THE FINE STRUCTURE OF THE KASPAROV GROUPS II: RELATIVE QUASIDIAGONALITY

C. L. Schochet

Mathematics Department
Wayne State University
Detroit, MI 48202

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Abstract. Quasidiagonality was introduced by P.R. Halmos for operators and quickly generalized to C^*-algebras. D. Voiculescu asked how quasidiagonality (in its various forms) was related to topological invariants. N. Salinas systematically studied the topology of the Kasparov groups $KK_*(A,B)$ and showed that this topology is related to relative quasidiagonality.

In this paper we identify $QD_*(A,B)$, the quasidiagonal elements in $KK_*(A,B)$, in terms of $K_*(A)$ and $K_*(B)$, and we use these results in various applications.

Here is our central result. Let $\mathcal{N}$ denote the bootstrap category.

Theorem. *Suppose that $A \in \mathcal{N}$ and A is quasidiagonal relative to B. Then there is a natural isomorphism*

$$QD_*(A,B) \cong Pext^1_{\mathbb{Z}}(K_*(A), K_*(B))_{*-1}.$$

Thus for $A \in \mathcal{N}$ relative quasidiagonality is indeed a topological invariant. We also settle a question raised by L.G. Brown on the relation between relative quasidiagonality and the kernel of the natural map

$$\theta^* : Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \longrightarrow Ext^1_{\mathbb{Z}}(K_*(A)_t, K_*(B)).$$

where $K_*(A)_t$ denotes the torsion subgroup of $K_*(A)$. Finally we establish a converse to a theorem of Davidson, Herrero, and Salinas, giving conditions under which the quasidiagonality of $A/\mathcal{K}$ implies the quasidiagonality of A.

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1. Introduction: Quasidiagonality and KK-theory

This is the second in a series of papers in which the topological structure of the Kasparov groups, systematically studied first by Salinas, is developed and put to use. The first paper [S:FI] is devoted to general structural results and serves as the theoretical background for the present work, which centers about quasidiagonality. From the point of view of [S:FI], this is an exploration of the closure of zero in the Kasparov groups, which we have labeled the *fine structure* subgroup.

Quasidiagonality was defined by P.R. Halmos [H] in 1970. A bounded operator on Hilbert space is *quasidiagonal* if it is a compact perturbation of a block-diagonal operator. This soon was generalized to C^*-algebras. Quasidiagonality is thus a finite dimensional approximation property. It is not well understood.

L.G. Brown, R.G. Douglas, and P.A. Fillmore [BDF] first recognized that the study of quasidiagonality for operators and for C^*-algebras might be approached by topological methods. They topologized their functor $\mathcal{E}xt(X)$ (which is known now to be $KK_1(C(X), \mathbb{C})$) and announced that the closure of zero corresponded to the quasidiagonal extensions (cf. 3.1 below). L.G. Brown pursued this theme, particularly in [B] (cf. §§6,7).

Salinas [Sal] studied the topology on the Kasparov groups $KK_*(A, B)$ and showed that this topology is related to relative quasidiagonality. The quasidiagonal elements $QD_*(A, B)$ (defined precisely in §3) constitute a certain graded subgroup of KK-theory:

$$QD_*(A, B) \subseteq KK_*(A, B).$$

If A is in the bootstrap category[1] $\mathcal{N}$ and B has a countable approximate unit then more can be said. The bootstrap hypothesis implies that the Universal Coefficient Theorem [RS] holds. The UCT is a natural short exact sequence

$$0 \to Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \xrightarrow{\delta} KK_*(A, B) \xrightarrow{\gamma} Hom_{\mathbb{Z}}(K_*(A), K_*(B)) \to 0 \tag{1.1}$$

which splits unnaturally and thus computes $KK_*(A, B)$ in terms of $K_*(A)$ and $K_*(B)$. In particular, it identifies a canonical subgroup of $KK_*(A, B)$, namely

$$Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \overset{\delta}{\cong} Ker(\gamma) \hookrightarrow KK_*(A, B)$$

Henceforth we suppress mention of the map δ.[2]

Salinas has shown [Sal Lemma 5.1] that

$$QD(A, B)_* \subseteq Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))_{*-1} \tag{1.2}$$

[1] The bootstrap category $\mathcal{N}$ [S:II, RS] is the smallest full subcategory of separable nuclear C^*-algebras which contains all separable Type I C^*-algebras and which is closed under strong Morita equivalence, direct limits, extensions, and crossed products by $\mathbb{Z}$ and $\mathbb{R}$. We may also require that if J is an ideal of A and if J, $A \in \mathcal{N}$ then so is A/J, and if A and A/J are in $\mathcal{N}$ then so is J.

[2] Note, though, that the map δ has degree one and so the elements of degree zero in the group $Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))$, denoted by $Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))_0$, are contained in $KK_1(A, B)$.

and in fact [Sal, Theorem 5.2a as reformulated by M. Dadarlat (private communication)] that in general,

$$QD(A,B)_* \subseteq Pext^1_{\mathbb{Z}}(K_*(A),K_*(B))_{*-1} \tag{1.3}$$

where $Pext^1_{\mathbb{Z}}(G,H)$ is the subgroup of $Ext^1_{\mathbb{Z}}(G,H)$ consisting of pure extensions.[3]

In the first paper of this series [S:FI, Theorem 6.11] we demonstrated that if $A \in \mathcal{N}$ then $KK_*(A,B)$ breaks apart as topological groups into three direct summands, one of which is the group

$$Pext^1_{\mathbb{Z}}(K_*(A),K_*(B))$$

and we showed there that the relative topology on this group induced by regarding it as a subgroup of $KK_*(A,B)$ coincided with the natural topology obtained on $Pext^1_{\mathbb{Z}}(K_*(A),K_*(B))$ via the isomorphism

$$Pext^1_{\mathbb{Z}}(K_*(A),K_*(B)) \;\cong\; \varprojlim{}^1\, Hom_{\mathbb{Z}}(K_*(A_i),K_*(B)).$$

We refer to this group as the *fine structure* subgroup of $KK_*(A,B)$.

The following theorem is a generalization of a theorem of Salinas ([Sal, Theorem 5.2b], and see also [B], [BD, Theorem 5]). He proved the theorem under certain stringent assumptions on A and $K_*(A)$ which we have removed.

Theorem 1.4. *Suppose that $A \in \mathcal{N}$ and A is quasidiagonal relative to B. Then there is a natural homeomorphism of topological groups*

$$QD(A,B)_* \;\cong\; Pext^1_{\mathbb{Z}}(K_*(A),K_*(B))_{*-1}$$

regarded as topological subgroups of $KK_*(A,B)$ via the canonical inclusion δ in the UCT. Further, $QD_*(A,B)$ is a direct summand of $KK_*(A,B)$ as topological groups.

Note as an immediate consequence of this theorem that relative quasidiagonality is a topological invariant for $A \in \mathcal{N}$, answering the relative form of a question of D. Voiculescu [V1, V2]. For instance (see Theorem 4.5), if

$$0 \to B \otimes \mathcal{K} \to E \to A \to 0$$

representing $\tau \in KK_1(A,B)$ is an essential extension such that $A \in \mathcal{N}$, A is quasidiagonal and B is separable, then E is quasidiagonal as a set relative to B if and only if both of the following topological conditions hold:

(1) $\gamma(\tau) = 0 : K_*(A) \to K_*(B)$, and

[3] A subgroup $G' \subseteq G$ is said to be *pure* if for each $n \in \mathbb{N}$,

$$nG' = nG \cap G'$$

and an extension of abelian groups

$$0 \to G' \to G \to G' \to 0$$

is said to be *pure* if G' is a pure subgroup of G. If G' is a direct summand of G then G' is pure, but the most interesting cases involve non-split pure extentions. For example, tG, the torsion subgroup of G, is always a pure subgroup of G but it is not necessarily a direct summand of G. See [FI §53].

(2)
$$\tau \in \bigcap_n nExt^1_{\mathbb{Z}}(K_*(A), K_*(B))_0 = Pext^1_{\mathbb{Z}}(K_*(A), K_*(B))_0.$$

If $K_*(A)$ is torsionfree then condition 2) is automatically satisfied, so that E is quasidiagonal as a set if and only if $\gamma(\tau) = 0$.

The remainder of the paper is organized as follows.

Section 2 contains the definition of the Salinas topology and a demonstration that if $A = \varinjlim A_j$ with each $K_*(A_j)$ finitely generated then the closure of zero in $KK_*(A, B)$ may be identified with $Pext$.

In Section 3 the definitions of quasidiagonality are recalled and Theorem 1.4 is established.

Theorem 1.4 has several corollaries which are developed in §4. Here is one. Suppose that $f : A \to B$ so that $[f] \in KK_0(A, B)$. When is $[f]$ quasidiagonal? It is easy to show that the following are necessary conditions:

(1) The induced homomorphism $f_* : K_*(A) \to K_*(B)$ is trivial; and,

(2) The associated short exact sequence
$$0 \to K_*(SB) \to K_*(Cf) \to K_*(A) \to 0$$
is pure exact, where Cf denotes the mapping cone of f.

It is shown that (for $A \in \mathcal{N}$) these conditions are also sufficient. If $K_*(A)$ is torsionfree then (2) is automatic, and so $[f]$ is quasidiagonal if and only if $f_* = 0$. The section concludes with Theorem 4.5, which demonstrates the topological nature of relative quasidiagonality in a concrete manner.

The remaining four sections are devoted to applications.

Section 5 is devoted to the application of some of the theory of infinite abelian groups to obtain results on quasidiagonality.

In Section 6 we answer a question raised by L.G. Brown [B] concerning the relation between quasidiagonality and the kernel of the map
$$\theta^* : Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \longrightarrow Ext^1_{\mathbb{Z}}(K_*(A)_t, K_*(B)).$$
induced by the natural inclusion of $K_*(A)_t$, the torsion subgroup of A, into $K_*(A)$.

Section 7 deals with another result of L.G. Brown. Brown constructed [B] an operator T which was not quasidiagonal but such that $T \oplus T$ was quasidiagonal. In Section 7 a complete analysis of such phenomena is provided.

Section 8 presents a converse to a theorem of Davidson, Herrero, and Salinas which deals with conditions under which the quasidiagonality of $A/\mathcal{K}$ implies the quasidiagonality of A.

It is a pleasure to acknowledge helpful correspondence and conversations regarding quasidiagonality and abelian groups with L.G. Brown, M. Dadarlat, J. Irwin, T. Loring, N. Salinas, and D. Voiculescu, with a special thanks to N. Salinas for his help and encouragement.

In this paper all C^*-algebras are assumed separable, with the obvious exceptions of multiplier algebras $\mathcal{M}(A)$, their quotients, and an occasional B that is assumed to have a countable approximate unit. Any C^*-algebra which appears in the first variable of KK is assumed nuclear, so that

$$\mathcal{E}xt_*(A,B) \cong KK_*(A,B).$$

A homeomorphism of topological groups is an algebraic isomorphism and a homeomorphism of topological spaces.

Note: This paper, and to a small extent [S:FI], replace and very substantially extend the preliminary preprint entitled "Continuity of the Kasparov pairing and relative quasidiagonality "which will not appear.

2. The Salinas topology

In this section the definition of the Salinas topology on $KK_*(A,B)$ is recalled and it is shown (Proposition 2.3) that if A is the direct limit of a sequence of subalgebras A_j with each $K_*(A_j)$ finitely generated then the closure of zero in the Salinas topology may be identified as a $Pext$ group.

Recall from [Sal] that the Salinas topology on the group $KK_1(A,B)$ is defined for A unital as follows. Given a countable dense sequence of non-zero elements $\{a_i\}$ in A and extensions τ and τ' regarded as maps

$$A \longrightarrow \mathcal{M}(B \otimes \mathcal{K})/B$$

let

$$\mu(\tau,\tau') \;=\; \sum_{i=1}^{\infty} \frac{\|(\tau-\tau')a_i\|}{2^i\|a_i\|}.$$

This yields a complete metric on the set of extensions. Different dense sequences yield equivalent metrics.

Given x, $x' \in KK_1(A,B)$, define

$$\mu(x,x') = \mathit{inf}\, \mu(\tau,\tau') \tag{2.1}$$

where the inf is taken over all $\tau \in x$, $\tau' \in x'$. Salinas shows [Sal, 3.1] that one obtains a pseudometric on $KK_1(A,B)$ whose topology is independent of the choice of the sequence $\{a_i\}$ and with respect to which $KK_1(A,B)$ is a topological group. It is easy to see that this defines a functor to the category of topological groups and continuous homomorphisms in each variable.

If A is not necessarily unital, let A^+ denote the unitalization of A. Then there is a short exact sequence

$$0 \to KK_1(\mathbb{C},B) \to KK_1(A^+,B) \to KK_1(A,B) \to 0$$

so $KK_1(A,B)$ is a quotient of the group $KK_1(A^+,B)$ and may be given the quotient topology. Thus the Salinas topology is defined here as well. Finally, declaring suspension in the A variable to be a homeomorphism, one obtains the structure of a $\mathbb{Z}/2$-graded topological group on $KK_*(A,B)$.

Let us assume that A is the closure of the union of an increasing sequence of subalgebras A_j in the bootstrap category $\mathcal{N}$. For convenience assume that A is unital and that this unit is contained in each A_j. For each of these subalgebras A_j, choose a countable dense sequence of non-zero elements $\{a_{ij}\}$. Given elements $x, x' \in KK_1(A_j,B)$, define

$$\nu_j(x,x') = \mathit{inf} \sum_{i=1}^{\infty} \frac{\|(\tau-\tau')a_{ij}\|}{2^{i+j}\|a_{ij}\|}.$$

where the inf is with respect to all $\tau \in x$ and $\tau' \in x'$. Then

$$\nu_j(x,x') \;=\; \frac{1}{2^j}\mu(x,x')$$

and so ν_j is a pseudometric yielding the Salinas topology on $KK_*(A_j, B)$.

Given elements $x, x' \in KK_1(A,B)$, define

$$\nu(x,x') = \sum_{j=1}^{\infty} \nu_j(x_j, x'_j) \;=\; inf \;\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \frac{\|(\tau_j - \tau'_j)a_{ij}\|}{2^{i+j}\|a_{ij}\|} \tag{2.2}$$

where the subscript j on x and τ denotes the restriction to A_j. The function ν is a pseudometric which is equivalent to a Salinas pseudometric on $KK_1(A,B)$.

The following Proposition is more or less the same as [Sal, Theorem 5.2] to whom we credit the result. The proof is our own.

Proposition 2.3. *(Salinas) Suppose that $A = \underrightarrow{lim}\, A_j$ is the direct limit of an increasing union of subalgebras with each $A_j \in \mathcal{N}$ and each $K_*(A_j)$ finitely generated. Then the closure of zero in the Salinas topology on the group $KK_*(A,B)$ is the group*

$$Pext^1_{\mathbb{Z}}(K_*(A), K_*(B))_{*-1}$$

regarded as a subgroup of $KK_*(A,B)$ via the canonical inclusion in the UCT.

Proof. By suspension in the A-variable, it suffices to consider $KK_1(A,B)$. First consider the case when A is unital and the unit is contained in each A_j. The closure of zero is a subgroup of $Pext^1_{\mathbb{Z}}(A,B)$ by (1.3), so it suffices to prove the opposite inclusion. Recall from [S:FI, Theorem 6.11] that the group $Pext^1_{\mathbb{Z}}(K_*(A), K_*(B))$ is homeomorphic as topological groups to the kernel of the canonical map

$$\rho : KK_*(A,B) \longrightarrow \underleftarrow{lim}\, KK_*(A_j, B).$$

induced by the various maps $A_j \to A$. Suppose that

$$x \in KK_1(A,B) \cap Ker(\rho).$$

Then $\rho(x) = 0$ and hence $x_j = 0$ for each j, so that x_j is distance zero from $0_j \in KK_1(A_j, B)$. In other words,

$$\nu_j(x_j, 0_j) = 0$$

for each j. Thus

$$\nu(x,0) = \sum \nu_j(x_j, 0_j) = 0$$

so that x is in the closure of zero of $KK_1(A,B)$ as required.

Next consider the general case with $A = \underrightarrow{lim}\, A_j$. It is easy to see that $A^+ = \underrightarrow{lim}\, (A_j^+)$. Further, since $K_*(A^+) \cong K_*(A) \oplus \mathbb{Z}$, it follows that the natural map induces an isomorphism

$$Pext^1_{\mathbb{Z}}(K_*(A^+), K_*(B))_* \;\cong\; Pext^1_{\mathbb{Z}}(K_*(A), K_*(B))_*.$$

Consider the commuting exact diagram

$$\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \downarrow \\
 & & & & KK_*(\mathbb{C}, B) \\
 & & & & \downarrow \\
0 & \longrightarrow & Pext^1_{\mathbb{Z}}(K_*(A^+), K_*(B))_{*-1} & \longrightarrow & KK_*(A^+, B) \\
 & & \downarrow \cong & & \downarrow \pi \\
0 & \longrightarrow & Pext^1_{\mathbb{Z}}(K_*(A), K_*(B))_{*-1} & \longrightarrow & KK_*(A, B) \\
 & & & & \downarrow \\
 & & & & 0
\end{array}$$

The group $KK_*(\mathbb{C}, B) \cong K_*(B)$ is countable, complete, and hence discrete. This implies that the map π is a local homeomorphism in a suitably small neighborhood of zero. Using the fact that

$$Pext^1_{\mathbb{Z}}(K_*(A^+), K_*(B))_{*-1}$$

is the closure of zero in $KK_*(A^+, B)$, it follows that

$$Pext^1_{\mathbb{Z}}(K_*(A), K_*(B))_{*-1}$$

is indeed the closure of zero in $KK_*(A, B)$ as required. $\square$

3. Quasidiagonality

In this section we define quasidiagonality and using the results of §2 we demonstrate Theorem 1.4, which answers Voiculescu's question on the topological nature of relative quasidiagonality and serves as the key result for the rest of the paper.

Suppose that A is a unital C^*-algebra and B has a countable approximate unit consisting of projections. An extension of C^*-algebras represented as

$$\tau : A \to \mathcal{M}(B \otimes \mathcal{K})/B \otimes \mathcal{K}$$

is said to be *quasidiagonal* if there exists a completely positive lifting

$$\phi : A \to \mathcal{M}(B \otimes \mathcal{K})$$

and a *quasicentral approximate unit*, that is, an increasing sequence of projections $\{p_n\}$ in $B \otimes \mathcal{K}$ that tend strictly to 1 in $\mathcal{M}(B \otimes \mathcal{K})$ and such that

$$\lim_{n\to\infty} ||\phi(a)p_n - p_n\phi(a)|| = 0$$

for each $a \in A$. A is said to be *quasidiagonal relative to* B if there exists an absorbing trivial quasidiagonal extension

$$A \to \mathcal{M}(B \otimes \mathcal{K}).$$

If A is quasidiagonal relative to $\mathcal{K}$ then A is said to be *quasidiagonal*. For instance, every commutative unital C^*-algebra has a faithful diagonal representation on $\mathcal{M}(\mathcal{K}) = \mathcal{L}(\mathcal{H})$ and hence is quasidiagonal.

An element $x \in KK_1(A, B)$ is *quasidiagonal* if $x = [\tau]$ for some quasidiagonal extension τ. Let $QD_1(A, B)$ denote the set of quasidiagonal elements of $KK_1(A, B)$. This set is non-empty if and only if A is quasidiagonal relative to B, by [Sal, Theorem 4.4]. Since taking suspensions in the A variable preserves quasidiagonality relative to B, it follows that $QD_1(SA, B)$ is non-empty if and only if $QD_1(A, B)$ is non-empty. Define

$$QD_0(A, B) = QD_1(SA, B) \subseteq KK_1(SA, B) = KK_0(A, B).$$

Salinas has shown that the quasidiagonal elements may be described in terms of his topology on $KK_*(A, B)$. For $B = \mathcal{K}$ the following theorem was established by L.G. Brown [B, p. 63, Remark 1.]

Theorem 3.1 (Salinas [Sal, Theorem 4.4]**).** *If A is quasidiagonal relative to B then the closure of zero in $KK_*(A, B)$ is the set $QD_*(A, B)$ of quasidiagonal extensions.*

□

Of course this implies that $QD_*(A, B)$ is a subgroup of $KK_*(A, B)$. Combining this theorem with our results yields Theorem 1.4:

Theorem 1.4. *Suppose that $A \in \mathcal{N}$ and A is quasidiagonal relative to B. Then there is a natural homeomorphism of topological groups*

$$QD_*(A,B) \cong Pext^1_{\mathbb{Z}}(K_*(A), K_*(B))_{*-1}$$

regarded as subgroups of $KK_*(A,B)$ via the canonical inclusion δ in the UCT. Further, $QD_*(A,B)$ is a direct summand of $KK_*(A,B)$ as topological groups.

Note that this shows that for $A \in \mathcal{N}$ that $QD_*(A,B)$ *is* a topological invariant, which answers the question of D. Voiculescu [V1, V2] on the topological invariance of relative quasidiagonality.

Theorem 1.4, together with Theorem 6.11 of [S:FI], imply that if $A \in \mathcal{N}$ and A is quasidiagonal relative to B then $KK_*(A,B)$ is topologically the direct sum of the group

$$Hom_{\mathbb{Z}}(K_*(A), K_*(B))$$

which houses index information, the group

$$\varprojlim Ext^1_{\mathbb{Z}}(K_*(A_i), K_*(B))$$

which is the $\mathbb{Z}$-adic completion of $Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))_*$, and the group

$$QD_*(A,B) \cong Pext^1_{\mathbb{Z}}(K_*(A), K_*(B))_{*-1} \cong \varprojlim{}^1 Hom_{\mathbb{Z}}(K_*(A), K_*(B))_{*-1}$$

which is the fine structure subgroup.

Proof. Choose a KK-filtration $\{A_j\}$ for A. This consists of an increasing sequence of separable commutative C^*-algebras

$$A_1 \hookrightarrow A_2 \hookrightarrow A_3 \hookrightarrow \dots$$

where each $A_j^+ \cong C(X_j)$ for some finite complex X_j, the induced maps

$$K_*(A_j) \to K_*(A_{j+1})$$

are injective, and $\varinjlim A_j$ is KK-equivalent to A. (KK-filtrations exist and are unique in a suitable sense for each $A \in \mathcal{N}$ by [S:D Theorem 1.5].) Then by Theorem 3.10 (a consequence of Theorem 1.4) there is a homeomorphism

$$KK_*(A,B) \cong KK_*(\varinjlim A_j, B).$$

which implies that the closure of zero in $KK_*(A,B)$ is homeomorphic to the closure of zero in $KK_*(\varinjlim A_j, B)$. Since A is quasidiagonal relative to B, the closure of zero in $KK_*(A,B)$ is isomorphic to $QD_*(A,B)$. On the other hand, the C^*-algebra $\varinjlim A_j$ is commutative, hence quasidiagonal, and hence quasidiagonal relative to B by [Sal, Lemma 4.3].

This implies that the closure of zero in $KK_*(\varinjlim A_i, B)$ is isomorphic to $QD_*(\varinjlim A_i, B)$. Combining results, there is a natural homeomorphism

$$QD_*(A,B) \cong QD_*(\varinjlim A_i, B)$$

and thus without loss of generality we may assume that $A = \varinjlim A_j$. Then

$$QD_*(A,B) \cong (closure\ of\ zero)$$

by Theorem 3.1 (Salinas)

$$\cong Pext^1_{\mathbb{Z}}(K_*(A), K_*(B))_{*-1}$$

by Proposition 2.3. This establishes the first part of the Theorem.

It remains to show that $QD_*(A,B)$ is a direct summand. From the first part of the Theorem it suffices to show that $Pext^1_{\mathbb{Z}}(K_*(A), K_*(B))_{*-1}$ is a direct summand, and this follows from [S:FI Theorem 6.11] together with the identification of $Pext^1_{\mathbb{Z}}(K_*(A), K_*(B))_{*-1}$ with the fine structure subgroup of $KK_*(A,B)$. □

Corollary 3.2. *Suppose that $A \in \mathcal{N}$ and that A is quasidiagonal. Then*

$$QD_*(A,\mathcal{K}) \cong Pext^1_{\mathbb{Z}}(K_{*-1}(A), \mathbb{Z}).$$

□

Theorem 1.4 and Corollary 3.2 may be used readily since much is known about computing the *Pext* groups shown. For instance, suppose that $K_*(A)$ and $K_*(B)$ are torsionfree and $QD_*(A,B) \neq 0$. Then $QD_*(A,B)$ is divisible and uncountable. For much more information on the computation of these groups, see [S:FIII].

Remark 3.3. The group $QD(A,B)$ may well have torsion. For instance, suppose that $A \in \mathcal{N}$, A is quasidiagonal and $K_*(A)$ is torsionfree. Then[4]

$$QD_1(A,\mathcal{K}) \cong Pext^1_{\mathbb{Z}}(K_0(A), \mathbb{Z}) \cong Ext^1_{\mathbb{Z}}(K_0(A), \mathbb{Z}).$$

Choose $A \in \mathcal{N}$ with $K_0(A) = \mathbb{Z}_p$. Then

$$QD_1(A,\mathcal{K}) \cong \mathbb{Q}^{\aleph_0} \oplus \mathbb{Z}(p^\infty).$$

where $\mathbb{Z}(p^\infty)$ denotes the p-torsion subgroup of $\mathbb{Q}/\mathbb{Z}$. This point was overlooked in [Sal, Cor. 5.4]. In that paper the primary interest was the case with $K_*(B)$ torsionfree and $K_*(A)$ finitely generated. If G is finitely generated then $Pext^1_{\mathbb{Z}}(G,H) = 0$ for all H, so if $A \in \mathcal{N}$ with $K_*(A)$ finitely generated then every quasidiagonal extension of the form

$$0 \to B \to E \to A \to 0$$

is trivial.

[4]since

$$Pext^1_{\mathbb{Z}}(G,H) \cong Ext^1_{\mathbb{Z}}(G,H)$$

whenever G is torsionfree

4. Homomorphisms and Split Morphisms

In this section the identification of the quasidiagonal elements of Theorem 1.4 is made concrete. One corollary answers a simple question: suppose that $f : A \to B$ so that $[f] \in KK_0(A, B)$. When is $[f]$ quasidiagonal? There are certain easy algebraic necessary conditions, and we show that (for $A \in \mathcal{N}$) these conditions are sufficient. Finally, we demonstrate the workings of our results on the topological nature of relative quasidiagonality in a concrete context.

In applications many of the most interesting classes in KK-theory come from $*$-homomorphisms $f : A \to B$. Recall that the *mapping cone* Cf is the C^*algebra

$$Cf = \{(\xi, a) \in B[0,1] \oplus A : \xi(0) = 0,\ \xi(1) = f(a)\}$$

with associated mapping cone sequence (cf. [S:III])

$$0 \to SB \to Cf \to A \to 0.$$

This gives rise to a class

$$[f] \in KK_1(A, SB) \cong KK_0(A, B).$$

Alternately, we may define

$$[f] = f_*[1_A]$$

where $[1_A] \in KK_0(A, A)$ is the ring identity. Thus $[f]$ is closely related to the structure of $KK_*(A, B)$ regarded as a left module over the ring $KK_*(A, A)$ with the action given by f.

When is this class quasidiagonal? Here is the answer.

Corollary 4.1. *Suppose that $A \in \mathcal{N}$ and A is quasidiagonal relative to B. Let $f : A \to B$ be a $*$-homomorphism. Then $[f] \in KK_0(A, B)$ is quasidiagonal if and only if both of the following conditions hold:*

(1) *The induced homomorphism $f_* : K_*(A) \to K_*(B)$ is trivial; and,*

(2) *The associated short exact sequence*

$$0 \to K_*(SB) \to K_*(Cf) \to K_*(A) \to 0$$

is pure exact.

If $K_*(A)$ is torsionfree then condition (2) is automatically satisfied, so that $[f]$ is quasidiagonal if and only if $f_* = 0$.

Note that conditions 1) and 2) are necessary by Salinas's result (1.3). The point here is that they suffice.

Proof. By Theorem 1.4 it suffices to show that

$$[f] \in Pext^1_{\mathbb{Z}}(K_*(A), K_*(SB))_1$$

if and only if conditions 1) and 2) hold. Condition 1) is necessary and sufficient for

$$[f] \in Ext^1_{\mathbb{Z}}(K_*(A), K_*(SB))_1$$

by the UCT, and condition 2) picks out those elements of $Ext^1_{\mathbb{Z}}(K_*(A), K_*(SB))_1$ which lie in $Pext^1_{\mathbb{Z}}(K_*(A), K_*(SB))_1$. If $K_*(A)$ is torsionfree then

$$Pext^1_{\mathbb{Z}}(K_*(A), K_*(SB))_1 \; = \; Ext^1_{\mathbb{Z}}(K_*(A), K_*(SB))_1$$

and the corollary follows. □

As remarked previously, every KK-class may be represented as the KK-product of a class induced by a homomorphism and by a split morphism. Corollary 4.1 tells us when a class induced by a homomorphism is quasidiagonal. The following Corollary answers the same question for split morphisms. It turns out that quasidiagonal split morphisms are trivial.

Corollary 4.2. *Suppose that $A \in \mathcal{N}$ and A is quasidiagonal relative to B. Further, suppose given an extension of C^*-algebras*

$$0 \to B \xrightarrow{i} A \xrightarrow{p} A/B \to 0 \tag{4.3}$$

which is split by a $*$-homomorphism $s : A/B \to A$. Let

$$\pi_s \in KK_0(A, B)$$

be the associated split morphism.[5]

Then the following are equivalent:

1. (1) π_s *is quasidiagonal;*
2. (2) $K_*(B) = 0$.
3. (3) $KK_*(A, B) = 0$.
4. (4) $\pi_s = 0$.

[5]This is the unique class in the group $KK_0(A, B)$ with the property that

$$i_*(\pi_s) = [1] - [sp] \in KK_0(A, A).$$

Proof. The implications 3) → 4) → 1) are obvious. We shall show that 1) → 2) → 3).

Suppose first that π_s is quasidiagonal. Then $\gamma(\pi_s) = 0$ by Theorem 1.4, and

$$1 - s_* p_* \;=\; \gamma([1] - [sp]) \;=\; \gamma(i_*(\pi_s)) \;=\; i_*(\gamma(\pi_s)) \;=\; 0$$

so that

$$0 = \gamma([1]) - \gamma([sp]) = 1 - s_* p_*$$

in the short exact sequence

$$0 \to K_*(B) \xrightarrow{i_*} K_*(A) \xrightarrow{p_*} K_*(A/B) \to 0$$

which implies that p_* is an isomorphism and thus that $K_*(B) = 0$ as required. Thus 1) → 2). If $K_*(B) = 0$ then $KK_*(A, B) = 0$ by the UCT. Thus 2) → 3). □

Corollary 4.1 tells when a class $[f]$ is quasidiagonal. Corollary 4.2 tells when a split morphism π_s is quasidiagonal. The following theorem describes when a class with factorization $x = [f] \otimes_D \pi_s = f^*(\pi_s)$ is quasidiagonal. As explained in [S:FI, 3.6], every KK-class has this form. Thus this theorem gives a complete solution to the quasidiagonality of particular KK-classes.

Theorem 4.4. *Suppose that $A \in \mathcal{N}$ and A is quasidiagonal relative to B. Let $x \in KK_0(A, B)$ with factorization*

$$x = [f] \otimes_D \pi_s = f^*(\pi_s)$$

with respect to the map

$$f : A \to D$$

and short exact sequence

$$0 \to B \otimes \mathcal{K} \to D \xrightarrow{p} A \to 0$$

with splitting $s : A \to D$ and $\pi_s \in KK_0(D, B)$.

Then:

(a) The following conditions are equivalent:

(1)

$$\gamma(x) = 0 \in Hom_{\mathbb{Z}}(K_*(A), K_*(B))_0$$

(2)

$$\begin{aligned} Im(f_* : K_*(A) \to K_*(D)) &\subseteq Ker(\gamma(\pi_s) : K_*(D) \to K_*(B) \\ &= Im(s_* : K_*(A) \to K_*(D)) \end{aligned}$$

(b) Suppose that $\gamma(x) = 0$, so that

$$x \in Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))_1 \subseteq KK_0(A, B).$$

Then

$$x = z \otimes_D \pi_s \tag{*}$$

for some

$$z \in Ext^1_{\mathbb{Z}}(K_*(A), K_*(D))_1 \subseteq KK_0(A, D).$$

The element z is unique modulo the subgroup

$$s_*(Ext^1_{\mathbb{Z}}(K_*(A), K_*(A)))_1.$$

Conversely, any element x of form (*) is in the group $Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))_1$.

(c) Suppose that $x \in Pext^1_{\mathbb{Z}}(K_*(A), K_*(B))_1 \subseteq KK_0(A, B)$. Then

$$x = z \otimes_D \pi_s \tag{**}$$

for some

$$z \in Pext^1_{\mathbb{Z}}(K_*(A), K_*(D))_1 \subseteq KK_0(A, D).$$

The element z is unique modulo the subgroup

$$s_*(Pext^1_{\mathbb{Z}}(K_*(A), K_*(A)))_1.$$

Conversely, any element of form (**) is in the group $Pext^1_{\mathbb{Z}}(K_*(A), K_*(B))_0$

Proof. For Part (a) we compute:

$$\gamma(x) = \gamma(\, [f] \otimes_D \pi_s \,) = \gamma(\pi_s) f_*$$

and hence $\gamma(x) = 0$ if and only if $Im(f_*) \subseteq Ker(\,\gamma(\pi_s)\,)$. The identification $Ker(\,\gamma(\pi_s)\,) = Im(s_*)$ is immediate from the definition of $\gamma(\pi_s)$. This proves Part (a).

In order to prove Parts (b) and (c) the following Lemma is required. It uses the notation and assumptions of Theorem 4.4.

Lemma 4.5. *The diagram*

$$\begin{array}{ccc} Ext^1_{\mathbb{Z}}(K_*(A), K_*(D))_1 & \xrightarrow{\delta_D} & KK_0(A, D) \\ \Big\downarrow{\scriptstyle \gamma(\pi_s)_*} & & \Big\downarrow{\scriptstyle (-)\otimes_D \pi_s} \\ Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))_1 & \xrightarrow{\delta_B} & KK_0(A, B) \end{array}$$

commutes, where the maps δ are the inclusion maps from the UCT.

Proof. This does not follow immediately from the naturality of the UCT, since the map $(-) \otimes_D \pi_s$ is not induced by a map of C^*-algebras. We argue as follows. Expand the diagram to the diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
Ext^1_{\mathbb{Z}}(K_*(A), K_*(A))_1 & \xrightarrow{\delta_A} & KK_0(A, A) \\
\downarrow{\scriptstyle s_*} & & \downarrow{\scriptstyle s_*} \\
Ext^1_{\mathbb{Z}}(K_*(A), K_*(D))_1 & \xrightarrow{\delta_D} & KK_0(A, D) \\
\downarrow{\scriptstyle \gamma(\pi_s)_*} & & \downarrow{\scriptstyle (-)\otimes_D \pi_s} \\
Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))_1 & \xrightarrow{\delta_B} & KK_0(A, B) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Each column is split exact, since $s_* : K_*(A) \to K_*(D)$ is split mono, and the upper square commutes by the naturality of the UCT (since the map s_* *is* induced by the map $s : A \to D$) and it is easy to see that the map δ_B is the quotient map, making the lower square commute. □

Returning to the proof of Part (b), we see from Lemma 4.5 that the restriction of the map $(-) \otimes_D \pi_s)$ to the map

$$(-) \otimes_D \pi_s : Ext^1_{\mathbb{Z}}(K_*(A), K_*(D)) \to Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))$$

is just the split surjection $\gamma(\pi_s)$. This implies that each element of the group $Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))$ is of the requisite form and the indeterminacy is as stated.

For Part (c), consider the commuting diagram

$$\begin{array}{ccc}
Pext^1_{\mathbb{Z}}(K_*(A), K_*(D)) & \xrightarrow{(-)\otimes_D \pi_s} & Pext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\
Ext^1_{\mathbb{Z}}(K_*(A), K_*(D)) & \xrightarrow{(-)\otimes_D \pi_s} & Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))
\end{array}$$

The lower horizontal map is just the map $\gamma(\pi_s)_*$ by the Lemma, and so the upper horizontal map (which is a further restriction, of course) is just the map $\gamma(\pi_s)_*$ as well. Then the conclusion of the Theorem is an easy consequence of previous results and the fact that both horizontal maps are split surjections with known kernels. □

Here is our solution to the relative quasidiagonality problem, applied to a concrete extension.

Theorem 4.6. *Suppose that $A \in \mathcal{N}$ is quasidiagonal and that B is separable. Suppose given an essential extension*

$$0 \to B \otimes \mathcal{K} \to E \to A \to 0$$

representing $\tau \in KK_1(A, B)$. Then E is quasidiagonal as a set relative to B if and only if both of the following conditions hold:

(1) *$\gamma(\tau) = 0$, or, equivalently, the boundary homomorphism $K_*(A) \to K_*(B)$ is trivial; and*

(2)
$$\tau \in \bigcap_n nExt^1_{\mathbb{Z}}(K_*(A), K_*(B))_0$$

or, equivalently,

$$\tau \in Ker[\phi : Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))_0 \longrightarrow Ext^1_{\mathbb{Z}}(K_*(A), K_*(B))_0{}^\wedge]$$

where $G^\wedge$ denotes the $\mathbb{Z}$-adic completion of G.

If $K_*(A)$ is torsionfree then condition (2) is satisfied automatically, so that E is quasidiagonal as a set relative to B if and only if the boundary homoorphism $K_*(A) \to K_*(B)$ is trivial.

Proof. Conditions 1) and 2) are exactly the conditions that guarantee that τ lies in the subgroup $QD_1(A, B)$. $\square$

5. Purity and Quasidiagonality

In this section we take advantage of standard results in infinite abelian groups to deduce results on quasidiagonality. Recall the following facts from [FI].

Proposition 5.1.
(a) *Suppose that H is a countable abelian group. Then the following are equivalent:*

(1) $Pext^1_{\mathbb{Z}}(G,H) = 0$ *for all countable abelian groups G.*

(2) $Pext^1_{\mathbb{Z}}(G,H) = 0$ *for the groups $G = \mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$.*

(3) *H is algebraically compact.*

(b) *Suppose that G is a countable abelian group. Then the following are equivalent:*

(1) $Pext^1_{\mathbb{Z}}(G,H) = 0$ *for all countable abelian groups H.*

(2) $Pext^1_{\mathbb{Z}}(G,H) = 0$ *for all countable direct sums of cyclic groups H.*

(3) *G is the direct sum of cyclic groups.*

Proof. First concentrate on Part (a). Of course (a1) implies (a2). The implication (a3) implies (a1) is immediate from [FI, 53.4]. The implication (a2) implies (a3) is the least obvious, but it is also found in Fuchs [FI, page 232.]

Turning to part (b), the implication (b1) implies (b2) is trivial, and the implication (b3) implies (b1) follows from [FI, 53.4] as well. For the following argument that (b2) implies (b3) I am endebted to John Irwin. Suppose that G satisfies condition (b2). Let $\tilde{G}$ be the free abelian group on the (countable) set

$$\{[g] : g \in G\}$$

modulo the relations given by

$$n[g] = 0$$

if $g \in G$ has order n. There is an obvious surjection $\tilde{G} \to G$ and hence a short exact sequence

$$\Theta : \qquad 0 \to K \to \tilde{G} \to G \to 0.$$

This sequence is pure, since every torsion element of G lifts to a torsion element of $\tilde{G}$ of the same order. The group $\tilde{G}$ is countable by construction, hence K is countable. Further, $\tilde{G}$ is a direct sum of cyclic groups. The group K is a subgroup of a direct sum of cyclic groups and by Kulikov's theorem (cf. [FI, 18.1]) K itself is a direct sum of cyclic groups. Thus

$$\Theta \in Pext^1_{\mathbb{Z}}(G,K) = 0$$

and hence the extension Θ must be split. This implies that G is isomorphic to a subgroup of a direct sum of cyclic groups and hence (by Kulikov's theorem) is itself a direct sum of cyclic groups. □

The following theorem is the KK-version of the preceeding, purely algebraic results. First a bit of notation. For G any countable abelian group, let C_G be a separable commutative C^*-algebra such that for each $j = 0, 1$,

$$K_j(C_G) = G.$$

Note that C_G exists by geometric realization and is unique up to KK-equivalence by the UCT.

Theorem 5.2.
(a) Suppose that B is a separable C^*-algebra. Then the following are equivalent:

(1) *For each $A \in \mathcal{N}$ with A quasidiagonal relative to B,*

$$QD_1(A, B) \;=\; 0.$$

(2) *For $G = \mathbb{Q}$ and $G = \mathbb{Q}/\mathbb{Z}$,*

$$QD_1(C_G, B) \;=\; 0.$$

(3) *$K_*(B)$ is algebraically compact.*

(b) Suppose given a quasidiagonal C^*-algebra $A \in \mathcal{N}$. Then the following are equivalent:

(1) *For each separable C^*-algebra B,*

$$QD_1(A, B) \;=\; 0.$$

(2) *For H any direct sum of cyclic groups,*

$$QD_1(A, C_H) \;=\; 0.$$

(3) *$K_*(A)$ is the direct sum of cyclic groups.*

Proof. First consider (a). The implication (a1) implies (a2) is immediate. The implication (a3) implies (a1) is elementary, since if $K_*(B)$ is algebraically compact then

$$(*) \qquad Pext^1_{\mathbb{Z}}(G, K_*(B)) = 0$$

for all groups G (cf. 5.1). If $A \in \mathcal{N}$ with A quasidiagonal relative to B then

$$QD_1(A, B) \cong Pext^1_{\mathbb{Z}}(K_*(A), K_*(B))_0$$

by Theorem 1.4, and

$$Pext^1_{\mathbb{Z}}(K_*(A), K_*(B))_0 \;=\; 0$$

by (*), completing the argument.

Next we show that (a2) implies (a3). The condition (a2) implies that

$$Pext^1_{\mathbb{Z}}(G, K_*(B)) \;=\; 0$$

for $G = \mathbb{Q}$ and $G = \mathbb{Q}/\mathbb{Z}$ and then Theorem 5.1a implies that $K_*(B)$ is algebraically compact. This completes the proof of part (a).

The proof of part (b) is quite similar, and we comment only on the deep implication (b2) implies (b3). Condition (b2) together with Theorem B imply that

$$Pext^1_{\mathbb{Z}}(K_*(A), H)_0 \;=\; 0$$

whenever H is a direct sum of cyclic groups, and then Theorem 5.1b implies that $K_*(A)$ is itself a direct sum of cyclic groups. This completes the proof of Theorem 5.2. $\square$

6. A problem of L.G. Brown

Let $\theta : K_*(A)_t \to K_*(A)$ be the canonical inclusion of the torsion subgroup of $K_*(A)$. L.G. Brown [B page 63] showed that (with $B = \mathcal{K}$) there is a relation between quasidiagonality and the kernel of the induced map

$$\theta^* : Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \longrightarrow Ext^1_{\mathbb{Z}}(K_*(A)_t, K_*(B)).$$

We generalize his result as follows. Recall [S:FI, Theorem 7.7] that given A, there is an associated exact sequence of C^*-algebras

$$0 \to A \otimes \mathcal{K} \to A_f \to SA_t \to 0$$

which realizes the short exact sequence

$$(*) \qquad 0 \to K_*(A)_t \xrightarrow{\theta} K_*(A) \to K_*(A)_f \to 0$$

In particular,

$$K_*(A_t) \cong K_*(A)_t \qquad \text{and} \qquad K_*(A_f) \cong K_*(A)_f$$

where G_t denotes the torsion subgroup of a group G and $G_f = G/G_t$ denotes the maximal torsionfree quotient of G.

Theorem 6.1. *Suppose that $A \in \mathcal{N}$, B is separable, and A is quasidiagonal relative to B. Then:*

(1) *There is a natural commutative diagram with exact columns:*

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
Ker(Q\theta^*)_{*-1} & \xrightarrow{\cong} & Ker(\theta^*)_{*-1} \\
\downarrow & & \downarrow \\
QD_*(A,B) & \xrightarrow{\cong} & Pext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \\
\downarrow{\scriptstyle Q\theta^*} & & \downarrow{\scriptstyle \theta^*} \\
QD_*(A_t,B) & \xrightarrow{\cong} & Pext^1_{\mathbb{Z}}(K_*(A)_t, K_*(B)) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

(2) *There is a natural exact sequence*

$$Hom_{\mathbb{Z}}(K_*(A)_t, K_*(B)) \xrightarrow{\delta'} QD_*(A_f, B) \to Ker(\theta^*)_{*-1} \to 0$$

where δ' is the boundary map in the Hom-$Pext$ long exact sequence associated to the pure short exact sequence (*).

(3) *If* $\mathcal{I}m(\delta') = 0$[6] *then there is a natural isomorphism*

$$QD(A_f, B) \cong Ker(\theta^*)_{*-1}.$$

Proof. The short exact sequence is pure exact. After identifying the pure extensions with the quasidiagonal elements as per Theorem 1.4, one is left with the following commutative diagram with exact columns:[7]

$$\begin{array}{ccc}
Hom_{\mathbb{Z}}(K_*(A)_t, K_*(B)) & \xrightarrow{\cong} & Hom_{\mathbb{Z}}(K_*(A)_t, K_*(B)) \\
\downarrow \delta' & & \downarrow \delta \\
QD_*(A_f, B) & \xrightarrow{\cong} & Ext^1_{\mathbb{Z}}(K_*(A)_f, K_*(B)) \\
\downarrow & & \downarrow \\
QD_*(A, B) & \longrightarrow & Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \\
\downarrow Q\theta^* & & \downarrow \theta^* \\
QD_*(A_t, B) & \longrightarrow & Ext^1_{\mathbb{Z}}(K_*(A)_t, K_*(B)) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

An easy diagram chase shows that $Ker(Q\theta^*) \cong Ker(\theta^*)$ from which 1) is immediate. Part 2) follows from expressing $Ker(Q\theta^*)$ as the quotient of $QD(A_f, B)$ modulo the group $\mathcal{I}m(\delta')$. $\square$

[6] This condition usually holds. For instance, it holds if $K_*(A)_t$ is a direct summand of $K_*(A)$ or if $K_*(B)$ is torsionfree, and of course it holds if $Hom_{\mathbb{Z}}(K_*(A)_t, K_*(B)) = 0$.

[7] The map

$$QD_*(A_f, B) \to Ext^1_{\mathbb{Z}}(K_*(A_f), K_*(B))$$

is an isomorphism since $K_*(A_f)$ is torsionfree.

7. Quasidiagonality and Torsion

One of the early applications of the Brown-Douglas-Fillmore theory was contained in work of L.G. Brown [B]. He exhibited an example of a bounded operator T which was not quasidiagonal but such that $T \oplus T$ was quasidiagonal. In fact Brown showed that $T \oplus T$ generated a trivial extension. The following theorem presents a complete analysis of such behavior at the level of KK-theory.

Theorem 7.1. *Suppose that $A \in \mathcal{N}$, A is relatively quasidiagonal relative to B, and $\tau \in KK_*(A, B)$ is not quasidiagonal. Then:*

1. *If τ has infinite order in $KK_*(A, B)$ then no multiple of τ is quasidiagonal.*
2. *If τ has finite order n in $KK_*(A, B)$ then $n\tau$ is quasidiagonal (since it is trivial), and no smaller multiple of τ is quasidiagonal.*

Proof. Theorem 1.4 implies that $QD_*(A, B)$ is a direct summand of $KK_*(A, B)$, so that $\tau \notin QD_*(A, B)$ implies that $n\tau \notin QD_*(A, B)$ for all nonzero multiples of τ.

□

8. Lifting Quasidiagonality

In this section we present a converse to the following theorem of Davidson, Herrero, and Salinas.

Theorem 8.1. [DHS] *Suppose that*

$$\tau : \qquad 0 \to \mathcal{K} \to A \to A/\mathcal{K} \to 0 \tag{8.2}$$

is an essential extension with A separable nuclear. If A is quasidiagonal then $A/\mathcal{K}$ is quasidiagonal.

There is an obvious obstruction to a converse, known already to Halmos [H]. Suppose that S is the unilateral shift. Let A be the C^*-algebra generated by the compacts, the identity and S. Then there is an extension

$$\tau_S : \qquad 0 \to \mathcal{K} \to A \to A/\mathcal{K} \cong C(S^1) \to 0$$

and the quotient $A/\mathcal{K} \cong C(S^1)$ is commutative, hence quasidiagonal. However the C^*-algebra A itself is not quasidiagonal since the unilateral shift is not quasidiagonal. Halmos demonstrates this by observing that S has non-trivial Fredholm index, whereas any Fredholm quasidiagonal operator must have trivial Fredholm index. In modern jargon, the map

$$\gamma : KK_1(C(S^1), \mathcal{K}) \to Hom_{\mathbb{Z}}(K^1(S^1), \mathbb{Z}) \cong \mathbb{Z}$$

satisfies

$$\gamma(\tau_S)(z) = -1 \neq 0$$

which is an obstruction to quasidiagonality.

Here is the complete story, at least within the bootstrap category.

Theorem 8.3. *Suppose given an essential extension (8.2) with $A/\mathcal{K} \in \mathcal{N}$ and quasidiagonal. Then A is quasidiagonal as a set if and only if the following two conditions hold:*

(1) $\gamma(\tau) = 0$.

(2) *The resulting K-theory short exact sequence*

$$0 \to \mathbb{Z} \to K_0(A) \to K_0(A/\mathcal{K}) \to 0$$

is a pure exact sequence.

If in addition $K_0(A/\mathcal{K})$ is torsionfree then condition 2) is automatically satisfied, so that A is quasidiagonal as a set (with respect to this embedding) if and only if condition 1) holds.

Proof. Theorem 1.4 reduces in this case to the identification

$$QD_1(A/\mathcal{K}, \mathcal{K}) \cong Pext^1_{\mathbb{Z}}(K_0(A/\mathcal{K}), \mathbb{Z}).$$

Condition 1) is equivalent to

$$\tau \in Ext^1_{\mathbb{Z}}(K_0(A/\mathcal{K}), \mathbb{Z})$$

and condition 2) is simply a statement that

$$\tau \in Pext^1_{\mathbb{Z}}(K_0(A/\mathcal{K}), \mathbb{Z}).$$

If in addition $K_0(A/\mathcal{K})$ is torsionfree then

$$Pext^1_{\mathbb{Z}}(K_0(A/\mathcal{K}), \mathbb{Z}) \;\cong\; Ext^1_{\mathbb{Z}}(K_0(A/\mathcal{K}), \mathbb{Z}).$$

This completes the proof. $\square$

References

[**Bl**] B. Blackadar, *K-Theory for Operator Algebras*, Math. Sci. Res. Inst. No. 5, Springer-Verlag, New York, 1986.

[**B**] L.G. Brown, *The Universal coefficient theorem for Ext and quasidiagonality,*, Operator Algebras and Group Representations, vol. 17, Pitman Press, Boston, 1983, pp. 60-64.

[**BD**] L.G. Brown and M. Dadarlat, *Extensions of C^*-algebras and quasidiagonality*, J. London Math. Soc. (2) **53** (1996), 582-600.

[**BDF**] L.G. Brown, R.G. Douglas, and P.A. Fillmore, *Extensions of C^*-algebras, operators with compact self-commutators, and K-homology*, Bull. Amer. Math. Soc. **79** (1973), 973-978.

[**DHS**] K.R. Davidson, D.A. Herrero, and N. Salinas, *Quasidiagonal operators, approximations, and C^*-algebras*, Indiana U. Math. J. **38** (1989), 973-998.

[**FI**] László Fuchs, *Infinite Abelian Groups,* Pure and Applied Mathematics No. 36 , vol. 1, Academic Press, New York, 1970.

[**H**] P.R. Halmos, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc. **76** (1970), 887-933.

[**J**] C. U. Jensen, *Les Foncteurs Dérivés de $\varprojlim$ et leur Applications en Théorie des Modules,* Lecture Notes in Mathematics , vol. 254, Springer, Verlag, New York, 1972.

[**PS**] V. Paulsen and N. Salinas.

[**P**] L.S. Pontryagin, *Topological Groups*, Second Edition, trans. Arlen Brown, Gordon and Breach, New York, 1966.

[**RS**] J. Rosenberg and C. Schochet, *The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor*, Duke Math. J. **55** (1987), 431-474.

[**Sal**] N. Salinas, *Relative quasidiagonality and KK-theory*, Houston J. Math. **18** (1992), 97-116.

[**S:II**] C. Schochet, *Topological methods for C^*-algebras II: geometric resolutions and the Künneth formula*, Pacific J. Math. **98** (1982), 443-458.

[**S:III**] C. Schochet, *Topological methods for C^*-algebras III: axiomatic homology*, Pacific J. Math. **114** (1984), 399-445.

[**S:D**] C. Schochet, *The UCT, the Milnor sequence, and a canonical decomposition of the Kasparov groups*, K-Theory **10** (1966), 49-72.

[**S:FI**] C. Schochet, *The fine structure of the Kasparov groups I: continuity of the KK-pairing*, preprint, 1996.

[**S:FIII**] C. Schochet, *The fine structure of the Kasparov groups III: calculating $\varprojlim^1$* , in preparation.

[**V1**] D. Voiculescu, *A Note on quasi-diagonal C^*-algebras and homotopy*, Duke Math. J. **62** (1991), 267-271.

[**V2**] D. Voiculescu, *Around quasidiagonal operators*, Integ. Equat. Oper. Th. **17** (1993), 137-149.