On an elliptic analogue of Zagier's conjecture.

Jörg Wildeshaus
Mathematisches Institut
Einsteinstr. 62
D–48149 Münster
e–mail: wildesh@math.uni-muenster.de

Introduction

This article is the elliptic version of "Interprétation motivique de la conjecture de Zagier reliant polylogarithmes et régulateurs" ([BD]).

The object of this work is a conjecture which predicts that certain formal linear combinations of elements of the Mordell–Weil group are homologically meaningful, i.e., yield elements in certain K–groups of symmetric powers of elliptic curves. It also predicts their images under the Beilinson regulator.
We refer to this as the *Weak Version of Zagier's Conjecture for elliptic curves*.

§ 1 contains the formulation of the conjecture. The most general context in which it can be stated is that of *families* of elliptic curves over a base B. However, we give the statement for the image under the regulator only when B is the spectrum of a number field, where Deligne cohomology admits an easy description.
Since we require functoriality with respect to pullbacks, this does still impose nontrivial restrictions on the regulator for arbitrary bases.

We remark that if the formalism predicted by the conjecture satisfied a certain surjectivity requirement, then Beilinson's conjecture would give a description of values of L–functions of symmetric powers of elliptic curves in terms of determinants of values of Eisenstein–Kronecker series.
For CM–elliptic curves satisfying Shimura's condition (S), this relation was shown in [De1] and [De2]. For $L(E, 2)$, which in our notation corresponds to the case $k = 3$, it has recently been proven for modular elliptic curves ([GL]).

In § 2 and § 3, we develop a machinery, that will allow us to construct extensions from certain formal linear combinations of elements in the Mordell–Weil group, as soon as we work in a category of smooth sheaves satisfying certain axioms, among which the existence of elliptic polylogarithmic extensions.

In § 4, we show that if a category of mixed motivic sheaves with these axioms exists, and if it has the right Ext–groups, then the weak version of Zagier's conjecture follows from the results of § 3, and the description of the Hodge version of the elliptic polylogarithm given elsewhere.

This work would not exist, were it not for N. Schappacher, who on more than one occasion insisted with vigour that the author concern himself with the elliptic Zagier conjecture. I thank him warmly.

Conversations with him, and with S. Bloch and T. Scholl helped me realize that an integrality criterion should be included in the conjecture. However, its sheaf theoretic interpretation remains a desideratum. The special shape of the criterion can be justified as follows: by the main result of [SaSo] on the boundary of the Eisenstein symbol, the condition is the correct one as long as we treat only formal linear combinations of torsion points (see Theorem 1.9.1). Furthermore, the above mentioned result of [GL] shows that for the first symmetric power of a modular elliptic curve, the criterion gives the right answer for arbitrary formal linear combinations (see Example 1.11.a)).

For the lowest nontrivial step ($k = 2$) of our formalism, local heights appear in the formulae for the regulator and for the integrality criterion. This interpretation of the explicit fomulae was not clear to me when I initially made available part of the material contained in this article. The first to mention heights to me in this context was R. de Jeu. The relation was amplified further when [GL] appeared (see 2.2 of loc. cit.). The precise formula became clear to me after I studied K. Rolshausen's thesis ([R], in particular V.4).

I am indebted to the referee for a number of helpful suggestions, particularly the one concerning the construction of the map $\varphi_2^\sharp$ in the proof of Theorem 1.9.2. Also, I wish to thank A. Werner, H. Gangl, A. Goncharov, C. Soulé and D. Zagier for useful conversations.

Finally, it is a pleasure to thank Gabi Weckermann for TEXing my manuscript.

§ 1 Formulation of the conjecture, and of the main results

As in the classical case, the weak version of Zagier's conjecture consists of several parts: the first gives a recipe to construct elements in certain K–groups from formal linear combinations of symbols on the Mordell–Weil group, which lie in the kernel of a certain differential. The second, which ensures that the first is not trivial, predicts the image of these elements in Deligne cohomology.
In contrast to the classical case, where the integral cohomology

$$H^1_{\mathcal{M}}(\mathrm{Spec}(K), \mathbb{Q}(k))_{\mathbb{Z}} = H^1_{\mathcal{M}}(\mathrm{Spec}(O_K), \mathbb{Q}(k))$$

equals the whole of $H^1_{\mathcal{M}}(\mathrm{Spec}(K), \mathbb{Q}(k))$ for $k \geq 2$, there is a genuine need for an integrality criterion for the elements in K–theory obtained via the hypothetical formalism (c.f. 1.4, where we discuss integrality for elliptic curves). This criterion is the content of the third part.

We found it reasonable to present only those aspects of the conjecture that do not require a large notational framework. This work therefore contains

- the formulation of the first part of the conjecture in full generality.
- the formulation of the second and third part in the absolute case, i.e., that of elliptic curves over number fields.

1.1. Let K be a number field, and B a smooth quasiprojective connected scheme over K or O_K, and let

$$\pi : \mathcal{E} \longrightarrow B$$

be an elliptic curve over B with zero section i. Set $\widetilde{\mathcal{E}} := \mathcal{E} - i(B)$.
The objects of our interest are the motivic cohomology groups

$$H^2_{\mathcal{M}}(\mathcal{E}, \mathbb{Q}(1))_- \quad \text{for } k = 1\,,$$

$$H^{k-1}_{\mathcal{M}}\left(\mathcal{E}^{(k-2)}, \mathbb{Q}(k-1)\right)_{\mathrm{sgn}} \quad \text{for } k \geq 2\,.$$

The first is the (-1)–eigenspace for the action of $[-1]$ on the group $H^2_{\mathcal{M}}(\mathcal{E}, \mathbb{Q}(1))$, which equals $\mathrm{Pic}(\mathcal{E}) \otimes_{\mathbb{Z}} \mathbb{Q}$, the latter are the sign character eigenspaces for the action on

$$H^{k-1}_{\mathcal{M}}\left(\mathcal{E}^{(k-2)}, \mathbb{Q}(k-1)\right) = \left(K_{k-1}(\mathcal{E}^{(k-2)}) \otimes_{\mathbb{Z}} \mathbb{Q}\right)^{k-1}$$

induced by the natural action of the symmetric group $\mathfrak{S}_{k-1}$ on

$$\mathcal{E}^{(k-2)} := \ker\left(\sum : \mathcal{E}^{k-1} \longrightarrow \mathcal{E}\right)\,.$$

Lemma: The natural isomorphism $s \longmapsto (s) - (i)$ between the Mordell–Weil group $\mathcal{E}(B)$ and the group $Pic^0_{\mathcal{E}/B}(B)$ of global sections of $Pic^0_{\mathcal{E}/B}$ induces an isomorphism between

$$\mathcal{E}(B) \otimes_{\mathbb{Z}} \mathbb{Q} \quad \text{and} \quad H^2_{\mathcal{M}}(\mathcal{E}, \mathbb{Q}(1))_-\,.$$

Proof: Since π admits a section and $\pi_* \mathcal{O}_{\mathcal{E}} = \mathcal{O}_B$, the natural map

$$\mathrm{Pic}(\mathcal{E})/\pi^*(\mathrm{Pic}(B)) \longrightarrow Pic_{\mathcal{E}/B}(B)$$

is an isomorphism. The fibrewise degree is a morphism

$$Pic_{\mathcal{E}/B} \longrightarrow \underline{\mathbb{Z}}\,,$$

whose kernel equals $Pic^0_{\mathcal{E}/B}$.
Thus we have an isomorphism

$$\mathrm{Pic}^0(\mathcal{E})/\pi^*(\mathrm{Pic}(B)) \xrightarrow{\sim} Pic^0_{\mathcal{E}/B}(B)\;.$$

On the other hand, we have a filtration

$$0 \subset \pi^*(\mathrm{Pic}(B)) \otimes_{\mathbb{Z}} \mathbb{Q} \subset \mathrm{Pic}^0(\mathcal{E}) \otimes_{\mathbb{Z}} \mathbb{Q} \subset H^2_{\mathcal{M}}(\mathcal{E}, \mathbb{Q}(1))\,,$$

whose graded pieces are $\pi^*(\mathrm{Pic}(B)) \otimes_{\mathbb{Z}} \mathbb{Q}$, $\mathcal{E}(B) \otimes_{\mathbb{Z}} \mathbb{Q}$, and $\mathbb{Q}$.
On both $\pi^*(\mathrm{Pic}(B)) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q}$, the map $[-1]$ acts trivially. q.e.d.

1.2. If $B = \mathrm{Spec}(K)$, i.e., $\mathcal{E} = E$ is an elliptic curve over K, and $k \geq 2$, we have the following description of the Deligne cohomology group

$$H^{k-1}_{\mathcal{D}}\left(E^{(k-2)} \otimes_{\mathbb{Q}} \mathbb{R}, \mathbb{R}(k-1)\right)_{\mathrm{sgn}} :$$

as $\mathbb{R}$–vector space, it is canonically isomorphic to

$$\left(H^{k-2}_B(E^{(k-2)} \otimes_{\mathbb{Q}} \mathbb{C}, \mathbb{C})/H^{k-2}_B(E^{(k-2)} \otimes_{\mathbb{Q}} \mathbb{C}, (2\pi i)^{k-1}\mathbb{R})\right)^+_{\mathrm{sgn}}\;,$$

the superscript $+$ denoting the $(+1)$–eigenspace of the action of complex conjugation on both $E^{(k-2)} \otimes_{\mathbb{Q}} \mathbb{C}$ and the coefficients ([Sn], long exact cohomology sequence on page 7, Lemma on page 8).
The Künneth isomorphism identifies the above with

$$\left(\mathrm{Sym}^{k-2} H^1_B(E \otimes_{\mathbb{Q}} \mathbb{C}, \mathbb{C})/(2\pi i \cdot \mathrm{Sym}^{k-2} H^1_B(E \otimes_{\mathbb{Q}} \mathbb{C}, 2\pi i\,\mathbb{R}))\right)^+$$

(see Remark c) below).
This isomorphism is canonical up to sign. It is dependent on the choice of a $(k-2)$–tuple $(i_1, \ldots, i_{k-2})$, with pairwise unequal $i_j \in \{1, \ldots, k-1\}$. Then the inverse of the isomorphism is induced by

$$v_1 \otimes \ldots \otimes v_{k-2} \longmapsto p^*_{i_1} v_1 \cup \ldots \cup p^*_{i_{k-2}} v_{k-2}\,,$$

where $p_{i_j} : E^{(k-2)} \subset E^{k-1} \longrightarrow E$ is the projection onto the i_j–th component.
In order to fix choices, we let $i_j := j+1$. This conforms with [De1], 8.7, and differs from the choice in [BL], 6.1.1.i) to the effect that our isomorphism is $(-1)^{k-2}$ times the one in loc.cit.

We end up with an isomorphism identifying $H^{k-1}_{\mathcal{D}}(E^{(k-2)} \otimes_{\mathbb{Q}} \mathbb{R}, \mathbb{R}(k-1))_{\rm sgn}$ and

$$\left(\bigoplus_{\sigma : K \hookrightarrow \mathbb{C}} \mathrm{Sym}^{k-2} H^1_B({}^\sigma E(\mathbb{C}), \mathbb{C}) / \left(2\pi i \cdot \mathrm{Sym}^{k-2} H^1_B({}^\sigma E(\mathbb{C}), 2\pi i\, \mathbb{R}) \right) \right)^+ ,$$

$+$ denoting the $(+1)$–eigenspace of the action of complex conjugation on both $\{\sigma : K \hookrightarrow \mathbb{C}\}$ and the coefficients.

Remarks: a) The same proof as of [J2], Lemma 9.2 shows that the group

$$\left(\bigoplus_{\sigma : K \hookrightarrow \mathbb{C}} \mathrm{Sym}^{k-2} H^1_B({}^\sigma E(\mathbb{C}), \mathbb{C}) / \left(2\pi i \cdot \mathrm{Sym}^{k-2} H^1_B({}^\sigma E(\mathbb{C}), 2\pi i\, \mathbb{R}) \right) \right)^+$$

equals the Yoneda–Ext group

$$\mathrm{Ext}^1 \left(\mathbb{R}(0), \bigoplus_{\sigma : K \hookrightarrow \mathbb{C}} \left(\mathrm{Sym}^{k-2} H^1_B({}^\sigma E(\mathbb{C}), \mathbb{R}(1)) \right) (1) \right)$$

in the category of $\mathbb{R}$–Hodge structures over $\mathbb{R}$, which consists of pairs (H, F_∞), with an $\mathbb{R}$–Hodge structure H, and an isomorphism F_∞ between H and $\overline{H}$, the $\mathbb{R}$–Hodge structure conjugate to H. By definition, $\overline{H}$ has the same underlying weight filtered $\mathbb{R}$–vector space as H, and the conjugate Hodge filtration.

$$H \longmapsto \overline{H}$$

is an involution on the category of $\mathbb{R}$–Hodge structures, and we demand that F_∞ behave involutively as well:

$$\overline{F_\infty} = F_\infty^{-1} .$$

b) In fact, the composite isomorphism between $H^{k-1}_{\mathcal{D}}(E^{(k-2)} \otimes_{\mathbb{Q}} \mathbb{R}, \mathbb{R}(k-1))_{\rm sgn}$ and

$$\mathrm{Ext}^1 \left(\mathbb{R}(0), \bigoplus_{\sigma : K \hookrightarrow \mathbb{C}} \left(\mathrm{Sym}^{k-2} H^1_B({}^\sigma E(\mathbb{C}), \mathbb{R}(1)) \right) (1) \right)$$

should be regarded as natural, while the isomorphism of 1.2 is merely a consequence of the explicit description of Ext^1 in the category of $\mathbb{R}$–Hodge structures over $\mathbb{R}$.

It is certainly a perfect motivation for expecting an interpretation of the motivic cohomology group $H^{k-1}_{\mathcal{M}}(E^{(k-2)}, \mathbb{Q}(k-1))_{\rm sgn}$ as

$$\mathrm{Ext}^1 \left(\mathbb{Q}(0), \left(\mathrm{Sym}^{k-2} h_1(E) \right) (1) \right)$$

in the category of mixed motives.

c) Note that the desire to identify the motive $\mathrm{Sym}^{k-2} h_1(E)$ as a direct summand of $h(E^{(k-2)})$ is the reason for us to consider the action of $\mathfrak{S}_{k-1}$ rather than just its subgroup $\mathfrak{S}_{k-2} = \{\tau \in \mathfrak{S}_{k-1} \mid \tau(1) = 1\}$. The sgn–component of $h(E^{k-2})$ for the natural action of $\mathfrak{S}_{k-2}$ contains a nontrivial summand of the shape

$$\mathrm{im}\left(h_0(E) \otimes h_1(E)^{\otimes(k-4)} \otimes h_2(E) \longrightarrow h(E^{k-2})\right)_{\mathrm{sgn}} .$$

However, any such summand lies in the fixed space of a suitable transposition in $\mathfrak{S}_{k-1} - \mathfrak{S}_{k-2}$, and we end up with an isomorphism

$$\mathrm{Sym}^{k-2} h_1(E) \stackrel{\sim}{\longrightarrow} h(E^{(k-2)})_{\mathrm{sgn}} .$$

For the details, see [De1], 8.7.

d) Projection onto the real (for k even) or imaginary part (for k odd) of the coefficients identifies $H^{k-1}_{\mathcal{D}}(E^{(k-2)} \otimes_{\mathbb{Q}} \mathbb{R}, \mathbb{R}(k-1))_{\mathrm{sgn}}$ with

$$\left(\bigoplus_{\sigma : K \hookrightarrow \mathbb{C}} \mathrm{Sym}^{k-2} H^1_B({}^{\sigma}E(\mathbb{C}), 2\pi i\, \mathbb{R})\right)^+ ,$$

and we shall describe elements of Deligne cohomology by their image under this isomorphism.

e) In the relative situation $\pi : \mathcal{E} \longrightarrow B$, we see as in c) that the Leray spectral sequence for $\pi^{(k-2)}$ computing

$$H^{k-1}_B\left(\mathcal{E}^{(k-2)} \otimes_{\mathbb{Z}} \mathbb{C}, \mathbb{R}(k-1)\right)_{\mathrm{sgn}}$$

degenerates, and that we get an identification of this $\mathbb{R}$–vector space with

$$\bigoplus_{\sigma : K \hookrightarrow \mathbb{C}} \left(\mathrm{Ext}^1\left(\mathbb{R}(0), \mathrm{Sym}^{k-2} R^1({}^{\sigma}\pi)_* \mathbb{R}(1)\right)(1)\right)$$

the Ext groups being the ones in the respective categories of local systems with $\mathbb{R}$–coefficients on ${}^{\sigma}B(\mathbb{C})$.

1.3. Let E be an elliptic curve over $\mathbb{C}$, and write

$$\eta : E(\mathbb{C}) \stackrel{\sim}{\longrightarrow} \mathbb{C}/L$$

for some lattice $L \subset \mathbb{C}$.

Definition:

a) $A(L) := 1/\pi \cdot Vol(L) = -1/2\pi i \cdot \int_{\mathbb{C}/L} dz \wedge d\overline{z}$.
So if L is generated by ω_1 and ω_2, with $\mathrm{Im}(\omega_2/\omega_1) > 0$, we have

$$A(L) = 1/\pi \cdot \mathrm{Im}(\omega_2/\omega_1) \cdot |\omega_1|^2 .$$

b) The *Pontryagin pairing*

$$(\, , \,)_L : (\mathbb{C}/L) \times L \longrightarrow \mathbb{C}$$

is defined by

$$(z,\gamma)_L := \exp(A(L)^{-1} \cdot (z\overline{\gamma} - \overline{z}\gamma)) \, .$$

Explicitly, if L is generated by ω_1 and ω_2, with $\mathrm{Im}(\omega_2/\omega_1) > 0$, and if $z = r_1\omega_1 - r_2\omega_2$ and $\gamma = n\omega_1 + m\omega_2$, then

$$(z,\gamma)_L = \exp(2\pi i(-nr_2 - mr_1)) \, .$$

c) The *Kronecker double series*

$$K_{a,b,L} : \mathbb{C} - L \longrightarrow \mathbb{C} \, ,$$

for $a, b \geq 1$, is defined by

$$K_{a,b,L}(z) := \sum_{\gamma \in L - \{0\}} \frac{(z,\gamma)_L}{\gamma^a \overline{\gamma}^b} \, .$$

For $a = b = 1$, where the sum does not converge absolutely, we define the right hand side to be

$$\lim_{s \to 1} \sum_{\gamma \in L - \{0\}} \frac{(z,\gamma)_L}{|\gamma|^{2s}} \, .$$

Note ([Si], VI, Theorem 3.4, [Za], Theorem 1) that the function

$$z \longmapsto \frac{A(L)}{2} K_{1,1,L}(z)$$

equals the local height function on $E(\mathbb{C}) - \{0\}$.

d) $\omega(L)$ denotes the standard generator dz of

$$F^1 H^1_{DR}(\mathbb{C}/L) \overset{\eta^*}{\hookrightarrow} H^1_B(E(\mathbb{C}), \mathbb{C}) \, .$$

Proposition: Let $k \geq 2$. For a point $s \in \widetilde{E}(\mathbb{C})$, the element

$$\begin{aligned} G_{E,k}(s) &:= \frac{A(L)}{2} \sum_{\alpha+\beta=k-2} \omega(L)^\alpha \overline{\omega(L)}^\beta K_{\alpha+1,\beta+1,L}(\eta(s)) \\ &= \frac{A(L)}{2} \sum_{\alpha+\beta=k-2} \omega(L)^\alpha \overline{\omega(L)}^\beta \sum_{\gamma \in L-\{0\}} \frac{(\eta(s),\gamma)_L}{\gamma^{\alpha+1}\overline{\gamma}^{\beta+1}} \in \mathrm{Sym}^{k-2} H^1_B(E(\mathbb{C}), \mathbb{C}) \end{aligned}$$

lies in $\mathrm{Sym}^{k-2} H^1_B(E(\mathbb{C}), 2\pi i \, \mathbb{R})$ and is independent of the choice of η.
Also, the conjugate of this element coincides with the value of $G_{\overline{E},k}$ at the point $\overline{s}$ of $\overline{E}(\mathbb{C})$.

Proof: Independence of the choice of η is immediate: for $\lambda \in \mathbb{C}^*$, we have

$$\begin{aligned} K_{\alpha+1,\beta+1,\lambda L}(\lambda z) &= (\lambda^{\alpha+1}\overline{\lambda}^{\beta+1})^{-1} K_{\alpha+1,\beta+1,L}(z)\,, \\ A(\lambda L) &= \lambda\overline{\lambda} A(L)\,, \\ \omega(\lambda L) &= \lambda\omega(L)\,. \end{aligned}$$

An elementary calculation shows that

$$\overline{K_{a,b,L}(z)} = (-1)^{a+b} K_{b,a,L}(z)\,,$$

hence $\overline{G_{E,k}(s)} = (-1)^{k-2} G_{E,k}(s)$.
This shows that $G_{E,k}(s)$ lies in

$$\mathrm{Sym}^{k-2} H^1_B(E(\mathbb{C}), 2\pi i\,\mathbb{R}) \subset \mathrm{Sym}^{k-2} H^1_B(E(\mathbb{C}), \mathbb{C})\,.$$

Similarly,

$$\overline{K_{a,b,L}(z)} = K_{a,b,\overline{L}}(\overline{z})\,.$$

Our claim follows from the fact that complex conjugation

$$\mathbb{C}/L \longrightarrow \mathbb{C}/\overline{L}$$

induces complex conjugation

$$E(\mathbb{C}) \longrightarrow \overline{E}(\mathbb{C})\,.$$

q.e.d.

1.4. Before stating the conjecture, we need to discuss integrality.
Let $B = \mathrm{Spec}(K)$, i.e., $\mathcal{E} = E$ an elliptic curve over K, and $k \geq 2$. Recall that the "integral part"

$$H^{k-1}_{\mathcal{M}}\left(E^{(k-2)}, \mathbb{Q}(k-1)\right)_{\mathbb{Z}}$$

of $H^{k-1}_{\mathcal{M}}(E^{(k-2)}, \mathbb{Q}(k-1))$ is defined to be the image of the motivic cohomology of a regular proper model of $E^{(k-2)}$ over O_K. (Such models exist: [D1], Lemme 5.4, or [So], section 2.)

We recall the description of $H^{k-1}_{\mathcal{M}}(E^{(k-2)}, \mathbb{Q}(k-1))_{\mathbb{Z}}$ given in section 6 of [SaSo]: Let v be a finite place of K. From the localization sequence for K–theory, we get a boundary map

$$\delta_v : H^{k-1}_{\mathcal{M}}\left(E^{(k-2)}, \mathbb{Q}(k-1)\right) \longrightarrow H^{k-2}_{\mathcal{M}}\left(E^{(k-2)}_v, \mathbb{Q}(k-2)\right)\,,$$

where E_v denotes the fibre at v of the connected component of the Néron model of E.

For $k = 2$, we have

$$\begin{aligned} H^{k-1}_{\mathcal{M}}\left(E^{(k-2)}, \mathbb{Q}(k-1)\right) &= K^* \otimes_{\mathbb{Z}} \mathbb{Q}\,, \\ H^{k-2}_{\mathcal{M}}\left(E^{(k-2)}_v, \mathbb{Q}(k-2)\right) &= \mathbb{Q}, \end{aligned}$$

and $\delta_v : K^* \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q}$ is simply the valuation associated to v. So

$$\delta_v(x) = -\frac{\log |x|_v}{\log N(v)} .$$

Furthermore,

$$H^{k-1}_{\mathcal{M}}\left(E^{(k-2)}, \mathbb{Q}(k-1)\right)_{\mathbb{Z}} = O_K^* \otimes_{\mathbb{Z}} \mathbb{Q}$$

is the kernel of the map

$$(\delta_v)_v : K^* \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \bigoplus_v \mathbb{Q} .$$

Now let $k \geq 3$. It is known that if E has good reduction at v, then the target space of δ_v is trivial. More precisely, we have the following

Theorem: ([SaSo], 6.3, 6.5.) Let $k \geq 3$.
a) $H^{k-1}_{\mathcal{M}}(E^{(k-2)}, \mathbb{Q}(k-1))_{\mathbb{Z}}$ is the kernel of the map

$$(\delta_v)_v : H^{k-1}_{\mathcal{M}}\left(E^{(k-2)}, \mathbb{Q}(k-1)\right) \longrightarrow \bigoplus_v H^{k-2}_{\mathcal{M}}\left(E_v^{(k-2)}, \mathbb{Q}(k-2)\right) ,$$

the sum being over all finite places of K where E has split multiplicative reduction if k is odd / potentially multiplicative reduction if k is even.
b) Let K'/K be a finite Galois extension with group G, and $E' := E \otimes_K K'$. Then

$$H^{k-1}_{\mathcal{M}}\left(E^{(k-2)}, \mathbb{Q}(k-1)\right)_{\mathbb{Z}} = H^{k-1}_{\mathcal{M}}\left(E'^{(k-2)}, \mathbb{Q}(k-1)\right)^G_{\mathbb{Z}} .$$

Hence for k even, one takes K' to be such that E' has good or split multiplicative reduction everywhere. An element

$$h \in H^{k-1}_{\mathcal{M}}\left(E^{(k-2)}, \mathbb{Q}(k-1)\right)$$

is integral if and only if it is integral in $H^{k-1}_{\mathcal{M}}(E'^{(k-2)}, \mathbb{Q}(k-1))$, i.e., if and only if it lies in the kernel of all $\delta_{v'}$, where the v' run through the set of finite places of K' where E' has split multiplicative reduction. In short:

$$\begin{aligned} & h \in H^{k-1}_{\mathcal{M}}\left(E^{(k-2)}, \mathbb{Q}(k-1)\right)_{\mathbb{Z}} \\ \Longleftrightarrow \; & h \in \bigcap_{v' \text{ bad}} \ker\left(\delta_{v'}\right) . \end{aligned}$$

1.5. Let v be a finite place of K where E has split multiplicative reduction. Fix an isomorphism between E_v and the multiplicative group over $\kappa(v)$. It allows to identify a generator $\Phi_v^{(k-2)}$ of the one dimensional space

$$H^{k-2}_{\mathcal{M}}\left(E_v^{(k-2)}, \mathbb{Q}(k-2)\right) ,$$

and also induces a bijection between some $\mathbb{Z}/N_v\mathbb{Z}$ and the group of connected components of the fibre at v of the Néron model.

For a section $s \in E(K)$ and $n \in \mathbb{Z}/N_v\mathbb{Z}$, define $d_s(v,n)$ to be the degree of the restriction of the flat extension of s to the component n.

1.6. We can now state the conjecture. As in the classical case (compare [BD], 1.7), there is a conjecture for each $k \geq 2$, and the k–th can only be formulated if the second to $(k-1)$–th are true.

Our aim is to construct, for each $k \geq 1$, a $\mathbb{Q}$–vector space $\mathcal{L}_k = \mathcal{L}_k(\mathcal{E})$, a map

$$\{\ \}_k = \{\ \}_k(\mathcal{E}) : \widetilde{\mathcal{E}}(B) \longrightarrow \mathcal{L}_k\,,$$

a homomorphism

$$d_k = d_k(\mathcal{E}) : \mathcal{L}_k \longrightarrow \left(\bigoplus_{l=1}^{k-1} \mathcal{L}_l\right)^{\otimes 2},$$

and a monomorphism

$$\varphi_k = \varphi_k(\mathcal{E}) : \ker(d_k) \hookrightarrow \begin{cases} H^2_{\mathcal{M}}(\mathcal{E}, \mathbb{Q}(1))_- & \text{if } k = 1 \\ H^{k-1}_{\mathcal{M}}\left(\mathcal{E}^{(k-2)}, \mathbb{Q}(k-1)\right)_{\mathrm{sgn}} & \text{if } k \geq 2 \end{cases}.$$

Furthermore, we require that these objects behave contravariantly with respect to base change: for any cartesian diagram

$$\begin{array}{ccc} \mathcal{E}' & \stackrel{f}{\longrightarrow} & \mathcal{E} \\ {\scriptstyle \pi'}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\ B' & \stackrel{f}{\longrightarrow} & B \end{array}$$

there is a *pullback map* $f^* : \mathcal{L}_k(\mathcal{E}) \to \mathcal{L}_k(\mathcal{E}')$ such that the obvious diagrams involving $\{\ \}_k, d_k$ and φ_k commute.

Finally, if $k \geq 2$, then for any étale isogeny

$$\begin{array}{ccc} \mathcal{E}_1 & \stackrel{\psi}{\longrightarrow} & \mathcal{E}_2 \\ & {\scriptstyle \pi_1}\searrow & \big\downarrow {\scriptstyle \pi_2} \\ & & B \end{array}$$

whose kernel $\ker(\psi)$ consists of sections of π_1, we require the following: for all $s_{1,\alpha} \in (\mathcal{E}_1 - \ker(\psi))(B)$ and $\lambda_\alpha \in \mathbb{Q}$, we have

$$d_k\left(\sum_\alpha \lambda_\alpha \{\psi(s_{1,\alpha})\}_k\right) = 0$$

if and only if

$$d_k\left(\sum_\alpha \lambda_\alpha \sum_{t \in \ker(\psi)(B)} \{s_{1,\alpha} + t\}_k\right) = 0\,,$$

and if this is the case, then the equality

$$\psi_* \circ \varphi_k \left(\sum_\alpha \lambda_\alpha \sum_{t \in \ker(\psi)(B)} \{s_{1,\alpha} + t\}_k \right) = \varphi_k \left(\sum_\alpha \lambda_\alpha \{\psi(s_{1,\alpha})\}_k \right)$$

holds. Here, ψ_* denotes the Gysin map

$$\psi_* : H^{k-1}_{\mathcal{M}} \left(\mathcal{E}_1^{(k-2)}, \mathbb{Q}(k-1) \right)_{\mathrm{sgn}} \longrightarrow H^{k-1}_{\mathcal{M}} \left(\mathcal{E}_2^{(k-2)}, \mathbb{Q}(k-1) \right)_{\mathrm{sgn}} .$$

We refer to this formula as *norm compatibility for* $\{\ \}_k$ *with respect to* d_k.

(A) For $k = 1$, let $\mathcal{L}_1 := \mathcal{E}(B) \otimes_{\mathbb{Z}} \mathbb{Q}$,

$$\{\ \}_1 : \widetilde{\mathcal{E}}(B) \longrightarrow \mathcal{L}_1 , \ s \longmapsto s \otimes 1 ,$$

$d_1 := 0$, and $\varphi_1 := \mathrm{id}$ under the identification of Lemma 1.1.

(B) For $k \geq 2$, let $\mathcal{L}_k^\sharp$ be the $\mathbb{Q}$–vector space with basis $(\{s\}_k^\sharp \mid s \in \widetilde{\mathcal{E}}(B))$. Define

$$d_k^\sharp : \mathcal{L}_k^\sharp \longrightarrow \mathcal{L}_{k-1} \otimes_{\mathbb{Q}} \mathcal{L}_1 \subset \left(\bigoplus_{l=1}^{k-1} \mathcal{L}_l \right)^{\otimes 2} ,$$
$$\{s\}_k^\sharp \longmapsto \{s\}_{k-1} \otimes \{s\}_1 ,$$

and define pullback maps in the obvious manner.

Conjecture:

Part 1: There exists a homomorphism

$$\varphi_k^\sharp : \ker(d_k^\sharp) \longrightarrow H^{k-1}_{\mathcal{M}} \left(\mathcal{E}^{(k-2)}, \mathbb{Q}(k-1) \right)_{\mathrm{sgn}} ,$$

which is functorial with respect to pullback maps, such that norm compatibility for $\{\ \}_k^\sharp$ with respect to $d_k^\sharp$ is satisfied.

Part 2 (absolute case): If $B = \mathrm{Spec}(K)$, i.e., $\mathcal{E} = E$ is an elliptic curve over K, then for any $S = \sum_\alpha \lambda_\alpha \{s_\alpha\}_k^\sharp$ in the kernel of $d_k^\sharp$, the regulator

$$r_{\mathcal{D}} : H^{k-1}_{\mathcal{M}} \left(E^{(k-2)}, \mathbb{Q}(k-1) \right)_{\mathrm{sgn}} \longrightarrow H^{k-1}_{\mathcal{D}} \left(E^{(k-2)} \otimes_{\mathbb{Q}} \mathbb{R}, \mathbb{R}(k-1) \right)_{\mathrm{sgn}}$$

maps $\varphi_k^\sharp(S)$ to the element

$$\left(G_{^\sigma E,k}(\sigma S) \right)_\sigma := \left(\sum_\alpha \lambda_\alpha G_{^\sigma E,k}(\sigma s_\alpha) \right)_\sigma ,$$

where we have used the identification

$$H^{k-1}_{\mathcal{D}} \left(E^{(k-2)} \otimes_{\mathbb{Q}} \mathbb{R}, \mathbb{R}(k-1) \right)_{\mathrm{sgn}} \stackrel{\sim}{\longrightarrow} \left(\bigoplus_{\sigma : K \hookrightarrow \mathbb{C}} \mathrm{Sym}^{k-2} H^1_B({}^\sigma E(\mathbb{C}), 2\pi i\, \mathbb{R}) \right)^+$$

of Remark 1.2.d). Proposition 1.3 ensures that the function

$$(G_{{}^\sigma E,k} \circ \sigma)_\sigma : \widetilde{E}(K) \longrightarrow \bigoplus_{\sigma: K \hookrightarrow \mathbb{C}} \mathrm{Sym}^{k-2} H^1_B({}^\sigma E(\mathbb{C}), 2\pi i\, \mathbb{R})$$

lands in the subspace fixed by complex conjugation.

Part 3 (absolute case): If $B = \mathrm{Spec}(K)$, i.e., $\mathcal{E} = E$ is an elliptic curve over K, then for any $S = \sum_\alpha \lambda_\alpha \{s_\alpha\}_2^\sharp$ in the kernel of $d_2^\sharp$, and for any finite place v of K, the boundary

$$\delta_v : K^* \otimes_{\mathbb{Z}} \mathbb{Q} = H^1_{\mathcal{M}}(\mathrm{Spec}(K), \mathbb{Q}(1)) \longrightarrow H^0_{\mathcal{M}}(\mathrm{Spec}(\kappa(v)), \mathbb{Q}(0)) = \mathbb{Q}$$

maps $\varphi_2^\sharp(S)$ to the element

$$-(\log N(v))^{-1} \cdot \sum_\alpha \lambda_\alpha h_v(s_\alpha)\,.$$

Here, $h_v : \widetilde{E}(K) \longrightarrow \log N(v) \cdot \mathbb{Q}$ is the local height function. (For a comprehensive discussion and explicit formulae, we refer to [Si], VI.)

For $k \geq 3$, any finite place v of K where E has split multiplicative reduction, and any $S = \sum_\alpha \lambda_\alpha \{s_\alpha\}_k^\sharp$ in the kernel of $d_k^\sharp$, the boundary

$$\delta_v : H^{k-1}_{\mathcal{M}}\left(E^{(k-2)}, \mathbb{Q}(k-1)\right)_{\mathrm{sgn}} \longrightarrow H^{k-2}_{\mathcal{M}}\left(E_v^{(k-2)}, \mathbb{Q}(k-2)\right)$$

maps $\varphi_k^\sharp(S)$ to the element

$$\pm \frac{k-1}{N_v^{k-2} \cdot k!} \cdot \left(\sum_\alpha \lambda_\alpha \sum_{n \in \mathbb{Z}/N_v\mathbb{Z}} d_{s_\alpha}(v,n) \cdot B_k(<\frac{n}{N_v}>) \right) \cdot \Phi_v^{(k-2)}\,.$$

Here, B_k denotes the k-th Bernoulli polynomial, and $0 \leq <x> < 1$ is the representative of $x \in \mathbb{R}/\mathbb{Z}$. Furthermore, the symbols N_v, $\Phi_v^{(k-2)}$ and $d_{s_\alpha}(v,n)$ are as in 1.5.

Let us discuss the case where $B = \mathrm{Spec}(K)$:
Part 3, together with Theorem 1.4, predicts in particular when $\varphi_k^\sharp$ will map an element $S = \sum_\alpha \lambda_\alpha \{s_\alpha\}_k^\sharp$ belonging to the kernel of $d_k^\sharp$ to

$$H^{k-1}_{\mathcal{M}}\left(E^{(k-2)}, \mathbb{Q}(k-1)\right)_{\mathbb{Z},\mathrm{sgn}} \subset H^{k-1}_{\mathcal{M}}\left(E^{(k-2)}, \mathbb{Q}(k-1)\right)_{\mathrm{sgn}} \quad :$$

For $k = 2$, this is the case if and only if

$$\sum_\alpha \lambda_\alpha h_v(s_\alpha) = 0$$

for all finite places v of K.
For odd $k \geq 3$, this is the case if and only if

$$\sum_\alpha \lambda_\alpha \sum_{n \in \mathbb{Z}/N_v\mathbb{Z}} d_{s_\alpha}(v,n) \cdot B_k(<\frac{n}{N_v}>) = 0$$

for all finite places v of K where E has split multiplicative reduction.
For even $k \geq 4$, one has to perform base change to a field K' where all reduction is either good or split multiplicative, and check the above criterion there.

We remark that the part of Beilinson's conjecture predicting injectivity of the regulator $r_{\mathcal{D}}$ on the integral part of motivic cohomology would imply that the homomorphism $\varphi_k^\sharp$ is unique if it exists.

1.7. Assume that there exists a natural transformation

$$\varphi_k^\sharp : \ker(d_k^\sharp) \longrightarrow \left(\mathcal{E} \longmapsto H_{\mathcal{M}}^{k-1}\left(\mathcal{E}^{(k-2)}, \mathbb{Q}(k-1)\right)_{\mathrm{sgn}}\right)$$

with the properties of 1.6.(B).
The $\mathbb{Q}$–vector space $\mathcal{L}_k := \mathcal{L}_k^\sharp / \ker(\varphi_k^\sharp)$, together with the induced maps $\{\ \}_k, d_k$, and φ_k, will then solve the problem posed in 1.6.

1.8. Note that although we did not formulate part 2 of the conjecture in full generality, the functoriality requirement of part 1 gives non–trivial constraints on the regulator also for families of elliptic curves.
Functoriality also implies that in order to construct elements in the group $H_{\mathcal{M}}^{k-1}(\mathcal{E}^{(k-2)}, \mathbb{Q}(k-1))_{\mathrm{sgn}}$, we don't have to restrict ourselves to elements of $\ker(d_k(\mathcal{E}))$: since we have Galois descent for motivic cohomology, we get a map

$$\varphi_k^G(\mathcal{E}') : \ker(d_k(\mathcal{E}'))^G \hookrightarrow H_{\mathcal{M}}^{k-1}\left(\mathcal{E}^{(k-2)}, \mathbb{Q}(k-1)\right)_{\mathrm{sgn}}$$

for any finite Galois covering

$$\begin{array}{ccc} \mathcal{E}' & \xrightarrow{f} & \mathcal{E} \\ {\scriptstyle \pi'}\downarrow & & {\scriptstyle \pi}\downarrow \\ B' & \xrightarrow{f} & B \end{array}$$

with Galois group G.
It would certainly be unreasonable to expect $\varphi_k(\mathcal{E})$ itself to be bijective – there are elliptic curves with trivial Mordell–Weil group. It is therefore sensible only to hope to be able to describe $\varinjlim_f \operatorname{im} \varphi_k^G(\mathcal{E}')$. However, in the general case, we have no good guess to offer.

We remark that we have the natural action of the Galois group on

$$\varinjlim_f \ker(d_k^\sharp(\mathcal{E}')) \subset \varinjlim_f \mathcal{L}_k^\sharp(\mathcal{E}'),$$

the latter being the vector space with basis the set of sections not meeting the zero section of the base change of $\mathcal{E}$ to the universal cover of B.
By construction, the projection

$$\varinjlim_f \ker(d_k^\sharp(\mathcal{E}')) \longrightarrow\!\!\!\!\rightarrow \varinjlim_f \ker(d_k(\mathcal{E}'))$$

is Galois–equivariant. A priori, there is no reason to expect the map

$$\varinjlim_f \ker\left(d_k^\sharp(\mathcal{E}')\right)^G \longrightarrow \varinjlim_f \ker\left(d_k(\mathcal{E}')\right)^G$$

to be surjective.

1.9. Let us summarize the main results of this work: as in [BD], it is shown (see § 4) that a certain motivic formalism, involving the elliptic polylogarithmic extension, implies part 1 and part 2 of Conjecture 1.6.(B).

We may consider the subspaces $\mathcal{L}^\sharp_{k,\mathrm{tors}}(\mathcal{E})$ of the $\mathcal{L}^\sharp_k(\mathcal{E})$ generated by the image of $\widetilde{\mathcal{E}}(B) \cap \mathcal{E}_{\mathrm{tors}}(B)$ under $\{\ \}^\sharp$. Clearly the differential $d_k^\sharp$ is trivial on $\mathcal{L}^\sharp_{k,\mathrm{tors}}(\mathcal{E})$, and it makes sense to talk about the

Torsion part of Conjecture 1.6.(B):

Part 1: There exists a homomorphism

$$\varphi^\sharp_{k,\mathrm{tors}} : \mathcal{L}^\sharp_{k,\mathrm{tors}}(\mathcal{E}) \longrightarrow H^{k-1}_{\mathcal{M}}\left(\mathcal{E}^{(k-2)}, \mathbb{Q}(k-1)\right)_{\mathrm{sgn}} ,$$

which is functorial with respect to pullback maps, such that norm compatibility for $\{\ \}^\sharp_k$ is satisfied.

Part 2 (absolute case): If $B = \mathrm{Spec}(K)$, i.e., $\mathcal{E} = E$ is an elliptic curve over K, then for any $S = \sum_\alpha \lambda_\alpha \{s_\alpha\}^\sharp_k$, with $s_\alpha \in \widetilde{E}(K) \cap E_{\mathrm{tors}}(K)$, the regulator $r_{\mathcal{D}}$ maps $\varphi^\sharp_{k,\mathrm{tors}}(S)$ to the element

$$\left(G_{{}^\sigma E,k}(\sigma S)\right)_\sigma = \left(\sum_\alpha \lambda_\alpha G_{{}^\sigma E,k}(\sigma s_\alpha)\right)_\sigma .$$

Part 3 (absolute case): If $B = \mathrm{Spec}(K)$, i.e., $\mathcal{E} = E$ is an elliptic curve over K, then for any $S = \sum_\alpha \lambda_\alpha \{s_\alpha\}^\sharp_2$, with $s_\alpha \in \widetilde{E}(K) \cap E_{\mathrm{tors}}(K)$, and any finite place v of K, the boundary

$$\delta_v : K^* \otimes_{\mathbb{Z}} \mathbb{Q} = H^1_{\mathcal{M}}(\mathrm{Spec}(K), \mathbb{Q}(1)) \longrightarrow H^0_{\mathcal{M}}(\mathrm{Spec}(\kappa(v)), \mathbb{Q}(0)) = \mathbb{Q}$$

maps $\varphi^\sharp_{2,\mathrm{tors}}(S)$ to the element

$$-\left(\log N(v)\right)^{-1} \cdot \sum_\alpha \lambda_\alpha h_v(s_\alpha) .$$

For $k \geq 3$, any finite place v of K where E has split multiplicative reduction, and any $S = \sum_\alpha \lambda_\alpha \{s_\alpha\}^\sharp_k$, with $s_\alpha \in \widetilde{E}(K) \cap E_{\mathrm{tors}}(K)$, the boundary

$$\delta_v : H^{k-1}_{\mathcal{M}}\left(E^{(k-2)}, \mathbb{Q}(k-1)\right)_{\mathrm{sgn}} \longrightarrow H^{k-2}_{\mathcal{M}}\left(E_v^{(k-2)}, \mathbb{Q}(k-2)\right)$$

maps $\varphi^\sharp_{k,\mathrm{tors}}(S)$ to the element

$$\pm \frac{k-1}{N_v^{k-2} \cdot k!} \cdot \left(\sum_\alpha \lambda_\alpha \sum_{n \in \mathbb{Z}/N_v\mathbb{Z}} d_{s_\alpha}(v,n) \cdot B_k(<\frac{n}{N_v}>)\right) \cdot \Phi_v^{(k-2)} .$$

By restricting one's attention to bases B which are smooth and quasiprojective over K, one can speak of the torsion part of 1.6.(B) for K–schemes.

Theorem 1: The torsion part of Conjecture 1.6.(B) holds for K–schemes.

Proof: For $k = 2$, we have the more general statement of Theorem 2.
So let $k \geq 3$. The proof essentially reduces to the observation that up to a factor, the Eisenstein symbol has the desired properties:
First, let $N \geq 3$, and assume that $\pi : \mathcal{E} \longrightarrow B$ is the universal family of elliptic curves $\mathcal{E}_N$ over the modular curve M_N over $\mathbb{Q}$ of level N. We have the Eisenstein symbol

$$\mathcal{E}is_N^{k-2} : \mathbb{Q}\left[\mathcal{E}_N[N]\right]^0 \longrightarrow H^{k-1}_{\mathcal{M}}\left(\mathcal{E}_N^{(k-2)}, \mathbb{Q}(k-1)\right)_{\mathrm{sgn}}$$

([B], § 3; see also [SaSo], section 7), and we define

$$\Phi^{\sharp}_{k,\mathrm{tors},N} : \mathbb{Q}\left[\mathcal{E}_N[N]\right]^0 \longrightarrow H^{k-1}_{\mathcal{M}}\left(\mathcal{E}_N^{(k-2)}, \mathbb{Q}(k-1)\right)_{\mathrm{sgn}}$$

by $\Phi^{\sharp}_{k,\mathrm{tors},N} := -N^{-2(k-2)} \cdot \mathcal{E}is_N^{k-2}$. Because of [SaSo], (1.2.2), the maps $\Phi^{\sharp}_{k,\mathrm{tors},N}$ glue together to give a map

$$\Phi^{\sharp}_{k,\mathrm{tors}} : \varinjlim_N \mathbb{Q}\left[\mathcal{E}_N[N]\right]^0 \longrightarrow \varinjlim_N H^{k-1}_{\mathcal{M}}\left(\mathcal{E}_N^{(k-2)}, \mathbb{Q}(k-1)\right)_{\mathrm{sgn}} .$$

Because of its construction via cup products of functions with prescribed divisors, and because of the formula

$$[d]_*(\mathrm{div}\, f) = \mathrm{div}\, ([d]_* f)$$

for any rational function f on $\mathcal{E}_N$, the map $\Phi^{\sharp}_{k,\mathrm{tors}}$ satisfies the distribution relation: for any $\sum_\alpha \lambda_\alpha \{s_\alpha\} \in \lim \mathbb{Q}\left[\mathcal{E}_N[N]\right]^0$, one has

$$\Phi^{\sharp}_{k,\mathrm{tors}}\left(\sum_\alpha \lambda_\alpha \{[d]s_\alpha\}\right) = d^{k-2} \cdot \Phi^{\sharp}_{k,\mathrm{tors}}\left(\sum_\alpha \lambda_\alpha \sum_{t \in \ker\, [d](M_d)} \{s_\alpha + t\}\right) .$$

We need to extend $\Phi^{\sharp}_{k,\mathrm{tors}}$ to the whole of $\lim \mathbb{Q}\left[\mathcal{E}_N[N]\right]$ in such a way that the distribution relation holds. Of course, there is at most one way of doing this: for any d, we must necessarily have

$$\Phi^{\sharp}_{k,\mathrm{tors}}\left(\{0\}\right) = d^{k-2} \cdot \Phi^{\sharp}_{k,\mathrm{tors}}\left(\sum_{t \in \ker\, [d](M_d)} \{t\}\right) .$$

The problem is well–definedness. For this, use the fact ([B], Theorem 3.17, or [SaSo], Theorem 7.4) that the boundary map

$$\delta : H^{k-1}_{\mathcal{M}}\left(\mathcal{E}_N^{(k-2)}, \mathbb{Q}(k-1)\right)_{\mathrm{sgn}} \longrightarrow \mathbb{Q}[\mathrm{cusps}]$$

is an isomorphism on the image of $\mathcal{E}is_N^{k-2}$, and that $\delta \circ \mathcal{E}is_N^{k-2}$ can be uniquely extended to a distribution, as can be read off formula (7.3.1) of [SaSo].
We end up with a homomorphism

$$\Phi_{k,\mathrm{tors}}^{\sharp} : \varinjlim_N \mathbb{Q}\left[\mathcal{E}_N[N]\right] \longrightarrow \varinjlim_N H_{\mathcal{M}}^{k-1}\left(\mathcal{E}_N^{(k-2)}, \mathbb{Q}(k-1)\right)_{\mathrm{sgn}} .$$

On any finite level $\pi_N : \mathcal{E}_N \longrightarrow M_N$, restriction to $\mathcal{L}_{k,\mathrm{tors}}^{\sharp}(\mathcal{E}_N) \subset \lim \mathbb{Q}[\mathcal{E}_N[N]]$ gives the desired map $\varphi_{k,\mathrm{tors}}^{\sharp}$. Note that since the system $(M_N)_N$ is cofinal in the set of all modular curves associated to arithmetic subgroups of $SL_2(\mathbb{Z})$, this rule allows to define $\varphi_{k,\mathrm{tors}}^{\sharp}$ for any universal family of elliptic curves over a modular curve. Because of the same reason as above, $\varphi_{k,\mathrm{tors}}^{\sharp}$ satisfies norm compatibility under isogenies.

In order to define $\varphi_{k,\mathrm{tors}}^{\sharp}$ for an arbitrary family $\pi : \mathcal{E} \longrightarrow B$, one *has to* proceed as follows: let $s \in \widetilde{\mathcal{E}}(B) \cap \mathcal{E}_{\mathrm{tors}}(B)$, and choose some $N \geq 3$ annihilating s. By Galois descent, we may assume that $\ker [N]$ consists of sections of π. Then there is a cartesian diagram

$$\begin{array}{ccc} \mathcal{E} & \stackrel{f}{\longrightarrow} & \mathcal{E}_N \\ \pi\big\downarrow & & \pi_N\big\downarrow \\ B & \stackrel{f}{\longrightarrow} & M_N \end{array}$$

and $s_N \in \mathcal{E}_N(M_N)$ such that $f^*(s_N) = s$. Define

$$\varphi_{k,\mathrm{tors}}^{\sharp}(s) := f^* \varphi_{k,\mathrm{tors}}^{\sharp}(s_N) .$$

With these definitions, part 1 of the conjecture is satisfied. Part 2 is [De1], Theorem 10.9, and part 3 is the main result of [SaSo] (see the remark at the end of section 4 of loc.cit.). q.e.d.

Given the theorem, it seems justified to think of 1.6.(B) as the natural extension of the Eisenstein symbol to non–torsion sections.

For $k = 2$, we have the equality

$$H_{\mathcal{M}}^{k-1}\left(\mathcal{E}^{(k-2)}, \mathbb{Q}(k-1)\right)_{\mathrm{sgn}} = \mathcal{O}^*(B) \otimes_{\mathbb{Z}} \mathbb{Q} ,$$

and hence 1.6.(B) amounts to the construction of units over the base from formal linear combinations $\sum_\alpha \lambda_\alpha \{s_\alpha\}_2^{\sharp}, \lambda_\alpha \in \mathbb{Q}, s_\alpha \in \widetilde{\mathcal{E}}(B)$ satisfying the relation

$$\sum_\alpha \lambda_\alpha \cdot (s_\alpha \otimes s_\alpha) = 0 \quad \text{in } \mathcal{E}(B) \otimes_{\mathbb{Z}} \mathcal{E}(B) \otimes_{\mathbb{Z}} \mathbb{Q} .$$

Theorem 2: Conjecture 1.6.(B) holds for $k = 2$.

We describe the construction of $\varphi_2^{\sharp}$:
Let $\mathcal{P}$ denote the Poincaré bundle on $\mathcal{E} \times_B \mathcal{E}$. It comes equipped with rigidifications along $i(B) \times_B \mathcal{E}$ and $\mathcal{E} \times_B i(B)$, and as such object does not admit any

nontrivial automorphisms.
Let $S = \sum_\alpha \lambda_\alpha \{s_\alpha\}_2^\sharp$ be an element of $\mathcal{L}_2^\sharp$, and $n \ge 1$ satisfying $n \cdot \lambda_\alpha \in \mathbb{Z}$ for all α. Consider the line bundle

$$\Phi_2^\sharp(2n \cdot S) := \bigotimes_\alpha <s_\alpha, s_\alpha>^{\otimes n\cdot\lambda_\alpha} ,$$

where $<s,t>$ denotes the bundle $(s,t)^*\mathcal{P}$ on B.
Because of the bimultiplicativity of the pairing $< \, , >$, the bundle $\Phi_2^\sharp(2n \cdot S)$ is equipped with a canonical trivialization

$$\theta_1 : \mathcal{O}_B \xrightarrow{\sim} \Phi_2^\sharp(2n \cdot S)$$

as soon as $n \cdot \sum_\alpha \lambda_\alpha \cdot (s_\alpha \otimes s_\alpha) = 0$.

On the other hand, because the *classes* of $\Delta^*\mathcal{P}$ and $\mathcal{O}(i(B))^{\otimes 2}$ in $\mathrm{Pic}(\mathcal{E})$ are the same, we have

$$\Delta^*\mathcal{P} = \mathcal{O}(i(B))^{\otimes 2} \otimes_{\mathcal{O}_\mathcal{E}} \pi^* i^* \mathcal{O}(i(B))^{\otimes -2}$$

as rigidified line bundles on $\mathcal{E}$. There is a canonical isomorphism

$$i^*\mathcal{O}(i(B))^{\otimes -2} \xrightarrow{\sim} \omega_{\mathcal{E}/B}^{\otimes 2} ,$$

where $\omega_{\mathcal{E}/B} := i^*\Omega^1_{\mathcal{E}/B} = \pi_*\Omega^1_{\mathcal{E}/B}$. In local Weierstraß coordinates (x,y), it can be described by sending x^{-1} to the square of the differential $\omega = dx/(2y+a_1x+a_3)$. Altogether, we end up with an isomorphism

$$\omega_{\mathcal{E}/B}^{\otimes 2} \xrightarrow{\sim} <s,s>$$

for any $s \in \widetilde{\mathcal{E}}(B)$. But $(\omega_{\mathcal{E}/B}^{\otimes 2})^{\otimes 6} = \omega_{\mathcal{E}/B}^{\otimes 12}$ can be trivialized by the discriminant: locally, the section is given by $\Delta \cdot \omega^{12}$.

We therefore have

$$\mathcal{O}_B \xrightarrow{\sim} \omega_{\mathcal{E}/B}^{\otimes 12} \xrightarrow{\sim} <s,s>^{\otimes 6}$$

(given locally by "$1 \longmapsto x^{-6} \cdot \Delta$"), and in particular, a trivialization

$$\theta_2 : \mathcal{O}_B \xrightarrow{\sim} \Phi_2^\sharp(2n \cdot S)^{\otimes 6} .$$

Composition yields an isomorphism

$$\theta_2^{-1} \circ \theta_1^{\otimes 6} : \mathcal{O}_B \xrightarrow{\sim} \mathcal{O}_B ,$$

i.e., an element of $\mathcal{O}^*(B)$. We define

$$\varphi_2^\sharp(S) := \left(\theta_2^{-1} \circ \theta_1^{\otimes 6}\right)^{1/12n} \in \mathcal{O}^*(B) \otimes_\mathbb{Z} \mathbb{Q} .$$

Norm compatibility is a consequence of the stronger statement

$$\Phi_2^\sharp\left(\{\psi(s_1)\}_2^\sharp\right) = \Phi_2^\sharp\left(\sum_{t \in \ker(\psi)(B)} \{s_1 + t\}_2^\sharp\right)$$

for all $s_1 \in \mathcal{E}_1(B)$, which follows from standard compatibility properties of the Poincaré bundles under isogenies.

We sketch an alternative construction of $\varphi_2^\sharp$, which uses the fact that there is a Hodge–de Rham version of the elliptic polylogarithm, and the machinery to be developed in 3.5. A priori, we get extensions

$$\varphi_2^\sharp(S) \in \mathrm{Ext}^1_{HDR^s_{\mathbb{Q}}(B)}\left(\mathbb{Q}(0), \mathbb{Q}(1)\right) ,$$

where $HDR^s_{\mathbb{Q}}$ denotes the category of variations of $\mathbb{Q}$–Hodge–de Rham structure (see [W2], chapter 3). There is an injection

$$\mathcal{O}^*(B) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathrm{Ext}^1_{HDR^s_{\mathbb{Q}}(B)}\left(\mathbb{Q}(0), \mathbb{Q}(1)\right) .$$

One uses a modular argument to see that the image of $\varphi_2^\sharp$ is contained in $\mathcal{O}^*(B) \otimes_{\mathbb{Z}} \mathbb{Q}$. For the details, we refer to [W3].

The precise relation with heights uses an explicit computation for $< \, , >$ in terms of multipliers of line bundles on tori. The details can be found in [GL], section 2, and we content ourselves with giving a sketch:
Let $B = \mathrm{Spec}(K)$, i.e., $\mathcal{E} = E$ an elliptic curve over K, and choose a Weierstraß equation

$$y^2 = x^3 + ax + b$$

for E of discriminant Δ. We have special elements of $\ker(d_2^\sharp)$ (cmp. [GL], 2.2):

$$\begin{aligned} \{s,t\}_2^\sharp &:= \{s+t\}_2^\sharp + \{s-t\}_2^\sharp - 2\{s\}_2^\sharp - 2\{t\}_2^\sharp \quad \text{for } s \neq t\,, \\ \{s,s\}_2^\sharp &:= \{2s\}_2^\sharp - 4\{s\}_2^\sharp \quad \text{for } 2s \neq 0\,, \\ \{s,-s\}_2^\sharp &:= \{2s\}_2^\sharp - 2\{s\}_2^\sharp - 2\{-s\}_2^\sharp \quad \text{for } 2s \neq 0\,, \\ \{s,s\}_2^\sharp &:= -4\{s\}_2^\sharp \quad \text{for } 2s = 0\,. \end{aligned}$$

By an elementary computation (cmp. [GL], Lemma 2.4), elements of this shape generate the whole of $\ker(d_2^\sharp)$. The following is proved by considering the analytic situation:

Proposition: ([GL], Proposition 2.1.)

$$\begin{aligned} \varphi_2^\sharp\left(\{s,t\}_2^\sharp\right) &= \left((x(t)-x(s))\Delta^{-1/6}\right)^{-1} \quad \text{for } s \neq t\,, \\ \varphi_2^\sharp\left(\{s,s\}_2^\sharp\right) = \varphi_2^\sharp\left(\{s,-s\}_2^\sharp\right) &= \left(-y(s)\Delta^{-1/4}\right)^{-1} \quad \text{for } 2s \neq 0\,, \\ \varphi_2^\sharp\left(\{s,s\}_2^\sharp\right) &= \left(-\frac{1}{4}(3x(s)^2+a)\Delta^{-1/3}\right)^{-1} \quad \text{for } 2s = 0\,. \end{aligned}$$

Using the formulae of [Si], VI, in particular, 4.1, and Exercises 6.3, 6.7.b) and 6.8, one can show the desired relation to the non–archimedian local heights.

As was observed in [R], V.4, it is possible to give a closed formula for $\varphi_2^\sharp$: let $\mathcal{H}^+$ denote the complex upper half plane. Define the following functions on $\mathbb{C} \times \mathcal{H}^+$:

$$r_2 : \mathbb{C} \times \mathcal{H}^+ \longrightarrow \mathbb{R}\,,\ (z,\tau) \longmapsto -\frac{\mathrm{Im}\,(z)}{\mathrm{Im}\,(\tau)}\,,$$

$$q_{\mathcal{H}} : \mathbb{C} \times \mathcal{H}^+ \longrightarrow \mathbb{C}^* \, , \, (z,\tau) \longmapsto \exp(2\pi i \tau) \, ,$$
$$q_{\mathbb{C}} : \mathbb{C} \times \mathcal{H}^+ \longrightarrow \mathbb{C}^* \, , \, (z,\tau) \longmapsto \exp(2\pi i z) \, .$$

Observe that for $(z,\tau) \in \mathbb{C} \times \mathcal{H}^+$, we have

$$z + r_2(z,\tau) \cdot \tau \in \mathbb{R} \, .$$

Definition: The Siegel function

$$Si : \mathbb{C} \times \mathcal{H}^+ - \{(z,\tau) \mid z \in \mathbb{Z} \oplus \mathbb{Z}\tau\} \longrightarrow \mathbb{C}$$

is given by

$$Si := -q_{\mathbb{C}}^{-\frac{1}{2}(r_2+1)} q_{\mathcal{H}}^{\frac{1}{12}} (1 - q_{\mathbb{C}}) \prod_{n=1}^{\infty} (1 - q_{\mathcal{H}}^n q_{\mathbb{C}})(1 - q_{\mathcal{H}}^n / q_{\mathbb{C}}) \, .$$

We then have the

Proposition: ([R], V, Proposition 4.4.)
Choose an embedding $\sigma : K \hookrightarrow \mathbb{C}$ and an isomorphism ${}^{\sigma}E(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}/L$, where L is of the shape $\mathbb{Z} \oplus \mathbb{Z}\tau$. Let $S \in \ker(d_2^{\sharp})$. Choose $T = \sum_\alpha \lambda_\alpha \{z_\alpha\} \in \mathbb{Q}[\mathbb{C} - L]$ mapping to σS under the projection

$$\mathbb{Q}[\mathbb{C} - L] \longrightarrow \mathbb{Q}\left[{}^{\sigma}E(\mathbb{C}) - \{0\}\right]$$

and lying in the kernel of

$$\mathbb{Q}[\mathbb{C} - L] \longrightarrow \mathbb{Q}[\mathbb{C}] \otimes_{\mathbb{Q}} \mathbb{Q}[\mathbb{C}] \, ,$$
$$\{z\} \longmapsto z \otimes z \, .$$

(One may e.g. employ [GL], Lemma 2.4 to see that such T exists.) Then we have

$$\varphi_2^{\sharp}(S) = \sum_\alpha Si^{-1}(z_\alpha) \otimes \lambda_\alpha \in \mathbb{C}^* \otimes_{\mathbb{Z}} \mathbb{Q} \, .$$

The relation to the archimedian local heights then follows from [Si], VI, Theorem 3.4.

1.10. As in [BD], 1.8, it is possible to state a *necessary* criterion for a linear combination $\sum_\alpha \lambda_\alpha \{s_\alpha\}_k^{\sharp}$ to belong to the kernel of $d_k^{\sharp}$:

1.10.1. For any homomorphism $X : \mathcal{E}(B) \longrightarrow \mathbb{Q}$, one has

$$\sum_\alpha \lambda_\alpha X(s_\alpha)^k = 0 \quad \text{in } \mathbb{Q} \, .$$

1.10.2. For any $2 \le l < k$ and $\alpha + \beta = l - 2$, $\alpha \le \beta$, any closed point b of the generic fibre of B with residue class field $K(b)$ and fibre $\mathcal{E}_b$ of $\mathcal{E}$, any embedding $\sigma : K(b) \hookrightarrow \mathbb{C}$ and homomorphism $X : \mathcal{E}_b(K(b)) \longrightarrow \mathbb{Q}$, one has

$$\sum_\alpha \lambda_\alpha X(s_{\alpha,b})^{k-l} K_{\alpha+1,\beta+1,L}(\eta(\sigma s_{\alpha,b})) = 0 \quad \text{in } \mathbb{C}$$

for some choice of $\eta : {}^{\sigma}\mathcal{E}_b(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}/L$.

1.10.3. For any $2 \leq l < k$, any closed point b of the generic fibre of B with residue class field $K(b)$ and fibre $\mathcal{E}_b$ of $\mathcal{E}$, and any homomorphism $X : \mathcal{E}_b(K(b)) \longrightarrow \mathbb{Q}$, one has

a) if $l = 2$, then

$$-(\log N(v))^{-1} \cdot \sum_\alpha \lambda_\alpha X(s_{\alpha,b})^{k-2} h_v(s_{\alpha,b}) = 0 \quad \text{in } \mathbb{Q}$$

for any finite place v of $K(b)$.

b) if $l \geq 3$, then

$$\sum_\alpha \lambda_\alpha X(s_{\alpha,b})^{k-l} \sum_{n \in \mathbb{Z}/N_v\mathbb{Z}} d_{s_{\alpha,b}}(v,n) \cdot B_l(<\frac{n}{N_v}>) = 0 \quad \text{in } \mathbb{Q}$$

for all finite places v of $K(b)$ where $\mathcal{E}_b$ has split multiplicative reduction if l is odd / all finite places v of $K(b)'$ where $\mathcal{E}_b \otimes_{K(b)} K(b)'$ has split multiplicative reduction if l is even. Here, $K(b)'$ denotes a finite extension of $K(b)$ where the reduction type of $\mathcal{E}_b \otimes_{K(b)} K(b)'$ is good or split multiplicative everywhere.

For $B = \operatorname{Spec}(K)$, validity of the part of Beilinson's conjecture predicting injectivity of

$$r_{\mathcal{D}} : H^{l-1}_{\mathcal{M}}\left(E^{(l-2)}, \mathbb{Q}(l-1)\right)_{\mathbb{Z},\mathrm{sgn}} \longrightarrow H^{l-1}_{\mathcal{D}}\left(E^{(l-2)} \otimes_{\mathbb{Q}} \mathbb{R}, \mathbb{R}(l-1)\right)_{\mathrm{sgn}}$$

for $2 \leq l < k$ would imply that 1.10.1–1.10.3 are in fact sufficient for $\sum_\alpha \lambda_\alpha \{s_\alpha\}^\sharp_k$ to belong to $\ker(d^\sharp_k)$.

1.11 Examples: a) If $k = 3$ and $B = \operatorname{Spec}(K)$, we have the following description of $\ker(d^\sharp_3)$:

$$\sum_\alpha \lambda_\alpha \{s_\alpha\}^\sharp_3 \in \ker(d^\sharp_3) \iff \text{for any homomorphism } X : E(K) \longrightarrow \mathbb{Q} \ :$$

i) $\sum_\alpha \lambda_\alpha X(s_\alpha)^3 = 0$ in $\mathbb{Q}$.

ii) $\sum_\alpha \lambda_\alpha X(s_\alpha) h_v(s_\alpha) = 0$ for any (finite or infinite) place v of K.

If i) and ii) are satisfied, then the conjecture predicts that

$$\varphi^\sharp_3\left(\sum_\alpha \lambda_\alpha \{s_\alpha\}^\sharp_3\right) \in H^2_{\mathcal{M}}\left(E^{(1)}, \mathbb{Q}(2)\right)_{\mathrm{sgn}}$$

is integral if and only if

iii) $\sum_\alpha \lambda_\alpha \sum_{n \in \mathbb{Z}/N_v\mathbb{Z}} d_{s_\alpha}(v,n) \cdot B_3(<\frac{n}{N_v}>) = 0$ for all places v of K where E has split multiplicative reduction.

For $K = \mathbb{Q}$, where $H^2_{\mathcal{M}}(E^{(1)}, \mathbb{Q}(2))_{\mathbb{Z},\mathrm{sgn}}$ is expected to be one–dimensional, Beilinson's conjecture predicts that if i)–iii) are satisfied for some $\sum_\alpha \lambda_\alpha \{s_\alpha\}^\sharp_3$, then there is a rational number q (possibly zero) such that

$$q \cdot L(E,2) = (2\pi i)^2 \cdot \frac{A(L)^2}{2} \sum_{\alpha} \lambda_\alpha K_{1,2,L}(\eta(s_\alpha))$$
$$= (2\pi i)^2 \cdot \frac{A(L)^2}{2} \sum_{\gamma \in L - \{0\}} \sum_{\alpha} \lambda_\alpha \frac{(\eta(s_\alpha), \gamma)_L}{\gamma \overline{\gamma}^2}$$

for an isomorphism $\eta : E(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}/L$ such that $\eta^*(dz)$ is real.
In [GL], it is proven that for any modular elliptic curve E, there is an element in

$$\left(\varinjlim_{K/\mathbb{Q}} \ker \left(d_3^\sharp (E \otimes_{\mathbb{Q}} K) \right) \right)^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$$

satisfying i)–iii), such that the above relation (with some $q \neq 0$) holds. In fact, it is shown (loc.cit., Theorem 1.2.i)) that for any element h in $H^2_{\mathcal{M}}(E^{(1)}, \mathbb{Q}(2))_{\mathrm{sgn}}$, there is

$$S_h \in \left(\varinjlim_{K/\mathbb{Q}} \ker \left(d_3^\sharp (E \otimes_{\mathbb{Q}} K) \right) \right)^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$$

such that

$$r_{\mathcal{D}}(h) = G_{E,3}(S_h) \quad \text{in } H^2_{\mathcal{D}} \left(E^{(1)} \otimes_{\mathbb{Q}} \mathbb{R}, \mathbb{R}(2) \right)_{\mathrm{sgn}} .$$

Conversely (loc.cit., Theorem 1.2.ii)), if

$$S \in \left(\varinjlim_{K/\mathbb{Q}} \ker \left(d_3^\sharp (E \otimes_{\mathbb{Q}} K) \right) \right)^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$$

is given, then there exists $h_S \in H^2_{\mathcal{M}}(E^{(1)}, \mathbb{Q}(2))_{\mathrm{sgn}}$ such that the above relation holds. We refer to [R], V for further discussion of the connection of [GL] to our approach.

Let us also mention the numerical tables of [R], VI. There, the Nekovář family

$$E_a : (3a+1)y^2 = x^3 - (3a^2+1)x + (2a^3+a+1) , \quad a \neq -\frac{1}{3}$$

of elliptic curves is considered. It is equipped with a section P. One can construct an element

$$S = 16 \cdot \{P\}_3^\sharp - 10 \cdot \{2P\}_3^\sharp + \{4P\}_3^\sharp$$

of $\mathcal{L}_3^\sharp$. For all a considered, the element S_a lies in $\ker(d_3^\sharp(E_a))$ (which may lead one to expect that S itself lies in $\ker(d_3^\sharp)$). For those a, it is possible to numerically detect a rational dependence between $L(E_a, 2)$ and

$$(2\pi i)^2 \cdot \frac{A(L)^2}{2} K_{1,2,L}(\eta(S))$$

if and only if the integrality obstruction iii) is satisfied.

b) Let $k = 4$ and $B = \mathrm{Spec}(K)$, and assume that

$$r_{\mathcal{D}} : H^2_{\mathcal{M}}\left(E^{(1)}, \mathbb{Q}(2)\right)_{\mathbb{Z},\mathrm{sgn}} \longrightarrow H^2_{\mathcal{D}}\left(E^{(1)} \otimes_{\mathbb{Q}} \mathbb{R}, \mathbb{R}(2)\right)_{\mathrm{sgn}}$$

is injective. We then have the following description of $\ker(d_4^\sharp)$:

$$\sum_\alpha \lambda_\alpha \{s_\alpha\}_4^\sharp \in \ker(d_4^\sharp) \iff \text{for any homomorphism } X : E(K) \longrightarrow \mathbb{Q} :$$

i) $\sum_\alpha \lambda_\alpha X(s_\alpha)^4 = 0$ in $\mathbb{Q}$.
ii) $\sum_\alpha \lambda_\alpha X(s_\alpha)^2 h_v(s_\alpha) = 0$ for any (finite or infinite) place v of K.
iii.a) For any embedding $\sigma : K \hookrightarrow \mathbb{C}$, one has

$$\sum_\alpha \lambda_\alpha X(s_\alpha) K_{1,2,L}(\eta(\sigma s_\alpha)) = 0 \quad \text{in } \mathbb{C}$$

for some choice of $\eta : {}^\sigma E(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}/L$,
iii.b) $\sum_\alpha \lambda_\alpha X(s_\alpha) \sum_{n \in \mathbb{Z}/N_v\mathbb{Z}} d_{s_\alpha}(v,n) \cdot B_3(<\dfrac{n}{N_v}>) = 0$
for all places v of K where E has split multiplicative reduction.

If i)–iii) are satisfied, then the conjecture predicts that

$$\varphi_4^\sharp\left(\sum_\alpha \lambda_\alpha \{s_\alpha\}_4^\sharp\right) \in H^3_{\mathcal{M}}\left(E^{(2)}, \mathbb{Q}(3)\right)_{\mathrm{sgn}}$$

is integral if and only if
iv) $\sum_\alpha \lambda_\alpha \sum_{n \in \mathbb{Z}/N_v\mathbb{Z}} d_{s_\alpha}(v,n) \cdot B_4(<\dfrac{n}{N_v}>) = 0$ for all places v of K' where E has split multiplicative reduction. As usual, K' denotes a finite extension of K where $E \otimes_K K'$ has good or split multiplicative reduction everywhere.

These conditions should be compared to the ones of Conjecture 1.1 of [G2].

c) In the special case where ψ is multiplication by an integer N, norm compatibility acquires the following shape:
For any N which is prime to the characteristic of B, such that $\ker[N]$ consists of sections of π, and for all $s_\alpha \in (\mathcal{E} - \ker[N])(B)$ and $\lambda_\alpha \in \mathbb{Q}$, we have

$$d_k\left(\sum_\alpha \lambda_\alpha \{[N]s_\alpha\}_k\right) = 0$$

if and only if

$$d_k\left(\sum_\alpha \lambda_\alpha \sum_{t \in (\ker[N])(B)} \{s_\alpha + t\}_k\right) = 0\,,$$

and if this is the case, then the equality

$$N^{k-2} \cdot \varphi_k\left(\sum_\alpha \lambda_\alpha \sum_{t \in (\ker[N])(B)} \{s_\alpha + t\}_k\right) = \varphi_k\left(\sum_\alpha \lambda_\alpha \{[N]s_\alpha\}_k\right)$$

holds. Note that $[N]_*$ acts by multiplication by N^{k-2} on

$$H^{k-1}_{\mathcal{M}}\left(\mathcal{E}^{(k-2)}, \mathbb{Q}(k-1)\right)_{\text{sgn}} \quad :$$

since $[N]_* \circ [N]^* = N^{2(k-2)}$, this claim is equivalent to the corresponding statement for $[N]^*$, which can be found e.g. in [De1], 8.16 or [BL], 5.5 and 6.2.1.

Of course, norm compatibility would be implied by the following statement: for all $s \in (\mathcal{E} - \ker[N])(B)$, we have

$$d_k\left(\{[N]s\}_k - N^{k-2} \cdot \sum_{t \in (\ker[N])(B)} \{s+t\}_k\right) = 0\,, \quad \text{and}$$

$$\varphi_k\left(\{[N]s\}_k - N^{k-2} \cdot \sum_{t \in (\ker[N])(B)} \{s+t\}_k\right) = 0\,.$$

Assuming injectivity of

$$r_{\mathcal{D}} : H^{l-1}_{\mathcal{M}}\left(E^{(l-2)}, \mathbb{Q}(l-1)\right)_{\mathbb{Z},\text{sgn}} \longrightarrow H^{l-1}_{\mathcal{D}}\left(E^{(l-2)} \otimes_{\mathbb{Q}} \mathbb{R}, \mathbb{R}(l-1)\right)_{\text{sgn}}$$

for $2 \le l < k$, and using the distribution property for both Kronecker double series and the function

$$x \longmapsto B_k(<x>)$$

on $\mathbb{Q}/\mathbb{Z}$, it can be shown inductively that for $B = \text{Spec}(K)$, the above statement is implied by Conjecture 1.6.(B).

1.12. We note the conjectural consequences of the following

Assumption: Let $B = \text{Spec}(K)$ and $k \ge 3$, and assume that Conjecture 1.6.(B) holds. Suppose in addition that for an elliptic curve E/K, the image of $\varinjlim_{M/K} \varphi_k^{\text{Gal}(M/K)}$ contains

$$H^{k-1}_{\mathcal{M}}\left(E^{(k-2)}, \mathbb{Q}(k-1)\right)_{\mathbb{Z},\text{sgn}} \;,$$

where the direct system runs over all Galois extensions M of K. Let r_1 and $2r_2$ be the number of real and complex embeddings of K respectively.

We then have the following, which supplements the discussion in [De2], § 5:

Theorem: Under the above hypothesis, the Beilinson conjecture for the leading term of $L(\text{Sym}^{k-2}h_1(E), s)$ at 0, or, equivalently, for the value of $L(\text{Sym}^{k-2}h_1(E), s)$ at $k-1$ implies the following statement:

There is a finite Galois extension M/K, and elements

$$S_j \in \mathbb{Q}[E(M) - 0]\,, \quad j = 1, \ldots, m := r_1 \cdot [\frac{k}{2}] + r_2 \cdot (k-1)$$

satisfying conditions 1.10.1–1.10.3, and the integrality obstruction of 1.6.(B), such that

$$L^*\left(\mathrm{Sym}^{k-2}h_1(E),0\right)\cdot\det\Omega = q_1\cdot\det\left(c_{\sigma,a}(S_j)\right)_{\substack{\sigma,a\\1\le j\le m}},$$
$$L\left(\mathrm{Sym}^{k-2}h_1(E),k-1\right)\cdot\det\Omega = q_2\cdot\varepsilon\cdot(2\pi i)^{n\frac{k(k-1)}{2}-m}\cdot\det\left(c_{\sigma,a}(S_j)\right)_{\substack{\sigma,a\\1\le j\le m}}$$

for suitable $q_1, q_2 \in \mathbb{Q}^*$.
Here, ε denotes the epsilon factor $\varepsilon\left(\mathrm{Sym}^{k-2}h_1(E),k-1\right)$, and σ runs through the first r_1+r_2 of the embeddings

$$\sigma_1,\dots,\sigma_{r_1},\sigma_{r_1+1},\dots,\sigma_{r_1+r_2},\overline{\sigma_{r_1+1}},\dots,\overline{\sigma_{r_1+r_2}}$$

of K into $\mathbb{C}$. For σ real and $1\le a\le[\frac{k}{2}]$ or σ complex and $1\le a\le k-1$, we choose an extension σ_M of σ to M, and define

$$c_{\sigma,a}(S_j) := \frac{A(L_\sigma)}{2}K_{a,k-a,L_\sigma}\left(\eta_\sigma(\sigma_M\,S_j)\right) = \frac{A(L_\sigma)}{2}\sum_{\gamma\in L_\sigma-\{0\}}\frac{(\eta_\sigma(\sigma_M\,S_j),\gamma)_{L_\sigma}}{\gamma^a\overline{\gamma}^{k-a}}$$

for an isomorphism $\eta_\sigma : {}^\sigma E(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}/L_\sigma$. For σ real, we choose η_σ such that $\omega(L_\sigma)=\eta_\sigma^*(dz)$ is fixed under the de Rham–conjugation.
The S_j can be chosen such that the $c_{\sigma,a}(S_j)$ are independent of the choice of the σ_M.
With the same notation as in [De2], § 5, the matrix Ω compares a basis b_1 of

$$H^{k-1}_{\mathcal{D}}\left(E^{(k-2)}\otimes_{\mathbb{Q}}\mathbb{R},\mathbb{R}(k-1)\right)_{\mathrm{sgn}} = \left(\bigoplus_{\sigma:K\hookrightarrow\mathbb{C}}\mathrm{Sym}^{k-2}H^1_B({}^\sigma E(\mathbb{C}),2\pi i\,\mathbb{R})\right)^+$$

given by the rational structure of the right hand side to a basis b_2 of

$$\left(\bigoplus_{\sigma:K\hookrightarrow\mathbb{C}}\mathrm{Sym}^{k-2}H^1_B({}^\sigma E(\mathbb{C}),2\pi i\,\mathbb{R})\right)^+ \subset \mathrm{Sym}^{k-2}H^1_{DR}(E\otimes_{\mathbb{Q}}\mathbb{C})$$

constructed from $(\omega(L_\sigma)^{a-1}\,\overline{\omega(L_\sigma)}^{k-1-a}\,|\,\sigma:K\hookrightarrow\mathbb{C}, 1\le a\le k-1)$ by applying the projection onto the $(-1)^{k-2}$–eigenspace of the conjugation of coefficients.
In explicit terms, we have

$$\det\Omega = (2\pi i)^{r_1(k-2)([\frac{k}{2}]-1)+r_2\frac{(k-2)(k-1)}{2}}\cdot\prod_{\sigma\mathrm{real}}A(L_\sigma)^{-(k-2)}\prod_{r_1+1\le l\le r_1+r_2}A(L_{\sigma_l})^{-\frac{(k-2)(k-1)}{2}}$$

up to a non-zero rational factor.

Remark: In [De1] and [De2], the statement on $L^*(\mathrm{Sym}^{k-2}h_1(E),0)$ is proven for CM–elliptic curves satisfying Shimura's condition (S). [GL] contains the proof for modular elliptic curves (hence $K=\mathbb{Q}$) and $k=3$. For $K=\mathbb{Q}$ and $k=4$, the statement should be compared to Conjecture 1.1 of [G2].

Proof: Observe that m is the real dimension of

$$H^{k-1}_{\mathcal{D}}\left(E^{(k-2)}\otimes_{\mathbb{Q}}\mathbb{R},\mathbb{R}(k-1)\right)_{\mathrm{sgn}}.$$

Using the notation of [J1], 4.6–4.10, we have $d^-(M) = m$. Thus our statements on $L^*(\mathrm{Sym}^{k-2} h_1(E), 0)$ and $L(\mathrm{Sym}^{k-2} h_1(E), k-1)$ are equivalent.
For $L^*(\mathrm{Sym}^{k-2} h_1(E), 0)$, we have to express the image of $r_{\mathcal{D}}$ in terms of the basis b_1 – hence the correction term $\det \Omega$ since the $c_{\sigma,a}(S_j)$ are the coefficients with respect to b_2.
We leave the calculation of $\det \Omega$ to the reader. q.e.d.

§ 2 Homological background

Assume given an exact sequence

$$1 \longrightarrow W \longrightarrow P \longrightarrow G \longrightarrow 1$$

of affine group schemes over a field F of characteristic 0, with W pro–unipotent, and a fixed splitting of $P \longrightarrow G$. Write $P = W \rtimes G$, and $W = \varprojlim_j W_j$ as the projective limit of its algebraic quotients. We get an action of G on $\mathrm{Lie}\,(W) = \varprojlim_j \mathrm{Lie}\,(W_j)$ by conjugation.

Proposition 2.1: There is a natural equivalence of categories between $\mathrm{Rep}_F(P)$ and the category of algebraic G–representations V together with a G–equivariant Lie algebra morphism

$$\rho : \mathrm{Lie}\,(W) \longrightarrow \mathrm{End}_F(V)$$

with nilpotent image.

Proof: We reduce easily to the case $G = 1$. There, our claim follows from [DG], IV, § 2, Corollaire 4.5.b). q.e.d.

2.2. Proposition 2.1 enables us to give a description of certain one–extensions in the category $\mathrm{Rep}_F(P)$: let $V_1, V_2 \in \mathrm{Rep}_F(G)$, considered as P–representations with trivial W–action. Assume given a class in

$$\ker\left(\mathrm{Ext}^1_{\mathrm{Rep}_F(P)}(V_1, V_2) \longrightarrow \mathrm{Ext}^1_{\mathrm{Rep}_F(G)}(V_1, V_2)\right) .$$

By 2.1, this class is determined by a linear map

$$\rho : \mathrm{Lie}\,(W) \longrightarrow \mathrm{Hom}_F(V_1, V_2) = \begin{pmatrix} 0 & 0 \\ V_1^\vee \otimes_F V_2 & 0 \end{pmatrix} \subset \mathrm{End}_F(V_1 \oplus V_2)$$

satisfying
i) the restriction of ρ to $[\mathrm{Lie}\,(W), \mathrm{Lie}\,(W)]$ is trivial.

ii) ρ is G–equivariant.
Furthermore, this morphism is unique if $\mathrm{Hom}_{\mathrm{Rep}_F(G)}(V_1, V_2) = 0$.

So if we follow [BD], 2.1 and denote by

$$d : \mathrm{Lie}\,(W)^\vee \longrightarrow \overset{2}{\Lambda}\, \mathrm{Lie}\,(W)^\vee$$

minus the dual map of the commutator, we have

Corollary: There is a canonical epimorphism between

$$\ker\left(d \otimes \mathrm{id} \,:\, (\mathrm{Lie}\,(W)^\vee \otimes_F V_1^\vee \otimes_F V_2)^G \longrightarrow (\overset{2}{\Lambda}\,\mathrm{Lie}\,(W)^\vee \otimes_F V_1^\vee \otimes_F V_2)^G\right)$$

and

$$\ker\left(\mathrm{Ext}^1_{\mathrm{Rep}_F(P)}(V_1, V_2) \longrightarrow \mathrm{Ext}^1_{\mathrm{Rep}_F(G)}(V_1, V_2)\right) .$$

Here, $\mathrm{Lie}\,(W)^\vee$ denotes the continuous dual $\varinjlim \mathrm{Lie}\,(W_j)^\vee$.
The morphism is bijective if $\mathrm{Hom}_{\mathrm{Rep}_F(G)}(V_1, V_2) = 0$.

Observe that the above epimorphism is well–behaved under the identification

$$\mathrm{Ext}^1_{\mathrm{Rep}_F(P)}(V_1, V_2) = \mathrm{Ext}^1_{\mathrm{Rep}_F(P)}(F, V_1^\vee \otimes_F V_2) .$$

2.3. As in [BD], 2.1, let us introduce coefficients:
Let $V_1, V_2 \in \mathrm{Rep}_F(G)$, and $V \in \mathrm{Rep}_F(P)$ together with G–morphisms

$$x : V_1 \longrightarrow V ,$$
$$y : V \longrightarrow V_2 .$$

These data define a regular, $V_1^\vee \otimes_F V_2$–valued function $C_{y,x}$ on W:

$$C_{y,x}(w) := ywx .$$

The same formula for $w \in \mathrm{Lie}\,(W)$ defines

$$c_{y,x} \in \mathrm{Hom}_G\,(\mathrm{Lie}\,(W), V_1^\vee \otimes_F V_2) = (\mathrm{Lie}\,(W)^\vee \otimes_F V_1^\vee \otimes_F V_2)^G ,$$

since $ygwg^{-1}x = gywxg^{-1}$ for all $g \in G$.

Definition 2.4: Let $\mathcal{T}$ be an abelian F–linear tensor category.
We call $\mathcal{T}$ a tensor category with weights, if the following hold:

i) Each object V of $\mathcal{T}$ is equipped with a finite ascending weight filtration $W.V$ by subobjects, indexed by the integers.

ii) Each morphism in $\mathcal{T}$ is strictly compatible with $W.$, i.e.,

$$V \longmapsto \mathrm{Gr}^W_\cdot V$$

is an exact functor.

iii) $W.$ is compatible with tensor products:

$$V \longmapsto \mathrm{Gr}^W_\cdot V$$

is a tensor functor

$$\mathcal{T} \longrightarrow \mathcal{T}^{\mathrm{pure}} ,$$

where $\mathcal{T}^{\mathrm{pure}}$ is the full tensor subcategory of $\mathcal{T}$, whose weight filtration is split.

Because of ii), there are no nontrivial morphisms between objects of disjoint weights. Therefore, the splitting of the weight filtration in $\mathcal{T}^{\text{pure}}$ is unique. Similarly, for V_1 of weights smaller than n and V_2 of weights greater or equal to n, we have

$$\operatorname{Ext}^1_{\mathcal{T}}(V_1, V_2) = 0\,.$$

2.5. In the situation of 2.4, assume in addition that $\mathcal{T}$ is neutral Tannakian. Choose a fibre functor

$$\omega : \mathcal{T}^{\text{pure}} \longrightarrow \operatorname{Vec}_F\,.$$

The functor

$$V \longmapsto \omega(\operatorname{Gr}^W_{\cdot} V)$$

is a fibre functor of $\mathcal{T}$, also denoted by ω.
Let G and P denote the Tannakian duals ([DM], Theorem 2.11) of $\mathcal{T}^{\text{pure}}$ and $\mathcal{T}$, respectively. The inclusion

$$\mathcal{T}^{\text{pure}} \hookrightarrow \mathcal{T}$$

and the projection

$$\mathcal{T} \longrightarrow\!\!\!\!\!\rightarrow \mathcal{T}^{\text{pure}}\,,\ V \longmapsto \operatorname{Gr}^W_{\cdot} V\,,$$

which by construction are compatible with ω, give a split exact sequence

$$1 \longrightarrow W \longrightarrow P \longrightarrow G \longrightarrow 1\,.$$

W admits an explicit description as those automorphisms of ω, that act trivially on objects of $\mathcal{T}^{\text{pure}}$.
It is therefore pro–unipotent, and $\operatorname{Lie}(W) \in \text{pro–}\operatorname{Rep}_F(G) = \text{pro–}\mathcal{T}^{\text{pure}}$ has weights smaller than zero. One may e.g. employ Proposition 2.1 to see that

$$\operatorname{Lie}_{\mathcal{T}} := \operatorname{Lie}(W) \in \text{pro–}\mathcal{T}^{\text{pure}}$$

is in fact independent of the choice of the fibre functor.
Observe that for objects V_1 and V_2 of $\mathcal{T}^{\text{pure}}$ of disjoint weights, Corollary 2.2 states that there is a canonical isomorphism between $\operatorname{Ext}^1_{\mathcal{T}}(V_1, V_2)$ and

$$\ker\left(d^* : \operatorname{Hom}_{\mathcal{T}}(\operatorname{Lie}_{\mathcal{T}}, V_1^\vee \otimes_F V_2) \longrightarrow \operatorname{Hom}_{\mathcal{T}}(\overset{2}{\Lambda} \operatorname{Lie}_{\mathcal{T}}, V_1^\vee \otimes_F V_2)\right)\,.$$

In the application we have in mind, V_1 and V_2 will be weight–graded parts of the elliptic polylogarithmic sheaf. Those linear combinations of the coefficients

$$c_{y,x} \in \operatorname{Hom}_{\mathcal{T}}(\operatorname{Lie}_{\mathcal{T}}, V_1^\vee \otimes_F V_2) = \operatorname{Hom}_{\mathcal{T}}(F(0), \operatorname{Lie}^\vee_{\mathcal{T}} \otimes_F V_1^\vee \otimes_F V_2)$$

with vanishing differential $d \otimes \operatorname{id}$ will yield extensions of V_1 by V_2. Here, we denoted by $F(0)$ the identity object of $\mathcal{T}$.

§ 3 Elliptic polylogarithmic sheaves

3.1. For the purpose of this paper, it would be sufficient to work in the category of admissible variations of Hodge structure, and of course the one of mixed motivic sheaves. But since the calculation of the differential holds as soon as certain axioms are satisfied, and because we have more applications in mind (e.g. [W3], or an l–adic or p–adic version of Zagier's conjecture), we assume that we work in a stack $\mathcal{T}$ (see e.g. [DM], page 221) of F–linear abelian categories on schemes B, which are smooth and quasiprojective over a base scheme S, provided with an associative and commutative tensor product. The topology is the étale topology, and F is a field of characteristic 0.
Note that this means in particular that for a finite Galois covering $f : B_1 \longrightarrow B_2$, the functor $f^* : \mathcal{T}(B_2) \longrightarrow \mathcal{T}(B_1)$ induces an equivalence of $\mathcal{T}(B_2)$ with the category of descent data in $\mathcal{T}(B_1)$.

Further:

(A) For B connected, $\mathcal{T}(B)$ is a neutral F–linear Tannakian category. For a morphism $f : B_1 \longrightarrow B_2$, the functor $f^* : \mathcal{T}(B_2) \longrightarrow \mathcal{T}(B_1)$ is exact.

(B) There is given an object of rank one, $F(1)$ in $\mathcal{T}(S)$. For every B, we still write $F(1)$ for the inverse image of $F(1)$ under the structural morphism.
For $k \in \mathbb{Z}$, write $F(k) := F(1)^{\otimes k}$.
For any $V \in \mathcal{T}(B)$, write $V(k) := V \otimes_F F(k)$.

(C) For any elliptic curve

$$\pi : \mathcal{E} \longrightarrow B\,,$$

there is given an object of rank two, $R^1\pi_*F$ in $\mathcal{T}(B)$, and an isomorphism

$$\cup_\pi : \overset{2}{\Lambda}\, R^1\pi_*F \overset{\sim}{\longrightarrow} F(-1)\,.$$

Both the formation of $R^1\pi_*F$ and $\cup_\pi$ are compatible with base change.
Any isogeny

$$\begin{array}{ccc} \mathcal{E}_1 & \overset{\psi}{\longrightarrow} & \mathcal{E}_2 \\ & \underset{\pi_1}{\searrow} & \downarrow \pi_2 \\ & & B \end{array}$$

induces an isomorphism

$$\psi_* : R^1(\pi_1)_*F \overset{\sim}{\longrightarrow} R^1(\pi_2)_*F\,,$$

and the formula

$$\deg\psi \cdot \cup_{\pi_1} = \cup_{\pi_2} \circ \overset{2}{\Lambda}\, \psi_* : \overset{2}{\Lambda}\, R^1(\pi_1)_*F \overset{\sim}{\longrightarrow} F(-1)$$

holds.

(D) $\mathcal{T}(B)$ is a tensor category with weights, and all morphisms f^* respect the weights. $F(1)$ is pure of weight -2, and for any elliptic curve

$$\pi : \mathcal{E} \longrightarrow B\,,$$

$R^1\pi_* F$ is pure of weight 1.

(E) For any elliptic curve

$$\pi : \mathcal{E} \longrightarrow B\,,$$

there is given a morphism

$$[\ ] : \mathcal{E}(B) \otimes_{\mathbb{Z}} F \longrightarrow \operatorname{Ext}^1_{\mathcal{T}(B)}\left(F(0), R^1\pi_* F(1)\right)\,,$$

which is compatible with base change.
Note that because of 2.4.ii), an extension representing a class in

$$\operatorname{Ext}^1_{\mathcal{T}(B)}(F(0), R^1\pi_* F(1))$$

is unique up to unique isomorphism.

(F) For any elliptic curve

$$\pi : \mathcal{E} \longrightarrow B\,,$$

we can form the base change

$$\mathrm{pr}_1 : \mathcal{E} \times_B \mathcal{E} \longrightarrow \mathcal{E}$$

and look at the extension $[\Delta] \in \operatorname{Ext}^1_{\mathcal{T}(\mathcal{E})}(F(0), \pi^* R^1\pi_* F(1))$, and the symmetric powers

$$\mathrm{Sym}^{N-1}[\Delta]\,,$$

whose graded objects are $\pi^*\mathrm{Sym}^l(R^1\pi_* F(1)), l = 0, \ldots, N-1$. These form a projective system, the so–called *logarithmic pro–sheaf* $\mathcal{L}\mathrm{og}$ on $\mathcal{E}$, and we require a projective system $(pol^N)_{N \in \mathbb{N}}$ of extensions in $\mathcal{T}(\widetilde{\mathcal{E}})$ of $\pi^* R^1\pi_* F(1)\,|_{\widetilde{\mathcal{E}}}$ by $\mathrm{Sym}^{N-1}[\Delta](1)\,|_{\widetilde{\mathcal{E}}}$, the *(small) polylogarithmic pro–extension*, such that

$$\begin{aligned} pol^1 \;&\in \operatorname{Ext}^1_{\mathcal{T}(\widetilde{\mathcal{E}})}\left(\pi^* R^1\pi_* F(1)\,|_{\widetilde{\mathcal{E}}}, F(1)\right) \\ &= \operatorname{Ext}^1_{\mathcal{T}(\widetilde{\mathcal{E}})}\left(F(0), (\pi^* R^1\pi_* F(1)\,|_{\widetilde{\mathcal{E}}})^\vee(1)\right) \\ &\underset{(C)}{=} \operatorname{Ext}^1_{\mathcal{T}(\widetilde{\mathcal{E}})}\left(F(0), \pi^* R^1\pi_* F(1)\,|_{\widetilde{\mathcal{E}}}\right) \end{aligned}$$

is given by the restriction of $[\Delta] \in \operatorname{Ext}^1_{\mathcal{T}(\mathcal{E})}(F(0), \pi^* R^1\pi_* F(1))$. Here, the isomorphism

$$R^1\pi_* F(1) \stackrel{\sim}{\longrightarrow} (R^1\pi_* F(1))^\vee(1)$$

is given by sending v to $\cup_\pi(v, \_)$.
Again, the formation of the polylogarithmic pro–extension is compatible with base change.

(G) (Norm compatibility of [ ] and *pol*.)
For any isogeny

$$\begin{array}{ccc} \mathcal{E}_1 & \xrightarrow{\psi} & \mathcal{E}_2 \\ & \searrow^{\pi_1} & \downarrow \pi_2 \\ & & B \end{array}$$

the diagram

$$\begin{array}{ccc} \mathcal{E}_1(B) \otimes_{\mathbb{Z}} F & \xrightarrow{[\,]} & \operatorname{Ext}^1_{\mathcal{T}(B)}\left(F(0), R^1(\pi_1)_* F(1)\right) \\ \psi \downarrow & & \wr\downarrow \psi_* \\ \mathcal{E}_2(B) \otimes_{\mathbb{Z}} F & \xrightarrow{[\,]} & \operatorname{Ext}^1_{\mathcal{T}(B)}\left(F(0), R^1(\pi_2)_* F(1)\right) \end{array}$$

commutes. In particular, the extensions $[\Delta_1]$ and $\psi^*[\Delta_2]$ coincide, and we get a canonical isomorphism

$$\mathcal{L}\mathrm{og}_1 \xrightarrow{\sim} \psi^* \mathcal{L}\mathrm{og}_2 \, .$$

In addition, if ψ is étale, and $\ker(\psi)$ consists of sections of π_1, we demand that the norm map

$$N_\psi : \operatorname{Ext}^1_{\mathcal{T}(\mathcal{E}_1 - \ker(\psi))}\left(\pi_1^* R^1(\pi_1)_* F(1) \,|_{\mathcal{E}_1 - \ker(\psi)}, \mathcal{L}\mathrm{og}_1(1) \,|_{\mathcal{E}_1 - \ker(\psi)}\right)$$
$$\longrightarrow \operatorname{Ext}^1_{\mathcal{T}(\widetilde{\mathcal{E}}_2)}\left(\pi_2^* R^1(\pi_2)_* F(1) \,|_{\widetilde{\mathcal{E}}_2}, \mathcal{L}\mathrm{og}_2(1) \,|_{\widetilde{\mathcal{E}}_2}\right) ,$$

whose existence follows from Galois descent, map the restriction of pol_1 to pol_2.

3.2. Examples of categories satisfying 3.1.(A)–(G) include

a) For K a number field and $S = \operatorname{Spec}(K)$ or $S = \operatorname{Spec}(O_K)$, and F finite over $\mathbb{Q}_l$, the category $\mathrm{Et}^{l,m}_F$ of lisse mixed étale F–sheaves (see [D2], 6.1) admitting a weight filtration.

b) For $S = \operatorname{Spec}(\mathbb{C})$ and $F \subset \mathbb{R}$, the full subcategory Var_F of those objects of the category of graded–polarizable variations of F–Hodge structure, that are admissible in the sense of [Ka].

c) For K a number field, $S = \operatorname{Spec}(K)$ and $F = \mathbb{Q}$, the category $HDR^s_{\mathbb{Q}}$ of variations of $\mathbb{Q}$–Hodge–de Rham structure (see [W2], chapter 3).

d) For K a number field, $S = \operatorname{Spec}(K)$ and $F = \mathbb{Q}$, the category $MS^s_{\mathbb{Q}}$ of mixed systems of smooth sheaves (see [W1], chapter 2).

Although we shall not use categories a), c) or d) here, we sketch how to show (E)–(G) in all of the cases: first, we give a description of the morphism [ ]:
In case a), the category $\mathrm{Et}_F^{l,m}$ is the category of smooth mixed objects of Et_F, the category of étale F–sheaves. By [E], Theorem 6.3, there is a triangulated category $D_c^b(\mathrm{Et}_F)$ equipped with a t–structure whose heart is Et_F. The category $D_c^b(\mathrm{Et}_F)$ admits the full formalism of Grothendieck's functors.
Similarly, in case b), the category Var_F is the category of smooth objects of MHM_F, the category of algebraic mixed F–Hodge modules ([S1], [S2], in particular Theorem 3.27 and § 4). Again, we have Grothendieck's functors on $D^b(MHM_F)$.
So in cases a) and b), given a section i' of π, we may form the complex

$$C: \; F(0)_{\mathcal{E}} \longrightarrow i_* F(0)_B \oplus i'_* F(0)_B \,,$$

with $F(0)_{\mathcal{E}}$ sitting in degree zero, and apply π_*. The short exact sequence

$$(*) \quad 0 \longrightarrow (i_* F(0)_B \oplus i'_* F(0)_B)[-1] \longrightarrow C \longrightarrow F(0)_{\mathcal{E}}[0] \longrightarrow 0$$

induces an exact triangle $\pi_*(*)$.
In case a), the sequence of cohomology objects shows that all $R^j \pi_* C$ are again objects of $\mathrm{Et}_F^{l,m}$.
In case b), a bit of care has to be applied: certainly, the exact triangle of complexes of constructible topological sheaves underlying $\pi_*(*)$ shows that the perverse cohomology objects of $\pi_* C_{\mathrm{top}}$ coincide, up to a shift by -1, with the usual cohomology objects, which are in fact locally constant. Hence all cohomology objects of $\pi_*(*)$ are admissible variations of Hodge structure. Writing $R^1\pi_*$ for $\mathcal{H}^0\pi_*$, we get in both cases an exact sequence

$$F(0)_B \longrightarrow F(0)_B \oplus F(0)_B \longrightarrow R^1\pi_* C \longrightarrow R^1\pi_* F(0)_{\mathcal{E}} \longrightarrow 0 \,.$$

Identification of the cokernel of the first map, which is given by the diagonal embedding, with $F(0)_B$ via projection onto the i–component yields an extension

$$0 \longrightarrow F(0)_B \longrightarrow R^1\pi_* C \longrightarrow R^1\pi_* F(0)_{\mathcal{E}} \longrightarrow 0 \,,$$

which we identify, as in 3.1.(F), with an extension

$$0 \longrightarrow R^1\pi_* F(1)_{\mathcal{E}} \longrightarrow (R^1\pi_* C)^\vee \longrightarrow F(0)_B \longrightarrow 0 \,,$$

which by definition is the value of [ ] at i'.
The same construction works in the category of $\mathcal{D}$–modules, giving an extension of algebraic vector bundles with integrable connection.

Since the direct and inverse images in the categories $D_c^b(\mathrm{Et}_F)$ and $D^b(MHM_F)$, and the bounded derived category of $\mathcal{D}$–modules, are compatible with the forgetful functors to the derived categories of perverse sheaves on $B \otimes_{O_K} \overline{K}$ (in case a)) and $B(\mathbb{C})$ respectively, our construction gives an extension $[i']$ in cases c) and d) as well.

For (F) and (G), we refer to [BL], 1.2, in particular 1.2.7, and 1.3, in particular 1.3.4 and 1.3.13. Concerning cases c) and d), the reader may find it useful to consult [W2], in particular the remark at the end of the second chapter.

3.3. From axiom (F), it follows that for any elliptic curve

$$\pi : \mathcal{E} \longrightarrow B \, ,$$

there are distinguished isomorphisms

$$\pi^* V_2 \,|_{\widetilde{\mathcal{E}}} \stackrel{\sim}{\longrightarrow} \mathrm{Gr}^W_{-1} pol \, ,$$
$$\mathrm{Gr}^W_{-k-1} pol \stackrel{\sim}{\longrightarrow} \pi^* \mathrm{Sym}^{k-1} V_2(1) \,|_{\widetilde{\mathcal{E}}} \, , \; k \geq 1 \, ,$$

where we set $V_2 := R^1 \pi_* F(1)$, in accordance with [W2].
We use these to define coefficients as in § 2:
For $\mathcal{F} \in \mathcal{T}(B)$, write $\Gamma(B, \mathcal{F}) := \mathrm{Hom}_{\mathcal{T}(B)}(F(0), \mathcal{F})$. Set $\mathrm{Lie}\,_B := \mathrm{Lie}\,_{\mathcal{T}(B)}$.
Fix $k \geq 1$, and let $s \in \widetilde{\mathcal{E}}(B)$. Define

$$x : V_2 \stackrel{\sim}{\longrightarrow} s^* \mathrm{Gr}^W_{-1} pol$$
$$\text{and} \quad y : s^* \mathrm{Gr}^W_{-k-1} pol \stackrel{\sim}{\longrightarrow} \mathrm{Sym}^{k-1} V_2(1)$$

to be the respective pullbacks via s^* of the isomorphisms above.

The coefficient $c_{y,x}$ lies in

$$\Gamma \left(B, \mathrm{Lie}\,^\vee_B \otimes_F V_2^\vee \otimes_F \mathrm{Sym}^{k-1} V_2(1) \right) \, .$$

For $k \geq 2$, there is an epimorphism

$$V_2^\vee \otimes_F \mathrm{Sym}^{k-1} V_2 \longrightarrow\!\!\!\!\rightarrow \mathrm{Sym}^{k-2} V_2$$

given by $\frac{1}{k}$ times "derivation". In any basis $(\varepsilon_1, \varepsilon_2)$ of $\omega(V_2)$, this epimorphism sends $\varepsilon_j^\vee \otimes f(\varepsilon_1, \varepsilon_2)$ to $\frac{1}{k} \frac{\partial}{\varepsilon_j} f(\varepsilon_1, \varepsilon_2)$.

Definition:

i) For $k = 1$, let $\{s\}_1 := c_{y,x} \in \Gamma \left(B, \mathrm{Lie}\,^\vee_B \otimes_F V_2^\vee(1) \right) \underset{\text{(C)}}{=} \Gamma(B, \mathrm{Lie}\,^\vee_B \otimes_F V_2)$.

ii) For $k \geq 2$, let $\{s\}_k \in \Gamma \left(B, \mathrm{Lie}\,^\vee_B \otimes_F \mathrm{Sym}^{k-2} V_2(1) \right)$ be the image of $c_{y,x}$ under the above epimorphism.

Remarks:

a) Since $\mathrm{Lie}\,_B$ has weights smaller than zero, the differential $d \otimes \mathrm{id}$ of 2.2 on $\Gamma \left(B, \mathrm{Lie}\,^\vee_B \otimes_F V_2 \right)$ is trivial, and we have

$$\Gamma \left(B, \mathrm{Lie}\,^\vee_B \otimes_F V_2 \right) = \mathrm{Ext}^1_{\mathcal{T}(B)}(F(0), V_2) \, .$$

Because of axiom (F), we have $\{s\}_1 = [s]$.

b) We shall explain later (4.3) why we do not lose any valuable information by forgetting the component of $c_{y,x}, k \geq 2$, corresponding to the direct summand $\mathrm{Sym}^k V_2(-1)$ of $V_2^\vee \otimes_F \mathrm{Sym}^{k-1} V_2$.

3.4. The vital ingredient in our formalism is of course the calculation of the differential

$$d := d \otimes \mathrm{id} \; : \Gamma\left(B, \mathrm{Lie}\,{}_B^\vee \otimes_F \mathrm{Sym}^{k-2} V_2(1)\right) \longrightarrow \Gamma\left(B, \overset{2}{\Lambda}\,\mathrm{Lie}\,{}_B^\vee \otimes_F \mathrm{Sym}^{k-2} V_2(1)\right) .$$

Theorem: For $k \geq 2$, we have

$$d\{s\}_k = \frac{k-1}{k} \mathrm{pr}_k (\{s\}_{k-1} \otimes \{s\}_1) ,$$

where pr_k denotes the morphism $\Lambda \otimes \mathrm{mult}$:

$$\mathrm{Lie}\,{}_B^\vee \otimes_F \mathrm{Sym}^{k-3} V_2(1) \otimes_F \mathrm{Lie}\,{}_B^\vee \otimes_F V_2 \longrightarrow \overset{2}{\Lambda}\,\mathrm{Lie}\,{}_B^\vee \otimes_F \mathrm{Sym}^{k-2} V_2(1) .$$

Proof: Define $\{s\}_k^\sim \in \Gamma\left(B, \mathrm{Lie}\,{}_B^\vee \otimes_F V_2^\vee \otimes_F \mathrm{Sym}^{k-1} V_2(1)\right)$ to be the coefficient $c_{y,x}$ of 3.3. Our claim follows if we show for $k \geq 2$:

$$d\{s\}_k^\sim = \mathrm{pr}_k^\sim (\{s\}_{k-1}^\sim \otimes \{s\}_1) ,$$

where $\mathrm{pr}_k^\sim$ denotes the morphism

$$\begin{aligned} \Lambda \otimes \mathrm{id} \otimes \mathrm{mult} \; : \mathrm{Lie}\,{}_B^\vee \otimes_F V_2^\vee \otimes \mathrm{Sym}^{k-2} V_2(1) \otimes_F \mathrm{Lie}\,{}_B^\vee \otimes_F V_2 \\ \longrightarrow \overset{2}{\Lambda}\,\mathrm{Lie}\,{}_B^\vee \otimes_F V_2^\vee \otimes \mathrm{Sym}^{k-1} V_2(1) . \end{aligned}$$

Fix a basis $(\varepsilon_1, \varepsilon_2)$ of $\omega(V_2)$. Note that because $\mathcal{L}\mathrm{og} = \varprojlim_N \mathrm{Sym}^{N-1}[\Delta]$, the image of $\mathrm{Lie}\,_B$ in $\mathrm{End}_F(\omega(pol))$ is contained in the Lie algebra $\mathfrak{v}$ generated by

$$e_1 := \left(\begin{array}{c|c|cc|ccc|cccc|c}
0\,0 & & & & & & & & & & \\
0\,0 & & & & & & & & & & \\
\hline
 & 0 & & & & & & & & & \\
\hline
 & 1 & 0 & 0 & & & & & & & \\
 & 0 & 0 & 0 & & & & & & & \\
\hline
 & & 1 & 0 & 0 & 0 & 0 & & & & \\
 & & 0 & 1 & 0 & 0 & 0 & & & & \\
 & & 0 & 0 & 0 & 0 & 0 & & & & \\
\hline
 & & & & 1 & 0 & 0 & & & & \\
 & & & & 0 & 1 & 0 & & & & \\
 & & & & 0 & 0 & 1 & & & & \cdots \\
 & & & & 0 & 0 & 0 & & & & \\
 & & & & & \vdots & & & & &
\end{array}\right)
\begin{array}{c}
V_2 \\ \hline F(1) \\ \hline V_2(1) \\ \hline \mathrm{Sym}^2 V_2(1) \\ \hline \mathrm{Sym}^3 V_2(1) \\ \vdots
\end{array}$$

$$e_2 := \left(\begin{array}{cc|c|cc|ccc|cccc}
0&0&&&&&&&&&&\\
0&0&&&&&&&&&&\\ \hline
&&0&&&&&&&&&\\ \hline
&&0&0&0&&&&&&&\\
&&1&0&0&&&&&&&\\ \hline
&&&0&0&0&0&0&&&&\\
&&&1&0&0&0&0&&&&\\
&&&0&1&0&0&0&&&&\\ \hline
&&&&&0&0&0&&&&\\
&&&&&1&0&0&\cdots&&&\\
&&&&&0&1&0&&&&\\
&&&&&0&0&1&&&&\\
&&&&&&\vdots&&&&&
\end{array}\right),$$

$$d_1 := \left(\begin{array}{cc|c|c}
0&0&&\\
0&0&&0\\ \hline
1&0&0&\\
&&&\\
0&&&0
\end{array}\right),$$

$$\text{and}\quad d_2 := \left(\begin{array}{cc|c|c}
0&0&&\\
0&0&&0\\ \hline
0&1&0&\\
&&&\\
0&&&0
\end{array}\right).$$

A basis of $\mathfrak{v}$ is given by

$$\Big(e_1, e_2, (\mathrm{ad}e_2)^l(\mathrm{ad}e_1)^m(d_i) \,|\, l, m \geq 0\Big) \;,$$

where $(\mathrm{ad}e_2)^l(\mathrm{ad}e_1)^m(d_i)$ is the matrix with zero everywhere except for an entry 1 in the i–th column, and the $(l+1)$–st row of the graded part $\mathrm{Sym}^{m+l}V_2(1)$.

By definition, $\{s\}_k^{\sim}$ is the pullback via $\rho_{pol} : \mathrm{Lie}_B \longrightarrow \mathfrak{v}$ of the morphism

$$\langle s\rangle_k : \mathfrak{v} \longrightarrow V_2^{\vee} \otimes_F \mathrm{Sym}^{k-1}V_2(1)$$

sending everything to zero except

$$(\mathrm{ad}e_2)^l(\mathrm{ad}e_1)^m(d_i) \quad \text{for } m + l = k - 1\,,$$

which is sent to the map

$$V_2 \ni \varepsilon_j \longmapsto \delta_{ij} \cdot 2\pi i \cdot \varepsilon_1^m \varepsilon_2^l \in \mathrm{Sym}^{k-1}V_2(1)\,.$$

Here, the factor $2\pi i$ should only be seen as a label for the Tate twist, which cannot be detected by Lie_B.
Remember that $d\langle s\rangle_k$ sends $v_1 \wedge v_2 \in \mathfrak{v}$ to $\langle s\rangle_k([v_2, v_1])$. It follows that

$$d\langle s\rangle_k : \overset{2}{\Lambda}\,\mathfrak{v} \longrightarrow V_2^{\vee} \otimes_F \mathrm{Sym}^{k-1}V_2(1)$$

annihilates everything except

$$(\mathrm{ad}e_2)^l(\mathrm{ad}e_1)^{m-1}(d_i)\wedge e_1 \quad \text{and}$$

$$(\mathrm{ad}e_2)^{l-1}(\mathrm{ad}e_1)^{m}(d_i)\wedge e_2 \quad \text{for } m+l=k-1\,,$$

both of which are sent to the map

$$V_2 \ni \varepsilon_j \longmapsto \delta_{ij}\cdot 2\pi i\cdot \varepsilon_1^m\varepsilon_2^l \in \mathrm{Sym}^{k-1}V_2(1)\,.$$

Hence we get

$$d\langle s\rangle_k = \mathrm{pr}_k^{\sim}(\langle s\rangle_{k-1}\otimes I)\,,$$

where $\mathrm{pr}_k^{\sim} := \Lambda\otimes\mathrm{id}\otimes\mathrm{mult}$ as above, and

$$I:\mathfrak{v}\longrightarrow V_2$$

is the map sending everything to zero except $e_i, i=1,2$, which is sent to ε_i. Because of axiom (F), we have $pol^1=[\Delta]$, which means precisely that we have the equality $\rho_{pol}(I)=\rho_{pol}(\langle s\rangle_1)=\{s\}_1$. q.e.d.

3.5. As an immediate consequence, we get that in any category "with elliptic polylogarithms" as in 3.1, we have a formalism as in 1.6:
Let $\mathcal{L}_1^\sharp := \mathcal{L}_1 := \mathcal{E}(B)\otimes_{\mathbb{Z}} F$,

$$\{\ \}_1^\sharp := \{\ \}_1 : \widetilde{\mathcal{E}}(B)\longrightarrow \mathcal{L}_1\,,\ s\longmapsto s\otimes 1\,,$$

$$\varphi_1^\sharp := \varphi_1 := [\ ] : \mathcal{L}_1 \longrightarrow \Gamma(B,\mathrm{Lie}_B^\vee\otimes_F V_2)\,.$$

For $k\geq 2$, let $\mathcal{L}_k^\sharp$ be the F–vector space with basis $(\{s\}_k^\sharp \mid s\in\widetilde{\mathcal{E}}(B))$.

Lemma: For $k\geq 2$, there is a commutative diagram

$$\begin{array}{ccc}
\mathcal{L}_k^\sharp & \xrightarrow{\varphi_k^\sharp} & \Gamma\left(B,\mathrm{Lie}_B^\vee\otimes_F \mathrm{Sym}^{k-2}V_2(1)\right) . \\
\{s\}_k^\sharp \mapsto \{s\}_{k-1}^\sharp\otimes\{s\}_1 \Big\downarrow & & \Big\downarrow d \\
\mathcal{L}_{k-1}^\sharp\otimes_F\mathcal{L}_1 & \xrightarrow{\frac{k-1}{k}\mathrm{pr}_k\circ(\varphi_{k-1}^\sharp\otimes\varphi_1)} & \Gamma\left(B,\overset{2}{\Lambda}\,\mathrm{Lie}_B^\vee\otimes_F\mathrm{Sym}^{k-2}V_2(1)\right)
\end{array}$$

the maps $\varphi_k^\sharp$ being given by $\{s\}_k^\sharp\longmapsto\{s\}_k$.

Define $d_1^{\sharp\sharp}:=0$, and

$$d_k^{\sharp\sharp}:\mathcal{L}_k^\sharp\longrightarrow\Gamma\left(B,\overset{2}{\Lambda}\,\mathrm{Lie}_B^\vee\otimes_F\mathrm{Sym}^{k-2}V_2(1)\right)$$

for $k\geq 2$ by

$$\{s\}_k^\sharp\longmapsto d(\{s\}_k)=\frac{k-1}{k}\mathrm{pr}_k(\{s\}_{k-1}\otimes\{s\}_1)\,.$$

Inductively, we define quotients $\mathcal{L}_k$ of $\mathcal{L}_k^\sharp$, through which the maps $\varphi_k^\sharp$ and $d_k^{\sharp\sharp}$ factor:

$$\varphi_k^\sharp : \mathcal{L}_k^\sharp \twoheadrightarrow \mathcal{L}_k \xrightarrow{\varphi_k} \Gamma\left(B, \mathrm{Lie}_B^\vee \otimes_F \mathrm{Sym}^{k-2} V_2(1)\right) ,$$

$$d_k^{\sharp\sharp} : \mathcal{L}_k^\sharp \twoheadrightarrow \mathcal{L}_k \xrightarrow{d_k} \mathcal{L}_{k-1} \otimes_F \mathcal{L}_1 \longrightarrow \Gamma\left(B, \overset{2}{\Lambda}\, \mathrm{Lie}_B^\vee \otimes_F \mathrm{Sym}^{k-2} V_2(1)\right) .$$

$\{\ \}_k : \widetilde{\mathcal{E}}(B) \longrightarrow \mathcal{L}_k$ will be defined by composition.

Let $k \geq 2$. If $\mathcal{L}_1, \ldots, \mathcal{L}_{k-1}$ are defined, we have the map

$$d_k^\sharp : \mathcal{L}_k^\sharp \longrightarrow \mathcal{L}_{k-1} \otimes \mathcal{L}_1 \ , \ \{s\}_k^\sharp \longmapsto \{s\}_{k-1} \otimes \{s\}_1 \, ,$$

and $\ker(d_k^\sharp) \subset \ker(d_k^{\sharp\sharp})$. Define

$$\mathcal{L}_k := \mathcal{L}_k^\sharp / \left(\ker(\varphi_k^\sharp) \cap \ker(d_k^\sharp)\right) .$$

This is the largest quotient of $\mathcal{L}_k^\sharp$, through which both $\varphi_k^\sharp$ and $d_k^\sharp$ factor:

$$\varphi_k : \mathcal{L}_k \longrightarrow \Gamma\left(B, \mathrm{Lie}_B^\vee \otimes_F \mathrm{Sym}^{k-2} V_2(1)\right) ,$$

$$d_k : \mathcal{L}_k \longrightarrow \mathcal{L}_{k-1} \otimes_F \mathcal{L}_1 \ , \ \{s\}_k \longmapsto \{s\}_{k-1} \otimes \{s\}_1 \, ,$$

and we conclude:

Corollary: For $k \geq 2$, there is a functorial monomorphism

$$\varphi_k = \varphi_k(\mathcal{E}) : \ker(d_k) \hookrightarrow \mathrm{Ext}^1_{\mathcal{T}(B)}\left(F(0), \mathrm{Sym}^{k-2} V_2(1)\right)$$

sending $\sum_\alpha \lambda_\alpha \{s_\alpha\}_k$ to $\sum_\alpha \lambda_\alpha \{s_\alpha\}_k$.
Also, norm compatibility for $\{\ \}_k$ with respect to d_k is satisfied: for any étale isogeny

$$\begin{array}{ccc} \mathcal{E}_1 & \xrightarrow{\ \psi\ } & \mathcal{E}_2 \\ & {\scriptstyle \pi_1} \searrow & \downarrow {\scriptstyle \pi_2} \\ & & B \end{array}$$

whose kernel consists of sections of π_1, we have:
for all $s_{1,\alpha} \in (\mathcal{E}_1 - \ker(\psi))(B)$ and $\lambda_\alpha \in F$, we have

$$d_k\left(\sum_\alpha \lambda_\alpha \{\psi(s_{1,\alpha})\}_k\right) = 0$$

if and only if

$$d_k\left(\sum_\alpha \lambda_\alpha \sum_{t \in \ker(\psi)(B)} \{s_{1,\alpha} + t\}_k\right) = 0 \, ,$$

and if this is the case, then the equality

$$\psi_* \circ \varphi_k \left(\sum_\alpha \lambda_\alpha \sum_{t \in \ker(\psi)(B)} \{s_{1,\alpha} + t\}_k \right) = \varphi_k \left(\sum_\alpha \lambda_k \{\psi(s_{1,\alpha})\}_k \right)$$

holds, where

$$\begin{aligned} &\psi_* : \operatorname{Ext}^1_{\mathcal{T}(B)}(F(0), \operatorname{Sym}^{k-2}(R^1(\pi_1)_* F(1))(1)) \\ &\longrightarrow \operatorname{Ext}^1_{\mathcal{T}(B)}(F(0), \operatorname{Sym}^{k-2}(R^1(\pi_2)_* F(1))(1)) \end{aligned}$$

is induced by the map ψ_* in axiom (C).

Proof: The first statement is just Corollary 2.2, applied to our situation. From axiom (G), it follows that we have the equality

$$\psi_* \left(\sum_{t \in \ker(\psi)(B)} \{s_1 + t\}_k \right) = \{\psi(s_1)\}_k$$

in $\Gamma\left(B, \operatorname{Lie}^\vee_B \otimes_F \operatorname{Sym}^{k-2}(R^1(\pi_2)_* F(1))(1)\right)$, for all $s_1 \in \mathcal{E}_1(B)$ disjoint from the kernel of ψ.
This shows that we have norm compatibility for $\{\ \}^\sharp_k$ with respect to

$$d^{\sharp\sharp}_k : \mathcal{L}^\sharp_k \longrightarrow \Gamma\left(B, \bigwedge^2 \operatorname{Lie}^\vee_B \otimes_F \operatorname{Sym}^{k-2} V_2(1)\right) .$$

We need to show norm compatibility fo $\{\ \}^\sharp_k$ with respect to

$$d^\sharp_k : \mathcal{L}^\sharp_k \longrightarrow \mathcal{L}_{k-1} \otimes \mathcal{L}_1 ,$$

and this we do by induction on k.
The following is necessary and sufficient for a linear combination $\sum_\alpha \lambda_\alpha \{s_\alpha\}^\sharp_k$ to belong to the kernel of $d^\sharp_k$: for any homomorphism $X : \mathcal{L}_1 \longrightarrow F$, one has

i) $d^\sharp_{k-1}(\sum_\alpha \lambda_\alpha X(s_\alpha)\{s_\alpha\}^\sharp_{k-1}) = 0$ in $\mathcal{L}_{k-2} \otimes_F \mathcal{L}_1$,
ii) $\varphi^\sharp_{k-1}(\sum_\alpha \lambda_\alpha X(s_\alpha)\{s_\alpha\}^\sharp_{k-1}) = 0$ in $\Gamma\left(B, \operatorname{Lie}^\vee_B \otimes_F \operatorname{Sym}^{k-3} V_2(1)\right)$.

Now let $\psi : \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ be an isogeny as in the statement. Observe that ψ induces an isomorphism

$$\psi : \mathcal{L}_1(\mathcal{E}_1) \xrightarrow{\sim} \mathcal{L}_1(\mathcal{E}_2) ,\ s \otimes 1 \longmapsto \psi(s) \otimes 1 .$$

Let $X : \mathcal{L}_1(\mathcal{E}_2) \longrightarrow F$ be a homomorphism, and $\psi^* X$ the induced homomorphism $\mathcal{L}_1(\mathcal{E}_1) \longrightarrow F$. So

$$\psi^* X(s_1 + t) = X(\psi(s_1))$$

for all $t \in \ker(\psi)(B)$ and $s_1 \in \mathcal{E}_1(B)$.
Now let $s_{1,\alpha} \in (\mathcal{E}_1 - \ker(\psi))(B)$ and $\lambda_\alpha \in F$. By induction, we have

$$d^\sharp_{k-1} \left(\sum_\alpha \lambda_\alpha X(\psi(s_{1,\alpha}))\{\psi(s_{1,\alpha})\}^\sharp_{k-1} \right) = 0$$

if and only if

$$d^{\sharp}_{k-1}\left(\sum_{\alpha}\lambda_\alpha X(\psi(s_{1,\alpha}))\sum_{t\in\ker(\psi)(B)}\{s_{1,\alpha}+t\}^{\sharp}_{k-1}\right)$$
$$=d^{\sharp}_{k-1}\left(\sum_{\alpha}\lambda_\alpha\sum_{t\in\ker(\psi)(B)}\psi^*X(s_{1,\alpha}+t)\{s_{1,\alpha}+t\}^{\sharp}_{k-1}\right)=0\,.$$

Norm compatibility for $\{\ \}^{\sharp}_{k-1}$ with respect to $d^{\sharp\sharp}_{k-1}$ implies that

$$d_k\left(\sum_{\alpha}\lambda_\alpha\{\psi(s_{1,\alpha})\}_k\right)=0$$

if and only if

$$d_k\left(\sum_{\alpha}\lambda_\alpha\sum_{t\in\ker(\psi)(B)}\{s_{1,\alpha}+t\}_k\right)=0\,.$$

Observe that for $k-1=1$, we have norm compatibility of $\{\ \}^{\sharp}_{k-1}$ up to a factor $\deg(\psi)$, which is just as well for our purposes.
Because of norm compatibility for $\{\ \}^{\sharp}_{k}$ with respect to $d^{\sharp\sharp}_{k}$, and since $\ker(d^{\sharp}_{k})$ is contained in $\ker(d^{\sharp\sharp}_{k})$, our claim is proven. q.e.d.

3.6. The careful reader will have observed that our construction does not a priori detect all the elements of $\mathcal{L}^{\sharp}_{k}$, $k\geq 2$, which are homologically meaningful, i.e., which lie in the kernel of

$$d^{\sharp\sharp}_{k}:\mathcal{L}^{\sharp}_{k}\longrightarrow\Gamma\left(B,\overset{2}{\Lambda}\,\mathrm{Lie}^{\vee}_{B}\otimes_F\mathrm{Sym}^{k-2}V_2(1)\right)\,.$$

For the sake of completeness, we therefore give a condition, which ensures that

$$\ker(d^{\sharp}_{k})=\ker(d^{\sharp\sharp}_{k})\,,$$

i.e., that the map

$$pr_k\circ(\varphi_{k-1}\otimes\varphi_1):\mathcal{L}_{k-1}\otimes_F\mathcal{L}_1\longrightarrow\Gamma\left(B,\overset{2}{\Lambda}\,\mathrm{Lie}^{\vee}_{B}\otimes_F\mathrm{Sym}^{k-2}V_2(1)\right)$$

is injective on the image of

$$d_k:\mathcal{L}_k\longrightarrow\mathcal{L}_{k-1}\otimes_F\mathcal{L}_1\,.$$

It will be fulfilled in particular for any category satisfying the axioms of the next paragraph:

Proposition: Assume that $F \subset \mathbb{R}$, and that the morphism

$$[\ ] : \mathcal{E}(B) \otimes_{\mathbb{Z}} F \longrightarrow \mathrm{Ext}^1_{\mathcal{T}(B)} \left(F(0), R^1\pi_* F(1) \right)$$

is injective. Assume furthermore that B has a $\mathbb{C}$–valued point, and that there is an F-linear tensor functor

$$\mathcal{T}(B) \longrightarrow \mathrm{Var}_{\mathbb{R}}(\mathbb{C})$$

into the category of mixed graded–polarizable $\mathbb{R}$–Hodge structures compatible with $[\ ]$, $\cup_\pi$ and $W.$, and mapping $F(1)$ to $\mathbb{R}(1)$, $R^1\pi_* F$ to $R^1\pi_*\mathbb{R}$, and *pol* to *pol*.
Then $\ker(d_k^\sharp) = \ker(d_k^{\sharp\sharp})$.

Remark: The Abel–Jacobi map for elliptic curves over $\mathbb{C}$ induces an isomorphism

$$[\ ] : E(\mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathrm{Ext}^1_{\mathrm{Var}_{\mathbb{Q}}(\mathbb{C})} \left(\mathbb{Q}(0), R^1\pi_*\mathbb{Q}(1) \right) .$$

It follows that the above assumptions are satisfied in particular if $F = \mathbb{Q}$, if B is integral, if its generic point admits an embedding into $\mathbb{C}$, and if there is a $\mathbb{Q}$–linear tensor functor

$$\mathcal{T}(B) \longrightarrow \mathrm{Var}_{\mathbb{Q}}(\mathbb{C})$$

compatible with the above structures.

Proof: The map $pr_k \circ (\varphi_{k-1} \otimes \varphi_1)$ factors over

$$\Gamma \left(B, \mathrm{Lie}^\vee_B \otimes_{\mathbb{Q}} \mathrm{Sym}^{k-3} V_2(1) \right) \otimes_{\mathbb{Q}} \Gamma(B, \mathrm{Lie}^\vee_B \otimes_{\mathbb{Q}} V_2) ,$$

and the proof will therefore consist of two parts. We shall show:

i) The map $\varphi_{k-1} \otimes \varphi_1$ to the above tensor product is injective on $\mathrm{im}(d_k)$.
ii) The map pr_k from the above tensor product to

$$\Gamma \left(B, \bigwedge^2 \mathrm{Lie}^\vee_B \otimes_{\mathbb{Q}} \mathrm{Sym}^{k-2} V_2(1) \right)$$

is injective on $\mathrm{im}((\varphi_{k-1} \otimes \varphi_1) \circ d_k)$.

By assumption, the map $\varphi_1 = [\ ]$ is injective. Therefore, the claim is trivial for $k = 2$. So let $k \geq 3$.

We have for $\sum_\alpha \lambda_\alpha \{s_\alpha\}_k \in \mathcal{L}_k$:

$$d_k \left(\sum_\alpha \lambda_\alpha \{s_\alpha\}_k \right) \in \ker(\varphi_{k-1} \otimes \varphi_1)$$

$\iff$ for all $X : \mathcal{L}_1 \longrightarrow \mathbb{Q}$:

$$\sum_\alpha \lambda_\alpha X(s_\alpha) \{s_\alpha\}_{k-1} \in \ker(\varphi_{k-1})$$

$\Longrightarrow$ for all $X : \mathcal{L}_1 \longrightarrow \mathbb{Q}$:

$$d_{k-1}\left(\sum_\alpha \lambda_\alpha X(s_\alpha)\{s_\alpha\}_{k-1}\right) \in \ker(\varphi_{k-2} \otimes \varphi_1)$$

$\Longleftrightarrow$ for all $X : \mathcal{L}_1 \longrightarrow \mathbb{Q}$:

$$d_{k-1}\left(\sum_\alpha \lambda_\alpha X(s_\alpha)\{s_\alpha\}_{k-1}\right) = 0\,,$$

the last equivalence being our induction hypothesis. But by definition, we have $\ker(d_{k-1}) \cap \ker(\varphi_{k-1}) = 0$.
Therefore, $d_k(\sum_\alpha \lambda_\alpha\{s_\alpha\}_k) \in \ker(\varphi_{k-1} \otimes \varphi_1)$ implies that for all $X : \mathcal{L}_1 \longrightarrow \mathbb{Q}$, we have

$$\sum_\alpha \lambda_\alpha X(s_\alpha)\{s_\alpha\}_{k-1} = 0\,,$$

which of course means that $d_k(\sum_\alpha \lambda_\alpha\{s_\alpha\}_k) = 0$.

For ii), we have

$$\operatorname{im}(\varphi_k) \subset \Gamma\left(B, \operatorname{Lie}_B^\vee \otimes_{\mathbb{Q}} \operatorname{Sym}^{k-2} V_2(1)\right) = \operatorname{Hom}_{\mathcal{T}(B)}\left(\operatorname{Lie}_B, \operatorname{Sym}^{k-2} V_2(1)\right)\,.$$

Our claim follows as soon as we prove:
Let $L \in \mathcal{T}(B)$ be any object, and

$$\mathcal{L}_k^\flat \subset \operatorname{Hom}_{\mathcal{T}(B)}\left(L, \operatorname{Sym}^{k-2} V_2(1)\right)$$

any subspace. Then for $k \geq 3$, the maps

$$\begin{aligned} \mathcal{L}_{k-1}^\flat \otimes_F \mathcal{L}_1^\flat &\longrightarrow \operatorname{Hom}_{\mathcal{T}(B)}\left(\stackrel{2}{\Lambda} L, \operatorname{Sym}^{k-2} V_2(1)\right)\,, \\ f \otimes g &\longmapsto pr_k(f \otimes g) \end{aligned}$$

are injective. For $k = 2$, the map $f \otimes g \mapsto pr_k(f \otimes g)$ is injective on $\operatorname{Sym}^2 \mathcal{L}_1^\flat$.

Observe that any tensor functor of neutral Tannakian categories is automatically faithful. So for the proof of the claim, we may work in the category $\operatorname{Var}_{\mathbb{R}}(\mathbb{C})$, and assume that

$$\mathcal{L}_k^\flat = \operatorname{Hom}_{\operatorname{Var}_{\mathbb{R}}(\mathbb{C})}\left(L, \operatorname{Sym}^{k-2} V_2(1)\right)\,.$$

$\operatorname{Var}_{\mathbb{R}}(\mathbb{C})$ is semisimple, and V_2 is a simple object. For $k = 2$, we may assume that $L = \bigoplus_{i=1}^n V_2$. So $\mathcal{L}_1^\flat$ has a basis $(p_1, \ldots, p_n)$, where p_i is the projection onto the i–th component. Also, we see that $\operatorname{Sym}^2 \mathcal{L}_1^\flat = \langle p_i p_j \,|\, i \leq j \rangle_{\mathbb{R}}$, while $\stackrel{2}{\Lambda} L = \bigoplus_{i=1}^n \stackrel{2}{\Lambda} V_2 \oplus \bigoplus_{i<j} (V_2 \otimes_{\mathbb{R}} V_2) = \bigoplus_{i=1}^n \mathbb{R}(1) \oplus \bigoplus_{i<j} (V_2 \otimes_{\mathbb{R}} V_2)$. The morphism

$$\operatorname{Sym}^2 \mathcal{L}_1^\flat \hookrightarrow \mathcal{L}_1^\flat \otimes_{\mathbb{R}} \mathcal{L}_1^\flat \longrightarrow \operatorname{Hom}_{\operatorname{Var}_{\mathbb{R}}(\operatorname{Spec}(\mathbb{C}))}\left(\stackrel{2}{\Lambda} L, \mathbb{R}(1)\right)$$

maps $p_i p_j$ to $(\stackrel{2}{\Lambda} L \twoheadrightarrow \mathbb{R}(1)_i)$ or to $(\stackrel{2}{\Lambda} L \twoheadrightarrow (V_2 \otimes_{\mathbb{R}} V_2)_{i,j} \twoheadrightarrow \mathbb{R}(1))$ according to whether $i = j$ or $i < j$. It is therefore visibly injective.

Now let $k \geq 3$. Then no direct summand of $\mathrm{Sym}^{k-3}V_2(1)$ is isomorphic to V_2. We see that we may assume that every simple summand of L is isomorphic to one of $\mathrm{Sym}^{k-3}V_2(1)$ or V_2, that we only need to show injectivity of

$$\mathcal{L}^\flat_{k-1} \otimes_{\mathbb{R}} \mathcal{L}^\flat_1 \longrightarrow \mathrm{Hom}_{\mathrm{Var}_{\mathbb{R}}(\mathbb{C})}\left(W_{k-1} \otimes_{\mathbb{R}} W_1, \mathrm{Sym}^{k-2}V_2(1)\right) ,$$

where W_{k-1} is the sum of the subobjects of L isomorphic to one of $\mathrm{Sym}^{k-3}V_2(1)$, and W_1 is the V_2–isotypical component of L, and finally, that we may assume that

$$W_1 = V_2 \quad \text{and} \quad W_{k-1} = \mathrm{Sym}^{k-3}V_2(1)\,,$$

hence

$$\mathcal{L}^\flat_1 = \mathbb{R} \quad \text{and} \quad \mathcal{L}^\flat_{k-1} = \mathrm{End}_{\mathrm{Var}_{\mathbb{R}}(\mathbb{C})}\left(\mathrm{Sym}^{k-3}V_2(1)\right) .$$

So our claim amounts to saying that the endomorphisms induced by projection onto the simple summands W of $\mathrm{Sym}^{k-3}V_2(1)$

$$\mathrm{Sym}^{k-3}V_2(1) \otimes_{\mathbb{R}} V_2 \longrightarrow\!\!\!\!\!\rightarrow W \otimes_{\mathbb{R}} V_2 \hookrightarrow \mathrm{Sym}^{k-3}V_2(1) \otimes_{\mathbb{R}} V_2\,,$$

composed with

$$\mathrm{mult}\ : \mathrm{Sym}^{k-3}V_2(1) \otimes_{\mathbb{R}} V_2 \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Sym}^{k-2}V_2(1)\,,$$

give linearly independent morphisms in

$$\mathrm{Hom}_{\mathrm{Var}_{\mathbb{R}}(\mathbb{C})}\left(\mathrm{Sym}^{k-3}V_2(1) \otimes_{\mathbb{R}} V_2, \mathrm{Sym}^{k-2}V_2(1)\right) .$$

For this, it clearly suffices to show that no

$$W \otimes_{\mathbb{R}} V_2 \hookrightarrow \mathrm{Sym}^{k-3}V_2(1) \otimes_{\mathbb{R}} V_2 \stackrel{\mathrm{mult}}{\longrightarrow\!\!\!\!\!\rightarrow} \mathrm{Sym}^{k-2}V_2(1)$$

is zero.

Now for any such W, there is a unique a between 0 and $[\frac{k-3}{2}]$ such that W is the simple $\mathbb{R}$–Hodge structure of type $\{(-a-1, a+2-k), (a+2-k, -a-1)\}$. If h is a base vector of $H^{0,-1}(V_2 \otimes_{\mathbb{R}} \mathbb{C})$, i.e., $V_2 \otimes_{\mathbb{R}} \mathbb{C} = \langle h, \overline{h} \rangle_{\mathbb{C}}$, then $W \subset \mathrm{Sym}^{k-3}V_2(1)$ is generated by $2\pi i h^a \overline{h}^{k-3-a}$ and $2\pi i h^{k-3-a} \overline{h}^a$. The image of $W \otimes_{\mathbb{R}} V_2$ under mult is therefore generated by non–zero vectors. q.e.d.

§ 4 Motivic proof of Conjecture 1.6

4.1. In this final paragraph we let K denote a number field, and $S = \mathrm{Spec}(K)$ or $S = \mathrm{Spec}(O_K)$. We shall show that parts 1 and 2 of Conjecture 1.6.(B) are implied by the following

Conjecture: There exists a $\mathbb{Q}$–linear theory $\mathcal{MM}^s$ of smooth motivic sheaves satisfying axioms (A) to (G) in 3.1, and in addition

1) There are canonical isomorphisms

$$H_{\mathcal{M}}^{k-1}(\mathcal{E}^{(k-2)}, \mathbb{Q}(k-1))_{\mathrm{sgn}} \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{MM}^s(B)}(\mathbb{Q}(0), \mathrm{Sym}^{k-2} V_2(1))$$

for any elliptic curve

$$\pi : \mathcal{E} \longrightarrow B\,,$$

which are compatible with pullbacks and maps ψ_* for isogenies ψ.

2) Similarly, there is a canonical isomorphism

$$H^2_{\mathcal{M}}(\mathcal{E}, \mathbb{Q}(1))_- \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{MM}^s(B)}(\mathbb{Q}(0), V_2)\,,$$

such that [ ] becomes the isomorphism in Lemma 1.1.

3) There is a $\mathbb{Q}$–linear tensor functor

$$\mathrm{real}(B) : \mathcal{MM}^s(B) \longrightarrow \mathrm{Var}_{\mathbb{Q}}(B \otimes_{\mathbb{Q}} \mathbb{C})$$

compatible with change of the base B, mapping $\mathbb{Q}(1)$ to $\mathbb{Q}(1)$ and $R^1\pi_*\mathbb{Q}$ to $R^1\pi_*\mathbb{Q}$, and compatible with [ ], $\cup_\pi$, $W_.$, and ψ_*. It is called the Hodge realization. Of course, the Hodge realization of $pol \in \mathcal{MM}^s(\widetilde{\mathcal{E}})$ is supposed to be the variation $pol \in \mathrm{Var}_{\mathbb{Q}}(\widetilde{\mathcal{E}} \otimes_{\mathbb{Q}} \mathbb{C})$.

4) For $B = \mathrm{Spec}(K)$, we require that the diagram

$$\begin{array}{ccc} H_{\mathcal{M}}^{k-1}(E^{(k-2)}, \mathbb{Q}(k-1))_{\mathrm{sgn}} & \xrightarrow{\mathrm{real} \otimes \mathbb{R}} & \mathrm{Ext}^1\left(\mathbb{R}(0), \mathrm{Sym}^{k-2} V_2(1)\right) \\ {\scriptstyle r_{\mathcal{D}}}\downarrow & & \downarrow\wr \\ H_{\mathcal{D}}^{k-1}(E^{(k-2)} \otimes_{\mathbb{Q}} \mathbb{R}, \mathbb{R}(k-1))_{\mathrm{sgn}} & \xhookrightarrow{1.2} & \mathrm{Sym}^{k-2} H^1_B(E \otimes_{\mathbb{Q}} \mathbb{C}, 2\pi i\,\mathbb{R}) \end{array}$$

be commutative.
Here, the upper right group is the group of one–extensions in the category $\mathrm{Var}_{\mathbb{R}}(\mathrm{Spec}(K) \otimes_{\mathbb{Q}} \mathbb{C})$, and the right vertical map is the isomorphism of

[J2], Lemma 9.2. Its inverse sends $h \in \mathrm{Sym}^{k-2} H^1_B(E \otimes_{\mathbb{Q}} \mathbb{C}, 2\pi i\, \mathbb{R})$ to the extension described by the matrix

$$\begin{pmatrix} 1 & 0 \\ -h & \mathrm{id} \end{pmatrix} .$$

This means that we equip $\mathbb{C} \oplus \mathrm{Sym}^{k-2} H^1_B(E \otimes_{\mathbb{Q}} \mathbb{C}, \mathbb{C})$ with the trivial weight and Hodge filtrations, and the real structure extending the real structure $\mathrm{Sym}^{k-2} H^1_B(E \otimes_{\mathbb{Q}} \mathbb{C}, \mathbb{R}(1))(1)$ by the vector $1-h$. Note that this convention of normalizing the above isomorphism differs from the one of [BD], 1.6, by the factor -1.

4.2. We recall the description of the $\mathbb{R}$–Hodge version of the elliptic polylogarithm for a curve E over $\mathbb{C}$. Choose an isomorphism

$$\eta : E(\mathbb{C}) \stackrel{\sim}{\longrightarrow} \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau) .$$

Via η, we may view E as a fibre of the universal elliptic curve over some modular curve. Then *pol* is just the fibre of the small polylogarithmic extension over the universal elliptic curve. By [J2], Lemma 9.2, the logarithmic variation with coefficients in $\mathbb{R}$ splits at *any* point, since

$$\mathrm{Ext}^1_{\mathrm{Var}_{\mathbb{R}}(\mathrm{Spec}(\mathbb{C}))}(\mathbb{R}(0), V_2) = (V_2 \otimes_{\mathbb{R}} \mathbb{C})/(V_2 + F^0(V_2 \otimes_{\mathbb{R}} \mathbb{C})) = 0 .$$

This in turn means that at any point $s \in \widetilde{E}(\mathbb{C})$, we have a push out map

$$\mathrm{Ext}^1_{\mathrm{Var}_{\mathbb{R}}(\mathrm{Spec}(\mathbb{C}))}(V_2, \mathcal{L}\mathrm{og}(1)) \longrightarrow \mathrm{Ext}^1_{\mathrm{Var}_{\mathbb{R}}(\mathrm{Spec}(\mathbb{C}))}(V_2, \mathrm{Sym}^{k-1} V_2(1)) .$$

In homological terms, this means that all the differentials in $\mathrm{Var}_{\mathbb{R}}(\mathrm{Spec}(\mathbb{C}))$ map $\{s\}_k$ to zero.

Theorem: For $\pi : E \longrightarrow \mathrm{Spec}(\mathbb{C})$ an elliptic curve, $\mathcal{T}(\mathbb{C}) = \mathrm{Var}_{\mathbb{R}}(\mathrm{Spec}(\mathbb{C}))$ and any point $s \in \widetilde{E}(\mathbb{C})$, we have for $k \geq 2$:

$$\varphi_k(\{s\}_k) \in \mathrm{Ext}^1_{\mathcal{T}(\mathbb{C})}(\mathbb{R}(0), \mathrm{Sym}^{k-2} V_2(1))$$

is mapped to

$$G_{E,k}(s) \in \mathrm{Sym}^{k-2} H^1_B(E(\mathbb{C}), 2\pi i\, \mathbb{R})$$

under the isomorphism of [J2], Lemma 9.2.

Proof: [BL], 3.3–3.6, or [W2], Corollary 4.10.a). q.e.d.

In particular, this result gives an alternative, and more conceptual proof of Proposition 1.3:
First of all, the element in the last line is well defined. Furthermore, the universal polylogarithmic extension is a variation, that is fixed under the involution ι coming from complex conjugation on the base and interchanging the Hodge filtration and its conjugate. This follows for example from the fact that there is

a Hodge–de Rham version of *pol*.
As a consequence, we get that for the complex conjugate $\overline{s}$ of s, we have

$$\varphi_k(\{\overline{s}\}_k) = \overline{\varphi_k(\{s\}_k)}\,.$$

Of course, this transforms into saying that the value of $G_{\overline{E},k}$ at $\overline{s}$ is the conjugate of $G_{E,k}(s)$.

4.3. Before showing that parts 1 and 2 of Conjecture 1.6.(B) are implied by Conjecture 4.1, let us indicate why the component of $c_{y,x}$ obtained by the G–equivariant epimorphism

$$V_2^\vee \otimes_F \mathrm{Sym}^{k-1} V_2 \xrightarrow{\sim} V_2(-1) \otimes_F \mathrm{Sym}^{k-1} V_2 \overset{\mathrm{mult}}{\twoheadrightarrow} \mathrm{Sym}^k V_2(-1)\,,$$

for $k \geq 2$, should not be expected to give non–trivial extensions in

$$\mathrm{Ext}^1_{\mathcal{MM}^s(B)}(\mathbb{Q}(0), \mathrm{Sym}^k V_2) :$$

Theorem: Let B be smooth over $\mathbb{C}$, and $\pi : \mathcal{E} \longrightarrow B$ an elliptic curve.
Let $F \subset \mathbb{R}$ be a field, and $k \geq 3$.
If π is not isotrivial, then any extension in $\mathrm{Ext}^1_{\mathrm{Var}_F(B)}(F(0), \mathrm{Sym}^k V_2)$ is split. If π is trivial, i.e., there is a cartesian diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{f} & E \\ \pi\big\downarrow & & \big\downarrow \\ B & \xrightarrow{f} & \mathrm{Spec}(\mathbb{C}) \end{array}$$

then any extension in $\mathrm{Ext}^1_{\mathrm{Var}_F(B)}(F(0), \mathrm{Sym}^k V_2)$ is constant, i.e., lies in the image of f^*.

Proof: Clearly we may replace B by a finite étale covering, so it constitutes no loss of generality to assume that π is the base change of some elliptic curve/the universal elliptic curve via a morphism with open image

$$f : B \longrightarrow M$$

to $M = \mathrm{Spec}(\mathbb{C})$/some modular curve M, depending on whether π is trivial or not.

It follows from the Leray spectral sequence for $B \longrightarrow \mathbb{C}$, the faithfulness of the forgetful functor of MHM_F to the category of perverse sheaves, and its compatibility with higher direct and inverse images that we have a rigidity property for extensions:
Any one–extension of $F(0)$ by an admissible variation of weights unequal to zero on a connected smooth variety splits as soon as the underlying extension of local systems, and the extension of Hodge structures at one point split.

Therefore, we may replace B by some open dense subscheme. So we may assume that $f : B \longrightarrow f(B)$ is smooth and equidimensional, that all cohomology objects of the complexes of topological sheaves Rf_*F and $Rf_*\mathrm{Sym}^k V_2$ are locally

constant, and that the number n of components of the fibres is constant.
It follows that if d is the relative dimension of f, then the Hodge module $\mathcal{H}^{-d+1} f_* \mathrm{Sym}^k V_2$ equals $\mathcal{H}^{-d+1} f_* F(0) \otimes_F \mathrm{Sym}^k V_2$, with $\mathcal{H}^{-d+1} f_* F(0)$ an admissible variation with underlying local system $R^1 f_* F$. Since f is smooth, the only weights possibly occurring in $\mathcal{H}^{-d+1} f_* F(0)$ are one and two. So $\mathcal{H}^{-d+1} f_* \mathrm{Sym}^k V_2$ has weights smaller than zero.
Now, we apply the Grothendieck spectral sequence for f to see that any extension of $F(0)$ by $\mathrm{Sym}^k V_2$ on B comes from an extension of $F(0)$ by the Hodge module $\mathcal{H}^{-d} f_* \mathrm{Sym}^k V_2 = \mathcal{H}^{-d} f_* F(0) \otimes_F \mathrm{Sym}^k V_2$ on $f(B)$.
This clearly takes care of the trivial case.

So assume we have an open immersion $j : f(B) \hookrightarrow M \hookrightarrow \overline{M}$, where $\overline{M}$ is the smooth compactification of M. The variation of Hodge structure $\mathcal{H}^{-d} f_* F(0)$ has rank n and is pure of weight zero, and $\pi_1(f(B))$ acts on the underlying local system via a finite quotient.
Thus we have $H^0(f(B), \mathcal{H}^{-d} f_* \mathrm{Sym}^k V_2) = 0$, and the Leray spectral sequence for the structural morphism gives an injection of $\mathrm{Ext}^1_{\mathrm{Var}_F(f(B))}(F(0), \mathcal{H}^{-d} f_* \mathrm{Sym}^k V_2)$ into

$$\mathrm{Hom}_{\mathrm{Var}_F(\mathrm{Spec}(\mathbb{C}))} \Big(F(0), H^1(f(B), \mathcal{H}^{-d} f_* \mathrm{Sym}^k V_2) \Big) \, .$$

Let us show that weight zero does not occur in $H^1(f(B), \mathcal{H}^{-d} f_* \mathrm{Sym}^k V_2)$.
For this, we use the exact cohomology sequence associated to p_* of the exact triangle

$$\begin{array}{ccc} j_! & \longrightarrow & j_* \\ [1] \nwarrow & & \swarrow \\ & i_* i^* j_* & \end{array}$$

in the derived category of algebraic mixed Hodge modules, where p denotes the structural morphism of $\overline{M}$. It gives an exact sequence

$$H^1_! \longrightarrow H^1(f(B), \mathcal{H}^{-d} f_* \mathrm{Sym}^k V_2) \longrightarrow H^0(\overline{M} - f(B), \mathcal{H}^0 j_* \mathcal{H}^{-d} f_* \mathrm{Sym}^k V_2)$$

of Hodge structures, where $H^1_! := H^1_!(f(B), \mathcal{H}^{-d} f_* \mathrm{Sym}^k V_2)$ is the image of cohomology with compact support. Recall that the vector space underlying the fibre of $\mathcal{H}^0 j_* \mathcal{H}^{-d} f_* \mathrm{Sym}^k V_2$ at a point $P \in \overline{M} - f(B)$ is the space of local coinvariants at P of $\mathrm{Sym}^k V_2$.
The first of these Hodge structures is pure of weight $1-k$, and the third is a direct sum of a pure Hodge structure of weight $2-k$, corresponding to the points of $M - f(B)$, and a pure Hodge structure of weight 2, corresponding to the cusps. We leave the details to the reader. q.e.d.

Clearly, a statement analogous to our theorem should hold for motivic sheaves. So any non–trivial extension will come from an extension on elliptic curves E over $B = \mathrm{Spec}(K)$.

Let e be an element of the group

$$\mathrm{Ext}^1_{\mathcal{MM}^s(\mathrm{Spec}(K))}(\mathbb{Q}(0), \mathrm{Sym}^k V_2) \, ,$$

which we think of as being isomorphic to

$$H^{k+1}_{\mathcal{M}}(E^{(k)}, \mathbb{Q}(k))_{\rm sgn} \ ,$$

and assume that e lies in fact in $H^{k+1}_{\mathcal{M}}(E^{(k)}, \mathbb{Q}(k))_{\mathbb{Z},{\rm sgn}}$. The regulator $r_{\mathcal{D}}$ maps e to $H^{k+1}_{\mathcal{D}}(E^{(k)} \otimes_{\mathbb{Q}} \mathbb{R}, \mathbb{R}(k))_{\rm sgn}$, which, as in Remark 1.2.b), is contained in

$$\mathrm{Ext}^1_{\mathrm{Var}_{\mathbb{R}}(K \otimes_{\mathbb{Q}} \mathbb{C})}(\mathbb{R}(0), \mathrm{Sym}^k V_2 \otimes_{\mathbb{Q}} \mathbb{R}) \, .$$

By [W2], Corollary 4.10.a), all extensions in this group obtained from evaluating the elliptic polylogarithm at points of $\widetilde{E} \otimes_{\mathbb{Q}} \mathbb{C}$ are trivial.
Since $k \geq 3$, the Beilinson conjectures predict that

$$r_{\mathcal{D}} : H^{k+1}_{\mathcal{M}}(E^{(k)}, \mathbb{Q}(k))_{\mathbb{Z},{\rm sgn}} \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow H^{k+1}_{\mathcal{D}}(E^{(k)} \otimes_{\mathbb{Q}} \mathbb{R}, \mathbb{R}(k))_{\rm sgn}$$

is an isomorphism and hence, that all elements in $H^{k+1}_{\mathcal{M}}(E^{(k)}, \mathbb{Q}(k))_{\mathbb{Z},{\rm sgn}}$ constructed via the elliptic polylogarithm are zero.

So assume finally that $\mathcal{E} \longrightarrow B$ is a trivial family. Let e be an element of

$$H^{k+1}_{\mathcal{M}}(\mathcal{E}^{(k)}, \mathbb{Q}(k))_{\mathbb{Z},{\rm sgn}} \ ,$$

which we think of being contained in

$$\mathrm{Ext}^1_{\mathcal{MM}^s(B)}(\mathbb{Q}(0), \mathrm{Sym}^k V_2) \, ,$$

and assume that e is obtained via the formalism of § 3. Because of the theorem, the underlying extension of local systems is trivial. The above predicts that all pointwise extensions split as well. So by rigidity, which one should also expect to hold in the motivic context, the extension e must be trivial.

4.4. For the sake of completeness, we also mention the following

Lemma: Let $k \geq 3$, and

$$\begin{array}{ccc} \mathcal{E} & \stackrel{f}{\longrightarrow} & \mathcal{E}_C \\ \pi \downarrow & & \pi_C \downarrow \\ B & \stackrel{f}{\longrightarrow} & C \end{array}$$

a cartesian diagram of smooth schemes over $\mathbb{C}$, with π_C a family of elliptic curves, and B of pure dimension $d+1$.
Assume the Hodge module $\mathcal{H}^{-d+1} f_* \mathrm{Sym}^{k-2} V_2(1)$ has weights smaller than zero, and that

$$\mathcal{H}^{-d} f_* \mathrm{Sym}^{k-2} V_2(1) = \mathrm{Sym}^{k-2} V_2(1) \, .$$

Then the natural map

$$\mathrm{Ext}^1_{\mathrm{Var}_F(C)} \Big(\mathbb{Q}(0), \mathrm{Sym}^{k-2} V_2(1) \Big) \longrightarrow \mathrm{Ext}^1_{\mathrm{Var}_F(B)} \Big(\mathbb{Q}(0), \mathrm{Sym}^{k-2} V_2(1) \Big)$$

is an isomorphism.

Proof: Apply the Grothendieck spectral sequence for f. q.e.d.

The first part of the proof of Theorem 4.3 shows that given a non–isotrivial $\pi : \mathcal{E} \longrightarrow B$, the hypothesis of the lemma is met for a Galois covering of a Zariski–dense open subset U of B. It follows that any extension of $\mathbb{Q}(0)$ by $\mathrm{Sym}^{k-2}V_2(1)$ on B is obtained, via pullback, Galois descent, and extension across $B - U$, from an extension on a curve.

4.5. We conclude with the following

Theorem: Conjecture 4.1 implies parts 1 and 2 of Conjecture 1.6.(B). The homomorphism

$$\varphi_k^\sharp : \ker(d_k^\sharp) \longrightarrow H_{\mathcal{M}}^{k-1}(\mathcal{E}^{(k-2)}, \mathbb{Q}(k-1))_{\mathrm{sgn}} = \mathrm{Ext}^1_{\mathcal{MM}^s(B)}(\mathbb{Q}(0), \mathrm{Sym}^{k-2}V_2(1))$$

is given by sending

$$\sum_\alpha \lambda_\alpha \{s_\alpha\}_k^\sharp \quad \text{to} \quad \sum_\alpha \lambda_\alpha \{s_\alpha\}_k$$

(in the notation of Corollary 3.5).

Proof: Part 1 of the conjecture follows from 3.5, and the axioms in 4.1. Part 2 (absolute case) is Theorem 4.2. q.e.d.

References

[B] A.A. Beilinson, "Higher regulators of modular curves", Contemp. Math. 55, Part I, AMS 1986, pp. 1–34.

[BD] A.A. Beilinson, P. Deligne, "Interprétation motivique de la conjecture de Zagier reliant polylogarithmes et régulateurs", in U. Jannsen, S.L. Kleiman, J.-P. Serre, "Motives", Proc. of Symp. in Pure Math. 55, Part II, AMS 1994, pp. 97–121.

[BL] A.A. Beilinson, A. Levin, "The Elliptic Polylogarithm", in U. Jannsen, S.L. Kleiman, J.-P. Serre, "Motives", Proc. of Symp. in Pure Math. 55, Part II, AMS 1994, pp. 123–190.

[D1] P. Deligne, "Formes modulaires et représentations l–adiques", LNM 179, Springer–Verlag 1968, pp. 139–172.

[D2] P. Deligne, "La Conjecture de Weil.II", Publ. Math. IHES 52 (1981), pp. 313–428.

[DM] P. Deligne, J.S. Milne, "Tannakian Categories", in P. Deligne, J.S. Milne, A. Ogus, K.-y. Shih, "Hodge Cycles, Motives, and Shimura varieties", LNM 900, Springer–Verlag 1982, pp. 101–228.

[DG] M. Demazure, P. Gabriel, "Groupes Algébriques", Tôme 1, North–Holland Publ. Comp. 1970.

[De1] C. Deninger, "Higher regulators and Hecke L–series of imaginary quadratic fields, I", Inv. math. 96 (1989), pp. 1–69.

[De2] C. Deninger, "Higher regulators and Hecke L–series of imaginary quadratic fields, II", Ann. of Math. 132 (1990), pp. 131–158.

[E] T. Ekedahl, "On the Adic Formalism", in P. Cartier et al. (eds.), "The Grothendieck Festschrift", Volume II, Birkhäuser 1990, pp. 197-218.

[G1] A. Goncharov, "Polylogarithms and Motivic Galois Groups", in U. Jannsen, S.L. Kleiman, J.-P. Serre, "Motives", Proc. of Symp. in Pure Math. 55, Part II, AMS 1994, pp. 43–96.

[G2] A. Goncharov, "The Eisenstein–Kronecker series and L–function of symmetric square of an elliptic curve at $s = 2$", preprint, october 1995.

[GL] A. Goncharov, A. Levin, "Zagier's conjecture on $L(E, 2)$", preprint, june 1996.

[J1] U. Jannsen, "Deligne homology, Hodge–$\mathcal{D}$–conjecture, and motives", in M. Rapoport, N. Schappacher, P. Schneider, "Beilinson's Conjectures on Special Values of L–Functions", Perspectives in Mathematics 4, Academic Press 1988, pp. 305–372.

[J2] U. Jannsen, “Mixed Motives and Algebraic K–Theory”, LNM 1400, Springer–Verlag 1990.

[Ka] M. Kashiwara, “A Study of Variation of Mixed Hodge Structure”, Publ. RIMS, Kyoto Univ. 22 (1986) pp. 991–1024.

[R] K. Rolshausen, “Eléments explicites dans K_2 d’une courbe elliptique”, thèse, Prépublication de l’Institut de Recherche Mathématique avancée 1996/05.

[S1] Morihiko Saito, “Modules de Hodge Polarisables”, Publ. RIMS, Kyoto Univ. 24 (1988), pp. 849–995.

[S2] Morihiko Saito, “Mixed Hodge Modules”, Publ. RIMS, Kyoto Univ. 26 (1990), pp. 221–333.

[SaSo] N. Schappacher, A.J. Scholl, “The boundary of the Eisenstein symbol”, Math. Ann. 290 (1991), pp. 303–321, 815.

[Si] J.H. Silverman, “Advanced Topics in the Arithmetic of Elliptic Curves”, GTM 151, Springer–Verlag 1994.

[Sn] P. Schneider, “Introduction to the Beilinson conjectures”, in M. Rapoport, N. Schappacher, P. Schneider, “Beilinson’s Conjectures on Special Values of L–Functions”, Perspectives in Mathematics 4, Academic Press 1988, pp. 1–35.

[So] A.J. Scholl, “Motives for modular forms”, Inv. math. 100 (1990), pp. 419–430.

[W1] J. Wildeshaus, “Mixed structures on fundamental groups”, in “Realizations of Polylogarithms”, submitted as Springer LNM.

[W2] J. Wildeshaus, “Polylogarithmic Extensions on Mixed Shimura varieties. Part III: The elliptic polylogarithm”, in “Realizations of Polylogarithms”, submitted as Springer LNM.

[W3] J. Wildeshaus, “Modular–elliptic units”, in preparation.

[Za] D. Zagier, “The Bloch–Wigner–Ramakrishnan polylogarithm function”, Math. Ann. 286 (1990), pp. 613–624.