Classical Motivic Polylogarithm according to Beilinson and Deligne

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Introduction

The aim of this work is to present the construction of the class of the cyclotomic, or classical polylogarithm in motivic cohomology. It maps to the elements in Deligne and l–adic cohomology defined and studied in Beilinson's "Polylogarithm and cyclotomic elements" ([B4]). The latter elements can be seen as being represented by a pro–variation of Hodge structure, or a pro–l–adic sheaf on the projective line minus three points.

Our main interest lies in the specialization of these sheaves to roots of unity: they represent the "cyclotomic" one–extensions of Tate twists already studied by Soulé ([Sou5]), Deligne ([D5]) and Beilinson ([B2]).

Let us be more precise: denote by μ_d^0 the set of primitive d–th roots of unity in $\mathbb{Q}(\mu_d) = \mathbb{Q}[T]/\Phi_d(T)$, $d \geq 1$. We get an alternative proof of the following theorem of Beilinson's:

Corollary 9.6.a). Assume $n \geq 0$, and denote by $r_{\mathcal{D}}$ the regulator map

$$H^1_{\mathcal{M}}(\operatorname{Spec}\mathbb{Q}(\mu_d), \mathbb{Q}(n+1)) \longrightarrow \bigoplus_{\sigma:\mathbb{Q}(\mu_d)\hookrightarrow\mathbb{C}} \mathbb{C}/(2\pi i)^{n+1}\mathbb{R} \; .$$

There is a map of sets

$$\epsilon_{n+1} : \mu_d^0 \longrightarrow H^1_{\mathcal{M}}(\operatorname{Spec}\mathbb{Q}(\mu_d), \mathbb{Q}(n+1))$$

such that

$$r_{\mathcal{D}} \circ \epsilon_{n+1} : \mu_d^0 \longrightarrow \bigoplus_{\sigma:\mathbb{Q}(\mu_d)\hookrightarrow\mathbb{C}} \mathbb{C}/(2\pi i)^{n+1}\mathbb{R}$$

maps a root of unity ω to $(-Li_{n+1}(\sigma\omega))_\sigma = \left(-\sum_{k\geq 1}\frac{\sigma\omega^k}{k^{n+1}}\right)_\sigma$.

Now fix a d–th primitive root of unity ζ in $\overline{\mathbb{Q}}$. This choice allows to identify continuous étale cohomology $H^1_{cont}(\operatorname{Spec}\mathbb{Q}(\mu_d), \mathbb{Q}_l(n+1))$ with a $\mathbb{Q}_l$–subspace of

$$\left(\left(\varprojlim_{r\geq 1} \mathbb{Q}(\mu_{l^\infty},\zeta)^*/(\mathbb{Q}(\mu_{l^\infty},\zeta)^*)^{l^r} \otimes \mu_{l^r}^{\otimes n}\right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l\right)^{\operatorname{Gal}(\mathbb{Q}(\mu_{l^\infty},\zeta)/\mathbb{Q}(\zeta))} .$$

Note that there is a distinguished root of unity T in $\mathbb{Q}(\mu_d)$. The main new result is the validity of [BlK], Conjecture 6.2:

Corollary 9.7. Let ϵ_{n+1} be the map constructed in 9.6. Under the above inclusion, the l–adic regulator

$$r_l : H^1_{\mathcal{M}}(\operatorname{Spec}\mathbb{Q}(\mu_d), \mathbb{Q}(n+1)) \longrightarrow H^1_{cont}(\operatorname{Spec}\mathbb{Q}(\mu_d), \mathbb{Q}_l(n+1))$$

maps $\epsilon_{n+1}(T^b)$ to

$$\frac{1}{d^n} \cdot \frac{1}{n!} \cdot \left(\sum_{\alpha^{l^r}=\zeta^b} [1-\alpha] \otimes (\alpha^d)^{\otimes n} \right)_r .$$

This compatibility shows in particular that Soulé's cyclotomic elements in $K_{2n+1}(F) \otimes_{\mathbb{Z}} \mathbb{Z}_l$ (for an abelian number field F and an odd prime l) are indeed induced by elements in K–theory itself, at least up to inversion of the prime 2 (Corollary 9.8). Furthermore, as noted by Bloch and Kato ([BlK], § 6), it implies the validity of the Tamagawa number conjecture modulo powers of 2 also for odd Tate twists $\mathbb{Q}(n)$, $n \geq 2$ (Corollary 9.9).

Even before giving a more detailed description of the content, the authors wish to make clear that the main ideas necessary for both the construction of the motivic polylogarithm and the proof of the compatibility with the realization classes already existent are due to others. Indeed, without any false modesty, we view this work as a somewhat extended exegesis of part of the last paragraph of the unpublished preprint "Motivic Polylogarithm and Zagier Conjecture" ([BD1]). We are grateful to both Beilinson and Deligne for not only letting us work with their ideas, but explicitly encouraging us to bring mathematics into a state in which the constructions of [BD1] can be performed. As a result of this effort, half of our paper consists of appendices, which we see as our main contribution to the subject. We hope they prove to be useful in other contexts than that treated in the main text.

We also need to mention in advance that we have not treated the "Zagier Conjecture" part of Beilinson's and Deligne's paper. One reason is of course the sheer size of the present article. More significantly, we feel that in this respect we do not have anything new to add to the original.

We see two main groups of papers related to polylogarithms:

The first deals with realizations of polylogarithms, and is therefore concerned with objects like variations of Hodge structure or l–adic mixed lisse sheaves. Maybe the nucleus of these papers is Deligne's observation that the analytic and topological properties of the dilogarithm Li_2, viewed as a multivalued holomorphic function on $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}$, can be coded by saying that Li_2 is an entry of the period matrix of a certain rank three variation of $\mathbb{Q}$–Tate–Hodge structure on $\mathbb{P}^1_{\mathbb{C}}\setminus\{0,1,\infty\}$.

This approach was extended in Beilinson's "Polylogarithm and Cyclotomic elements" ([B4]), where he defined pro–objects in the categories of variations and l–adic sheaves on $\mathbb{P}^1\setminus\{0,1,\infty\}$ respectively. These pro–objects, the cyclotomic, or classical polylogarithms, enjoy a remarkable property, called rigidity: up to unique isomorphism, they are uniquely determined by the underlying pro–local system on $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}$ / pro–sheaf on $\mathbb{P}^1_{\overline{\mathbb{Q}}}\setminus\{0,1,\infty\}$. Their fibres at roots of unity different from 1 coincide with the cyclotomic extensions mentioned above.

Soon afterwards, Beilinson and Levin succeeded in constructing a pro–variation of Hodge structure in another geometric situation, which behaved similarly: as is established in "The Elliptic Polylogarithm" ([BL]), this object again satisfies rigidity. Its fibres at torsion points on the elliptic curve which are different from zero turn out to be homologically meaningful, i.e., are known to lie in the image of the regulator from K–theory to absolute Hodge cohomology. In [W2], the definition of polylogarithmic extensions was generalized to the setting of mixed Shimura varieties. However, up to now, no examples other than the classical and elliptic polylogarithms seem to have been described by means of multivalued functions, i.e., higher dimensional higher logarithms, or l–adic torsors. Meanwhile, as in the known examples, one may reasonably expect rigidity to play an important role in the explicit construction of the polylogarithm.

The hope and indeed, the motivation underlying these papers is that once a satisfactory formalism of motivic sheaves is developed, the definition of polylogarithms should basically carry over. We would thus obtain polylogarithmic classes in Ext groups of motives, these groups being supposedly closely connected to K–theory, of which everything already defined on the level of realizations would turn out to be the respective regulator. In the two known examples, this hope has proved to be very well compatible with Beilinson's conjectures on special values of L–functions.

Nowhere is this hope documented more manifestly than in Beilinson's and Deligne's "Interprétation motivique de la conjecture de Zagier reliant polylogarithmes et régulateurs" ([BD2]): if there is such a motivic formalism, then the weak version of Zagier's conjecture necessarily holds: not only the values at roots of unity of higher logarithms, but also appropriate linear combinations of *arbitrary* values must lie in the image of the regulator.

Even with the recent work of Voevodsky and Suslin available, some more effort is required to establish the foundations necessary for this hope to come true. For the time being, and in each case separately, honest work is needed to perform the K–theoretic constructions, and calculate their images under the regulators.

The second class of papers is concerned with precisely that task. In analogy with the above, one should first mention Bloch's "Application of the dilogarithm function in algebraic K–theory and algebraic geometry" ([Bl]).

Beilinson's "Higher regulators and values of L–functions" ([B2]) provided the K–theoretic construction of cyclotomic elements. In terms of polylogs, this appears as a comparatively easy case. Still, the proof of the fact that the regulator to Deligne cohomology maps these elements to the classes of the fibres at roots of unity of the cyclotomic polylogarithm (loc. cit., Theorem 7.1.5) required quite a bit of secondary literature ([Neu]), and can be regarded as complete only since Esnault ([E]) produced a proof of the "crucial lemma" in [Neu].

As for Zagier's conjecture, we mention Goncharov's "Polylogarithms and Motivic Galois Groups" ([Go]), where Zagier's conjecture, including the surjectivity statement is proved for K_5 of a number field, and de Jeu's "Zagier's Conjecture and Wedge Complexes in Algebraic K–theory" ([Jeu]), which contains the proof of the weak version of Zagier's conjecture, independently of motivic considerations, for K_{2n-1} of a number field, and arbitrary $n \geq 2$.

Typically, the objects of interest in this class of papers are complexes, cocycles, and symbols, i.e., objects which do not constantly afford a geometric, or sheaf–theoretic interpretation. It is by no means easy to see, say, how a concrete element in some Deligne cohomology group can be interpreted as an extension of variations of $\mathbb{R}$–Hodge structure. These and similar difficulties present themselves to the reader willing to translate from one class to the other.

The authors like to think of the present article as an attempt to bridge the gap between the two disciplines.

In a sense, the coarse structure of the article follows the above scheme: sections 1–6 are entirely sheaf–theoretic. Anything we say there is therefore a priori restricted to the level of realizations, i.e., non–motivic. In sections 7–9, K–theory enters.

In order to ensure that the infinitesimal step from the end of section 6 to the beginning of section 7 proceed without serious turbulences, one needs to be able to talk about K–theory of simplicial schemes, regulators to absolute (Hodge and l–adic) cohomology of simplicial schemes, and an interpretation of the latter groups as Ext groups of Hodge modules / l–adic sheaves. The appendices provide the necessary foundations.

Once we work in a category of "mixed sheaves" satisfying enough axioms, most arguments are purely formal. One such category is given by mixed Hodge modules. We use perverse l–adic sheaves on schemes over number fields and rings of integers ([H2]), which are the correct l–adic analogue. Readers not familiar with mixed perverse sheaves will still find themselves able to understand the constructions of the present article: everything can be translated to "usual" constructible sheaves, by shifting the degree of cohomology objects appropriately.

We now turn to the description of the finer structure of this work:

In section 1, we normalise the sheaf theoretic notations used throughout the whole article.

Section 2 gives a quick axiomatic description of the logarithmic sheaf $\mathcal{L}og$, and the (small) polylogarithmic extension *pol*. The main results on higher direct images (2.3) and the splitting principle (2.4) are reproved in later sections. The universal property (2.1) is needed only to connect the general definition of the logarithmic sheaf as a solution of a representability problem to the somewhat ad hoc, but much more geometric definition of section 4. A reader prepared to accept the results on the shape of the Hodge theoretic and l–adic incarnation of the polylogarithm (2.5, 2.6) may therefore take the constructions in sections 4 and 6 as a definition of both $\mathcal{L}og$ and *pol*, and view section 2 as an extended introduction providing background material.

In section 3, we establish the geometric situation used thereafter. As section 1, it is mainly intended for easier reference. We mention that basically the same geometric arrangement was used in [Jeu]. More precisely, writing down the iterated cone construction of loc. cit., one arrives at a simplicial object which is homotopy equivalent to Beilinson's and Deligne's construction used here.

The main result of section 4 is Theorem 4.11: the logarithmic sheaf $\mathcal{L}og$ (or rather, its pullback to $\mathbb{U} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$) is identified with a projective limit of relative cohomology objects with coefficients in Tate twists of certain subschemes of products of $\mathbb{G}_m$ over $\mathbb{U}$.

Section 5 contains a geometric proof of the splitting principle (5.2). Since we need a proof which translates easily to the motivic situation, we return to Beilinson's original approach to the splitting principle, which consists of an analysis of the action of the multiplication by natural numbers on our absolute cohomology groups.

The main result of the next section is 6.6. The Leray spectral sequence together with Theorem 4.11 suggest that one–extensions of $\mathbb{Q}(0)$ by $\mathcal{L}og$ should be described as elements of the projective limit of absolute cohomology with Tate coefficients of certain simplicial schemes. 6.6 makes explicit this identification for *pol*.

Given that quite a lot has been said about the l–adic and Hodge theoretic incarnations of the classical polylogarithm ([B4], [BD2], [W3]), the reader may wonder why sheaf–theoretic considerations still take up one third of this work.

Indeed, the constructions of sections 4–6 would not be necessary if a satisfactory formalism of mixed motivic sheaves were available, whose associated absolute cohomology groups with Tate coefficients, i.e., Ext groups of the neutral object by Tate twists, would equal motivic cohomology. The necessity to replace a simple geometric situation by a rather complicated, even simplicial one, in order to replace complicated coefficients like $\mathcal{L}og$ by Tate twists, should be seen as the main source of difficulty in any attempt to the construction of motivic versions of polylogarithms.

In Section 7 our main tool, the "residue sequence" is constructed in the setting of motivic cohomology (Proposition 7.2 and Lemma 7.3). We have to use simplicial objects (more precisely simplicial objects in the big Zariski–topos) in order to be able to use the machinery developed in Appendix B. The arguments are very much parallel to those used for absolute cohomology of realizations in section 4. Everything is compatible with what was done before under the regulator maps.

Section 8 is the K–theoretic analogue of section 6. We consider a certain projective system of relative motivic cohomology groups. In order to show that its projective limit can be identified with $H^0_{\mathcal{M}}(\operatorname{Spec}\mathbb{Z}, 0) = \mathbb{Q}$, we use bijectivity or at least controlled injectivity of the regulator to Deligne cohomology. Once the identification is accomplished (Corollary 8.8) we then define in 8.9 the universal motivic polylog as the element of the projective system corresponding to $1 \in \mathbb{Q}$. Again compatibility with the polylog element in absolute cohomology will be clear from the construction.

In the final section 9 the motivic version of the splitting principle is shown (9.3). Again we strongly use the known behaviour of the regulator to show that the action of multiplication by natural numbers splits into eigenspaces. Applied to the preimage of the universal motivic polylogarithm this induces the cyclotomic elements in motivic cohomology. In the light of section 5 it is clear from their very construction that they induce the right elements not only in Deligne but also in continuous étale cohomology. We conclude by drawing the corollaries which are the main results announced at the beginning (9.6–9.9).

The Appendices can be read independently of the main text and of each other. They are meant to be used as a reference, but a careful reader might actually want to read them first.

Appendix A provides a natural interpretation of absolute Hodge cohomology over $\mathbb{R}$ as Ext groups with Tate coefficients in the category of algebraic Hodge modules over $\mathbb{R}$ (Theorem A.2.7). That such a sheaf–theoretic interpretation should be possible was already anticipated by Beilinson ([B1], 0.3), long before Hodge modules were defined.

The appendix is divided into two subsections. The first (A.1) starts with a summary of those parts of Saito's theory relevant to us. The central result is A.1.8, were we prove that for a smooth scheme $a : U \to \operatorname{Spec}(\mathbb{C})$, the polarizable Hodge complex $\underline{R\Gamma}(U, F)$ of [D3], (8.1.12) and [B1], § 4 is a representative for $a_*F(0)_U$, the object in the derived category of polarizable F–Hodge structures defined via Saito's formalism ([S2], 4.3). As a consequence, we are able (A.1.10) to identify absolute Hodge cohomology of a smooth scheme U over $\mathbb{C}$, as defined in [B1], § 5: it equals the Ext groups of Tate twists in the category of algebraic Hodge modules on U. The compatibility between the approaches of Deligne–Beilinson and of Saito will come as no surprise to the experts (see e.g. [S3], (2.8)). However, we were unable to find a quotable reference.

In A.2, we turn to the variant of the theory we really need: algebraic Hodge modules over $\mathbb{R}$. These live on the complexification of separated, reduced schemes of finite type over $\mathbb{R}$, and are basically the objects fixed by the natural involution on the category of mixed Hodge modules given by complex conjugation. The comparison statement for absolute Hodge cohomology over $\mathbb{R}$ (Theorem A.2.7) then follows formally from A.1.10.

Appendix B is an exposition of K–cohomology of simplicial schemes or more precisely of simplicial objects in the big Zariski–topos over an appropriate base scheme (cf. B.2.1). Applied to a regular scheme, we get back its K–groups (cf. B.2.3.a)). The discussion is based on the extremely useful (unfortunately unpublished) paper [GSo1] by Gillet and Soulé. More often than not the results in Appendix B will be due to them. An enlightening investigation was carried out by Jardine, in particular [Jr2]. De Jeu was the first to use this setting to define motivic cohomology of simplicial objects and check the key properties. In his article [Jeu] a brief summary of the main definitions and the theorems relevant to us can be found. He then proves Riemann-Roch in this setting. The wish for a complete published reference made us go over the material again.

Using simplicial methods has two rather different reasons: primarily this allows to use results in homotopical algebra to construct long exact sequences for relative K–cohomology in B.1.6. They will end up being the main computational tool we use in the main text. On the other hand a number of K–theoretic properties is well–understood only for regular schemes. We will replace some singular schemes by explicit simplicial schemes with regular components. For questions of motivic cohomology we shift our point of view: we see these simplicial schemes as the primary objects. In terms of realizations the singular and the simplicial object will be equivalent.

It is possible to define λ–operations on K–cohomology groups with negative index (B.2.8, they correspond to K–groups with positive indices) and hence motivic cohomology as Adams eigenspaces. This allows to define relative motivic cohomology. We show a variant of a Riemann–Roch theorem (B.2.16) for these. The above mentioned long exact sequences for relative K–cohomology hence induce long exact sequences for relative motivic cohomology (see also B.2.17).

The construction of a regulator map into cohomology of sheaves does not pose a serious problem using the results for schemes. We carry this out for continuous étale cohomology (B.4) and for absolute Hodge cohomology (B.5).

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