A Absolute Hodge Cohomology with Coefficients

The aim of this appendix is to provide a natural interpretation of absolute Hodge cohomology as extension groups in the category of algebraic Hodge modules over $\mathbb{R}$ (A.2.7). We use the opportunity to reconsider the definition of absolute Hodge cohomology. In doing so, we follow the spirit of [B1], 0.3.

The first subsection therefore starts with a brief summary of the results of the theory of algebraic Hodge modules ([S1], [S2]) needed in the sequel. We prove the comparison results A.1.3, A.1.6 and A.1.7 needed for the identification of the absolute Hodge cohomology groups as defined in [B1] with the Ext groups for Tate twists in the category of Hodge modules (A.1.10).

In the second subsection, we give the definition of algebraic Hodge modules over $\mathbb{R}$ (A.2.4), and of absolute Hodge cohomology with coefficients in such (A.2.6). Again, we show that our definition coincides with the one given in [B1], § 7 for Tate coefficients (A.2.7).

A.1 Algebraic Mixed Hodge Modules

In [S2], § 4, the category $\mathrm{MHM}_A(X)$ of *algebraic mixed A–Hodge modules* is defined, where A is a field contained in $\mathbb{R}$, and X a separated reduced scheme of finite type over $\mathbb{C}$.

Saito's construction admits the full formalism of Grothendieck's functors $\pi_!$, $\pi^!$, π^*, π_*, $\underline{\mathrm{Hom}}$, $\otimes$, $\mathbb{D}$ on the level of bounded derived categories $D^b\,\mathrm{MHM}_A$ ([S2], 4.3, 4.4) and a forgetful functor

$$\mathrm{rat} : \mathrm{MHM}_A(X) \longrightarrow \mathrm{Perv}_A(\overline{X})$$

to the category of perverse sheaves on the topological space $\overline{X}$ underlying $X(\mathbb{C})$, which have algebraic stratifications such that the restrictions of their cohomology sheaves to the strata are local systems. By the definition of MHM_A, which we shall partly sketch in a moment, rat is faithful and exact. The functor rat on the level of derived categories is compatible with Grothendieck's functors ([S2], 4.3, 4.4).

For smooth X, one constructs $\mathrm{MHM}_A(X)$ as an abelian subcategory ([S1], Proposition 5.1.14) of the category $\mathrm{MF}_h\mathrm{W}(\mathcal{D}_X, A)$, whose objects are

$$((M, F^{\cdot}, W_{\cdot}), (K, W_{\cdot}), \alpha) \ ,$$

where $(M, F^{\cdot})$ is an object of the category $\mathrm{MF}_h(\mathcal{D}_X)$, i.e., a regular holonomic algebraic $\mathcal{D}_X$–module M together with a good filtration $F^{\cdot}$, and $K \in \mathrm{Perv}_A(\overline{X})$. $W.$ is a locally finite ascending filtration, and α is an isomorphism

$$DR(M) \xrightarrow{\sim} K \otimes_A \mathbb{C}$$

respecting $W.$. Here, DR denotes the de Rham functor from the category of $\mathcal{D}_X$–modules to the category of perverse sheaves.

We note that by definition, the weight graded objects of all algebraic Hodge modules satisfy a certain polarizability condition (see [S1], 5.2.10).

Call an algebraic Hodge module on a smooth variety *smooth* if the underlying perverse sheaf is a local system up to a shift.

Theorem A.1.1 (Saito). *Let X be smooth and separated. Then there is an equivalence*

$$\mathrm{Var}_A(X) \xrightarrow{\sim} \mathrm{MHM}_A(X)^s$$

between the category of admissible variations of mixed A–Hodge structure ([Ks]) and the category of smooth algebraic A–Hodge modules on X.

Proof. This is the remark following [S2], Theorem 3.27. ∎

In particular, we see that $\mathrm{MHM}_A(\mathrm{Spec}(\mathbb{C}))$ is the category MHS_A of polarizable mixed A–Hodge structures.

If $\mathbb{V}$ is a variation on X with underlying local system $\mathrm{For}(\mathbb{V})$, then the perverse sheaf underlying the Hodge module $\mathbb{V}$ under the correspondence of A.1.1 is

$$\mathrm{For}(\mathbb{V})[d]$$

if X is of pure dimension d.

It turns out that the definition of Tate twists in $\mathrm{MHM}_A(X)$ is compatible with the above equivalence only up to shift:

Definition A.1.2 ([S2], (4.5.5)). *Let $n \in \mathbb{Z}$, and $A(n) \in \mathrm{MHS}_A$ the usual Tate twist. For a separated reduced scheme $a : X \to \mathrm{Spec}(\mathbb{C})$, define*

$$A(n)_X := a^* A(n) \in D^b \, \mathrm{MHM}_A(X) \, .$$

If X is smooth and of pure dimension d, then $A(n)_X[d]$ is the variation of Hodge structure, which one denotes $A(n)$.

For arbitrary X, the complex $A(n)_X$ will not even be the shift of a Hodge module, but a proper element of $D^b\,\mathrm{MHM}_A(X)$, whose cohomology objects $\mathcal{H}^p A(n)_X$ are a priori trivial only for $p > \dim X$ ([S2], (4.5.6)).

We note again that we follow Saito's convention and write e.g. π_* for the functor on derived categories

$$D^b\,\mathrm{MHM}_A(X) \longrightarrow D^b\,\mathrm{MHM}_A(Y)$$

induced by a morphism $\pi : X \to Y$.

In order to compare the Hodge structures on Betti cohomology given by Saito's and Deligne's constructions, we need to go into the details of [S2]:

Theorem A.1.3 (Saito). *Let $j : U \hookrightarrow X$ be an open immersion of smooth separated schemes over $\mathbb{C}$, with $Y := X \setminus U$ a divisor with normal crossings. If X is of pure dimension d, then*

$$j_*A(0)_U[d] = \mathcal{H}^d j_*A(0)_U \in \mathrm{MHM}_A(X) \subset \mathrm{MF}_h\mathrm{W}(\mathcal{D}_X, A)$$

equals the object

$$(w_X(*Y), (j_{\mathrm{top}})_*A_U[d], \alpha)\ ,$$

where $w_X(*Y)$ denotes the $\mathcal{D}_X$–module $\Omega^d_X(\log Y)$, and $(j_{\mathrm{top}})_*$ the direct image for the derived category of perverse sheaves.

The de Rham complex with logarithmic singularities is quasi-isomorphic to $w_X(*Y) \overset{L}{\otimes}_{\mathcal{D}_X} \mathcal{O}_X[-d] = DR(w_X(*Y))[-d]$, hence

$$DR(w_X(*Y)) = \Omega^{\cdot}_X(\log Y)[d]$$

(compare [Bo3], VIII, 13.1), and

$$\alpha : \Omega^{\cdot}_X(\log Y)[d] \overset{\sim}{\longrightarrow} (j_{\mathrm{top}})_*\mathbb{C}\,[d]$$

is the usual quasi-isomorphism

$$\Omega^{\cdot}_X(\log Y) \overset{\sim}{\longrightarrow} (j_{\mathrm{top}})_*\Omega^{\cdot}_U \overset{\sim}{\longleftarrow} (j_{\mathrm{top}})_*\mathbb{C}$$

(compare [D2], 3.1), shifted by d.

The Hodge filtration $F^{\cdot}$ on $w_X(*Y)$ is induced from the stupid filtration, while the weight filtrations $W_{\cdot}$ on $w_X(*Y)$ and $(j_{\mathrm{top}})_* \mathbb{C}\,[d]$ are those induced from the canonical filtration on $(j_{\mathrm{top}})_* \Omega_U$, shifted by d.

Proof. The equation $j_* A(0)_U[d] = \mathcal{H}^d j_* A(0)_U$ follows from the faithfulness of rat and the fact that the corresponding statement for $(j_{\mathrm{top}})_*$ is true since j is affine.

In our geometric situation, the explicit construction of j_* of *any* admissible variation of A–Hodge structure is carried out in the proof of [S2], Theorem 3.27. For $A(0)_U$, it specializes to our claim. □

In [B1], 3.9, Beilinson extends Deligne's notion of Hodge complexes ([D3], 8.1) to the polarizable situation:

Definition A.1.4 (Beilinson). *A mixed A–Hodge complex*

$$K = ((K_{\mathbb{C}}, F^{\cdot}, W_{\cdot}), (K, W_{\cdot}), \alpha)$$

is called polarizable if the cohomology objects of the weight n Hodge complexes $\mathrm{Gr}_n^W(K)$ are polarizable A–Hodge structures.

Remark: The weight filtration $W_{\cdot}$ of a mixed Hodge complex K induces mixed Hodge structures on its cohomology. Observe however that $\mathrm{Gr}_n^W(H^i K)$ is of weight $n+i$.

As in the non–polarizable situation, Beilinson proves:

Theorem A.1.5 ([B1], Lemma 3.11). *There is an equivalence of categories between $D^b\,\mathrm{MHS}_A$ and the derived category of polarizable A–Hodge complexes.*

Let X be smooth and separated over $\mathbb{C}$. Forgetting part of the structure of a Hodge module yields a functor

$$\mathrm{For} : C^b\,\mathrm{MHM}_A(X) \longrightarrow T(X)\ .$$

Here, $T(X)$ is the category of triples

$$M^{\cdot} = ((M^{\cdot}, F^{\cdot\cdot}, W^{\cdot}_{\cdot}), (K^{\cdot}, W^{\cdot}_{\cdot}), \alpha^{\cdot})\ ,$$

where $(M^{\cdot}, F^{\cdot\cdot}, W^{\cdot}_{\cdot})$ is a class in the filtered derived category $D^b W(\mathrm{MF}_h(\mathcal{D}_X))$ of $\mathrm{MF}_h(\mathcal{D}_X)$, and $(K^{\cdot}, W^{\cdot}_{\cdot})$ a class in the filtered derived category of sheaves of A–vector spaces on $X(\mathbb{C})$, denoted by $D^bW(X(\mathbb{C}),A)$. Furthermore, the map $\alpha^\cdot$ is an isomorphism

$$DR(M^\cdot) \xrightarrow{\sim} K^\cdot \otimes_A \mathbb{C}$$

respecting $W^\cdot_\cdot$.

Recall that in order to obtain a class in $D^bW(X(\mathbb{C}),A)$ from a complex of perverse sheaves, one applies the realization functor of [BBD], 3.1.9.

The global section functor Γ can be derived on $D^bW(X(\mathbb{C}),A)$. By [S1], 2.3, we have a functor $R\Gamma$ on $D^bW(\mathrm{MF}_h(\mathcal{D}_X))$ if X is proper, and the two constructions are compatible with the comparison isomorphism $\alpha^\cdot$ of any object in $T(X)$ ([S1], 2.3.7). We end up with an object

$$R\Gamma M^\cdot = (R\Gamma(M^\cdot, F^{\cdot\cdot}, W^\cdot_\cdot), R\Gamma(K^\cdot, W^\cdot_\cdot), R\Gamma\alpha^\cdot)$$

of $T(\mathrm{Spec}(\mathbb{C}))$. The functor

$$\underline{R\Gamma} := R\Gamma \circ \mathrm{For} : C^b\,\mathrm{MHM}_A(X) \longrightarrow T(\mathrm{Spec}(\mathbb{C}))$$

factorizes through $D^b\,\mathrm{MHM}_A(X)$.

Our second comparison result is the following:

Theorem A.1.6. *Let $a : X \to \mathrm{Spec}(\mathbb{C})$ be smooth and proper, and $M^\cdot$ an object of $D^b\,\mathrm{MHM}_A(X)$. Write*

$$\mathrm{For}\, M^\cdot = ((M^\cdot, F^{\cdot\cdot}, W^\cdot_\cdot), (K^\cdot, W^\cdot_\cdot), \alpha^\cdot) \in T(X)\ .$$

a)

$$\underline{R\Gamma} M^\cdot = (R\Gamma(M^\cdot, F^{\cdot\cdot}, W^\cdot_\cdot), R\Gamma(K^\cdot, W^\cdot_\cdot), R\Gamma\alpha^\cdot)$$

is a mixed polarizable A–Hodge complex.

b) The class of $\underline{R\Gamma} M^\cdot$ in the derived category of polarizable Hodge complexes is canonically isomorphic, under the identification of A.1.5, to

$$a_* M^\cdot \in D^b\,\mathrm{MHS}_A\ .$$

c) Let $f : Y \to X$ be a (proper) morphism of smooth and proper schemes over $\mathbb{C}$, and let b denote the structure morphism of Y, such that

$$b = a \circ f\ .$$

For any $N^\cdot \in D^b \operatorname{MHM}_A(Y)$ *together with a morphism* $\eta : M^\cdot \to f_* N^\cdot$ *in* $D^b \operatorname{MHM}_A(X)$, *the morphism*

$$a_*\eta : a_* M^\cdot = \underline{R\Gamma} M^\cdot \longrightarrow \underline{R\Gamma} N^\cdot = b_* N^\cdot = a_* f_* N^\cdot$$

equals, under the isomorphism of a), the morphism

$$(R\Gamma\eta, R\Gamma\eta, R\Gamma\eta)$$

of A–Hodge complexes.

Proof. a) We may assume that $M^\cdot$ is pure of some weight. Using [S2], (4.5.4), we are reduced to the case where $M^\cdot = M$ is a Hodge module of weight n, and we have to show that $\underline{R\Gamma} M$ is a polarizable Hodge complex of the same weight.

Axiom (CH 1) of [D3], (8.1.1) follows from [S2], Proposition 2.16, in particular (2.16.5), applied to $\operatorname{pr}^* M$, where

$$\operatorname{pr} : X \times_{\mathbb{C}} \mathbb{A}^1_{\mathbb{C}} \longrightarrow X \ .$$

Furthermore, by the remark following [S2], (4.2.9), and by loc. cit., 2.15, we have isomorphisms in $\operatorname{MF}_h\operatorname{W}(\mathcal{D}_{\operatorname{Spec}(\mathbb{C})}, A)$

$$\underline{R^i\Gamma} M := \big(R^i\Gamma(M, F^\cdot, W_\cdot[i]), R^i\Gamma(K, W_\cdot[i]), R^i\Gamma\alpha\big) \xrightarrow{\sim} \mathcal{H}^i a_* M \ .$$

Since the right hand side is a polarizable Hodge structure of weight $i + n$ ([S2], (4.5.2)), we have (CH 2), and in addition, polarizability.

b) In the proof of a), we constructed a functor

$$a_*^\sim := \underline{R\Gamma} : D^b \operatorname{MHM}_A(X) \longrightarrow D^b \operatorname{MHS}_A \ ,$$

such that

$$\mathcal{H}^i a_*^\sim = \mathcal{H}^i a_* : \operatorname{MHM}_A(X) \hookrightarrow D^b \operatorname{MHM}_A(X) \longrightarrow \operatorname{MHS}_A$$

for all i. Composition with $j_* : D^b \operatorname{MHM}_A(U) \to D^b \operatorname{MHM}_A(X)$ for open immersions $j : U \hookrightarrow X$ defines

$$(a \circ j)_*^\sim := a_*^\sim \circ j_* : D^b \operatorname{MHM}_A(U) \longrightarrow D^b \operatorname{MHS}_A \ .$$

But for affine U, $(a \circ j)_*$ is the left derived functor of

$$\mathcal{H}^0 (a \circ j)_* : \operatorname{MHM}_A(U) \longrightarrow \operatorname{MHS}_A$$

([S2], proof of Theorem 4.3.). If U is affine, then so is $j : U \hookrightarrow X$, and hence j_* is exact. Therefore,

$$\mathcal{H}^0(a \circ j)_* = \mathcal{H}^0 a_* \circ j_* : \mathrm{MHM}_A(U) \xrightarrow{j_*} \mathrm{MHM}_A(X) \xrightarrow{\mathcal{H}^0 a_*} \mathrm{MHS}_A$$

coincides with $\mathcal{H}^0(a \circ j)^{\sim}_*$, and we get a natural transformation

$$(a \circ j)_* \longrightarrow (a \circ j)^{\sim}_* \ ,$$

which is an isomorphism, since this is true on the level of cohomology objects, as one checks on the level of vector spaces. Observe that this natural transformation is compatible with restriction to smaller affine subschemes of X.

Now recall ([S2], proof of 4.3) that the functor a_* is constructed using the Čech complex associated to an affine covering of X (for details, see [B3], 3.4). In the same way, the functor $a^{\sim}_*$ is recoverable from the $(a \circ j)^{\sim}_*$. We end up with an isomorphism of a_* and $a^{\sim}_*$, which is independent of the covering.

c) In the proof of b), we constructed a natural isomorphism

$$\kappa : a_* \xrightarrow{\sim}$$

of functors from $D^b \mathrm{MHM}_A(X)$ to $D^b \mathrm{MHS}_A$. For $f = \mathrm{id}$, our claim is therefore proved.

For the general situation, we use the same techniques as in the proof of b) to first construct a natural isomorphism

$$b_* \xrightarrow{\sim} a^{\sim}_* \circ f_*$$

of functors from $D^b \mathrm{MHM}_A(Y)$ to $D^b \mathrm{MHS}_A$, and then to see that the triangle

$$\begin{array}{ccc} b_* & \longrightarrow & a_* \circ f_* \\ & \searrow & \downarrow \kappa \\ & & a^{\sim}_* \circ f_* \end{array}$$

commutes. ☐

Corollary A.1.7 (cf. [S3], (2.8)). *Let $j : U \hookrightarrow X$ be a smooth compactification of a smooth and separated scheme $a : U \to \mathrm{Spec}(\mathbb{C})$, such that $Y := X \setminus U$ is a divisor with normal crossings.*

a) $a_* A(0)_U \in D^b \operatorname{MHS}_A$ *is isomorphic, under the identification of A.1.5, to the class of the mixed polarizable A–Hodge complex*

$$\underline{R\Gamma}(U, A) := \underline{R\Gamma}(DR^{-1}\Omega^{\cdot}_X(\log Y), (j_{\mathrm{top}})_* A_U, \alpha)$$

of [D3], (8.1.12) and [B1], § 4 (with the same notation).

b) *If* $f : X \to X'$ *is a morphism of compactifications* $j : U \hookrightarrow X$ *and* $j' : U \hookrightarrow X'$ *of* U *as in a), then* f *induces an isomorphism*

$$\underline{R\Gamma}(DR^{-1}\Omega^{\cdot}_{X'}(\log Y'), (j'_{\mathrm{top}})_* A_U) \xrightarrow{\sim} \underline{R\Gamma}(DR^{-1}\Omega^{\cdot}_X(\log Y), (j_{\mathrm{top}})_* A_U)$$

([D3], remark preceding (8.1.17)), so $\underline{R\Gamma}(U, A)$ *depends only on* U*.*

The isomorphism in a) also depends only on U*.*

c) *In particular, the Hodge structures on*

$$\operatorname{rat}(\mathcal{H}^n a_* A(n)_U) = H^n_B(U(\mathbb{C}), (2\pi i)^n A)$$

given by Deligne's and Saito's constructions coincide.

Proof. a) Combine A.1.3 and A.1.6.b).
b) Use A.1.6.c).
c) follows from a) and b). ☐

Actually, the statement A.1.6.c) implies the functoriality property we were after: we have two functors

$$(Sm/\mathbb{C})^0 \longrightarrow D^b \operatorname{MHS}_A ,$$

where $(Sm/\mathbb{C})$ denotes the category of smooth separated schemes over $\mathbb{C}$:

$$\underline{R\Gamma}(\_, A) : U \longmapsto \underline{R\Gamma}(U, A) ,$$
$$_*(A) : (a : U \longrightarrow \operatorname{Spec}(\mathbb{C})) \longmapsto a_* \left(A(0)_U\right) .$$

Corollary A.1.8. *The isomorphism of A.1.7.a) is functorial in* $U \in Sm/\mathbb{C}$*. In other words, there is a natural isomorphism*

$$_*(A) \xrightarrow{\sim} \underline{R\Gamma}(\_, A)$$

of functors from $(Sm/\mathbb{C})^0$ *to* $D^b \operatorname{MHS}_A$*.*

Proof. Let

$$\begin{array}{ccc} U' & \overset{j'}{\hookrightarrow} & X' \\ f\big\downarrow & & \big\downarrow f \\ U & \overset{j}{\hookrightarrow} & X \end{array}$$

be a commutative diagram of smooth and separated schemes over $\mathbb{C}$, where X' and X are proper, and $Y' := X' \setminus U'$ and $Y := X \setminus U$ are divisors with normal crossings. We have a morphism

$$(*) \qquad j_* A(0)_U \longrightarrow f_*(j'_* A(0)_{U'}) \ .$$

Application of $(a_X)_*$ gives the morphism

$$(a_U)_* A(0)_U \longrightarrow (a_{U'})_* A(0)_{U'}$$

belonging to the functoriality requirement for $_*(A)$. Our claim follows from A.1.6.c), applied to a shift of the morphism $(*)$. ☐

Definition A.1.9. *Let $X/\mathbb{C}$ be separated, reduced and of finite type, and $M^{\cdot}$ an object of $D^b \operatorname{MHM}_A(X)$.*

a) The absolute Hodge complex of X with coefficients in $M^{\cdot}$ is

$$R\Gamma_{\mathfrak{H}^p}(X, M^{\cdot}) := R \operatorname{Hom}_{D^b \operatorname{MHM}_A(X)}(A(0)_X, M^{\cdot}) \ .$$

b) Its cohomology groups

$$H^i_{\mathfrak{H}^p}(X, M^{\cdot}) := H^i R\Gamma_{\mathfrak{H}^p}(X, M^{\cdot})$$

are called absolute Hodge cohomology groups of X with coefficients in $M^{\cdot}$.

c) We denote absolute Hodge cohomology with coefficients in Tate twists by

$$H^i_{\mathfrak{H}^p}(X, n) := H^i_{\mathfrak{H}^p}(X, A(n)_X) \ .$$

d) For a closed reduced subscheme Z of X with complement $j : U \hookrightarrow X$, we define relative absolute Hodge cohomology with coefficients in Tate twists as

$$H^i_{\mathfrak{H}^p}(X \ rel \ Z, n) := H_{\mathfrak{H}^p}(X, j_! A(n)_U) \ .$$

Note that if X is smooth and of pure dimension d, and if

$$M^{\cdot} = M \in \mathrm{MHM}_A(X) \ ,$$

then the right hand side of A.1.9.b), being equal to

$$\mathrm{Hom}_{D^b \, \mathrm{MHM}_A(X)}(A(0)_X[d], M[d+i]) \, ,$$

admits an interpretation as the group of $(d+i)$–extensions of Hodge modules modulo Yoneda equivalence.

Corollary A.1.10. *If X is smooth and separated over $\mathbb{C}$, and $n \in \mathbb{Z}$, then*

$$R\Gamma_{\mathfrak{H}^p}(X, n) = R\Gamma_{\mathfrak{H}^p}(X, A(n)_X) \quad \textit{and} \quad H^{\cdot}_{\mathfrak{H}^p}(X, n) = H^{\cdot}_{\mathfrak{H}^p}(X, A(n)_X)$$

coincide functorially with the same noted objects of [B1], § 5.

Proof. This follows from A.1.8 and the adjunction formula

$$R\,\mathrm{Hom}_{D^b \, \mathrm{MHM}_A(X)}(A(0)_X, M^{\cdot}) = R\,\mathrm{Hom}_{D^b \, \mathrm{MHS}_A}(A(0), a_* M^{\cdot}) \ .$$

□

Remark: The Leray spectral sequence for $a : X \to \mathrm{Spec}(\mathbb{C})$ yields exact sequences

$$0 \to \mathrm{Ext}^1_{\mathrm{MHS}_A}\left(A(0), H^{i-1}\right) \to H^i_{\mathfrak{H}^p}(X, A(n)_X) \to \mathrm{Hom}_{\mathrm{MHS}_A}\left(A(0), H^i\right) \to 0$$

(with $H^k := H^k_B(X(\mathbb{C}), (2\pi i)^n A)$) since MHS_A has cohomological dimension one ([B1], Corollary 1.10). Comparing them with the analogous sequences for $H^i_{\mathfrak{H}}$, we see that

$$H^i_{\mathfrak{H}^p}(X, A(n)_X) = H^i_{\mathfrak{H}}(X, A(n)_X)$$

(in the notation of [B1], § 5) if $H^{i-1}_B(X(\mathbb{C}), (2\pi i)^n A)$ has weights smaller than zero, which is the case if $i \le n$ ($i \le 2n$ if X is proper).

Observe that this is the same range of indices where Deligne cohomology coincides with $H^i_{\mathfrak{H}}(X, \mathbb{R}(n)_X)$ ([N], (7.1)): we have natural morphisms

$$H^i_{\mathfrak{H}^p}(X, \mathbb{R}(n)_X) \longrightarrow H^i_{\mathfrak{H}}(X, \mathbb{R}(n)_X) \longrightarrow H^i_{\mathcal{D}}(X, \mathbb{R}(n)_X) \, ,$$

both of which are isomorphisms if $i \le n$ ($i \le 2n$ if X is proper).

A.2 Algebraic Mixed Hodge Modules over $\mathbb{R}$

Algebraic Hodge modules over $\mathbb{R}$ are defined as the category of Hodge modules fixed under a certain involution given by complex conjugation. We start by constructing this involution:

Let $X/\mathbb{C}$ be smooth, and let ${}^{\iota}X$ denote the complex conjugate scheme. We have an equivalence

$$\iota^* : \mathrm{Var}_A({}^{\iota}X) \xrightarrow{\sim} \mathrm{Var}_A(X)$$

of the categories of admissible variations, induced by complex conjugation

$$\iota : X(\mathbb{C}) \longrightarrow {}^{\iota}X(\mathbb{C}),$$

and defined as follows:

The local system and the weight filtration on $X(\mathbb{C})$ are the pullbacks via ι of the local system and the weight filtration on ${}^{\iota}X(\mathbb{C})$, and the Hodge filtration on $X(\mathbb{C})$ is the pullback of the conjugate of the Hodge filtration on ${}^{\iota}X(\mathbb{C})$.

ι^* preserves admissibility, and behaves, in an obvious sense, involutively. In particular, if X is defined over $\mathbb{R}$, we get an involution ι^* on $\mathrm{Var}_A(X \otimes_{\mathbb{R}} \mathbb{C})$.

Definition A.2.1. *Let $X/\mathbb{R}$ be smooth and separated.*

a) The category $\mathrm{Var}_A^{\sim}(X/\mathbb{R})$ consists of pairs $(\mathbb{V}, F_\infty)$, where $\mathbb{V}$ is an object of $\mathrm{Var}_A(X \otimes_{\mathbb{R}} \mathbb{C})$, and F_∞ is an isomorphism

$$\mathbb{V} \xrightarrow{\sim} \iota^* \mathbb{V}$$

of variations such that $\iota^* F_\infty = F_\infty^{-1}$.

In the category $\mathrm{Var}_A^{\sim}(X/\mathbb{R})$, we may define Tate twists $A(n)$: F_∞ acts via multiplication by $(-1)^n$.

b) $\mathrm{Var}_A(X/\mathbb{R})$, the category of admissible variations of mixed A–Hodge structure over $\mathbb{R}$, is the full subcategory of $\mathrm{Var}_A^{\sim}(X/\mathbb{R})$ of pairs $(\mathbb{V}, F_\infty)$ which are graded–polarizable: for $n \in \mathbb{Z}$, there is a morphism

$$\mathrm{Gr}_n^W(\mathbb{V}, F_\infty) \otimes_A \mathrm{Gr}_n^W(\mathbb{V}, F_\infty) \longrightarrow A(-n)$$

in $\mathrm{Var}_A^{\sim}(X/\mathbb{R})$*, such that the induced morphism*

$$\mathrm{Gr}_n^W \mathbb{V} \otimes_A \mathrm{Gr}_n^W \mathbb{V} \longrightarrow A(-n)$$

is a polarization in the usual sense.

Remark: We note that implicit in our definition is a descent datum over $\mathbb{R}$ of the bifiltered flat vector bundle on $X \otimes_{\mathbb{R}} \mathbb{C}$ underlying any admissible variation $(\mathbb{V}, F_\infty)$ of mixed A–Hodge structure over $\mathbb{R}$:

For this claim to make sense, recall first ([D1], II, Théorème 5.9) that any flat analytic vector bundle on $X(\mathbb{C})$ carries a canonical algebraic structure. If the vector bundle underlies an admissible variation, then the Hodge filtration is a filtration by algebraic subbundles ([Ks], Proposition 1.11.3).

Now the descent datum is given by the anti-linear isomorphism

$$c_{DR} := F_{\mathrm{diff}}(F_\infty) \circ c_\infty = c_\infty \circ F_{\mathrm{diff}}(F_\infty) : F_{\mathrm{diff}}(\mathbb{V}) \xrightarrow{\sim} F_{\mathrm{diff}}(\iota^* \mathbb{V})$$

of the C^∞–bundles underlying $\mathbb{V}$ and $\iota^*\mathbb{V}$. Here, c_∞ denotes the anti-linear involutions given by complex conjugation of coefficients, and F_{diff} is the forgetful functor to C^∞–bundles.

Lemma A.2.2. *The category* $\mathrm{Var}_A(\mathrm{Spec}(\mathbb{R})/\mathbb{R})$ *equals the category* MHS_A^+ *of mixed polarizable* A*–Hodge structures over* $\mathbb{R}$ *([B1], § 7).*

Proof. Straightforward. □

Our aim is to generalize our definition of sheaves over $\mathbb{R}$ to algebraic Hodge modules.

For smooth and separated $X/\mathbb{C}$, recall that $\mathrm{MHM}_A(X)$ is an abelian subcategory of $\mathrm{MF}_h\mathrm{W}(\mathcal{D}_X, A)$. Objects of the latter are

$$((M, F^{\cdot}, W.), (K, W.), \alpha) \ ,$$

where $(M, F^{\cdot})$ is an object of the category $\mathrm{MF}_h(\mathcal{D}_X)$ of regular holonomic algebraic $\mathcal{D}_X$–modules with a good filtration, and $K \in \mathrm{Perv}_A(\overline{X})$. $W.$ is a locally finite ascending filtration, and α is an isomorphism

$$DR(M) \xrightarrow{\sim} K \otimes_A \mathbb{C}$$

respecting W.

The equivalence

$$\iota^* : \mathrm{MF}_h\mathrm{W}(\mathcal{D}_{{}^\iota X}, A) \xrightarrow{\sim} \mathrm{MF}_h\mathrm{W}(\mathcal{D}_X, A)$$

is constructed componentwise:

The perverse sheaf and the weight filtration on $X(\mathbb{C})$ are the pullbacks via $\iota : X(\mathbb{C}) \to {}^\iota X(\mathbb{C})$ of the perverse sheaf and the weight filtration on ${}^\iota X(\mathbb{C})$.

The equivalence

$$\iota^* : \mathrm{Mod}_{\mathcal{D}_{{}^\iota X}} \xrightarrow{\sim} \mathrm{Mod}_{\mathcal{D}_X} ,$$

which by construction will respect holonomicity, comes about as follows:

Given a $\mathcal{D}_{{}^\iota X}$–module N, we may form the inverse image (in the sense of sheaves of abelian groups) $\iota^{-1}N$, which is a $\iota^{-1}\mathcal{D}_{{}^\iota X}$–module. All we therefore need is an isomorphism $c_\infty : \iota^{-1}\mathcal{D}_{{}^\iota X} \xrightarrow{\sim} \mathcal{D}_X$ of sheaves of rings extending the isomorphism $c_\infty : \iota^{-1}\mathcal{O}_{{}^\iota X} \xrightarrow{\sim} \mathcal{O}_X$ given by complex conjugation of coefficients – we then define

$$\iota^* N := \iota^{-1}N \otimes_{\iota^{-1}\mathcal{D}_{{}^\iota X}} \mathcal{D}_X .$$

Of course, the map c_∞ is itself given by conjugation of coefficients: in local coordinates $x_1, \ldots, x_n$, we have

$$c_\infty \left(\sum_\alpha f_\alpha \partial_x^\alpha \right) = \sum_\alpha (c_\infty \circ f_\alpha \circ \iota) \partial_x^\alpha .$$

Altogether, we get

$$\iota^* : \mathrm{MF}_h\mathrm{W}(\mathcal{D}_{{}^\iota X}, A) \xrightarrow{\sim} \mathrm{MF}_h\mathrm{W}(\mathcal{D}_X, A) ,$$

which again behaves involutively.

Going through the definition, one checks that ι^* induces

$$\iota^* : \mathrm{MHM}_A({}^\iota X) \xrightarrow{\sim} \mathrm{MHM}_A(X) .$$

Using local embeddings as in [S2], 2.1, we can define ι^* for any scheme X, which is separated, reduced and of finite type over $\mathbb{C}$. Furthermore, if X is defined over $\mathbb{R}$, we get an involution ι^* on $\mathrm{MHM}_A(X \otimes_\mathbb{R} \mathbb{C})$.

Theorem A.2.3. *Let X and Y be separated and reduced schemes of finite type over $\mathbb{C}$.*

a) *ι^* is compatible with $\underline{\mathrm{Hom}}$, $\otimes$, and $\mathbb{D}$: e.g., for $M^\cdot, N^\cdot \in D^b \mathrm{MHM}_A({}^\iota X)$, we have*
$$\underline{\mathrm{Hom}}_X(\iota^* M^\cdot, \iota^* N^\cdot) = \iota^* \underline{\mathrm{Hom}}_{{}^\iota X}(M^\cdot, N^\cdot)\ .$$

b) *If $\pi : X \to Y$ is a morphism, then ι^* is compatible with $\pi_!$, $\pi^!$, π^*, π_*: e.g., for $M^\cdot \in D^b \mathrm{MHM}_A({}^\iota X)$, we have*
$$\iota^*({}^\iota\pi)_* M^\cdot = \pi_*(\iota^* M^\cdot) \in D^b \mathrm{MHM}_A(Y)\,.$$

Proof. This follows from the definitions. □

Definition A.2.4. a) *Let $a : X \to \mathrm{Spec}(\mathbb{R})$ be smooth and separated. The category $\mathrm{MHM}_A^{\sim}(X/\mathbb{R})$ consists of pairs (M, F_∞), where M is an object of $\mathrm{MHM}_A(X \otimes_\mathbb{R} \mathbb{C})$, and F_∞ is an isomorphism*
$$M \xrightarrow{\ \sim\ } \iota^* M$$
such that $\iota^* F_\infty = F_\infty^{-1}$.

By A.2.3.b), we have $a^! A(n) \in \mathrm{MHM}_A^{\sim}(X/\mathbb{R})$.

b) *Let $a : X \to \mathrm{Spec}(\mathbb{R})$ be smooth and separated. $\mathrm{MHM}_A(X/\mathbb{R})$, the category of algebraic mixed A–Hodge modules over $\mathbb{R}$ on X, is the full subcategory of $\mathrm{MHM}_A^{\sim}(X/\mathbb{R})$ of pairs (M, F_∞) which are graded–polarizable: for any $n \in \mathbb{Z}$, there is a morphism*
$$\mathrm{Gr}_n^W(M, F_\infty) \otimes_A \mathrm{Gr}_n^W(M, F_\infty) \longrightarrow a^! A(-n)$$
in $\mathrm{MHM}_A^{\sim}(X/\mathbb{R})$, such that the induced morphism
$$\mathrm{Gr}_n^W M \otimes_A \mathrm{Gr}_n^W M \longrightarrow a^! A(-n)$$
is a polarization in the sense of [S1], 5.2.10.

As in A.1.1, we identify the category of smooth objects in $\mathrm{MHM}_A(X/\mathbb{R})$ with $\mathrm{Var}_A(X/\mathbb{R})$.

c) *For an arbitrary separated and reduced scheme X of finite type over $\mathbb{R}$, one defines the category $\mathrm{MHM}_A(X/\mathbb{R})$ using local embeddings as in [S2], 2.1.*

Remark: a) As in the case of variations over $\mathbb{R}$, we get a descent datum over $\mathbb{R}$ for the bifiltered $\mathcal{D}_{X\otimes_{\mathbb{R}}\mathbb{C}}$–module underlying any Hodge module over $\mathbb{R}$ on a smooth and separated scheme X over $\mathbb{R}$.
b) As in [S2], (4.2.7), the category $\mathrm{MHM}_A(Z/\mathbb{R})$, for any closed reduced subscheme Z of X, is equivalent to the category of Hodge modules over $\mathbb{R}$ on X with support in Z.

Theorem A.2.5. *There is a formalism of Grothendieck's functors $\pi_!$, $\pi^!$, π^*, π_*, $\underline{\mathrm{Hom}}$, $\otimes$, $\mathbb{D}$ on $D^b\,\mathrm{MHM}_A(\cdot/\mathbb{R})$. It is compatible with the forgetful functor*

$$D^b\,\mathrm{MHM}_A(\cdot/\mathbb{R}) \longrightarrow D^b\,\mathrm{MHM}_A(\cdot \otimes_{\mathbb{R}} \mathbb{C}) \ .$$

Proof. By A.2.3, we may e.g. define

$$\pi_!(M^{\cdot}, F^{\cdot}_{\infty}) := (\pi_! M^{\cdot}, \pi_! F^{\cdot}_{\infty}) \ .$$

□

Definition A.2.6. *Let $X/\mathbb{R}$ be separated, reduced and of finite type, and $M^{\cdot}$ an object of $D^b\,\mathrm{MHM}_A(X/\mathbb{R})$.*

a) The absolute Hodge complex of $X/\mathbb{R}$ with coefficients in $M^{\cdot}$ is

$$R\Gamma_{\mathfrak{H}^p}(X/\mathbb{R}, M^{\cdot}) := R\,\mathrm{Hom}_{D^b\,\mathrm{MHM}_A(X/\mathbb{R})}(A(0)_X, M^{\cdot}) \ .$$

b) Its cohomology groups

$$H^i_{\mathfrak{H}^p}(X/\mathbb{R}, M^{\cdot}) := H^i R\Gamma_{\mathfrak{H}^p}(X/\mathbb{R}, M^{\cdot})$$

are called absolute Hodge cohomology groups of $X/\mathbb{R}$ with coefficients in $M^{\cdot}$.

c) We denote absolute Hodge cohomology with coefficients in Tate twists by

$$H^i_{\mathfrak{H}^p}(X/\mathbb{R}, n) := H^i_{\mathfrak{H}^p}(X/\mathbb{R}, A(n)_X) \ .$$

d) For a closed reduced subscheme Z of X with complement $j : U \hookrightarrow X$, we define relative absolute Hodge cohomology with coefficients in Tate twists as

$$H^i_{\mathfrak{H}^p}(X \ \mathit{rel}\ Z/\mathbb{R}, n) := H_{\mathfrak{H}^p}(X/\mathbb{R}, j_! A(n)_U) \ .$$

Again, if X is smooth and of pure dimension d, and $M^\cdot = M \in \mathrm{MHM}_A(X)$, we have

$$H^i_{\mathfrak{H}^p}(X/\mathbb{R}, M) = \mathrm{Ext}^{d+i}_{\mathrm{MHM}_A(X/\mathbb{R})}(A(0)_X[d], M) \; .$$

We have statements analogous to A.1.1–A.1.10 for the situation over $\mathbb{R}$. For reference, we note explicitly:

Theorem A.2.7. *If X is smooth and separated over $\mathbb{R}$, and $n \in \mathbb{Z}$, then*

$$R\Gamma_{\mathfrak{H}^p}(X/\mathbb{R}, n) \quad \textit{and} \quad H^\cdot_{\mathfrak{H}^p}(X/\mathbb{R}, n)$$

coincide functorially with the absolute Hodge complex and cohomology groups of [B1], § 7.

Next, we have

Lemma A.2.8. *Let $X/\mathbb{R}$ be separated, reduced and of finite type, and $M^\cdot$ an object of $D^b\,\mathrm{MHM}_A(X/\mathbb{R})$. Then the forgetful functor*

$$D^b\,\mathrm{MHM}_A(X/\mathbb{R}) \longrightarrow D^b\,\mathrm{MHM}_A(X \otimes_{\mathbb{R}} \mathbb{C})$$

induces functorial isomorphisms

$$\begin{array}{rcl} R\Gamma_{\mathfrak{H}^p}(X/\mathbb{R}, M^\cdot) & \stackrel{\sim}{\longrightarrow} & R\Gamma_{\mathfrak{H}^p}(X \otimes_{\mathbb{R}} \mathbb{C}, M^\cdot)^+ \; , \\ H^\cdot_{\mathfrak{H}^p}(X/\mathbb{R}, M^\cdot) & \stackrel{\sim}{\longrightarrow} & H^\cdot_{\mathfrak{H}^p}(X \otimes_{\mathbb{R}} \mathbb{C}, M^\cdot)^+ \; . \end{array}$$

Here, the superscript + denotes the fixed part of the action of the involution ι^* on

$$R\,\mathrm{Hom}_{D^b\,\mathrm{MHM}_A(X \otimes_{\mathbb{R}} \mathbb{C})}(A(0)_{X \otimes_{\mathbb{R}} \mathbb{C}}, M^\cdot) \; .$$

In particular, the category MHS^+_A has cohomological dimension one since this is true for MHS_A. Furthermore, observe that the above action of $\mathbb{Z}/2\mathbb{Z}$ on $R\Gamma_{\mathfrak{H}^p}(X \otimes_{\mathbb{R}} \mathbb{C}, A(n)_{X \otimes_{\mathbb{R}} \mathbb{C}})$ is precisely that of [B1], § 7.

Corollary A.2.9. *Let $X/\mathbb{R}$ be separated, reduced and of finite type. The forgetful functor*

$$\mathrm{rat} : \mathrm{MHM}_A(X/\mathbb{R}) \longrightarrow \mathrm{Perv}_A(\overline{X \otimes_{\mathbb{R}} \mathbb{C}})$$

is faithful and exact.

Remark: Again we have

$$H^i_{\mathfrak{H}^p}(X/\mathbb{R}, A(n)_X) = H^i_{\mathfrak{H}}(X/\mathbb{R}, A(n)_X)$$

if $i \leq n$ ($i \leq 2n$ if X is proper). We have natural morphisms

$$H^i_{\mathfrak{H}^p}(X/\mathbb{R}, \mathbb{R}(n)_X) \longrightarrow H^i_{\mathfrak{H}}(X/\mathbb{R}, \mathbb{R}(n)_X) \longrightarrow H^i_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(n)_X)\,,$$

which are isomorphisms in the same range of indices.

We conclude with an explicit formula for Ext^1 in $\mathrm{MHM}_A(X/\mathbb{R})$ of a finite scheme $X/\mathbb{R}$.

Theorem A.2.10. *For any $H \in \mathrm{MHS}_A^+$, there is a canonical isomorphism*

$$\begin{aligned}\big(W_0H_\mathbb{C}/(W_0H_A + W_0F^0H_\mathbb{C})\big)^+ &\xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{MHS}_A^+}(A(0), H)\\ &= \ H^1_{\mathfrak{H}^p}(\mathrm{Spec}(\mathbb{R})/\mathbb{R}, H)\,,\end{aligned}$$

where the superscript $+$ on the left hand side denotes the fixed part of the de Rham–conjugation

$$\begin{aligned}W_0H_\mathbb{C}/(W_0H_A + W_0F^0H_\mathbb{C}) &\xrightarrow{c_\infty} W_0H_\mathbb{C}/(W_0H_A + W_0\overline{F}^0H_\mathbb{C})\\ &= \ W_0\,\iota^*H_\mathbb{C}/(W_0\,\iota^*H_A + W_0F^0\iota^*H_\mathbb{C})\\ &\xrightarrow{F_\infty} W_0H_\mathbb{C}/(W_0H_A + W_0F^0H_\mathbb{C})\,.\end{aligned}$$

The isomorphism is given by sending the class of $h \in W_0H_\mathbb{C}$ to the extension described by the matrix

$$\begin{pmatrix} 1 & 0 \\ -h & \mathrm{id}_H \end{pmatrix}.$$

This means that we equip $\mathbb{C} \oplus H_\mathbb{C}$ with the diagonal weight and Hodge filtrations, and the A–rational structure extending the A–rational structure H_A of $H_\mathbb{C}$ by the vector

$$1 - h \in \mathbb{C} \oplus H_\mathbb{C}\,,$$

thereby obtaining an extension E of $A(0)$ by H in the category MHS_A.

The conjugate extension $\iota^*E \in \mathrm{Ext}^1_{\mathrm{MHS}_A}(A(0), \iota^*H)$ is given, with the same notation, by the matrix

$$\begin{pmatrix} 1 & 0 \\ -F_\infty(h) & \mathrm{id}_{\iota^*H} \end{pmatrix},$$

and the extension of F_∞ to an isomorphism

$$F_\infty : E \xrightarrow{\sim} \iota^*E$$

sends $1 - h$ to $1 - F_\infty(h)$. Thus

$$(F_\infty)_\mathbb{C} = \mathrm{id} \oplus (F_\infty)_\mathbb{C} : \mathbb{C} \oplus H_\mathbb{C} \longrightarrow \mathbb{C} \oplus \iota^*H_\mathbb{C}\,.$$

Proof. Using [B1], § 1 or [Jn3], Lemma 9.2 and Remark 9.3.a), we see that there is an isomorphism

$$W_0 H_{\mathbb{C}}/(W_0 H_A + W_0 F^0 H_{\mathbb{C}}) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{MHS}_A}(A(0), H)\,.$$

Note that our normalization follows that of Jannsen, and therefore differs from that of Beilinson by the factor -1.

In general, if $h \in W_0 H_{\mathbb{C}}$ corresponds to an extension E in MHS_A, then $c_\infty h \in W_0 \iota^* H_{\mathbb{C}}$ corresponds to $\iota^* E$, and its pullback via

$$F_\infty : \iota^* H \longrightarrow H\ ,$$

is described by $F_\infty c_\infty h$.

Therefore, the action of the involution on $\mathrm{Ext}^1_{\mathrm{MHS}_A}(A(0), H)$ corresponds to $F_\infty c_\infty$ on the left hand side of the above isomorphism. ☐

Corollary A.2.11. *Let $X/\mathbb{R}$ be finite and reduced, and $M \in \mathrm{MHM}_A(X/\mathbb{R})$. Then there is a canonical isomorphism*

$$\begin{aligned} &\left(\bigoplus_{x \in X(\mathbb{C})} W_0 M_{x,\mathbb{C}}/(W_0 M_{x,A} + W_0 F^0 M_{x,\mathbb{C}})\right)^+ \\ \xrightarrow[A.2.10]{\sim}\ & \mathrm{Ext}^1_{\mathrm{MHS}_A^+}\left(A(0), \bigoplus_{x \in X(\mathbb{C})} M_x\right) \\ =\ \ & H^1_{\mathfrak{H}^p}(X/\mathbb{R}, M)\ . \end{aligned}$$

Proof. The last isomorphism is given by the observation that we have

$$\mathrm{MHM}_A(X) = \bigoplus_{x \in X(\mathbb{C})} \mathrm{MHS}_A\ .$$

☐

Corollary A.2.12. *For $X/\mathbb{R}$ finite and reduced, and $n \geq 1$, we have*

$$\begin{aligned} \left(\bigoplus_{x \in X(\mathbb{C})} \mathbb{C}/(2\pi i)^n A\right)^+ &\xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{MHM}_A(X/\mathbb{R})}(A(0)_X, A(n)_X) \\ &=\ H^1_{\mathfrak{H}^p}(X/\mathbb{R}, n)\ . \end{aligned}$$

Here, the superscript + denotes the fixed part with respect to the conjugation on both $X(\mathbb{C})$ and $\mathbb{C}/(2\pi i)^n A$, and the isomorphism associates to $(z_x)_{x\in X(\mathbb{C})}$ the extension, whose stalk at $x \in X(\mathbb{C})$ is given by the matrix

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{(2\pi i)^n} \cdot z_x & 1 \end{pmatrix} :$$

if e_0 and e_n are the base vectors $1 \in F \subset \mathbb{C}$ and $(2\pi i)^n \in (2\pi i)^n A \subset \mathbb{C}$, then the Hodge structure is specified by

$$F^0 := \langle e_0 \rangle_{\mathbb{C}} \ , \quad W_{-2n} \otimes_A \mathbb{C} = \langle e_n \rangle_{\mathbb{C}} \ ,$$

and the A–rational structure is generated by e_n and

$$e_0 - \frac{1}{(2\pi i)^n} \cdot z_x e_n \ .$$

Proof. This is A.2.11 and A.2.10 with respect to the basis (e_n) of $A(n)$. □

B K-Theory of Simplicial Schemes and Regulators

This appendix contains the presentation of K-theory (B.2.1) and motivic cohomology (B.2.9) for simplicial schemes. We then construct regulators (i.e., Chern classes) from K-cohomology to continuous étale cohomology (B.4) and to absolute Hodge cohomology (B.5) in this situation. Our main interest is the construction of a long exact sequence for relative K-cohomology of simplicial schemes as well as for their motivic cohomology which is mapped to the corresponding long exact sequences in sheaf cohomology (B.3.8).

A first introduction to the necessary simplicial methods is [M].

B.1 Generalized Cohomology Theories

We want to have a framework which is general enough to treat K-theory and the usual cohomology theories in parallel. It turns out such a framework is given by homotopical algebra as axiomatized by Quillen in [Q1].

We define cohomology of spaces (=simplicial sheaves of sets) with coefficients in another space (B.1.4). We then construct a long exact sequence for relative cohomology in this context (B.1.6). Finally we deduce the spectral sequence relating generalized cohomology of a space to generalized cohomology of its components (B.1.7).

We work in the setting of [Jeu] originally introduced in the unpublished article [GSo1] by Gillet and Soulé. The method seems to be due to Brown ([Br], [BrG]). A systematic investigation of generalized cohomology for Grothendieck topologies was carried out by Jardine, in particular [Jr2]. We recapitulate the definitions for the convenience of the reader.

We fix a regular affine irreducible base scheme B of finite Krull dimension. In our applications B is either a field or an open subscheme of the ring of integers of a number field. We fix a small category of noetherian finite dimensional B–schemes which is closed under finite disjoint unions and contains all open subschemes of all its objects. We turn it into a site using the Zariski topology. Typically this will be a subcategory of all smooth schemes over the base B.

Let $\mathbf{T}$ be the topos of sheaves of sets on our Zariski site over B. Let $s\mathbf{T}$ be the category of pointed simplicial $\mathbf{T}$-objects. Its objects will be called *spaces* in the sequel. We denote the final and initial object of $s\mathbf{T}$ by $\star$.

Remark: A space is given by a simplicial sheaf of sets $X_\cdot$ and a simplicial map ι from $\star$ (the constant simplicial sheaf all of whose components are given by the constant sheaf $\tilde{\star}$ attached to the set with one element) to $X_\cdot$. Equivalently we can consider it as a simplicial object in the category of sheaves pointed by $\tilde{\star}$.

Let X be a scheme. We can also see it as an object of $\mathbf{T}$. The corresponding constant simplicial object pointed by a disjoint base point,

$$U \mapsto \mathrm{Mor}_B(U, X) \cup \{\star\} \qquad \text{for connected } U \in \mathbf{T},$$

will also be denoted X.

Definition B.1.1. *A space is said to be constructed from schemes if all components are representable by a scheme in the site plus a disjoint base point.*

Note that any simplicial scheme gives rise to a space constructed from schemes but there are many spaces constructed from schemes which do not come from simplicial schemes. The main example is the mapping cone of a map of schemes taken in $s\mathbf{T}$ (cf. B.1.5 below).

If $\mathbf{P}$ is a property of schemes and if the space X is constructed from schemes, we say X has $\mathbf{P}$ if the scheme parts of the components have $\mathbf{P}$.

The easiest way to define the homotopy sets $\pi_n(X)$ of a simplicial set X is to take the homotopy sets of its geometric realization. $\pi_n(X)$ is a group for $n \geq 1$, even abelian for $n \geq 2$. If X is a space and K a finite simplicial set (i.e., all K_n are finite), then we define the space $X \otimes K$ componentwise as the sum of pointed sheaves

$$n \mapsto \bigvee_{\sigma \in K_n} X_n \ .$$

Definition B.1.2 (Gillet, Soulé). *Let X be a space. The homotopy sheaf $\underline{\pi}_n(X)$ is the sheaf of pointed sets attached to the presheaf*

$$U \longmapsto \pi_n(X(U), *)$$

on $\mathbf{T}$. Let $f : X \to Y$ be a map of spaces.

a) *f is called a weak equivalence if all $f_* : \underline{\pi}_n(X) \to \underline{\pi}_n(Y)$ are isomorphisms for $n \geq 0$.*

b) *f is called a cofibration if for all schemes U in $\mathbf{T}$ the induced map $f(U) : X(U) \to Y(U)$ is injective.*

c) *f is called a fibration if it has the following lifting property: given a commutative diagram*

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & Y \end{array}$$

where i is a cofibration and a weak equivalence, there exists a map $B \to X$ that makes the diagram commute.

d) *For two spaces X and Y, let $\mathrm{Hom}_.(X, Y)$ be the pointed simplicial set*

$$n \mapsto \mathrm{Hom}_{s\mathbf{T}}(X \otimes \Delta(n), Y)$$

where $\Delta(n)$ is the standard simplicial n-simplex (e.g. [M] 5.4) pointed by zero.

This is the pointed version of the global theory discussed in [Jr2] §2.

Technical Remark: Note that the unique map $\star \longrightarrow X$ is always a cofibration, i.e., all spaces are cofibrant. A space will be called fibrant if the unique map $X \longrightarrow \star$ is a fibration. The sections $X(U)$ of a fibrant space X over a scheme U form a simplicial set satisfying Kan's extension condition (cf. [M] 1.3). However, this property does not suffice to make X fibrant. All the same, the functor Ex_∞, which assigns to any simplicial set a homotopy equivalent one which satisfies the extension condition ([Kn]) can be used to construct functorially a fibrant resolution of a space ([GSo1] 1.4.3), i.e., a weak equivalence $X \longrightarrow \tilde{X}$ where $\tilde{X}$ is fibrant.

Quillen's notion of a closed model category axiomatizes the properties which are needed in order to pass to a homotopy category which behaves similar to the homotopy category of CW–spaces.

Proposition B.1.3 (Joyal). *$s\mathbf{T}$ is a pointed closed simplicial model category in the sense of Quillen [Q1].*

Proof. The category is pointed by $\star$. The simplicial structure ([Q1] II Def. 1) is given by B.1.2.d). For a model category we need fibrations, cofibrations and weak equivalences satisfying a set of axioms ([Q1] I Def. 1). This is [GSo1] Theorem 1. Gillet and Soulé attribute this theorem to Joyal (letter to Grothendieck). A published proof of all properties can be found in [Jr2] Cor. 2.7. □

Let Ho($s\mathbf{T}$) be the homotopy category associated to the model category $s\mathbf{T}$ by localizing at the class of weak equivalences. As usual we will write $[X, Y]$ for the morphisms from X to Y in the homotopy category. If Y is fibrant, then this set is given by the set of morphisms from X to Y in $s\mathbf{T}$ up to simplicial homotopy. For general Y, we compute $[X, Y]$ by $[X, \tilde{Y}]$ where $\tilde{Y}$ is a fibrant resolution of Y.

Remark: By loc. cit., Lemma 2.6 we could use the category of presheaves instead of the category of sheaves in B.1.2. The map from a presheaf to its sheafification would be a weak equivalence and we would get the same homotopy category.

If X is a space, then its suspension SX is given by $X \otimes \Delta(1)/\sim$ where $\sim$ is the usual equivalence relation generated by $(x, 0) \sim (x, 1)$. By [Q1] Ch.

I 2, the loop space functor Ω is right adjoint to S on the homotopy category.

There are two natural ways of thinking about $\mathrm{Ho}(s\mathbf{T})$. From the point of view of algebraic topology it corresponds to the category of CW-complexes with morphisms up to homotopy. From the point of view of homology theory it corresponds to the category of homological complexes which are concentrated in positive degrees with morphisms up to homotopy. S and Ω shift the complexes. This second point of view is not quite precise - note that in general morphisms in $\mathrm{Ho}(s\mathbf{T})$ form pointed sets rather than groups.

Definition B.1.4. *For any space A we define cohomology of spaces with coefficients in A by setting*

$$H^{-m}_{s\mathbf{T}}(X,A) = [S^m X, A] \quad \textit{for } m \geq 0 \quad .$$

This is only a pointed set for $m = 0$, a group for $m > 0$ and even abelian for $m > 1$. If A belongs to an infinite loop spectrum, i.e., if there are spaces A_i for $i \geq 0$ with $A_0 = A$ and weak equivalences $A_i \longrightarrow \Omega A_{i+1}$, then we also define cohomology groups with positive indices by setting

$$H^{n-m}_{s\mathbf{T}}(X,A) = [S^m X, A_n] \quad \textit{for } m,n \geq 0 \quad .$$

Note that the set only depends on $n - m$ because the suspension S and the loop functor Ω are adjoint.

Definition B.1.5. *Let $f : X \longrightarrow Y$ be a map of spaces. Then the mapping cone of f is the space*

$$C(f) = X \otimes \Delta(1) \amalg Y / \sim$$

where $\sim$ is the usual equivalence relation of the mapping cone (i.e., $(x,1) \sim f(x)$, $(x,0) \sim \star$). For any map of spaces $f : X \longrightarrow Y$, we define relative cohomology by

$$H^{-m}_{s\mathbf{T}}(Y \textit{ rel } X, A) = H^{-m}_{s\mathbf{T}}(C(f), A) \quad .$$

Proposition B.1.6. *For any morphism $f : X \longrightarrow Y$ of spaces there is a long exact cohomology sequence:*

$$\longrightarrow H^{-m}_{s\mathbf{T}}(Y \textit{ rel } X, A) \longrightarrow H^{-m}_{s\mathbf{T}}(Y, A) \longrightarrow H^{-m}_{s\mathbf{T}}(X, A) \longrightarrow H^{-m+1}_{s\mathbf{T}}(Y \textit{ rel } X, A) \quad .$$

Proof. By [Q1] Ch. I 3 we have the above long exact sequence attached to the triple of spaces

$$X \xrightarrow{i} Y' \longrightarrow Y' \vee_X \star$$

if i is a cofibration. The mapping cylinder of f is defined as $X \otimes \Delta(1) \vee_X Y$. It is weakly equivalent to Y, and the induced mapping $X \longrightarrow X \otimes \Delta(1) \vee_X Y$ is a cofibration. The mapping cone of f is nothing but the cofibre of this inclusion. Hence the long exact sequence of the lemma is a special case of Quillen's with $Y' = X \otimes \Delta(1) \vee_X Y$. □

If A is only a space, then the sequence will end at the index zero. There is no reason for it to be right exact. The $H^0_{s\mathbf{T}}$ are only pointed sets. The $H^{-1}_{s\mathbf{T}}$ are groups, all others are even abelian groups. However, if A is an infinite loop spectrum, then all cohomology groups will be abelian groups and the sequence is unbounded in both directions.

We will consider a couple of spectral sequences which are constructed by means of homotopical algebra. Their differentials are

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q+r-1} \quad .$$

We refer to this behaviour as homological spectral sequence as opposed to a cohomological spectral sequences with differentials

$$d_r : E_r^{p,q} \longrightarrow E_r^{p-r,q-r+1} \quad .$$

In the same way as with the long exact sequences which involve pointed sets we also have to be careful about our spectral sequences. They will be constructed by the method of Bousfield-Kan (cf. [BouK] Ch. IX §§4-5). We refer to them as spectral sequences of Bousfield-Kan type. We give an overview over their properties. They look like this:

$$E_r^{p,q} \Rightarrow L^{q-p} \qquad q \geq p \geq 0,\ r \geq 1$$

with homological differentials.

$$L^{q-p}, E_r^{p,q} = \begin{cases} \text{are abelian groups} & \text{if } q-p \geq 2; \\ \text{are groups} & \text{if } q-p = 1; \\ \text{are pointed sets} & \text{if } q-p = 0. \end{cases}$$

We have $E_{r+1}^{p,q} = \operatorname{Ker} d_r^{p,q} / \operatorname{im} d^{p-r,q-r+1}$. (Treat non-existing $E_r^{p,q}$ as zero for this formation.) By [BouK] IX 4.2.iv) this makes also sense for $p = q$.

Let

$$E_\infty^{p,q} = \varprojlim_r E_r^{p,q} = \bigcap_{r>p} E_r^{p,q} \ .$$

There is a descending cofiltration Q_* on the limit term L^n (i.e., $Q_i L^n$ is a quotient of L^n). Let

$$e_\infty^{p,q} = \operatorname{Ker}\left(Q_p L^{q-p} \longrightarrow Q_{p-1} L^{q-p}\right) \ .$$

In general, there will be an injection $e_\infty^{p,q} \longrightarrow E_\infty^{p,q}$. Convergence is a more complicated question. The spectral sequence stabilizes if all projective systems $(E_r^{p,q})_{r>p}$ become eventually stable. Then we have complete convergence ([BouK] IX 5.3). Hence the cofiltration on the limit term is exhaustive ($\varprojlim Q_s L^n = L^n$), and we have isomorphisms

$$e_\infty^{p,q} \xrightarrow{\cong} E_\infty^{p,q} \qquad \text{for } p - q > 0.$$

Note that even then the case $p = q$ has to be discussed separately. We refer to this problem and more generally the fact that pointed sets rather then groups appear as the fringe effect.

Proposition B.1.7. a) *Let X and A be spaces. The filtration of X by its skeletons $sq_n X$ induces a spectral sequence of Bousfield-Kan type for its A-cohomology*

$$E_1^{p,q} = H_{s\mathbf{T}}^{-q}(X_p, A) \Rightarrow H_{s\mathbf{T}}^{-(q-p)}(X, A) \qquad \textit{for } q \geq p \geq 0 \ .$$

It converges completely if X is degenerate above some degree (i.e., if there is N such that for $n \geq N$, X_n is covered by the image of the degeneracy maps.).

b) *If A is an infinite loop spectrum and X as in a), then we have a converging homological spectral sequence*

$$E_1^{p,q} = H_{s\mathbf{T}}^{-q}(X_p, A) \Rightarrow H_{s\mathbf{T}}^{-(q-p)}(X, A) \qquad \textit{for } p \geq 0 \ .$$

Proof. This is the hypercohomology spectral sequence of [GSo1] 1.2.3. We sketch their proof: We can assume A to be fibrant. We can construct a weak equivalence $X' \longrightarrow X$ such that $sk_p X'/sk_{p-1}X' \cong S^p X_p$. The $\operatorname{Hom}_{.}(sk_p X', A)$ form a tower of fibrations of simplicial sets converging to $\operatorname{Hom}_{.}(X, A)$. The attached Bousfield-Kan spectral sequence ([BouK] §4 -§5) has starting terms

$$\begin{aligned} E_1^{p,q} &= \pi_{q-n} \operatorname{Hom}_{.}(sk_p X'/sk_{p-1}X', A) \\ &= \pi_{q-p} \operatorname{Hom}_{.}(S^p X_p, A) = H_{s\mathbf{T}}^{-q}(X_p, A) \ . \end{aligned}$$

This finishes the construction of the spectral sequence. In order to discuss convergence we consider the same spectral sequence attached to X itself. It stabilizes by the assumption on degeneracy (see [BouK] §5). Both spectral sequences agree from $r = 2$ on.

For b) we consider the spectral sequence in a) for each space in the spectrum. By shifting q accordingly we get a direct system of spectral sequences whose limit is the one we are interested in. □

Remark: Working with spectra helps to get rid of the fringe effect. However, the question of convergence does not get easier, the reason behind this being that all these spectral sequences are constructed for some kind of homotopy limit, and projective limits are not exact.

It would be much nicer to work with spectra and their homotopy category throughout. It would be a triangulated category. However, the literature we want to use is in the setting of spaces. The reason is that we want to use the λ-ring structure in order to define motivic cohomology and the λ-operators do not deloop.

B.2 K-theory

We now introduce higher algebraic K–theory of spaces as a generalized cohomology theory. It gives back usual K–theory in the case of regular schemes (B.2.3). We then define λ-operators on these K–cohomology groups (B.2.8). This allows definition of motivic cohomology of spaces as graded parts of the γ-filtration (B.2.9). We then prove a Grothendieck-Riemann-Roch type theorem (B.2.16). As a consequence we get a long exact localization sequence for motivic cohomology (B.2.17).

Recall that all schemes in the site underlying $\mathbf{T}$ are assumed to be noetherian and finite dimensional.

Let $\mathbf{K}$ be the space $\mathbb{Z} \times \mathbb{Z}_\infty BGl$ where BGl is the simplicial sheaf associated to the simplicial presheaf $U \mapsto BGl(U) = \varinjlim BGl_n(U)$. It is in fact part of an infinite loop spectrum. We also need the "unstable" spaces $\mathbf{K}^N = \mathbb{Z} \times \mathbb{Z}_\infty BGl_N$. There are natural transition maps $\mathbf{K}^N \to \mathbf{K}^{N+1} \to \mathbf{K}$.

Remark: Even though it is well-known that K-theory is defined by a spectrum, it is not completely trivial to define it as a functor from schemes to spectra (rather than just a functor up to homotopy). We refer to [GSo2], 5.1.2 for the details of this construction.

Definition B.2.1 (Gillet, Soulé). *For any space X in $s\mathbf{T}$ we define its K-cohomology*

$$H^{-m}_{s\mathbf{T}}(X,\mathbf{K}) = [S^m X,\mathbf{K}] \qquad \text{for } m\in\mathbb{Z}$$

and the unstable K-groups $H^{-m}_{s\mathbf{T}}(X,\mathbf{K}^N)$ for $m\geq 0$. Following [GSo1] we call a space K-coherent if $\varinjlim H^{-m}_{s\mathbf{T}}(X,\mathbf{K}^N)\to H^{-m}_{s\mathbf{T}}(X,\mathbf{K})$ for $m\geq 0$ is an isomorphism.

Proposition B.2.2 (Brown). *Let $\mathcal{K}_q$ be the sheafification of the presheaf $Y\mapsto H^{-q}_{s\mathbf{T}}(Y,\mathbf{K})$. Let X be a scheme in $\mathbf{T}$. There is a homological spectral sequence*

$$E_2^{pq}\Rightarrow H^{-(q-p)}_{s\mathbf{T}}(X,\mathbf{K})$$

with

$$E_2^{pq} = H^p_{\mathrm{ZAR}}(X,\mathcal{K}_q)$$

It converges completely.

Proof. For $q-p\geq 0$ this is the spectral sequence [GSo1] Prop. 2 (the indices are different there). The basic version for the small Zariski site was constructed in [BrG] Theorem 3. Our generalization follows from the proof of [Jr2] 3.4 and 3.5, which deals with the étale topology. The key is to construct a Postnikov-tower for $\mathbf{K}$. This is done as in in the proof of [BrG] Thm 3. We then have to check that the homotopy sheaves of $\mathbf{K}$ are isomorphic to the homotopy sheaves of the limit of its Postnikov-tower. It suffices to check this for the small Zariski site Zar/Y for all schemes Y in $\mathbf{T}$. Hence we are reduced to the situation considered in loc. cit. Note that Y was assumed to be noetherian and finite dimensional.

We extend to arbitrary p,q using the full K-theory spectrum. Convergence follows because X has finite cohomological dimension. □

Remark: We could generalize the spectral sequence to arbitrary spaces X. $H^p_{\mathrm{ZAR}}(X,\mathcal{K}_q)$ would have to be understood as in B.3. Convergence would not be guaranteed anymore.

The most important application of this proposition is that it allows to transport properties which are well-known for cohomology with coefficients in an abelian sheaf to cohomology with coefficients in a space. One such property is the comparison between different Zariski sites.

Proposition B.2.3 (Gillet, Soulé, de Jeu). a) *Let X be a regular noetherian finite dimensional scheme in the site. Then $H^{-m}_{s\mathbf{T}}(X, \mathbf{K}) = K_m(X)$ where the right hand side means Quillen K-theory of the scheme X. In particular, $H^{-m}_{s\mathbf{T}}(X, \mathbf{K}) = 0$ for $m < 0$.*

b) *Let X be a space constructed from schemes. Assume that all components are regular Noetherian finite dimensional schemes and that X is degenerate above some simplicial degree. Then X is K–coherent.*

Proof. The constant case is proved in [GSo1] 2.2.2 Prop. 5. We sketch a slightly different argument: We use the converging Brown spectral sequence and comparison theorems for sheaf cohomology to show that it suffices to prove the proposition in the case of $\mathbf{T} = \mathrm{Zar}/X$. (Note that the existence of the whole spectrum means we do not have to worry about fringe effects.) In this case we have a Mayer-Vietoris sequence for K-theory ([Q2] Rem. 3.5) and hence the presheaf defining K-cohomology is pseudo-flasque in the sense of Brown and Gersten ([BrG] p. 285). By loc. cit. Thm. 4 this implies a) for the site Zar/X.
The vanishing property is [Ba] Chapter XII Proposition 10.1 with the notation of loc. cit. p. 664. The generalization to spaces constructed from schemes using the skeletal spectral sequence was carried out in [Jeu] 2.1 (1) and Lemma 2.1. □

If all components of X are regular, we will often write $K_m(X)$ instead of $H^{-m}_{s\mathbf{T}}(X, \mathbf{K})$.

Corollary B.2.4. *If X is a space meeting the conditions of part b) of the proposition, then its K-cohomology does not depend on the category of schemes underlying the topos.*

Proof. If X is constant, then we always get its K-theory. For more general X we have to use the converging skeletal spectral sequence. There are no fringe problems because $\mathbf{K}$ is an infinite loop spectrum. □

Let S^0 be the simplicial version of the 0-sphere, i.e., the constant simplicial sheaf associated to $\{0, 1\}$ pointed by 0. We will use the notation $K_0(s\mathbf{T})$ for $H^0_{s\mathbf{T}}(S^0, \mathbb{Z})$. It is a ring with unity where the ring structure is induced by the ring structure on $\mathbb{Z}$.

Lemma B.2.5. *If the site underlying* $\mathbf{T}$ *has a final object* X, *then*

$$K_0(X) \cong K_0(s\mathbf{T}) \ .$$

Proof. If X is the final object of the site, then the space we denote by X is equal to S^0. Note also that the space $\mathbb{Z}_\infty BGl_.$ is connected. Hence

$$H^0_{s\mathbf{T}}(S^0, \mathbb{Z}) = H^0_{s\mathbf{T}}(X, \mathbb{Z}) = H^0_{s\mathbf{T}}(X, \mathbf{K}) = K_0(X) \quad .$$

□

$\mathbf{K}$ is an H–space where addition is induced by the direct sum $\oplus$ of matrices. This structure allows to define a map of spaces

$$pr_{\mathbb{Z}} : \mathbb{Z} \wedge \mathbf{K} \to \mathbf{K} \quad .$$

The following lemma generalizes an operation of $K_0(X)$ which was explained to us by de Jeu in the case where Y is constructed from X–schemes.

Lemma B.2.6. *Let* Y *be a space in* $s\mathbf{T}$. *Then the ring* $K_0(s\mathbf{T})$ *operates on* $H^{-n}_{s\mathbf{T}}(Y, \mathbf{K})$ *for* $n \geq 0$.

Proof. If Y is a space in $s\mathbf{T}$, then there is canonical isomorphism $Y \cong S^0 \wedge Y$. The product $\alpha \in K_0(s\mathbf{T})$ with $\beta \in H^{-n}_{s\mathbf{T}}(Y, \mathbf{K})$ is defined by the composition

$$Y \longrightarrow S^0 \wedge Y \xrightarrow{\alpha \wedge \beta} \mathbb{Z} \wedge \mathbf{K} \xrightarrow{pr_{\mathbb{Z}}} \mathbf{K} \quad .$$

□

The construction of the Loday-product [L] 2.1.5

$$\mathbb{Z}_\infty BGl_N(U) \wedge \mathbb{Z}_\infty BGl_N(U) \to \mathbb{Z}_\infty BGl(U)$$

is functorial in U. Together with the product $pr_{\mathbb{Z}}$ on the factor $\mathbb{Z}$ it defines a system of maps

$$pr_{\mathbf{K}} : \mathbf{K}^N \wedge \mathbf{K}^N \to \mathbf{K}$$

(compatible up to homotopy), which defines a product

$$[Y, \mathbf{K}] \times [Y, \mathbf{K}] \to [Y, \mathbf{K}]$$

for all K–coherent spaces Y. It turns all $H^{-n}_{s\mathbf{T}}(Y, \mathbf{K})$ for $n \geq 0$ into a ring, possibly without unity. The product structure is compatible with the operation of $K_0(s\mathbf{T})$.

Remark: Note that this product on $[Y, \mathbf{K}]$ is zero on $H^{-n}_{s\mathbf{T}}(Y, \mathbf{K})$ for $n > 0$ (cf. [Kr] Ex. 1 p. 243). The same map $pr_{\mathbf{K}}$ of spaces also induces a non-trivial product

$$[S^n Y, \mathbf{K}] \times [S^m Y, \mathbf{K}] \to [S^{n+m} Y, \mathbf{K}] \quad .$$

This is the one which is usually called Loday-product. We do not need it in the sequel.

Lemma B.2.7 (Gillet, Soulé). *Let G be a group over $\mathbb{Z}$. Let $R_{\mathbb{Z}}(G)$ be the Grothendieck group of representations of G on free $\mathbb{Z}$–modules of finite type. Then there is an algebra homomorphism*

$$r : R_{\mathbb{Z}}(G) \to \operatorname{im} \varinjlim [\mathbb{Z} \times \mathbb{Z}_\infty BG, \mathbf{K}^N] \subset [\mathbb{Z} \times \mathbb{Z}_\infty BG, \mathbf{K}] \ .$$

Proof. We follow [GSo1] 3.2 or the affine case [Kr] 3. We define the map on generators of $R_{\mathbb{Z}}(G)$. Any representation of G induces a map of sheaves

$$G \to Gl_N \to Gl$$

It induces a well-defined map of spaces

$$\mathbb{Z}_\infty BG \to \{N\} \times \mathbb{Z}_\infty BGl_N \to \mathbf{K} \ .$$

It extends to the factor $\mathbb{Z}$ using the above product $pr_{\mathbb{Z}}$. Now we have to check compatibility with the ring operations. This works precisely as in the affine case, see [Kr] Cor. 3.2. □

$K_0(s\mathbf{T})$ is a λ–ring, i.e., the axioms in [Kr] Def. 4.1 are satisfied. If R is a $K_0(s\mathbf{T})$–algebra, then it is called a $K_0(s\mathbf{T})$–λ–algebra if it is equipped with operators λ^i for $i \geq 1$ such that $K_0(s\mathbf{T}) \oplus R$ is a λ–ring (cf. [Kr] 5.). Note that λ^0 has to have the constant value 1. If R itself does not have a unity, then it cannot be a λ–ring.

Theorem B.2.8 (Gillet, Soulé). *Let Y be a K–coherent space. For $k \geq 1$ and $m \geq 0$ there are maps*

$$\lambda^k : H^{-m}_{s\mathbf{T}}(Y, \mathbf{K}) \to H^{-m}_{s\mathbf{T}}(Y, \mathbf{K}) \quad .$$

They turn $H^{-m}_{s\mathbf{T}}(Y, \mathbf{K})$ into a $K_0(s\mathbf{T})$–λ–algebra.

Proof. This is essentially [GSo1] Prop. 8. Put $G = Gl_n$ in the previous lemma. Let $\tilde{\mathbb{Z}}^n = [\mathbb{Z}^n_{id}] - [n \cdot 1] \in R_{\mathbb{Z}}(Gl_n)$ where $\mathbb{Z}^n_{id}$ is the canonical representation of Gl_n on $\mathbb{Z}^n$ and 1 is the trivial representation. We define $\lambda^k_n = r(\lambda^k(\tilde{S}^n))$. By composition it induces a map $\lambda^k_n : H^{-m}_{s\mathbf{T}}(Y, \mathbf{K}^n) \to H^{-m}_{s\mathbf{T}}(Y, \mathbf{K})$. These form a projective system and hence define an operation on K–cohomology of a K–coherent space. Well-definedness and all properties of a λ–ring can be checked on the universal level and hence as in the affine case [Kr] Thm 5.1. □

Technical Remark: When we try to define λ^0 in the same way, then we still get a map

$$\lambda^0 : \mathbb{Z}_\infty BGl_N \longrightarrow \mathbb{Z} \times \mathbb{Z}_\infty BGl\ .$$

It does not extend to the factor $\mathbb{Z}$ because $\lambda^0 : \mathbb{Z} \longrightarrow \mathbb{Z}$ does not respect the base point - in fact it maps 0 to 1. This reflects the fact that the ring $K_0(Y)$ does not have a unity for a general space Y. The most striking example is $Y = C(i)$ where $i : Z \longrightarrow X$ is a morphism between regular schemes (cf. [Sou4] 4.3). Then $K_0(Y) = \mathrm{Ker}\,(K_0(X) \longrightarrow K_0(Z))$ does not contain 1.
Gillet and Soulé ([GSo1] Prop. 8) consider the structure as a $H^0_{s\mathbf{T}}(Y, \mathbf{K})$–$\lambda$–algebra. This only makes sense if $H^0_{s\mathbf{T}}(Y, \mathbf{K})$ happens to have a unity. However, we can check in general that the operation of $H^0_{s\mathbf{T}}(Y, \mathbf{K})$ on $H^{-m}_{s\mathbf{T}}(Y, \mathbf{K})$ is compatible with the $K_0(s\mathbf{T})$–λ–algebra structure of both groups.

Note that the λ-structure is compatible with the contravariant functoriality of K-cohomology. This means that the long exact sequences for relative K-theory are compatible with the λ-operation where it is defined.

Once we have λ-operations we get as usual a γ-filtration and Adams-operators on the λ-module $H^n_{s\mathbf{T}}(Y, \mathbf{K})$ for $n \leq 0$. If the γ-filtration is locally finite, then we have in particular the Chern character

$$ch : H^n_{s\mathbf{T}}(Y, \mathbf{K})_{\mathbb{Q}} \longrightarrow \bigoplus_{j \in \mathbb{N}_0} \mathrm{Gr}^j_\gamma\, H^n_{s\mathbf{T}}(Y, \mathbf{K})_{\mathbb{Q}} \qquad \text{for } n \leq 0,$$

which is an isomorphism. For a quick survey cf. [T] pp. 117–123.

Definition B.2.9. *Let Y be a K-coherent space. Suppose that the γ-filtration is locally finite and hence that rationally K-cohomology splits into Adams-eigenspaces. Then we put for $j \geq n/2$*

$$H^n_{\mathcal{M}}(Y, j) = \mathrm{Gr}^j_\gamma\, H^{n-2j}_{s\mathbf{T}}(Y, \mathbf{K})_{\mathbb{Q}}\ ,$$

the motivic cohomology of the space Y. If $i : X \to Y$ is a morphism of spaces then we define relative motivic cohomology by

$$H^n_{\mathcal{M}}(Y \text{ } rel \text{ } X, j) = H^n_{\mathcal{M}}(\mathrm{Cone}(i), j) \ .$$

Remark: We restrict to this range of indices because we did not define Adams-eigenspaces for K-cohomology with positive indices (=K-theory with negative indices). However, if these K-groups vanish we can simply define the corresponding motivic cohomology groups to be zero. This is the case if X is a regular scheme.

The long exact sequence for relative cohomology (B.1.6) together with the above remarks on the λ-operation give a long exact sequence for relative motivic cohomology

$$\to H^{-m}_{\mathcal{M}}(Y \text{ rel } X, A) \to H^{-m}_{\mathcal{M}}(Y, A) \to H^{-m}_{\mathcal{M}}(X, A) \to H^{-m+1}_{\mathcal{M}}(Y \text{ rel } X, A) \quad .$$

Lemma B.2.10. *Let X be a space degenerate above some simplicial degree. We assume the conditions of the previous definition. Fix an integer j. There is a cohomological spectral sequence with starting terms*

$$E_1^{s,t} = \begin{cases} H^t_{\mathcal{M}}(X_s, j) & \text{for } s \geq 0,\ 2j \geq t, \\ 0 & \text{else.} \end{cases}$$

It converges to $H^{s+t}_{\mathcal{M}}(X, j)$ for $2j \geq s + t$.

Proof. Consider the skeletal spectral sequence B.1.7.b) with coefficients in the spectrum $\mathbf{K}$. It reads

$$E_1^{p,q} = H^{-q}_{s\mathbf{T}}(X_p, \mathbf{K}) \Rightarrow H^{-(q-p)}_{s\mathbf{T}}(X, \mathbf{K})$$

for $p \geq 0$. The problem is that only those $E_1^{p,q}$ with $q \geq 0$ are λ-modules. For the following $E_r^{p,q}$ fewer and fewer terms are λ-modules and finally on the limit terms this is only true for the indices $q \geq p \geq 0$. If we restricted to this range of indices in the first place, i.e., if we only considered the skeletal spectral sequence for the space $\mathbf{K}$, then we would not get information on the limit terms on the $p = q$–line.

Hence we argue more carefully beginning with the above spectral sequence for $p \geq 0$ and arbitrary q. We dispose of the $q < 0$-terms. This does not influence any $E_r^{p,q}$ for $q - p \geq 0$. The spectral sequence is completely convergent to the correct limit terms in this range.

All starting terms of the spectral sequence have a λ-operation. By carefully checking the construction of the spectral sequence, we see that all differentials $d_r^{p,q}$ are induced by functoriality in the first argument. Hence they are morphisms of λ-modules. For $q-p \geq 0$ the limit terms are also λ-modules and by construction the morphisms $e_\infty^{p,q} \xrightarrow{\cong} E_\infty^{p,q}$ are compatible with this structure. This allows to take Adams-eigenspaces. By reindexing $s = p, t = -q + 2j$ we get a cohomological spectral sequence as stated. Note that we use the terms below the $p = q$-diagonal to compute the terms on it but we do not consider their limit terms. □

The same spectral sequence also shows that the conditions in the definition of motivic cohomology hold if X is a space constructed from schemes and degenerate above some degree.

The next thing we need is pushout at least for certain closed immersions and a Riemann-Roch theorem. Push-forward was defined by de Jeu in [Jeu] 2.2. We adapt his method to our needs in the proof of B.2.16.

Definition B.2.11. *Let S be a regular irreducible Noetherian affine scheme. Let X be smooth and quasi-projective over S.*

A finite diagram $\mathcal{D}_X$ over X is a category of finitely many smooth quasi-projective S-schemes with final object X such that all $\mathrm{Mor}_{\mathcal{D}_X}(Y, Y')$ are finite sets and such that all morphisms in $\mathcal{D}_X$ are of finite Tor-dimension.

By the small Zariski site $\mathrm{Zar}_{\mathcal{D}_X}$ we mean the category of all finite disjoint unions of open subschemes of objects in $\mathcal{D}_X$ with the induced morphisms between them. It is equipped with the Zariski-topology. The corresponding topos will be denoted $\mathbf{T}_X$.

An easy case of such a diagram is a single morphism $Y \longrightarrow X$ that meets the conditions.

We consider the following situation: Let $i : Z \longrightarrow X$ be a closed immersion of smooth quasi-projective S-schemes and $\mathcal{D}_X$ a finite diagram over X. We assume the following conditions, corresponding to the ones formulated by de Jeu in [Jeu] 2.2:

(TC) For all X' in $\mathcal{D}_X$, the pullback $X' \times_X Z$ is S–smooth. If $f : X_1 \longrightarrow X_2$

is a morphism in $\mathcal{D}_X$, then in the cartesian diagram

$$\begin{array}{ccc} Z_1 = X_1 \times_X Z & \longrightarrow & X_1 \\ {\scriptstyle f\times_X Z}\downarrow & & \downarrow{\scriptstyle f} \\ Z_2 = X_2 \times_X Z & \xrightarrow{i} & X_2 \end{array}$$

the maps f and i are tor-independent, i.e.,

$$\underline{\mathrm{Tor}}^k_{\mathcal{O}_{X_2}}(\mathcal{O}_{Z_2}, \mathcal{O}_{X_1}) = 0$$

for $k > 0$. ($\underline{\mathrm{Tor}}^k$ denotes the sheaf of tor-groups.)

Lemma B.2.12. *The pullback $\mathcal{D}_Z$ of $\mathcal{D}_X$ by Z satisfies the conditions for a finite diagram over Z.*

Proof. Finite Tor-dimension in $\mathcal{D}_Z$ follows from Tor-independence and the same property in $\mathcal{D}_X$. □

Let $Y_.$ be a space in $s\mathbf{T}_X$. Let $j : U \longrightarrow X$ be the open complement of Z in X. Let $Y_. \times_X U$ be the pointed version of $j_! j^* Y_.$, i.e., the sheaf associated to the presheaf

$$V \mapsto \begin{cases} Y_.(V) & \text{if } V \to U \subset X, \\ 0 & \text{else.} \end{cases}$$

It is a space in $s\mathbf{T}_X$. Let $Y_. \times_X Z = i^{-1}Y_.$, a space in $s\mathbf{T}_Z$. If $Y_.$ is constructed from schemes, then so are $Y_. \times_X U$ and $Y_. \times_X Z$. The scheme components are given by the base change with U or Z respectively. Note that $i^{-1}(Y_. \times_X U)$ is empty, i.e., only consists of the base point.

Proposition B.2.13 (de Jeu). *Let $i : Z \to X$ be a closed immersion with open complement U. Let $\mathcal{D}_X$ be a finite diagram over X such that (TC) holds with respect to i. Then:*

a) *There is a natural pushout map*

$$H^k_{s\mathbf{T}_Z}(Y_. \times_X Z, \mathbf{K}) \to H^k_{s\mathbf{T}_X}(Y_., \mathbf{K})\,.$$

b) *Let $Y_.$ be a space in $s\mathbf{T}_X$ which is constructed from schemes. We assume that it is degenerate above some simplicial degree. Then*

$$Y_. \times_X Z = C(Y_. \times_X U \subset Y_.) \times_X Z$$

and the pushout

$$H^k_{s\mathbf{T}_Z}(Y_. \times_X Z, \mathbf{K}) \to H^k_{s\mathbf{T}_X}(Y_. \text{ rel } Y_. \times_X U, \mathbf{K})$$

is an isomorphism.

Proof. For an object V of the site $\mathrm{Zar}_{\mathcal{D}_X}$ let $M(V)$ be the category of all coherent sheaves on V. In it let $P(V, \mathcal{D}_X)$ be the subcategory of those sheaves $\mathcal{F}$ satisfying

$$\underline{\mathrm{Tor}}^j_{\mathcal{O}_V}(\mathcal{O}_{V'}, \mathcal{F}) = 0$$

for all $j > 0$ and all $V' \to V$ in $\mathcal{D}_X$. Note that there are only finitely many conditions as our diagram is finite. The nice thing about $P(V, \mathcal{D}_X)$ is that it is contravariantly functorial. Hence Quillen's $\Omega BQP(\,\cdot\,, \mathcal{D}_X)$ (loop space of the classifying space of the Q-construction) defines a presheaf of simplicial sets on the site by [Q2] §7 2.5. It is here where we use the fact that all schemes are quasi-projective. Let $\Omega BQP'_X$ be the space in $s\mathbf{T}_X$ defined by its sheafification. By Quillen's Resolution Theorem ([Q2] Thm 3, Cor 3, p. 27) there is a weak equivalence of spaces $\Omega BQP'_X \to \mathbf{K}_X$. (Basically this is the fact that K'-theory and K-theory agree for regular schemes.)

We also have the space $\Omega BQP'_Z$ in $s\mathbf{T}_Z$. For the closed immersion $i : V \times_X Z \to V$ the pushout i_* is exact on the category of coherent sheaves. Because of (TC), it maps the the subcategory $P(V \times Z, \mathcal{D}_Z)$ to $P(V, \mathcal{D}_X)$. In fact we get a morphism of spaces in $s\mathbf{T}_X$

$$i_*(\Omega BQP'_Z) \xrightarrow{i_*} \Omega BQP'_X \; .$$

Using the weak equivalences to $\mathbf{K}_?$ this defines a map in the homotopy category

$$i_*(\mathbf{K}_Z) \xrightarrow{i_*} \mathbf{K}_X \; .$$

If $Y_.$ is a space in $s\mathbf{T}_X$, then we get the map in a) as

$$H^k_{s\mathbf{T}_Z}(i^{-1}Y_., \mathbf{K}_Z) \to H^k_{s\mathbf{T}_X}(i_* i^{-1} Y_., i_*\mathbf{K}_Z) \to H^k_{s\mathbf{T}_X}(Y_., \mathbf{K}_X) \; .$$

In the special case of a scheme Y part b) is nothing but Quillen's pushout isomorphism

$$K_n(i^{-1}Y) \to K_n(Y \text{ rel } Y \times_X U)$$

for regular schemes [Q2] §7 Prop. 3.2 (recall that all schemes in the site are regular). This generalizes to the case of spaces constructed from schemes by the skeletal spectral sequence. □

Lemma B.2.14. *We consider a cartesian diagram of quasi-projective smooth S-schemes*

$$\begin{array}{ccc} Z' & \xrightarrow{i'} & X' \\ {\scriptstyle f_Z}\downarrow & & {\scriptstyle f_X}\downarrow \\ Z & \xrightarrow{i} & X \end{array}$$

where i is a closed immersion. Let $\mathcal{D}_X$ be a finite diagram on X. Assume that the pullback $\mathcal{D}_{X'}$ defines a finite diagram over X' and that both i and i' satisfy (TC). We also assume that for all V in $\mathcal{D}_X$ the maps

$$V \times_X X' \to V$$

and

$$V \times_X Z \to V$$

are tor-independent.

Then for all spaces $Y_.$ in $s\mathbf{T}_X$ there is a commutative diagram

$$\begin{array}{ccc} H^k_{s\mathbf{T}_{Z'}}(f_Z^* i^* Y_., \mathbf{K}) & \xrightarrow{i'_*} & H^k_{s\mathbf{T}_{X'}}(f_X^* Y_., \mathbf{K}) \\ {\scriptstyle f_Z^*}\uparrow & & \uparrow{\scriptstyle f_X^*} \\ H^k_{s\mathbf{T}_Z}(i^* Y_., \mathbf{K}) & \xrightarrow{i_*} & H^k_{s\mathbf{T}_X}(Y_., \mathbf{K}) \end{array} .$$

Proof. We have to refine the categories $P(V, \mathcal{D}_Z)$ used in the proof of B.2.13 further. Let $P''(V, \mathcal{D}_Z)$ be the subcategory of $P'(V, \mathcal{D}_Z)$ of those coherent sheaves $\mathcal{F}$ satisfying

$$\underline{\mathrm{Tor}}^j_{\mathcal{O}_Z}(\mathcal{O}_{Z'}, \mathcal{F}) = 0 .$$

The induced space $\Omega BQP''_Z$ is again weakly equivalent to $\mathbf{K}_Z$. By [Q2] §7 2.11 there is a commutative diagram of spaces in $s\mathbf{T}_X$

$$\begin{array}{ccc} i'_* f_{X,*} \Omega BQP'_{Z'} & \longrightarrow & f_{X,*} \Omega P'_{X'} \\ \uparrow & & \uparrow \\ i_* \Omega BQP''_Z & \longrightarrow & \Omega BQP'_X \end{array} .$$

This proves the lemma. ☐

We also need the following lemma from algebraic geometry.

Lemma B.2.15. *Suppose we are given a cartesian diagram*

$$\begin{array}{ccc} Z' & \xrightarrow{i'} & X' \\ \downarrow & & \downarrow f \\ Z & \xrightarrow{i} & X \end{array}$$

of smooth S-schemes where i is a closed embedding, then the blow-up of X' in Z' is the base change by f of the blow-up of X in Z provided i and f are tor-independent.

Proof. In order to see this, note that by [EGAII] 3.5.3 we have to check that $f^*(\mathcal{I}^n) = \mathcal{I}^n \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}$ is isomorphic to $\mathcal{J}^n$ where $\mathcal{I}$ is the sheaf of ideals of Z in X and $\mathcal{J}$ the one of Z' in X'. This follows from tor-independence in the case $n = 1$. Note that in general we have a surjection $f^*\mathcal{I}^n \to \mathcal{J}^n$. Let K_n be the kernel.

Pull-back by f^* is right exact, i.e., we have an exact sequence

$$f^*\mathcal{I}^2 \to \mathcal{J} \to f^*(\mathcal{I}/\mathcal{I}^2) \to 0 \,.$$

Together with the above surjectivity this implies $f^*(\mathcal{I}/\mathcal{I}^2) \cong \mathcal{J}/\mathcal{J}^2$. As X respectively X' are regular and Z respectively Z' are locally given by regular sequences, the structural theorem [Ha] II Theorem 8.21A e) implies

$$f^*(\mathcal{I}^n/\mathcal{I}^{n+1}) \cong \mathcal{J}^n/\mathcal{J}^{n+1} \,.$$

By the snake lemma $K_{n+1} \to K_n$ is surjective and hence $f^*(\mathcal{I}^n/\mathcal{I}^{n+k}) \cong \mathcal{J}^n/\mathcal{J}^{n+k}$ for all k. But then

$$\mathcal{J}^n \cong \varprojlim \mathcal{J}^n/\mathcal{J}^{n+k} \cong \varprojlim f^*\mathcal{I}^n/\operatorname{im} f^*\mathcal{I}^{n+k} \cong \varprojlim f^*\mathcal{I}^n/\mathcal{J}^k f^*\mathcal{I}^n \cong f^*\mathcal{I}^n \,.$$

□

Push-forward is not a λ-ring morphism but it does respect the γ-filtration up to a shift, at least under good conditions. This is made precise in the following Riemann-Roch Theorem, which is a slight generalization of de Jeu's in [Jeu] 2.3.

Theorem B.2.16 (Grothendieck-Riemann-Roch). *Let S be a regular irreducible Noetherian affine scheme S. Let $i : Z \to X$ be a closed immersion of constant codimension d of quasi-projective smooth S-schemes.*

For ? = X, Z let $td(?) \in \operatorname{Gr}^*_\gamma K_0(?)_\mathbb{Q}$ be the usual Todd classes (e.g. [T] p. 135). Let a finite diagram $\mathcal{D}_X$ be given that satisfies the conditions (TC) with respect to i. Finally let $Y_.$ be a space constructed from schemes in $s\mathbf{T}_X$.

a) *The homomorphism $i_* : K_n(i^{-1}Y_.)_\mathbb{Q} \longrightarrow K_n(Y_.)_\mathbb{Q}$ has degree $-d$ with respect to the γ-filtration, i.e.,*

$$F^j K_n(i^{-1}Y_.)_\mathbb{Q} \xrightarrow{i_*} F^{j-d} K_n(Y_.)_\mathbb{Q} \quad .$$

b) *The following diagram commutes:*

$$\begin{array}{ccc} K_n(i^{-1}Y_.)_\mathbb{Q} & \xrightarrow{td(Z)ch} & \operatorname{Gr}^*_\gamma K_n(i^{-1}Y_.)_\mathbb{Q} \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i_*} \\ K_n(Y_.)_\mathbb{Q} & \xrightarrow{td(X)ch} & \operatorname{Gr}^*_\gamma K_n(Y_.)_\mathbb{Q} \end{array}$$

Remark: $td(?)$ is a unit with augmentation 1. Hence the horizontal maps in b) are isomorphisms.

Proof. We essentially have to prove classical Riemann-Roch for the inclusion $Z \longrightarrow X$. The conditions on our situation are chosen in a way that the diagrams we drag along do not make any difficulties. We want to follow de Jeu's arguments in [Jeu] 2.3 who in turn follows [T] Theorem 1.1. In contrast to them we allow more general base S than a field. However, the arguments work without any changes. Indeed, the original article [Sou4] Thm 3 treated the more general case.

Note also that we can replace $Y_.$ by the cone of $Y_. \times U \longrightarrow Y_.$, i.e., we can assume that all pushout maps are isomorphisms. Having observed this we can follow the classical case [Sou4] Thm 3. The first step is to prove the analogue of [T] Theorem 1.2 or [Jeu] Proposition 2.5 ("Riemann-Roch without denominators"). We only sketch the idea: Because of functoriality B.2.14 and the homotopy property of K'-theory we can make the transformation to the normal cone. Hence we can assume without loss of generality that i is a section of a projective bundle over Z. The existence of the projection p which is a left-inverse of i allows to make explicit calculations. All details of the argument can be found in [Jeu] 2.5 when replacing $K_0(Y_0)$ ($= K_0(X_0)$ there) by $K_0(X) = K_0(s\mathbf{T}_X)$. The necessary compatibility of blow-up and base change is guaranteed by the previous lemma.

We then show that up to multiplication with the appropriate Todd class i_* has the required behaviour with respect to Adams eigenspaces. The argument is the same as in [Jeu] Proposition 2.3 or [T] Lemma 2.2. Now the theorem follows by the same formal manipulations as in the proof of [T] Lemma 2.3. □

Corollary B.2.17. *Let $i : Z \to X$ (closed immersion of constant codimension d) and $Y_.$ be as in the theorem. Let $U = X \setminus Z$. Then there is a natural localization sequence*

$$\begin{aligned} \ldots \to K_m(Z \times_X Y_.)_{\mathbb{Q}} \to K_m(Y_.)_{\mathbb{Q}} \to K_m(U \times_X Y_.)_{\mathbb{Q}} \\ \to K_{m-1}(Z \times_X Y_.)_{\mathbb{Q}} \to \ldots \end{aligned}$$

or in terms of motivic cohomology

$$\begin{aligned} \ldots \to H^{i-2d}_{\mathcal{M}}(Z \times_X Y_., j-d) \to H^i_{\mathcal{M}}(Y_., j) \to H^i_{\mathcal{M}}(U \times Y_., j) \\ \to H^{i-2d+1}_{\mathcal{M}}(Z \times_X Y_., j-d) \to \ldots \end{aligned}$$

Proof. Part b) of Theorem B.2.16 implies that

$$i_* : \bigoplus_{j \in \mathbb{N}_0} \mathrm{Gr}^j_\gamma K_m(Y_. \text{ rel } Y_. \times U) \to \bigoplus_{j \in \mathbb{N}_0} \mathrm{Gr}^{j-d}_\gamma K_m(Y_. \times Z)$$

is an isomorphism, i.e., $H^i_{\mathcal{M}}(Y_. \text{ rel } Y_. \times U, j) \cong H^{i-2d}_{\mathcal{M}}(Z \times_X Y_., j-d)$.

We consider the long exact sequence of relative K-cohomology or relative motivic cohomology for the open embedding $U \times Y_. \subset Y_.$. We can use i_* to identify the relative cohomology with cohomology of the closed complement. □

Only a few K-groups are known. However, the ranks of the K-groups of number fields are understood.

Theorem B.2.18 (Borel). *Let K be a number field with ring of S-integers $\mathcal{O}_S$ where S is a finite set of primes of K. Let $B = \operatorname{Spec} \mathcal{O}_S$. As usual r_1 is the number of real places of K and r_2 the number of complex places. Then the motivic cohomology has the following ranks:*

$H^0_{\mathcal{M}}(B,0)$	1	
$H^1_{\mathcal{M}}(B,1)$	$\#S + r_1 + r_2 - 1$	
$H^1_{\mathcal{M}}(B,n)$	r_2	$n > 1$, *even*;
$H^1_{\mathcal{M}}(B,n)$	$r_1 + r_2$	$n > 1$, *odd*;
$H^i_{\mathcal{M}}(B,j)$	0	*else*.

Proof. The computation of $K_0(B)$ and $K_1(B)$ is classical ([Ba] Ch. IX, Prop. 3.2 and Ch. X, Cor. 3.6). The higher K-groups for the ring of integers $\mathcal{O}_K$ were calculated by Borel ([Bo1], Prop 12.2). It follows from Quillen's computation of the K-groups of finite fields that the ranks are not changed by localizing at finite primes. □

B.3 Cohomology of Abelian Sheaves

We now show how the usual cohomology theories fit in the set-up of generalized cohomology. This is well documented in the literature [BrG], [G], [Jeu]. In the case of a cohomology theory defined by a pseudo-flasque complex of presheaves $\mathcal{F}$, we compare the different possible points of view. These are Zariski-cohomology of the associated complex of sheaves, generalized cohomology of the associated space or simply cohomology of the sections. We always get the same cohomology groups (B.3.2 and B.3.4). If the complex of presheaves $\mathcal{F}$ is part of a twisted duality theory (B.3.7), we define Chern classes from K–cohomology of spaces to cohomology with coefficients in $\mathcal{F}$. Finally we check compatibility of the localization sequence in K–cohomology with the one for cohomology of spaces with coefficients in $\mathcal{F}$ (B.3.8).

By a complex we always mean a cohomological complex. Of course it can also be considered as a homological complex by inverting the signs of the indices.

The Dold-Puppe functor [M] Thm 22.4 attaches to a complex of abelian groups G which is concentrated in non-positive degrees a simplicial abelian group $K(G)$ whose homotopy groups $\pi_i K(G)$ agree with the cohomology groups $h^{-i}(G)$. It induces an equivalence between the homotopy category of simplicial abelian groups and the homotopy category of complexes of abelian groups concentrated in non-positive degrees. By construction of the functor K there is a natural weak equivalence of spaces

$$\mathrm{Cone}(K(G) \to *) \longrightarrow K(\mathrm{Cone}(G \to 0)) = K(G[1])$$

and hence a natural map $\Omega K(G[1]) \to K(G)$ in the homotopy category of pointed simplicial sets, which is a homotopy equivalence. If G is an arbitrary complex of abelian groups, let $\tau_{\leq N} G$ be the canonical subcomplex in degrees less or equal to N. We put

$$K(G)_N = K(\tau_{\leq N} G[N]) \; .$$

The natural map $\tau_{\leq N-1}G[N] \to \tau_{\leq N}G[N]$ induces

$$K(G)_{N-1} \cong \Omega K(\tau_{\leq N-1}G[N]) \to \Omega K(G)_N \ ,$$

which is a weak equivalence. This means the $K(G)_N$ form an infinite loop spectrum whose homotopy groups reflect all cohomology groups of the complex.

Definition B.3.1. *Let $\mathcal{G}$ be a cohomological complex of sheaves of abelian groups on the big Zariski site. The sheafified version of the above construction yields an infinite loop spectrum of spaces $K(\mathcal{G})$ with homotopy sheaves*

$$\underline{\pi}_i(K(\mathcal{G})) \cong h^{-i}(\mathcal{G}) \ .$$

As a spectrum $K(\mathcal{G})$ defines generalized cohomology groups with indices in $\mathbb{Z}$ for any space X.

Proposition B.3.2. *Let $\mathcal{G}$ be a bounded below complex of sheaves on the big Zariski site. Let X be a scheme. Then*

$$H^i_{s\mathbf{T}}(X, K(\mathcal{G})) \cong H^i_{\mathrm{ZAR}}(X, \mathcal{G}) \ .$$

Proof. As $\mathcal{G}$ is bounded below it has a bounded below resolution by flasque sheaves. Now the proof proceeds as in [BrG] Prop. 2. The main ingredient is that $K(\mathcal{I})$ is a fibrant space if $\mathcal{I}$ is a flasque sheaf. □

Definition B.3.3. *a) Following [BrG] a complex of abelian presheaves $\mathcal{F}$ on the big Zariski site is called pseudo-flasque if it has the Mayer-Vietoris property, i.e., for open subschemes U and V of some scheme X, we have a long exact sequence of abelian groups*

$$\begin{aligned} \ldots \to h^i\left(\mathcal{F}(U \cup V)\right) \to h^i\left(\mathcal{F}(U) \oplus \mathcal{F}(V)\right) \to h^i\left(\mathcal{F}(U \cap V)\right) \\ \to h^{i+1}\left(\mathcal{F}(U \cup V)\right) \to \ldots \end{aligned}$$

b) Let $\mathcal{F}$ be a complex of abelian presheaves. For the object $\star \amalg U$ in $\mathbf{T}$ where U is a scheme, we put

$$\mathcal{F}(\star \amalg U) = \mathcal{F}(U) \ .$$

Let X be a space constructed from schemes. Then we put

$$\mathcal{F}(X) = \mathrm{Tot}_i\, \mathcal{F}(X_i) \ .$$

the total complex of the cosimplicial complex $\mathcal{F}(X_i)_{i \in \mathbb{N}_0}$.

Taking the total complex of a bicomplex as in b) of course involves a choice of signs which we fix once and for all. Different choices of signs differ by a canonical isomorphism of the total complex.

Lemma B.3.4. *Let $\mathcal{F}$ be a bounded below pseudo-flasque complex of abelian presheaves. Let $\tilde{\mathcal{F}}$ be its sheafification. Then*

$$H^i_{s\mathbf{T}}(X, K(\tilde{\mathcal{F}})) = h^i(\mathcal{F}(X))$$

for all spaces X constructed from schemes.

Proof. Let $\mathcal{I}$ be a (bounded below) flasque resolution of $\tilde{\mathcal{F}}$. This is in particular a pseudo-flasque complex of presheaves that is quasi-isomorphic to $\mathcal{F}$ as a complex of presheaves because both compute Zariski-cohomology of $\tilde{\mathcal{F}}$. As in the proof of [BrG] Theorem 4, the simplicial sheaf $K(\mathcal{I})$ is a fibrant resolution of $K(\tilde{\mathcal{F}})$. Hence we can assume without loss of generality that $\mathcal{F}$ itself is a complex of flasque sheaves.

For the case of a scheme X the lemma is the reformulation of [BrG] Theorem 4 in the easier case of simplicial presheaves that come from a complex of abelian presheaves. In the general case

$$\begin{aligned}
H^i_{s\mathbf{T}}(X, K(\tilde{\mathcal{F}})) &= \pi_{-i} \operatorname{Hom}_{.}(X, K(\tilde{\mathcal{F}})) \\
&= \pi_{-i} \operatorname{Hom}_{.}(\operatorname{hocolim} X_j, K(\tilde{\mathcal{F}})) \\
&= \pi_{-i} \operatorname{holim} \operatorname{Hom}_{.}(X_j, K(\tilde{\mathcal{F}})) \qquad \text{[BouK] XII Prop. 4.1} \\
&= h^i(\operatorname{Tot} \mathcal{F}(X_i)) = h^i(\mathcal{F}(X)) \quad .
\end{aligned}$$

□

This means if we define a cohomology theory by a pseudo-flasque complex of presheaves on the big Zariski site we can freely change from the point of view of generalized cohomology to ordinary Zariski-cohomology or cohomology of the sections of the presheaf.

If $X \to Y$ is a morphism of schemes, we consider as usual its Čech-nerve $\operatorname{cosk}_0(X/Y)$, i.e., the simplicial Y-scheme given by

$$\operatorname{cosk}_0(X/Y)_n = (X \times_Y \cdots \times_Y X) \qquad n+1\text{-fold product}$$

with the natural boundary and degeneracy morphisms.

Definition B.3.5. *We say that a morphism $X \to Y$ of schemes has cohomological descent for the cohomology theory given by the complex of abelian Zariski-sheaves $\mathcal{G}$ if the natural morphisms*

$$H^i_{s\mathbf{T}}(Y, K(\mathcal{G})) \to H^i_{s\mathbf{T}}(\operatorname{cosk}_0(X/Y), K(\mathcal{G}))$$

are isomorphisms for all $i \in \mathbb{Z}$.

This is of course an ad hoc version of the general notion of cohomological descent.

Lemma B.3.6. *Let $j : U \to X$ be an open immersion with closed complement Y. Let $\mathcal{F}$ be a pseudo-flasque complex of presheaves on ZAR_X with sheafification $\tilde{\mathcal{F}}$.*

a) *There are natural isomorphisms*

$$H^i_{s\mathbf{T}}(X \ \mathit{rel}\, Y, K(\tilde{\mathcal{F}})) \to H^i_{\mathrm{ZAR}}(X, j_! j^* \tilde{\mathcal{F}})) \ .$$

b) *If $\tilde{Y} \to Y$ is a morphism with cohomological descent for $\tilde{\mathcal{F}}$, then we get a natural isomorphism*

$$H^i_{s\mathbf{T}}(X \ \mathit{rel} \ \operatorname{cosk}_0(\tilde{Y}/Y), K(\tilde{\mathcal{F}})) \xrightarrow{\cong} H^i_{\mathrm{ZAR}}(X, j_! j^* \tilde{\mathcal{F}}) \ .$$

Proof. By B.3.4 the left-hand side of a) is canonically isomorphic to the cohomology of

$$\mathcal{F}(C(Y \xrightarrow{i} X)) \cong \operatorname{Cone}\left(\mathcal{F}(X) \xrightarrow{\mathcal{F}(i)} \mathcal{F}(Y)\right)[-1]$$

where the right hand side is the cone in the category of cohomological complexes. We assume without loss of generality that $\tilde{\mathcal{F}}$ is a flasque complex. The key point is the short exact sequence of complexes of sheaves on X

$$0 \to j_! j^* \tilde{\mathcal{F}} \to \tilde{\mathcal{F}} \to i_* i^* \tilde{\mathcal{F}} \to 0 \ \ .$$

It induces a canonical quasi-isomorphism of complexes

$$j_! j^* \tilde{\mathcal{F}} \to \operatorname{Cone}\left(\tilde{\mathcal{F}} \to i_* i^* \tilde{\mathcal{F}}\right)[-1] \ .$$

We now take $R\Gamma_{\mathrm{Zar}}(X, \ \cdot\ )$ of the right-hand side. Because $\mathcal{F}$ was assumed to be pseudo-flasque the morphism

$$\operatorname{Cone}\left(\mathcal{F}(X) \to \mathcal{F}(Y)\right) \longrightarrow \operatorname{Cone}\left(\tilde{\mathcal{F}}(X) \to \tilde{\mathcal{F}}(Y)\right) \ .$$

is a quasi-isomorphism. This last fact follows from B.3.4 and B.3.2. (Of course it can also be proved, even more easily, in terms of complexes of abelian groups rather than simplicial abelian groups.) In the case of a morphism $\tilde{Y} \to Y$ with cohomological descent the left hand side of the statement is by B.3.4 given by the cohomology of

$$\mathrm{Cone}\left(\mathcal{F}(X) \to \mathcal{F}(\mathrm{cosk}_0(\tilde{Y}/Y)\right)[-1] \ .$$

The natural morphism $\mathcal{F}(Y) \to \mathcal{F}(\mathrm{cosk}_0(\tilde{Y}/Y))$ is a quasi-isomorphism by definition and Lemma B.3.4. □

Theorem B.3.7 (Gillet, de Jeu). *Let $\mathcal{F} = \bigoplus_{i\in\mathbb{Z}} \mathcal{F}(i)$ be a pseudo-flasque complex of abelian presheaves on the big Zariski site. Assume that $\mathcal{F}$ defines a twisted duality theory, i.e., the extra data of [G] Def. 1.1 exist and all conditions of loc. cit. Def. 1.2 are fulfilled. Then:*

- *There are Chern class maps of spaces*

  $$c_j : \mathbf{K} \to K(\tilde{\mathcal{F}}(j)[2j]) \ .$$

  They induce morphisms

  $$c_j : H^i_{s\mathbf{T}}(Y, \mathbf{K}) \to H^{i+2j}_{s\mathbf{T}}(Y, K(\tilde{\mathcal{F}}(j)))$$

  for all spaces Y in $s\mathbf{T}$.

- *If Y is a K-coherent space, then the total Chern class c_Γ is a morphism of λ-algebras on K–cohomology of Y.*

- *Let $i : Z \to X$ a closed immersion of smooth S–schemes with open complement U. The map $i_! : i_*\mathcal{F}(r) \mid_Z \to \mathcal{F}(r+d) \mid_X [2d]$ required in [G] Def. 1.2. induces push-forward on generalized cohomology. If $Y_.$ is a space over X as in B.2.16, then the diagram*

  $$\begin{array}{ccc} \mathrm{Gr}^j_\gamma K_n(Y_. \times_X Z)_\mathbb{Q} & \xrightarrow{i_*} & \mathrm{Gr}^{j+d} K_n(Y_.)_\mathbb{Q} \\ \downarrow{\scriptstyle c_j} & & \downarrow{\scriptstyle c_{j+d}} \\ H^{2j-n}_{s\mathbf{T}}(Y_. \times_X Z, K(\tilde{\mathcal{F}}(j)))_\mathbb{Q} & \xrightarrow{i_!} & H^{2j+2d-n}_{s\mathbf{T}}(Y_., K(\tilde{\mathcal{F}}(j+d)))_\mathbb{Q} \end{array}$$

  is commutative.

Proof. The construction of the Chern classes is [G] Thm 2.2. Gillet's formulation is for schemes but he constructs in fact a morphism of spaces (loc. cit. p. 225) so the results hold for more general spaces (see also [GSo1] 4.1). The assertion on the λ-ring structure is [GSo1] Thm. 7. We sketch the idea: Everything is defined on the level of coefficients, so it does not depend on Y. Compatibility with multiplication is [G] 2.3.2. Compatibility with γ-operators can be checked on the level of universal Chern classes, i.e., for elements $C_{i,N} \in H^{2i}_{s\mathbf{T}}(BGl_n, \tilde{\mathcal{F}}(i))$. Now use the splitting principle ([G] 2.4).

The last part of the proposition is a generalization of Gillet's Riemann-Roch Theorem [G] 4.1 to spaces of our special type. The proof carries over by the same method as in the proof of Riemann-Roch for K-cohomology B.2.16. Mutis mutanda the statement can be found in [Jeu] Lemma 2.13. □

Remark: This will allow to define regulator maps from K-cohomology to the cohomology theories we are interested in.

Corollary B.3.8. *Let X, Z, d, $Y_.$ and $\mathcal{F}$ be as in the theorem. In addition assume that $\mathcal{F}$ is pseudo-flasque. Let U be the complement of Y in X. We abbreviate $Y_U = Y_. \times_X U$, $Y_Z = Y_. \times_X Z$ and $F_j = K(\tilde{\mathcal{F}}(j))$. Then there is a natural morphism of long exact sequences*

$$\begin{array}{ccccccc}
H^{i-1}_{\mathcal{M}}(Y_U,j) & \longrightarrow & H^{i-2d}_{\mathcal{M}}(Y_Z,j-d) & \longrightarrow & H^i_{\mathcal{M}}(Y_.,j) & \longrightarrow & H^i_{\mathcal{M}}(Y_U,j) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{i-1}_{s\mathbf{T}}(Y_U,F_j) & \longrightarrow & H^{i-2d}_{s\mathbf{T}}(Y_Z,F_{j-d}) & \longrightarrow & H^i_{s\mathbf{T}}(Y_.,F_j) & \longrightarrow & H^i_{s\mathbf{T}}(Y_U,F_j) \\
{=}\downarrow & & {=}\downarrow & & {=}\downarrow & & {=}\downarrow \\
h^{i-1}\mathcal{F}(j)(Y_U) & \longrightarrow & h^{i-2d}\mathcal{F}(j-d)(Y_Z) & \longrightarrow & h^i\mathcal{F}(j)(Y_.) & \longrightarrow & h^i\mathcal{F}(j)(Y_U)
\end{array} .$$

Proof. We start with the long exact sequences for relative cohomology (B.1.7) with coefficients in the spectrum $\mathbf{K}$ and in the spectrum $K(\tilde{\mathcal{F}})$. Their compatibility is nothing but functoriality. Relative cohomology is replaced by cohomology of $Y_. \times_X Z$ using B.3.7. Finally we pass to graded pieces of the γ-filtration. Note that the indices in the definition of motivic cohomology are chosen in a way that they agree with the indices of other cohomology theories under Chern class maps. Equality of the last two lines is B.3.4 □

Note that the last line has nothing to do with generalized cohomology or spaces.

B.4 Continuous Etale Cohomology

There are different ways of defining continuous étale cohomology. We will see that they all give the same thing.

Fix a number field K and a prime l. Let B be an open subscheme of $\operatorname{Spec}\mathcal{O}_K[1/l]$ where $\mathcal{O}_K$ is the ring of integers of K.

Proposition B.4.1 (Deligne, Ekedahl). *Let $f : Y \longrightarrow X$ be a morphism of B-schemes of finite type. Then there are triangulated categories $D^b_c(X - \mathbb{Z}_l)$ and $D_c(Y - \mathbb{Z}_l)$ admitting the following: there is a t-structure whose heart are the constructible l-adic systems. There are functors*

$$f_!, f_* : D^b_c(Y - \mathbb{Z}_l) \longrightarrow D^b_c(X - \mathbb{Z}_l)$$

and

$$f^*, f^! : D^b_c(X - \mathbb{Z}_l) \longrightarrow D^b_c(Y - \mathbb{Z}_l)$$

having all the usual properties of Grothendieck functors.

Proof. This is [Ek] Thm 6.3. In the case $B = \operatorname{Spec}\mathcal{O}_K[1/l]$ the category was already constructed in [D4], 1.1.2. □

Remark: $D^b_c(X-\mathbb{Z}_l)$ should be thought of as the bounded derived categories of constructible l-adic sheaves on X_{et}. By Ekedahl's construction $D^b_c(X-\mathbb{Z}_l)$ is a subcategory of a localization of a subcategory of the derived category of the abelian category $(X_{et})^{\mathbb{N}} - \mathbb{Z}_l$. By this notation Ekedahl means the category of projective systems of étale sheaves on X ringed by the projective system $\mathbb{Z}/l^n$. The four functors are defined on the level of this last derived category. Ekedahl then shows that they induce well-defined functors on $D^b_c(X-\mathbb{Z}_l)$. In the case B open in $\operatorname{Spec}\mathcal{O}_K[1/l]$, we get away with Deligne's more straightforward construction.

Definition B.4.2 (1. Version). **a)** *For $k \in \mathbb{Z}$ let $\mathbb{Z}_l(k)$ be the constructible l-adic sheaf on B given by the projective system $\mu_{l^n}^{\otimes k}$.*

b) *We define continuous étale cohomology of $s : X \longrightarrow B$ by*

$$H^i_{cont}(X, k) = \operatorname{Hom}_{D^b_c(X-\mathbb{Z}_l)}(s^*\mathbb{Z}_l(0), s^*\mathbb{Z}_l(k)[i]) \quad .$$

c) *If $j : U \longrightarrow X$ is an open immersion with complement Y we define relative continuous étale cohomology by*

$$H^i_{cont}(X \ \mathit{rel}\ Y, k) = \operatorname{Hom}_{D^b_c(X-\mathbb{Z}_l)}(s^*\mathbb{Z}_l(0), j_!(s\circ j)^*\mathbb{Z}_l(k)[i]) \quad .$$

d) *More generally, let $\mathcal{M}$ be an object of $D^b_c(X - \mathbb{Z}_l)$. We define continuous étale cohomology of X with coefficients in $\mathcal{M}$ as*

$$H^i_{cont}(X, \mathcal{M}) = \mathrm{Hom}_{D^b_c(X-\mathbb{Z}_l)}(s^*\mathbb{Z}_l(0), \mathcal{M}[i]) \quad .$$

This definition allows to derive all the usual spectral sequences from the calculus of the Grothendieck functors.

Remark: As checked in [H2] §4 this definition coincides with Jannsen's original one in [Jn1] sect. 3. In our case continuous étale cohomology with coefficients in a constructible l-adic sheaf $(\tilde{\mathcal{F}}_n)_n$ is nothing but the naive $\varprojlim H^n_{et}(X, \tilde{\mathcal{F}}_n)$ because all $H^n_{et}(X, \tilde{\mathcal{F}}_n)$ are finite.

Let us now define continuous étale cohomology in a way that fits in the setting of the previous section.

Definition B.4.3 (2. Version). *Consider the projective system of sheaves $\left(\mu_{l^n}^{\otimes k}\right)_{n\in\mathbb{N}}$ on the big étale site over B. Let $\mathcal{I}$ be an injective resolution in the category of projective systems. It is given by a projective system $\mathcal{I}_n$ of injective resolutions of $\mu_{l^n}^{\otimes j}$ on the big étale site with split surjective transition morphisms ([Jn1] 1.1). By taking sections we get a projective system of complexes of Zariski-presheaves $R\Gamma\left(\mu_{l^n}^{\otimes j}\right)_{n\in\mathbb{N}}$. The functor $R\varprojlim$ turns it into a complex $\mathcal{F}_l(k)$ of Zariski-presheaves. For any space X put*

$$H^i_{cont}(X, k) = H^i_{s\mathbf{T}}(X, K(\tilde{\mathcal{F}}_l(k))) \quad .$$

In particular if $\iota : Y \to X$ is a morphism of spaces, then we put

$$H^i_{cont}(X \text{ rel } Y, k) = H^i_{s\mathbf{T}}(C(\iota), K(\tilde{\mathcal{F}}_l(k))) \quad .$$

Lemma B.4.4. *If X is a B-scheme, then both versions of the definition of continuous étale cohomology agree canonically. If $Z \to X$ is a closed immersion, then the same is true for both definitions of relative continuous étale cohomology.*

Proof. $\mathcal{F}_l(X)$ is nothing but an explicit version of the derived functor $R\varprojlim R\Gamma(X, \cdot)$ from the derived category of projective systems of étale sheaves to the derived category of abelian groups. Hence the complex $\mathcal{F}_l(X)$ computes the first version of continuous étale cohomology. In particular it has the Mayer-Vietoris property. Hence we can apply the lemmas of the previous section (B.3.4) and get

$$H^i_{s\mathbf{T}}(X, K(\tilde{\mathcal{F}}_l)) = h^i(\mathcal{F}_l(X)) \quad .$$

To extend the result to relative étale cohomology we use essentially the same argument as in B.3.6.b). □

Remark: When we say that the isomorphism is canonical, we think in particular of the following situation: The cartesian diagram of schemes

$$\begin{array}{ccccc} U' & \xrightarrow{j'} & X' & \xleftarrow{i'} & Y' \\ \downarrow & & f\downarrow & & \downarrow \\ U & \xrightarrow{j} & X & \xleftarrow{i} & Y \\ \uparrow & & g\uparrow & & \uparrow \\ U'' & \xrightarrow{j''} & X'' & \xleftarrow{i''} & Y'' \end{array}$$

(f, j open, g, i closed complements) induces a map

$$H^i_{cont}(X \text{ rel } Y, n) \xrightarrow{f^*} H^i_{cont}(X' \text{ rel } Y', n) \quad ,$$

which is compatible with the identification. If all schemes are smooth and X'' intersects Y transversally, then we also get the same long exact sequence

$$\cdots \longrightarrow H^{i-2d}_{cont}(X'' \text{ rel } Y'', n-d) \longrightarrow H^i_{cont}(X \text{ rel } Y, n) \longrightarrow H^i_{cont}(X' \text{ rel } Y', n)$$
$$\longrightarrow H^{i+1-2d}_{cont}(X'' \text{ rel } Y'', n-d) \longrightarrow \cdots$$

using either definition of relative cohomology. If the Y's are not smooth, we might still be able to use the following lemma.

Lemma B.4.5. *If $\tilde{Y} \longrightarrow Y$ is a proper covering (i.e., a proper and surjective map), then it has cohomological descent for continuous étale cohomology. In particular if $Y \longrightarrow X$ is a closed embedding and $\tilde{Y}$ a proper covering of Y, then there is a natural isomorphism*

$$H^i_{cont}(X \textit{ rel } Y, j) \longrightarrow H^i_{cont}(X \textit{ rel } \operatorname{cosk}_0(\tilde{Y}/Y), j)$$

where the right hand side is taken in the sense of spaces.

Proof. Cohomological descent is a consequence of the same descent for étale cohomology with torsion coefficients prime to the characteristic of the schemes ([SGA4,II], Exp. Vbis, 4.1.6). By B.3.6.b) the second part follows. $\square$

Proposition B.4.6. *On the Zariski site of smooth schemes over B, the presheaf $\tilde{\mathcal{F}}_l$ has the properties of a twisted duality theory. There are regulator maps from K-cohomology to continuous étale cohomology*

$$H^i_{\mathcal{M}}(Y,j) \to H^i_{cont}(Y,j)$$

for all K–coherent spaces Y. They are compatible with pullback, i.e., if $f : Y \to Y'$ is a map of K–coherent spaces, we get commutative diagrams

$$\begin{array}{ccc} H^i_{\mathcal{M}}(Y',j) & \xrightarrow{f^*} & H^i_{\mathcal{M}}(Y,j) \\ {\scriptstyle c_j}\downarrow & & \downarrow{\scriptstyle c_j} \\ H^i_{cont}(Y',j) & \xrightarrow{f^*} & H^i_{cont}(Y,j) \end{array} .$$

If $i : Z \to X$ is a closed immersion of smooth schemes (constant codimension d) with open complement U and $Y_.$ a space constructed form schemes over X as in B.2.16, then the regulator is compatible with pushout, i.e., the diagram

$$\begin{array}{ccc} H^{n-2d}_{\mathcal{M}}(Y_. \times_X Z, j-d) & \xrightarrow{i_*} & H^n_{\mathcal{M}}(Y_., j) \\ {\scriptstyle c_{j-d}}\downarrow & & \downarrow{\scriptstyle c_j} \\ H^{n-2d}_{cont}(Y_. \times_X Z, j-d) & \xrightarrow{i_!} & H^n_{cont}(Y_., j) \end{array}$$

is commutative.

Proof. We restrict to smooth schemes for simplicity. We have to define the extra-structure from [G] 1.1 and 1.2. We put

$$H_i(X,j) = H^{2d-i}_{cont}(X, d-j)$$

for a d-dimensional smooth connected scheme. Pull-back on cohomology and pushout on homology are induced from the functors on sheaves on the étale site. We do not work out the details. For a single étale sheaf μ_{l^n} this is actually one of Gillet's examples 1.4 (iii). $\square$

There is really only one case when this regulator is understood.

Lemma B.4.7. *Let K be a number field, $\mathcal{O}_K$ be its ring of integers and l an odd prime. Assume $2i-k \geq 2$, then Soulé's l-adic regulator from $K_{2i-k}(\mathcal{O}_K[1/l]) \otimes \mathbb{Z}_l$ to $H^k_{cont}(\operatorname{Spec} \mathcal{O}_K[1/l], i)$ agrees with the one obtained from Prop. B.4.6.*

Proof. Put $A = \mathcal{O}_K[1/l]$. Soulé's definition in [Sou2] is the composition

$$K_{2i-k}(A) \to \varprojlim K_{2i-k}(A, \mathbb{Z}/l^\nu) \xrightarrow{\varprojlim \overline{c}_{i,k}} \varprojlim H^k_{et}(A, \mathbb{Z}/l^\nu(i))$$

where $\overline{c}_{i,k}$ is as in [Sou1] II.2.3.

There is a natural map of presheaves $\mathcal{F}_l(i) \to R\Gamma(\,\cdot\,, \mathbb{Z}/l^\nu(i))$. Hence in Gillet's definition of Chern classes, we get a commutative diagram

$$\begin{array}{ccc} K_{2i-k}(A) & \xrightarrow{c_i} & H^k_{cont}(\operatorname{Spec} A, i) \\ & \searrow & \downarrow \\ & & H^k_{et}(\operatorname{Spec} A, \mathbb{Z}/l^\nu(i)) \end{array} .$$

Hence we only have to consider finite coefficients. Furthermore in this simple case of a regular commutative ring, we do not really need to consider the sheafified versions and generalized cohomology. Gillet's construction boils down to a composition of the Hurewicz-map with universal Chern classes. For $2i-k \geq 2$, the map $\overline{c}_{i,k}$ is defined by the same type of composition ([Sou2] II 2.3.) with the same universal Chern classes.
By the definition of K-theory with coefficients, we have a commutative diagram (loc. cit. II.2.2) with $X = \mathbb{Z}_\infty BGl(A)$:

$$\begin{array}{ccccccccc} \longrightarrow & \pi_n(X) & \xrightarrow{\times l^\nu} & \pi_n(X) & \longrightarrow & \pi_n(X, \mathbb{Z}/l^\nu) & \longrightarrow & \\ & \downarrow h & & \downarrow h & & \downarrow h_q & & \\ \longrightarrow & H_n(X, \mathbb{Z}) & \xrightarrow{\times l^\nu} & H_n(X, \mathbb{Z}) & \longrightarrow & H_n(X, \mathbb{Z}/q) & \longrightarrow & . \end{array}$$

□

Theorem B.4.8 (Soulé). *Let K be a number field, $\mathcal{O}_K$ be its ring of integers and l any prime. Let S' be a finite set of prime ideals of $\mathcal{O}_K$ and $S = S' \cup \{l\}$. Let $\mathcal{O}_S$ be the localization of $\mathcal{O}_K$ at S. The regulator map*

$$c_j : H^i_{\mathcal{M}}(\operatorname{Spec} \mathcal{O}_{S'}, j) \otimes_{\mathbb{Q}} \mathbb{Q}_l \to H^i_{cont}(\operatorname{Spec} \mathcal{O}_S, j)_{\mathbb{Q}}$$

is always injective and an isomorphism for $i = 1$ and $j > 1$. We have the following behaviour for pairs of indices (i, j):

$(0, j)$	$j \in \mathbb{Z}$	*isomorphism*
$(1, j)$	$j < 1$	*mot. coh. vanishes, l-adic does not in general*
$(1, 1)$		*injective of finite codimension*
$(1, j)$	$j > 1$	*isomorphism*
$(2, j)$	$j < 1$	*conjectured to be isom., i.e., etale coh. to vanish*
$(2, 1)$		*injective of finite codimension*
$(2, j)$	$j > 1$	*isomorphism, i.e., both vanish*
(i, j)	*else*	*both vanish*

Proof. We have

$$H^i_{cont}(\operatorname{Spec}\mathcal{O}_S, j)_{\mathbb{Q}} = H^i(G_S, \mathbb{Q}_l(j))$$

where G_S is the Galois group of the maximal extension of K that is unramified outside of S. We first check that these groups vanish for $i > 2$: By [Mi] I Cor. 4.15 all $H^i(G_S, \mu_{l^n}^{\otimes j})$ are finite. This means that the projective systems for varying n are Artin-Rees. We do not get a $\varprojlim^1$-contribution to continuous cohomology. Moreover, by loc. cit. I. 4.10.c) the $H^i(G_S, \mu_{l^n}^{\otimes j})$ for $i \geq 3$ are 2-torsion. This implies that their projective limit is 2-torsion. In total we have vanishing cohomology $H^i(G_S, \mathbb{Q}_l(j))$ for $i \geq 3$.

The case $i = 0$ is trivial. $H^1(G_S, \mathbb{Q}_l(1)) = E_S \otimes \mathbb{Q}_l$ where E_S are the S-units, while $H^1_{\mathcal{M}}(\operatorname{Spec}\mathcal{O}_{S'}, 1) = \mathcal{O}^*_{S'} \otimes \mathbb{Q}_l$.
For $H^2(G_S, \mathbb{Q}_l(1))$ (the S-Brauer-group) the codimension is the same as in the $(1, 1)$–case by Euler-Poincaré duality (cf. the discussion in [Jn2] Lemma 2 and Cor. 1.).

In the remaining cases, neither motivic (B.2.18) nor continuous étale cohomology ([Jn3] Lemma 4) is changed by the inversion of S', at least up to torsion. We assume $S' = \emptyset$. For odd l, the cases $(1, j)$ and $(2, j)$ for $j > 1$ are Soulé's result in [Sou2] Theorem 1. Note that we are in the range where the previous lemma applies.
For $l = 2$, we have to refine the argument. Assume the ranks of the $\mathbb{Q}_2$-vector spaces still have the right dimension. If K contains $\sqrt{-1}$, then [DwF] Theorem 8.7 and the succeeding remark give surjectivity even for $l = 2$. On the level of $\mathbb{Q}_2$-coefficients we can pass from $K(\sqrt{-1})$ to K by taking Galois-invariants – note that the only prime which can possibly ramify in

this quadratic extension has been inverted, and hence we get an étale extension of rings.

To conclude, we need to show that the $\mathbb{Q}_2$–vector spaces have the right dimension. Let $j > 1$. By [Jn2], proof of Lemma 1, the dimension of

$$H^i_{cont}(\operatorname{Spec}\mathcal{O}_K[1/2], j)_{\mathbb{Q}}$$

equals the corank of

$$H^i_{cont}(\operatorname{Spec}\mathcal{O}_K[1/2], \mathbb{Q}_2/\mathbb{Z}_2(j))\,.$$

By [Sou3], 1.2 and Proposition 2, this corank, for $i = 1$, equals the rank of the K–group if and only if

$$H^2_{cont}(\operatorname{Spec}\mathcal{O}_K[1/2], \mathbb{Q}_2/\mathbb{Z}_2(j))$$

is torsion. This in turn follows from [Sou1], Théorème 5, whose proof can be modified to give an analogous statement for $l = 2$, with $k!$ possibly replaced by $2^m k!$. □

B.5 Absolute Hodge Cohomology

Let $B = \operatorname{Spec}\mathbb{C}$ or $B = \operatorname{Spec}\mathbb{R}$ in this section.

In A.1.9 a definition of absolute Hodge cohomology and relative cohomology for general varieties over $\mathbb{C}$ was given. The variant over $\mathbb{R}$ was A.2.6.

By A.1.10 resp. A.2.7 absolute Hodge cohomology of smooth varieties is given functorially by Beilinson's complexes $R\Gamma_{\mathfrak{H}^p}(\,\cdot\,/B, n)$.

Lemma B.5.1. *These form a pseudo-flasque presheaf on the Zariski site of smooth B-schemes.*

Proof. By construction [B1] they form a presheaf on pairs $(U, \overline{U})$ where $\overline{U}$ is a compactification with complement an NC-divisor. (For more details cf. [H1] Prop. 8.3.3.) Taking the limit over all choices of $\overline{U}$ we get the desired presheaf. To say it is pseudo-flasque means that absolute Hodge cohomology has the Mayer-Vietoris property. In the context of A.1.9 and A.2.6 it is a formal consequence of the existence of triangles $(i_*i^!, id, j_*j^*)$ for open immersions j with closed complement i. In the context of [B1] it follows from the Mayer-Vietoris property of De Rham-cohomology and singular cohomology. □

We now consider the corresponding generalized cohomology.

Definition B.5.2 (2. Version). *If X is a space over B, then we define absolute Hodge cohomology by*

$$H^i_{\mathfrak{H}^p}(X/B,n) = H^i_{s\mathbf{T}}(X, K(\widetilde{R\Gamma}_{\mathfrak{H}^p}(\,\cdot\,/B,n))\ .$$

If $f : Z \longrightarrow X$ is a morphism of spaces, then we define relative cohomology

$$H^i_{\mathfrak{H}^p}(X\ rel\ Z/B,n) = H^i_{s\mathbf{T}}(\mathrm{Cone}(f), K(\widetilde{R\Gamma}_{\mathfrak{H}^p}(\,\cdot\,/B,n))\ .$$

Lemma B.5.3. *There is a functorial isomorphism between both definitions of absolute Hodge cohomology for a smooth variety X.*

If $Y \longrightarrow X$ is a closed immersion of smooth schemes, then the same is true for relative cohomology.

Proof. Lemma B.3.4 and Lemma B.3.6.a). □

In order to get the same equalities at least for some singular varieties we have to check a descent property for Hodge modules. For this we need functoriality of i_*i^* with values in complexes of Hodge modules rather than objects in the derived category.

Lemma B.5.4. *Let $X/\mathbb{C}$ be smooth and $i : Y \longrightarrow X$ a closed reduced subscheme of pure codimension 1. Let $Y = \bigcup_{i=0}^n Y_i$. For $I \subset \{0,\dots,n\}$ and $M \in \mathrm{MHM}_F(X)$ let*

$$\begin{aligned} i_I : Y_I &= \bigcap_{i\in I} Y_i \longrightarrow X \\ j_I : U_I &= X \smallsetminus \bigcup_{i\in I} Y_i \longrightarrow X \\ M_I &= j_{I!}j_I^* M \in \mathrm{MHM}_F(X)\ . \end{aligned}$$

All Y_I are equipped with the reduced structure. Then $i_{I*}i_I^*M$ defines a functor

$$\{\textit{subsets of } \{0,\dots,n\}\} \longrightarrow C^b(\mathrm{MHM}_F(X))\ .$$

Proof. As j_I is affine both j_I^* and $j_{I!}$ map Hodge modules to such. Note that locally each Y_i is given by a function f_i on X. The functor i_*i^* has an explicit description for closed subschemes of the type Y_I given in the proof of [S2] Prop. 2.19. In fact

$$i_{I*}i_I^*M = \ldots \to \bigoplus_{I'\subset I;|I'|=2} M_{I'} \to \bigoplus_{I'\subset I;|I'|=1} M_{I'} \to M$$

where the complex sits in degrees less or equal to zero. □

Proposition B.5.5. *Let $X/\mathbb{C}$ be smooth and $Y \to X$ a closed subscheme as in the lemma. Let $\tilde{Y} = Y_0 \amalg \cdots \amalg Y_n$ and*

$$\tilde{Y}_\cdot = \operatorname{cosk}_0(\tilde{Y}/Y) \xrightarrow{s} Y \ ,$$

i.e.,

$$\tilde{Y}_k = \tilde{Y} \times_Y \cdots \times_Y \tilde{Y} \qquad k+1 \textit{ factors} \ .$$

Then the functor s_*s^* defined by the total complex of the cosimplicial complex $(s_{n*}s_n^*)_{n\in\mathbb{N}_0}$ is isomorphic to i_*i^*.

Proof. Note that

$$\tilde{Y}_k = \coprod_{I\in\{1,\ldots,n\}^{k+1}} Y_I$$

where $Y_I = Y_{\{i_0,\ldots,i_k\}}$ in the notation of the previous lemma. Let M be in $\mathrm{MHM}_F(X)$. By the previous lemma we get indeed a cosimplicial complex hence s_*s^*M is a well-defined complex of Hodge modules. Let $\tilde{Y}_\cdot^{\le}$ be the simplicial subscheme given by

$$\tilde{Y}_k^{\le} = \coprod_{I=(i_0\le i_1\le\cdots\le i_k)} Y_I \xrightarrow{s_k^{\le}} Y \ .$$

By the Hodge module version of the combinatorial Lemma B.6.2, the morphism $s_*s^*M \to s_*^{\le}s^{\le *}M$ is a quasi-isomorphism. By definition ([S2] 2.19)

$$i_*i^*M = M_{\{0,\ldots,n\}} \to M \ ,$$

and this complex is canonically quasi-isomorphic to the total complex of the constant cosimplicial complex i_*i^*M. It is easy to see that the natural morphism

$$\operatorname{Tot} i_*i^*M \to s_*^{\le}s^{\le *}M$$

is a quasi-isomorphism. □

Corollary B.5.6. *Let X/B be smooth. Suppose $Y \to X$ is an NC-divisor over B all of whose irreducible components are smooth over B. Then $H^i_{\mathfrak{H}^p}(Y/B, j)$ as defined in A.1.9 resp. A.2.6 is isomorphic to the generalized cohomology group $H^i_{\mathfrak{H}^p}(\tilde{Y}_{\cdot}/B, j)$ and to the same noted group in [B1].*

Proof. The condition on Y makes sure that $\tilde{Y}_{\cdot}$ is indeed a smooth simplicial scheme and hence gives rise to a space over B. Cohomological descent for the coefficients as in B.5.5 implies cohomological descent for their global sections in the sense of B.3.5. We can use $\tilde{Y}_{\cdot}$ as the smooth proper hypercovering needed in Beilinson's definition. Equality to the generalized cohomology version is again B.3.4. □

This is of course cohomological descent for a closed Čech-covering. We have restricted to this case which is built into the very definition of Hodge modules for simplicity. There is no reason why there should not be cohomological descent in the same generality as for constructible sheaves.

Lemma B.5.7. *Let X/B be smooth, and $Z \subset X$ a closed immersion of an NC-divisor all of whose irreducible components are smooth over B. Let $\tilde{Z}_{\cdot}$ be the smooth simplicial scheme of B.5.5, then there is a canonical isomorphism*

$$H^i_{\mathfrak{H}^p}(X \ \mathit{rel}\ Z/B, n) = H^i_{\mathfrak{H}^p}(X \ \mathit{rel}\ \tilde{Z}_{\cdot}/B, n)$$

where we use the original definition on the left and the second on the right.

Proof. This follows by the general method of B.3.6.b) from the descent property that we have just established. □

Remark: If we had checked cohomological descent in general, then we would get B.5.6 for arbitrary varieties and B.5.7 for arbitrary closed immersions.

Theorem B.5.8 (Beilinson). *On the site of smooth schemes over B, the presheaves $R\Gamma_{\mathfrak{H}^p}(\,\cdot\,/B, n)$ have the properties of a twisted duality theory. There are regulator maps from K-cohomology to absolute Hodge cohomology*

$$H^i_{\mathcal{M}}(Y, j) \to H^i_{\mathfrak{H}^p}(Y/B, j)$$

for all K–coherent spaces Y. They are compatible with pullback, i.e., if $f : Y \to Y'$ is a map of K–coherent spaces, we get commutative diagrams

$$\begin{array}{ccc} H^i_{\mathcal{M}}(Y', j) & \xrightarrow{f^*} & H^i_{\mathcal{M}}(Y, j) \\ {\scriptstyle c_j}\downarrow & & \downarrow{\scriptstyle c_j} \\ H^i_{\mathfrak{H}^p}(Y'/B, j) & \xrightarrow{f^*} & H^i_{\mathfrak{H}^p}(Y/B, j) \end{array} .$$

If $i : Z \to X$ is a closed immersion of smooth schemes (constant codimension d) with open complement U and $Y_.$ a space constructed form schemes over X as in B.2.16, then the regulator is compatible with pushout, i.e., the diagram

$$\begin{array}{ccc} H^{n-2d}_{\mathcal{M}}(Y_. \times_X Z, j-d) & \xrightarrow{i_*} & H^n_{\mathcal{M}}(Y_., j) \\ \downarrow{\scriptstyle c_{j-d}} & & \downarrow{\scriptstyle c_j} \\ H^{n-2d}_{\mathfrak{H}^p}(Y_. \times_X Z/B, j-d) & \xrightarrow{i_!} & H^n_{\mathfrak{H}^p}(Y_./B, j) \end{array}$$

is commutative.

Proof. [B1] 5.5. We can also use Gillet's method B.3.7. All axioms hold e.g. [H1] Ch. 15. □

Theorem B.5.9 (Borel). *Let K be a number field with r_1 real and r_2 pairs of complex embeddings into $\mathbb{C}$. We consider the ring of integers $\mathcal{O}_K$ as a scheme over $\mathbb{Z}$. Then the Beilinson regulator*

$$H^i_{\mathcal{M}}(\operatorname{Spec}\mathcal{O}_K, j) \otimes_{\mathbb{Q}} \mathbb{R} \to H^i_{\mathfrak{H}^p}((\operatorname{Spec}\mathcal{O}_K)_{\mathbb{R}}/\mathbb{R}, j)$$

is an isomorphism for all pairs $(i, j) \neq (0, 0), (1, 1)$. It is injective of codimension $r_1 + r_2 - 1$ for $(i, j) = (0, 0)$, and injective of codimension one in the case $(i, j) = (1, 1)$.

Proof. Note that the cohomological dimension of the category of Hodge structures is 1. The case $i = 0$ is trivial, and the case $(1, 1)$ is Dirichlet's classical result.
In [Bo2], our claim (and much more) is proved for the Borel regulator instead of the Beilinson regulator. By [R], Corollary 4.2, the two regulators coincide up to a non–vanishing rational factor. □

B.6 A Combinatorial Lemma

This section gives a purely combinatorial proof why two conceivable definitions of the Čech-nerve of a covering are homotopically equivalent. This is well-known at least for open coverings and Čech-cohomology (and probably in general). But for lack of finding an appropriate reference we work out the combinatorics here.

Let $C(n)$ be the following simplicial set:

$$C(n)_k = \{1, \ldots, n\}^{k+1}$$

with the obvious face and degeneracy maps. Let $C(n)^{\leq}$ be the simplicial subset of simplices whose entries are ordered by $\leq$. In fact this is the simplicial version of the n-simplex.

Suppose we are given a covariant functor from the category of subsets of $\{1, \dots, n\}$ to the category of sets. We get simplicial sets by setting

$$A(n)_k = \dot{\bigcup}_{I \in C(n)_k} A_I$$
$$A(n)_k^{\leq} = \dot{\bigcup}_{I \in C(n)_k^{\leq}} A_I$$

where A_I is the value of our functor on the set $I = \{i_0, \dots, i_k\}$. Note that the elements of $C(n)_k$ are ordered tuples but the value of A_I does not depend on the ordering.

Lemma B.6.1. *If the functor has constant value A, then both simplicial sets have the homotopy*

$$\pi_i\left(A(n)^?_\cdot, \star\right) = \begin{cases} A & \text{if } i = 0, \\ 0 & \text{else.} \end{cases}$$

Proof. Obviously it is enough to consider the case $A = \star$, i.e., of the simplicial sets $C(n)^{\leq} \to C(n)$ themselves. Both simplicial sets satisfy the extension condition [M] 1.3 rather trivially. Hence we can use the combinatorial computation of the homotopy groups given in [M] Def. 3.6. We immediately get the result. □

Proposition B.6.2. *For a general functor A the injection $A(n)^{\leq} \to A(n)$ of simplicial sets is a weak homotopy equivalence.*

Proof. We filter the simplicial sets $C(n)^?$ by the simplicial subsets $F^iC(n)^?$ of simplices in which at most i different integers occur. This induces a filtration of the simplicial sets $A(n)^?$. Let $G^iA(n)^?$ be the cofibre of the cofibration $F^{i-1}A(n)^? \subset F^iA(n)^?$. It consists of simplices in which precisely i different integers occur. We argue by induction on i for all functors A at the same time. There is a long exact homotopy sequence attached to the cofibration sequence

$$F^{i-1}A(n)^? \to F^iA(n)^? \to G^iA(n)^? \; .$$

By induction it suffices to show that all cofibres $G^iA(n)/(G^iA(n)^{\leq})$ are weakly equivalent to the final object $\star$.

The cofibre decomposes into a union of simplicial sets corresponding to a different choice of i elements in $\{1,\ldots,n\}$ each. If suffices to prove acyclicity for one choice e.g for the subset $\{1,\ldots,i\}$. Hence we only have to consider $G^iA(i)/G^iA(i)^{\leq}$. But this last cofibre is isomorphic to $G^iB(i)/G^iB(i)^{\leq}$ where B is the functor with constant value $A_{\{1,\ldots,i\}}$. For $i>1$ it is easy to see that $\pi_0 G^iB(i)/G^iB(i)^{\leq} = \star$. By B.6.1 the quotients $B(n)/B(n)^{\leq}$ are acyclic for all n. Using the same cofibration sequence as for A and the inductive hypothesis this implies that all $G^iB(i)/G^iB(i)^{\leq}$ are acyclic. □

Note that A could also be a functor to the category of abelian groups or to the dual of the category of abelian groups.

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