
& \mapsto \begin{pmatrix} L_3 & V_{34} & \cdots & \cdots & V_{3,k+1} & V_{31} & V_{32} \\ tV_{43} & L_4 & \cdots & \cdots & V_{4,k+1} & V_{41} & V_{42} \\ & & \ddots & & & & \\ tV_{k,3} & tV_{k,4} & & L_k & V_{k,k+1} & V_{k,1} & V_{k,2} \\ tV_{k+1,3} & tV_{k+1,4} & & & L_{k+1} & V_{k+1,1} & V_{k+1,2} \\ tV_{13} & tV_{14} & & & V_{1,k+1} & L_1 & V_{12} \\ tV_{23} & tV_{24} & & & V_{2,k+1} & V_{21} & L_2 \end{pmatrix} \\
& \mapsto \begin{pmatrix} L_3 & & & & & & \\ & L_4 & & & & & \\ & & \ddots & & & & \\ & & & L_k & & & \\ & & & & \begin{pmatrix} L_{k+1} & V_{k+1,1} & V_{k+1,2} \\ V_{1,k+1} & L_1 & V_{12} \\ V_{2,k+1} & V_{21} & L_2 \end{pmatrix} & & \end{pmatrix} \\
& \mapsto (\det L_3, \dots, \det L_k).
\end{aligned}$$

Hence, the map $(F^\times)^{k-1} \longrightarrow (F^\times)^{k-2}$ is given by

$$(\alpha_1, \dots, \alpha_{k-1}) \mapsto (\alpha_2, \dots, \alpha_{k-1}).$$

This completes the proof of Lemma 6.3. $\square$

Denote the element $(\alpha_1, \dots, \alpha_{k-1})$ of $H_1(\Gamma_{i_1, \dots, i_k})$ by $\sigma_{i_1 \dots i_k} \otimes [\alpha_1, \dots, \alpha_{k-1}]$. Then the d^1 -map is given by the formula

$$\begin{aligned}
(12) \quad d^1 : \sigma_{i_1 \dots i_k} \otimes [\alpha_1, \dots, \alpha_{k-1}] &\mapsto \\
&\sigma_{i_2 \dots i_k} \otimes [\alpha_2, \dots, \alpha_{k-1}] \\
&+ \sum_{l=2}^{k-1} (-1)^{l-1} \sigma_{i_1 \dots \widehat{i_l} \dots i_k} \otimes [\alpha_1, \dots, \alpha_{l-1} \alpha_l, \widehat{\alpha_l}, \dots, \alpha_{k-1}] \\
&+ (-1)^{k-1} \sigma_{i_1 \dots i_{k-1}} \otimes [\alpha_1, \dots, \alpha_{k-2}].
\end{aligned}$$

Let A be an abelian group (written additively). Denote by $Q_\bullet^{(n)}$ the chain complex defined as follows. To each $(k-1)$ -simplex $\sigma_{i_1 \dots i_k}$ of $\mathcal{C}$ we assign the group A^{k-1} . The boundary map $d : Q_{k-1}^{(n)} \longrightarrow Q_{k-2}^{(n)}$ is given by formula (12) above. We will compute the homology of $Q_\bullet^{(n)}$ for any abelian group A . Taking $A = F^\times$ we obtain the terms $E_{*,1}^2$ of the spectral sequence (4).

To compute the homology of the complex $Q_\bullet^{(n)}$, we realize $Q_\bullet^{(n)}$ as a quotient of another complex $C_\bullet^{(n)}$. We shall then compute $H_\bullet(C_\bullet^{(n)})$ and use this along with a long exact homology sequence to obtain $H_\bullet(Q_\bullet^{(n)})$.

Construct the chain complex $C_{\bullet}^{(n)}$ by assigning to each $(k-1)$ -simplex $\sigma_{i_1 \dots i_k}$ of $\mathcal{C}$ the group A^k . Define the boundary map ∂ by

$$(13) \quad \partial : \sigma_{i_1 \dots i_k} \otimes (a_1, \dots, a_k) \mapsto \sum_{l=1}^k (-1)^{l-1} \sigma_{i_1 \dots \widehat{i_l} \dots i_k} \otimes (a_1, \dots, \widehat{a_l}, \dots, a_k).$$

Observe that for each $n \geq 2$, $C_{\bullet}^{(n)}$ is a subcomplex of $C_{\bullet}^{(n+1)}$.

Denote by $B_{\bullet}^{(n)}$ the standard simplicial chain complex for $\mathcal{C}$ with coefficients in A . Embed the complex $B_{\bullet}^{(n)}$ into $C_{\bullet}^{(n)}$ via

$$\sigma_{i_1 \dots i_k} \otimes a \mapsto \sigma_{i_1 \dots i_k} \otimes (a, \dots, a).$$

Then we have the following.

Lemma 6.4. *The quotient complex $C_{\bullet}^{(n)}/B_{\bullet}^{(n)}$ is isomorphic to the complex $Q_{\bullet}^{(n)}$.*

Proof. Denote the quotient complex by $D_{\bullet}^{(n)}$. In $D_{\bullet}^{(n)}$, we have assigned to each simplex $\sigma_{i_1 \dots i_k}$ the group $A^k/A \cdot (1, \dots, 1) \cong A^{k-1}$. We need only check that the boundary map is the same as that for $Q_{\bullet}^{(n)}$. We take our isomorphism $A^k/A \cdot (1, \dots, 1) \cong A^{k-1}$ to be the map

$$(a_1, \dots, a_k) \mapsto (a_2 - a_1, a_3 - a_2, \dots, a_k - a_{k-1}).$$

To compute the boundary map in $D_{\bullet}^{(n)}$, we lift elements to $C_{\bullet}^{(n)}$, apply ∂ , and then project back to $D_{\bullet}^{(n)}$. Denote the projection map $C_{\bullet}^{(n)} \rightarrow D_{\bullet}^{(n)}$ by π . Then we have

$$\pi : \sigma_{i_1 \dots i_k} \otimes (0, a_1, a_1 + a_2, \dots, a_1 + a_2 + \dots + a_{k-1}) \mapsto \sigma_{i_1 \dots i_k} \otimes [a_1, \dots, a_{k-1}]$$

and

$$\begin{aligned} \partial : \sigma_{i_1 \dots i_k} \otimes (0, a_1, a_1 + a_2, \dots, a_1 + a_2 + \dots + a_{k-1}) \mapsto \\ \sum_{l=1}^k (-1)^{l-1} \sigma_{i_1 \dots \widehat{i_l} \dots i_k} \otimes (0, a_1, \dots, a_1 + \widehat{\dots} + a_{l-1}, \dots, a_1 + \dots + a_{k-1}). \end{aligned}$$

Applying π to the right hand side of this equation, we see that the boundary map in $D_{\bullet}^{(n)}$ is the map

$$\begin{aligned} \sigma_{i_1 \dots i_k} \otimes [a_1, \dots, a_{k-1}] \mapsto \\ \sigma_{i_2 \dots i_k} \otimes [a_2, \dots, a_{k-1}] \\ + \sum_{l=2}^{k-1} (-1)^{l-1} \sigma_{i_1 \dots \widehat{i_l} \dots i_k} \otimes [a_1, \dots, a_{l-1} + a_l, \widehat{a_l}, \dots, a_{k-1}] \\ + (-1)^{k-1} \sigma_{i_1 \dots i_{k-1}} \otimes [a_1, \dots, a_{k-2}]. \end{aligned}$$

It follows that $D_{\bullet}^{(n)}$ is isomorphic to $Q_{\bullet}^{(n)}$. $\square$

We now have a short exact sequence of chain complexes

$$0 \longrightarrow B_{\bullet}^{(n)} \longrightarrow C_{\bullet}^{(n)} \longrightarrow Q_{\bullet}^{(n)} \longrightarrow 0.$$

The homology of $B_{\bullet}^{(n)}$ is easily computed (since $\mathcal{C}$ is contractible). We now compute the homology of $C_{\bullet}^{(n)}$.

Proposition 6.5. *The complex $C_{\bullet}^{(n)}$ is contractible. Hence, $H_{\bullet}(C_{\bullet}^{(n)}) = 0$.*

Proof. If n is even, we define a contracting homotopy h for $C_{\bullet}^{(n)}$ by

$$\begin{aligned} h : \sigma_{i_1 \dots i_k} \otimes (a_1, \dots, a_k) &\mapsto \\ &\sum_{l=1}^{i_1-1} \sigma_{li_1 \dots i_k} \otimes (0, (-1)^{i_1+l+1} a_1, (-1)^{i_2+l+1} a_2, \dots, (-1)^{i_k+l+1} a_k) \\ &- \sum_{l=i_1+1}^{i_2-1} \sigma_{i_1 li_2 \dots i_k} \otimes ((-1)^{i_1+l+1} a_1, 0, (-1)^{i_2+l+1} a_2, \dots, (-1)^{i_k+l+1} a_k) \\ &+ \dots \\ &+ (-1)^k \sum_{l=i_k+1}^n \sigma_{i_1 \dots i_k l} \otimes ((-1)^{i_1+l+1} a_1, \dots, (-1)^{i_k+l+1} a_k, 0). \end{aligned}$$

If n is odd, then $n-1$ is even. So if $\sigma_{i_1 \dots i_k}$ is a simplex in $\mathcal{C}$ with $i_k < n$, then we may view $\sigma_{i_1 \dots i_k} \otimes (a_1, \dots, a_k)$ as belonging to the subcomplex $C_{\bullet}^{(n-1)}$. Thus, we may use the formula above. We extend h to simplices with $i_k = n$ as follows. If $i_{k-1} < n-1$, then we define h to be

$$\begin{aligned} h : \sigma_{i_1 \dots i_{k-1} n} \otimes (a_1, \dots, a_k) &\mapsto \\ &\sum_{l=1}^{i_1-1} \sigma_{li_1 \dots i_{k-1} n} \otimes (0, (-1)^{i_1+l+1} a_1, \dots, (-1)^{n+l+1} a_k) \\ &- \sum_{l=i_1+1}^{i_2-1} \sigma_{i_1 li_2 \dots i_{k-1} n} \otimes ((-1)^{i_1+l+1} a_1, 0, (-1)^{i_2+l+1} a_2, \dots, (-1)^{n+l+1} a_k) \\ &+ \dots \\ &+ (-1)^{k-1} \sum_{l=i_{k-1}+1}^{n-1} \sigma_{i_1 \dots i_{k-1} l n} \otimes ((-1)^{i_1+l+1} a_1, \dots, 0, (-1)^{n+l+1} a_k) \\ &- \sum_{l=1}^{i_1-1} \sigma_{li_1 \dots i_{k-1} n} \otimes (0, \dots, 0, (-1)^l a_k) \\ &+ \sum_{l=i_1+1}^{i_2-1} \sigma_{i_1 li_2 \dots i_{k-1} n} \otimes (0, \dots, 0, (-1)^l a_k) \\ &+ \dots \\ &+ (-1)^k \sum_{l=i_{k-1}+1}^{n-2} \sigma_{i_1 \dots i_{k-1} l n} \otimes (0, \dots, 0, (-1)^l a_k). \end{aligned}$$

If $i_{k-1} = n - 1$, then

$$\begin{aligned}
h : \sigma_{i_1 \dots i_{k-2}, n-1, n} \otimes (a_1, \dots, a_k) \mapsto & \\
& \sum_{l=1}^{i_1-1} \sigma_{li_1 \dots i_{k-2}, n-1, n} \otimes (0, (-1)^{i_1+l+1} a_1, \dots, (-1)^{n+l+1} a_k) \\
& - \sum_{l=i_1+1}^{i_2-1} \sigma_{i_1 li_2 \dots i_{k-2}, n-1, n} \otimes ((-1)^{i_1+l+1} a_1, 0, \dots, (-1)^{n+l+1} a_k) \\
& + \dots \\
& + (-1)^{k-2} \sum_{l=i_{k-2}+1}^{n-2} \sigma_{i_1 \dots i_{k-2}, n-1, n} \\
& \quad \otimes ((-1)^{i_1+l+1} a_1, \dots, 0, (-1)^{(n-1)+l+1} a_{k-1}, (-1)^{n+l+1} a_k) \\
& - \sum_{l=1}^{i_1-1} \sigma_{li_1 \dots i_{k-2}, n-1, n} \otimes (0, \dots, 0, (-1)^l a_k) \\
& + \sum_{l=i_1+1}^{i_2-1} \sigma_{i_1 li_2 \dots i_{k-2}, n-1, n} \otimes (0, \dots, 0, (-1)^l a_k) \\
& + \dots \\
& + (-1)^{k-1} \sum_{l=i_{k-2}+1}^{n-2} \sigma_{i_1 \dots i_{k-2} l, n-1, n} \otimes (0, \dots, 0, (-1)^l a_k).
\end{aligned}$$

One checks that $\partial h + h \partial = \text{identity}$. This completes the proof of the proposition. $\square$

Corollary 6.6. *The homology of the complex $Q_\bullet^{(n)}$ is given by*

$$H_k(Q_\bullet^{(n)}) = \begin{cases} A & k = 1 \\ 0 & k \neq 1. \end{cases}$$

Proof. Since $C_\bullet^{(n)}$ is contractible, the long exact homology sequence implies that

$$H_k(Q_\bullet^{(n)}) \cong H_{k-1}(B_\bullet^{(n)}).$$

The result follows since

$$H_k(B_\bullet^{(n)}) = \begin{cases} A & k = 0 \\ 0 & k \neq 0. \end{cases} \quad \square$$

Taking $A = F^\times$, we obtain the following.

Corollary 6.7. *The spectral sequence (4) satisfies*

$$E_{p,1}^2 = \begin{cases} F^\times & p = 1 \\ 0 & p \neq 1. \end{cases} \quad \square$$

6.3. The Second Homology and Cohomology Groups.

Corollary 6.8. *There is an exact sequence*

$$0 \longrightarrow \operatorname{coker}\{d_{1,2}^1 : E_{1,2}^1 \rightarrow E_{0,2}^1\} \longrightarrow H_2(SL_n(F[t, t^{-1}])) \longrightarrow F^\times \longrightarrow 1.$$

Proof. Since $E_{p,0}^2 = E_{p,1}^2 = 0$ for $p > 1$, we have $E_{0,2}^2 = E_{0,2}^\infty$. The group $E_{0,2}^2$ is precisely the cokernel of $d^1 : E_{1,2}^1 \longrightarrow E_{0,2}^1$. Since $E_{1,1}^2 = F^\times$, the result follows. $\square$

Corollary 6.9. *Let F be a number field and denote the number of real embeddings of F by r_1 . Then*

$$H_2(SL_2(F[t, t^{-1}]), \mathbb{Q}) \cong (F^\times \otimes \mathbb{Q}) \oplus \mathbb{Q}^{2r_1}.$$

Proof. By Borel–Yang [3], we have

$$H_2(SL_2(F), \mathbb{Q}) = \mathbb{Q}^{r_1}.$$

It follows that $E_{0,2}^1 = \mathbb{Q}^{2r_1}$. By Proposition 6.2, the map $d^1 : E_{1,2}^1 \rightarrow E_{0,2}^1$ is trivial. Hence, we have an exact sequence

$$0 \longrightarrow \mathbb{Q}^{2r_1} \longrightarrow H_2(SL_2(F[t, t^{-1}]), \mathbb{Q}) \longrightarrow F^\times \otimes \mathbb{Q} \longrightarrow 0. \quad \square$$

We now investigate the map $d_{1,2}^1$.

Proposition 6.10. *If $n \geq 3$, then the cokernel of the map $d_{1,2}^1 : E_{1,2}^1 \longrightarrow E_{0,2}^1$ is isomorphic to $H_2(SL_n(F), \mathbb{Z})$.*

Proof. The term $E_{0,2}^1$ is equal to

$$\bigoplus_{i=1}^n H_2(\Gamma_i).$$

Since each Γ_1 is conjugate to $SL_n(F[t])$ in $GL_n(F[t, t^{-1}])$, by Theorem 3.4 we have

$$E_{0,2}^1 \cong H_2(SL_n(F), \mathbb{Z})^{\oplus n}.$$

Consider the map

$$p : H_2(SL_n(F), \mathbb{Z})^{\oplus n} \longrightarrow H_2(SL_n(F), \mathbb{Z})$$

defined by

$$p(a_1, \dots, a_n) = \sum_{i=1}^n a_i.$$

The map p is surjective with kernel consisting of those elements of $H_2(SL_n(F), \mathbb{Z})^{\oplus n}$ whose entries sum to zero. We show that the image of $d_{1,2}^1$ coincides with the kernel of p . Given a pair of integers i, j with $1 \leq i < j \leq n$, we have maps

$$H_2(\Gamma_{ij}) \longrightarrow H_2(\Gamma_i) \quad \text{and} \quad H_2(\Gamma_{ij}) \longrightarrow H_2(\Gamma_j)$$

induced by inclusion. The map $d_{1,2}^1$ is the alternating sum of these maps. To compute the image of $d_{1,2}^1$ as a subgroup of $H_2(SL_n(F), \mathbb{Z})^{\oplus n}$, we make use of the diagrams

$$\begin{array}{ccccc} H_2(\Gamma_{ij}) & \xrightarrow{\cong} & H_2(\Gamma_{1,j-i+1}) & \longrightarrow & H_2(\Gamma_1) \\ & & \searrow & & \uparrow \cong \\ & & & & H_2(\Gamma_{j-i+1}) \end{array}$$

to see that the image of $H_2(\Gamma_{ij})$ in $H_2(\Gamma_i)$ is isomorphic (via the identifications $\Gamma_i \cong \Gamma_1$) to the image of $H_2(\Gamma_{ij})$ in $H_2(\Gamma_j)$. Since $d_{1,2}^1$ maps $H_2(\Gamma_{ij})$ to $H_2(\Gamma_i)$ with a negative sign and to $H_2(\Gamma_j)$ with a positive sign, we see that the image of $d_{1,2}^1$ in $H_2(SL_n(F), \mathbb{Z})^{\oplus n}$ lies in the kernel of p .

To see that the image is all of the kernel, we use a result of Hutchinson [10, p. 200] which states that if F is an infinite field, then the map

$$H_2(\Gamma_{12}) \longrightarrow H_2(\Gamma_1)$$

is surjective for $n \geq 3$. It follows that the maps

$$H_2(\Gamma_{i,i+1}) \longrightarrow H_2(\Gamma_i) \quad \text{and} \quad H_2(\Gamma_{i,i+1}) \longrightarrow H_2(\Gamma_{i+1})$$

are surjective for $i = 1, \dots, n-1$. Thus, the image of $d_{1,2}^1$ contains all elements of the form

$$(-a, a, 0, \dots, 0), (0, -a, a, 0, \dots, 0), \dots, (0, \dots, 0, -a, a)$$

and it follows that the image of $d_{1,2}^1$ coincides with the kernel of p . $\square$

Corollary 6.11. *If F is an infinite field, then for $n \geq 3$,*

$$H_2(SL_n(F[t, t^{-1}]), \mathbb{Z}) = H_2(SL_n(F), \mathbb{Z}) \oplus F^\times.$$

Proof. The spectral sequence (4) gives an exact sequence

$$0 \longrightarrow H_2(SL_n(F), \mathbb{Z}) \xrightarrow{\phi} H_2(SL_n(F[t, t^{-1}]), \mathbb{Z}) \longrightarrow F^\times \longrightarrow 0.$$

Observe that the map $p : E_{1,2}^1 \longrightarrow E_{0,2}^1$ is split by inclusion onto the first factor. It follows that the map ϕ is induced by the canonical inclusion $SL_n(F) \longrightarrow SL_n(F[t, t^{-1}])$. Observe that this map is split by the map

$$SL_n(F[t, t^{-1}]) \xrightarrow{t=1} SL_n(F).$$

It follows that $H_2(SL_n(F), \mathbb{Z})$ is a direct summand of $H_2(SL_n(F[t, t^{-1}]), \mathbb{Z})$. This proves the corollary. $\square$

Remark. Since $K_2(F[t, t^{-1}]) = K_2(F) \oplus K_1(F)$ and since $K_2(F) = H_2(SL_n(F), \mathbb{Z})$ for $n \geq 3$, Corollary 6.11 implies that $H_2(SL_n(F[t, t^{-1}]), \mathbb{Z})$ stabilizes at $n = 3$; i.e., for $n \geq 3$ we have an isomorphism

$$H_2(SL_n(F[t, t^{-1}]), \mathbb{Z}) \cong K_2(F[t, t^{-1}]).$$

Corollary 6.12. *If $n \geq 3$, then*

$$H^2(SL_n(F[t, t^{-1}]), \mathbb{Z}) \cong H^2(SL_n(F), \mathbb{Z}) \oplus \text{Hom}_{\mathbb{Z}}(F^\times, \mathbb{Z}).$$

Proof. By the Universal Coefficient Theorem,

$$\begin{aligned}
 H^2(SL_n(F[t, t^{-1}]), \mathbb{Z}) &\cong \operatorname{Hom}_{\mathbb{Z}}(H_2(SL_n(F[t, t^{-1}]), \mathbb{Z}), \mathbb{Z}) \oplus \\
 &\quad \operatorname{Ext}_{\mathbb{Z}}(H_1(SL_n(F[t, t^{-1}]), \mathbb{Z}), \mathbb{Z}) \\
 &\cong \operatorname{Hom}_{\mathbb{Z}}(H_2(SL_n(F), \mathbb{Z}), \mathbb{Z}) \oplus \operatorname{Hom}_{\mathbb{Z}}(F^{\times}, \mathbb{Z}) \oplus 0 \\
 &\cong H^2(SL_n(F), \mathbb{Z}) \oplus \operatorname{Hom}_{\mathbb{Z}}(F^{\times}, \mathbb{Z}). \quad \square
 \end{aligned}$$

REFERENCES

1. A. Borel, *Stable real cohomology of arithmetic groups*, Ann. Scient. Éc. Norm. Sup. (4) **7** (1974), 235–272.
2. A. Borel, *Stable real cohomology of arithmetic groups II*, Prog. Math. **14** (1981), 21–55, Birkhäuser–Boston.
3. A. Borel, J. Yang, *The rank conjecture for number fields*, Mathematical Research Letters **1** (1994), 689–699.
4. K. Brown, *Buildings*, Springer–Verlag, Berlin, Heidelberg, New York, 1989.
5. K. Brown, *Cohomology of Groups*, Springer–Verlag, Berlin, Heidelberg, New York, 1982.
6. F. Bruhat, J. Tits, *Groupes réductifs sur un corps local I: Données radicielles valuées*, Publ. IHES **41** (1972), 5–252.
7. R. Charney, *Homology stability for GL_n of a Dedekind domain*, Invent. Math. **56** (1980), 1–17.
8. J. Dupont, H. Sah, *Scissors congruences II*, J. Pure and Appl. Algebra **25** (1982), 159–195.
9. J. Humphreys, *Linear Algebraic Groups*, Springer–Verlag, Berlin, Heidelberg, New York, 1981.
10. K. Hutchinson, *A new approach to Matsumoto’s Theorem*, K–Theory **4** (1990), 181–200.
11. W. van der Kallen, *Homology stability for linear groups*, Invent. Math. **60** (1980), 269–295.
12. K. Knudson, *The homology of $SL_2(F[t, t^{-1}])$* , J. Alg. **180** (1996), 87–101.
13. H. Nagao, *On $GL(2, K[x])$* , J. Poly. Osaka Univ. **10** (1959), 117–121.
14. D. Quillen, *On the cohomology and K–theory of the general linear groups over a finite field*, Ann. Math. **96** (1972), 552–586.
15. D. Quillen, *Higher algebraic K–theory: I*, in K–Theory, H. Bass, ed., Springer Lecture Notes in Mathematics **341** (1976), 85–147.
16. J-P. Serre, *Trees*, Springer–Verlag, Berlin, Heidelberg, New York, 1980.
17. C. Soulé, *Chevalley groups over polynomial rings*, in Homological Group Theory (C.T.C. Wall, ed.), London Math. Soc. Lecture Notes **36**, Cambridge University Press, Cambridge (1979), 359–367.
18. C. Soulé, *The cohomology of $SL_3(\mathbb{Z})$* , Topology **17** (1978), 1–22.
19. A. Suslin, *Homology of GL_n , characteristic classes and Milnor K–theory*, Springer Lecture Notes in Mathematics **1046** (1984), 357–375.
20. A. Suslin, *K_3 of a field and Bloch’s group*, Proceedings of the Steklov Institute of Mathematics, 1991, Issue 4.
21. A. Suslin, *On the structure of the special linear group over polynomial rings*, Math. USSR Izvestija **11** (1977), 221–238.

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