ON THE WITT RING OF A RELATIVE PROJECTIVE LINE

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ABSTRACT. A ring homomorphism $e^0 : W(X) \to E^+(X)$ from the Witt ring of a scheme X into proper subquotient $E^+(X)$ of the Grothendieck ring $K_0(X)$ is a natural generalisation of dimension index for Witt ring of a field. In the case of projective line X over affine quadric Spec$S$ the value of e^0 on the Witt class of bundle $\mathcal{M}$ with a symmetric bilinear form ϕ is outside of the image of composition $W(S) \to W(X) \to E^+(X)$. Therefore the Witt ring of a projective line over a regular ring S may be greater than the Witt ring $W(S)$ of the ring S itself. Thus Arason theorem on Witt ring of a projective space over a field can not be generalized to projective spaces over regular rings.

INTRODUCTION.

After Witt the classical algebraic theory of quadratic forms deals with symmetric bilinear spaces (V, β) where V is a finitely generated vector space over a field K of characteristic different to 2 and $\beta : V \longrightarrow V\hat{}$ is an self-dual ($\beta = \beta\hat{}$, or, equivalently, $\beta(u)(v) = \beta(v)(u)$ for arbitrary $u, v \in V$) isomorphism of V with its dual space $V\hat{}$. Ignoring trivial in some sense (e.g. for the problem of representability of elements of K by quadratic form $v \longmapsto q(v) = \beta(v)(v)$) hyperbolic spaces

$$(M \oplus M\hat{}, \begin{bmatrix} 0 & 1_M \\ 1_{M\hat{}} & 0 \end{bmatrix})$$

yields Witt ring $W(K)$ of a field K, consisting of classes of symmetric bilinear forms up to hyperbolic direct summands, with addition induced by the direct sum and multiplication induced by the tensor product. This theory has numerous applications in algebraic number theory, theory of algebras, fields theory, Galois theory, cohomology theory, algebraic K-theory and algebraic geometry. Conversly, there are numerous applications of algebraic number theory, theory of algebras, fields theory, Galois theory, cohomology theory, algebraic K-theory and algebraic geometry in algebraic theory of quadratic forms. Extent bibliography may be found in [8]. There are several generalizations obtained by changing main objects: skew-symmetric bilinear forms, hermitian forms, algebras with involution and Ranicki formations. On the other hand there is natural way to generalize the notion of Witt ring: take a ring in place of field K and finitely generated projective (i.e. locally free) modules in place of vector spaces. For local rings with 2 invertible the theory is similar to the classical one. In general a difficult theory for fields becomes more difficult for, say, hermitian forms over group rings, which has applications in geometry and topology (e.g. group ring $\mathbb{Z}[\pi(X)]$ of a fundamental group in surgery

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theory). The next step is due to Knebusch [4], [5]: consider schemes in place of (spectra of) rings and vector bundles (locally free coherent sheaves of $\mathcal{O}_X$-modules) in place of projective modules. Let $(X, \mathcal{O}_X)$ be a scheme and let $\mathcal{L}$ be a line bundle over X. From here onwards we write $\pm$ to indicate two possibilities: a $+$ one and a $-$ one. A "$+$-symmetric" means simply "symmetric", while "$-$-symmetric" should be read as "skew-symmetric".

A $\pm$-symmetric $\mathcal{L}$-valued bilinear space (V, β) consists of a vector bundle V and an isomorphism $\beta : V \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(V, \mathcal{L}) = V\hat{} \otimes \mathcal{L}$ such that $\beta\hat{}^{\mathcal{L}} = (\beta\hat{} \otimes 1_{\mathcal{L}}) \circ (1 \otimes \mu^{-1}) = \pm\beta$, where $\mu : \mathcal{L}\hat{} \otimes \mathcal{L} \longrightarrow \mathcal{O}_X$ is the isomorphism of evaluation .

For a subbundle $\iota : W \rightarrowtail V$ its orthogonal complement $W^\perp$ is a subbundle of V defined as $W^\perp = \mathrm{Ker}(i\hat{}^{\mathcal{L}} \circ \beta)$.

A subbundle W of a bilinear space is totally isotropic or sublagrangian, iff $W = W^\perp$, and is lagrangian if $W = W^\perp$. Equivalently: a lagrangian subbundle of a bundle (V, β) is a totally isotropic subbundle of rank equal to half of rank V.

A bilinear space (V, β) is metabolic iff it posseses a lagrangian subbundle, i.e. if there exists exact sequence

$$0 \longrightarrow W \xrightarrow{\iota} V \xrightarrow{\iota' \circ \beta} W\hat{}^{\mathcal{L}} \longrightarrow 0$$

of vector bundles, where $\iota' = \iota\hat{}^{\mathcal{L}} : \mathcal{H}om_{\mathcal{O}_X}(V, \mathcal{L}) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(W, \mathcal{L})$ is the restriction to W.

Two $\pm$-symmetric $\mathcal{L}$-valued bilinear spaces (V, β) and (W, γ) are Witt equivalent iff there exist metabolic $\pm$-symmetric $\mathcal{L}$-valued bilinear spaces (M, μ) and (N, ν) such that

$$(V, \beta) \oplus (M, \mu) \cong (W, \gamma) \oplus (N, \nu).$$

Witt group $W^\pm(X, \mathcal{L})$ of $\pm$-symmetric $\mathcal{L}$-valued bilinear spaces consists of classes of Witt equivalence of $\pm$- symmetric $\mathcal{L}$-valued bilinear spaces with direct sum as addition. In the case of trivial line bundle $\mathcal{L} = \mathcal{O}_X$ and symmetric forms we write

$$W(X) = W^+(X, \mathcal{O}_X).$$

The tensor product induces multiplication on $W(X)$, so $W(X)$ is a ring, the Witt ring of the scheme X. Witt ring is a (co)functor: for morphism $f : X \longrightarrow Y$ of schemes the inverse image functor f^* induces a ring homomorphism $f^* : W(Y) \longrightarrow W(X)$. Arason proved [1, Satz] that for a field K, $\mathrm{char}(K) \neq 2$, the canonical map $W(K) \longrightarrow W(\mathbb{P}^n_K)$, induced by the structure map $\mathbb{P}^n_K \longrightarrow \mathrm{Spec}K$, is an isomorphism. This proof depends on one result of Horrocks on representing bundles as a direct sum of line bundles and properties of bundles Ω^r of differential forms. In 1991 M. Ojanguren asked, if Arasons theorem may be generalized to the case of a projective space over a ring. The only possible answer was that the question is difficult. Now there are tools to construct an infinite sequence of regular rings R, $\dim R \equiv 6 \pmod 8$, such that the canonical map $W(R) \longrightarrow W(\mathbb{P}^1_R)$ is not surjective. The idea consists in study of a group $E^+(X)$, a subfactor of $K_0(X)$, closely related to $W(X)$ and much easier to compute. The group $E(X)$ together with the homomorphism $e^0 : W(X) \longrightarrow E^+(X)$ and its generalizations $E^\pm(X, \mathcal{L})$ are introduced in section 1. General Theorem 2.1 in section 2 describes E-groups of a projective bundle. This description shows the way to construct an element of $E^+(\mathbb{P}^n_R)$ which is outside the image of the map $E^+(\mathrm{Spec}R) \longrightarrow E^+(\mathbb{P}^n_R)$ and special properties of (projective modules over) R that provide the construction. Rings with required property are coordinate rings of affine split quadrics (section 4), and computatins are possible in framework of Swan K-theory of quadrics (section 3). Thus commutativity of diagram provides that it is enough to find a bilinear space $(\mathcal{M},\beta)$ with prescribed value $e^0(\mathcal{M},\beta)$ to give negative answer to Ojanguren question. This is done for projective line in section 5 by means of theory of Ranicki formations developed by W. Pardon for rings [6] and by F. Fernández-Carmena for schemes [2]. A theorem due to Fernández-Carmena provides a construction of a symmetric bilinear form over a scheme for given formation over a closed subscheme of codimension one. The only reason of restricting this study to projective lines instead of general projective spaces is lack of methods of constructing symmetric bilinear spaces.

1. E-groups and invariant e^0.

Any line bundle $\mathcal{L}$ on scheme X defines an exact involutive contravariant functor ${}^{\hat{}\mathcal{L}}$ on the category of vector bundles on X

$$\begin{aligned} M &\longmapsto M^{\hat{}\mathcal{L}} = M^{\hat{}} \otimes \mathcal{L}, \\ \varphi^{\hat{}\mathcal{L}} &= \varphi^{\hat{}} \otimes 1_{\mathcal{L}} \quad \text{for} \quad \varphi : M \longrightarrow N. \end{aligned}$$

This involution induces analogous involution on Q-construction. Since its geometric realization interchanges ends of paths, there is induced involutive automorphism ${}^{\hat{}\mathcal{L}}$ of K-groups (homotopy groups of Q-construction) which acts on $K_0(X)$ as $[M] \longmapsto -[M^{\hat{}\mathcal{L}}]$. Nevertheless we define

$$[M]^{\hat{}\mathcal{L}} = [M^{\hat{}\mathcal{L}}].$$

We are interested in Tate cohomology of two-element group $\{1, {}^{\hat{}\mathcal{L}}\}$ with values in $K_0(X)$. Denote $C(X,\mathcal{L})$ the complete resolution

$$C(X.\mathcal{L}) : \cdots \xrightarrow{1-{}^{\hat{}\mathcal{L}}} K_0(X) \xrightarrow{1+{}^{\hat{}\mathcal{L}}} K_0(X) \xrightarrow{1-{}^{\hat{}\mathcal{L}}} K_0(X) \xrightarrow{1-{}^{\hat{}\mathcal{L}}} \cdots \tag{1.1}$$

Definition 1.1.

$$\begin{aligned} E^+(X,\mathcal{L}) &= Ker(1-{}^{\hat{}\mathcal{L}})/Im(1+{}^{\hat{}\mathcal{L}}), \\ E^-(X,\mathcal{L}) &= Ker(1+{}^{\hat{}\mathcal{L}})/Im(1-{}^{\hat{}\mathcal{L}}). \end{aligned}$$

We will refer to E-groups meaning collection of $E^+(X,\mathcal{L})$ and $E^-(X,\mathcal{L})$ for all line bundles $\mathcal{L}$. However, types of E-groups of a scheme X correspond to elements of the factor group $\mathrm{Pic}(X)/2\mathrm{Pic}(X)$.

Proposition 1.1. *For arbitrary line bundle K there are isomorphisms*

$$\begin{aligned} E^+(X,\mathcal{L}\otimes\mathcal{K}^{\otimes 2}) &\cong E^+(X,\mathcal{L}); \\ E^-(X,\mathcal{L}\otimes\mathcal{K}^{\otimes 2}) &\cong E^-(X,\mathcal{L}). \end{aligned}$$

Proof. Tensoring with K induces isomorphism of complexes
$C(X,\mathcal{L}) \xrightarrow{[\mathcal{K}]\cdot} C(X,\mathcal{L}\otimes\mathcal{K}^{\otimes 2})$:

$$\begin{array}{ccccccc} \cdots & \xrightarrow{1-\alpha} & K_0(X) & \xrightarrow{1+\alpha} & K_0(X) & \xrightarrow{1-\alpha} & \cdots \\ & & {\scriptstyle [K]\cdot}\uparrow & & {\scriptstyle [K]\cdot}\uparrow & & \\ \cdots & \xrightarrow{1-\beta} & K_0(X) & \xrightarrow{1+\beta} & K_0(X) & \xrightarrow{1-\beta} & \cdots \end{array}$$

where $\alpha(\mathcal{P}) = \mathcal{L}\otimes\mathcal{K}^{\otimes 2}\otimes\mathcal{P}^{\hat{}}$ and $\beta(\mathcal{P}) = \mathcal{L}\otimes\mathcal{P}^{\hat{}}$. $\square$

Definition 1.2. *The forgetful functor induces a group homomorphism*

$$e^0_{\mathcal{L}} : W^+(X,\mathcal{L}) \oplus W^-(X,\mathcal{L}) \longrightarrow E^+(X,\mathcal{L}),$$
$$e^0_{\mathcal{L}}(\mathcal{P},\beta) = [\mathcal{P}](\operatorname{mod} \operatorname{Im}(1 + \hat{}^{\mathcal{L}})).$$

The inverse image functor f^* for a morphism $f : Y \longrightarrow X$ of schemes induces a homomorphism $f^* : E(X,\mathcal{L}) \longrightarrow E(Y, f^*\mathcal{L})$. As an example we prove homotopy property of E - groups.

Proposition 1.2 (homotopy property). *If $f : X \longrightarrow Y$ is a flat morphism of regular noetherian separated schemes whose fibres are affine spaces, then $E^+(Y,\mathcal{L}) \cong E^+(X, f^*\mathcal{L})$ and $E^-(Y,\mathcal{L}) \cong E^-(X, f^*\mathcal{L})$.*

Proof. By the homotopy property of K-groups the map $f^* : K_0(Y) \longrightarrow K_0(X)$ induced by the inverse image functor f^* provides an isomorphism of complexes

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{1-\alpha} & K_0(X) & \xrightarrow{1+\alpha} & K_0(X) & \xrightarrow{1-\alpha} & \cdots \\
 & & {\scriptstyle f^*}\uparrow & & {\scriptstyle f^*}\uparrow & & \\
\cdots & \xrightarrow{1-\beta} & K_0(Y) & \xrightarrow{1+\beta} & K_0(Y) & \xrightarrow{1-\beta} & \cdots
\end{array}$$

where $\alpha = \hat{}^{f^*\mathcal{L}}$ and $\beta = \hat{}^{\mathcal{L}}$. □

2. E-groups of a projective bundle.

Theorem 2.1 (Projective bundle theorem). *Let $\mathcal{E}$ be a vector bundle on a scheme S, rank $\mathcal{E} = n$, and $X = \mathbb{P}(\mathcal{E}) = \operatorname{Proj}(\mathrm{S}(\mathcal{E}\hat{}\,))$ be the associated projective bundle. Let $\mathcal{O}_X(-1)$ be the tautological line bundle on X and $f : X \longrightarrow S$ the structure map. Let $\mathcal{L}$ be arbitrary line bundle on S.*
i) If $n = 2k+2$ is even, then there is exact hexagon

$$\begin{array}{ccccc}
 & & E^+(X, f^*\mathcal{L}) & & \\
 & {\scriptstyle f^*}\nearrow & & \searrow & \\
E^+(S,\mathcal{L}) & & & & E^-(S, \bigwedge^n \mathcal{E}\hat{}\otimes \mathcal{L}) \\
{\scriptstyle [\bigwedge^{k+1}\mathcal{E}]\cdot}\uparrow & & & & \downarrow{\scriptstyle [\bigwedge^{k+1}\mathcal{E}]\cdot} \\
E^+(S, \bigwedge^n \mathcal{E}\hat{}\otimes \mathcal{L}) & & & & E^-(S,\mathcal{L}) \\
 & \nwarrow & & \swarrow{\scriptstyle f^*} & \\
 & & E^+(X, f^*\mathcal{L}) & &
\end{array}$$

and $E^{\pm}(X, f^*\mathcal{L}\otimes \mathcal{O}_X(-1)) = 0$.
ii) If $n = 2k+1$ is odd, then $E^{\pm}(X, f^*\mathcal{L}) \cong E^{\pm}(S,\mathcal{L})$, and $E^{\pm}(X, f^*\mathcal{L}\otimes\mathcal{O}_X(-1)) \cong E^{\pm}(S, \mathcal{L}\otimes \bigwedge^n \mathcal{E}\hat{}\,)$.

Proof. $K_0(X) \cong (K_0(S))[t]/(\sum_{i=0}^{n}(-1)^i[\bigwedge^i \mathcal{E}]t^i)$ where t corresponds to $\xi = \mathcal{O}_X(-1)$ (see [7], Sect. 8, 1.5) and $\xi\hat{} = \xi^{-1}$. $K_0(X)$ is a free $K_0(S)$ - module with free base $\mathcal{O}_X(i) = \xi^{-i}$ for $i = \left[\frac{n}{2}\right], \left[\frac{n}{2}\right] - 1, ..., \left[\frac{n}{2}\right] - n$, where [] means an integer part of a number. There is identity

$$\sum_{i=-k}^{n-k}(-1)^{i+k}[f^*\bigwedge^{i+k}\mathcal{E}]\xi^i = 0$$

in $K_0(X)$ for arbitrary integer k. We shall alter the base of $K_0(X)$ to obtain triangular matrix of involution under consideration.

- In the case of even $n = 2k+2$ and forms with values in the line bundle $f^*\mathcal{L}\otimes\mathcal{O}_X(-1)$, the initial base ξ^i, $i = -k, -k, 1-k, ..., -1, 0, 1, ..., k, k+1$ may be transformed into $\xi^{i+1}+\xi^{-i}$, ξ^{-i} for $i = 0, 1, ..., k$. If A denotes the span (with coefficients in $K_0(S)$) of all $\xi^{i+1}+\xi^{-i}$ for $i = 0, 1, ..., k$, then the exact sequence

$$0 \longrightarrow A \xrightarrow{\epsilon} K_0(X) \xrightarrow{\kappa} K_0(X)/A \longrightarrow 0$$

in instance of formulas

$$(f^*(\alpha)(\xi^{i+1}+\xi^{-i}))^{\hat{}}\cdot[f^*\mathcal{L}]\cdot\xi = f^*(\alpha^{\hat{}}\cdot[\mathcal{L}])(\xi^{i+1}+\xi^{-i})$$

$$(f^*(\alpha)\xi^{-i})^{\hat{}}\cdot[f^*\mathcal{L}]\cdot\xi = f^*(\alpha^{\hat{}}\cdot[\mathcal{L}])(\xi^{i+1}+\xi^{-i}) - f^*(\alpha^{\hat{}}\cdot[\mathcal{L}])\xi^{-i}$$

for arbitrary $\alpha\in K_0(S)$ yields exact sequence of Tate cohomology

$$\cdots \xrightarrow{\hat{}\mathcal{L}} E^+(S,\mathcal{L})^{k+1} \xrightarrow{\epsilon} E^+(X, f^*\mathcal{L}\otimes\mathcal{O}_X(-1)) \xrightarrow{\kappa} E^-(S,\mathcal{L})^{k+1}$$
$$\xrightarrow{\hat{}\mathcal{L}} E^-(S,\mathcal{L})^{k+1} \xrightarrow{\epsilon} E^-(X, f^*\mathcal{L}\otimes\mathcal{O}_X(-1)) \xrightarrow{\kappa}$$
$$E^+(S,\mathcal{L})^{k+1} \xrightarrow{\hat{}\mathcal{L}} E^+(S,\mathcal{L})^{k+1} \xrightarrow{\epsilon} \cdots.$$

Connecting homomorphisms are induced by involution $\hat{}^{\mathcal{L}}$ acting componentwise, and are isomorphisms. Hence $E^{\pm}(X, f^*\mathcal{L}\otimes\mathcal{O}_X(-1)) = 0$.

- In the case of odd $n = 2k+1$ and line bundle of the form $f^*\mathcal{L}$, the elements $1 = \xi^0, \xi^i+\xi^{-i}, \xi^i$ for $i = 1, ..., k$ form another base of $K_0(X)$. If:
  A**:** $=$ the span (with coefficients in $K_0(S)$) of $\xi^i+\xi^{-i}$ for $i = 1, ..., k$;
  B**:** $=$ the span of ξ^i for $i = 1, ..., k$;
  C**:** $=$ the span of 1,

  then $K_0(X) = A\oplus B\oplus C$, and for any α in $K_0(S)$ formulas

$$(f^*(\alpha)(\xi^i+\xi^{-i}))^{\hat{}}\cdot[f^*\mathcal{L}] = (f^*(\alpha^{\hat{}}\cdot[\mathcal{L}])(\xi^i+\xi^{-i}),$$

$$(f^*(\alpha)\xi^i)^{\hat{}}\cdot[f^*\mathcal{L}] = -f^*(\alpha^{\hat{}}\cdot[\mathcal{L}])\xi^i + f^*(\alpha^{\hat{}}\cdot[\mathcal{L}])(\xi^i+\xi^{-i})$$

hold. Therefore regarding $A\subset A\oplus B\subset A\oplus B\oplus C$ as a filtration of the complex $C(X,\mathcal{L})$ yields that f^* induces an isomorphism on Tate cohomology: $E^{\pm}(X, f^*\mathcal{L})\cong E^{\pm}(S,\mathcal{L})$.

- In the case of odd $n = 2k+1$ and line bundle of the form $f^*\mathcal{L}\otimes\mathcal{O}_X(-1)$, the elements $\xi^{-k}, \xi^i+\xi^{1-i}, \xi^i$ for $i = 1, ..., k$ form another base of $K_0(X)$. If:
  A**:** $=$ the span (with coefficients in $K_0(S)$) of $\xi^i+\xi^{1-i}$ for $i = 1, ..., k$;
  B**:** $=$ the span of ξ^i for $i = 1, ..., k$;
  C**:** $=$ the span of ξ^{-k},

then $K_0(X) = A \oplus B \oplus C$, and for any α in $K_0(S)$ formulas

$$(f^*(\alpha)(\xi^i + \xi^{1-i}))^\wedge \cdot [f^*\mathcal{L}]\xi = (f^*(\alpha^\wedge \cdot [\mathcal{L}])(\xi^i + \xi^{1-i}),$$

$$(f^*(\alpha)\xi^i)^\wedge \cdot [f^*\mathcal{L}]\xi = -f^*(\alpha^\wedge \cdot [\mathcal{L}])\xi^i + f^*(\alpha^\wedge \cdot [\mathcal{L}])(\xi^i + \xi^{1-i}),$$

$$(f^*(\alpha)\xi^{-k})^\wedge \cdot [f^*\mathcal{L}] \cdot \xi \quad = \quad f^*(\alpha^\wedge \cdot [\mathcal{L}])\xi^{k+1} =$$

$$= f^*(a^\wedge \cdot [\mathcal{L}]) \sum_{i=-k}^{k} (-1)^{i+k} [f^* \bigwedge^{k+1-i} \mathcal{E}^\wedge] \cdot \xi^i =$$

$$= f^*(a^\wedge \cdot [\mathcal{L}]) \left([f^* \bigwedge^{n} \mathcal{E}^\wedge] \cdot \xi^{-k} + \sum_{i=1-k}^{k} (-1)^{i+k} [f^* \bigwedge^{k+1-i} \mathcal{E}^\wedge] \cdot \xi^i \right) =$$

$$= f^*(a^\wedge \cdot [\mathcal{L}]) \left([f^* \bigwedge^{n} \mathcal{E}^\wedge] \cdot \xi^{-k} + \sum_{i=1-k}^{0} (-1)^{i+k} [f^* \bigwedge^{k+1-i} \mathcal{E}^\wedge] \cdot \xi^i \right) +$$

$$+ f^*(a^\wedge \cdot [\mathcal{L}]) \left(\sum_{i=1}^{k} (-1)^{i+k} [f^* \bigwedge^{k+1-i} \mathcal{E}^\wedge] \cdot \xi^i \right) =$$

$$= f^*(a^\wedge \cdot [\mathcal{L}]) \left([f^* \bigwedge^{n} \mathcal{E}^\wedge] \cdot \xi^{-k} + \sum_{i=0}^{k-1} (-1)^{k-i} [f^* \bigwedge^{k+1+i} \mathcal{E}^\wedge] \cdot \xi^{-i} \right) +$$

$$+ f^*(a^\wedge \cdot [\mathcal{L}]) \left(\sum_{i=1}^{k} (-1)^{i+k} [f^* \bigwedge^{k+1-i} \mathcal{E}^\wedge] \cdot \xi^i \right) =$$

$$f^*(a^\wedge \cdot [\mathcal{L}]) \left([f^* \bigwedge^{n} \mathcal{E}^\wedge] \cdot \xi^{-k} + \sum_{i=1}^{k} (-1)^{k-i-1} [f^* \bigwedge^{k+i} \mathcal{E}^\wedge] \cdot (\xi^{1-i} + \xi^i) \right) +$$

$$+ f^*(a^\wedge \cdot [\mathcal{L}]) \left(\sum_{i=1}^{k} (-1)^{i+k} ([f^* \bigwedge^{k+1-i} \mathcal{E}^\wedge] + [f^* \bigwedge^{k+i} \mathcal{E}^\wedge]) \cdot \xi^i \right)$$

hold. Thus in the E_2 part of the spectral sequence associated with the filtration $A \subset A \oplus B \subset A \oplus B \oplus C = K_0(X)$ of the complex $C(X, f^*\mathcal{L} \otimes \mathcal{O}_X(-1))$ differentials $d : \mathrm{E}_2^{p,1} = (E^{(-1)^p}(S, \mathcal{L}))^k \longrightarrow (E^{(-1)^p}(S, \mathcal{L}))^k = \mathrm{E}_2^{p+2,0}$ are isomorphisms induced by ${}^{\wedge\mathcal{L}}$. Therefore $E^{(-1)^p}(X, f^*\mathcal{L} \otimes \mathcal{O}_X(-1)) = \mathrm{E}_2^{p-2,2} \cong E^{(-1)^p}(S, \mathcal{L} \otimes \bigwedge^n \mathcal{E}^\wedge)$.

- In the most complicated case $n = 2k+2$, line bundle of the type $f^*\mathcal{L}$ the initial base ξ^i, $i = -k, ..., k+1$ should be replaced by $1, \xi^{k+1}, \xi^i + \xi^{-i}, \xi^i$ for $i = 1, 2, ..., k$.

Formulas

$$(f^*\alpha(\xi^i + \xi^{-i}))^\wedge \cdot [f^*\mathcal{L}] = (f^*\alpha^{\wedge\mathcal{L}})(\xi^i + \xi^{-i})$$

$$(f^*\alpha\xi^i)^\wedge \cdot [f^*\mathcal{L}] = (f^*\alpha^{\wedge\mathcal{L}})\xi^{-i} = -(f^*\alpha^{\wedge\mathcal{L}})\xi^i + (f^*\alpha^{\wedge\mathcal{L}})(\xi^i + \xi^{-i})$$

$$(f^*\alpha\xi^{k+1})\hat{} \cdot [f^*\mathcal{L}] = (f^*\alpha^{\hat{}\mathcal{L}})\xi^{-k-1} = f^*\alpha^{\hat{}\mathcal{L}}\sum_{i=-k}^{k+1}(-1)^{k+i}[f^*\bigwedge^{k+1+i}\mathcal{E}]\cdot\xi^i =$$

$$= f^*\alpha^{\hat{}\mathcal{L}}\left(-[f^*\bigwedge^{n}\mathcal{E}]\cdot\xi^{k+1} + \sum_{i=1}^{k}(-1)^{k-i}[f^*\bigwedge^{k+1-i}\mathcal{E}]\cdot\xi^{-i}\right) +$$

$$+ f^*\alpha^{\hat{}\mathcal{L}}\left(\sum_{i=1}^{k}(-1)^{k+i}[f^*\bigwedge^{k+1+i}\mathcal{E}]\cdot\xi^i + (-1)^k[f^*\bigwedge^{k+1}\mathcal{E}]\right) =$$

$$= -f^*\alpha^{\hat{}\mathcal{L}}[f^*\bigwedge^{n}\mathcal{E}]\cdot\xi^{k+1} + f^*\alpha^{\hat{}\mathcal{L}}\sum_{i=1}^{k}(-1)^{k-i}[f^*\bigwedge^{k+1-i}\mathcal{E}]\cdot(\xi^i+\xi^{-i}) +$$

$$+ f^*\alpha^{\hat{}\mathcal{L}}\sum_{i=1}^{k}(-1)^{k+i}([f^*\bigwedge^{k+1+i}\mathcal{E}] - [f^*\bigwedge^{k+1-i}\mathcal{E}])\cdot\xi^i + (-1)^k f^*\alpha^{\hat{}\mathcal{L}}[f^*\bigwedge^{k+1}\mathcal{E}]$$

allow to define filtration $A \subset A\oplus B \subset A\oplus B\oplus C \subset A\oplus B\oplus C\oplus D = K_0(X)$ on the complex $C(X, f^*\mathcal{L})$ where

A**:** $=$ span of $\xi^i + \xi^{-i}$ for $i = 1, 2, ..., k$,
B**:** $=$ span of ξ^i for $i = 1, 2, ..., k$,
C**:** $= f^*K_0(S)\cdot 1$
D**:** $= f^*K_0(S)\cdot\xi^{k+1}$.

In the E_2 - term of associated spectral sequence the differentials $d_2^{\cdot,1} : \mathrm{E}_2^{\cdot,1} = (E^{\pm}(S,\mathcal{L}))^k \longrightarrow (E^{\pm}(S,\mathcal{L}))^k = \mathrm{E}_2^{\cdot+2,0}$ are isomorphisms induced by $\hat{}^{\mathcal{L}}$, while differentials $d_2^{\cdot,3} : \mathrm{E}_2^{\cdot,3} = E^{\pm}(S, \mathcal{L}\otimes\bigwedge^{n}\mathcal{E}) \longrightarrow E^{\pm}(S,\mathcal{L}) = \mathrm{E}_2^{\cdot+2,2}$ are induced by multiplication by $[\bigwedge^{k+1}\mathcal{E}]$, hence the theorem. ☐

If the bundle $\mathcal{E}$ in the theorem is trivial, then $[\bigwedge^{k+1}\mathcal{E}]$ is in the image of $1 + \hat{}^{\mathcal{L}}$, so maps $[\bigwedge^{k+1}\mathcal{E}]\cdot$ in the hexagon of the theorem are zero maps. In this case more detailed description of E-groups of a projective space may be given.

Proposition 2.2. *For arbitrary scheme S let $X = \mathbb{P}^d_S$ and let $p_1 : X \longrightarrow \mathbb{P}^d$, $p_2 : X \longrightarrow S$ be the projections. Then for arbitrary line bundle $\mathcal{L}$ on S and arbitrary line bundle $\mathcal{M}$ on $\mathbb{P}^d$*

$$E^+(X, \mathcal{M}\boxtimes\mathcal{L}) = E^+(\mathbb{P}^d,\mathcal{M})\boxtimes E^+(S,\mathcal{L}) \oplus E^-(\mathbb{P}^d,\mathcal{M})\boxtimes E^-(S,\mathcal{L})$$
$$E^-(X, \mathcal{M}\boxtimes\mathcal{L}) = E^+(\mathbb{P}^d,\mathcal{M})\boxtimes E^-(S,\mathcal{L}) \oplus E^-(\mathbb{P}^d,\mathcal{M})\boxtimes E^+(S,\mathcal{L}),$$

where $\boxtimes$ is induced by operation $\mathcal{F}\boxtimes\mathcal{G} = p_1^*(\mathcal{F})\otimes p_2^*(\mathcal{G})$.

Proof. By the projective bundle theorem for K-theory the maps p_1^*, p_2^* yield identification $K_0(X) = K_0(\mathbb{P}^d)\otimes K_0(S)$. Denote

$$\begin{aligned} A &= \mathrm{Ker}(K_0(\mathbb{P}^d) \xrightarrow{1-\hat{}^{\mathcal{M}}} K_0(\mathbb{P}^d)), \\ B &= (1-\hat{}^{\mathcal{M}})K_0(\mathbb{P}^d) \\ \mathcal{K} &= p_1^*\mathcal{M}\otimes p_2^*\mathcal{L}. \end{aligned}$$

The complex 1.1

$$\cdots \longrightarrow K_0(X) \xrightarrow{1+\hat{}^{\mathcal{K}}} K_0(X) \xrightarrow{1-\hat{}^{\mathcal{K}}} K_0(X) \xrightarrow{1+\hat{}^{\mathcal{K}}} K_0(X) \longrightarrow \cdots$$

for $X = \mathbb{P}^d \times S$ fits into the short exact sequence of complexes:

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow \\
A \otimes K_0(S) & \rightarrowtail & K_0(X) & \overset{(1-\hat{}^{\mathcal{M}})\otimes 1}{\twoheadrightarrow} & B \otimes K_0(S) \\
{\scriptstyle 1-\hat{}^{\mathcal{K}}}\Big\uparrow & & {\scriptstyle 1-\hat{}^{\mathcal{K}}}\Big\uparrow & & {\scriptstyle 1-\hat{}^{\mathcal{K}}}\Big\uparrow \\
A \otimes K_0(S) & \rightarrowtail & K_0(X) & \overset{(1-\hat{}^{\mathcal{M}})\otimes 1}{\twoheadrightarrow} & B \otimes K_0(S) \\
{\scriptstyle 1-\hat{}^{\mathcal{K}}}\Big\uparrow & & {\scriptstyle 1-\hat{}^{\mathcal{K}}}\Big\uparrow & & {\scriptstyle 1-\hat{}^{\mathcal{K}}}\Big\uparrow \\
A \otimes K_0(S) & \rightarrowtail & K_0(X) & \overset{(1-\hat{}^{\mathcal{M}})\otimes 1}{\twoheadrightarrow} & B \otimes K_0(S) \\
\uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots
\end{array}
$$

Note that $1 \pm \hat{}^{\mathcal{K}}$ restricted to $A \otimes K_0(S)$ coincides with $1 \otimes (1 \pm \hat{}^{\mathcal{L}})$ and induces $1 \otimes (1 \mp \hat{}^{\mathcal{L}})$ on $B \otimes K_0(S)$. Therefore the exact hexagon in homology breaks into short split exact sequences:

$$
\begin{array}{l}
0 \longrightarrow E^+(\mathbb{P}^d, \mathcal{M}) \otimes E^-(S, \mathcal{L}) \longrightarrow E^-(X, \mathcal{K}) \longrightarrow E^-(\mathbb{P}^d, \mathcal{M}) \otimes E^+(S, \mathcal{L}) \longrightarrow 0 \\
0 \longrightarrow E^+(\mathbb{P}^d, \mathcal{M}) \otimes E^+(S, \mathcal{L}) \longrightarrow E^+(X, \mathcal{K}) \longrightarrow E^-(\mathbb{P}^d, \mathcal{M}) \otimes E^-(S, \mathcal{L}) \longrightarrow 0.
\end{array}
$$

□

To identify explicite generators of groups under consideration, for absolute projective space $Y = \mathbb{P}^d$ denote:

(2.1) $\quad 1 \quad = \quad [\mathcal{O}_Y]$ - the unit element in $K_0(Y)$;

(2.2) $\quad H \quad = \quad 1 - [\mathcal{O}_Y(-1)]$ - the class of hyperplane section in $K_0(Y)$.

We summarize some technicalities as follows:

Lemma 2.3. *If $Y = \mathbb{P}^d$, then*
i) $H^{d+1} = 0$;
ii) $[\mathcal{O}_Y(1)] = (1-H)^{-1} = \sum_{i=0}^{d} H^i$ *in* $K_0(Y)$ *(here* $H^0 = 1$*)*;
iii) $H\hat{} = \dfrac{-H}{1-H} = -\sum_{i=1}^{d} H^i$;
iv) $(H^k)\hat{} = \left(\dfrac{-H}{1-H}\right)^k = (-1)^k H^k \sum_{i=0}^{d-k} \binom{k+i-1}{i} H^i$;
v) $(H^d)\hat{} = (-1)^d H^d$ *is a class of rational point.*

Proof. $H = 1 - [\mathcal{O}_Y(-1)]$, so $[\mathcal{O}_Y(-1)] = 1 - H$, $[\mathcal{O}_Y(1)] = (1-H)^{-1}$, H being nilpotent. Thus $H\hat{} = 1 - [\mathcal{O}_Y(1)] = ([\mathcal{O}_Y(-1)] - 1)[\mathcal{O}_Y(1)] = -H(1-H)^{-1}$ and $(H^k)\hat{} = (1-H)^{-k}(-H)^k$. □

Corollary 2.4. *If $Y = \mathbb{P}^d$, the projective space, then*

$$\begin{aligned} E^+(Y) &= E^+(Y, \mathcal{O}_Y) = \mathbb{Z}/2\mathbb{Z}[\mathcal{O}_Y] \\ E^-(Y) &= E^-(Y, \mathcal{O}_Y) = \begin{cases} 0 & \textit{for even } d \\ \mathbb{Z}/2\mathbb{Z}[H^d] & \textit{for odd } d \end{cases} \\ E^+(Y, \mathcal{O}_Y(-1)) &= \begin{cases} \mathbb{Z}/2\mathbb{Z}[H^d] & \textit{for even } d \\ 0 & \textit{for odd } d \end{cases} \\ E^-(Y, \mathcal{O}_Y(-1)) &= 0. \end{aligned}$$

3. Swan K-theory of projective quadrics.

To compute E-groups of affine quadrics we need some facts on dualization of vector bundles on a projective quadrics. All needed information is known in fact, since indecomposable components of Swan sheaf correspond to spinor representations. Nevertheless we give here complete proofs of needed facts. We shall apply results of [9] in the simplest possible case of split quadric: X is a projective quadric hypersurface over a field F, char$F \neq 2$, defined by the quadratic form of maximal index. Consider a vector space V with base $v_0, v_1, ..., v_{d+1}$ over a field F, char$F \neq 2$. Let $z_0, z_1, ..., z_{d+1}$ be the dual base of $V\hat{}$ and let q be the quadratic form

$$q = \sum_{i=0}^{d+1} (-1)^i z_i^2.$$

Moreover, let $e_i = \frac{1}{2}(v_{2i} - v_{2i+1})$, $f_i = \frac{1}{2}(v_{2i} + v_{2i+1})$ for all possible values of i. Thus if d is even, $d = 2m$, then $e_0, f_0, e_1, f_1, ..., e_m, f_m$ form a base of V with the dual base $x_0, y_0, x_1, y_1, ..., x_m, y_m$ and

$$q = \sum_{i=0}^{m} x_i y_i.$$

If d is odd, $d = 2m + 1$, then $e_0, f_0, e_1, f_1, ..., e_m, f_m, v_{d+1}$ form a base of V with the dual base $x_0, y_0, x_1, y_1, ..., x_m, y_m, z_{d+1}$ and

$$q = \sum_{i=0}^{m} x_i y_i + z_{d+1}^2.$$

We shall prove several properties of dualisation functor on the category of vector bundles on the d-dimensional projective quadric X defined by equation $q = 0$ in $\mathbb{P}_F^{d+1}$, i.e. for

$$X = \text{Proj S}(V\hat{})/(q) \cong \text{Proj } F[z_0, z_1, ..., z_{d+1}]/(q).$$

to compute E-groups of an affine part of this quadric in the following section. In the case of odd $d = 2m + 1$ the even part $C_0 = C_0(q)$ of the Clifford algebra $C(q)$ is isomorfic to the matrix algebra $M_{2N}(F)$, where $N = 2^m$. In particular $K_p(C_0) \cong K_p(F)$. In the case of even $d = 2m$, the algebra C_0 has the centre $F \oplus F\delta$, where $d = v_0 \cdot v_1 \cdot ... \cdot v_{d+1}$ and $\delta^2 = 1$. Thus $\frac{1}{2}(1+\delta)$, $\frac{1}{2}(1-\delta)$ are othogonal central idempotents of C_0, so

$$C_0 = \frac{1}{2}(1+\delta)C_0 \oplus \frac{1}{2}(1-\delta)C_0$$

where each direct summand is isomorphic to the matrix algebra $M_N(F)$. In fact, in this case there exists an isomorphism $C(q) \cong M_{2N}(F)$ of algebras, which identifies

C_0 with the subalgebra of block-diagonal matrices $\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$ and maps $\frac{1}{2}(1+\delta)C_0$ onto the set of matrices of the form $\begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix}$, and $\frac{1}{2}(1-\delta)C_0$ onto the set of matrices of the form $\begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}$. This observation provides some motivation for what follows. Such a matrix representation of Clifford algebra may be found in [9], lemma 4.3. More "classical" construction, based upon minimal orthogonal idempotents, may be easy deduced from the proof of Proposition 4.6 below.

For even $d = 2m$ consider the principal antiauthomorphism $\Im : C_0 \longrightarrow C_0$:

$$\Im(w_1 \cdot w_2 \cdot ... \cdot w_k) = (-1)^k w_k \cdot w_{k-1} \cdot ... \cdot w_1$$

for $w_1, w_2, ..., w_k \in V$. Note that

$$\Im(\delta) = (-1)^{m+1}\delta.$$

Moreover for every anisotropic vector $w \in V$ the reflexion $\alpha \mapsto -w\alpha w^{-1}$ in V induces an automorphism ρ_w of C_0, which iterchanges the δ with its opposite for even d:

$$\rho_w(\delta) = (-1)^{d-1}\delta.$$

Regarding subscripts i mod2 denote

$$P_i = (1 + (-1)^i\delta)C_0 \text{ for even } d.$$

Lemma 3.1. *In the case of even $d = 2m$:*

i) *the involution $\Im$ of the algebra C_0 provides an identification of the left C_0 - module $P_i\hat{}= \mathrm{Hom}_F(Pi, F)$ with the right C_0 - module P_{i+m+1};*
ii) *for any anisotropic vector $w \in V$ the reflexion ρ_w interchanges P_i's: $\rho_w(P_i) = P_{i+1}$.*

Note that as *left* C_0-modules P_0 and P_1 are *not* isomorphic.

Recall basic facts and notation of [9]. Denote C_1 the odd part of the Clifford algebra $C(q)$. We shall use mod 2 subscripts in C_i. Recall the definition of the Swan bundle $\mathcal{U}$. Put $\varphi = \sum_{i=0}^{d+1} z_i v_i \in \Gamma(X, \mathcal{O}_X \otimes V)$. The complex

(3.1)
$$\cdots \xrightarrow{\varphi\cdot} \mathcal{O}_X(-n) \otimes C_{n+d+1} \xrightarrow{\varphi\cdot} \mathcal{O}_X(1-n) \otimes C_{n+d} \xrightarrow{\varphi\cdot} \mathcal{O}_X(2-n) \otimes C_{n+d-1} \xrightarrow{\varphi\cdot} \cdots$$

is exact and locally splits ([9], Prop. 8.2.(a)).

Definition 3.1.

$$\begin{aligned} \mathcal{U}_n &= \mathrm{Coker}(\mathcal{O}_X(-n-2) \otimes C_{n+d+3} \xrightarrow{\varphi\cdot} \mathcal{O}_X(-n-1) \otimes C_{n+d+2}), \\ \mathcal{U} &= \mathcal{U}_{d-1}. \end{aligned}$$

Since the complex 3.1 is - up to twist - periodical with period two, it follows that

$$\mathcal{U}_{n+2} = \mathcal{U}_n(-2).$$

Consider exact sequences $\mathcal{O}_X(-n-2) \otimes C_{n+d+3} \xrightarrow{\varphi\cdot} \mathcal{O}_X(-n-1) \otimes C_{n+d+2} \longrightarrow \mathcal{U}_n \longrightarrow 0$ for two consecutive values n; twist the first one by 1. For any anisotropic vector $w \in V$ the isomorphism given by right multiplication by $1 \otimes w$ fits into commutative diagram:

$$\begin{array}{ccccccc} \mathcal{O}_X(-n-2) \otimes C_{n+d+4} & \xrightarrow{\varphi\cdot} & \mathcal{O}_X(-n-1) \otimes C_{n+d+3} & \longrightarrow & \mathcal{U}_{n+1}(1) & \longrightarrow & 0 \\ \downarrow \cdot 1 \otimes w & & \downarrow \cdot 1 \otimes w & & & & \\ \mathcal{O}_X(-n-2) \otimes C_{n+d+3} & \xrightarrow{\varphi\cdot} & \mathcal{O}_X(-n-1) \otimes C_{n+d+2} & \longrightarrow & \mathcal{U}_n & \longrightarrow & 0 \end{array}.$$

Thus we have proved following lemma:

Lemma 3.2. *$\mathcal{U}_{n+1} \cong \mathcal{U}_n(-1)$ and $\mathcal{U}_n \cong \mathcal{U}_0(-n)$ for arbitrary integer n.*

There is exact sequence

$$0 \longrightarrow \mathcal{U}_0 \xrightarrow{\varphi\cdot} \mathcal{O}_X \otimes C_0 \longrightarrow \mathcal{U}_{-1} \longrightarrow 0 \tag{3.2}$$

where isomorphism $\cdot 1 \otimes w$ was used to replace $\mathcal{O}_X \otimes C_1$ by $\mathcal{O}_X \otimes C_0$ for even d.

Lemma 3.3. *$\mathrm{End}_X(\mathcal{U}_n) \cong C_0$ acts on $\mathcal{U}_n$ from the right.*

Proof. [9], Lemma 8.7. □

The main Theorem 9.1 of [9] states that for arbitrary regular ring R and arbitrary generalized Azumaya algebra A over R and arbitrary projective quadric X of dimension d over R, defined by nonsingular quadratic form q, the family of functors

$$\begin{aligned} u_i(M) &= M \otimes \mathcal{O}_X(-i) \text{ for } i = 0, 1, ..., d-1, \\ u(M) &= \mathcal{U} \otimes_{C_0(q)} M \end{aligned}$$

defines the isomorphism

$$(u_0, u_1, ..., u_{d-1}, u) : K_*(A)^d \oplus K_*(A \otimes C_0(q)) \longrightarrow K_*(X).$$

Important argument is that for large enough class of sheaves (namely regular sheaves) $\mathcal{F}$ there exists truncated canonical resolution

$$\begin{aligned} TCan.(\mathcal{F}) : 0 \to \mathcal{U} \otimes_{C_0(q)} T(\mathcal{F}) \to \mathcal{O}_X(1-d) \otimes T_{d-1}(\mathcal{F}) \to ... \\ ... \to \mathcal{O}_X \otimes T_0(\mathcal{F}) \to \mathcal{F} \to 0 \end{aligned}$$

([9], section 6). In the special case when $R = A$ is a field F, char$F \neq 2$, $T_i(\mathcal{F})$ are vector spaces over F, so $\mathcal{O}_X(-i) \otimes T_i(\mathcal{F})$ is a direct sum of $\dim T_i(\mathcal{F})$ copies of $\mathcal{O}_X(-i)$. Thus in $K_0(X)$ the equality

$$\begin{aligned} [\mathcal{F}] &= \dim T_0(\mathcal{F})[\mathcal{O}_X] - \dim T_1(\mathcal{F})[\mathcal{O}_X(-1)] + ... \\ &\quad + (-1)^{d-1} \dim T_{d-1}(\mathcal{F})[\mathcal{O}_X(1-d)] + (-1)^d [\mathcal{U} \otimes_{C_0(q)} T(\mathcal{F})] \end{aligned}$$

holds.

We are now ready to compute $\mathcal{U}_n\hat{}$.

Lemma 3.4. *$\mathcal{U}_n\hat{} \cong \mathcal{U}_n(2n+1)$, in particular $\mathcal{U}\hat{} \cong \mathcal{U}(2d-1)$.*

Proof. We have chosen a base $v_0, v_1, ..., v_{d+1}$ of V above. The set of naturally ordered products of several v_i's in an even number forms a base of C_0. Define a quadratic form Q on C_0 as follows: let distinct base products are orthogonal to each other and

$$Q(v_{i_1} \cdot v_{i_2} \cdot \ldots \cdot v_{i_k}) = q(v_{i_1}) \cdot q(v_{i_2}) \cdot \ldots \cdot q(v_{i_k})$$

The form Q is nonsingular and defines - by scalar extension - nonsingular symmetric bilinear form Δ on $\mathcal{O}_X \otimes C_0$. Since $(q(v_i))^2 = 1$, so that

$$Q(v_{i_1} \cdot v_{i_2} \cdot \ldots \cdot v_{i_l} \cdot \ldots \cdot v_{i_k}) = Q(v_{i_1} \cdot v_{i_2} \cdot \ldots \cdot v_{i_l}) Q(v_{i_{l+1}} \cdot \ldots \cdot v_{i_k}),$$

direct computation shows that $\mathrm{Im}(\mathcal{O}_X(-1) \otimes C_0 \xrightarrow{\varphi\cdot} \mathcal{O}_X \otimes C_0) = \varphi\mathcal{U}_0 \cong \mathcal{U}_0$ is a totally isotropic subspace of $\mathcal{O}_X \otimes C_0$. Therefore

$$\mathcal{U}_0 \cong \varphi\mathcal{U}_0 = (\varphi\mathcal{U}_0)^\perp \cong ((\mathcal{O}_X \otimes C_0)/(\varphi\mathcal{U}_0))\hat{} \cong \mathcal{U}_{-1}\hat{}.$$

Thus

$$\mathcal{U}_0\hat{} \cong \mathcal{U}_{-1} \cong \mathcal{U}_0(1)$$

and, in general

$$\mathcal{U}_n\hat{} \cong (\mathcal{U}_0(-n))\hat{} \cong \mathcal{U}_0\hat{}(n) \cong \mathcal{U}_0(n+1) \cong \mathcal{U}_n(2n+1).$$

□

Corollary 3.5.
i) $[\mathcal{U}\hat{}] = [\mathcal{U}(2d-1)]$ *and* $[\mathcal{U}(d-1)] + [\mathcal{U}(d-1)]\hat{} = 2d+1$ *in* $K_0(X)$*;*
ii) $\mathrm{rank}\mathcal{U} = \frac{1}{2} \dim C_0 = 2^d$*.*

In the case of even $d = 2m$ the algebra $\mathrm{End}_X(\mathcal{U}) = C_0$ splits into direct product of algebras P_i defined above: $C_0 = P_0 \times P_1$.

Definition 3.2. *In the case of even d:*

$$\begin{aligned} \mathcal{U}'_n &= \mathcal{U}_n \otimes_{C_0} P_0, \\ \mathcal{U}''_n &= \mathcal{U}_n \otimes_{C_0} P_1, \\ \mathcal{U}' &= \mathcal{U} \otimes_{C_0} P_0, \\ \mathcal{U}'' &= \mathcal{U} \otimes_{C_0} P_1. \end{aligned}$$

Note that $\mathcal{U}_n = \mathcal{U}'_n \oplus \mathcal{U}''_n$, $\mathcal{U} = \mathcal{U}' \oplus \mathcal{U}''$. $\mathcal{U}'_0$ and $\mathcal{U}''_0$ correspond to spinor representations and we shall copy here standard argument on dualization (compare [3], sect. 4.3). In the case of even $d = 2m$ another property of φ and the quadratic form Q introduced in the proof of Lemma 3.4 may be verified by direct computation:

Lemma 3.6. *In the case of even* $d = 2m$

i) *if* m *is even, then* $P_i = (1 \pm \delta)C_0$ *are orthogonal to each other, hence self-dual;*

ii) *if* m *is odd, then* $P_i = (1 \pm \delta)C_0$ *are totally isotropic, hence dual to each other;*

iii) $\varphi(1 \pm \delta) = (1 \mp \delta)\varphi$.

Corollary 3.7. *In the case of even* $d = 2m$

i) $\mathcal{U}'\hat{} \cong \mathcal{U}'(2d-1)$ *and* $\mathcal{U}''\hat{} \cong \mathcal{U}''(2d-1)$ *for even* m*;*

ii) $\mathcal{U}'\hat{} \cong \mathcal{U}''(2d-1)$ *and* $\mathcal{U}''\hat{} \cong \mathcal{U}'(2d-1)$ *for odd* m*;*

iii) $\mathrm{End}_X(\mathcal{U}') \cong \mathrm{End}_X(\mathcal{U}'') \cong M_{2^m}(F)$*;*

iv) *the exact sequence 3.2 splits into two exact parts*

$$0 \longrightarrow \mathcal{U}_0' \xrightarrow{\varphi\cdot} \mathcal{O}_X \otimes P_0 \longrightarrow \mathcal{U}_0''(1) \longrightarrow 0$$
$$0 \longrightarrow \mathcal{U}_0'' \xrightarrow{\varphi\cdot} \mathcal{O}_X \otimes P_1 \longrightarrow \mathcal{U}_0'(1) \longrightarrow 0.$$

The standard way to determine indecomposable components is tensoring with the simple left module over appropriate endomorphism algebra. We will use (from here onwards) superscript as a notation for direct sum of identical objects.

Definition 3.3. i) *in the case of odd* $d = 2m + 1$ *:*

$$\mathcal{V} = \mathcal{U} \otimes_{C_0} F^{2N}$$

ii) *in the case of even* $d = 2m$ *:*

$$\mathcal{V}_0 = \mathcal{U}' \otimes_{M_N(F)} F^N,$$
$$\mathcal{V}_1 = \mathcal{U}'' \otimes_{M_N(F)} F^N$$

where $N = 2^m$*.*

For convenience we will use mod 2 subscripts in $\mathcal{V}_i$. Since $M_N(F) = (F^N)^N$ as a left $M_N(F)$-module, indecomposable components inherit properties of the Swan bundle: we have

Proposition 3.8. a) *In the case of odd* $d = 2m + 1$*:*

i) $\mathcal{U} = \mathcal{V}^{2N}$ *, where* $N = 2^m$*;*
ii) $\mathcal{V}\hat{} = \mathcal{V}(2d - 1)$*;*
iii) $\mathrm{End}_X(\mathcal{V}) \cong F$ *and* $\mathrm{rank}\mathcal{V} = 2^m$*.*

b) *In the case of even* $d = 2m$*:*

i) $\mathcal{U}' = \mathcal{V}_0^N$ *and* $\mathcal{U}'' = V_1^N$ *, where* $N = 2^m$*;*
ii) $\mathcal{V}_i\hat{} = \mathcal{V}_{i+m}(2d - 1)$*;*
iii) $\mathrm{End}_X(\mathcal{V}_i) \cong F$ *and* $\mathrm{rank}\mathcal{V}_i = 2^{m-1}$*,*
iv) $[\mathcal{V}_i(d - 1)] + [\mathcal{V}_{i+1}(d)] = 2^m$ *in* $K_0(X)$*.*

In particular there is no global morphism $\mathcal{V}_i \longrightarrow \mathcal{V}_{i+1}$, since
$\mathrm{End}_X(\mathcal{U}) = \mathrm{End}_X(\mathcal{V}_0^N \oplus \mathcal{V}_1^N) = M_N(\mathrm{End}_X(\mathcal{V}_0)) \times M_N(\mathrm{End}_X(\mathcal{V}_1))$.

Example 3.8.1. Split projective quadric $X = \mathrm{Proj}\ S$
$S = F[x_0, y_0, x_1, y_1, ..., x_m, y_m]/(\sum_{i=0}^{m} x_i y_i)$ of dimension $d = 2m$ contains two projective spaces of dimension m:
$Y = \mathrm{Proj}\ F[y_0, y_1, ..., y_m]$ defined by equations $x_0 = x_1 = ... = x_m = 0$ and $Z = \mathrm{Proj}\ F[x_0, y_1, ..., y_m]$ defined by equations $y_0 = x_1 = ... = x_m = 0$. These subvarieties are not rationally equivalent. It may be shown that their structural sheaves define two distinct elements of $K_0'(X) = K_0(X)$:

$$\sum_{i=0}^{d-1} \left(\sum_{p=0}^{m} \binom{m}{p} \right) (-1)^i [\mathcal{O}_X(-i)] + [\mathcal{V}_0]$$

and

$$\sum_{i=0}^{d-1} \left(\sum_{p=0}^{m} \binom{m}{p} \right) (-1)^i [\mathcal{O}_X(-i)] + [\mathcal{V}_1]$$

(see [10], Theorem 4.1). If, in particular, $d = 2$, then X is isomorphic to the product of two projective lines, and Y and Z are generatrices.

4. E-groups of affine quadrics.

Following theorem due to Swan [9], (Theorem 10.5, Corollary 10.7) describes K-theory of a (relative) affine quadric.

Theorem 4.1. *Let R be a regular ring, $q \cong < -1 > \oplus q'$ - a nonsingular quadratic form over R defined on the projective R-module $M = M' \oplus R$, S - the coordinate ring of affine quadric $q' = 1$ over R, $S = \mathrm{S}(M\hat{\ })/(q'-1)$, A - a generalised Azumaya algebra over R. Let moreover*
$g : K_*(A \otimes_R C_0(q')) \longrightarrow K_*(A)$ be the norm map,
$h : K_*(A \otimes_R C_0(q')) \longrightarrow K_*(A \otimes_R C_0(q))$ be the scalar extension map,
$r : K_*(A) \longrightarrow K_*(A \otimes_R S)$ be the scalar extension map, and let
$s : K_*(A \otimes_R C_0(q)) \longrightarrow K_*(A \otimes_R S)$ be the map, induced by the exact functor $\mathcal{U}(\mathrm{Spec}S) \otimes_{C_0(q)} -$. Then the sequence

$$\cdots \longrightarrow K_{i+1}(A \otimes_R S) \xrightarrow{\partial} K_i(A \otimes_R C_0(q')) \xrightarrow{\alpha}$$
$$K_i(A) \oplus K_i(A \otimes_R C_0(q)) \xrightarrow{[r,s]} K_i(A \otimes_R S) \xrightarrow{\partial} \cdots$$

where $\alpha = \left[\begin{smallmatrix} -g \\ h \end{smallmatrix}\right]$, is exact.

Let X be the affine quadric $\mathrm{Spec}S$.

Proposition 4.2. *If under assumptions of the Theorem 4.1 if in addition $A = R = F$ is a field of characteristic different from 2, $d = 2m$ is even and $q = \sum_{i=0}^{d+1}(-1)^i z_i^2$, $q' = \sum_{i=0}^{d}(-1)^i z_i^2$, then*

- *$g : K_0(C_0(q')) \longrightarrow K_0(F)$ is the map $2^m\cdot\ : \mathbb{Z} \longrightarrow \mathbb{Z}$,*
- *$h : K_0(C_0(q')) \longrightarrow K_0(C_0(q)) = K_0(P_0) \oplus K_0(P_1)$ is the diagonal map $\mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$,*
- *$r : K_0(F) \longrightarrow K_0(S)$ maps the generator $[F^1]$ onto the generator $[S^1]$,*
- *$s : K_0(C_0(q)) = K_0(P_0) \oplus K_0(P_1) \longrightarrow K_0(S)$ maps generators $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ onto $[V_0]$, $[V_1]$ respectively, where $V_i = \mathcal{V}_i(\mathrm{Spec}S)$ for $i = 0, 1$.*

Proof. Direct computation on the level of functors. The (reduced) norm map g is induced by the forgetful functor from $C_0(q')$ - modules to F-modules, which maps the generator of $K_0(C_0(q'))$ (the class of the simple $C_0(q')$ - module) onto the class of the same module. The scalar extension map h is induced by the scalar extension functor , which in turn produces identity if composed with projections onto P_i's. The map r is induced by the scalar extension functor $S \otimes_F -$. The definition of s is the definition of V_0, V_1 in fact. □

Proposition 4.3. *Under assumptions of the Theorem 4.1 if in addition $A = R = F$ is a field of characteristic different from 2, $d = 2m$ is even and $q = \sum_{i=0}^{d+1}(-1)^i z_i^2$, $q' = \sum_{i=0}^{d}(-1)^i z_i^2$, then $K_0(S) \cong \mathbb{Z} \oplus \mathbb{Z}$ has generators $1 = [S^1], [V_0], [V_1]$ subject to defining identity $[V_0] + [V_1] = 2^m[S^1]$.*

Proof. It is an immediate consequence of Proposition 4.2. $\square$

Proposition 4.4. *Under assumptions of the Theorem 4.1 if in addition $A = R = F$ is a field of characteristic different from 2, $d = 2m$ is even and $q = \sum_{i=0}^{d+1}(-1)^i z_i^2$, $q' = \sum_{i=0}^{d}(-1)^i z_i^2$, then:*

i) *if m is even, then*

$$\begin{aligned} E^+(X) &= E^+(S) = \mathbb{Z}/2\mathbb{Z}[S^1] \oplus \mathbb{Z}/2\mathbb{Z}(2^{m-1}[S^1] - [V_0]), \\ E^-(X) &= E^-(S) = 0; \end{aligned}$$

ii) *if m is odd, then*

$$\begin{aligned} E^+(X) &= E^+(S) = \mathbb{Z}/2\mathbb{Z}[S^1], \\ E^-(X) &= E^-(S) = \mathbb{Z}/2\mathbb{Z}(2^{m-1}[S^1] - [V_0]). \end{aligned}$$

Proof. By Proposition 3.8 b) ii) and iv), the involution $\hat{}$ on $K_0(X)$ has the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & (-1)^m \end{bmatrix}$$

with respect to the base $[S^1], 2^{m-1}[S^1] - [V_0]$ of the group $K_0(X)$. $\square$

Example 4.4.1. For $d = 4$ the consider the affine quadric defined by the equation $x_0y_0 + x_1y_1 + z^2 = 1$. The sequence

$$0 \longleftarrow F[y_0, y_1] \longleftarrow S \overset{\alpha}{\longleftarrow} S^3 \overset{\beta}{\longleftarrow} S^4 \overset{\gamma}{\longleftarrow} S^4 \longleftarrow V_0 \longleftarrow 0$$

where

$$\begin{aligned} \alpha &= [1 - z, -x_1, x_0], \\ \beta &= \begin{bmatrix} 1 + z & x_1 & -x_0 & 0 \\ y_1 & 1 - z & 0 & x_0 \\ -y_0 & 0 & 1 - z & x_1 \end{bmatrix}, \\ \gamma &= \frac{1}{2}\begin{bmatrix} 1 - z & -x_1 & x_0 & 0 \\ -y_1 & 1 + z & 0 & -x_0 \\ y_0 & 0 & 1 + z & -x_1 \\ 0 & -y_0 & -y_1 & 1 - z \end{bmatrix}, \end{aligned}$$

is exact and is the sequence of sections over complement to hyperplane section of the truncated canonical resolution for the structural sheaf of Y from the example 3.8.1. The map γ is chosen to be a projection of S^4 onto a direct summand. Hence V_0 is a submodule of S^4 generated by columns of the matrix

$$1 - \gamma = \frac{1}{2}\begin{bmatrix} 1 + z & x_1 & -x_0 & 0 \\ y_1 & 1 - z & 0 & x_0 \\ -y_0 & 0 & 1 - z & x_1 \\ 0 & y_0 & y_1 & 1 + z \end{bmatrix}.$$

Note that $m = 2$, so $E^-(S)$ is trivial. In fact, $\mathrm{rank} V_0 = 2$, so the module V_0 carries the canonical nonsingular skew-symmetric bilinear form, defined by the exterior multiplication $V_0 \times V_0 \longrightarrow \bigwedge^2 V_0 \cong S^1$.

Proposition 4.5. *Under assumptions of the Theorem 4.1 in addition $A = R = F$ is a field of characteristic different from 2, $d = 2m$ is even, m is odd, then in the ring $K_0(S) = \mathbb{Z}[S^1] \oplus \mathbb{Z}(2^{m-1}[S^1] - [V_0])$ the equality*

$$(2^{m-1}[S^1] - [V_0])^2 = 0$$

holds.

Proof. The rank map $K_0(S) \longrightarrow \mathbb{Z}$ is a ring homomorphism with the kernel $\mathbb{Z}(2^{m-1}[S^1]-[V_0])$. Moreover, $(2^{m-1}[S^1]-[V_0])\hat{} = -(2^{m-1}[S^1]-[V_0])$, so $(2^{m-1}[S^1]-[V_0])^2$ is a fixed point of $\hat{}$, hence it is a multiply of $[S^1]$, and it is an element of rank zero, hence it is a multiply of $2^{m-1}[S^1] - [V_0]$. □

It will be crucial for the construction in the next section that the identity $[V_0] + [V_1] = 2^m[S^1]$ arises from a direct decomposition with special properties.

Proposition 4.6. *If under assumptions of the Theorem 4.1 if in addition $A = R = F$ is a field of characteristic different from 2, $d = 2m$ is even and $q = \sum_{i=0}^{d+1}(-1)^i z_i^2$, $q' = \sum_{i=0}^{d}(-1)^i z_i^2$, $N = 2^m$ then*

a) *$S^N \cong V_0 \oplus V_1$;*
b) *if $m \equiv 2 \mod 4$, then there exists a direct decomposition $S^N = S^{N/2} \oplus S^{N/2}\hat{}$ such that for associated symmetric hyperbolic form $\overline{\chi}$ on S^N the direct summands V_0, V_1 are totally isotropic.*

Proof. In the ring S the last variable z_{d+1} is set to be 1:

$$z_{d+1} = 1.$$

More precisely, affine coordinates z_i are obtained as $\dfrac{z_i}{z_{d=1}}$ form homegoneous coordinates z_i. Therefore the multiplier $\varphi \in S \otimes_F V$ is equal to $\sum_{i=0}^{d} z_i \otimes v_i + v_{d+1}$. The exact sequence 3.1 restricted to Spec$S$ reduces to the following exact sequence of free S-modules:

$$\cdots \xrightarrow{\varphi\cdot} S \otimes_F C_{n+d+1} \xrightarrow{\varphi\cdot} S \otimes_F C_{n+d} \xrightarrow{\varphi\cdot} S \otimes_F C_{n+d-1} \xrightarrow{\varphi\cdot} \cdots .$$

There is one projective S-module $U = \Gamma(\mathrm{Spec}S, \mathcal{U}_{-1})$ such that

$$U \cong \mathrm{Coker}(S \otimes_F C_0 \xrightarrow{\varphi\cdot} S \otimes_F C_1) \cong \mathrm{Coker}(S \otimes_F C_1 \xrightarrow{\varphi\cdot} S \otimes_F C_0).$$

Moreover $U' = \Gamma(\mathrm{Spec}S, \mathcal{U}'_{-1}) = U \otimes_{C_0} P_0$, $U'' = \Gamma(\mathrm{Spec}S, \mathcal{U}''_{-1}) = U \otimes_{C_0} P_1$. In addition to central idempotents $\frac{1}{2}(1 \pm \delta)$ consider a family of minimal orthogonal idempotents of the even Clifford algebra C_0 defined for sequences $I = (i_0, i_1, ..., i_m)$ of ± 1's as follows:

$$e_I = \frac{1}{2^{m+1}}(1 + i_0 v_0 v_1) \cdot \ldots \cdot (1 + i_m v_{2m} v_{2m+1}).$$

It follows that

$$\begin{aligned}\delta\varepsilon_I &= v_0\cdot v_1\cdot\ldots\cdot v_{2m}\cdot v_{2m+1}\cdot\frac{1}{2^{m+1}}(1+i_0v_0v_1)\cdot\ldots\cdot(1+i_mv_{2m}v_{2m+1})\\ &= \frac{1}{2^{m+1}}v_0\cdot v_1\cdot(1+i_0v_0v_1)\cdot\ldots\cdot v_{2m}\cdot v_{2m+1}\cdot(1+i_mv_{2m}v_{2m+1})\\ &= \frac{1}{2^{m+1}}(v_0v_1+i_0)\cdot\ldots\cdot(v_{2m}v_{2m+1}+i_m)\\ &= i_0i_1...i_m\frac{1}{2^{m+1}}(1+i_0v_0v_1)\cdot\ldots\cdot(1+i_mv_{2m}v_{2m+1}).\end{aligned}$$

Thus

- $\frac{1}{2}(1+\delta)\varepsilon_I=\varepsilon_I$ if the number of -1's in the sequece I is even ,
- $\frac{1}{2}(1+\delta)\varepsilon_I=0$ if the number of -1's in the sequence I is odd ,
- $\frac{1}{2}(1-\delta)\varepsilon_I=\varepsilon_I$ if the number of -1's in the sequence I is odd ,
- $\frac{1}{2}(1-\delta)\varepsilon_I=0$ if the number of -1's in the sequece I is even .

Each left ideal $C_0\cdot\varepsilon_I$ is a minimal left ideal of C_0, and it is isomorphic to F^N as a C_0-module. Moreover, P_0 is a direct sum of half of these ideals, while P_1 is a direct sum of the other half. Denote for short $\varepsilon_0=\varepsilon_{(1,1,\ldots,1)}$, $\varepsilon_1=\varepsilon_{(1,1,\ldots,-1)}$, $\varepsilon_{-1}=\varepsilon_{(-1,-1,\ldots,-1)}$. Then

$$S^N=S\otimes_F F^N\cong S\otimes_F C_0\varepsilon_0\cong S\otimes_F C_0\otimes_{C_0}P_0\otimes_{P_0}F^N.$$

The identity $a=(1+\frac{1}{2}\varphi\cdot v_{d+1})a-\frac{1}{2}\varphi\cdot v_{d+1}\cdot a$ yields direct decomposition

$$S^N\cong S\otimes_F C_0\varepsilon_0=(1+\frac{1}{2}\varphi\cdot v_{d+1})\cdot(S\otimes_F C_0\varepsilon_0)\oplus\varphi\cdot(S\otimes_F C_1\varepsilon_0)$$

since $v_{d+1}C_0=C_1$. Therefore

$$\begin{aligned}V_0 &= \operatorname{Coker}(S\otimes_F C_1\xrightarrow{\varphi\cdot}S\otimes_F C_0)\otimes_F C_0\otimes_{C_0}P_0\otimes_{P_0}F^N\\ &\cong (1+\frac{1}{2}\varphi\cdot v_{d+1})\cdot(S\otimes_F C_0\varepsilon_0),\end{aligned}$$

so

$$S^N\cong V_0\oplus\varphi\cdot(S\otimes_F C_1\varepsilon_0).$$

Analogously

$$S^N\cong S\otimes_F C_0\varepsilon_1=(1+\frac{1}{2}\varphi\cdot v_{d+1})\cdot(S\otimes_F C_0\varepsilon_1)\oplus\varphi\cdot(S\otimes_F C_1\varepsilon_1)$$

where $(1+\frac{1}{2}\varphi\cdot v_{d+1})\cdot(S\otimes_F C_0\varepsilon_1)\cong V_1$, and

$$S^N=S\otimes_F F^N\cong S\otimes_F C_0\varepsilon_1\cong S\otimes_F C_0\otimes_{C_0}P_1\otimes_{P_1}F^N.$$

Left multiplication by φ and right multiplication by v_{d+1} yield

$$\begin{aligned}V_1 &\cong \varphi V_1v_{d+1}\cong\varphi(1+\frac{1}{2}\varphi v_{d+1})\cdot(S\otimes_F C_0\varepsilon_1)\cdot v_{d+1}=\varphi\cdot(S\otimes_F C_0\varepsilon_1)\cdot v_{d+1}\\ &= \varphi\cdot(S\otimes_F C_0v_{d+1}\varepsilon_0)=\varphi\cdot(S\otimes_F C_1\varepsilon_0),\end{aligned}$$

since $v_{d+1}\varepsilon_0=\varepsilon_1v_{d+1}$. Thus

$$S^N\cong V_0\oplus V_1.$$

To prove claim b) consider the principal antiauthomorphism $\Im$ of $S\otimes_F C_0$.

First of all note that $\Im(\varepsilon_0)=\varepsilon_{-1}$. Next, the set of all products $v_{2i_1+1}\cdot v_{2i_2+1}\cdot\ldots\cdot v_{2i_k+1}$ with $0\leqslant i_1<i_2<\ldots<i_k\leqslant m$ form a base of the minimal ideal $C\varepsilon_0$

of the Clifford algebra $C = C(q)$, and set of such a products with an even number of factors form a base of $C_0\varepsilon_0$. Moreover,

$$\Im((\prod_{i\in A} v_{2i+1})\cdot\varepsilon_0)\cdot(\prod_{i\in B} v_{2i+1})\cdot\varepsilon_0 =$$
$$\begin{cases} 0 & \text{if } A\cap B\neq\emptyset \\ & \text{or } A\cup B\neq\{0,1,...,m\} \\ \pm v_1 v_3\cdot\ldots\cdot v_{d+1}\varepsilon_0 & \text{if } A\cap B=\emptyset \\ & \text{and } A\cup B=\{0,1,...,m\}\end{cases}.$$

Thus for odd m the map $(\alpha\varepsilon_0, \beta\varepsilon_0)\mapsto$coefficient of $v_1\cdot v_3\cdot\ldots\cdot v_{d+1}\varepsilon_0$ in $\Im(\alpha\varepsilon_0)\cdot\beta\varepsilon_0$ defines a nonsingular pairing on S^N, such that the adjoint of it is an isomorphism of direct summands in suitably chosen direct decomposition $S^N = S^{N/2}\oplus S^{N/2}$. This pairing is symmetric for $m\equiv 3 \bmod 4$, and is skew-symmetric for $m\equiv 1 \bmod 4$.

Finally,

$$\Im(\varphi\cdot(S\otimes_F C_1\varepsilon_0))\cdot\varphi\cdot(S\otimes_F C_1\varepsilon_0) = (\varepsilon_{-1}S\otimes_F C_1)\varphi^2(S\otimes_F C_1\varepsilon_1) = 0,$$

$$\Im((1+\frac{1}{2}\varphi\cdot v_{d+1})\cdot(S\otimes_F C_0\varepsilon_0))\cdot(1+\frac{1}{2}\varphi\cdot v_{d+1})\cdot(S\otimes_F C_0\varepsilon_0) =$$

$$(\varepsilon_{-1}S\otimes_F C_0)(1+\frac{1}{2}v_{d+1}\cdot\varphi)(1+\frac{1}{2}\varphi\cdot v_{d+1})\cdot(S\otimes_F C_0\varepsilon_0) =$$

$$(\varepsilon_{-1}S\otimes_F C_0)(1+\frac{1}{2}(v_{d+1}\cdot\varphi+\varphi\cdot v_{d+1}))\cdot(S\otimes_F C_0\varepsilon_0) = 0$$

so V_0 and V_1 are totally isotropic. □

Example 4.6.1. Consider the case $d = 2$. As in Example 4.4.1 there is exact sequence

$$0\longleftarrow F[y_0]\longleftarrow S\overset{\alpha}{\longleftarrow} S^2\overset{\beta}{\longleftarrow} S^2\overset{\gamma}{\longleftarrow} S^2\overset{\beta}{\longleftarrow}\cdots$$

where

$$S = F[x_0,y_0,z_2]/(x_0y_0+z_2^2-1)$$

$$\begin{aligned}\alpha &= [z_2-1, x_0],\\ \beta &= \frac{1}{2}\begin{bmatrix}1+z_2 & x_0\\ y_0 & 1-z_2\end{bmatrix},\\ \gamma &= \frac{1}{2}\begin{bmatrix}1-z_2 & -x_0\\ -y_0 & 1+z_2\end{bmatrix}.\end{aligned}$$

Thus $\beta+\gamma = 1$, $\beta\gamma=\gamma\beta=0$, hence $\beta^2=\beta$, $\gamma^2=\gamma$. V_0 is the submodule of S^2 spanned by columns of γ, $V_1$is the submodule of S^2 spanned by columns of β and the form $\overline{\chi}$ has matrix $\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$:

$$\begin{bmatrix}1+z_2 & x_0\\ y_0 & 1-z_2\end{bmatrix}^T\cdot\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\cdot\begin{bmatrix}1+z_2 & x_0\\ y_0 & 1-z_2\end{bmatrix} =$$
$$\begin{bmatrix}0 & -y_0x_0-z_2^2+1\\ y_0x_0+z_2^2-1 & 0\end{bmatrix} = 0.$$

Example 4.6.2. Interesting case of smallest dimension is the one for $d = 6$. Then $N = 2^m = 8$,

$$S = F[x_0, y_0, x_1, y_1, x_2, y_2, z_6]/(x_0y_0 + x_1y_1 + x_2y_2 + z_6^2 - 1).$$

V_0 and V_1 are submodules of the free module S^8 spanned by columns of matrices

$$\beta = \frac{1}{2}\begin{bmatrix} 1-z_6 & 0 & 0 & 0 & -y_2 & y_1 & -y_0 & 0 \\ 0 & 1-z_6 & 0 & 0 & -x_1 & -x_2 & 0 & y_0 \\ 0 & 0 & 1-z_6 & 0 & x_0 & 0 & -x_2 & y_1 \\ 0 & 0 & 0 & 1-z_6 & 0 & x_0 & x_1 & y_2 \\ -x_2 & -y_1 & y_0 & 0 & 1+z_6 & 0 & 0 & 0 \\ x_1 & -y_2 & 0 & y_0 & 0 & 1+z_6 & 0 & 0 \\ -x_0 & 0 & -y_2 & y_1 & 0 & 0 & 1+z_6 & 0 \\ 0 & x_0 & x_1 & x_2 & 0 & 0 & 0 & 1+z_6 \end{bmatrix}$$

$$\gamma = \frac{1}{2}\begin{bmatrix} 1+z_6 & 0 & 0 & 0 & y_2 & -y_1 & y_0 & 0 \\ 0 & 1+z_6 & 0 & 0 & x_1 & x_2 & 0 & -y_0 \\ 0 & 0 & 1+z_6 & 0 & -x_0 & 0 & x_2 & -y_1 \\ 0 & 0 & 0 & 1+z_6 & 0 & -x_0 & -x_1 & -y_2 \\ x_2 & y_1 & -y_0 & 0 & 1-z_6 & 0 & 0 & 0 \\ -x_1 & y_2 & 0 & -y_0 & 0 & 1-z_6 & 0 & 0 \\ x_0 & 0 & y_2 & -y_1 & 0 & 0 & 1-z_6 & 0 \\ 0 & -x_0 & -x_1 & -x_2 & 0 & 0 & 0 & 1-z_6 \end{bmatrix}$$

which meet analogous conditions: $\beta + \gamma = 1$, $\beta\gamma = \gamma\beta = 0$. The symmetric bilinear form on S^8, which has V_0 and V_1 as totally isotropic submodules has the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

5. Witt ring of the projective line.

Consider the ring $S = F[z_0, z_1, ..., z_d]/(\sum_{i=0}^{d}(-1)^i z_i^2 - 1)$ and the projective line $X = \mathbb{P}^1_S = \text{Proj}S[t_0, t_1]$. By the Proposition 2.2, corollary 2.4 and Proposition 4.4 above, for $d \equiv 2 \pmod 4$ the group $E^+(X)$ has four elements:

$$E^+(X) = \mathbb{Z}/2\mathbb{Z}[\mathcal{O}_X] \oplus \mathbb{Z}/2\mathbb{Z}(2^{m-1}[S^1] - [V_0]) \boxtimes H,$$

where $\boxtimes$ denotes the tensor product of inverse images by projections of X onto Spec$S$ and onto $\mathbb{P}^1$. We need to find a self-dual locally free sheaf $\mathcal{M}$ on X which maps onto $(2^{m-1}[S^1] - [V_0]) \boxtimes H$ in $E^+(X)$ and a symmetric bilinear form on $\mathcal{M}$. Note that in $K_0(X)$ the element $(2^{m-1}[S^1] - [V_0]) \boxtimes H$ is the direct image of

$2^{m-1}[S^1] - [V_0]$ for immersion ι of a S-rational point Y into X since there is exact sequence

$$0 \longleftarrow \iota_* \mathcal{O}_Y \longleftarrow \mathcal{O}_X \xleftarrow{t_0 \cdot} \mathcal{O}_X(-1) \longleftarrow 0$$

of sheaves ($\mathcal{O}_Y = S$). Twist does not change the sheaf $\iota_* \mathcal{O}_Y$, since for arbitrary integer k the sequence

$$0 \longleftarrow \iota_* \mathcal{O}_Y \longleftarrow \mathcal{O}_X(k) \xleftarrow{t_0 \cdot} \mathcal{O}_X(k-1) \longleftarrow 0$$

is exact.

Theorem 5.1. *If $d = 2m$, $m = 4k + 3$ and $S = F[z_0, z_1, ..., z_d]/(\sum_{i=0}^{d}(-1)^i z_i^2 - 1)$, then there exists a vector bundle $\mathcal{M}$ on $\mathbb{P}^1_S$ and a symmetric bilinear form $\phi : \mathcal{M} \longrightarrow \mathcal{M}\hat{}$ such that*

$$e^0(\mathcal{M}, \phi) = (2^{m-1}[S^1] - [V_0]) \boxtimes H.$$

In particular the Witt ring $W(\mathbb{P}^1_S)$ is greater than the Witt ring $W(S)$.

Proof. Let, like in Proposition 4.6, $N = 2^m$. Fix a direct decompositions

$$S^N = S^{N/2} \oplus S^{N/2}\hat{} = V_0 \oplus V_1$$

such that V_0 is a totally isotropic subbundle of the hyperbolic space $(S^N, \overline{\chi})$. These data define a symmetric formation $(\iota_* \mathcal{O}_Y^N, \overline{\chi}; \iota_* \mathcal{O}_Y^{N/2}, \iota_* V_0)$ in $\mathbb{M}_1$ in the sense of [2]. This formation has a resolution (in the sense of Definition 15 p. 464 of [2]) - following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_X(-1)^{N/2} & \to & \mathcal{O}_X(-1)^{N/2} \oplus \mathcal{O}_X^{N/2} & \to & \mathcal{O}_X^{N/2} & \to & 0 \\
 & & \downarrow & & \downarrow h & & \downarrow & & \\
0 & \to & \mathcal{O}_X^{N/2} & \to & \mathcal{O}_X^{N/2} \oplus \mathcal{O}_X(1)^{N/2} & \to & \mathcal{O}_X(1)^{N/2} & \to & 0 \\
 & & \downarrow & & \downarrow \pi & & \downarrow & & \\
0 & \to & \iota_* \mathcal{O}_Y^{N/2} & \to & \iota_* \mathcal{O}_Y^{N/2} \oplus \iota_*(\mathcal{O}_Y^{N/2}\hat{}) & \to & \iota_*(\mathcal{O}_Y^{N/2}\hat{}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

such that each column is a direct sum of resolutions $0 \longleftarrow \iota_* \mathcal{O}_Y \longleftarrow \mathcal{O}_X(k) \longleftarrow \mathcal{O}_X(k-1) \longleftarrow 0$ for appropriate integer k. Moreover, there is nonsingular $\mathcal{O}_X(1)$-valued symmetric bilinear form

$$\mathcal{O}_X^{N/2} \oplus \mathcal{O}_X(1)^{N/2} \xrightarrow{\chi} (\mathcal{O}_X^{N/2} \oplus \mathcal{O}_X(1)^{N/2})\hat{} \otimes \mathcal{O}_X(1)$$

which reduces to $\overline{\chi}$ on $\iota_* \mathcal{O}_Y^N$. The following is (with adjusted notation) the Lemma 16 of [2] with its proof:

Lemma 5.2. *Let (r) be a resolution of a formation as above. Let $\mathcal{M}$ be the subsheaf of $\mathcal{O}_X^{N/2} \oplus \mathcal{O}_X(1)^{N/2}$ defined by $\pi^{-1}(V_0)$ (this means that $\mathcal{M}$ is the pullback*

$$\begin{array}{ccc}
\mathcal{M} & \to & \mathcal{O}_X^{N/2} \oplus \mathcal{O}_X(1)^{N/2} \\
\downarrow & & \downarrow \\
\iota_* V_0 & \to & \iota_* \mathcal{O}_Y^{N/2} \oplus \iota_*(\mathcal{O}_Y^{N/2}\hat{})
\end{array}$$

).

Then $\mathcal{M}$ is a locally free sheaf, and the restriction ϕ of of the form χ to $\mathcal{M}$ defines a non-singular symmetric bilinear form:

$$\mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{O}_X.$$

Proof. Since this is a local statement, the proof is the same as in [6] 7.3, p. 374. □

It remains to compute $e^0(\mathcal{M}, \phi)$. $\mathcal{M}$ is a product of $\mathcal{O}_X^{N/2} \oplus \mathcal{O}_X(1)^{N/2}$ and $\iota_* V_0$ over $\iota_* \mathcal{O}_Y^{N/2} \oplus \iota_*(\mathcal{O}_Y^{N/2}\hat{})$. Thus in the group $K_0(X)$

$$\begin{aligned}[\mathcal{M}] = [\mathcal{O}_X^{N/2} \oplus \mathcal{O}_X(1)^{N/2}] + [\iota_* V_0] - [\iota_* \mathcal{O}_Y^{N/2} \oplus \iota_*(\mathcal{O}_Y^{N/2}\hat{})] = \\ N/2[\mathcal{O}_X] + N/2[\mathcal{O}_X(1)] + [V_0]([\mathcal{O}_X] - [\mathcal{O}_X(-1)]) \\ - N/2([\mathcal{O}_X] - [\mathcal{O}_X(-1)]) - N/2([\mathcal{O}_X(1)] - [\mathcal{O}_X]) = \\ [V_0]([\mathcal{O}_X] - [\mathcal{O}_X(-1)]) + N/2[\mathcal{O}_X(-1)] + N/2[\mathcal{O}_X] = \\ N[\mathcal{O}_X] + ([V_0] - N/2[\mathcal{O}_X])H.\end{aligned}$$

$N = 2^{4k+3}$ is even, so $e^0(\mathcal{M}, \phi) = ([V_0] - 2^{m-1}) \boxtimes H$. □

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