Rigidity of K-theory under deformation quantization

Jonathan Rosenberg*

Dedicated to Calvin C. Moore on his 60th birthday

Abstract

Quantization, at least in some formulations, involves replacing some algebra of observables by a (more non-commutative) deformed algebra. In view of the fundamental role played by K-theory in non-commutative geometry and topology, it is of interest to ask to what extent K-theory remains "rigid" under this process. We show that some positive results can be obtained using ideas of Gabber, Gillet-Thomason, and Suslin. From this we derive that the algebraic K-theory with finite coefficients of a deformation quantization of the functions on a compact symplectic manifold, *forgetting the topology*, recovers the topological K-theory of the manifold.

Key words: deformation quantization, star-product, algebraic K-theory, K-theory with finite coefficients, power series ring.

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Notation. If A is a ring, $\mathbb{K}(A)$ will denote its (connective) K-theory spectrum, the spectrum associated to the infinite loop space $K_0(A) \times BGL(A)^+$, where $BGL(A)^+$ is the result of applying the Quillen $+$-construction to the classifying space of the infinite general linear group over A. By definition, the (algebraic) K-groups $K_i(A)$ of A are (at least in positive degrees) the homotopy groups of $\mathbb{K}(A)$, and the K-groups of A with finite coefficients $\mathbb{Z}/(m)$, $K_i(A; \mathbb{Z}/(m))$, are defined (at least in positive degrees) to be the homotopy groups of $S(\mathbb{Z}/(m)) \wedge \mathbb{K}(A)$, where $S(\mathbb{Z}/(m))$ is the $\mathbb{Z}/(m)$ Moore spectrum. These come with universal coefficient short exact sequencees

$$0 \to K_i(A) \otimes_{\mathbb{Z}} \mathbb{Z}/(m) \to K_i(A; \mathbb{Z}/(m)) \to \mathrm{Tor}_{\mathbb{Z}}(K_{i-1}(A), \mathbb{Z}/(m)) \to 0.$$

(This is almost, but not quite, the definition of Browder in [1]; for an explanation of the difference between the two definitions, see [11], pp. 285–286.)

In the one case below where confusion might be possible between algebraic and topological K-groups, we denote these by K_j^{alg} and K_j^{top}, respectively.

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Now we begin with a very general definition of (formal) deformation quantization. Intuitively, this is a formal deformation of the multiplication on an "algebra of observables," the deformation parameter being identified with "Planck's constant" $\hbar$.

Definition 1 Let $\overline{A}_0 = (A_0, \cdot\,)$ be an algebra over a commutative ring k (with unit), where A_0 is the underlying k-module of $\overline{A}_0$ and $\cdot$ is the multiplication in A. A (formal) **deformation quantization** of $\overline{A}_0$ will mean an (associative) algebra $A = (A_0[[\hbar]], \star\,)$ over $k[[\hbar]]$ (the commutative ring of formal power series over k in a variable $\hbar$) with underlying $k[[\hbar]]$-module $A_0[[\hbar]]$, where the multiplication $\star$ in A is defined by perturbing the multiplication $\cdot$ in $\overline{A}_0$ to a new product $\star$ via

$$a \star b = a \cdot b + \hbar\phi_1(a, b) + \hbar^2\phi_2(a, b) + \cdots, \qquad a,\, b \in A_0, \tag{1}$$

and then extending to series in the obvious way:

$$\left(\sum_{j=0}^{\infty} a_j\hbar^j\right) \star \left(\sum_{l=0}^{\infty} b_l\hbar^l\right) = \sum_{j,\,l=0}^{\infty} \hbar^{j+l}\left(a_j \cdot b_l + \sum_{p=1}^{\infty} \hbar^p\phi_p(a_j, b_l)\right).$$

Here ϕ_j, $j = 1, 2, \ldots$ are k-bilinear maps $A_0 \times A_0 \to A_0$. Note that $\overline{A}_0 \cong A/(\hbar)$ *as algebras*, so that one has a natural algebra map $e_0 : A \twoheadrightarrow \overline{A}_0$ ("setting $\hbar$ to 0"). We call the map e_0 the **classical limit map**.

Example 2 A trivial but still important example is the case where the multiplication on $\overline{A}_0$ is undeformed. In this case $A = \overline{A}_0[[\hbar]]$ is simply a ring of formal power series in one variable over the ring $\overline{A}_0$. $\square$

Example 3 In one of the most important examples, $k = \mathbb{C}$ and $\overline{A}_0 = C^\infty(M)$, where M is a symplectic manifold. Then there exist non-commutative algebras A satisfying Definition 1 for which

$$\phi_1(f,\, g) - \phi_1(g,\, f) = \{f,\, g\},$$

where $\{\ ,\ \}$ is the Poisson bracket on M. This was shown in [14] and [3]. The $\star$-product is obtained by a patching procedure using the Weyl quantization of $C^\infty(\mathbb{R}^{2n})$. $\square$

In this generality, it turns out that K_0 is preserved under deformation quantization.

Theorem 4 *Let k be a commutative ring with unit, let $\overline{A}_0$ be an algebra (with unit) over k, let A be a deformation quantization of $\overline{A}_0$ in the sense of Definition 1, and let $e_0 : A \twoheadrightarrow \overline{A}_0$ be the associated classical limit map. Then the map $(e_0)_* : K_0(A) \to K_0(\overline{A}_0)$ induced by e_0 is an isomorphism.*

We will need the following simple lemma.

Lemma 5 *Let k be a commutative ring with unit, let A_0 be a k-algebra, and let A be a deformation quantization of $\overline{A}_0$ in the sense of Definition 1. Then an element $a = \sum_{j=0}^{\infty} a_j \hbar^j$ of A $(a_j \in A_0)$ is invertible if and only if $e_0(a) = a_0$ is invertible in $\overline{A}_0$. Similarly, an element $a = \sum_{j=0}^{n-1} a_j \hbar^j$ of $A/(\hbar^n)$ $(a_j \in A_0)$ is invertible if and only if $e_0(a) = a_0$ is invertible in $\overline{A}_0$.*

Proof. The "only if" direction is trivial, and the "if" direction in the case of $A/(\hbar^n)$ follows from the result for A. The proof (in the case of A) for the "if" direction is the usual algorithm for inversion of power series. More specifically, suppose a_0 is invertible for the multiplication $\cdot$ in $\overline{A}_0$, and let $a = \sum_{j=0}^{\infty} a_j \hbar^j \in A$ $(a_j \in A_0)$. We can construct an inverse $b = \sum_{l=0}^{\infty} b_l \hbar^l$ for a with respect to the product $\star$ in A by letting $b_0 = a_0^{-1}$ (the inverse of a_0 in $\overline{A}_0$) and then solving for the coefficients b_l by iteration in the equation

$$1 = a \star b = \left(\sum_{j=0}^{\infty} a_j \hbar^j \right) \star \left(\sum_{l=0}^{\infty} b_l \hbar^l \right) = \sum_{j,\, l=0}^{\infty} \hbar^{j+l} \left(a_j \cdot b_l + \sum_{p=1}^{\infty} \hbar^p \phi_p(a_j, b_l) \right). \tag{2}$$

Equating coefficients of powers of $\hbar$ on the two sides of (2) gives for each $q \geq 1$ an equation (in A_0)

$$\sum_{j+l+p=q} \phi_p(a_j, b_l) = 0. \tag{3}$$

where for convenience we let $\phi_0(a_j, b_l) = a_j \cdot b_l$. To show these equations are (uniquely) solvable, note that assuming we have solved for $b_0, \ldots, b_{q-1}$, $q \geq 1$, (3) reduces to

$$a_0 \cdot b_q + \sum_{\substack{j+l+p=q \\ l \leq q-1}} \phi_p(a_j, b_l) = 0,$$

or

$$b_q = - \sum_{\substack{j+l+p=q \\ l \leq q-1}} a_0^{-1} \cdot \phi_p(a_j, b_l).$$

Thus, by induction on q, (2) has a unique solution which is a right $\star$-inverse to a. Similarly, a has a unique left $\star$-inverse. By the usual argument, these must be equal, so a is invertible in A. □

Proof of Theorem 4. For the injectivity, it is enough to show that if M and N are (left) A-modules with $M \oplus N = A^n$ for some n, and if $\overline{A}_0 \otimes_{e_0} M$ and $\overline{A}_0 \otimes_{e_0} N$ are free $\overline{A}_0$-modules, then M and N are free A-modules. Since the kernel $(\hbar)$ of e_0 is contained in the radical of A (this follows immediately from Lemma 5), the proof of Theorem 1.3.11 in [11] applies without change.

The proof of surjectivity is based on a version of Hensel's Lemma. Since $A = \varprojlim A/(\hbar^j)$ and we can replace A by $M_n(A)$, the $n \times n$ matrices over A, if necessary, it is enough to show that for $j \geq 1$, any idempotent $\overline{a}$ in $A/(\hbar^j)$ can be lifted to an idempotent in $A/(\hbar^{j+1})$. (Then an idempotent in $\overline{A}_0 = A/(\hbar)$

defining an element of $K_0(\overline{A}_0)$ can be lifted by induction to an idempotent in A.) Consider the exact sequence of k-algebras

$$0 \to I \to A/(\hbar^{j+1}) \to A/(\hbar^j) \to 0,$$

where as a vector space, $I = \hbar^j A_0$, but the multiplication on I vanishes since $2j \geq j+1$. Lift the idempotent $\overline{a} \in A/(\hbar^j)$ to any element $a \in A/(\hbar^{j+1})$. Then $a^2 - a \in I$. But $2\overline{a} - 1$ is invertible in $A/(\hbar^j)$, hence $2a-1$ is invertible in $A/(\hbar^{j+1})$ by Lemma 5. Let $x = (2a-1)^{-1} \star (a - a^2) \in I$, and observe that

$$(a+x)^2 - (a+x) = (a^2 + 2a \star x) - (a+x) = (a^2 - a) + (2a-1) \star x = 0,$$

so that $a + x$ is an idempotent lifting $\overline{a}$. □

The situation for higher K-theory is more complicated. But in the remarkable paper [13], based on [5] and [6], Suslin computed the K-theory with finite coefficients for certain discrete valuation rings, and used the results to study the comparison map from algebraic to topological K-theory in the case of $\mathbb{R}$ and $\mathbb{C}$. Some of the same techniques can be used to prove rigidity of algebraic K-theory with finite coefficients under deformation quantization. Our results are basically the same as Theorem 1 in [5], but without requiring the rings involved to be commutative.

Theorem 6 *Let k be a field of characteristic zero, let $\overline{A}_0$ be an algebra (with unit) over k, let A be a deformation quantization of $\overline{A}_0$ in the sense of Definition 1, and let $e_0 : A \twoheadrightarrow \overline{A}_0$ be the associated classical limit map. Then e_0 induces isomorphisms $K_j(A; \mathbb{Z}/(m)) \xrightarrow{\cong} K_j(\overline{A}_0; \mathbb{Z}/(m))$ on K-theory with finite coefficients for any $m > 1$, $j > 0$.*

The motto of the theorem is: *passage to the classical limit preserves K-theory with finite coefficients*. But perhaps a few words of explanation for the peculiar formulation are in order.

1. We certainly cannot expect e_0 to induce isomorphisms of K-groups *integrally*, since this is false even in the case of Example 2. If $\overline{A}_0 = k$, $\star = \cdot$, and $A = k[[\hbar]]$, then A is a commutative local ring and thus (see for instance [11], Corollary 2.2.6) $K_1(A) = A^\times$, which is vastly bigger than $K_1(\overline{A}_0) = k^\times$, and in fact the kernel of the map induced by $(e_0)_*$ on π_1 may be identified with a k-vector space of uncountable dimension.

2. There is some subtlety in the result since A is as a k-vector space an infinite product of copies of A_0, but the K-theory groups of an infinite product of rings are in general *not* the products of the K-groups of the factors. For a simple counterexample, let $R_j = C(S^{2j})$ (the continuous complex-valued functions on a sphere), $j = 1, 2, \ldots$. By Bott periodicity, $\widetilde{K}_0(R_j) \cong \mathbb{Z}$. Let $b_j \in K_0(R_j)$ have non-trivial projection into $\widetilde{K}_0(R_j)$. Then the element $(b_1, b_2, \ldots)$ of $\prod_j K_0(R_j)$ does *not* lie in the image of $K_0(\prod_j R_j)$, since realizing b_j as a formal difference of idempotent matrices requires matrices of increasing size as $j \to \infty$, so that $(b_1, b_2, \ldots)$ cannot

> come from matrices of finite size over $\prod_j R_j$. The K-theory of *categories* does commute with infinite products [2], but for quite non-trivial reasons. However, if $\mathcal{P}(R)$ denotes the category of finitely generated projective R-modules for a ring R (the relevant category for K-theory of rings), then $\mathcal{P}(\prod_j R_j)$ is not generally equivalent to $\prod_j \mathcal{P}(R_j)$.

Before giving the proof, we need two preliminaries.

Lemma 7 *Let k be a field, let A_0 be a k-algebra, and let A be a deformation quantization of $\overline{A}_0$ in the sense of Definition 1. Then for any $n \geq 1$, the natural maps $GL(n, A/(\hbar^{j+1})) \to GL(n, A/(\hbar^j))$ $(j = 1, 2, \ldots)$ are all surjective, and $GL(n, A) = \varprojlim GL(n, A/(\hbar^j))$.*

Proof. This follows immediately from Lemma 5, applied not to A but to $M_n(A)$, the $n \times n$ matrices over A. $\Box$

Proposition 8 *Let k, $\overline{A}_0$, and A be as in Theorem 6. Then for any integers $n, j \geq 1$, $m > 1$, the natural map $GL(n, A/(\hbar^j)) \to GL(n, \overline{A}_0)$ induces an isomorphism on homology with $\mathbb{Z}/(m)$ coefficients.*

Proof. We fix n and prove this by induction on j. The statement is trivially true when $j = 1$. So assume $j \geq 1$ and the statement is true for j; we'll prove it for $j + 1$. Consider the exact sequence of k-algebras

$$0 \to I \to A/(\hbar^{j+1}) \to A/(\hbar^j) \to 0,$$

where as a vector space, $I = \hbar^j A_0$, but the multiplication on I vanishes since $2j \geq j + 1$. By the previous lemma, the induced map $GL(n, A/(\hbar^{j+1})) \to GL(n, A/(\hbar^j))$ is surjective, and the kernel K consists of matrices of the form $1 + x$, $x \in M_n(I)$. Since $I^2 = 0$, multiplication in K is given by $(1 + x)(1 + y) = 1 + x + y$, i.e., $K \cong M_n(I)$ with its additive group structure. Since k is of characteristic zero, K is therefore isomorphic to the underlying additive group of a $\mathbb{Q}$-vector space, which is uniquely divisible. Hence K is $\mathbb{Z}/(m)$-acyclic, and the Hochschild-Serre spectral sequence for

$$1 \to K \to GL(n, A/(\hbar^{j+1})) \to GL(n, A/(\hbar^j)) \to 1$$

collapses to give $H_\bullet(GL(n, A/(\hbar^{j+1})); \mathbb{Z}/(m)) \cong H_\bullet(GL(n, A/(\hbar^j)); \mathbb{Z}/(m))$. This gives the inductive step. $\Box$

Proof of Theorem 6. By Lemma 7, $GL(n, A) = \varprojlim GL(n, A/(\hbar^j))$ (for any n). Hence the $\mathbb{Z}/(m)$-homology of $GL(n, A)$ can be computed from that of the $GL(n, A/(\hbar^j))$ by the Milnor $\varprojlim^1$ sequence. But by Proposition 8, the maps $GL(n, A/(\hbar^{j+1})) \to GL(n, A/(\hbar^j))$ are all $\mathbb{Z}/(m)$-homology isomorphisms. Hence the inverse system $H_\bullet(GL(n, A/(\hbar^j)); \mathbb{Z}/(m))$ (for fixed n) satisfies the Mittag-Leffler criterion, and

$$H_\bullet(GL(n, A); \mathbb{Z}/(m)) \cong H_\bullet(GL(n, \overline{A}_0); \mathbb{Z}/(m)).$$

Now pass the to the limit as $n \to \infty$. We deduce that the map of groups $GL(A) \to GL(\overline{A}_0)$ induces a $\mathbb{Z}/(m)$-homology isomorphism. Applying the classifying space functor and the Quillen +-construction yields that $BGL(A)^+ \to BGL(\overline{A}_0)^+$ is a $\mathbb{Z}/(m)$-homology equivalence (and of course also an infinite loop map). Now the usual connective K-theory spectrum of A, $\mathbb{K}(A)$, is just the spectrum associated to the infinite loop structure on $K_0(A) \times BGL(A)^+$, and K-theory with finite coefficients (in positive degrees, at least) is computed by taking the homotopy groups of $\mathbb{K}(A;\, \mathbb{Z}/(m)) = S(\mathbb{Z}/(m)) \wedge \mathbb{K}(A)$. Combining the fact that $BGL(A)^+ \to BGL(\overline{A}_0)^+$ is a $\mathbb{Z}/(m)$-homology equivalence with the fact that $K_0(A) \to K_0(\overline{A}_0)$ is an isomorphism (Theorem 4), we see $\mathbb{K}(A;\, \mathbb{Z}/(m)) \to \mathbb{K}(\overline{A}_0;\, \mathbb{Z}/(m))$ is a homology equivalence, hence a homotopy equivalence by the Hurewicz Theorem (which applies to *connective* spectra). (This argument bypasses the sort of reasoning used in [13], Proposition 1.5, but one could use that here instead.) So $\pi_j(\mathbb{K}(A;\, \mathbb{Z}/(m))) \xrightarrow{\cong} \pi_j(\mathbb{K}(\overline{A}_0;\, \mathbb{Z}/(m)))$, i.e., $K_j(A;\, \mathbb{Z}/(m)) \xrightarrow{\cong} K_j(\overline{A}_0;\, \mathbb{Z}/(m))$, for $j > 0$. □

Corollary 9 (Cf. [5], Theorem 1, for the commutative case.) *If k is a field of characteristic zero and if B is a k-algebra, then for $j > 0$ and any $m > 1$, $K_j(B[[t]];\, \mathbb{Z}/(m)) \xrightarrow{\cong} K_j(B;\, \mathbb{Z}/(m))$.*

Proof. Apply Theorem 6 to Example 2. □

Corollary 10 *Let M be a compact symplectic manifold, let $\overline{A}_0 = C^\infty(M)$ with its usual Poisson structure, and let A be a deformation quantization of $\overline{A}_0$. Then for $j > 0$ and any $m > 1$, $K_j^{\mathrm{alg}}(A;\, \mathbb{Z}/(m)) \cong K_{\mathrm{top}}^{-j}(M;\, \mathbb{Z}/(m))$, the topological K-theory of M with finite coefficients.*

Proof. We apply our results to Example 3. By Theorem 6, $K_0(A) \times BGL(A)^+ \to K_0(\overline{A}_0) \times BGL(\overline{A}_0)^+$ is a $\mathbb{Z}/(m)$-homotopy equivalence, so for $j > 0$,

$$K_j^{\mathrm{alg}}(A;\, \mathbb{Z}/(m)) \cong K_j^{\mathrm{alg}}(C^\infty(M);\, \mathbb{Z}/(m)).$$

The group on the right is known to coincide with $K_{\mathrm{top}}^{-j}(M;\, \mathbb{Z}/(m))$ by [4]. This requires comment: Fischer's theorem is stated for the algebra of continuous functions on a compact space X, but since the proof is sheaf-theoretic, when X is a manifold M, one can replace the sheaf of germs of continuous functions by the sheaf of germs of C^∞ functions, and all the arguments go through. The essential facts needed to make everything work are:

1. the local ring of germs of C^∞ functions at a point of a smooth manifold is Henselian;
2. for G a Lie group (in particular, for $G = GL(n,\, \mathbb{C})$), the group $C^\infty(M,\, G)$ is a "locally convex" topological group in the sense of [4], that is, that it is a topological group in the C^∞ topology, and that functions $M \to G$ which are close in the C^∞ topology can be joined by a smooth path; and

3. the topological K-theory of $C^\infty(M)$ coincides with that of $C(M)$ (a well-known consequence of C^∞ approximation).

□

Remark 11 Exactly the same statement as in Theorem 6 works when the ground ring k is a field of characteristic p, except that in this case one has to assume $(m, p) = 1$. The only difference in the proof is that in the proof of Proposition 8, one should substitute the fact that if $(m, p) = 1$, then a $\mathbb{Z}/(p)$-vector space (regarded as a group under addition) is $\mathbb{Z}/(m)$-acyclic. In fact one can even take $k = \mathbb{Z}[\frac{1}{m}]$ and the argument still works (see [13], Lemma 1.1). □

Remark 12 In fact the connective K-theory spectrum is the connective cover of a non-connective K-theory spectrum $\mathbb{K}^{\text{non-conn}}(A)$, whose homotopy groups in non-negative degrees are the same as those of $K_0(A) \times BGL(A)^+$ (in other words, the Quillen K-groups), and whose negative homotopy groups are the negative K-groups of Bass. (One of the many constructions of this spectrum may be found in [9], and a proof that it is equivalent to all the other standard definitions of this spectrum may be found in [10], §§5–6.) An optimal statement along the lines of Theorem 6—I am not sure whether this is correct or not—would thus be that

$$(e_0)_* : \mathbb{K}^{\text{non-conn}}(A; \mathbb{Z}/(m)) \xrightarrow{\cong} \mathbb{K}^{\text{non-conn}}(\overline{A}_0; \mathbb{Z}/(m)),$$

so that one gets isomorphisms similar to those of Theorem 6 for negative K-theory as well, but we have been unable to prove this. The difficulty is that the natural way to deloop the equivalence of Theorem 6 would be to replace $\overline{A}_0$ by $\overline{B}_0 = \overline{A}_0[t, t^{-1}]$ and define B from B_0 by the obvious formula derived from (1), keeping t central in B. The problem is that the resulting B is *not* just $A[t, t^{-1}]$ (which is not $(\hbar)$-adically complete), but rather its $(\hbar)$-adic completion, and it's not clear what effect the completion process has on K-groups. Other delooping techniques run into similar problems having to do with the failure of products and coproducts to commute. □

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Jonathan Rosenberg
Department of Mathematics
University of Maryland
College Park, MD 20742
email: `jmr@math.umd.edu`