1 Mixed sheaves

We start by defining the sheaf categories which will be relevant for us. For our purposes, it will be necessary to work in the settings of mixed l–adic perverse sheaves ([H2]), and of algebraic mixed Hodge modules over $\mathbb{R}$ (A.2). Since the procedures are entirely analogous, we introduce, for economical reasons, the following rules: whenever an area of paper is divided by a vertical bar

|

the text on the left of it will concern the Hodge theoretic setting, while the text on the right will deal with the l–adic setting. Of course, we hope that before long, there will be a satisfactory formalism of mixed motivic sheaves providing a third setting to which our constructions can be applied. We let

$A := \mathbb{R}$, $F := \mathbb{Q}$,	$l :=$ a fixed prime number , $A := \mathbb{Z}\left[\frac{1}{l}\right]$, $F := \mathbb{Q}_l$,

and set $B := \mathrm{Spec}(A)$.

For any reduced, separated and flat scheme X of finite type over B, we let

$X_{\mathrm{top}} := X(\mathbb{C})$ as a topol. space , $\mathrm{Sh}(X_{\mathrm{top}}) := \mathrm{Perv}(X_{\mathrm{top}}, \mathbb{Q})$,	$X_{\mathrm{top}} := X \otimes_A \overline{\mathbb{Q}}$, $\mathrm{Sh}(X_{\mathrm{top}}) := \mathrm{Perv}(X_{\mathrm{top}}, \mathbb{Q}_l)$

the latter categories denoting the respective categories of perverse sheaves on X_{top} ([BBD], 2.2).

Next we define the category $\mathrm{Sh}(X)$: in the l–adic setting, we fix a pair $(\mathbf{S}, L)$ consisting of a horizontal stratification $\mathbf{S}$ of X ([H2], § 2) and a collection $L = \{L(S) \mid S \in \mathbf{S}\}$, where each $L(S)$ is a set of irreducible lisse l–adic sheaves on S. For all $S \in \mathbf{S}$ and $F \in L(S)$, we require that for the inclusion $j : S \hookrightarrow X$, all higher direct images $R^n j_* F$ are $(\mathbf{S}, L)$*–constructible*, i.e., have lisse restrictions to all $S \in \mathbf{S}$, which are extensions of objects of $L(S)$.

We assume that all $F \in L(S)$ are pure.

We can make this more explicit: in our computations X will always be a locally closed subscheme of some $\mathbb{A}^n$; the stratification is the pullback of the one defined before Lemma 4.5; $L(S)$ is the set of all Tate sheaves on S.

Following [H2], § 3, we define $D^b_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$ as the full subcategory of $D^b_c(X, \mathbb{Q}_l)$ of complexes with $(\mathbf{S}, L)$–constructible cohomology objects. Note that all objects will be mixed. By [H2], § 3, $D^b_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$ admits a perverse t–structure, whose heart we denote by $\mathrm{Perv}_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$.

$\mathrm{Sh}(X) := \mathrm{MHM}_{\mathbb{Q}}(X/\mathbb{R})$ (see A.2.4) .	$\mathrm{Sh}(X) := \mathrm{Perv}_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$.

Because of the horizontality requirement in the l–adic situation we have the full formalism of Grothendieck's functors only on the direct limit $D^b_m(\mathfrak{U}_X, \mathbb{Q}_l)$ of the $D^b_{(\mathbf{S},L)}(X_U, \mathbb{Q}_l)$, for U open in B, and $(\mathbf{S}, L)$ as above (see [H2], § 2). However, for a fixed morphism

$$\pi : X \longrightarrow Y ,$$

we have a notion of e.g. π_*–admissibility for a pair $(\mathbf{S}, L)$: this is the case if

$$D^b_{(\mathbf{S},L)}(X, \mathbb{Q}_l) \hookrightarrow D^b_m(\mathfrak{U}_X, \mathbb{Q}_l) \xrightarrow{\pi_*} D^b_m(\mathfrak{U}_Y, \mathbb{Q}_l)$$

factors through some $D^b_{(\mathbf{T},K)}(Y, \mathbb{Q}_l)$. Our computations will show, at least a posteriori, that for our choice of $(\mathbf{S}, L)$ all functors which appear are admissible. We will not stress these technical problems and even suppress $(\mathbf{S}, L)$ in our notation.

As in [BBD], we denote by $\pi_*, \pi^*, \mathrm{Hom}$ etc. the respective functors on the categories

$D^b\,\mathrm{Sh}(X) := D^b\,\mathrm{MHM}_{\mathbb{Q}}(X/\mathbb{R})$,	$D^b\,\mathrm{Sh}(X) := D^b_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$,

and $\mathcal{H}^q$ for the (perverse) cohomology functors.

We refer to objects of $\mathrm{Sh}(X)$ as sheaves, and to objects of $\mathrm{Sh}(X_{\mathrm{top}})$ as topological sheaves. Let us denote by

$$\mathbb{V} \longrightarrow \mathbb{V}_{\mathrm{top}}$$

the forgetful functor from $\mathrm{Sh}(X)$ to $\mathrm{Sh}(X_{\mathrm{top}})$.
If we use the symbol $W.$, it will always refer to the weight filtration.

If X is smooth, we let

$\mathrm{Sh}^s(X) := \mathrm{Var}_{\mathbb{Q}}(X/\mathbb{R}) \subset \mathrm{Sh}(X)$ (see A.2.1) , $\mathrm{Sh}^s(X_{\mathrm{top}}) :=$ the category of $\mathbb{Q}$–local systems on X_{top}.	$\mathrm{Sh}^s(X) := \mathrm{Et}^{l,m}_{\mathbb{Q}_l}(X) \subset \mathrm{Sh}(X)$, the category of lisse mixed $\mathbb{Q}_l$–sheaves on X, $\mathrm{Sh}^s(X_{\mathrm{top}}) :=$ the category of lisse $\mathbb{Q}_l$–sheaves on X_{top}.

We refer to objects of $\mathrm{Sh}^s(X)$ as smooth sheaves, and to objects of $\mathrm{Sh}^s(X_{\mathrm{top}})$ as smooth topological sheaves. Denote by $U\mathrm{Sh}^s(X)$ the category of unipotent objects of $\mathrm{Sh}^s(X)$, i.e., those smooth sheaves admitting a filtration, whose graded parts are pullbacks of smooth sheaves of $\mathrm{Sh}^s(B)$ via the structure morphism. Similarly, one defines $U\mathrm{Sh}^s(X_{\mathrm{top}})$.

Remark: Note that in the l–adic situation, the existence of a weight filtration, i.e., an ascending filtration $W.$ by subsheaves indexed by the integers, such that Gr^W_m is of weight m, is not incorporated in the definition of Sh^s – compare the warnings in [H2], § 3. In the Hodge theoretic setting, the existence of a weight filtration is part of the data.

Remark: We have to deal with a shift of the index when viewing e.g. a variation as a Hodge module, which occurs either in the normalization of the embedding

$$\mathrm{Var}_{\mathbb{Q}}(X/\mathbb{R}) \longrightarrow D^b \,\mathrm{MHM}_{\mathbb{Q}}(X/\mathbb{R})$$

or in the numbering of cohomology objects of functors induced by morphisms between schemes of different dimension. In order to conform with the conventions laid down in appendix A and [W1], chapter 4, we chose the second possibility: a variation *is* a Hodge module, not just a shift of one such. Similarly, a lisse mixed $\mathbb{Q}_l$–sheaf *is* a perverse mixed sheaf. Therefore, if X is of pure relative dimension d over B, then the embedding

$$\mathrm{Et}^{l,m}_{\mathbb{Q}_l}(X) \longrightarrow D^b_m(\mathfrak{U}_X, \mathbb{Q}_l)$$

associates to $\mathbb{V}$ the complex concentrated in degree $-d$, whose only non–trivial cohomology object is $\mathbb{V}$.

As a consequence, the numbering of cohomology objects of the direct image (say) will differ from what the reader might be used to: e.g., the cohomology of a curve is concentrated in degrees -1, 0, and 1 instead of 0, 1, and 2. Similarly, one has to distinguish between the "naive" pullback $(\pi^s)^*$ of a smooth sheaf and the pullback π^* on the level of $D^b \operatorname{Sh}(X)$: $(\pi^s)^*$ lands in the category of smooth sheaves, while π^* of a smooth sheaf yields only a smooth sheaf up to a shift.

In the special situation of pullbacks, we allow ourselves one notational inconsistency: if there is no danger of confusion (e.g. in Theorem 2.1), we use the notation π^* also for the naive pullback of smooth sheaves. Similar remarks apply for smooth topological sheaves.

For a scheme $a : X \to B$, we define

$$F(n)_X := a^* F(n) \in D^b \operatorname{Sh}(X) \ ,$$

where $F(n)$ is the usual Tate twist on B.

If X is smooth, we also have the naive Tate twist

$$F(n) \in \operatorname{Sh}^s(X) \subset \operatorname{Sh}(X)$$

on X. If X is of pure dimension d, then we have the equality

$$F(n) = F(n)_X[d] \ .$$

In order to keep our notation transparent, we have the following

Definition 1.1. *For any morphism $\pi : X \longrightarrow S$ of reduced, separated and flat B–schemes we let*

$$\mathcal{R}_S(X, \ \cdot \ ) := \pi_* : D^b \operatorname{Sh}(X) \longrightarrow D^b \operatorname{Sh}(S) \ ,$$
$$\mathcal{H}^i_S(X, \ \cdot \ ) := \mathcal{H}^i \pi_* : D^b \operatorname{Sh}(X) \longrightarrow \operatorname{Sh}(S) \ .$$

Definition 1.2. *For a closed reduced subscheme Z of a separated, reduced, flat B–scheme X of finite type, with complement $j : U \hookrightarrow X$, and an object $M^\cdot$ of $D^b \operatorname{Sh}(X)$, define*

a)
$$R\Gamma_{\text{abs}}(X, M^\cdot) := R \operatorname{Hom}_{D^b \operatorname{Sh}(X)}(F(0)_X, M^\cdot) \ ,$$
$$H^i_{\text{abs}}(X, M^\cdot) := H^i R\Gamma_{\text{abs}}(X, M^\cdot) \ ,$$

the absolute complex and absolute cohomology groups of X with coefficients in $M^{\cdot}$.

b) $$\begin{aligned} R\Gamma_{\mathrm{abs}}(X,n) &:= R\Gamma_{\mathrm{abs}}(X,F(n)_X)\ , \\ H^i_{\mathrm{abs}}(X,n) &:= H^i_{\mathrm{abs}}(X,F(n)_X)\ . \end{aligned}$$

c) $$\begin{aligned} R\Gamma_{\mathrm{abs}}(X \ \mathit{rel}\ Z,n) &:= R\Gamma_{\mathrm{abs}}(X,j_!F(n)_X)\ , \\ H^i_{\mathrm{abs}}(X \ \mathit{rel}\ Z,n) &:= H^i_{\mathrm{abs}}(X,j_!F(n)_X)\ , \end{aligned}$$

the relative absolute complex and relative absolute cohomology with Tate coefficients.

Remark: If X is a scheme over S, then we have the formulae

$$\begin{aligned} R\Gamma_{\mathrm{abs}}(X,\ \cdot\ ) &= R\Gamma_{\mathrm{abs}}\left(S,\mathcal{R}_S(X,\ \cdot\ )\right)\ , \\ H^i_{\mathrm{abs}}(X,\ \cdot\ ) &= H^i_{\mathrm{abs}}(S,\mathcal{R}_S(X,\ \cdot\ ))\ . \end{aligned}$$

2 The Logarithmic Sheaf, and the Polylogarithmic Extension

We aim at a sheaf theoretic description of the (small) classical polylogarithm on $\mathbb{P}^1 \setminus \{0,1,\infty\}$. Our first aim is a quick axiomatic definition of the *logarithmic pro–sheaf.* We need the following result:

Theorem 2.1. *Let X be the complement in a smooth, projective B–scheme of an NC–divisor relative to B ([SGA1], Exp. XIII, 2.1), all of whose irreducible components are smooth over B. Let $x \in X(B)$, and write $a : X \to B$. The functor*

$$x^* : U\mathrm{Sh}^s(X) \longrightarrow \mathrm{Sh}^s(B)$$

is representable in the following sense:

a) *There is a pro–object*

$$\mathcal{G}en_x \in \textit{pro–}U\mathrm{Sh}^s(X)\ ,$$

the generic pro–unipotent sheaf with basepoint x on X, which has a weight filtration satisfying

$$\mathcal{G}en_x / W_{-n}\mathcal{G}en_x \in U\mathrm{Sh}^s(X) \quad \textit{for all}\, n\ .$$

Note that this implies that the direct system

$$\left(R^0 a_* \underline{\mathrm{Hom}}(\mathcal{G}en_x / W_{-n}\mathcal{G}en_x, \mathbb{V})\right)_{n\in\mathbb{N}}$$

of smooth sheaves on B becomes constant for any $\mathbb{V} \in U\mathrm{Sh}^s(X)$.

This constant value is denoted by

$$R^0 a_* \underline{\mathrm{Hom}}(\mathcal{G}en_x, \mathbb{V})\ .$$

b) *There is a section*

$$1 \in \Gamma\left(B, x^* \mathcal{G}en_x\right)\ .$$

c) *The natural transformation of functors from $U\mathrm{Sh}^s(X)$ to $\mathrm{Sh}^s(B)$*

$$\begin{array}{rcl} ev : R^0 a_* \underline{\mathrm{Hom}}(\mathcal{G}en_x, \_) & \longrightarrow & x^*\ , \\ \varphi & \longmapsto & (x^*\varphi)(1) \end{array}$$

is an isomorphism. Similarly for the transformation of functors from $U\mathrm{Sh}^s(X_{\mathrm{top}})$ to $\mathrm{Sh}^s(B_{\mathrm{top}})$

$$\begin{array}{rcl} ev : R^0 a_* \underline{\mathrm{Hom}}\left((\mathcal{G}en_x)_{\mathrm{top}}, \_\right) & \longrightarrow & x^*\ , \\ \varphi & \longmapsto & (x^*\varphi)(1)\ . \end{array}$$

Consequently, the pairs $(\mathcal{G}en_x, 1)$ and $((\mathcal{G}en_x)_{\mathrm{top}}, 1)$ are unique up to unique isomorphism.

d) *The natural transformations of functors*

$$\begin{array}{rcl} Hom_{U\mathrm{Sh}^s(X)}(\mathcal{G}en_x, \_) & \longrightarrow & Hom_{\mathrm{Sh}^s(B)}(F(0), x^*\_) \quad \textit{and} \\ Hom_{U\mathrm{Sh}^s(X_{\mathrm{top}})}\left((\mathcal{G}en_x)_{\mathrm{top}}, \_\right) & \longrightarrow & \Gamma(B_{\mathrm{top}}, x^*\_) \end{array}$$

from $U\mathrm{Sh}^s(X)$ and $U\mathrm{Sh}^s(X_{\mathrm{top}})$ respectively are isomorphisms.

Proof. For a)–c), we refer to

[W1], Remark d) after Theorem 3.6, | [W1], Theorem 3.5.i),

and loc. cit., Theorem 3.5.ii).

For d), simply apply the functors $Hom_{\mathrm{Sh}^s(B)}(F(0), \_)$ and $\Gamma(B_{\mathrm{top}}, \_)$ to the result in c). ☐

Now let

$$\begin{gathered} \mathbb{G}_m := \mathbb{G}_{m,B} \ , \quad \mathbb{U} := \mathbb{P}^1_B \setminus \{0,1,\infty\}_B \ , \\ j : \mathbb{U} \hookrightarrow \mathbb{G}_m \ , \\ p : \mathbb{G}_m \longrightarrow B \ , \quad \widetilde{p} := p \circ j : \mathbb{U} \longrightarrow B \ . \end{gathered}$$

We may form the generic pro–unipotent sheaf with basepoint 1 on $\mathbb{G}_m$.

Definition 2.2. *$\mathcal{L}og := \mathcal{G}en_1 \in$ pro–$U\mathrm{Sh}^s(\mathbb{G}_m)$ is called the logarithmic pro–sheaf.*

We need to know $\mathcal{H}^{\cdot}_B(\mathbb{U}, j^*\mathcal{L}og(1))$:

Theorem 2.3. *a) $\mathcal{H}^q_B(\mathbb{U}, j^*\mathcal{L}og(1)) = 0$ for $q \neq 0$.*

b) $\mathcal{H}^0_B(\mathbb{U}, j^*\mathcal{L}og(1))$ has a weight filtration, and $W_{-1}\left(\mathcal{H}^0_B(\mathbb{U}, j^*\mathcal{L}og(1))\right)$ is isomorphic to

$$\prod_{k \geq 1} F(k) \ .$$

Remark: By these statements on the higher direct images of the pro–sheaf $j^*\mathcal{L}og(1)$, we mean the following:

a) For $q \neq 0$, the projective system

$$\mathcal{H}^q_B\left(\mathbb{U}, j^*(\mathcal{L}og/W_{-n}\mathcal{L}og)(1)\right)_{n \geq 1}$$

is ML–zero.

b) There is a morphism of projective systems

$$\mathcal{H}^0_B\left(\mathbb{U}, j^*(\mathcal{L}og/W_{-2m}\mathcal{L}og)(1)\right)_{m \geq 1} \longrightarrow \left(\prod_{k=0}^{m} F(k)\right)_{m \geq 1}$$

of sheaves with a weight filtration, such that the weight ≤ -1–parts of the projective systems of kernels and co-kernels are ML–zero.

Proof. One uses the exact triangle

$$\begin{tikzcd} 1_*1^! \arrow[rr] & & \mathrm{id}_{\mathbb{G}_m} \arrow[dl] \\ & j_*j^* \arrow[ul, "{[1]}"] & \end{tikzcd}$$

or rather, $\mathcal{H}^{\cdot}_B(\mathbb{G}_m, \_)$ of it, and the fact that $\mathcal{H}^{\cdot}_B(\mathbb{G}_m, \mathcal{L}og)$ is easily computable. For the details, see [W2], Theorem 1.3. Or use 4.11 and 6.2, whose proof is independent of 2.3. ☐

The theorem enables one to define the *small polylogarithmic extension* as the extension

$$pol \in \mathrm{Ext}^1_{U\mathrm{Sh}^s(\mathbb{U})}\left(F(1), \mathcal{L}og(1)\,|_{\mathbb{U}}\right) = \mathrm{Ext}^1_{\mathrm{Sh}(\mathbb{U})}\left(F(1), \mathcal{L}og(1)\,|_{\mathbb{U}}\right)$$

mapping to the natural inclusion $F(1) \hookrightarrow \prod_{k \geq 1} F(k)$ under the isomorphism

$$\begin{aligned} \mathrm{Ext}^1_{\mathrm{Sh}(\mathbb{U})}\left(F(1), \mathcal{L}og(1)\,|_{\mathbb{U}}\right) &= \mathrm{Hom}_{D^b\,\mathrm{Sh}(\mathbb{U})}\left(F(1)_{\mathbb{U}}, \mathcal{L}og(1)\,|_{\mathbb{U}}\right) \\ &= \mathrm{Hom}_{D^b\,\mathrm{Sh}(\mathbb{U})}\left(\widetilde{p}^* F(1), j^* \mathcal{L}og(1)\right) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Sh}(B)}\left(F(1), \prod_{k \geq 1} F(k)\right) \end{aligned}$$

induced by the projective limit of the edge homomorphisms in the Leray spectral sequence for $\widetilde{p}$, and the isomorphism of 2.3.b). Note that the definition of *pol* is independent of the choice of isomorphism

$$W_{-1}\left(\mathcal{H}^0_B(\mathbb{U}, j^* \mathcal{L}og(1))\right) \xrightarrow{\sim} \prod_{k \geq 1} F(k) \ .$$

For the details, we refer to [W2], Theorem 1.5 – as there, we define

$$\mathrm{Ext}^1_{\mathrm{Sh}(\mathbb{U})}\left(F(1), \mathcal{L}og(1)\,|_{\mathbb{U}}\right) := \varprojlim_n \mathrm{Ext}^1_{\mathrm{Sh}(\mathbb{U})}\left(F(1), (\mathcal{L}og/W_{-n}\mathcal{L}og)(1)\,|_{\mathbb{U}}\right) \ .$$

A description of $\mathcal{L}og$ and *pol*, in both incarnations, was given by Beilinson and Deligne; see [B4], 2.1, 3.1 and [BD1], § 1 for the Hodge version and [B4], 3.3 for the l–adic setting. The reader may find it useful to also consult [W3], chapters 3 and 4, setting $N = 1$ in the notation of loc. cit.

We recall the "values" of *pol* at spectra of cyclotomic fields: let $d \geq 2$, and $C := \mathrm{Spec}(R)$, where $R := A\left[\frac{1}{d}, T\right] / \Phi_d(T)$, where $\Phi_d(T)$ is the d-th cyclotomic polynomial.

C is canonically a closed, reduced subscheme of $\mathbb{G}_m \otimes_A A\left[\frac{1}{d}\right]$. For any integer b prime to d, there is an embedding

$$\begin{aligned} i_b : C &\xrightarrow{\sim} C \hookrightarrow \mathbb{G}_m \otimes_A A\left[\frac{1}{d}\right] \ , \\ \zeta &\longmapsto \zeta^b \ . \end{aligned}$$

Since d is invertible on C, the image of i_b is actually contained in $\mathbb{U}$, and hence we may form the pullback of *pol* via i_b,

$$pol_b \in \mathrm{Ext}^1_{\mathrm{Sh}^s(C)}\left(F(1), \mathcal{L}og_b(1)\right) \ ,$$

where $\mathcal{L}og_b$ denotes the pullback of $\mathcal{L}og$.

Now we have the following

Theorem 2.4 (Splitting Principle). *$\mathcal{L}og_b$ splits (uniquely) into a direct product*

$$\mathcal{L}og_b = \prod_{k \geq 0} \mathrm{Gr}^W_{-2k} \left(\mathcal{L}og_b \right) \ ,$$

and $\mathrm{Gr}^W_{-2k} \left(\mathcal{L}og_b \right)$ is isomorphic to $F(k)$ for any $k \geq 0$.

Proof. [B4], 4, or [BD1], 3.6, or [W3], Lemma 3.10. Or use 4.11 and 5.2, whose proof is independent of 2.4. ☐

In order to identify pol_b with an element of

$$\prod_{k \geq 1} \mathrm{Ext}^1_{\mathrm{Sh}^s(C)} \left(F(1), F(k) \right) \ ,$$

we need to fix an isomorphism

$$\kappa_b : \mathrm{Gr}^W_{\cdot} \, \mathcal{L}og_b \xrightarrow{\sim} \prod_{k \geq 0} F(k) \ .$$

By definition, κ_b is the pullback via i_b of the isomorphism

$$\kappa : \mathrm{Gr}^W_{\cdot} \, \mathcal{L}og \xrightarrow{\sim} \prod_{k \geq 0} F(k)$$

of pro–sheaves on $\mathbb{G}_m$ of [W3], chapters 3 and 4, which we briefly describe now:

By 2.1.d), there is a canonical projection

$$\varepsilon : \mathcal{L}og \longrightarrow F(0) \ .$$

Furthermore, there is a canonical isomorphism

$$\gamma : \mathrm{Gr}^W_{-2} \, \mathcal{L}og \xrightarrow{\sim} p^* \mathcal{H}^0_B \left(\mathbb{G}_m, F(0) \right)^{\vee}$$

given by the fact that both sides are equal to p^* of the mixed structure on the (abelianized) fundamental group $\pi_1(\mathbb{G}_{m,\mathrm{top}}, 1)$ (see [W1], chapter 2).

Observe that there is an isomorphism

$$\mathrm{res} : \mathcal{H}^0_B\left(\mathbb{G}_m, F(0)\right) \xrightarrow{\sim} F(1)$$

given by the map "residue at 0".

Finally, both $\mathrm{Gr}^W_{\cdot}\,\mathcal{L}og$ and $\prod_{k \geq 0} F(k)$ carry a canonical multiplicative structure: for $\mathrm{Gr}^W_{\cdot}\,\mathcal{L}og$, this is a formal consequence of

[W1], Corollary 3.4.ii) | [W1], Corollary 3.2.ii)

(see Remark b) at the end of chapter 3 of loc. cit.).

Our isomorphism

$$\kappa : \mathrm{Gr}^W_{\cdot}\,\mathcal{L}og \xrightarrow{\sim} \prod_{k \geq 0} F(k)$$

is the unique isomorphism compatible with ε, $(\mathrm{res})^\vee \circ \gamma$, and the multiplicative structure of both sides.

Using the framing of $\mathcal{L}og_b$ given by κ_b, we may identify pol_b with an element of

$$\prod_{k \geq 1} \mathrm{Ext}^1_{\mathrm{Sh}^s(C)}\left(F(1), F(k)\right) \ ,$$

or, after twisting and forgetting the component "$k = 0$", as an element of

$$\prod_{k \geq 1} \mathrm{Ext}^1_{\mathrm{Sh}^s(C)}\left(F(0), F(k)\right) \ .$$

Note that in the Hodge setting we do not lose any information by forgetting the component "$k = 0$" as there are no non–trivial extensions in $\mathrm{Sh}^s(C)$ of $F(0)$ by itself. This latter statement fails to hold in the l–adic context.

It is however true that the zero–component of pol_b is trivial. One way to see this is via [W2], Corollary 2.2, where it is proved that there is in fact a mixed realization pol_b of which the above extensions are merely the Hodge and l–adic components. In the category of mixed realizations, there is a good concept of polarization, which ensures that there are no non–trivial extensions of pure realizations of the same weight.

Alternatively, one uses Theorem 9.5, where it is proved that our pol_b lie in the image of the respective regulators. The claim then follows from the vanishing of $H^1_{\mathcal{M}}(C, 0)$.

Theorem 2.5 (Beilinson). *Under the isomorphism of A.2.12, we have in the Hodge setting:*

$$pol_b = \left((-1)^k \operatorname{Li}_k(\omega^b)\right)_{\omega,k} \in \prod_{k\geq 1} \left(\bigoplus_{\omega\in C(\mathbb{C})} \mathbb{C}/(2\pi i)^k\mathbb{Q} \right)^+ ,$$

where $\operatorname{Li}_k(z) := \sum_{n\geq 1} \frac{z^n}{n^k}$ *for* $|z| \leq 1$ *and* $z \neq 1$.

Proof. [B4], 4.1, or [BD1], 3.6.3.i), or [W3], Theorem 3.11. ☐

Note that one may identify $C(\mathbb{C})$ with $\{\sigma : \mathbb{Q}(\mu_d) \hookrightarrow \mathbb{C}\}$ by associating to ω the unique embedding mapping $T \in \mathbb{Q}(\mu_d) = \mathbb{Q}[T]/\Phi_d(T)$ to ω.

In the l–adic situation, choose a geometric point $\zeta \in C(\overline{\mathbb{Q}})$. It allows to identify C and

$$\operatorname{Spec}\left(\mathcal{O}_{\mathbb{Q}(\zeta)} \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{ld}\right]\right) ,$$

and, furthermore, the category of continuous $\mathbb{Q}_l$–modules under the Galois group of $\mathbb{Q}(\zeta)$ that are mixed and unramified outside ld, and the category $\operatorname{Sh}^s(C) = \operatorname{Et}^{l,m}_{\mathbb{Q}_l}(C)$.

Given this, we think of $\operatorname{Ext}^1_{\operatorname{Sh}^s(C)}(\mathbb{Q}_l(0), \mathbb{Q}_l(k))$ as sitting inside

$$H^1_{cont}\left(\mathbb{Q}(\zeta), \mathbb{Q}_l(k)\right) ,$$

which is contained in

$$\left(\left(\varprojlim_{r\geq 1} \mathbb{Q}(\mu_{l^\infty},\zeta)^*/(\mathbb{Q}(\mu_{l^\infty},\zeta)^*)^{l^r} \otimes \mu_{l^r}^{\otimes(k-1)}\right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l\right)^{\operatorname{Gal}(\mathbb{Q}(\mu_{l^\infty},\zeta)/\mathbb{Q}(\zeta))}$$

(compare the discussion preceding [W3], Theorem 4.5).

Theorem 2.6 (Beilinson). *Under the above inclusion, we have in the* l*–adic setting:*

$$pol_b = \left((-1)^{k-1} \cdot \frac{1}{d^{k-1}} \cdot \frac{1}{(k-1)!} \cdot \sum_{\alpha^{l^r}=\zeta^b} ([1-\alpha] \otimes (\alpha^d)^{\otimes(k-1)})\right)_{r,k\geq 1} .$$

Proof. [B4], 4.1, or [BD1], 3.6.3.ii), or [W3], Theorem 4.5. ☐

Remark: One of the main results of this work will be (Theorem 9.5) that the elements in 2.5 and 2.6, for fixed b and d, are the respective regulators of one and the same element in motivic cohomology. This implies that Soulé's construction of cyclotomic elements in the K–theory with $\mathbb{Z}_l$–coefficients of an abelian number field ([Sou2], Lemma 1, [Sou5]), for an odd prime l, actually factors over the image of K–theory proper, or at least its tensor product with $\mathbb{Z}_{(2)}$ (Corollary 9.8). Together with the results of [BlK], § 6, Theorem 9.5 also implies that the Tamagawa number conjecture modulo powers of 2 is also true for odd Tate twists (Corollary 9.9).

Remark: There are relative versions of 2.1 and 2.3 for schemes over a base scheme S smooth over B. They allow to directly define the small polylogarithmic extension pol_S on $\mathbb{U} \times_B S$, which however turns out to be the base change to S of pol.

3 The Geometric Set-Up

For easier reference, we assemble the notation used in the next sections.

As before, we let

$$\begin{array}{c|c} & l := \text{a fixed prime number,} \\ A := \mathbb{R} \ , & A := \mathbb{Z}\left[\dfrac{1}{l}\right] \ , \end{array}$$

$$\begin{aligned} &B := \mathrm{Spec}(A) \ , \\ &\mathbb{G}_m := \mathbb{G}_{m,B} \ , \ \mathbb{U} := \mathbb{P}^1_B \setminus \{0, 1, \infty\}_B \ . \end{aligned}$$

Furthermore, we let $\underline{S}$ denote a smooth separated scheme over B of pure relative dimension $d(\underline{S})$,

$$\underline{\alpha}, \underline{\beta} \in \mathbb{G}_m(\underline{S}) \ ,$$

$S \subset \underline{S}$ the open subscheme of $\underline{S}$ where $\underline{\alpha}$ and $\underline{\beta}$ are disjoint. We assume S to be dense in $\underline{S}$.

$$\begin{aligned} &j : S \hookrightarrow \underline{S} \ , \\ &i : \underline{S} \setminus S \hookrightarrow \underline{S} \ , \end{aligned}$$

where $\underline{S} \setminus S$ is equipped with the reduced scheme structure.

$$\underline{Z} := \underline{\alpha}(\underline{S}) \cup \underline{\beta}(\underline{S})$$

with the reduced scheme structure,

$$\underline{V} := \mathbb{G}_{m,\underline{S}} \setminus \underline{Z} \ .$$

For $n \geq 0$, define

$$\begin{aligned}
&\underline{p}^n : \mathbb{G}_{m,\underline{S}}^n \to \underline{S} \ , \\
&\underline{v}^n : \underline{V}^n \hookrightarrow \mathbb{G}_{m,\underline{S}}^n \ , \\
&\underline{z}^{(n)} : \underline{Z}^{(n)} := \mathbb{G}_{m,\underline{S}}^n \setminus \underline{V}^n \hookrightarrow \mathbb{G}_{m,\underline{S}}^n \ ,
\end{aligned}$$

where $\underline{Z}^{(n)}$ carries the reduced scheme structure. (So $\underline{p}^0 = \underline{v}^0 = \mathrm{id}_{\underline{S}}$, and $\underline{Z}^{(0)} = \emptyset$.)
The base change of the above objects and morphisms to S is denoted by the same letters not underlined:

$$\begin{aligned}
&\alpha, \beta : S \to \mathbb{G}_{m,S} \ , \\
&Z := \alpha(S) \amalg \beta(S) \ , \\
&V := \mathbb{G}_{m,S} \setminus Z \ , \\
&p^n : \mathbb{G}_{m,S}^n \to S \ , \\
&v^n : V^n \hookrightarrow \mathbb{G}_{m,S}^n \ , \\
&z^{(n)} : Z^{(n)} \hookrightarrow \mathbb{G}_{m,S}^n \ .
\end{aligned}$$

Also, we define partial compactifications of p^n:

$$\begin{aligned}
&g^n : \mathbb{G}_{m,S}^n \hookrightarrow \mathbb{A}_S^n \ , \\
&h^{(n)} : H^{(n)} := \mathbb{A}_S^n \setminus \mathbb{G}_{m,S}^n \hookrightarrow \mathbb{A}_S^n \ ,
\end{aligned}$$

where again $H^{(n)}$ has the reduced structure,

$$\begin{aligned}
&\overline{p}^n : \mathbb{A}_S^n \to S \ , \\
&\overline{V} := \mathbb{A}_S^1 \setminus Z \ , \\
&\overline{v}^n : \overline{V}^n \hookrightarrow \mathbb{A}_S^n \ , \\
&\overline{z}^{(n)} : \overline{Z}^{(n)} := \mathbb{A}_S^n \setminus \overline{V}^n \hookrightarrow \mathbb{A}_S^n \ ,
\end{aligned}$$

where $\overline{Z}^{(n)}$ is equipped with the reduced structure. (So $\overline{Z}^{(1)} = Z^{(1)} = Z$.)

Remarks: a) The underlined objects should remind the reader that the partial compactification comes from the compactification j of the base S. The overlined objects refer to compactification upstairs, induced from g^n.
b) For fixed n, we have a natural action of the symmetric group $\mathfrak{S}_n$ on our geometric situation.

For the purposes of K-theory in section 7 we will have to replace the singular scheme $Z^{(n)}$ by some smooth simplicial scheme. Put

$$Z_0^{(n)} = Z \times_S \mathbb{G}_{m,S}^{n-1} \amalg \mathbb{G}_{m,S} \times Z \times_S \mathbb{G}_{m,S}^{n-2} \amalg \ldots \amalg \mathbb{G}_{m,S}^{n-1} \times_S Z$$

Note that $Z_0^{(n)}$ is a proper covering of $Z^{(n)}$. This is the easiest case of a morphism of schemes with cohomological descent, meaning that for any reasonable cohomology theory the cohomology of $Z^{(n)}$ will agree with the cohomology of the smooth simplicial scheme

$$Z_\cdot^{(n)} = \operatorname{cosk}_0(Z_0^{(n)}/\mathbb{G}_{m,S}^n) \,,$$

i.e.,

$$Z_k^{(n)} = Z_0^{(n)} \times_{\mathbb{G}_{m,S}^n} \cdots \times_{\mathbb{G}_{m,S}^n} Z_0^{(n)} \qquad (k+1\text{-fold product}).$$

Put $Z_\cdot^{(0)} = \star$ (corresponding to the empty scheme). We will also use the simplicial scheme $\overline{Z}_\cdot^{(n)}$ which is attached to $\overline{Z}^{(n)}$ sitting in $\mathbb{A}_S^n$ in the same way. Finally let

$$\begin{aligned} \mathbb{G}_{m,S}^{\vee n} &= \operatorname{Cone}(Z_\cdot^{(n)} \to \mathbb{G}_{m,S}^n) \\ \mathbb{A}_S^{\vee n} &= \operatorname{Cone}(\overline{Z}_\cdot^{(n)} \to \mathbb{A}_S^n) \end{aligned}$$

where the cone is taken in the category of pointed simplicial sheaves on the big Zariski site (cf. the discussion in appendix B.1).

4 Geometric Origin of the Logarithmic Sheaf

In section 2, we defined a pro–sheaf

$$\mathcal{L}og \in \text{pro}-U\,\mathrm{Sh}^s(\mathbb{G}_m)$$

and an element

$$\begin{aligned} pol &\in \operatorname{Ext}^1_{\mathrm{Sh}(\mathbb{U})}(F(1), \mathcal{L}og(1)\,|_{\mathbb{U}}) \\ &= \varprojlim_n \operatorname{Ext}^1_{\mathrm{Sh}(\mathbb{U})}(F(1), (\mathcal{L}og/W_{-n}\mathcal{L}og)(1)\,|_{\mathbb{U}})\,. \end{aligned}$$

The aim of this section is to identify $\mathcal{L}og\,|_{\mathbb{U}}$, or rather, its Noetherian quotients, as relative cohomology objects with coefficients in Tate twists of certain schemes over $\mathbb{U}$ (Theorem 4.11).

Recall that due to our conventions, we have

$$F(0) = F(0)_{\mathbb{U}}[1] \quad ,$$

and hence we may view *pol* as an element of

$$\operatorname{Hom}_{D^b\,\mathrm{Sh}(\mathbb{U})}(F(0)_{\mathbb{U}}, \mathcal{L}og\,|_{\mathbb{U}}) = H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{L}og\,|_{\mathbb{U}}) \ ,$$

where we have used the notation introduced in Definition 1.2.

With the notation of section 3, we have the following

Definition 4.1. *For $n \ge 0$,*

$$\mathcal{G}^{(n)} := \mathcal{H}^0_S\left(\mathbb{G}^n_{m,S}, v^n_! F(n)\right)^{sgn} = \mathcal{H}^{n+d(\underline{S})}_S\left(\mathbb{G}^n_{m,S}, v^n_! F(n)_{V^n}\right)^{sgn} \quad ,$$

where the superscript sgn refers to the sign–eigenspace under the natural action of the symmetric group $\mathfrak{S}_n$ on $\mathbb{G}^n_{m,S}$ and V^n.
Observe in particular that $\mathcal{G}^{(0)} = F(0)$.

The following is an immediate consequence of the Künneth formula:

Lemma 4.2. *There is a canonical isomorphism*

$$\mathcal{G}^{(n)} \xrightarrow{\sim} \operatorname{Sym}^n \mathcal{G}^{(1)} \ .$$

We want to compute $\mathcal{G}^{(n)}$, and simultaneously construct, for each $n \ge 1$, a projection

$$\mathcal{G}^{(n)} \twoheadrightarrow \mathcal{G}^{(n-1)}$$

via the “residue at 0”, whose projective limit over n we shall then identify, for special α and β, and $S = \mathbb{U}$, with the restriction $\mathcal{L}og\,|_{\mathbb{U}}$ of the logarithmic pro–sheaf to $\mathbb{U}$.

Let $H^{(n)}_{\mathrm{sing}}$ be the singular part of $H^{(n)}$ and $H^{(n)}_{\mathrm{reg}} := H^{(n)} \setminus H^{(n)}_{\mathrm{sing}}$ the smooth part. For any subscheme of $\mathbb{A}^n_S$, the subscript reg will mean the complement of $H^{(n)}_{\mathrm{sing}}$. We work with the following geometric arrangement:

$$\begin{tikzcd}
\overline{V}^n_{\mathrm{reg}} \cap H^{(n)}_{\mathrm{reg}} \arrow[r] \arrow[d, "\overline{v}^n_{H,\mathrm{reg}}"'] & \overline{V}^n_{\mathrm{reg}} \arrow[d, "\overline{v}^n_{\mathrm{reg}}"'] & V^n \arrow[l] \arrow[d, "v^n"'] \\
H^{(n)}_{\mathrm{reg}} \arrow[r, "h^{(n)}_{\mathrm{reg}}"] & \mathbb{A}^n_{S,\mathrm{reg}} & \mathbb{G}^n_{m,S} \arrow[l, "g^n_{\mathrm{reg}}"']
\end{tikzcd}$$

Both squares are cartesian. All maps are either open or closed immersions, and each line gives in fact a smooth pair of S–schemes.

Lemma 4.3. *For any complex $M \in D^b \operatorname{Sh}(\mathbb{A}^n_{S,\mathrm{reg}})$ such that $(\overline{v}^n_{\mathrm{reg}})^* M$ is a shift of a smooth sheaf on $\overline{V}^n_{\mathrm{reg}}$, there is an exact triangle*

$(*)$

$$\begin{tikzcd}
(h^{(n)}_{\mathrm{reg}})_* \, (\overline{v}^n_{H,\mathrm{reg}})_! \left(\overline{v}^n_{H,\mathrm{reg}} \circ h^{(n)}_{\mathrm{reg}}\right)^* M(-1)[-2] \arrow[rr] & & (\overline{v}^n_{\mathrm{reg}})_! \, (\overline{v}^n_{\mathrm{reg}})^* M \arrow[dl] \\
& (g^n_{\mathrm{reg}})_* \; v^n_! \left(v^n \circ g^n_{\mathrm{reg}}\right)^* M \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

Proof. This is $(\overline{v}^n_{\mathrm{reg}})_!$ applied to the exact triangle obtained from purity for the closed immersion

$$\overline{V}^n_{\mathrm{reg}} \cap H^{(n)}_{\mathrm{reg}} \to \overline{V}^n_{\mathrm{reg}}$$

of smooth schemes. □

We apply this lemma to $M = F(n)_{\mathbb{A}^n_{S,\mathrm{reg}}}$, and evaluate the cohomological functors $H^i_{\mathrm{abs}}(\mathbb{A}^n_{S,\mathrm{reg}}, \,\cdot\,)^{\mathrm{sgn}}$ on the triangle $(*)$. Following 1.2.c), we write everything as relative cohomology with Tate coefficients:

$$\begin{aligned}
\ldots \to & H^i_{\mathrm{abs}}(\mathbb{A}^n_{S,\mathrm{reg}} \text{ rel } \overline{Z}^{(n)}_{\mathrm{reg}}, n)^{\mathrm{sgn}} \to H^i_{\mathrm{abs}}(\mathbb{G}^n_{m,S} \text{ rel } Z^{(n)}, n)^{\mathrm{sgn}} \\
& \qquad \to H^{i-1}_{\mathrm{abs}}(H^{(n)}_{\mathrm{reg}} \text{ rel } (\overline{Z}^{(n)} \cap H^{(n)}_{\mathrm{reg}}), n-1)^{\mathrm{sgn}} \\
\to & H^{i+1}_{\mathrm{abs}}(\mathbb{A}^n_{S,\mathrm{reg}} \text{ rel } \overline{Z}^{(n)}_{\mathrm{reg}}, n)^{\mathrm{sgn}} \to \ldots
\end{aligned}$$

We refer to this as the *absolute residue sequence.*

Application of the cohomological functors $\mathcal{H}^i_S(\mathbb{A}^n_{S,\mathrm{reg}}, \,\cdot\,)^{\mathrm{sgn}}$ to the same exact triangle yields a long exact sequence of sheaves on S that we call the *relative residue sequence*:

$$\begin{aligned} \ldots \to &\mathcal{H}^i_S\left(\mathbb{A}^n_{S,\mathrm{reg}}, (\overline{v}^n_{\mathrm{reg}})_! F(n)_{\overline{V}^n_{\mathrm{reg}}}\right)^{\mathrm{sgn}} \to \mathcal{H}^i_S\left(\mathbb{G}^n_{m,S}, v^n_! F(n)_{V^n}\right)^{\mathrm{sgn}} \\ &\qquad\qquad \to \mathcal{H}^{i-1}_S\left(H^{(n)}_{\mathrm{reg}}, (\overline{v}^n_{H,\mathrm{reg}})_! F(n-1)_{\overline{V}^n_{\mathrm{reg}} \cap H^{(n)}_{\mathrm{reg}}}\right)^{\mathrm{sgn}} \\ \to &\mathcal{H}^{i+1}_S\left(\mathbb{A}^n_{S,\mathrm{reg}}, (\overline{v}^n_{\mathrm{reg}})_! F(n)_{\overline{V}^n_{\mathrm{reg}}}\right)^{\mathrm{sgn}} \to \ldots \end{aligned}$$

Note that $\mathcal{G}^{(n)} = \mathcal{H}^{n+d(\underline{S})}_S\left(\mathbb{G}^n_{m,S}, v^n_! F(n)_{V^n}\right)^{\mathrm{sgn}}$ occurs in this sequence.

We are now going to further analyze, and reshape these sequences. The final form will be achieved in Proposition 4.8 and Theorem 4.9.

First, we need to identify the terms

$$\begin{aligned} H^{i-1}_{\mathrm{abs}}(H^{(n)}_{\mathrm{reg}} \text{ rel } (\overline{Z}^{(n)} \cap H^{(n)}_{\mathrm{reg}}), n-1)^{\mathrm{sgn}} &, \quad n \geq 1 , \\ \mathcal{H}^{i-1}_S\left(H^{(n)}_{\mathrm{reg}}, (\overline{v}^n_{H,\mathrm{reg}})_! F(n-1)_{\overline{V}^n_{\mathrm{reg}} \cap H^{(n)}_{\mathrm{reg}}}\right)^{\mathrm{sgn}} &, \quad n \geq 1 . \end{aligned}$$

The complement of $\overline{Z}^{(n)} \cap H^{(n)}_{\mathrm{reg}}$ in $H^{(n)}_{\mathrm{reg}}$ is given by

$$\overline{v}^n_{H,\mathrm{reg}} : \overline{V}^n_{\mathrm{reg}} \cap H^{(n)}_{\mathrm{reg}} \to H^{(n)}_{\mathrm{reg}} .$$

Since $\overline{V}^n_{\mathrm{reg}} \cap H^{(n)}_{\mathrm{reg}} = \coprod_{k=1}^n V^{n-1}$ under the identification

$$H^{(n)}_{\mathrm{reg}} = \coprod_{k=1}^n \mathbb{G}^{n-1}_{m,S} ,$$

and these components are permuted transitively by $\mathfrak{S}_n$, we conclude

Lemma 4.4.

a) $$(\overline{v}^n_{H,\mathrm{reg}})_! F(n-1)_{\overline{V}^n_{\mathrm{reg}} \cap H^{(n)}_{\mathrm{reg}}} = \left(\coprod_{k=1}^n v^{n-1}\right)_! F(n-1)_{\coprod_{k=1}^n V^{n-1}} .$$

b) $$H^{i-1}_{\mathrm{abs}}(H^{(n)}_{\mathrm{reg}} \text{ } \mathit{rel} \text{ } (\overline{Z}^{(n)} \cap H^{(n)}_{\mathrm{reg}}), n-1) = \bigoplus_{k=1}^n H^{i-1}_{\mathrm{abs}}(\mathbb{G}^{n-1}_{m,S} \text{ } \mathit{rel} \text{ } Z^{(n-1)}, n-1) ,$$

and hence the sign–eigenspace $H^{i-1}_{\mathrm{abs}}(H^{(n)}_{\mathrm{reg}}$ *rel* $(\overline{Z}^{(n)} \cap H^{(n)}_{\mathrm{reg}}), n-1)^{sgn}$ *is isomorphic to*

$$H^{i-1}_{\mathrm{abs}}(\mathbb{G}^{n-1}_{m,S} \textit{ rel } Z^{(n-1)}, n-1)^{sgn} ,$$

where the last sgn refers to the action of $\mathfrak{S}_{n-1}$*, and the isomorphism is given by projection onto the k–th component, for some* choice $k \in \{1, \ldots , n\}$*. This isomorphism is independent of the choice of* k*.*

c)

$$\mathcal{R}_S\left(H^{(n)}_{\mathrm{reg}}, (\overline{v}^{\,n}_{H,\mathrm{reg}})_! F(n-1)_{\overline{V}^{\,n}_{\mathrm{reg}} \cap H^{(n)}_{\mathrm{reg}}}\right) = \bigoplus_{k=1}^{n} \mathcal{R}_S\left(\mathbb{G}^{n-1}_{m,S}, v^{n-1}_! F(n-1)_{V^{n-1}}\right) .$$

As in b), the sign–eigenspace $\mathcal{H}^{i-1}_S\left(H^{(n)}_{\mathrm{reg}}, (\overline{v}^{\,n}_{H,\mathrm{reg}})_! F(n-1)_{\overline{V}^{\,n}_{\mathrm{reg}} \cap H^{(n)}_{\mathrm{reg}}}\right)^{sgn}$ *is canonically isomorphic to*

$$\mathcal{H}^{i-1}_S\left(\mathbb{G}^{n-1}_{m,S}, v^{n-1}_! F(n-1)_{V^{n-1}}\right)^{sgn} .$$

For $i = n + d(\underline{S})$*, the latter equals* $\mathcal{G}^{(n-1)}$*.*

Proof. The only point that remains to be shown is the independence of the isomorphisms in b) and c) of the choice of k. Recall the identity

$$R\Gamma_{\mathrm{abs}}\left(\mathbb{G}^n_{m,S} \text{ rel } Z^{(n)}, n\right)^{\mathrm{sgn}} = R\Gamma_{\mathrm{abs}}\left(S, \mathcal{R}_S\left(\mathbb{G}^n_{m,S}, v^n_! F(n)_{V^n}\right)\right)^{\mathrm{sgn}} .$$

We are going to prove in 4.6.d) that $\mathcal{H}^q_S(\mathbb{G}^n_{m,S}, v^n_! F(n)_{V^n})^{\mathrm{sgn}} = 0$ for $q \neq n + d(\underline{S})$. So the associated spectral sequence degenerates, and shows that the independence of the map in b) follows from that of the map in c).

For c), we only need to consider $\mathcal{G}^{(n)} = \mathcal{H}^{n+d(\underline{S})}_S(\mathbb{G}^n_{m,S}, v^n_! F(n)_{V^n})^{\mathrm{sgn}}$. There, our claim follows from Lemma 4.2, and the graded–compatibility of the cup product with boundary morphisms ([GH], Proposition 2.2 and Corollary 2.3). ☐

Remark: The arguments of this section would become simpler if we could use an object $\mathcal{R}^{\mathrm{sgn}}_S$ in c). However, we do not know whether it is possible to make a decomposition into eigenspaces in our triangulated categories.

By the identification of the lemma, the residue sequences define canonical *residue maps*

$$\begin{aligned} \mathrm{res}: H^i_{\mathrm{abs}}(\mathbb{G}^n_{m,S}\ \mathrm{rel}\ Z^{(n)}, n)^{\mathrm{sgn}} &\longrightarrow H^{i-1}_{\mathrm{abs}}(\mathbb{G}^{n-1}_{m,S}\ \mathrm{rel}\ Z^{(n-1)}, n-1)^{\mathrm{sgn}}\,, \\ \mathrm{res}: \mathcal{H}^i_S\left(\mathbb{G}^n_{m,S}, v^{\ n}_! F(n)_{V^n}\right)^{\mathrm{sgn}} &\longrightarrow \mathcal{H}^{i-1}_S\left(\mathbb{G}^{n-1}_{m,S}, v^{\ n-1}_! F(n-1)_{V^{n-1}}\right)^{\mathrm{sgn}} \end{aligned}$$

fitting into the relative and absolute residue sequences. In particular, observe that we have a residue map

$$\mathrm{res}: \mathcal{G}^{(n)} \longrightarrow \mathcal{G}^{(n-1)}\,.$$

Now we concern ourselves with the identification of the remaining terms

$$\begin{gathered} H^i_{\mathrm{abs}}(\mathbb{A}^n_{S,\mathrm{reg}}\ \mathrm{rel}\ \overline{Z}^{(n)}_{\mathrm{reg}}, n)^{\mathrm{sgn}}\ , \ n \geq 0\,, \\ \mathcal{H}^i_S\left(\mathbb{A}^n_{S,\mathrm{reg}}, (\overline{v}^n_{\mathrm{reg}})_! F(n)_{\overline{V}^n_{\mathrm{reg}}}\right)^{\mathrm{sgn}}\ , \ n \geq 0 \end{gathered}$$

of the residue sequences.

We use the following filtration of $\mathbb{A}^n_S$ by open subschemes:

$$F_k\mathbb{A}^n_S := \{(x_1, \ldots, x_n) \in \mathbb{A}^n_S \,|\, \text{at most } k \text{ coordinates vanish}\}\,.$$

So we have $F_n\mathbb{A}^n_S = \mathbb{A}^n_S$ and $F_0\mathbb{A}^n_S = \mathbb{G}^n_{m,S}$.

The “graded pieces” of this filtration are

$$\begin{aligned} G_k\mathbb{A}^n_S &:= F_k\mathbb{A}^n_S \setminus F_{k-1}\mathbb{A}^n_S \\ &= \{(x_1, \ldots, x_n) \in \mathbb{A}^n_S \,|\, \text{precisely } k \text{ coordinates vanish}\}\,. \end{aligned}$$

$G_k\mathbb{A}^n_S$ is equipped with the reduced scheme structure. Note that it splits into several disjoint pieces. For $k \geq 2$ and any such piece, there is a transposition of $\mathfrak{S}_n$ acting trivially. By using triangles similar to $(*)$ for the inclusions

$$G_k\mathbb{A}^n_S \hookrightarrow F_k\mathbb{A}^n_S \hookleftarrow F_{k-1}\mathbb{A}^n_S\,,$$

we conclude inductively that the sign–eigenpart of the cohomology of $H^{(n)}_{\mathrm{sing}}$ is trivial:

Lemma 4.5. *The adjunction morphism induces isomorphisms*

$$\begin{aligned} H^i_{\mathrm{abs}}(\mathbb{A}^n_S\ \mathit{rel}\ \overline{Z}^{(n)}, n)^{sgn} &\xrightarrow{\sim} H^i_{\mathrm{abs}}(\mathbb{A}^n_{S,\mathrm{reg}}\ \mathit{rel}\ \overline{Z}^{(n)}_{\mathrm{reg}}, n)^{sgn}\,, \\ \mathcal{H}^i_S\left(\mathbb{A}^n_S, \overline{v}^{\ n}_! F(n)_{\overline{V}^n}\right)^{sgn} &\xrightarrow{\sim} \mathcal{H}^i_S\left(\mathbb{A}^n_{S,\mathrm{reg}}, (\overline{v}^n_{\mathrm{reg}})_! F(n)_{\overline{V}^n_{\mathrm{reg}}}\right)^{sgn}\,. \end{aligned}$$

By 4.4.b) and 4.5, the absolute residue sequence takes the form

$$\begin{aligned}\cdots \to & H^i_{\mathrm{abs}}(\mathbb{A}^n_S \text{ rel } \overline{Z}^{(n)}, n)^{\mathrm{sgn}} \to H^i_{\mathrm{abs}}(\mathbb{G}^n_{m,S} \text{ rel } Z^{(n)}, n)^{\mathrm{sgn}} \\ & \qquad\qquad \xrightarrow{\mathrm{res}} H^{i-1}_{\mathrm{abs}}(\mathbb{G}^{n-1}_{m,S} \text{ rel } Z^{(n-1)}, n-1)^{\mathrm{sgn}} \\ \to & H^{i+1}_{\mathrm{abs}}(\mathbb{A}^n_S \text{ rel } \overline{Z}^{(n)}, n)^{\mathrm{sgn}} \to \ldots\end{aligned}$$

Similarly, the relative residue sequence looks as follows:

$$\begin{aligned}\ldots \to & \mathcal{H}^i_S\left(\mathbb{A}^n_S, \overline{v}_!{}^n F(n)_{\overline{V}^n}\right)^{\mathrm{sgn}} \to \mathcal{H}^i_S\left(\mathbb{G}^n_{m,S}, v^n_! F(n)_{V^n}\right)^{\mathrm{sgn}} \\ & \qquad\qquad \xrightarrow{\mathrm{res}} \mathcal{H}^{i-1}_S\left(\mathbb{G}^{n-1}_{m,S}, v^{n-1}_! F(n-1)_{V^{n-1}}\right)^{\mathrm{sgn}} \\ \to & \mathcal{H}^{i+1}_S\left(\mathbb{A}^n_S, \overline{v}_!{}^n F(n)_{\overline{V}^n}\right)^{\mathrm{sgn}} \to \ldots\end{aligned}$$

For the computation of the term

$$\mathcal{H}^i_S\left(\mathbb{A}^n_S, \overline{v}_!{}^n F(n)_{\overline{V}^n}\right)^{\mathrm{sgn}} ,$$

we use the Künneth formula:

Lemma 4.6. *a)* $\mathcal{R}_S\left(\mathbb{A}^n_S, \overline{v}_!{}^n F(n)\right) = \mathcal{H}^0_S\left(\mathbb{A}^n_S, \overline{v}_!{}^n F(n)\right)$*, and the Künneth formula gives an isomorphism*

$$\mathcal{H}^0_S\left(\mathbb{A}^n_S, \overline{v}_!{}^n F(n)\right) = \mathcal{H}^0_S\left(\mathbb{A}^n_S, \overline{v}_!{}^n F(n)\right)^{sgn} \xrightarrow{\sim} \mathrm{Sym}^n \mathcal{H}^0_S\left(\mathbb{A}^1_S, \overline{v}_!{}^1 F(1)\right) .$$

b) The choice of an ordering of the sections α *and* β *gives an isomorphism*

$$\mathcal{R}_S\left(\mathbb{A}^1_S, \overline{v}_!{}^1 F(1)\right) = \mathcal{H}^0_S\left(\mathbb{A}^1_S \overline{v}_!{}^1 F(1)\right) \xrightarrow{\sim} F(1) .$$

Up to sign, it is canonical.
c) The isomorphisms of a) and b) induce an isomorphism

$$\mathcal{H}^{n+d(\underline{S})}_S\left(\mathbb{A}^n_S, \overline{v}_!{}^n F(n)_{\overline{V}^n}\right) = \mathcal{H}^0_S\left(\mathbb{A}^n_S, \overline{v}_!{}^n F(n)\right) \xrightarrow{\sim} F(n) .$$

It depends on the choice made in b) only up to the sign $(-1)^n$*. The group* $\mathfrak{S}_n$ *acts on these objects via the sign character.*
d) For $i \neq 0$*, we have*

$$\mathcal{H}^{i+n+d(\underline{S})}_S\left(\mathbb{G}^n_{m,S}, v^n_! F(n)_{V^n}\right)^{sgn} = \mathcal{H}^i_S\left(\mathbb{G}^n_{m,S}, v^n_! F(n)\right)^{sgn} = 0 .$$

Proof. For b), consider the long exact cohomology sequence associated to the triangle

(**)

$$\begin{tikzcd} \overline{v}^1_! F(1) \arrow[rr] & & F(1) \arrow[dl] \\ & \overline{z}^{(1)}_* F(1) \arrow[ul, "{[1]}"] & \end{tikzcd}$$

We have

$$\mathcal{H}^i_S\left(\mathbb{A}^1_S, F(1)\right) = \begin{cases} F(1)\ , & i = -1 \\ 0\ , & i \neq -1 \end{cases}$$

and

$$\mathcal{H}^i_S\left(\mathbb{A}^1_S, \overline{z}^{(1)}_* F(1)\right) = \begin{cases} \bigoplus\limits_{\alpha,\beta} F(1)\ , & i = 0 \\ 0\ , & i \neq 0 \end{cases} .$$

The long exact cohomology sequence thus reads

$$0 \to \mathcal{H}^{-1}_S\left(\mathbb{A}^1_S, \overline{v}^1_! F(1)\right) \to F(1) \xrightarrow{\Delta} \bigoplus_{\alpha,\beta} F(1) \to \mathcal{H}^0_S\left(\mathbb{A}^1_S, \overline{v}^1_! F(1)\right) \to 0\ .$$

If we let $\{\alpha, \beta\} = \{s_1, s_2\}$, then we identify the cokernel of

$$\Delta : F(1) \longrightarrow \bigoplus_{\alpha,\beta} F(1) = \bigoplus_{i=1}^{2} F(1)$$

with $F(1)$ by mapping $(f_{s_1}, f_{s_2}) \in \bigoplus_{i=1}^2 F(1)$ to $f_{s_2} - f_{s_1}$.
a) follows from b) since $\bigotimes^n F(1) = \operatorname{Sym}^n F(1)$.
c) is a consequence of a) and b).
d) follows from a) and the relative residue sequence by induction on n. $\square$

On the level of absolute cohomology, the isomorphism of 4.6.c) induces an isomorphism

$$H^{i+n}_{\text{abs}}\left(\mathbb{A}^n_S \text{ rel } \overline{Z}^{(n)}, n\right) = H^{i+n}_{\text{abs}}\left(\mathbb{A}^n_S \text{ rel } \overline{Z}^{(n)}, n\right)^{\text{sgn}} \xrightarrow{\sim} H^i_{\text{abs}}(S, n)\ .$$

This gives the final shape of the absolute residue sequence:

$$\begin{aligned} \cdots \longrightarrow & H^i_{\text{abs}}(S,n) \xrightarrow{\delta} H^{i+n}_{\text{abs}}(\mathbb{G}^n_{m,S} \text{ rel } Z^{(n)}, n)^{\text{sgn}} \\ & \qquad\qquad \xrightarrow{\text{res}} H^{i+n-1}_{\text{abs}}(\mathbb{G}^{n-1}_{m,S} \text{ rel } Z^{(n-1)}, n-1)^{\text{sgn}} \\ \longrightarrow & H^{i+1}_{\text{abs}}(S,n) \xrightarrow{\delta} \cdots \end{aligned}$$

By 4.6.d), the relative residue sequence collapses into the short exact sequence of sheaves on S:

$$0 \longrightarrow F(n) \longrightarrow \mathcal{G}^{(n)} \xrightarrow{\text{res}} \mathcal{G}^{(n-1)} \longrightarrow 0 .$$

In order to identify the long exact absolute cohomology sequence associated to this sequence with the absolute residue sequence, we need the following:

Lemma 4.7. *Let $K \in D^b\,\mathrm{Sh}(X)$ be a complex of sheaves on a separated, reduced and flat B–scheme X. Suppose there is an action of a finite group G on K. Let χ be the character of an absolutely irreducible representation of G over F. For any object $\mathbb{V}$ with a G–action of an F-linear abelian category, denote by $\mathbb{V}(\chi)$ the χ–isotypical component of $\mathbb{V}$, i.e., the image under the projector*

$$e_\chi := \frac{1}{\# G} \sum_{g \in G} \chi(g^{-1}) \cdot g \, .$$

Suppose that $(\mathcal{H}^i K)(\chi)$ vanishes for all $i \neq 0$. Then

$$\mathrm{Hom}_{D^b}(F, K[i])(\chi) = \mathrm{Hom}_{D^b}(F, (\mathcal{H}^0 K)(\chi)[i])$$

Proof. By applying e_χ and $1 - e_\chi$, one checks the statement for a complex of the special form $K \cong \mathcal{H}^0 K$.

For the general case, consider the spectral sequence for $\mathrm{Hom}_{D^b}(F, \,\cdot\, [i])$ induced by the truncation functors $\tau_{\leq n}$. It degenerates after applying e_χ. $\square$

Now that we know that formation of absolute cohomology commutes with formation of sign eigenspaces, we have:

Proposition 4.8. *The absolute residue sequence is the long exact sequence in absolute cohomology attached to the short exact sequence*

$$0 \longrightarrow F(n) \longrightarrow \mathcal{G}^{(n)} \xrightarrow{\text{res}} \mathcal{G}^{(n-1)} \longrightarrow 0 .$$

We conclude the computational part of this section by collecting our results:

Theorem 4.9. *a) For $n \geq 0$, we have*

$$\mathcal{H}^0_S \left(\mathbb{G}^n_{m,S}, v^n_! F(n) \right)^{sgn} = \mathcal{G}^{(n)} \, ,$$

and $\mathcal{H}^i_S\left(\mathbb{G}^n_{m,S}, v_!^n F(n)\right)^{sgn} = 0$ *for* $i \neq 0$.
b) The residue at 0*, i.e., the boundary map of* $(*)$*, gives an epimorphism*

$$\mathrm{res} : \mathcal{G}^{(n)} \twoheadrightarrow \mathcal{G}^{(n-1)}$$

for $n \geq 1$.
c) The Künneth formula gives an isomorphism

$$\mathcal{H}^0_S\left(\mathbb{A}^n_S, \overline{v}_!^n F(n)\right) = \mathcal{H}^0_S\left(\mathbb{A}^n_S, \overline{v}_!^n F(n)\right)^{sgn} \xrightarrow{\sim} \ker(\mathrm{res})$$

for $n \geq 1$*. A choice of an ordering of the sections* α *and* β *induces an isomorphism*

$$F(n) \xrightarrow{\sim} \ker(\mathrm{res}) \ ,$$

which depends on this choice only up to the sign $(-1)^n$.
d) Let $\mathcal{G}^{(n)} \xrightarrow{\sim} \mathrm{Sym}^n \mathcal{G}^{(1)}$ *be the canonical isomorphism of 4.2, and*

$$\mathrm{Sym}^n F(0) \xrightarrow{\sim} F(0) \ ,$$
$$\mathrm{Sym}^n F(1) \xrightarrow{\sim} F(n)$$

the isomorphisms given by multiplication. Then the diagrams

$$\begin{tikzcd}
\mathcal{G}^{(n)} \arrow[r] \arrow[d, "\wr"] & F(0) \\
\mathrm{Sym}^n \mathcal{G}^{(1)} \arrow[r] & \mathrm{Sym}^n F(0) \arrow[u, "\wr"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
F(n) \arrow[r] & \mathcal{G}^{(n)} \arrow[d, "\wr"] \\
\mathrm{Sym}^n F(1) \arrow[r] \arrow[u, "\wr"'] & \mathrm{Sym}^n \mathcal{G}^{(1)}
\end{tikzcd}$$

commute. Here, the horizontal maps are given by the successive residue maps, and by c) respectively.
e) Let $W_{-2n-1}\,\mathcal{G}^{(n)} := 0$,

$$W_{-2k}\,\mathcal{G}^{(n)} := W_{-2k+1}\,\mathcal{G}^{(n)} := \ker(\mathcal{G}^{(n)} \to \mathcal{G}^{(k-1)}) \quad \mathit{for} \quad 1 \leq k \leq n \,,$$

and $W_0\,\mathcal{G}^{(n)} := \mathcal{G}^{(n)}$*. The choice in c) induces isomorphisms*

$$\mathrm{Gr}^W_{\bullet}\,\mathcal{G}^{(n)} \xrightarrow{\sim} \bigoplus_{i=0}^{n} F(i) \ ,$$

which by their construction fit into commutative diagrams

$$\begin{tikzcd}
\mathrm{Gr}^W_\cdot \, \mathcal{G}^{(n)} \arrow[r, "\sim"] \arrow[d, "\mathrm{Gr}^W_\cdot \, \mathrm{res}"'] & \bigoplus_{i=0}^{n} F(i) \arrow[d, "\mathrm{can}"] \\
\mathrm{Gr}^W_\cdot \, \mathcal{G}^{(n-1)} \arrow[r, "\sim"] & \bigoplus_{i=0}^{n-1} F(i)
\end{tikzcd}$$

The filtration $W.$ is therefore the weight filtration of $\mathcal{G}^{(n)}$.

Proof. a), b) and c) follow from the previous results. The commutativity of the first diagram in d) follows from the definition of the residue map. For the second diagram, we use the fact that the Künneth formula of 4.2 is compatible with the Künneth formula of the proof of 4.6.a). For e), apply induction on n. $\square$

Recall that S is the open subscheme of $\underline{S}$ where the sections $\underline{\alpha}$ and $\underline{\beta}$ of $\mathbb{G}_{m,\underline{S}}$ are disjoint. The main step towards the identification, for special S, α and β, of the projective limit of the $\mathcal{G}^{(n)}$ with the restriction $\mathcal{L}og\,|_{\mathbb{U}}$ of the logarithmic sheaf, is the following

Lemma 4.10. *a) There is a unique smooth sheaf $\underline{\mathcal{G}}^{(n)}$ on $\underline{S}$ extending $\mathcal{G}^{(n)}$. It has a weight filtration.*
b) There is a canonical isomorphism

$$\underline{\mathcal{G}}^{(n)} \xrightarrow{\sim} \mathrm{Sym}^n \underline{\mathcal{G}}^{(1)} \ ,$$

and a unique isomorphism

$$\eta^{(n)} : \mathrm{Gr}^W_\cdot \, \underline{\mathcal{G}}^{(n)} \xrightarrow{\sim} \bigoplus_{i=0}^{n} F(i) \ ,$$

which is compatible with the isomorphism of 4.9.e).
c) The weight filtration of $i^* \underline{\mathcal{G}}^{(n)}$ is split: there is a canonical isomorphism

$$i^* \underline{\mathcal{G}}^{(n)} \xrightarrow{\sim} \mathrm{Gr}^W_\cdot \, i^* \underline{\mathcal{G}}^{(n)} \xrightarrow[b)]{\sim} \bigoplus_{i=0}^{n} F(i) \ .$$

Here, i denotes the inclusion of $\underline{S} \setminus S$ into $\underline{S}$.
d) There is an exact sequence

$$0 \longrightarrow i_* F(1) \longrightarrow \mathcal{H}^0_{\underline{S}} \left(\mathbb{G}_{m,\underline{S}}, \underline{v}_!^1 F(1) \right) \longrightarrow \underline{\mathcal{G}}^{(1)} \longrightarrow 0$$

of sheaves on $\underline{S}$.

Proof. If there is any smooth sheaf as in a), then it will automatically be unique, and hence b) follows from a), and 4.9.d), e). Also, it will suffice, because of 4.9.d), to show the lemma for the case $n = 1$.
There we have the following diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & & \\
 & & & i_*F(1) \arrow[d] & & & \\
0 \arrow[r] & \mathcal{H}^{-1} \arrow[r] & F(1) \arrow[r, "\Delta"] & K \arrow[r] \arrow[d] & \mathcal{H}^0 \arrow[r] & F(0) \arrow[r] & 0 \\
 & & & \bigoplus_{\alpha,\beta} F(1) \arrow[d] & & & \\
 & & & 0 & & &
\end{tikzcd}$$

where

$$K = \mathcal{H}^{-1}\operatorname{Cone}(\delta : \bigoplus_{\alpha,\beta} F(1)_{\underline{S}}[d(\underline{S})] \to i_*F(1)_{\underline{S}\setminus S}[d(\underline{S})])$$

with $\delta(v_1, v_2) := v_1 - v_2$ (in terms of constructible sheaves this is just $\operatorname{Ker}\delta$ shifted in the appropriate degree to define a perverse sheaf). The horizontal sequence is, as in the proof of 4.6.b), the long exact cohomology sequence on $\underline{S}$ associated to the short exact sequence on $\mathbb{G}_{m,\underline{S}}$

$$(**) \qquad 0 \to \underline{z}_*^{(1)} F(1) \to \underline{v}_!^1 F(1) \to F(1) \to 0\ ,$$

where we have set

$$\mathcal{H}^i := \mathcal{H}^i_{\underline{S}}\left(\mathbb{G}_{m,\underline{S}}, \underline{v}_!^1 F(1)\right)\ .$$

We thus get the equality

$$\mathcal{R}_{\underline{S}}\left(\mathbb{G}_{m,\underline{S}}, \underline{v}_!^1 F(1)\right) = \mathcal{H}^0_{\underline{S}}\left(\mathbb{G}_{m,\underline{S}}, \underline{v}_!^1 F(1)\right)[0]\ ,$$

and an exact sequence of sheaves on $\underline{S}$

$$0 \to K/\Delta\left(F(1)\right) \to \mathcal{H}^0_{\underline{S}}\left(\mathbb{G}_{m,\underline{S}}, \underline{v}_!^1 F(1)\right) \to F(0) \to 0\ ,$$

whose restriction to S is isomorphic, via the choice of an ordering of α and β, to

$$0 \to F(1) \to \mathcal{G}^{(1)} \to F(0) \to 0 \ .$$

Push out of the above via the morphism

$$K/\Delta(F(1)) \to \left(\bigoplus_{\alpha,\beta} F(1) \right) /\Delta(F(1)) \ ,$$

whose kernel is $i_*F(1)$ (recall again that we use perverse indices), gives the desired extension $\underline{\mathcal{G}}^{(1)}$. By construction b) and d) hold. Applying i^* to the pushout diagram and taking cohomology, we see that the sheaf $i^*\underline{\mathcal{G}}^{(1)}[-1]$ is the pushout of $F(0)$ via

$$0 \hookrightarrow F(1) \ ,$$

and we get c). $\square$

We now specialize our geometric situation: we let

$$\begin{aligned} &\underline{S} := \mathbb{G}_{m,B} \ , \\ &\underline{\alpha} := 1 : \mathbb{G}_{m,B} \twoheadrightarrow B \hookrightarrow \mathbb{G}_{m,B} \ , \\ &\underline{\beta} := \mathrm{id} : \mathbb{G}_{m,B} \to \mathbb{G}_{m,B} \ . \end{aligned}$$

So we have $S = \mathbb{U}$ and $\underline{S} \setminus S = 1_B$, the closed subscheme of $\mathbb{G}_{m,B}$ given by the immersion 1 of B into $\mathbb{G}_{m,B}$.

After having made precise which choice of normalization we have and in how far it affects our identifications, we now fix it: we let

$$s_1 := \alpha = 1 \text{ and } s_2 := \beta = \mathrm{id} \ \text{ in 4.9.c).}$$

We thus get a projective system $(\underline{\mathcal{G}}^{(n)})_{n \geq 0}$ of smooth Tate sheaves on $\mathbb{G}_{m,B}$ with

$$\underline{\mathcal{G}}^{(n)} \,|_{1_B} = \bigoplus_{i=0}^{n} F(i) \ .$$

By the universal property of $\mathcal{L}og$ (Theorem 2.1.d)), there is a unique morphism

$$\varphi : \mathcal{L}og \to \underline{\mathcal{G}} := \varprojlim_n \underline{\mathcal{G}}^{(n)}$$

such that $\varphi \,|_{1(B)}$ sends $1 \in \Gamma(B, \mathcal{L}og \,|_{1_B})$ to

$$1 : F(0) \hookrightarrow \prod_{i=0}^{\infty} F(i) = \underline{\mathcal{G}} \,|_{1(B)} \ .$$

Theorem 4.11. *φ is an isomorphism.*

Proof. The claim can be shown on the level of the underlying topological sheaves. The l–adic statement follows from the statement for the topological spaces of $\mathbb{C}$–valued points by comparison – recall that we are dealing with locally constant sheaves.

Over $\mathbb{C}$, the fibre at 1 of the pro–local system $\mathcal{L}og_{\mathrm{top}}$ equals the completion of the group ring $\mathbb{Q}[\pi_1]$ of $\pi_1 := \pi_1(\mathbb{G}_m(\mathbb{C}), 1) \cong \mathbb{Z}$ with respect to the augmentation ideal a. The representation of π_1 is given by multiplication; compare the general construction in [W1], 2.5–2.7. In particular, we have

$$\mathcal{L}og_{\mathrm{top}} = \varprojlim_n \mathrm{Sym}^n(\mathcal{L}og_{\mathrm{top},\geq -2}) \; ,$$

where $\mathcal{L}og_{\mathrm{top},\geq -2} := \mathcal{L}og_{\mathrm{top}}/a^2$ is of dimension two. Now in the category of unipotent local systems on $\mathbb{G}_m(\mathbb{C})$, the pro–sheaf $\mathcal{L}og_{\mathrm{top}}$ has the universal property of Theorem 2.1.d).

We apply this universal property to $\underline{\mathcal{G}}_{\mathrm{top},\geq -2} := \underline{\mathcal{G}}^{(1)}_{\mathrm{top}}$. The resulting map factors over φ_{top}. Since $\underline{\mathcal{G}}_{\mathrm{top},\geq -2}$ is two-dimensional, the representation of $\mathbb{Q}[\pi_1]$ is necessarily trivial on a^2, and we get a morphism of local systems

$$\varphi_{\mathrm{top},\geq -2} : \mathcal{L}og_{\mathrm{top},\geq -2} \longrightarrow \underline{\mathcal{G}}_{\mathrm{top},\geq -2}$$

giving rise to a morphism

$$\varprojlim_n \mathrm{Sym}^n(\varphi_{\mathrm{top},\geq -2}) : \mathcal{L}og_{\mathrm{top}} \longrightarrow \underline{\mathcal{G}}_{\mathrm{top}} \; .$$

Again because of the universal property of $\mathcal{L}og_{\mathrm{top}}$, this morphism is identical to φ_{top}.

It therefore suffices to show that $\varphi_{\mathrm{top},\geq -2}$ is bijective, which amounts to saying that the coinvariants of $\underline{\mathcal{G}}_{\mathrm{top},\geq -2}$ under the action of π_1 are one–dimensional.

But taking coinvariants under π_1 of a unipotent variation $\mathbb{V}$ amounts to computing singular cohomology

$$H^1(\mathbb{G}_m(\mathbb{C}), \mathbb{V}) = \mathcal{H}^0_{\mathrm{Spec}(\mathbb{R})}(\mathbb{G}_{m,\mathbb{R}}, \mathbb{V}) \; .$$

Firstly, we claim that

$$\mathcal{H}^i_{\mathrm{Spec}(\mathbb{R})}\left(\mathbb{G}_{m,\mathbb{R}} \times \mathbb{G}_{m,\mathbb{R}}, \underline{v}^1_! F(1)\right) = \begin{cases} F(-1) \; , & i = 0 \\ 0 \; , & i \neq 0 \end{cases} \; :$$

e.g., identify the left hand side with

$$\begin{aligned} & H^{i+2}\left(\mathbb{G}_m(\mathbb{C}) \times \mathbb{G}_m(\mathbb{C}), \Delta(\mathbb{G}_m(\mathbb{C})) \cup (\{1\} \times \mathbb{G}_m(\mathbb{C})), F(1)\right) \\ \cong \; & H^{i+2}\left(\mathbb{G}_m(\mathbb{C}) \times \mathbb{G}_m(\mathbb{C}), (\mathbb{G}_m(\mathbb{C}) \times \{1\}) \cup (\{1\} \times \mathbb{G}_m(\mathbb{C})), F(1)\right) \; , \end{aligned}$$

and apply the Künneth formula.

From the proof of 4.10, we recall – remember that we have $\underline{S} = \mathbb{G}_m$:

$$\mathcal{R}_{\mathbb{G}_{m,\mathbb{R}}}\left(\mathbb{G}_{m,\mathbb{R}} \times \mathbb{G}_{m,\mathbb{R}}, \underline{v}^1_! F(1)\right) = \mathcal{H}^0_{\mathbb{G}_{m,\mathbb{R}}}\left(\mathbb{G}_{m,\mathbb{R}} \times \mathbb{G}_{m,\mathbb{R}}, \underline{v}^1_! F(1)\right)[0] \; ,$$

from which we conclude:

$$\mathcal{H}^i_{\mathrm{Spec}(\mathbb{R})}\left(\mathbb{G}_{m,\mathbb{R}}, \mathcal{H}^0_{\mathbb{G}_{m,\mathbb{R}}}\left(\mathbb{G}_{m,\mathbb{R}} \times \mathbb{G}_{m,\mathbb{R}}, \underline{v}^1_! F(1)\right)\right) = \begin{cases} F(-1) \; , & i = 0 \\ 0 \; , & i \neq 0 \end{cases} \; .$$

The long exact sequence obtained by applying $\mathcal{R}_{\mathrm{Spec}(\mathbb{R})}(\mathbb{G}_{m,\mathbb{R}}, \_)$ to the exact sequence of 4.10.d)

$$0 \longrightarrow 1_* F(1) \longrightarrow \mathcal{H}^0_{\mathbb{G}_{m,\mathbb{R}}}\left(\mathbb{G}_{m,\mathbb{R}} \times \mathbb{G}_{m,\mathbb{R}}, \underline{v}^1_! F(1)\right) \longrightarrow \underline{\mathcal{G}}^{(1)} \longrightarrow 0$$

then shows that

$$\mathcal{H}^0_{\mathrm{Spec}(\mathbb{R})}(\mathbb{G}_{m,\mathbb{R}}, \underline{\mathcal{G}}^{(1)}) = F(-1) \; .$$

□

5 The Splitting Principle Revisited

In order to be able to translate easily to the motivic context, we recall Beilinson's original proof ([B4], 4) of the splitting of the logarithmic pro-sheaf over spectra of cyclotomic fields (Theorem 2.4).

First, we return to the general situation considered at the beginning of section 4. For $N \geq 1$, we have the morphism of S-schemes

$$\begin{aligned} \phi : \mathbb{G}_{m,S} & \longrightarrow \mathbb{G}_{m,S} \; , \\ x & \longmapsto x^N \; , \end{aligned}$$

and for each $n \geq 0$, the induced morphism

$$\phi^n : \mathbb{G}_{m,S}^n \longrightarrow \mathbb{G}_{m,S}^n \; .$$

We work under the additional assumption

$$\text{(A)} \qquad \phi \circ \alpha = \alpha \; , \quad \phi \circ \beta = \beta \; .$$

If this is the case, we have $(\phi^n)^{-1}(V^n) \subset V^n$, and hence get a morphism

$$(\phi^n)^* v_!^n F(n) \longrightarrow v_!^n F(n) \, ,$$

and hence a morphism

$$(\phi^n)^\sharp : v_!^n F(n) \longrightarrow \phi_*^n v_! F(n) \, ,$$

which after application of p_*^n and projection onto the sign-eigenpart induces

$$(\phi^n)^\sharp : \mathcal{G}^{(n)} \longrightarrow \mathcal{G}^{(n)} \; .$$

We need to understand the action of $(\phi^n)^\sharp$ on $\mathcal{G}^{(n)}$, and on absolute cohomology. First, we establish in how far $(\phi^n)^\sharp$ is compatible with the residue at 0:

Lemma 5.1. *a) Under any isomorphism*

$$\operatorname{Gr}_\cdot^W \mathcal{G}^{(n)} \xrightarrow{\sim} \bigoplus_{i=0}^{n} F(i) \; ,$$

the map $\operatorname{Gr}_\cdot^W (\phi^n)^\sharp$ *is multiplication by* N^{n-i} *on* $F(i)$.
b) For any $n \geq 1$, *the diagram*

$$\begin{tikzcd}
\mathcal{G}^{(n)} \arrow[r, "(\phi^n)^\sharp"] \arrow[d, "\mathrm{res}_n"'] & \mathcal{G}^{(n)} \arrow[d, "\mathrm{res}_n"] \\
\mathcal{G}^{(n-1)} \arrow[r, "N \cdot (\phi^{n-1})^\sharp"] & \mathcal{G}^{(n-1)}
\end{tikzcd}$$

commutes.

Proof. Since the morphisms in b) are strict with respect to the weight filtration, it suffices to check that

$$\operatorname{Gr}_\cdot^W(\mathrm{res}_n) \circ \operatorname{Gr}_\cdot^W (\phi^n)^\sharp = N \cdot \operatorname{Gr}_\cdot^W (\phi^{n-1})^\sharp \circ \operatorname{Gr}_\cdot^W(\mathrm{res}_n) \; .$$

But if we choose the isomorphism of 4.9.e), then $\mathrm{Gr}_{\cdot}^{W}(\mathrm{res}_n)$ is simply the canonical projection

$$\bigoplus_{i=0}^{n} F(i) \twoheadrightarrow \bigoplus_{i=0}^{n-1} F(i) \ ,$$

and therefore b) follows from a).

For a), we note first that it suffices to show the statement for one choice of isomorphism

$$\mathrm{Gr}_{\cdot}^{W} \mathcal{G}^{(n)} \xrightarrow{\sim} \bigoplus_{i=0}^{n} F(i) \ .$$

This time, we use the isomorphism on graded objects induced by 4.2, thereby reducing ourselves to the case $n = 1$. There, we consider the long exact cohomology sequence associated to the exact sequence

$$0 \to z_*^{(1)} F(1) \to v_!^1 F(1) \to F(1) \to 0 \ ,$$

and the cohomological functors $\mathcal{H}_S^i(\mathbb{G}_{m,S}, \ \cdot \ )$. We know the cohomology of $\mathbb{G}_m$:

$$\mathcal{H}_S^i\left(\mathbb{G}_{m,S}, F(1)\right) = \begin{cases} F(1) & , \quad i = -1 \\ F(0) & , \quad i = 0 \\ 0 & , \quad i \notin \{-1, 0\} \end{cases} \ .$$

Of course, we know the cohomology of two points:

$$\mathcal{H}_S^i\left(\mathbb{G}_{m,S}, z_*^{(1)} F(1)\right) = \begin{cases} \bigoplus_{\alpha,\beta} F(1) & , \quad i = 0 \\ 0 & , \quad i \neq 0 \end{cases} \ .$$

We get an exact sequence

$$0 \to F(1) \xrightarrow{\Delta} \bigoplus_{\alpha,\beta} F(1) \to \mathcal{G}^{(1)} = \mathcal{H}_S^0\left(\mathbb{G}_{m,S}, v_!^1 F(1)\right) \to F(0) \to 0 \ .$$

and because of assumption (A), it carries an action of $(\phi^n)^\sharp$. But this action can be identified on $\mathcal{H}_S^i\left(\mathbb{G}_{m,S}, F(1)\right)$ and $\mathcal{H}_S^i\left(\mathbb{G}_{m,S}, z_*^{(1)} F(1)\right)$: it is trivial on the $F(1)$, and multiplication by N on $F(0)$. □

Certainly (A) is only satisfied in very special situations, namely if α and β are supported in the schemes of $(N-1)$-torsion of $\mathbb{G}_{m,S}$.

Let again $d \geq 2$, $C := \mathrm{Spec}(R)$, where $R := A[\frac{1}{d}, T]/\Phi_d(T)$ as in section 2. For b prime to d, consider

$$\begin{array}{rcl} i_b : C & \overset{\sim}{\longrightarrow} & C \hookrightarrow \mathbb{G}_m \ , \\ \zeta & \longmapsto & \zeta^b \ . \end{array}$$

The pullback $\mathcal{L}og_b$ of the pro-sheaf $\mathcal{L}og\,|_{\mathbb{U}}$ on $\mathbb{U}$ via i_b is identical to the projective limit of the sheaves $\mathcal{G}_b^{(n)}$ obtained by setting

$$\begin{aligned} &\underline{S} := C \ , \\ &\underline{\alpha} := 1 : C \to B \hookrightarrow \mathbb{G}_m \ , \\ &\underline{\beta} := i_b \ . \end{aligned}$$

Since (A) is satisfied with $N = d + 1$, we may apply 5.1, and conclude:

Corollary 5.2. *$\mathcal{G}_b^{(n)}$ splits into a direct sum*

$$\mathcal{G}_b^{(n)} = \bigoplus_{i=0}^{n} \mathrm{Gr}_{-2i}^W \, \mathcal{G}_b^{(n)} \ .$$

Therefore, there is a unique isomorphism

$$\eta_b^{(n)} : \mathcal{G}_b^{(n)} \overset{\sim}{\longrightarrow} \bigoplus_{i=0}^{n} F(i) \ ,$$

which is compatible with the isomorphism $\eta^{(n)}$ of 4.10.b).

Proof. $F(i) \subset \mathcal{G}_b^{(n)}$ is the eigenspace of $(d+1)^{n-i}$ under the morphism $(\phi^n)^\sharp$. $\square$

We conclude with the implications of 5.1 and 5.2 for absolute cohomology with coefficients. For this, recall the absolute residue sequence for $n \geq 1$

$$\ldots \to H^{\cdot}_{\mathrm{abs}}(C, n) \to H^{\cdot + n}_{\mathrm{abs}}(\mathbb{G}_{m,C}^{\vee n}, n)^{\mathrm{sgn}} \overset{\mathrm{res}}{\to} H^{\cdot + n-1}_{\mathrm{abs}}(\mathbb{G}_{m,C}^{\vee n-1}, n-1)^{\mathrm{sgn}} \to \ldots$$

introduced after 4.9, where we have set

$$H^{\cdot + n}_{\mathrm{abs}}(\mathbb{G}_{m,C}^{\vee n}, n)^{\mathrm{sgn}} := H^{\cdot + n}_{\mathrm{abs}}(\mathbb{G}_{m,C}^{n} \ \mathrm{rel}\ Z^{(n)}, n)^{\mathrm{sgn}} \ ,$$

thus saving enough space to get the above sequence into a single line.

Corollary 5.3. *a) For $n \geq 1$, the absolute residue sequence splits into short exact sequences*

$$0 \to H^{\cdot}_{\mathrm{abs}}(C,n) \to H^{\cdot+n}_{\mathrm{abs}}(\mathbb{G}^{\vee n}_{m,C},n)^{sgn} \to H^{\cdot+n-1}_{\mathrm{abs}}(\mathbb{G}^{\vee n-1}_{m,C},n-1)^{sgn} \to 0 \; .$$

b) For $N = d+1$, the map $(\phi^n)^*$ acts on the short exact sequences of a): there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^{\cdot}_{\mathrm{abs}}(C,n) \arrow[r] \arrow[d, "\mathrm{id}"'] & H^{\cdot+n}_{\mathrm{abs}}(\mathbb{G}^{\vee n}_{m,C},n)^{sgn} \arrow[r] \arrow[d, "(\phi^n)^*"'] & H^{\cdot+n-1}_{\mathrm{abs}}(\mathbb{G}^{\vee n-1}_{m,C},n-1)^{sgn} \arrow[r] \arrow[d, "(d+1)\cdot(\phi^{n-1})^*"'] & 0 \\
0 \arrow[r] & H^{\cdot}_{\mathrm{abs}}(C,n) \arrow[r] & H^{\cdot+n}_{\mathrm{abs}}(\mathbb{G}^{\vee n}_{m,C},n)^{sgn} \arrow[r] & H^{\cdot+n-1}_{\mathrm{abs}}(\mathbb{G}^{\vee n-1}_{m,C},n-1)^{sgn} \arrow[r] & 0
\end{tikzcd}$$

Proof. By 4.8, the absolute residue sequence is the absolute cohomology sequence for the exact sequence of sheaves on C

$$0 \to F(n) \to \mathcal{G}^{(n)}_b \overset{\mathrm{res}_b}{\to} \mathcal{G}^{(n-1)}_b \to 0 \; .$$

Therefore, a) follows from 5.2, while b) follows from 5.1.b) and the fact that under the identification of 4.9.a)

$$H^{\cdot}_{\mathrm{abs}}(C,\mathcal{G}^{(n)}) \xrightarrow{\sim} H^{\cdot+n}_{\mathrm{abs}}(\mathbb{G}^{\vee n}_{m,C},n)^{\mathrm{sgn}} \; ,$$

the map induced by

$$(\phi^n)^\sharp : \mathcal{G}^{(n)}_b \to \mathcal{G}^{(n)}_b$$

is the map $(\phi^n)^*$ of the absolute cohomology groups. $\square$

It follows that the eigenvalues of $(\phi^n)^*$ on $H^{n+1}_{\mathrm{abs}}(\mathbb{G}^n_{m,C} \text{ rel } Z^{(n)},n)^{\mathrm{sgn}}$ are $1, d+1, \ldots, (d+1)^n$. The eigenspace decomposition yields

$$\eta^{(n)}_b : H^{n+1}_{\mathrm{abs}}(\mathbb{G}^n_{m,C} \text{ rel } Z^{(n)},n)^{\mathrm{sgn}} \xrightarrow{\sim} \bigoplus_{i=0}^{n} H^1_{\mathrm{abs}}(C,i) \; ,$$

which in sheaf theoretic terms corresponds to the decomposition

$$\eta^{(n)}_b : \mathrm{Ext}^1_{\mathrm{Sh}(C)}(F(0),\mathcal{G}^{(n)}_b) \xrightarrow{\sim} \bigoplus_{i=0}^{n} \mathrm{Ext}^1_{\mathrm{Sh}(C)}(F(0),F(i))$$

given by Corollary 5.2.

The pullback pol_b of the small polylogarithmic extension pol on $\mathbb{U}$ is an element of

$$\begin{aligned}
\varprojlim_{n\geq 1} \mathrm{Ext}^1_{\mathrm{Sh}(C)}(F(0),\mathcal{G}^{(n)}_b) &= \varprojlim_{n\geq 1} H^{n+1}_{\mathrm{abs}}(\mathbb{G}^n_{m,C} \text{ rel } Z^{(n)},n)^{\mathrm{sgn}} \\
&= \varprojlim_{n\geq 1} H^{n+1}_{\mathrm{abs}}(\mathbb{G}^{\vee n}_{m,C},n)^{\mathrm{sgn}} \; .
\end{aligned}$$

We have shown that, using the eigenspace decomposition for the action of the $(\phi^n)^\sharp$, these groups are isomorphic to

$$\prod_{k\geq 0} \mathrm{Ext}^1_{\mathrm{Sh}(C)}(F(0), F(k)) = \prod_{k\geq 0} H^1_{\mathrm{abs}}(C, k) \ .$$

2.5 and 2.6 describe pol_b as an element in this group.

Actually, in order to compare the above decomposition to the one used for 2.5 and 2.6, we shall need to copare the isomorphism

$$\eta := \varprojlim_{n\geq 1} \eta^{(n)} : \mathrm{Gr}^W_\cdot \underline{\mathcal{G}} \xrightarrow{\sim} \prod_{k\geq 0} F(k)$$

of 4.10.b) to the isomorphism

$$\kappa : \mathrm{Gr}^W_\cdot \underline{\mathcal{G}} = \mathrm{Gr}^W_\cdot \mathcal{L}og \xrightarrow{\sim} \prod_{k\geq 0} F(k)$$

of section 2.

A priori, we know that the isomorphisms

$$\eta_{-2k} \ , \ \kappa_{-2k} : \mathrm{Gr}^W_{-2k} \underline{\mathcal{G}} \xrightarrow{\sim} F(k)$$

satisfy an identity of the type

$$\eta_{-2k} = q_{-2k} \cdot \kappa_{-2k} \ ,$$

for a constant $q_{-2k} \in F^*$.

We remark that in order to prove the main results announced in the introduction, all one needs to know is that q_{-2k} is a rational number, which is independent of whether we work in the Hodge or the l–adic setting.

In order to exhibit the precise relation of the motivic analogue of *pol* (see section 8) to the cyclotomic elements in K–theory (see Corollary 9.6.b)), we need to identify q_{-2k}.

Proposition 5.4. *We have the equality*

$$\eta_{-2k} = k! \cdot \kappa_{-2k} \ .$$

Proof. Because of the compatibility of κ_0 with the canonical projection

$$\varepsilon : \underline{\mathcal{G}} \longrightarrow F(0) ,$$

we have $\eta_0 = \kappa_0$.

In order to show $\eta_{-2} = \kappa_{-2}$ we compare the classes of $\underline{\mathcal{G}}^{(1)}$ in

$$\operatorname{Ext}^1_{\mathrm{Sh}(\mathbb{G}_m)} (F(0), F(1))$$

induced by η_{-2} and κ_{-2} respectively. Let

$K := \mathbb{C}$,	$K := \mathbb{Q}$,

and choose any K–valued point t of $\mathbb{U}$. Of course, the value of q_{-2} can still be detected from the extensions of

mixed $\mathbb{Q}$–Hodge structures	Galois modules

given by the pullback $t^*\underline{\mathcal{G}}^{(1)}$ of $\underline{\mathcal{G}}^{(1)}$ via t. In both settings, there is a natural morphism of $K^* \otimes_{\mathbb{Z}} F$ into the respective $\operatorname{Ext}^1 (F(0), F(1))$ (see e.g.

[W3], Theorem 3.7).	[W3], Theorem 4.6).

By

[W3], Proposition 3.13.a),	[W3], Proposition 4.7.a),

the class of $t^*\underline{\mathcal{G}}^{(1)}$, calculated in the framing given by κ_{-2}, equals the image of $t \in K^*$ under this morphism.

By [Sch], 2.7, the same holds for the framing given by η_{-2} – note that here it is vital to choose the ordering of the sections $\underline{\alpha}$ and $\underline{\beta}$ in the way we did before 4.11.

For $k \geq 2$, let

$$\varphi_0^{(k)} : \underline{\mathcal{G}}^{(k)} \xrightarrow{\sim} \operatorname{Sym}^k \underline{\mathcal{G}}^{(1)}$$

be the isomorphism of 4.10.b). By 4.9.d), the diagram

$$\begin{tikzcd}
\underline{\mathcal{G}}^{(k)} \arrow[r] \arrow[d, "\varphi_0^{(k)}"', "\wr"] & F(0) \\
\operatorname{Sym}^k \underline{\mathcal{G}}^{(1)} \arrow[r] & \operatorname{Sym}^k F(0) \arrow[u, "\wr"']
\end{tikzcd}$$

commutes. By [W3], Theorem 3.12.a), the commutativity of this diagram characterizes $\varphi_0^{(k)}$ uniquely. From loc. cit., Theorem 3.12.b) and c), we know that the diagram

$$\begin{tikzcd}
F(k) \arrow[r, "\frac{1}{k!}\cdot\kappa^{-1}"] & \underline{\mathcal{G}}^{(k)} \arrow[d, "\wr"', "\varphi_0^{(k)}"] \\
\mathrm{Sym}^k F(1) \arrow[u, "\wr"'] \arrow[r, "\mathrm{Sym}^k\,\kappa^{-1}"] & \mathrm{Sym}^k \underline{\mathcal{G}}^{(1)}
\end{tikzcd}$$

commutes. So our identity

$$\eta_{-2k} = k! \cdot \kappa_{-2k}$$

follows from 4.9.d). $\square$

6 Polylogs in Absolute Cohomology Theories

In section 4, we showed that the logarithmic pro–sheaf is the projective limit of relative cohomology objects with coefficients in Tate twists of certain schemes over $\mathbb{U}$. The Leray spectral sequence suggests that is should be possible to recover *pol* as a projective limit of elements in absolute cohomology with Tate coefficients of these schemes, and indeed this is what we do in Theorem 6.6. That the coefficients are Tate is of course the central point: it allows us, in section 7, to imitate the construction of this section, and thus to define a motivic version of *pol*. This detour is necessary because we know, up to date, of no satisfactory formalism of mixed motivic sheaves, whose absolute cohomology with Tate coefficients would give back motivic cohomology defined via K-theory.

We return to the geometric situation set up before 4.11, and start by computing the higher direct images of the restriction of $\mathcal{L}og$ to $\mathbb{U}$:

Lemma 6.1. *a) The inclusion $F(1) \hookrightarrow \underline{\mathcal{G}}^{(1)}$ and the projection $\underline{\mathcal{G}}^{(1)} \twoheadrightarrow F(0)$ induce natural isomorphisms*

$$\begin{aligned}
F(1)_B &\xrightarrow{\sim} \mathcal{H}_B^{-1}\left(\mathbb{G}_m, \underline{\mathcal{G}}^{(1)}\right) , \\
\mathcal{H}_B^{0}\left(\mathbb{G}_m, \underline{\mathcal{G}}^{(1)}\right) &\xrightarrow{\sim} \mathcal{H}_B^{0}\left(\mathbb{G}_m, F(0)\right) ,
\end{aligned}$$

and the latter group is isomorphic to $F(-1)_B$ via the map "residue at 0".
b) The inclusion $F(n) \hookrightarrow \underline{\mathcal{G}}^{(n)}$ and the projection $\underline{\mathcal{G}}^{(n)} \twoheadrightarrow F(0)$ induce natural

identifications

$$\mathcal{H}^i_B\left(\mathbb{G}_m, \underline{\mathcal{G}}^{(n)}\right) = \begin{cases} F(n)_B \ , & i = -1 \\ F(-1)_B \ , & i = 0 \\ 0 \ , & i \notin \{-1, 0\} \end{cases} \ .$$

Proof. The statements need only be checked on the level of local systems. Part a) is shown in the proof of 4.11. From there, we also recall that we have to compute the invariants and coinvariants under the action of the group $\pi_1 := \pi_1(\mathbb{G}_m(\mathbb{C}), 1)$, or equivalently, of a generator of π_1. Using 4.10.b), we may deduce b) from a). ☐

Corollary 6.2.

$$\mathcal{H}^i_B\left(\mathbb{U}, \mathcal{G}^{(n)}\right) = \begin{cases} F(n)_B & , \ i = -1 \\ 0 & , \ i \notin \{-1, 0\} \end{cases} \ .$$

For $i = 0$*, the sheaf* $\mathcal{H}^0_B\left(\mathbb{U}, \mathcal{G}^{(n)}\right)$ *is the direct sum of* $\bigoplus_{k=1}^{n} F(k-1)_B$ *and an object which is an extension of* $F(-1)_B$ *by itself.*

Proof. By [W1], Theorem 4.3, there is a weight filtration on $\mathcal{H}^i_B\left(\mathbb{U}, \mathcal{G}^{(n)}\right)$. Now use the exact triangle

$$\begin{tikzcd} 1_*1^! \arrow[rr] & & \mathrm{id}_{\mathbb{G}_{m,B}} \arrow[dl] \\ & j_*j^* \arrow[ul, "{[1]}"] & \end{tikzcd}$$

purity, and 4.10.c). ☐

Remark: In the setting of Hodge modules, where a concept of polarization is available, any extension of pure objects of the same weight is necessarily split.

The map $\mathcal{H}^0_B\left(\mathbb{U}, \mathcal{G}^{(n)}\right) \to F(0)$ of the corollary yields in particular a map "residue at 1_B", for $n \geq 1$,

$$\mathrm{res} : H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)}) = H^0_{\mathrm{abs}}\left(B, \mathcal{R}_B(\mathbb{U}, \mathcal{G}^{(n)})\right) \to H^0_{\mathrm{abs}}(B, 0) \ .$$

Definition 6.3. *Let* $n \geq 1$*. The map*

$$\mathrm{res} : H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)}) = H^{n+1}_{\mathrm{abs}}(\mathbb{G}^n_{m,\mathbb{U}} \ rel \ Z^{(n)}, n)^{sgn} \to H^0_{\mathrm{abs}}(B, 0)$$

is called the total residue map.

For later reference, we note

Corollary 6.4. $H^1_{\mathrm{abs}}(\mathbb{G}^1_{m,\mathbb{U}}$ *rel* $Z^{(1)}, 1) = 0$.

Proof. We have

$$H^1_{\mathrm{abs}}(\mathbb{G}^1_{m,\mathbb{U}} \text{ rel } Z^{(1)}, 1) = H^{-1}_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(1)}) \ ,$$

which because of 6.2 equals $H^0_{\mathrm{abs}}(B, F(1)) = 0$. ☐

Next we have

Lemma 6.5. *i) The transition morphism*

$$\mathrm{res} : \mathcal{G}^{(n)} \twoheadrightarrow \mathcal{G}^{(n-1)}$$

satisfies

$$\mathcal{H}^{-1}_B(\mathbb{U}, \mathrm{res}) = 0 : F(n)_B \to F(n-1)_B \ ,$$
$$\mathcal{H}^0_B(\mathbb{U}, \mathrm{res}) : \mathcal{H}^0_B(\mathbb{U}, \mathcal{G}^{(n)}) \to \mathcal{H}^0_B(\mathbb{U}, \mathcal{G}^{(n-1)})$$

is surjective with kernel $F(n-1)_B$.

In particular, the total residue for $n \geq 2$ *factors over the total residue for* $n-1$*: there is a commutative diagram*

$$\begin{tikzcd}
H^0_{\mathrm{abs}}(B, 0) & H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)}) \arrow[l, "\mathrm{res}"'] \arrow[d, "\mathrm{res}"] \\
& H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n-1)}) \arrow[ul, "\mathrm{res}"]
\end{tikzcd}$$

ii) The Leray spectral sequences, for $n \geq 0$*, give exact sequences*

$$0 \longrightarrow H^1_{\mathrm{abs}}(B, n) \xrightarrow{\delta} H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)}) \xrightarrow{\mathrm{res}} H^0_{\mathrm{abs}}(B, 0) \longrightarrow 0 \ .$$

The map

$$\delta : H^1_{\mathrm{abs}}(B, n) \to H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)})$$

is the composition of $H^1_{\mathrm{abs}}(B, n) \to H^1_{\mathrm{abs}}(\mathbb{U}, n) = H^0_{\mathrm{abs}}(\mathbb{U}, F(n))$ *and the map induced by the inclusion of* $F(n)$ *into* $\mathcal{G}^{(n)}$*, in other words, the same noted map of the residue sequence.*

The projective limit of the above sequences identifies

$$H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{L}og\,|_{\mathbb{U}}) := \varprojlim_n H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)})$$

and $H^0_{\mathrm{abs}}(B,0)$*.*

iii) There are unique splittings

$$s_n : H^0_{\mathrm{abs}}(B,0) \hookrightarrow H^0_{\mathrm{abs}}(\mathbb{U},\mathcal{G}^{(n)})$$

of the sequences in ii), for any $n \geq 0$*, such that for any* $n \geq 1$ *we have a commutative diagram*

$$\begin{tikzcd}
H^0_{\mathrm{abs}}(B,0) \arrow[r, "s_n"] \arrow[dr, "s_{n-1}"'] & H^0_{\mathrm{abs}}(\mathbb{U},\mathcal{G}^{(n)}) \arrow[d, "\mathrm{res}"] \\
& H^0_{\mathrm{abs}}(\mathbb{U},\mathcal{G}^{(n-1)})
\end{tikzcd}$$

Proof. i) The first statement is clear. For the second, either go through the construction or observe that the direct image of the morphism $\mathbb{U}_{\mathrm{top}} \to B_{\mathrm{top}}$ has cohomological dimension one, hence $\mathcal{H}^0_B(\mathbb{U}, \,\cdot\,)$ is right exact on smooth sheaves.

ii) We have the Leray spectral sequence

$$E_2^{p,q} = H^p_{\mathrm{abs}}\left(B, \mathcal{H}^q_B(\mathbb{U},\mathcal{G}^{(n)})\right) \Rightarrow H^{p+q}_{\mathrm{abs}}(\mathbb{U},\mathcal{G}^{(n)}) ,$$

whose low-term sequence reads

$$0 \to H^1_{\mathrm{abs}}(B,n) \to H^0_{\mathrm{abs}}(\mathbb{U},\mathcal{G}^{(n)}) \to H^0_{\mathrm{abs}}(B,0) \overset{d_2^{(n)}}{\to} H^2_{\mathrm{abs}}(B,n) .$$

By i), the Mittag–Leffler condition is satisfied for the projective system $(H^1_{\mathrm{abs}}(B,n))_{n\geq 0}$, and therefore,

$$H^0_{\mathrm{abs}}(\mathbb{U},\mathcal{L}og|_{\mathbb{U}}) = \varprojlim_n \ker(d_2^{(n)}) = H^0_{\mathrm{abs}}(B,0)$$

since the projective system $(\mathrm{im}(d_2^{(n)}))_{n\geq 0} \subset (H^2_{\mathrm{abs}}(B,n))_{n\geq 0}$ is ML-zero. But then any of the

$$H^0_{\mathrm{abs}}(\mathbb{U},\mathcal{G}^{(n)}) \to H^0_{\mathrm{abs}}(B,0)$$

must be surjective as well.

iii) Apply ii). ☐

Denote by $pol^{(n)}$ the image of the small polylogarithmic extension pol under

$$H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{L}og\,|_{\mathbb{U}}) \to H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)}) \ .$$

Theorem 6.6. *a) Under the isomorphism*

$$H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{L}og\,|_{\mathbb{U}}) \xrightarrow{\sim} H^0_{\mathrm{abs}}(B, 0)$$

of 6.5 ii), the small polylogarithmic extension pol is mapped to 1.
b) For each $n \geq 0$, the map

$$s_n : H^0_{\mathrm{abs}}(B, 0) \to H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)})$$

maps 1 *to* $pol^{(n)}$.

Proof. This is the definition of pol and the s_n. □

Recall (4.9.a)) that we may identify

$$\begin{aligned} H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)}) &= H^0_{\mathrm{abs}}(\mathbb{G}^n_{m,\mathbb{U}}, v^n_! F(n))^{\mathrm{sgn}} \\ &= H^{n+1}_{\mathrm{abs}}(\mathbb{G}^n_{m,\mathbb{U}}, v^n_! F(n)_{V^n})^{\mathrm{sgn}} \\ &= H^{n+1}_{\mathrm{abs}}(\mathbb{G}^n_{m,\mathbb{U}} \text{ rel } Z^{(n)}, n)^{\mathrm{sgn}} \ . \end{aligned}$$

In section 8, we are going to construct a motivic version of pol by proving a motivic analogue of 6.5.ii), and then *defining pol* as the element in

$$\varprojlim_n H^{n+1}_{\mathcal{M}}(\mathbb{G}^n_{m,\mathbb{U}} \text{ rel } Z^{(n)}, n)^{\mathrm{sgn}}$$

mapping to 1 under the isomorphism to $H^0_{\mathcal{M}}(B, 0)$.

In order to prove an analogue of 6.5.ii), we shall frequently use injectivity of the Beilinson regulator on certain motivic cohomology groups, and two technical results on $H^{\cdot}_{\mathrm{abs}}$, that will occupy the rest of this section.
While this may appear artificial at first sight, we remind the reader that in order to prove 6.5.ii) for motivic cohomology, we cannot make use of any sheaf theoretic means like Leray spectral sequences.

An important means will be the *localization sequence* associated to the geometric situation

$$\{0,1\}_B \hookrightarrow \mathbb{A}^1_B \hookleftarrow \mathbb{U} \ .$$

It is the result of the degeneration of the Leray spectral sequence and reads

$$\begin{aligned} \cdots \to & H^{\cdot}_{\mathrm{abs}}(\mathbb{A}^1_B, p) \to H^{\cdot}_{\mathrm{abs}}(\mathbb{U}, p) \to H^{\cdot -1}_{\mathrm{abs}}(\{0,1\}_B, p-1) \\ \to & H^{\cdot +1}_{\mathrm{abs}}(\mathbb{A}^1_B, p) \to \ldots \end{aligned}$$

Lemma 6.7. *a) The structure morphism is an isomorphism*

$$H^{\cdot}_{\mathrm{abs}}(B,p) \xrightarrow{\sim} H^{\cdot}_{\mathrm{abs}}(\mathbb{A}^1_B,p) \ .$$

b) The boundary map is trivial, i.e., we have short exact sequences

$$0 \to H^{\cdot}_{\mathrm{abs}}(B,p) \to H^{\cdot}_{\mathrm{abs}}(\mathbb{U},p) \to \bigoplus_{i=0}^{1} H^{\cdot-1}_{\mathrm{abs}}(B,p-1) \to 0 \ .$$

Proof. For a), note that $\mathcal{R}_B\left(\mathbb{A}^1_B, F(p)_{\mathbb{A}^1_B}\right) = F(p)_B[0]$.
b) follows from the fact that there are B-valued points of $\mathbb{U}$. □

In particular, for $p=1$, we have the exact sequence

$$0 \longrightarrow H^1_{\mathrm{abs}}(B,1) \longrightarrow H^1_{\mathrm{abs}}(\mathbb{U},1) \xrightarrow{\partial} \bigoplus_{i=0}^{1} H^0_{\mathrm{abs}}(B,0) \longrightarrow 0 \ .$$

The last map equals the map of Ext groups

$$\partial : \mathrm{Ext}^1_{\mathrm{Sh}(\mathbb{U})}\left(F(0),F(1)\right) \longrightarrow \mathrm{Hom}_{\mathrm{Sh}(B)}\left(F(0),\mathcal{H}^0_B\left(\mathbb{U},F(1)\right)\right)$$

obtained from the Leray spectral sequence; observe that the residues at 0_B and 1_B provide an isomorphism

$$\mathcal{H}^0_B\left(\mathbb{U},F(1)\right) \xrightarrow{\sim} \bigoplus_{i=0}^{1} F(0) \ .$$

We have a natural map

$$\mathcal{O}(\mathbb{U})^* \to H^1_{\mathrm{abs}}(\mathbb{U},1) \ .$$

Its composition with

$$\partial : H^1_{\mathrm{abs}}(\mathbb{U},1) \longrightarrow \bigoplus_{i=0}^{1} H^0_{\mathrm{abs}}(B,0)$$

associates to a function on $\mathbb{U}$ its orders at 0 and 1 respectively.

We need to understand the composition

$$\begin{aligned}\mathrm{res}\circ\partial : H^1_{\mathrm{abs}}(\mathbb{U},1) &= \mathrm{Ext}^1_{\mathrm{Sh}(\mathbb{U})}\left(F(0),F(1)\right) \\ &\longrightarrow \mathrm{Hom}_{\mathrm{Sh}(B)}\left(F(0),\mathcal{H}^0_B\left(\mathbb{U},\mathcal{G}^{(1)}\right)\right) \ .\end{aligned}$$

Observe that due to 6.2, the last group is equal to $H^0_{\mathrm{abs}}(B,0)$. Furthermore, we recall from the proof of 6.2 and the definition of res that the composition

$$\bigoplus_{i=0}^{1} F(0) = \mathcal{H}^0_B\left(\mathbb{U}, F(1)\right) \longrightarrow \mathcal{H}^0_B\left(\mathbb{U}, \mathcal{G}^{(1)}\right) \xrightarrow{\mathrm{res}} F(0)$$

is given by projection onto the "1"–component of $\bigoplus_{i=0}^{1} F(0)$. We have thus proved:

Lemma 6.8. *Consider the non–vanishing functions t and $1-t$ on $\mathbb{U}$. We have*

$$\mathrm{res} \circ \partial(t) = 0 \ , \quad \mathrm{res} \circ \partial(1-t) = 1 \ .$$

In particular, the map

$$\delta : H^1_{\mathrm{abs}}(\mathbb{U}, 1) \longrightarrow H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(1)}) = H^2_{\mathrm{abs}}\left(\mathbb{G}^1_{m,\mathbb{U}} \ rel \ Z^{(1)}, 1\right)$$

does not map $1-t \in \mathcal{O}(\mathbb{U})^*$ to zero.

Proof. Observe that $\mathrm{res} \circ \partial$ factorizes through δ. □

7 Calculations in K-theory

The next step is to do the constructions of section 4 with K-groups, or more precisely, with relative K-cohomology as introduced in appendix B.2. For technical reasons we will have to use simplicial schemes to replace the singular schemes that appeared before. All constructions will be compatible with the regulator maps to absolute Hodge cohomology (appendix A and B.5.8) and to continuous étale cohomology (appendix B.4.6).

A priori these regulators have values in absolute cohomology groups for the same simplicial object (cf. B.4.2 and B.5.2). Using B.4.5 and B.5.7 these absolute cohomology groups are then identified with (relative) cohomology of singular schemes. This identification is made tacitly.

Let $B = \mathrm{Spec}(\mathbb{Z})$ and S a smooth affine B-scheme. We will work in the category of smooth S-schemes. K-cohomology is taken on the Zariski site over B.

Before returning to the geometric situation introduced in section 3, we have to check a technical lemma. Let us consider the following general construction: Let X be a smooth quasi-projective S-scheme and Y a closed subscheme of X which is itself also smooth over S. Put

$$Y_0^{(n)} = Y \times_S X^{n-1} \amalg X \times_S Y \times X^{n-2} \amalg \ldots \amalg X^{n-1} \times_S Y \ .$$

Note that $Y_0^{(n)}$ is a proper covering of the singular scheme

$$Y^{(n)} = X^n \smallsetminus (X \smallsetminus Y)^n \ .$$

This is the easiest case of a morphism of schemes with cohomological descent, meaning that for any reasonable cohomology theory the cohomology of $Y^{(n)}$ will agree with the cohomology of the smooth simplicial scheme

$$Y_\cdot^{(n)} = \operatorname{cosk}_0(Y_0^{(n)}/X^n) \ ,$$

i.e.,

$$Y_k^{(n)} = Y_0^{(n)} \times_{X^n} \cdots \times_{X^n} Y_0^{(n)} \qquad (k+1\text{-fold product}).$$

For étale cohomology and absolute Hodge cohomology, the corresponding results are B.4.5 and B.5.6 respectively.

We will work in the setting of spaces, i.e., pointed simplicial sheaves of sets on the Zariski site of smooth B-schemes. We refer to appendix B.1 for details and terminology. We use the notation

$$X_\cdot^{\vee n} = \operatorname{Cone}(Y_\cdot^{(n)} \longrightarrow X^n)$$

for the space that computes relative cohomology for the closed embedding (cf. B.1.5).

The space $Y_\cdot^{(n)}$ does not become degenerate above any simplicial degree. However, we have:

Lemma 7.1. a) *$Y_\cdot^{(n)}$ is isomorphic in* $\operatorname{Ho} s\mathbf{T}$ *to a simplicial scheme which is degenerate above degree $n-1$.*

b) *In particular,. $Y_\cdot^{(n)}$ and $X_\cdot^{\vee n}$ are K-coherent.*

c) *$X_\cdot^{\vee n}$ is a space constructed from schemes in a finite diagram over X^n in the sense of B.2.11.*

d) *If T is another closed subscheme of X which is smooth over S and disjoint of Y, then the inclusions*

$$T^i \times_S X^{n-i} \to X^n$$

are tor-independent of all morphisms in the diagram in c).

Proof. By definition

$$Y_0^{(n)} = Y_1 \amalg \cdots \amalg Y_n$$

where Y_i is the reduced closed subscheme of X^n of those points, whose i-th coordinate lies in Y. This induces a decomposition of $Y_k^{(n)}$ into disjoint subschemes of the form $Y_{i_1} \times_{X^n} \cdots \times_{X^n} Y_{i_k}$. Actually this subscheme is canonically isomorphic to

$$Y_{i_1} \cap \cdots \cap Y_{i_k} = \{(x_1, \ldots, x_n) \in X^n \mid x_{i_j} \in Y \text{ for } 1 \leq j \leq k\} .$$

We get the following more familiar form of the simplicial scheme

$$Y_k^{(n)} = \coprod_{I \in \{1,\ldots,n\}^k} \bigcap_{i \in I} Y_i .$$

Let $\Delta(n)$ be the simplicial set with

$$\Delta(n)_k = \{(i_0, \ldots, i_k) \mid 1 \leq i_0 \leq \cdots \leq \ldots i_k \leq n\} .$$

We define the simplicial scheme $Y_\cdot^{\Delta(n)}$ by

$$Y_k^{\Delta(n)} = \coprod_{I \in \Delta(n)_k} \bigcap_{i \in I} Y_i$$

It is degenerate above the simplicial degree $n-1$ and from our previous considerations we see that it is a natural subspace of $Y_\cdot^{(n)}$. We consider these simplicial schemes as spaces in the sense of appendix B.1 by adding a disjoint base point $\star$.

For a scheme U in the big Zariski site over B we consider the morphism of simplicial sets

$$Y_\cdot^{\Delta(n)}(U) \to Y_\cdot^{(n)}(U) .$$

By the combinatorial Lemma B.6.2 it induces an isomorphism of homotopy sets. Hence the inclusion is a weak homotopy equivalence of spaces.

b) is an immediate consequence of a) and B.2.3.b). Recall that Y and X were assumed smooth over B.

We already have seen that all components of $X^{\vee n}$ are disjoint unions of X^n-schemes of the form $Y_{i_1} \cap \cdots \cap Y_{i_k}$ and a disjoint base point. All morphisms between the scheme components are given by the natural closed immersions between them. The condition on the tor-dimension required in B.2.11 follows because are schemes are regular.

T, Y and X are all flat over S, hence the maps in the diagram

$$\begin{tikzcd} & X \times_S Y \arrow[d] \\ T \times_S X \arrow[r] & X \times_S X \end{tikzcd}$$

are easily seen to be tor-independent. The inclusions of T and Y into X are trivially tor-independent because this is a local condition. □

Basically this lemma tells us that all conditions hold that are needed to apply the machinery of appendix B.2. We have a well-behaved relative motivic cohomology theory (cf. B.2.9).

Now we return to the geometric situation set up in section 3. We consider

$$\begin{tikzcd} Z_{\cdot}^{(n)} \arrow[r] \arrow[d] & \mathbb{G}_{m,S}^n \arrow[d] \\ \overline{Z}_{\cdot}^{(n)} \arrow[r] & \mathbb{A}_S^n \end{tikzcd}$$

where $Z = \overline{Z} = \alpha(S) \amalg \beta(S)$ with disjoint S-rational points α and β of $\mathbb{G}_{m,S}$. There is a simplicial operation of $\mathfrak{S}^n$ on the situation which induces an operation on relative K-cohomology and on motivic cohomology.

Proposition 7.2. *There is a natural residue map*

$$H^i_{\mathcal{M}}(\mathbb{G}^n_{m,S} \textit{ rel } Z_{\cdot}^{(n)}, j)^{sgn} \xrightarrow{\mathrm{res}_n} H^{i-1}_{\mathcal{M}}(\mathbb{G}^{n-1}_{m,S} \textit{ rel } Z_{\cdot}^{(n-1)}, j-1)^{sgn}$$

where sgn means the sign eigen-space under the operation of the respective symmetric group.

Moreover, there is a long exact sequence

$$\begin{aligned}\cdots \to & H^{i-2}_{\mathcal{M}}(\mathbb{G}^{n-1}_{m,S} \textit{ rel } Z^{(n-1)}_{\cdot}, j-1)^{sgn} \to H^i_{\mathcal{M}}(\mathbb{A}^n_S \textit{ rel } \overline{Z}^{(n)}_{\cdot}, j)^{sgn} \\ & \qquad\qquad \to H^i_{\mathcal{M}}(\mathbb{G}^n_{m,S} \textit{ rel } Z^{(n)}_{\cdot}, j)^{sgn} \\ \to & H^{i-1}_{\mathcal{M}}(\mathbb{G}^{n-1}_{m,S} \textit{ rel } Z^{(n-1)}_{\cdot}, j-1)^{sgn} \to \cdots \ .\end{aligned}$$

Under the regulators, the long exact sequences are compatible with the ones in absolute cohomology (after 4.5).

Remark: Recall that $Z^{(0)}_{\cdot} = \star$ and hence $H^k_{\mathcal{M}}(\mathbb{G}^0_{m,S} \text{ rel } Z^{(0)}_{\cdot}, j) = H^k_{\mathcal{M}}(S, j)$ by definition.

Proof. We filter $\mathbb{A}^n_S$ by the open subschemes $F_k\mathbb{A}^n_S$ defined just before Lemma 4.5. In particular, $F_0\mathbb{A}^n_S = \mathbb{G}^n_{m,S}$. Again $G_k\mathbb{A}^n_S = F_k\mathbb{A}^n_S \smallsetminus F_{k-1}\mathbb{A}^n_S$. We use the notation $F_k\mathbb{A}^{\vee n}_{\cdot}$ and $G_k\mathbb{A}^{\vee n}_{\cdot}$ for the induced open respectively locally closed subspaces of $\mathbb{A}^{\vee n}_{\cdot}$.

Note that the situation is still symmetric under permutation of coordinates. Hence there is a compatible operation of the symmetric group on the space constructed from schemes $F_k\mathbb{A}^{\vee n}_{\cdot}$.

The closed immersion $G_k\mathbb{A}^{\vee n}_{\cdot} \to F_k\mathbb{A}^{\vee n}_{\cdot}$ satisfies the first condition in (TC) in B.2.11. The maps we have to consider for the rest of (TC) are locally of the form considered in 7.1.d). Hence B.2.17 applies, i.e., we can use the localization sequences for motivic cohomology induced by the triples $F_{k-1}\mathbb{A}^{\vee n}_{\cdot} \to F_k\mathbb{A}^{\vee n}_{\cdot} \leftarrow G_k\mathbb{A}^{\vee n}_{\cdot}$. We get

$$\begin{aligned}\ldots \to & H^i_{\mathcal{M}}(G_k\mathbb{A}^{\vee n}_{\cdot}, j) \to H^{i+2}_{\mathcal{M}}(F_k\mathbb{A}^{\vee n}_{\cdot}, j+1) \to H^{i+2}_{\mathcal{M}}(F_{k-1}\mathbb{A}^{\vee n}_{\cdot}, j+1) \\ \to & H^{i+1}_{\mathcal{M}}(G_k\mathbb{A}^{\vee n}_{\cdot}, j) \to \ldots\end{aligned}$$

The sequence remains exact when we take sign–eigenspaces. Now let us compute one of the groups involved.

$$H^i_{\mathcal{M}}(G_k\mathbb{A}^{\vee n}_{\cdot}, j) = \bigoplus_{\{1 \le a_1 < a_2 < \cdots < a_k \le n\}} H^i_{\mathcal{M}}(\mathbb{A}^{\vee n}_{\cdot} \times_{\mathbb{A}^n} \mathbb{G}_{m,S}(a_1, \ldots, a_k), j)$$

where

$$\mathbb{G}_{m,S}(a_1, \ldots, a_k) = \{(x_1, \ldots, x_n) \mid x_i = 0 \text{ if } i = a_j \text{ for some } j;\ x_i \neq 0 \text{ else } \}$$

The decomposition corresponds to the decomposition of $G_k\mathbb{A}^n$ into its connected components. The notation $\mathbb{A}^{\vee n}_{\cdot} \times_{\mathbb{A}^n} \mathbb{G}_{m,S}(a_1, \ldots, a_k)$ means the open subspace lying over the locally closed scheme.

Now consider the operation of the symmetric group. If $k > 1$, then there is for each component some transposition which acts trivially, namely one that interchanges two vanishing coordinates. Hence the sign–eigenspace vanishes altogether.

For $k = 1$, the decomposition has the form

$$H^i_{\mathcal{M}}(G_1\mathbb{A}^{\vee n}_{\cdot}, j) = \bigoplus_{a=1,\cdots,n} H^i_{\mathcal{M}}(\mathbb{A}^{\vee n}_{\cdot} \times_{\mathbb{A}^n} (\mathbb{G}^{a-1}_{m,S} \times \{0\} \times \mathbb{G}^{n-a}_{m,S}), j) \ .$$

The operation of the symmetric group permutes the factors transitively. The stabilizer of one summand is the symmetric group $\mathfrak{S}^{n-1}$. We get

$$H^i_{\mathcal{M}}(G_1\mathbb{A}^{\vee n}_{\cdot}, j)^{\mathrm{sgn}} \cong H^i_{\mathcal{M}}(\mathbb{G}^{\vee n-1}_{m,S}, j)^{\mathrm{sgn}}$$

where the sign eigenspace on the right hand side is taken with respect to the smaller symmetric group $\mathfrak{S}^{n-1}$. We have a choice of isomorphism here and use the one that identifies $\mathbb{G}^{n-1}_{m,S}$ with $\mathbb{G}^{n-1}_{m,S} \times \{0\}$.

Putting these results in the long exact sequences we get iteratively

$$H^i_{\mathcal{M}}(\mathbb{A}^n \text{ rel } \overline{Z}^{(n)}_{\cdot}, j)^{\mathrm{sgn}} = H^i_{\mathcal{M}}(F_n\mathbb{A}^{\vee n}_{\cdot}, j)^{\mathrm{sgn}} \xrightarrow{\cong} \cdots H^i_{\mathcal{M}}(F_1\mathbb{A}^{\vee n}_{\cdot}, j)^{\mathrm{sgn}} \ .$$

So the above sequence, for $k = 1$, gives the desired residue sequence.

We can do the same construction for absolute cohomology (Hodge or l-adic) considered as generalized cohomology theories. By B.4.6, B.5.8 and B.3.7, the long exact sequences for motivic cohomology will be compatible via the regulator with the ones in generalized cohomology. The next step is to pass from generalized cohomology to cohomology of abelian sheaves. By B.4.5 and B.5.7 this can be done. In fact we get precisely the residue sequence for absolute cohomology constructed in section 4. ☐

Remark: a) By B.2.17, we have the same maps and long exact sequences for the K-cohomology groups themselves. However, note that there is a Riemann-Roch hidden in the compatibility of the localization sequence in K-cohomology and absolute cohomology.

b) We shall show injectivity of the Beilinson regulator on

$$H^{n+1}_{\mathcal{M}}(\mathbb{G}^n_{m,S} \text{ rel } Z^{(n)}_{\cdot}, n)^{\mathrm{sgn}}$$

in Proposition 8.7. Together with Lemma 4.4.b), it shows that the residue map on

$$H^i_{\mathcal{M}}(\mathbb{G}^n_{m,S} \text{ rel } Z^{(n)}_{\cdot}, j)^{\mathrm{sgn}}$$

does not depend on the choice of embedding of $\mathbb{G}^{n-1}_{m,S}$ in

$$\bigcup_{a=1,\cdots,n} \mathbb{G}^{a-1}_{m,S} \times \{0\} \times \mathbb{G}^{n-a}_{m,S}$$

of the above proof, if $(i,j) = (n+1,n)$. Since we are only interested in these special indices, we chose to exclude from the statement of 7.2 the dependence of res_n in the general case from the above choice.

Lemma 7.3. *Let $2j \geq k$. Then*

$$H^k_{\mathcal{M}}(\mathbb{A}^n_S \textit{ rel } \overline{Z}^{(n)}_{\cdot}, j) \cong H^{k-n}_{\mathcal{M}}(S,j)$$

where the isomorphism is induced by a choice of ordering of the sections α and β. It is compatible with the identification in 4.6 under the regulator map. $\mathfrak{S}_n$ operates by sign on the left hand side.

Remark: Here and in the sequel we put $H^i_{\mathcal{M}}(S,j) = 0$ if $j < 2i$. This makes sense as S is regular and the corresponding K–group vanishes (see B.2.3).

Proof. Fix j. We consider the skeletal spectral sequence B.2.10. We have

$$E^{p,q}_1 = H^q_{\mathcal{M}}((\mathbb{A}^{\vee n}_S)_p, j) \ .$$

We will show that the only non-trivial E_2-terms are concentrated in one vertical line

$$E^{n,q}_2 = H^q_{\mathcal{M}}(S,j) \ .$$

This means that the spectral sequence converges in the strongest possible way. This yields isomorphisms as stated.

Before we can check this we need some preparation. If $X_{\cdot}$ is a space constructed from schemes, we denote by $Cp(X_{\cdot})$ the simplicial set of its connected components. $Cp(\overline{Z}^{(n)}_{\cdot})$ has the same singular cohomology as $Cp(\overline{Z}^{\Delta(n)}_{\cdot})$ (cf. proof of 7.1) which is the simplicial set attached to a CW-complex dual to the boundary of the n-dimensional hypercube (note that $\overline{Z}$ has two disjoint components). This means that $Cp(\overline{Z}^{\Delta(n)}_{\cdot})$ has a 1-vertex for every $(n-1)$-cell of the cube etc. In particular we see that it has the homotopy type of an $(n-1)$-sphere. $Cp(\mathbb{A}^n_S)$ is of course contractible. It follows that $Cp(\mathbb{A}^{\vee n}_{\cdot})$ has singular cohomology concentrated in degree n where it is one-dimensional.

Let us make this more explicit:
In order to compute the cohomology of a cosimplicial group it suffices to consider the subcomplex corresponding to nondegenerate simplices. $Cp(\overline{Z}^{\Delta(n)}_{\cdot})$ is completely degenerate from cosimplicial degree n on. In degree $n-1$, there is one nondegenerate simplex for each vertex of the hypercube. They are indexed by $\{\alpha, \beta\}^n$. Hence any element of $H^n Cp(\mathbb{A}^{\vee n}_{\cdot}) = H^{n-1}(Cp(\overline{Z}^{\Delta(n)}_{\cdot}))$ is represented by an element of

$$K^{n-1} = \bigoplus_{\{\alpha,\beta\}^n} \mathbb{Q} \, .$$

Let g be a generator of the cohomology group. $Cp(\overline{Z}^{(n)}_{\cdot})$ does not become degenerate. The nondegenerate part in degree $n-1$ is given by one copy of $\{\alpha, \beta\}^n$ for each possible permutation of the numbers $0, \ldots, n-1$. It is easy to see that $((-1)^{\mathrm{sgn}(\sigma)} g)_\sigma$ is in the kernel of the differential. It represents the generator of cohomology of $Cp(\overline{Z}^{(n)}_{\cdot})$. We see that $\mathfrak{S}_n$ operates by the sign of the permutation.

We choose the generator g of cohomology given by the tuple

$$(-1)^{s(i_1)+\cdots+s(i_n)} \in \mathbb{Q}_{i_1 \times \cdots \times i_n}$$

where $i_k \in \{\alpha, \beta\}$ and $s(\alpha) = 1$, $s(\beta) = 0$. This choice of generator amounts to picking the ordering $\alpha < \beta$ and extending it by the Künneth-formula.

Now let us analyze our E_1-term: For fixed q we have the complex attached to the cosimplicial abelian group $H^q_{\mathcal{M}}((\mathbb{A}^{\vee n}_S)_p, j)_{p \in \mathbb{N}_0}$. All connected components of $\mathbb{A}^{\vee n}_{\cdot}$ are isomorphic to a copy of some power of $\mathbb{A}^1_S$. By the homotopy property of K-theory we have

$$H^q_{\mathcal{M}}((\mathbb{A}_S)^{\vee n}_p, j)_{p \in \mathbb{N}_0} = H^q_{\mathcal{M}}(S, j) \otimes_{\mathbb{Q}} C^{\vee n}_{\cdot}$$

where $C^{\vee n}_{\cdot}$ is the cosimplicial vector space computing singular cohomology of $Cp(\mathbb{A}^{\vee n}_{\cdot})$. By the previous considerations we already know its cohomology. It also follows that the operation of $\mathfrak{S}_n$ on our motivic cohomology is by the sign.

Now compare our isomorphism to the one constructed in the realization. We have the same spectral sequence there (attached to the weight filtration). The identification of the E_2-term also uses Künneth-formula and choice of an ordering of the sections. $\square$

Using this identification we obtain the familiar residue sequence for motivic cohomology for $2j \geq k$

$$\begin{aligned} \ldots \to &H^{k-n}_{\mathcal{M}}(S,j) \to H^{k}_{\mathcal{M}}(\mathbb{G}^{\vee n}_{m,S},j)^{\mathrm{sgn}} \to H^{k-1}_{\mathcal{M}}(\mathbb{G}^{\vee n-1}_{m,S},j-1)^{\mathrm{sgn}} \\ \to &H^{k-n+1}_{\mathcal{M}}(S,j) \to \ldots \end{aligned}$$

This long exact sequence maps to the one before 4.7 by the regulators.

Note that these long exact sequences for all indices k and n organize into a spectral sequence connecting the relative motivic cohomology of $\mathbb{A}^{\vee ?}_{\cdot}$ and the relative motivic cohomology of $\mathbb{G}^{\vee ?}_{m}$. In particular for each n there is the converging cohomological spectral sequence

$$E_1^{pq} = H^{p+q-n}_{\mathcal{M}}(S,p) \Rightarrow H^{p+q}_{\mathcal{M}}(\mathbb{G}^{n}_{m,S} \text{ rel } Z^{(n)}_{\cdot}, n) \ .$$

This is the motivic version of the weight spectral sequence in absolute cohomology.

8 Universal Motivic Polylogarithm

We now return to the special situation used in section 6. Let $B = \mathrm{Spec}(\mathbb{Z})$. We consider now the case $S = \mathbb{U}$. Let $\alpha = 1$, and β the diagonal section of $\mathbb{U} \times_B \mathbb{G}_{m,B}$.

First we compute the motivic cohomology of $\mathbb{U}$. We use the embedding of $\mathbb{U}$ into $\mathbb{A}^1$ to do so. The long exact localization sequence B.2.16 reads

$$\begin{aligned} \ldots \to &H^{n-2}_{\mathcal{M}}(0(B) \amalg 1(B), j-1) \to H^{n}_{\mathcal{M}}(\mathbb{A}^1_B, j) \to H^{n}_{\mathcal{M}}(\mathbb{U}, j) \\ \to &H^{n-1}_{\mathcal{M}}(0(B) \amalg 1(B), j-1) \to \ldots \end{aligned}$$

By the homotopy property of K-theory we get

$$\begin{aligned} \ldots \to &H^{n}_{\mathcal{M}}(B,j) \to H^{n}_{\mathcal{M}}(\mathbb{U},j) \to H^{n-1}_{\mathcal{M}}(B,j-1) \oplus H^{n-1}_{\mathcal{M}}(B,j-1) \\ \to &H^{n+1}_{\mathcal{M}}(B,j) \to \ldots \end{aligned}$$

The Gysin map for the inclusion of a point in the affine line vanishes by [Q2] Thm 8 ii. Hence we are actually dealing with a system of short exact sequences. As all motivic cohomology groups of B vanish for $n > 1$ this sequence only gives non-trivial cohomology of $\mathbb{U}$ for $n = 0, 1, 2$.

Lemma 8.1. *For $B = \mathrm{Spec}(\mathbb{Z})$ we have*

$$\begin{aligned}
H^0_{\mathcal{M}}(\mathbb{U}, i) &= \begin{cases} \mathbb{Q} & \text{if } i = 0, \\ 0 & \text{else,} \end{cases} \\
H^1_{\mathcal{M}}(\mathbb{U}, j) &= \begin{cases} 0 & \text{for } j < 1, \\ \mathbb{Q} \oplus \mathbb{Q} & \text{for } j = 1, \\ H^1_{\mathcal{M}}(B, j) & \text{for } j > 1, \end{cases} \\
H^2_{\mathcal{M}}(\mathbb{U}, j) &= H^1_{\mathcal{M}}(B, j-1) \oplus H^1_{\mathcal{M}}(B, j-1) \ , \\
H^n_{\mathcal{M}}(\mathbb{U}, j) &= 0 \text{ if } n > 2.
\end{aligned}$$

Proof. Clear from the above using B.2.18 ☐

By Borel's Theorem (B.5.9) the Beilinson regulator

$$H^i_{\mathcal{M}}(X, j) \otimes_{\mathbb{Q}} \mathbb{R} \to H^i_{\mathfrak{H}^p}(X_{\mathbb{R}}/\mathbb{R}, j)$$

is injective for $X = \mathrm{Spec}(\mathbb{Z})$, even an isomorphism but in the one case $H^1_{\mathcal{M}}(B, 1)$ where the codimension is one. (We call Beilinson regulator what strictly speaking is its tensor product with $\mathbb{R}$.) This implies that it is also an isomorphism for $H^i_{\mathcal{M}}(\mathbb{U}, k)$ with the exception of the indices $(1, 1)$ and $(2, 2)$ where the codimension is 1 resp. 2.

This means that many of the residue maps are actually isomorphisms. The following computations are carried out in the case $B = \mathrm{Spec}(\mathbb{Z})$. With a little more effort they generalize to the case of the ring of integers of a number field.

Consider the residue sequence for $n = j = 1$ and $S = \mathbb{U}$.

$$\begin{aligned}
0 = H^0_{\mathcal{M}}(\mathbb{U}, 1) \to H^1_{\mathcal{M}}(\mathbb{G}^{\vee 1}_{m,\mathbb{U}}, 1) \to H^0_{\mathcal{M}}(\mathbb{U}, 0) \\
\to H^1_{\mathcal{M}}(\mathbb{U}, 1) \xrightarrow{\delta} H^2_{\mathcal{M}}(\mathbb{G}^{\vee 1}_{m,\mathbb{U}}, 1) \to H^1_{\mathcal{M}}(\mathbb{U}, 0) = 0 \ .
\end{aligned}$$

The Beilinson regulator induces a map between the above sequence and the residue sequence in section 4. On $H^0_{\mathcal{M}}(\mathbb{U}, 0) \otimes \mathbb{R}$, the regulator is an isomorphism, and on $H^1_{\mathcal{M}}(\mathbb{U}, 1) \otimes \mathbb{R}$ it is injective of codimension one. By 6.4, the absolute Hodge cohomology group $H^1_{\mathfrak{H}^p}(\mathbb{G}^{\vee 1}_{m,\mathbb{U}_\mathbb{R}}/\mathbb{R},1)$ vanishes. Hence the map from the first to the second line is injective and the regulator is injective of codimension one on $H^2_\mathcal{M}(\mathbb{G}^{\vee 1}_{m,\mathbb{U}},1)$. Furthermore, this last group is one dimensional.

The image of δ under the Beilinson regulator is the map occurring in 6.8 for $n=1$.

Definition 8.2. *Let s_1 be the composition of the maps*

$$\mathbb{Q} \;=\; H^0_\mathcal{M}(B,0) \;\xrightarrow{i_1}\; \bigoplus_{i=0,1} H^0_\mathcal{M}(B,0) \;=\; H^1_\mathcal{M}(\mathbb{U},1) \;\xrightarrow{\delta}\; H^2_\mathcal{M}(\mathbb{G}^{\vee 1}_{m,\mathbb{U}},1)$$

where i_1 is the inclusion of the 1*-summand and δ is the map of the residue sequence.*

Lemma 8.3. *s_1 is an isomorphism.*

Proof. Because of dimension reasons we only have to check that δ does not vanish on the image of i_1. This follows from 6.8. □

Definition 8.4. *Let* res_1 *be the inverse of s_1. We define the total residue map*

$$\mathrm{res}: H^{n+1}_\mathcal{M}(\mathbb{G}^{\vee n}_{m,\mathbb{U}},n)^{sgn} \longrightarrow \mathbb{Q}\ .$$

by composition of the residue maps in our long exact sequence 7.2 with res_1*.*

We now have to check that the total residue map deserves its name. By definition and 6.5.i) it suffices to consider res_1.

Lemma 8.5. *The regulators map the motivic* res_1 *to* res_1 *in absolute cohomology.*

Proof. Let us consider the situation of 6.8. The morphism

$$\mathcal{O}(\mathbb{U})^* \longrightarrow H^1_{\mathfrak{H}^p}(\mathbb{U}_\mathbb{R}/\mathbb{R},1)$$

factors through $H^1_\mathcal{M}(\mathbb{U},1) = K_1(\mathbb{U})_\mathbb{Q}$. There is a commutative diagram

$$\begin{tikzcd}
 & H^1_{\mathfrak{H}^p}(\mathbb{U}_\mathbb{R}/\mathbb{R},1) \arrow[r] & \bigoplus_{i=0,1} H^0_{H_{\mathfrak{H}^p}}(B_\mathbb{R}/\mathbb{R},0) \\
\mathcal{O}(\mathbb{U})^* \arrow[r] & H^1_\mathcal{M}(\mathbb{U},1) \arrow[u] \arrow[r, "\cong"] & \bigoplus_{i=0,1} H^0_\mathcal{M}(B,0) \arrow[u, "\cong"']
\end{tikzcd},$$

hence the functions t and $1-t$ on $\mathbb{U}$ correspond to the canonical generators of the two summands. We consider the commutative diagram for absolute Hodge cohomology

$$\begin{tikzcd}
H^1_{\mathfrak{H}^p}(\mathbb{U}_{\mathbb{R}}/\mathbb{R},1) \arrow[r, "\delta_{\mathbb{R}}"] & H^2_{\mathfrak{H}^p}(\mathbb{G}^{\vee 1}_{m,\mathbb{U}_{\mathbb{R}}}/\mathbb{R},1) \arrow[r, "\mathrm{res}_{\mathbb{R}}"] & H^0_{\mathfrak{H}^p}(B_{\mathbb{R}}/\mathbb{R},0) \\
\bigoplus\limits_{i=0,1} H^0_{\mathcal{M}}(B,0) \arrow[u] \arrow[r, "\delta"] & H^2_{\mathcal{M}}(\mathbb{G}^{\vee 1}_{m,\mathbb{U}},1) \arrow[u, "r"'] &
\end{tikzcd} ,$$

By 6.8 the composition from the bottom left to the top right corner is given by the projection to the 1-component tensored by $\mathbb{R}$. It follows that $(\mathrm{res}_R \circ r)\otimes\mathbb{R}$ is an isomorphism. In turn δ vanishes on the 0-component and is an isomorphism on the 1-component. But then by definition $\mathrm{res}_1 \circ \delta$ is also the projection to the 1-summand. As δ is surjective, this suffices. The same argument works in the étale situation. ☐

Lemma 8.6. *There is a short exact sequence*

$$0 \to H^1_{\mathcal{M}}(B,2) \to H^3_{\mathcal{M}}(\mathbb{G}^{\vee 2}_{m,\mathbb{U}},2)^{sgn} \xrightarrow{\mathrm{res}} \mathbb{Q} \to 0$$

and the Beilinson regulator is an isomorphism on the middle term.

Proof. This is nothing but the residue sequence using our computation of $H^2_{\mathcal{M}}(\mathbb{G}^{\vee 1}_{m,\mathbb{U}},1)$. The zeroes on both sides come from vanishing cohomology groups. Comparison with the short exact sequence 6.5.ii) shows that the regulator is an isomorphism. ☐

Proposition 8.7. *There are short exact sequences*

$$0 \to H^1_{\mathcal{M}}(B,n) \xrightarrow{\delta_n} H^{n+1}_{\mathcal{M}}(\mathbb{G}^{\vee n}_{m,\mathbb{U}},n)^{sgn} \xrightarrow{\mathrm{res}} \mathbb{Q} \to 0 .$$

The Beilinson regulator is injective on all $H^{n+1}_{\mathcal{M}}(\mathbb{G}^{\vee n}_{m,\mathbb{U}},n)^{sgn}$. It is even an isomorphism for $n>1$.

Proof. The $n=1$ and $n=2$ cases are the previous lemmas. We argue by induction for $n \geq 2$.

By induction, one checks that all $H^n_{\mathcal{M}}(\mathbb{G}^{\vee n}_{m,\mathbb{U}},n)^{\mathrm{sgn}}$ vanish for $n \geq 1$. Hence the residue sequence reads

$$0 \to H^1_{\mathcal{M}}(B,n) \xrightarrow{\delta_n} H^{n+1}_{\mathcal{M}}(\mathbb{G}^{\vee n}_{m,\mathbb{U}},n)^{\mathrm{sgn}} \to H^n_{\mathcal{M}}(\mathbb{G}^{\vee n-1}_{m,\mathbb{U}},n-1)^{\mathrm{sgn}} \to H^2_{\mathcal{M}}(\mathbb{U},n) .$$

By the five lemma and inductive hypothesis we see that the regulator is an isomorphism on the middle term for n. We need the previous lemma to get started.

Now consider the sequences of the proposition. All maps are well-defined. It follows from 6.5.ii) that the sequence is exact. $\square$

Remark: We relied heavily on the Beilinson regulator. Part of the argument actually does not need it but could be done directly using geometric properties and knowledge of the ranks of the K-groups.

Corollary 8.8. *There are canonical splittings* $s_n : \mathbb{Q} \longrightarrow H^{n+1}_{\mathcal{M}}(\mathbb{G}^{\vee n}_{m,\mathbb{U}}, n)^{sgn}$ *such that the diagram*

$$\begin{tikzcd}
H^0_{\mathcal{M}}(B,0) \arrow[r, "s_n"] \arrow[dr, "s_{n-1}"'] & H^{n+1}_{\mathcal{M}}(\mathbb{G}^n_{m,\mathbb{U}} \textit{ rel } Z^{(n)}_{\cdot}, n)^{sgn} \arrow[d, "\mathrm{res}"] \\
 & H^n_{\mathcal{M}}(\mathbb{G}^{n-1}_{m,\mathbb{U}} \textit{ rel } Z^{(n)}_{\cdot}, n-1)^{sgn}
\end{tikzcd}$$

commutes. They are compatible with the ones in 6.5.iii). Furthermore, the group $\varprojlim H^{n+1}_{\mathcal{M}}(\mathbb{G}^n_{m,\mathbb{U}} \textit{ rel } Z^{(n)}_{\cdot}, n)^{sgn}$ *is canonically isomorphic to* $\mathbb{Q}$.

Proof. $\mathrm{Im}(res_n)$ is isomorphic to $\mathbb{Q}$ by the total residue on $H^{n+1}_{\mathcal{M}}(\mathbb{G}^{\vee n}_{m,\mathbb{U}}, n)^{\mathrm{sgn}}$. This induces the same splitting as in 6.5. $\square$

Definition 8.9. *For* $n \in \mathbb{N}$ *the system* $pol_n = s_n(1)$ *defines the universal motivic polylogarithm.*

By construction pol_n is mapped to the polylogarithmic system in absolute Hodge cohomology and continuous étale cohomology.

9 The Cyclotomic Case

Let $d \geq 2$. As before let $R = A[1/d, T]/\Phi_d(T)$ the ring of d-integers of the cyclotomic field of d-th roots of unity. Put $C = \operatorname{Spec} R$. Let ζ be a primitive d-th root of unity in $\overline{\mathbb{Q}}$, and b an integer prime to d. We work in the situation $S = C$, $\alpha = 1 \in \mathbb{G}_m(C)$, and $\beta = i_b \in \mathbb{G}_m(C)$ as in section 5.

Lemma 9.1. **a)** *For $n \geq 0$ we have*

$$H^n_{\mathcal{M}}(\mathbb{G}^n_{m,C} \textit{ rel } Z^{(n)}_{\cdot}, n)^{sgn} = \mathbb{Q} \quad .$$

The Beilinson and the l-adic regulators are isomorphisms.

b) *For $n \geq 1$, the residue sequence induces short exact sequences*

$$\begin{aligned} 0 \longrightarrow H^1_{\mathcal{M}}(C, n) \longrightarrow H^{n+1}_{\mathcal{M}}(\mathbb{G}^n_{m,C} \textit{ rel } Z^{(n)}_{\cdot}, n)^{sgn} \\ \longrightarrow H^n_{\mathcal{M}}(\mathbb{G}^{n-1}_{m,C} \textit{ rel } Z^{(n-1)}_{\cdot}, n-1)^{sgn} \longrightarrow 0 \quad . \end{aligned}$$

The l-adic regulator is injective on the group $H^{n+1}_{\mathcal{M}}(\mathbb{G}^n_{m,C} \textit{ rel } Z^{(n)}_{\cdot}, n)^{sgn}$ for $n \geq 1$.

Proof. For $n = 0$ we have $H^0_{\mathcal{M}}(\mathbb{G}^0_{m,C} \text{ rel } Z^{(0)}_{\cdot}, 0) = H^0_{\mathcal{M}}(C, 0)$, which is canonically isomorphic to $\mathbb{Q}$ by B.2.18. In particular both regulator are isomorphisms.
$H^1_{\mathcal{M}}(\mathbb{G}^0_{m,C} \text{ rel } Z^{(0)}_{\cdot}, 0)$ and its counterpart in absolute cohomology vanish. Consider the following bit of the residue sequence for $n \geq 1$:

$$0 \longrightarrow H^{n+1}_{\mathcal{M}}(\mathbb{G}^{n+1}_{m,C} \text{rel}\, Z^{(n+1)}_{\cdot}, n{+}1)^{\text{sgn}} \longrightarrow H^n_{\mathcal{M}}(\mathbb{G}^n_{m,C} \text{rel}\, Z^{(n)}_{\cdot}, n)^{\text{sgn}} \longrightarrow H^1_{\mathcal{M}}(C, n) \; .$$

The 0 on the left is induced by $H^0_{\mathcal{M}}(C, n) = 0$. The l-adic regulator is always injective on the last term by B.4.8. By inductive hypothesis it is an isomorphism on the middle term. By Cor. 5.3 the last map vanishes in absolute cohomology. This implies a) for $n + 1$. In the next bit of the long exact sequence

$$\begin{aligned} H^1_{\mathcal{M}}(C, n) \longrightarrow H^{n+1}_{\mathcal{M}}(\mathbb{G}^n_{m,C} \text{ rel } Z^{(n)}_{\cdot}, n)^{\text{sgn}} \longrightarrow \\ H^n_{\mathcal{M}}(\mathbb{G}^{n-1}_{m,C} \text{ rel } Z^{(n-1)}_{\cdot}, n-1)^{\text{sgn}} \longrightarrow H^2_{\mathcal{M}}(C, n) \quad . \end{aligned}$$

the first map is injective by a) and we have $H^2_{\mathcal{M}}(C, n) = 0$. Hence this is precisely the short exact sequence in b) . The regulator maps it to the short exact sequence 5.3. By induction and B.4.8 we can control the injectivity of the l–adic regulator. Note that $H^1_{\mathcal{M}}(\mathbb{G}^0_{m,C} \text{ rel } Z^{(0)}_{\cdot}, 0) = H^1_{\mathcal{M}}(C, 0)$. $\square$

Remark: The Beilinson regulator is not injective on $H^1_{\mathcal{M}}(C, 1)$ because d is inverted in C.

Consider the morphism $\phi : \mathbb{G}_{m,C} \longrightarrow \mathbb{G}_{m,C}$ that raises points to the $d+1$-th power. As in section 5 it induces a morphism of spaces $\phi^n : \mathbb{A}^{\vee n}_C \longrightarrow \mathbb{A}^{\vee n}_C$. By contravariance it induces an operation on motivic cohomology.

Lemma 9.2. *$(\phi^n)^*$ operates on the short exact sequence of the previous lemma by*

$$\begin{tikzcd}
0 \arrow[r] & H^1_{\mathcal{M}}(C,n) \arrow[r] \arrow[d, "id"'] & H^{n+1}_{\mathcal{M}}(\mathbb{G}_{m,c}^{\vee n}, n)^{sgn} \arrow[r] \arrow[d, "(\phi^n)^*"'] & H^n_{\mathcal{M}}(\mathbb{G}_{m,C}^{\vee n-1}, n-1)^{sgn} \arrow[r] \arrow[d, "(d+1)(\phi^{n-1})^*"] & 0 \\
0 \arrow[r] & H^1_{\mathcal{M}}(C,n) \arrow[r] & H^{n+1}_{\mathcal{M}}(\mathbb{G}_{m,C}^{\vee n}, n)^{sgn} \arrow[r] & H^n_{\mathcal{M}}(\mathbb{G}_{m,C}^{\vee n-1}, n-1)^{sgn} \arrow[r] & 0
\end{tikzcd}.$$

Proof. This description follows immediately from the injectivity of the l-adic regulator and Cor. 5.3.b). □

Remark: The operation $(\phi^n)^*$ on $H^1_{\mathcal{M}}(C,n)$ is given by the operation on $H^{n+1}_{\mathcal{M}}(\mathbb{A}_C^{\vee n}, n)$. It is easy to check that it is trivial by considering the operation on the starting terms of the degenerating skeletal spectral sequence. To understand the compatibility with the residue map in terms of K-theory is much harder. The factor $d+1$ is induced by a push-forward from a non-reduced scheme to its reduction. The theory in Appendix B is not even set up to handle such schemes.

As in the case of absolute cohomology it follows that the eigenvalues of $(\phi^n)^*$ on $H^{n+1}_{\mathcal{M}}(\mathbb{G}_{m,C} \text{ rel } Z^{(n)}, n)^{\text{sgn}}$ are $1, d+1, \ldots, (d+1)^n$.

Lemma 9.3. *The eigenspace decomposition yields a splitting*

$$\eta_b^{(n)} : H^{n+1}_{\mathcal{M}}(\mathbb{G}^n_{m,C} \textit{ rel } Z^{(n)}, n)^{sgn} \xrightarrow{\sim} \bigoplus_{0 \le i \le n} H^1_{\mathcal{M}}(C,i) \ ,$$

which is compatible with the splitting $\eta_b^{(n)}$ after Cor. 5.3. There is a canonical isomorphism

$$\eta_b : \varprojlim H^{n+1}_{\mathcal{M}}(\mathbb{G}^n_{m,C} \textit{ rel } Z^{(n)}, n)^{sgn} \xrightarrow{\sim} \prod_{i \ge 1} H^1_{\mathcal{M}}(C,i) \ .$$

Proof. The first assertion is clear by construction. The second follows because the eigenspace decomposition is compatible with the residue map. □

Definition 9.4. *Let $i_b : C \longrightarrow \mathbb{U}$ be as before. Let pol_b be the pullback of the universal polylogarithm system pol defined in 8.9 to the inverse limit $\varprojlim H^{n+1}_{\mathcal{M}}(\mathbb{G}^{(n)}_{m,C} \textit{ rel } Z^{(n)}_{\cdot}, n)^{sgn}$. Via the isomorphism η_b of the previous lemma, we have constructed an element in $\prod_{i \ge 1} H^1_{\mathcal{M}}(C,i)$.*

Theorem 9.5. *Under the regulators, the element*

$$pol_b \in \varprojlim H^{n+1}_{\mathcal{M}}(\mathbb{G}^n_{m,C} \text{ rel } Z^{(n)}_{\cdot}, n)^{sgn} = \prod_{i \geq 1} H^1_{\mathcal{M}}(C, i)$$

is mapped to the elements

$$pol_b \in \varprojlim H^{n+1}_{\text{abs}}(\mathbb{G}^n_{m,C} \text{ rel } Z^{(n)}_{\cdot}, n)^{sgn} = \prod_{i \geq 1} H^1_{\text{abs}}(C, i)$$

constructed at the end of section 5.

Proof. This follows from the construction. □

We list the consequences of this result: denote by μ^0_d the set of primitive d–th roots of unity in $\mathbb{Q}(\mu_d)$.

Firstly, the description of the regulator to absolute Hodge cohomology yields an alternative proof of the following:

Corollary 9.6. *Assume $n \geq 0$.*

a) ([B2], 7.1.5, [Neu], II.1.1, [E], 3.9.)
There is a map of sets

$$\begin{aligned} \epsilon_{n+1} : \mu^0_d &\longrightarrow H^1_{\mathcal{M}}(C, n+1) \\ &\qquad (= H^1_{\mathcal{M}}(\operatorname{Spec} \mathbb{Q}(\mu_d), n+1) \textit{ for } n \geq 1) \end{aligned}$$

such that

$$\begin{aligned} r_{\mathcal{D}} \circ \epsilon_{n+1} : \mu^0_d &\longrightarrow H^1_{\mathfrak{H}^p}(\operatorname{Spec} \mathbb{Q}(\mu_d)_{\mathbb{R}}/\mathbb{R}, n+1) \\ &\xrightarrow[A.2.12]{\sim} \left(\bigoplus_{\sigma : \mathbb{Q}(\mu_d) \hookrightarrow \mathbb{C}} \mathbb{C}/(2\pi i)^{n+1}\mathbb{R} \right)^+ \end{aligned}$$

maps a root of unity ω to $(-Li_{n+1}(\sigma\omega))_\sigma$. For $n \geq 1$, this property characterizes the map ϵ_{n+1} uniquely.

b) For a root of unity $T^b \in \mathbb{Q}(\mu_d) = \mathbb{Q}[T]/\Phi_d(T)$, the element

$$\epsilon_{n+1}(T^b) \in H^1_{\mathcal{M}}(C, n+1)$$

is given by

$$\epsilon_{n+1}(T^b) := (-1)^n \cdot \frac{1}{(n+1)!} \cdot \left(\, (n+1)\textit{–component of } pol_b \right) \, .$$

Proof. Note that a) really *is* Beilinson's formulation of the result: his normalization of the isomorphism

$$H^1_{\mathfrak{H}^p}(\operatorname{Spec}\mathbb{Q}(\mu_d)_{\mathbb{R}}/\mathbb{R}, n+1) \xrightarrow{\sim} \left(\bigoplus_{\sigma} \mathbb{C}/(2\pi i)^{n+1}\mathbb{R}\right)^+$$

differs from ours by the factor -1.

The unicity assertion is a direct consequence of the injectivity of the regulator. So our claim follows from 2.5, and from 5.4. $\square$

Recall that the l–adic regulator r_l factorizes as follows:

$$\begin{aligned} K_{2n+1}(C) \otimes_{\mathbb{Z}} \mathbb{Q} = H^1_{\mathcal{M}}(C, n+1) &\hookrightarrow H_{\mathcal{M}}(C_{(l)}, n+1) \\ &\hookrightarrow H^1_{cont}(C_{(l)}, n+1) \\ &\hookrightarrow H^1_{cont}(\operatorname{Spec}\mathbb{Q}(\mu_d), n+1)\ , \end{aligned}$$

where we let $C_{(l)} := C \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{l}]$.

For the rest of this section, we fix $\zeta \in C(\overline{\mathbb{Q}})$.

Corollary 9.7 ([BlK], Conjecture 6.2). *Let d, n and ϵ_{n+1} be as in 9.6. Let l be a prime. Under the embedding of 2.6, the l–adic regulator*

$$r_l : H^1_{\mathcal{M}}(C, n+1) \longrightarrow H^1_{cont}(\operatorname{Spec}\mathbb{Q}(\mu_d), n+1)$$

maps $\epsilon_{n+1}(T^b)$ to

$$\frac{1}{d^n} \cdot \frac{1}{n!} \cdot \left(\sum_{\alpha^{l^r} = \zeta^b} [1-\alpha] \otimes (\alpha^d)^{\otimes n} \right)_r \ .$$

Proof. This is 2.6 and 5.4. $\square$

In order to get the comparison statement of [BlK], Conjecture 6.2 for the root of unity 1 as well, observe the relations

$$\begin{aligned} c_{n+1}(1) &= \frac{2^n}{1-2^n} c_{n+1}(-1)\ , \\ c_{n+1,2}(1) &= \frac{2^n}{1-2^n} c_{n+1,2}(-1) \end{aligned}$$

in the notation of loc. cit., if $n \geq 1$ ([D5], Proposition 3.13.1.i)).

Recall ([Sou2], Lemma 1, [Sou5], 4.1–4.3) Soulé's construction of maps

$$\varphi_l : \mu_d^0 \to K_{2n+1}(C_{(l)}) \otimes_{\mathbb{Z}} \mathbb{Z}_l$$

for any odd prime l: we have

$$K_{2n+1}(C_{(l)}) \otimes_{\mathbb{Z}} \mathbb{Z}_l = K_{2n+1}(C_{(l)}, \mathbb{Z}_l) = \varprojlim K_{2n+1}(C_{(l)}, \mathbb{Z}/l^r\mathbb{Z}) \ ,$$

and using the formalism of norm compatible units developed in [Sou2], one lets $\varphi_l(\omega)$ denote the projective system

$$\Big(N_r((1-\alpha_r) \cup (\alpha_r^d)^{\cup n})\Big)_r \in \varprojlim K_{2n+1}(C_{(l)}, \mathbb{Z}/l^r\mathbb{Z})$$

for some choice $(\alpha_r)_{r \geq 1} \in \varprojlim \mu_{dl^r}$ satisfying $\alpha_1^l = \omega$.

By construction, the l–adic regulator

$$r_l : K_{2n+1}(C_{(l)}) \otimes_{\mathbb{Z}} \mathbb{Q}_l \to H^1_{cont}(\operatorname{Spec} \mathbb{Q}(\mu_d), n+1)$$

takes $\varphi_l(T^b)$ to the cyclotomic element in continuous Galois cohomology

$$\left(\sum_{\alpha^{l^r} = \zeta^b} [1-\alpha] \otimes (\alpha^d)^{\otimes n} \right)_r$$

defined by Soulé and Deligne (cf. [Sou2], page 384, [D5], 3.1, 3.3).

Corollary 9.8. *For each d and n, there is a unique map*

$$\varphi : \mu_d^0 \to K_{2n+1}(\operatorname{Spec} \mathbb{Q}(\mu_d)) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{2}\right]$$

such that for each odd l, the map

$$\begin{aligned} \varphi_l : \mu_d^0 &\to K_{2n+1}(C_{(l)}) \otimes_{\mathbb{Z}} \mathbb{Z}_l \\ &\hookrightarrow K_{2n+1}(\operatorname{Spec} \mathbb{Q}(\mu_d)) \otimes_{\mathbb{Z}} \mathbb{Z}_l \end{aligned}$$

equals the composition of φ and the natural map

$$K_{2n+1}(\operatorname{Spec} \mathbb{Q}(\mu_d)) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{2}\right] \to K_{2n+1}(\operatorname{Spec} \mathbb{Q}(\mu_d)) \otimes_{\mathbb{Z}} \mathbb{Z}_l \ .$$

Furthermore, the map $\varphi \otimes_{\mathbb{Z}} \mathbb{Q}$ agrees with

$$\epsilon'_{n+1} : \mu_d^0 \to H^1_{\mathcal{M}}(\operatorname{Spec} \mathbb{Q}(\mu_d), n+1)$$

given by $d^n \cdot n! \cdot \epsilon_{n+1}$.

Proof. The uniqueness assertion is a formal consequence of the finite generation of $K_{2n+1}(\operatorname{Spec}\mathbb{Q}(\mu_d))$: to give an element in a finitely generated $\mathbb{Z}[\frac{1}{N}]$–module M is the same as giving elements in $M \otimes_{\mathbb{Z}} \mathbb{Q}$ and all $M \otimes_{\mathbb{Z}} \mathbb{Z}_l$, for l not dividing N, which coincide in $M \otimes_{\mathbb{Z}} \mathbb{Q}_l$. By 9.7, the maps $r_l \circ \varphi_l$ and $r_l \circ \epsilon'_{n+1}$ agree for all $l \neq 2$. From Theorem B.4.8, we conclude that φ_l and ϵ'_{n+1} agree as maps to $K_{2n+1} \otimes_{\mathbb{Z}} \mathbb{Q}_l$. ☐

Remark: It would obviously be desirable to construct a map φ_l also for $l = 2$, and prove a version of 9.8 for $K_{2n+1}(\operatorname{Spec}\mathbb{Q}(\mu_d))$ itself. For the construction (see [Sou5], 4.1–4.3), one needs a cup product on K–theory with $\mathbb{Z}/2^r\mathbb{Z}$–coefficients. Such a product exists, at least for $r \geq 2$ ([Brw], Proposition 1.4). It is induced by the homotopy class of a continuous map $Y^{m+n} \longrightarrow Y^m \bigwedge Y^n$, for $Y^k := S^{k-1} \cup_{2^r} e^k$, which unlike in the case of odd primes seems not known to be uniquely characterizable. To us, this point indicated that it might constitute a problem to define a cup product in a way which is well behaved under reduction of coefficients.

Finally, as observed by Bloch and Kato, Corollary 9.7 implies the validity of the following also for even n:

Corollary 9.9. *Let* $n \geq 1$.
Then the Tamagawa number conjecture ([BlK], Conjecture 5.15) is true modulo a power of 2 *for the motif* $\mathbb{Q}(n+1)$.

Proof. [BlK], Theorem 6.1.i) gives the complete proof for odd n, which is independent of anything said in the present article. In loc.cit., Theorem 6.1.ii), it is shown that the conjecture holds for even n if [BlK], 6.2 holds. But the latter is the content of 9.7. ☐