# LEIBNIZ COHOMOLOGY FOR DIFFERENTIABLE MANIFOLDS

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## Introduction

The goal of this paper is to extend Loday's Leibniz cohomology [L,P] from a Lie algebra invariant to an invariant for differentiable manifolds so that Leibniz cohomology is a non-commutative version of de Rham cohomology. The non-commutativity arises by considering a cochain complex of tensors (from differential geometry) which are not necessarily skew-symmetric. Leibniz cohomology, $HL^*$, is no longer a homotopy invariant—in fact the first obstruction to the homotopy invariance of $HL^*(\mathbf{R}^n)$ is the universal Godbillon-Vey invariant in dimension $2n+1$. The main calculational result of the paper is then a calculation of $HL^*(\mathbf{R}^n)$ in terms of (i) certain universal invariants of foliations, and (ii) Loday's product structure on $HL^*$. Unlike Lie algebra or Gelfand-Fuks cohomology, the Leibniz cohomology of vector fields on $\mathbf{R}^n$ contains infinite families of elements which support non-trivial products.

## §1. Foundations of non-commutative cohomologies

We begin by reviewing Loday's definition of Leibniz (co)homology [L1], [L2], [L,P]. Let $k$ be a commutative ring and $\mathfrak{g}$ a $k$-module together with a bilinear map $[\ ,\ ] : \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$. Then, by definition, $\mathfrak{g}$ is a Leibniz algebra if the bracket satisfies the "Leibniz identity"

$$[x,\,[y,\,z]] = [[x,\,y],\,z] - [[x,\,z],\,y] \tag{1.1}$$

for all $x,\ y,\ z$ in $\mathfrak{g}$. If in addition to (1.1), the bracket is skew-symmetric, $[x,\,y]=-[y,\,x]$, then (1.1) is equivalent to the Jacobi identity, and $\mathfrak{g}$ itself becomes a Lie algebra. In this sense, a Leibniz algebra is a non-commutative version of a Lie algebra. We also need the notion of a representation of a Leibniz algebra [L,P], which is a $k$-module $N$ equipped with left and right actions of $\mathfrak{g}$

$$[\ ,\ ] : \mathfrak{g}\times N\to N,\qquad [\ ,\ ] : N\times\mathfrak{g}\to N$$

which are bilinear and satisfy the three properties

$$\begin{aligned}[m,\,[x,\,y]] &= [[m,\,x],\,y] - [[m,\,y],\,x]\\ [x,\,[m,\,y]] &= [[x,\,m],\,y] - [[x,\,y],\,m]\\ [x,\,[y,\,m]] &= [[x,\,y],\,m] - [[x,\,m],\,y]\end{aligned} \tag{1.2}$$

for all $x,\ y \in \mathfrak{g}$ and $m \in N$. Of course, if $N$ is a representation of a Lie algebra $\mathfrak{g}$, then the above three conditions are equivalent, since a left representation of a Lie algebra, $[\ ,\ ] : \mathfrak{g} \times N \to N$ determines a right representation of the same Lie algebra $[\ ,\ ] : N \times \mathfrak{g} \to N$ via $[x,\ m] = -[m,\ x]$.

Returning to the setting of a Leibniz algebra $\mathfrak{g}$ and a representation $N$, the Leibniz cohomology of $\mathfrak{g}$ with coefficients in $N$, written $HL^*(\mathfrak{g};\ N)$, is the homology of the cochain complex

$$C^n(\mathfrak{g};\ N) = \mathrm{Hom}_k(\mathfrak{g}^{\otimes n},\ N), \quad n \geq 0.$$

To streamline the notation for the coboundary map

$$d : C^n(\mathfrak{g};\ N) \to C^{n+1}(\mathfrak{g};\ N)$$

let $(g_1,\ g_2,\ \ldots,\ g_n)$ denote the element $g_1 \otimes g_2 \otimes \ldots \otimes g_n \in \mathfrak{g}^{\otimes n}$. For $f \in C^n(\mathfrak{g};\ N)$, we have

$$(1.3) \qquad (df)(g_1,\ g_2,\ \ldots,\ g_{n+1}) = \sum_{1 \leq i < j \leq n+1} (-1)^{j+1} f(g_1,\ \ldots,\ g_{i-1},\ [g_i,\ g_j],\ g_{i+1},\ \ldots,\ \hat{g}_j,\ \ldots,\ g_{n+1}) + [g_1,\ f(g_2,\ \ldots,\ g_{n+1})] + \sum_{i=2}^{n+1} (-1)^i [f(g_1,\ \ldots,\ \hat{g}_i,\ \ldots,\ g_{n+1}),\ g_1].$$

Loday and Pirashvili [L,P] prove that $d \circ d = 0$ in (1.3), thus establishing that $C^*(\mathfrak{g};\ N)$ is a cochain complex. For a Lie algebra $\mathfrak{h}$ and a representation $N$, the projection to the exterior product $\mathfrak{h}^{\otimes n} \to \mathfrak{h}^{\wedge n}$ induces a natural homomorphism

$$H^*_{Lie}(\mathfrak{h};\ N) \to HL^*(\mathfrak{h};\ N),$$

where $H^*_{Lie}$ is Lie-algebra cohomology.

We wish to extend Leibniz cohomology to an invariant for differentiable manifolds so that in a certain sense $HL^*$ is a non-commutative version of de Rham cohomology. In particular the non-commutativity arises by considering a cochain complex of tensors (from differential geometry) which are not necessarily skew-symmetric. Let $M$ be a differentiable manifold and $\alpha$ an $n$-tensor on $M$. Then, by definition, $\alpha$ is a differentiable section

$$\alpha : M \to \big(T^*M\big)^{\otimes n},$$

where $T^*M$ is the cotangent bundle of $M$. To define the de Rham cohomology groups, $H^*_{DR}(M)$, however, one uses forms $\omega \in \Omega^n(M)$, where $\omega$ is a differentiable section of the $n$-th exterior power of $T^*M$,

$$\omega : M \to \big(T^*M\big)^{\wedge n}.$$

Any tensor $\alpha$ determines a form $\omega_\alpha$ by first symmetrizing over the symmetric group $\Sigma_n$, i.e.

$$\omega_\alpha(v_1 \wedge v_2 \wedge \ldots \wedge v_n) = \sum_{\sigma \in \Sigma_n} (\mathrm{sgn}\,\sigma)\, \alpha\big(v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \ldots \otimes v_{\sigma(n)}\big).$$

This is the step which yields a graded commutative ring structure on

$$\Omega^*(M) = \sum_{n \geq 0} \Omega^n(M).$$

Recall E. Cartan's global formulation for the exterior derivative of a differential form in terms of vector fields (see Spivak [S, p.289] for a modern treatment). Let $X_1,\ X_2,\ \ldots,\ X_{n+1}$ be differentiable vector fields on $M$ and let $\omega \in \Omega^n(M)$. Then there is a unique $(n+1)$-form $d\omega$ such that

(1.4)

$$d\omega(X_1 \wedge X_2 \wedge \ldots \wedge X_{n+1}) = \sum_{i=1}^{n+1}(-1)^{i+1} X_i\big(\omega(X_1 \wedge \ldots \hat{X}_i \ldots \wedge X_{n+1})\big) +$$

$$+ \sum_{1 \le i < j \le n+1} (-1)^{j+1}\omega\big(X_1 \wedge \ldots \wedge X_{i-1} \wedge [X_i,\, X_j] \wedge X_{i+1} \wedge \ldots \hat{X}_j \ldots \wedge X_{n+1}\big),$$

where [ , ] is the Lie bracket of vector fields and

$$X_i\big(\omega(X_1 \wedge \ldots \hat{X}_i \ldots \wedge X_{n+1})\big)$$

is the directional derivative of the function

$$\omega(X_1 \wedge \ldots \hat{X}_i \ldots \wedge X_{n+1})\big) : M \to \mathbf{R}.$$

Although $d\omega$ is defined on vector fields in (1.4), the value of the exterior derivative

$$d\omega\big(X_1(p) \wedge X_2(p) \wedge \ldots \wedge X_{n+1}(p)\big)$$

depends only on one point $p \in M$, since $d\omega$ is linear over $C^\infty$ functions on $M$. Thus, $d\omega$ determines a well-defined $(n+1)$-form. See Spivak [S, pp.162, 289] for more details. The following lemma is immediate.

**1.5 Lemma.** *If in (1.4) $\omega$ is an $n$-tensor and the exterior product $\wedge$ is replaced with the tensor product $\otimes$, then $(d \circ d)(\omega) = 0$.*

*proof.* Let $\chi(M)$ be the vector space of all differentiable vector fields on $M$. Then $\chi(M)$ is a Lie algebra under the Lie bracket of vector fields, and, hence a Leibniz algebra. Let $C^\infty(M)$ be the vector space of $C^\infty$ functions $f : M \to \mathbf{R}$. Clearly $C^\infty(M)$ is a representation of $\chi(M)$ by

$$[X,\, f] = X(f),$$

where $X(f)$ is the derivative of $f$ in the direction $X$. Setting $[f,\, X] = -[X,\, f]$, we see that $C^\infty(M)$ becomes a representation of the Leibniz algebra $\chi(M)$. The lemma follows since $d \circ d = 0$ in (1.3). $\square$

It must be noted, however, that when using tensors, $d\omega$ is no longer linear over $C^\infty$ functions. The obstruction to $C^\infty(M)$-linearity is precisely the failure of a tensor to be skew-symmetric.

We propose the following definition.

**1.6 Definition.** *The Leibniz cohomology of a differentiable manifold $M$, written*

$$HL^*(\chi(M);\, C^\infty(M)),$$

*is the homology of the complex of continuous cochains*

$$Hom^{\text{cont}}_{\mathbf{R}}(\chi(M)^{\otimes n},\, C^\infty(M)), \quad n \geq 0,$$

*in the $C^\infty$ topology, where $\chi(M)$ is the Lie algebra of differentiable vector fields on $M$, and $C^\infty(M)$ denotes the ring of $C^\infty$ real-valued functions on $M$. The coboundary map $d$ is given in (1.3).*

Several observations are in order.

1.7 The continuous Lie-algebra cohomology of $\chi(M)$ is called Gelfand-Fuks cohomology, $H^*_{GF}(\chi(M);\, \mathbf{R})$, and is essentially the subject of the book *Cohomology of Infinite Dimensional Lie Algebras* [F].

1.8 There is a commutative diagram

$$\begin{array}{ccc} H^*_{DR}(M) & \xrightarrow{\ i\ } & H^*_{GF}(\chi(M);\, C^\infty(M)) \\ {\scriptstyle \pi\circ i}\big\downarrow & \swarrow \pi & \\ HL^*(\chi(M);\, C^\infty(M)) & & \end{array}$$

which arises from the fact that $H^*_{DR}(M)$ is the homology of the cochain complex

$$\text{Hom}^{\text{cont}}_{C^\infty(M)}(\chi(M)^{\wedge n},\, C^\infty(M)), \quad n \geq 0,$$

with boundary map given by (1.4). In fact there is a natural isomorphism of cochain complexes

$$\Omega^n(M) \to \text{Hom}^{\text{cont}}_{C^\infty(M)}(\chi(M)^{\wedge n},\, C^\infty(M)) \quad \text{given by}$$

$$\omega \mapsto \omega(X_1 \wedge X_2 \wedge \, \ldots \, \wedge X_n) : M \to \mathbf{R}$$
$$\omega(X_1 \wedge X_2 \wedge \, \ldots \, \wedge X_n)(p) = \omega(X_1(p) \wedge X_2(p) \wedge \, \ldots \, \wedge X_n(p)).$$

The map $i : H^*_{DR}(M) \to H^*_{GF}(\chi(M);\, C^\infty(M))$ is then induced by the inclusion

$$\text{Hom}^{\text{cont}}_{C^\infty(M)}(\chi(M)^{\wedge n},\, C^\infty(M)) \to \text{Hom}^{\text{cont}}_{\mathbf{R}}(\chi(M)^{\wedge n},\, C^\infty(M)).$$

See [F, p.21] for further details. The map

$$\pi : H^*_{GF}(\chi(M);\, C^\infty(M)) \to HL^*(\chi(M);\, C^\infty(M))$$

is induced by the projection $\chi(M)^{\otimes n} \to \chi(M)^{\wedge n}$.

1.9 If $M = G$ is a Lie group, then $\chi(G)^G$, the left-invariant vector fields on $G$ form the Lie algebra $\mathfrak{g}$ of $G$. For invariant functions on $G$, we have $C^\infty(G)^G = \mathbf{R}$. Thus,

$$T(\mathfrak{g}) = \sum_{n \geq 0} \mathfrak{g}^{\otimes n}$$

is a subcomplex of

$$\left\{C^\infty(G) \otimes \chi(G)^{\otimes n}\right\}_{n \geq 0}.$$

In fact $T(\mathfrak{g})$ is the original complex proposed by Loday [L1, p.324] for the Leibniz homology of a Lie algebra. Of course, for $G$ compact and connected, there is the Chevalley-Eilenberg [C,E] isomorphism

$$H^*_{Lie}(\mathfrak{g};\, \mathbf{R}) \simeq H^*_{DR}(G;\, \mathbf{R}).$$

Such an isomorphism appears not to hold in the setting of Leibniz cohomology however.

1.10 We wish now to compare, at least philosophically, Leibniz cohomology with cyclic homology, both of which serve as certain non-commutative versions of de Rham cohomology. For cyclic homology, what has been generalized to a non-commutative setting is the coefficient ring, i.e. $C^\infty(M)$ is replaced with a non-commutative algebra. For Leibniz cohomology the differential operator (exterior derivative) has been cast in a non-commutative framework. That these are in fact genuinely different generalizations can be seen from the calculation of $HP_*(C^\infty(M))$, the periodic cyclic homology of $C^\infty(M)$. We have [C]

$$HP_*(C^\infty(M)) \simeq H^*_{DR}(M)$$

for $M$ $\sigma$-compact.

The notions of non-commutativity can be summarized in the following diagram, where the column headings describe the coefficients, and the row headings describe the operators.

| | COMMUTATIVE | NON-COMMUTATIVE |
|---|---|---|
| COMMUTATIVE | de Rham (singular cohomology) | cyclic homology (K-theory) |
| NON-COMMUTATIVE | Leibniz cohomology of Lie algebras (foliations) | Leibniz cohomology of Leibniz algebras (quantum field theory) |

The parenthetical remarks refer to the type of invariants which can be computed by each theory. For the relation between cyclic homology and K-theory, see for example Goodwillie's paper [Gw]. In this paper we capture the Godbillon-Vey invariant for foliations via a Leibniz cohomology computation of a certain Lie algebra. In the next remark we observe how a Leibniz algebra arises naturally in V. Kač's work [K] on vertex operator algebras. The Leibniz cohomology groups of this algebra would then provide new invariants of quantum field theory.

1.11 We now review a construction of Kač [K] concerning vertex algebras, and show that a Lie algebra occuring in [K]—formed by a certain quotient—is genuinely a Leibniz (non-Lie) algebra if the quotient is not formed. Let $U$ be an associative, but not necessarily commutative algebra, such as $\mathrm{End}(V)$ for a vector space $V$. Let $a(z)$ be a formal power series in $z$ and $z^{-1}$ with coefficients in $U$, i.e. $a(z)$ is a formal distribution. For

$$a(z),\ b(z) \in U[[z,\, z^{-1}]],$$

the 0-th order product, $a(z)_{(0)}b(z)$, is defined as

$$a(z)_{(0)}b(z) = \mathrm{Res}_z\big(a(z)b(w) - b(w)a(z)\big),$$

where for $c(z) = \sum_{k\in\mathbf{Z}} c_k z^k$, we set $\mathrm{Res}_z(c(z)) = c_{-1}$. Then

$$a(z)_{(0)}b(z) = \sum_{k\in\mathbf{Z}} (a_{-1}b_k - b_k a_{-1}) z^k \in U[[z,\, z^{-1}]].$$

By the work of Kač, the quotient $\mathcal{A}/\partial\mathcal{A}$ is a Lie algebra with respect to the 0-th order product, where $\mathcal{A} = U[[z,\, z^{-1}]]$. Setting $[a(z),\, b(z)] = a(z)_{(0)}b(z)$, we see that this bracket is not skew-symmetric on $\mathcal{A}$, but satisfies what Loday and Pirashvili [L,P] call the left Leibniz identity:

$$[[a(z),\, b(z)],\, c(z)] = [a(z),\, [b(z),\, c(z)]] - [b(z),\, [a(z),\, c(z)]]. \tag{1.12}$$

Thus, $U[[z,\, z^{-1}]]$ is a left Leibniz algebra. There is a one-to-one correspondence between left Leibniz algebras $(\mathfrak{g},\ [\ ,\ ]_L)$ for which the the bracket satisfies (1.12) and right Leibniz algebras $(\mathfrak{g},\ [\ ,\ ]_R)$ for which the bracket satisfies (1.1). The correspondence is simply given by

$$\mathfrak{g} \longleftrightarrow \mathfrak{g} \qquad\qquad [y,\, x]_L \longleftrightarrow [x,\, y]_R.$$

Also, if $V$ is a $\mathbf{Z}/2$-graded vector space (i.e. a super-space), and $U = \mathrm{End}(V)$, then there are corresponding statements for $\mathbf{Z}/2$-graded Lie algebras and $\mathbf{Z}/2$-graded Leibniz algebras.

## §2. **Leibniz Cohomology of Formal Vector Fields**

In this section we begin the calculation of $HL^*(\chi(\mathbf{R}^n);\, \mathbf{R})$, the Leibniz cohomology of $\mathbf{R}^n$ with trivial coefficients. These groups are isomorphic to the continuous Leibniz cohomology of formal vector fields $W_n$, which we now describe. Let $E$ be the real vector space with basis

$$\left\{ \frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_2},\ \ldots,\ \frac{\partial}{\partial x_n} \right\},$$

and let $E' = \mathrm{Hom}_{\mathbf{R}}(E,\, \mathbf{R})$ be the dual space with dual basis $\{x_1,\, x_2,\, \ldots,\, x_n\}$. Then [F] [Gb]

$$W_n = \Big( \prod_{k\geq 0} S^k(E') \Big) \otimes E,$$

where $S^k$ denotes the $k$-symmetric power, and $W_n$ becomes a Lie algebra with the bracket given by

$$\Big[ \sum_{i=1}^{n} P_i \frac{\partial}{\partial x_i},\ \sum_{i=1}^{n} Q_i \frac{\partial}{\partial x_i} \Big] = \sum_{k=1}^{n} \Big( \sum_{j=1}^{n} P_j \frac{\partial Q_k}{\partial x_j} - Q_j \frac{\partial P_k}{\partial x_j} \Big) \frac{\partial}{\partial x_k}$$

where $P_i$ and $Q_i$ are formal power series in $x_1,\ x_2,\ \ldots,\ x_n$. Furthermore $W_n$ is a topological space in the $\mathcal{M}$-adic topology, where

$$||x_i^k|| = c^{-k}, \quad i = 1,\, 2,\, \ldots,\, n,$$

for some fixed integer $c > 1$. By $HL^*(W_n)$ we mean the continuous Leibniz cohomology with coefficients in $\mathbf{R}$. We now define a Taylor series map $\phi : \chi(\mathbf{R}^n) \to W_n$. For $X \in \chi(\mathbf{R}^n)$, let $X = \sum_{i=1}^{n} f_i e_i$, where the $f_i : \mathbf{R}^n \to \mathbf{R}$ are $C^\infty$ functions and the $e_i$ are the canonical vector fields on $\mathbf{R}^n$ (the unit vector fields following the coordinate axes). Then

$$\phi(X) = \sum_{i=1}^{n} \sum_{J} \frac{1}{j_1! j_2! \cdots j_n!} \frac{\partial^J (f_i)}{\partial x^J}(0)\ x_1^{j_1} x_2^{j_2} \ldots x_n^{j_n} \frac{\partial}{\partial x_i},$$

where $J$ is the multi-index $j_1,\ j_2,\ \ldots,\ j_n \geq 0$. It can be checked that $\phi$ is a continuous homomorphism of Lie algebras.

**2.1 Lemma.** *The induced map* $\phi^* : HL^*(W_n) \to HL^*(\chi(\mathbf{R}^n);\, \mathbf{R})$ *is an isomorphism.*

*Proof.* The proof follows essentially from the work of Bott and Segal [B,S] who prove that for continuous Lie algebra cohomology

$$\phi^* : H^*_{Lie}(W_n) \to H^*_{GF}(\chi(\mathbf{R}^n);\, \mathbf{R})$$

is an isomorphism. We do not offer complete details of their proof here, but instead state how a few ideas are extended from Lie cochains to Leibniz cochains. Let

$$\chi = \chi(\mathbf{R}^n), \quad C^q = \mathrm{Hom}^{\mathrm{cont}}_{\mathbf{R}}(\chi^{\otimes q},\, \mathbf{R}),$$

and define $P_q$ to be the subspce of $\chi^{\otimes q}$ spanned by $\mathbf{R}$-linear combinations of

$$f_{i_1} e_{i_1} \otimes f_{i_2} e_{i_2} \otimes \ldots \otimes f_{i_q} e_{i_q},$$

where each $f_{i_j}$ is a polynomial and

$$\sum_{j=1}^{q} \mathrm{degree}(f_{i_j}) \le q.$$

Let $C^q_P = \mathrm{Hom}^{\mathrm{cont}}_{\mathbf{R}}(P_q,\, \mathbf{R})$. Then $d : C^q_P \to C^{q+1}_P$, and we have a short exact sequence

$$0 \to B^* \to C^* \to C^*_P \to 0,$$

where $B^*$ is defined by $C^*_P = C^*/B^*$. As in the case for Lie algebra cohomology [B,S], we show that $B^*$ is acyclic, thus proving that $C^* \to C^*_P$ induces an isomorphism on $HL^*$.

Let $T_t : \mathbf{R}^n \to \mathbf{R}^n$ be the contraction defined by $T_t(x) = tx$, $0 < t \le 1$. Then $T_t$ acts on $C^q$ by

$$(T_t\alpha)(X_1 \otimes \ldots \otimes X_q) = t^{-q}\alpha\big(T^*_t(X_1) \otimes \ldots \otimes T^*_t(X_q)\big),$$

where $\alpha \in C^q$ and $X_i \in \chi$. Since the one-parameter family of diffeomorphisms $T_t$ is generated by a vector field $\rho$ on $\mathbf{R}^n$, we have

$$T_t^{-1} \cdot t(d/dt)T_t = \theta(\rho) : C^* \to C^*,$$

where for the Leibniz cochain complex

$$\theta(\rho)(\alpha)(X_1 \otimes X_2 \otimes \ldots \otimes X_q)$$
$$= -\sum_{i=1}^{q} \alpha(X_1 \otimes \ldots \otimes X_{i-1} \otimes [X_i,\, \rho] \otimes X_{i+1} \otimes \ldots \otimes X_q).$$

¿From the work of Loday and Pirashvili [L,P]

$$\theta(\rho) = i(\rho)d + d\, i(\rho),$$

where $i(\rho) : C^{q+1} \to C^q$ is given by

$$i(\rho)\alpha(X_1 \otimes \ldots \otimes X_q) = (-1)^{q+1}\alpha(X_1 \otimes X_2 \otimes \ldots \otimes X_q \otimes \rho).$$

It now follows as in [B,S] that for $\alpha \in B^*$,

$$\alpha = K\, d\alpha + d\, K\alpha, \quad \text{where}$$
$$K(\alpha) = i(\rho) \int_0^1 t^{-1} T_t(\alpha)\, dt.$$

Thus $K$ is a contracting chain homotopy for $B^*$ regardless of whether $\alpha$ is skew-symmetric. $\square$

We now turn to the calculation of $HL^*(W_n)$ in terms of certain universal properties of foliations. Let $\mathfrak{F}$ be a codimension $n$, $C^\infty$ foliation on a manifold $M$ with trivial normal bundle. Then there is a characteristic map associated to $\mathfrak{F}$ [B] [F] [H]

$$\text{(2.2)} \qquad \text{char}_{\mathfrak{F}} : H^*_{Lie}(W_n) \to H^*_{DR}(M).$$

If $\mathfrak{F}_t$ is a smooth one-parameter family of such foliations, then one can compute the derivative

$$\frac{d}{dt}\left[\text{char}_{\mathfrak{F}_t}(\alpha)\right|_{t=0} \in H^q_{DR}(M;\, \mathbf{R}).$$

The class $\alpha \in H^q_{Lie}(W_n)$ is called variable if there exists a family $\mathfrak{F}_t$ for which

$$\frac{d}{dt}\left[\text{char}_{\mathfrak{F}_t}(\alpha)\right|_{t=0} \neq 0.$$

Otherwise, $\alpha$ is called rigid. From Fuks [F] a necessary condition for the variability of $\alpha$ is that $\text{var}(\alpha) \neq 0$, where var is the homomorphism

$$\text{(2.3)} \qquad \text{var} : H^q_{Lie}(W_n) \to H^{q-1}_{Lie}(W_n;\, W_n')$$

given on the cochain level by

$$\text{var}(\alpha)(g_1,\, g_2, \ldots, g_{q-1})(g_0) = (-1)^{q-1} \alpha(g_0,\, g_1,\, g_2,\, \ldots,\, g_{q-1}).$$

Here and further $H^*(W_n;\, W_n')$ denotes the continuous Lie algebra cohomology with coefficients in the co-adjoint representation

$$W_n' = \text{Hom}^{\text{cont}}_{\mathbf{R}}(W_n,\, \mathbf{R}).$$

Our calculation of $HL^*(W_n)$ is in terms of the homomorphism var and the product structure on Leibniz cohomology.

Recall that for a Leibniz algebra $\mathfrak{g}$, Loday [L3] has defined a non-commutative, non-associative product on $HL^*(\mathfrak{g})$ which affords $HL^*(\mathfrak{g})$ the structure of a dual Leibniz algebra, where dual is taken in the sense of Koszul duality for operads [G,K] [L4]. More specifically, let

$$C^p = \text{Hom}(\mathfrak{g}^{\otimes p},\, \mathbf{R}), \quad \alpha \in C^p, \quad \beta \in C^q,$$

and define $\alpha\beta \in C^{p+q}$ by

(2.4)
$$(\alpha\beta)(g_1 \otimes g_2 \otimes \ldots \otimes g_{p+q})$$
$$= \sum_{\sigma \in Sh_{p-1,q}} \mathrm{sgn}(\sigma)\, \alpha(g_1 \otimes g_{\sigma(2)} \otimes \ldots \otimes g_{\sigma(p)}) \cdot \beta(g_{\sigma(p+1)} \otimes \ldots \otimes g_{\sigma(p+q)}),$$

where the sum is over all $(p-1,\, q)$-shuffles of the symmetric group $\Sigma_{p+q-1}$, i.e.

$$\sigma(2) < \sigma(3) < \ldots < \sigma(p),$$
$$\sigma(p+1) < \sigma(p+2) < \ldots < \sigma(p+q).$$

Then from [L3] $d(\alpha\beta) = (d\alpha)\beta + (-1)^{|\alpha|}\alpha(d\beta)$, and for $x \in HL^p(\mathfrak{g})$, $y \in HL^q(\mathfrak{g})$, $z \in HL^r(\mathfrak{g})$, we have
$$(xy)z = x(yz) + (-1)^{qr} x(zy).$$

In the sequel we will need the following formula for products of homogeneous elements in $HL^*(\mathfrak{g})$

(2.5)
$$\{a_0(a_1(a_2 \ldots (a_{p-1}a_p) \ldots))\} \cdot \{a_{p+1}(a_{p+2}( \ldots (a_{p+q-1}a_{p+q}) \ldots))\}$$
$$= \sum_{\sigma \in Sh_{p,q}} \pm\, a_0(a_{\sigma^{-1}(1)}(a_{\sigma^{-1}(2)}( \ldots (a_{\sigma^{-1}(p+q-1)}a_{\sigma^{-1}(p+q)}) \ldots))).$$

Note that we are using $\sigma^{-1}$ in the above formula. Related to this formula is Loday's observation [L3] that for a vector space $V$, the reduced tensor module

$$\bar{T}(V) = \sum_{k \geq 1} V^{\otimes k}$$

carries the structure of a free dual Leibniz algebra for which

(2.6)
$$(v_0,\, v_1,\, \ldots,\, v_p) \cdot (v_{p+1},\, \ldots,\, v_{p+q})$$
$$= \sum_{\sigma \in Sh_{p,q}} (v_0,\, v_{\sigma^{-1}(1)},\, v_{\sigma^{-1}(2)}, \ldots,\, v_{\sigma^{-1}(p+q)})$$

for homogeneous elements $v_i$. Signs can be used in (2.6) if $\bar{T}(V)$ is given a grading.

An essential technique for the calculation of $HL^*(\mathfrak{g})$ in the special case of a Lie algebra is the Pirashvili spectral sequence [P] which uses $H^*_{Lie}(\mathfrak{g})$ and $H^*_{Lie}(\mathfrak{g};\, \mathfrak{g}')$ to glean information about $HL^*(\mathfrak{g})$. In [P], Pirashvili provides the details for Leibniz homology. Here we outline the construction for cohomology and show that Loday's product is defined on the filtration which yields the cohomology spectral sequence. To begin, let $\mathfrak{g}$ be a Lie algebra and define $\Omega^n(\mathfrak{g})$ to be the real vector space of skew-symmetric (alternating) homomorphisms $\mathfrak{g}^{\otimes n} \to \mathbf{R}$. If $\mathfrak{g}$ has a topology, then all morphisms are taken to be continuous. For $\alpha \in \Omega^n(\mathfrak{g})$, it is evident that

$$\alpha(g_1,\, \ldots,\, g_i,\, \ldots,\, g_j,\, \ldots,\, g_n) = -\alpha(g_1,\, \ldots,\, g_j,\, \ldots,\, g_i,\, \ldots,\, g_n),$$

and $\Omega^*(\mathfrak{g})$ is the cochain complex for $H^*_{Lie}(\mathfrak{g})$. We have a short exact sequence

$$0 \longrightarrow \Omega^*(\mathfrak{g}) \longrightarrow C^*(\mathfrak{g}) \longrightarrow C^*(\mathfrak{g})/\Omega^*(\mathfrak{g}) \longrightarrow 0,$$

where $C^n(\mathfrak{g}) = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{g}^{\otimes n}, \mathbf{R})$. Clearly $\Omega^0(\mathfrak{g}) = C^0(\mathfrak{g})$ and $\Omega^1(\mathfrak{g}) = C^1(\mathfrak{g})$. Following Pirashvili's grading, set

$$C^*_{rel}[2] = C^*(\mathfrak{g})/\Omega^*(\mathfrak{g}).$$

We then have a long exact sequence

$$(2.7) \qquad \ldots \longrightarrow H^n_{Lie}(\mathfrak{g}) \longrightarrow HL^n(\mathfrak{g}) \longrightarrow H^{n-2}_{rel}(\mathfrak{g}) \longrightarrow H^{n+1}_{Lie}(\mathfrak{g}) \longrightarrow \ \ldots .$$

Define a left representation of $\mathfrak{g}$ on $\mathfrak{g}' = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{g}, \mathbf{R})$ by

$$(g\gamma)(h) = \gamma([h, g])$$

where $g,\ h \in \mathfrak{g}$ and $\gamma \in \mathfrak{g}'$. Let $\Omega^n(\mathfrak{g}; \mathfrak{g}')$ denote the vector space of skew-symmetric homomorphisms $\alpha : \mathfrak{g}^{\otimes n} \to \mathfrak{g}'$. Thus, $\Omega^*(\mathfrak{g}; \mathfrak{g}')$ is the cochain complex for $H^*_{Lie}(\mathfrak{g}; \mathfrak{g}')$. There are inclusions of cochain complexes

$$i_1 : \Omega^{n+1}(\mathfrak{g}) \to \Omega^n(\mathfrak{g}; \mathfrak{g}'), \quad i_2 : \Omega^n(\mathfrak{g}; \mathfrak{g}') \to C^{n+1}(\mathfrak{g})$$

given by

$$\big(i_1(\alpha)(g_1, g_2, \ldots, g_n)\big)(g_0) = (-1)^n\, \alpha(g_0, g_1, g_2, \ldots, g_n)$$
$$\big(i_2(\beta)\big)(g_0, g_1, \ldots, g_n) = (-1)^n\, \beta(g_1, \ldots, g_n)(g_0).$$

Note that $i_1$ has occured as var in (2.3), and $i_2$ allows us to consider an element of $\Omega^n(\mathfrak{g}; \mathfrak{g}')$ as an element of $C^{n+1}(\mathfrak{g})$ which is alternating on the last $n$ tensor factors of $\mathfrak{g}^{\otimes(n+1)}$. The short exact sequence

$$0 \longrightarrow \Omega^*(\mathfrak{g}) \longrightarrow \Omega^{*-1}(\mathfrak{g}; \mathfrak{g}') \longrightarrow CR^*(\mathfrak{g})[1] \longrightarrow 0,$$

for which $CR^*(\mathfrak{g})[1] = \Omega^{*-1}(\mathfrak{g}; \mathfrak{g}')/\Omega^*(\mathfrak{g})$, yields the long exact sequence

$$(2.8) \qquad \ldots \longrightarrow H^n_{Lie}(\mathfrak{g}) \longrightarrow H^{n-1}_{Lie}(\mathfrak{g}; \mathfrak{g}') \longrightarrow HR^{n-2}(\mathfrak{g}) \longrightarrow H^{n+1}_{Lie}(\mathfrak{g}) \longrightarrow \ \ldots .$$

Consider now the decreasing filtration of $C^*_{rel}$ given by $F^0_* = C^*_{rel}$, and for $s \geq 1$

$$F^s_* = \{\, f \in C^*(\mathfrak{g}) \,|\, f \text{ is alternating in the last } (s+1)\text{-many tensor factors} \,\}/\Omega^*(\mathfrak{g}).$$

**2.9 Lemma.** *If $f : \mathfrak{g}^{\otimes(n+s)} \to \mathbf{R}$ is alternating in the last $s$-many factors, then*

$$df : \mathfrak{g}^{\otimes(n+1+s)} \to \mathbf{R}$$

*is alternating in the last $s$-many factors.*

*Proof.* This can be checked by hand. Compare with [P].

**2.10 Theorem.** *There is a spectral sequence converging to* $H^*_{rel}(\mathfrak{g})$ *with*

$$E_2^{s,n} \simeq HL^n(\mathfrak{g}) \otimes HR^s(\mathfrak{g})$$

*provided that* $HL^*(\mathfrak{g})$ *and* $HR^*(\mathfrak{g})$ *are finite dimensional vector spaces in each dimension. The completed tensor product can be used if this finiteness condition is not satisfied.*

*Proof.* Let $\hat{\otimes}$ denote the completed tensor product. Then

$$\begin{aligned}
&\left(F^s/F^{s+1}\right)_n \simeq A/B \\
&A = C^n(\mathfrak{g}) \,\hat{\otimes}\, \{\, f \in C^{s+2}(\mathfrak{g}) \,|\, f \text{ is alternating in last } (s+1) \text{ tensor factors} \,\} \\
&B = C^n(\mathfrak{g}) \,\hat{\otimes}\, \Omega^{s+2}(\mathfrak{g}).
\end{aligned}$$

Thus, $E_0^{s,n} \simeq C^n(\mathfrak{g}) \,\hat{\otimes}\, CR^s(\mathfrak{g})$, and $E_1^{s,n} \simeq HL^n(\mathfrak{g}) \,\hat{\otimes}\, CR^s(\mathfrak{g})$. Since the co-adjoint action of $\mathfrak{g}$ on $HL^*(\mathfrak{g})$ is trivial [L1, 10.1.7], we conclude that

$$E_2^{s,n} \simeq HL^n(\mathfrak{g}) \,\hat{\otimes}\, HR^s(\mathfrak{g}).$$

If $HL^*(\mathfrak{g})$ and $HR^*(\mathfrak{g})$ are finitely generated in each dimension, then

$$E_2^{s,n} \simeq HL^n(\mathfrak{g}) \otimes HR^s(\mathfrak{g}).$$

□

**2.11 Theorem.** *If* $\alpha : \mathfrak{g}^{\otimes p} \to \mathbf{R}$ *and* $\beta : \mathfrak{g}^{\otimes q} \to \mathbf{R}$ *are alternating in the last* $s$ *factors, where* $s \leq p$ *and* $s \leq q$*, then* $\alpha\beta : \mathfrak{g}^{\otimes(p+q)} \to \mathbf{R}$ *is alternating in the last* $s$ *factors, i.e.* $F^{s-1} \cdot F^{s-1} \subset F^{s-1}$.

*Proof.* It suffices to show that for $p+q-s < i < p+q$, we have

$$\alpha\beta(g_1,\, \ldots,\, g_i,\, g_{i+1},\, \ldots,\, g_{p+q}) = -\alpha\beta(g_1,\, \ldots,\, g_{i+1},\, g_i,\, \ldots,\, g_{p+q}).$$

Let $\sigma$ be a $(p-1,\, q)$-shuffle of $\{\, 2,\, 3,\, \ldots,\, p+q\}$. There are four possibilities to consider.
(i) Suppose that both $i$ and $i+1$ appear in the sequence

$$\sigma(p+1),\ \sigma(p+2),\ \ldots,\ \sigma(p+q).$$

Since there are at most $p+q-(i+1)$-many possible choices for $j$ so that $i+1 < \sigma(j)$, the integers $i$ and $i+1$ must appear among the last $s$-many terms of the increasing sequence

$$\sigma(p+1),\ \sigma(p+2),\ \ldots,\ \sigma(p+q).$$

(ii) Suppose that $i$ appears in the sequence

$$\sigma(2),\ \sigma(3),\ \ldots,\ \sigma(p)$$

and $i+1$ appears in the sequence

$$\sigma(p+1),\ \sigma(p+2),\ \ldots,\ \sigma(p+q).$$

Let $\sigma(x) = i$ and $\sigma(y) = i+1$. Then $\hat{\sigma}$ defined by

$$\hat{\sigma}(j) = \begin{cases} \sigma(j), & j \neq x,\ j \neq y \\ \sigma(x), & j = y \\ \sigma(y), & j = x, \end{cases}$$

is a $(p-1,\, q)$-shuffle of $\{\,2,\, 3,\, \ldots,\, p+q\,\}$ having the opposite sign of $\sigma$.
(iii) If $i$ appears in the sequence

$$\sigma(p+1),\ \sigma(p+2),\ \ldots,\ \sigma(p+q),$$

and $i+1$ appears in the sequence $\sigma(2),\ \sigma(3),\ \ldots,\ \sigma(p)$, then proceed as in (ii).
(iv) If both $i$ and $i+1$ appear in the sequence $\sigma(2),\ \sigma(3),\ \ldots,\ \sigma(p)$, then argue as in (i). □

It may well be that certain terms in the sum for $\alpha\beta$ lie in filtration degree greater than $s$, which is useful in the computation of higher differentials for the Pirashvili spectral sequence.

For the reader's convenience we record the results of the calculation of $H^*_{Lie}(W_n)$ [B] [F] [Gb]. First observe that there is a monomorphism of Lie algebras

$$\varphi : \mathfrak{gl}_n(\mathbf{R}) \to W_n$$

given by $\varphi(E_{ij}) = x_i \frac{\partial}{\partial x_j}$, where $E_{ij}$ is the elementary matrix with 1 in the $i$-th row and $j$-th column, and zeroes everywhere else. The image of $\varphi$ consists of the one jets in $W_n$. Using the Hochschild-Serre spectral sequence associated to a subalgebra of a Lie algebra [H,S] and invariant theory [W], one has

**2.12 Theorem.** *[B] [F] [Gb] The $E_2$-term of the Hochschild-Serre spectral sequence converging to $H^*_{Lie}(W_n)$ has form*

$$E_2^{*,*} \simeq H^*_{Lie}(\mathfrak{gl}_n(\mathbf{R})) \otimes \big(\mathbf{R}[P_1,\, \ldots,\, P_n]/I\big),$$

*where $\mathbf{R}[P_1,\, \ldots,\, P_n]$ is the polynomial algebra with deg$(P_j) = 2j$, and $I$ is the ideal of $\mathbf{R}[P_1,\, \ldots,\, P_n]$ generated by polynomials of degree greater than $2n$. Moreover, letting*

$$H^*_{Lie}(\mathfrak{gl}_n(\mathbf{R})) = \Lambda(u_1,\, u_2,\, \ldots,\, u_n),$$

*for certain generators $u_i$ with deg$(u_i) = 2i-1$, the differentials are determined by*

$$d_{2r}(u_r) = P_r, \quad r = 1,\, 2,\, \ldots,\, n.$$

**2.13 Corollary.** *[Gb] We have $H^i_{Lie}(W_n) = 0$ for $1 \leq i \leq 2n$ and $i > n^2 + 2n$.*

**2.14 Corollary.** *[Gb] The Vey basis for $H^*_{Lie}(W_n)$ is given by the monomials*

$$u_{i_1} u_{i_2} \ldots u_{i_r} P_{j_1} P_{j_2} \ldots P_{j_s}, \quad \textit{where}$$

$$\begin{aligned} &1 \leq i_1 < i_2 < \ \ldots \ < i_r \leq n, \\ &1 \leq j_1 \leq j_2 \leq \ldots \ \leq j_s \leq n, \\ &j_1 + j_2 + \cdots + j_s \leq n, \\ &i_1 + j_1 + j_2 + \cdots + j_s > n, \\ &i_1 \leq j_1. \end{aligned}$$

*The above monomial is then in degree*

$$2(i_1 + i_2 + \cdots + i_r + j_1 + j_2 + \cdots + j_s) - r.$$

The following computation of Feigin, Fuks and Gelfand [F] is quite useful

**2.15 Theorem.** *[F] There is a natural isomorphism*

$$H^q_{Lie}(W_n;\, W'_n) \xrightarrow{\simeq} H^{2n+1}_{Lie}(W_n) \otimes H^{q-2n}_{Lie}(\mathfrak{gl}_n(\mathbf{R})),$$

*and*

$$var : H^{2n+1}_{Lie}(W_n) \to H^{2n}_{Lie}(W_n;\, W'_n)$$

*is an isomorphism, where var is given in (2.3).*

For its applications to the Godbillon-Vey invariant, we prove the the following

**2.16 Lemma.** *The natural map*

$$H^{2n+1}_{Lie}(W_n) \to HL^{2n+1}(W_n)$$

*is an isomorphism.*

*Proof.* From the Vey basis, we see that $H^{2n+2}_{Lie}(W_n) = 0$. Since

$$\mathrm{var} : H^{2n+1}_{Lie}(W_n) \to H^{2n}_{Lie}(W_n;\, W'_n)$$

is an isomorphism, it follows from (2.8) that $HR^{2n-1}(W_n) = 0$. Actually, $HR^i(W_n) = 0$ for $i = 0,\, 1,\, 2,\, \ldots,\, 2n-1$, and, therefore, $H^i_{rel}(W_n) = 0$ for $i = 0,\, 1,\, \ldots,\, 2n-1$. The lemma follows from (2.7). □

**2.17 Corollary.** *If $\alpha_{GV}$ is a cochain representing the universal Godbillon-Vey invariant in $H^{2n+1}_{Lie}(W_n)$, then $\alpha_{GV}$ represents a non-zero class in $HL^{2n+1}(W_n)$.*

*Proof.* The corollary follows from (2.16) □

We may now state the main calculational result of the paper.

**2.18 Theorem.** *For $n = 1,\, 2,\, 3$, there is an isomorphism of dual Leibniz algebras*

$$\begin{aligned} HL^*(W_n) &\simeq \mathbf{R} \oplus [Im \oplus A] \otimes T(A \oplus B), \\ Im &= Im(var) \\ A &= coker(var)[1] \\ B &\simeq ker(var)[-1], \end{aligned}$$

*where $T$ denotes the tensor algebra (as a dual Leibniz algebra), and $zx = 0$ for $x \in Im$, $z \in Im$ or $z \in A$. Also, $\mathbf{R}$ is in dimension zero.*

*Proof.* We illustrate several techniques germane to the proof by considering the special case $n = 1$ for which

$$\overline{H}^q(W_1) = \begin{cases} \mathbf{R}, & q = 3 \\ 0, & \text{otherwise}, \end{cases}$$

and ker(var) $= 0$. Note that $HR^2(W_1)$ is generated (as a vector space) by a tensor $h : W_1^{\otimes 4} \to \mathbf{R}$,

$$h \in \mathrm{coker}[\mathrm{var} : \Omega^4(W_1) \to \Omega^3(W_1;\, W'_1)],$$

and $HR^i(W_1) = 0$ for $i \neq 2$. Moreover the alternating tensor $f : W_1^{\otimes 3} \to \mathbf{R}$ corresponding to the Godbillon-Vey invariant generates $HL^3(W_1)$. In the Pirashvili spectral sequence $d_r = 0$ for $r \geq 2$, whence as vector spaces

$$HL^*(W_1) \simeq \mathbf{R} \oplus [HL^3(W_1) \oplus HR^2(W_1)] \otimes T(HR^2(W_1)). \tag{2.19}$$

Certainly $f^2 = 0$ in $HL^*(W_1)$, since $HL^6(W_1) = 0$, but also since $f^2 \in F^4$ (the fourth filtration degree), and

$$H_*(F^4/F^5) = 0.$$

We now show that $h \otimes h$ is represented by $h^2$ (Loday's product) in $HL^*(W_1)$. This would follow by showing that $h \otimes h$ is represented by $h^2$ in $H_*(F^2/F^3)$. From (2.11) we conclude that $h^2 - h \otimes h \in F^2$. By omitting the identity permutation in the definition of Loday's product, we have

$$h^2 - h \otimes h \in F^3. \tag{2.20}$$

The proof of this requires four cases as in (2.11) and the fact that no permutation shuffles a factor into the first tensor position, where $h$ is not alternating. (See (2.4) for the formula of the shuffles.) Thus, $h^2 - h \otimes h = 0$ in $E_1$ and in $HL^*(W_1)$. By a similar argument,

$$h(hh), \quad h \otimes h^2, \quad \text{and} \quad h \otimes h \otimes h$$

represent the same element in $HL^*(W_1)$. By induction, the cochains

$$\underbrace{h(h(h \ldots (hh) \ldots))}_{n\text{-many}} \quad \text{and} \quad \underbrace{h \otimes h \otimes \ldots \otimes h}_{n\text{-many}}$$

represent the same element in $HL^*(W_1)$. A similar statement is valid for

$$f(h(h \ldots (hh) \ldots)) \quad \text{and} \quad f \otimes h \otimes h \otimes \ldots \otimes h.$$

Of course, $hf$ represents zero in $HL^*(W_1)$, since $hf \in F^5$.

For any positive integer $n$ and $q > n^2 + 2n$, we have $H^q_{Lie}(W_n) = 0$, and, therefore,

$$HL^q(W_n) \simeq H^{q-2}_{rel}(W_n).$$

Since $H^q_{Lie}(W_n) = 0$ for $q = 1, 2, \ldots, 2n$, and $H^q(W_n; W_n') = 0$ for $q = 0, 1, 2, \ldots, 2n-1$, we see that $HL^q(W_n) = 0$ for $q = 1, 2, \ldots, 2n$. Also, $HR^q(W_n) = 0$ for $q = 0, 1, 2, \ldots, 2n-1$ and for $q \geq n^2 + 2n$. It follows that for $0 \leq q \leq 4n$

$$H^q_{rel}(W_n) \simeq HR^q(W_n) \simeq \Big(\ker(\mathrm{var})[-3] \oplus \mathrm{coker}(\mathrm{var})[-1]\Big)_q.$$

Thus, for $1 \leq q \leq 4n + 2$

$$HL^q(W_n) \simeq \mathrm{Im}(\mathrm{var}) \oplus \mathrm{coker}(\mathrm{var})[+1].$$

We now show that for $r \geq 2$, $d_r(\alpha \otimes \beta) = 0$, where $\alpha \in HL^q(W_n)$, $1 \leq q \leq 4n+2$, and

$$\beta \in HR^j(W_n) \simeq \ker(\text{var})[-3] \oplus \text{coker}(\text{var})[-1], \quad j \geq 0.$$

In the $E^2$ term represent $\alpha \otimes \beta$ as $\alpha\beta$ (Loday's product) in a fashion similar to (2.20). Then

$$d(\alpha\beta) = (d\alpha)\beta + (-1)^{|\alpha|}\alpha(d\beta).$$

Necessarily $d\alpha = 0$. If $\beta$ coresponds to an element of coker(var), then $d\beta = 0$. If $\beta$ corresponds to an element of ker(var), then

$$d\beta \in \Omega^*(W_n),$$

and $d\beta$ represents zero in $HL^*(W_n)$. Thus, $\alpha \cdot d\beta$ is zero in $HL^*(W_n)$, and the map

$$HL^*(W_n) \to H^{*-2}_{rel}(W_n)$$

sends the Leibniz cohomology class of $\alpha \cdot d\beta$ to zero. It follows that $d_r(\alpha \otimes \beta) = 0$, $r \geq 2$.

Now, $H^{4n+1}_{rel}(W_n) \simeq HL^{2n+1}(W_n) \otimes HR^{2n}(W_n)$, and the boundary map

$$\partial : H^{4n+1}_{rel}(W_n) \to H^{4n+4}_{Lie}(W_n)$$

is zero for $n^2 + 2n < 4n + 4$, i.e. $n = 1, 2, 3$. (Recall that $H^q_{Lie}(W_n) = 0$ for $q > n^2 + 2n$.) Specializing to the cases $n = 1, 2, 3$, the theorem will follow once we show that

$$d_r\big(\alpha_1(\alpha_2(\ldots(\alpha_{k-1}\alpha_k)\ldots))\big) = 0, \quad r \geq 2,$$

where $\alpha_1 \in \text{Im}(\text{var})$ or $\alpha_1 \in \text{coker}(\text{var})$, and for $i \geq 2$, either $\alpha_i \in \text{coker}(\text{var})$ or $\alpha_i \in \ker(\text{var})$. This follows from

$$\begin{aligned} d\big(\alpha_1(\alpha_2(\ldots(\alpha_{k-1}\alpha_k)\ldots))\big) &= (d\alpha_1)(\alpha_2(\ldots(\alpha_{k-1}\alpha_k)\ldots)) \\ &\quad + (-1)^{|\alpha_1|}\alpha_1 d(\alpha_2(\alpha_3\ldots(\alpha_{k-1}\alpha_k)\ldots)). \end{aligned}$$

As a cochain, $d\alpha_1 = 0$. Letting

$$\gamma = \alpha_2(\alpha_3(\ldots(\alpha_{k-1}a_k)\ldots)),$$

we note that $d\gamma$ represents zero in $HL^*(W_n)$. Thus, $\alpha_1 \cdot d\gamma$ is zero in $HL^*(W_n)$, and the map

$$HL^*(W_n) \to H^{*-2}_{rel}(W_n)$$

sends the class of $\alpha_1 \cdot d\gamma$ to zero.

We conclude with a few words about products $zx$, where $x \in \text{Im}(\text{var})$, and either $z \in \text{coker}(\text{var})$ or $z \in \text{Im}(\text{var})$. For $n = 2$, the smallest non-zero dimension of $x$ is 5, and the smallest filtration in which $z$ could lie is $F^4$. Since $zx \in F^9$, and all non-zero elements of $HL^*(W_2)$ are in filtration degree at most 7, we conclude that $zx$ represents zero in $HL^*(W_2)$. For $n = 3$, the smallest non-zero dimension of $x$ is 7, and the smallest filtration degree of $z$ is 6. Although $HR^{13}(W_3) \neq 0$, it can be checked that

$$HR^{13}(W_3) \simeq \text{coker}[H^{15}_{Lie}(W_3) \to H^{14}_{Lie}(W_3;\, W_3')].$$

In [F, p.101], it is shown that the $H^*_{Lie}(W_n)$-module structure in $H^*_{Lie}(W_n;\, W_n')$ is trivial, i.e. the product

$$H^*_{Lie}(W_n;\, W_n') \otimes H^*_{Lie}(W_n) \to H^*_{Lie}(W_n;\, W_n')$$

is zero. Since this product gives precisely $zx$, this element represents zero in $H^{14}_{Lie}(W_3;\, W_3')$. It should also be noted that all non-zero elements of $HL^*_{Lie}(W_3)$ are in filtration degree at most 14. □

## §3. **The Characteristic Map for Leibniz Cohomology**

Recall that for a $C^\infty$, codimension $n$ foliation $\mathfrak{F}$ on $M$ with trivial normal bundle, there is a characteristic homomorphism [B] [F] [H]

$$\mathrm{char}_{\mathfrak{F}} : H^*_{Lie}(W_n) \to H^*_{DR}(M).$$

In this section we conjecture the existence of a homomorphism

$$L_{\mathfrak{F}} : HL^*(W_n) \to HL^*(\chi(M);\, C^\infty(M))$$

so that the following diagram commutes

$$\begin{array}{ccc} H^*_{Lie}(W_n) & \xrightarrow{\mathrm{char}_{\mathfrak{F}}} & H^*_{DR}(M) \\ \downarrow & & \downarrow \\ HL^*(W_n) & \xrightarrow[L_{\mathfrak{F}}]{} & HL^*(\chi(M);\, C^\infty(M)), \end{array} \tag{3.1}$$

where $H^*_{DR}(M) \to HL^*(\chi(M);\, C^\infty(M))$ is given in (1.8). A fixed foliation determines a splitting of tangent bundle

$$TM \simeq T_L M \oplus T_\eta M, \tag{3.2}$$

where $T_L M$ is the tangent space to the leaves and $T_\eta M$ is the normal bundle. For the special case $n = 1$, let $\omega$ be a determining one-form for the foliation $\mathfrak{F}$. Then $\omega(v_p) = 0$ for $v_p \in T_L M$, and $\omega(v_p) = 1$,where $v_p$ is chosen to be a unit vector with positive orientation in $T_\eta M$. Moreover, the splitting of the tangent bundle (3.2) yields a vector space isomorphism

$$\chi(M) \simeq \chi_L(M) \oplus \chi_\eta(M),$$

where $\chi_L(M)$ are vector fields along the leaves and $\chi_\eta(M)$ are vector fields perpendicular to the leaves. Given any $X \in \chi(M)$, we may write $X$ uniquely as a sum $X_L + X_\eta$, where

$$X_L \in \chi_L(M) \quad \text{and} \quad X_\eta \in \chi_\eta(M).$$

Define $L_\omega : \chi(M) \to C^\infty(M)$ by $L_\omega(X) = f$, where $f(p) = \omega(X_\eta(p))$ for all $p \in M$. Then $L_\omega$ is simply the image of $\omega$ under the inclusion

$$\Omega^1(M) \to \mathrm{Hom}^{\mathrm{cont}}_{\mathbf{R}}(\chi(M),\, C^\infty(M))$$

given in (1.8). In this way, all canonical one-forms used to define $\mathrm{Char}_{\mathfrak{F}}$ in the commutative framework occur as elements of

$$\mathrm{Hom}_{\mathbf{R}}^{\mathrm{cont}}(\chi(M), C^{\infty}(M)).$$


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