SIMPLICIAL RAPID DECAY ALGEBRAS ASSOCIATED TO A DISCRETE GROUP

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§0 Introduction

In [1], A. Connes introduced a fundamental technique for studying the topological K-theory of $C^*\pi$. The idea is to pass to a subalgebra of $C^*(\pi)$ which has the same topological K-theory, and which additionally satisfies some rapid decay conditon. These algebras are the non-commutative analogue of the algebra of smooth functions on a manifold, hence the occasional name "smooth algebras". In the presence of such rapid decay constraints, the topological cyclic homology of the algebra is often not far different from the cyclic homology of the complex group algebra.

One such algebra is $H_L^\infty(\pi)$, originally introduced by Haagarup for free groups and later studied by Jolissaint and de la Harpe for hyperbolic groups. When $H_L^\infty(\pi)$ is a subalgebra of $C_r^*(\pi)$, the group π is said to have property RD (for rapid decay; c.f. [3], [4], [5], [6]). This is usually not the case, even for discrete subgroups of Lie groups. In the non-RD case it is possible to construct a sub-algebra of $C_r^*(\pi)$ with the same topological K-theory which also embeds as a subspace of $H_L^\infty(\pi)$. One such example is the Connes "technical algebra" described in [5] and [6]. Another is the Connes-Moscovici algebra which appears in [2]. Because of the embedding into $H_L^\infty(\pi)$, it is natural to refer to these different algebras as rapid decay algebras.

In this paper we extend the construction of rapid decay algebras to the simplicial setting. Our starting point is a triple of the form $(\Gamma, \mathbb{X}, L)$ where Γ is an augmented simplicial group, $\mathbb{X}$ a generating set (closed under degeneracies) and L a simplicial word-length function on Γ. We say $(\Gamma, \mathbb{X}, L)$ is p-bounded if all of the face and degeneracy maps of Γ are polynomially bounded in word-length, and type P if in addition $\widetilde{\Gamma} = \ker(\Gamma \twoheadrightarrow \pi_0(\Gamma))$ admits a contracting degeneracy $s.'$ which is also polynomially bounded in word-length. The resolution is type B if the polynomials in question can always be taken to be the identity. In section 2, we show that any countable discrete group π admits a type B resolution where Γ is degreewise free, and all of the degeneracies, including $s.'$, preserve word-length. In section 3, we show how the existence of an extra degeneracy on $\widetilde{\Gamma}$ yields a chain contraction $\widetilde{s}_*$ of the chain complex $C^a_*(\Gamma)$ associated to $\mathbb{C}[\Gamma]$; this is another way of showing that $\mathbb{C}[\Gamma]$ is a resolution of $\pi_0(\mathbb{C}[\Gamma]) = \mathbb{C}[\pi_0(\Gamma)] = \mathbb{C}[\Gamma_{-1}]$. In section 4 we prove the main result of the paper, summarized in the following theorem.

Theorem 1. *Let $(\Gamma, \mathbb{X}, L)$ be a type B resolution of $\pi = \Gamma_{-1}$, with Γ_n free for all $n \geq 0$. Then associated to $(\Gamma, \mathbb{X}, L)$ is an augmented simplicial Fréchet algebra $\widetilde{T}^\infty_L(\Gamma)$ containing $\mathbb{C}[\Gamma]$. $\widetilde{T}^\infty_L(\Gamma)_{-1}$ is a subalgebra of $C^*_r(\pi)$ with the same topological K-groups above dimension 0. In addition, the contraction $\widetilde{s}_*$ of $C^a_*(\Gamma)$ extends to a bounded contraction of $C^t_*(\Gamma)$ = the associated chain complex of $\widetilde{T}^\infty_L(\Gamma)$.*

The construction involves a combination of all three of the rapid decay algebras mentioned above.

§1 Preliminaries

We recall certain results and constructions from simplicial topology. Throughout we will be working with augmented simplicial objects. The basic example is that of an augmented simplicial group, which consists of a collection of groups $\{\Gamma_n\}$ indexed on the integers $n \geq -1$, together with face maps $\partial_i : \Gamma_n \to \Gamma_{n-1}$ and degeneracy maps $s_j : \Gamma_n \to \Gamma_{n+1}$, defined for $0 \leq i \leq n$ with $n \geq 0$ which are group homomorphisms and satisfy the simplicial identities. The face map $\partial_0 : \Gamma_0 \to \Gamma_{-1}$ is called the augmentation map, and alternatively denoted by ε. For each $n \geq 1$, composing ε with any iterated sequence of face maps $\partial_I : \Gamma_n \to \Gamma_0$ defines a surjection $\varepsilon_n : \Gamma_n \to \Gamma_{-1}$ which is independent of the choice of I in ∂_I. Taking $\varepsilon_{-1} = id$, the collection of homomorphisms $\{\varepsilon\}_{n\geq -1}$ defines a surjection of simplicial groups $\varepsilon. : \Gamma \twoheadrightarrow \Gamma_{-1}$ where Γ_{-1} is viewed as a simplicial group with trivial simplicial structure. Given an augmented simplicial group $\Gamma = \{\Gamma_n\}_{n\geq -1}$, its associated simplicial group $\widetilde{\Gamma} = \{\Gamma_n\}_{n\geq 0}$ is gotton by forgetting Γ_{-1}. We write Γ_n^j for

$$\text{(1.1)} \qquad \Gamma_n^j = \cap_{i=0}^j (\ker \partial_i : \Gamma_n \to \Gamma_{n-1}) \qquad -1 \leq j \leq n\,,\ n \geq -1$$

By convention, $\Gamma_n^{-1} = \Gamma_n$. The <u>combinatorial homotopy groups</u> of $\widetilde{\Gamma}$, which we write as $\pi_*(\widetilde{\Gamma})$ are defined for $* \geq 0$ by the equality

$$\text{(1.2)} \qquad \begin{aligned} \pi_n(\widetilde{\Gamma}) &= \Gamma_n^n / \partial_{n+1}(\Gamma_{n+1}^n) \qquad n \geq 1, \\ \pi_0(\Gamma) &= \Gamma_0 / \partial_1(\Gamma_1^0) \end{aligned}$$

When Γ is augmented, we set $\pi_*(\Gamma) = \pi_*(\widetilde{\Gamma})$. Then $\widetilde{\Gamma}$ is a (simplicial) resolution if $\pi_i(\Gamma) = 0$ for $i > 0$, and Γ is an (augmented simplicial) resolution if $\widetilde{\Gamma}$ is a resolution and $\pi_0(\widetilde{\Gamma}) \underset{\varepsilon}{\xrightarrow{\cong}} \Gamma_{-1}$. Alternatively, Γ is an augmented simplicial resolution if and only if the augmentation map $\varepsilon : \Gamma \twoheadrightarrow \Gamma_{-1}$ is a weak equivalence. We call either $\widetilde{\Gamma}$ or Γ a free (simplicial) resolution if Γ_n is a free group for all $n \geq 0$. A basic fact is that for any discrete group π there is a free augmented simplicial resolution Γ with $\Gamma_{-1} = \pi$. In the following discussion all chain complexes C_* are assumed to be connective ($C_* = 0$ if $* < 0$) unless otherwise stated. An augmented simplicial chain complex F will be written as $F = \{F_n\}_{n\geq -1}$, where for each n the corresponding chain complex is $F_n = ((F_n)_k, d_{n,k})_{k\geq 0}$. Note that for each k, the augmented simplicial structure on F makes $((F_n)_k)_{n\geq -1}$ an augmented simplicial abelian group.

Now suppose $A.$ is an augmented simplicial abelian group and $\widetilde{A}.$ the corresponding simplicial abelian group. The associated chain complex of $\widetilde{A}.$, written as $\widetilde{A}_*$, is degreewise the same as $\widetilde{A}.$ but with the differential $d_n : \widetilde{A}_n \to \widetilde{A}_{n-1}$ given as the alternating sum of face maps $d_n = \sum_{i=0}^{n}(-1)^i\partial_i$. That $\widetilde{A}_* = (\widetilde{A}_n, d_n)_{n\geq 0}$ is actually a chain complex follows from the simplicial identities. A classical result (due to Moore) says that there is an isomorphism $\pi_*(\widetilde{A}.) \cong H_*(\widetilde{A}_*)$ for all simplicial abelian groups $\widetilde{A}.$. The associated complex of A_* is defined to be the same as that of $\widetilde{A}. : A_* = \widetilde{A}_*$.

If F is a discrete group and $\mathbb{X}$ a set of generators for F, we will denote this data by the pair $(F, \mathbb{X})$. A word-length function L on F with respect to $\mathbb{X}$ is called admissable if it is bounded below by the standard word-length function on $(F, \mathbb{X})$. L is proper if in addition the sets $\{x \in F \mid L(x) = n\}$ are finite for all $n \geq 0$. Any admissable word-length function on F is proper when F is finitely generated, but not in general. All word-length functions appearing in this paper are assumed proper unless explicitly stated otherwise.

Suppose that F resp. F' are two groups equipped with word-length functions L resp. L' . A map $\varphi : F \to F'$ is called polynomially bounded (or p-bounded) if there exists a polynomial function in one variable P such that

$$L'(\varphi(x)) \leq P(L(x))$$

for each element x in F . This definition applies not just to homomorphisms, but also to set maps from F to F' . The notion of p-boundedness obviously depends on the word-length functions L , L' being used, and is not a property intrinsic to the map φ . We will say that φ is 1-bounded, or simply bounded, if P may be taken to be the identity function.

Let $\mathbb{X} = \{\mathbb{X}_n\}_{n\geq -1}$ be a set of generators of the augmented simplicial group Γ and L a word-length function on $(\Gamma, \mathbb{X})$; thus $L = \{L_n\}_{n\geq -1}$ where L_n denotes the word-length function on $(\Gamma_n, \mathbb{X}_n)$. We call $(\Gamma, \mathbb{X}, L)$ p-bounded resp. bounded if all the face and degeneracy maps of Γ are p-bounded resp. bounded with respect to L . For convenience, we will additionally require of the generating set $\mathbb{X}$ that it be closed under degeneracies (this can always be arranged).

Assume that $(\Gamma, \mathbb{X}, L)$ is a p-bounded resp. bounded simplicial group which is also a resolution. Let

$$\Gamma' = \ker (\Gamma \overset{\varepsilon_*}{\twoheadrightarrow} \Gamma_{-1}) \tag{1.3}$$

Then $\Gamma' = \{\Gamma'_n = \ker (\Gamma_n \overset{\varepsilon_n}{\twoheadrightarrow} \Gamma_{-1})\}_{n\geq -1}$ is a contractible (augmented) simplicial group with $\Gamma'_{-1} = \{id\}$ (for notational convenience, we set $\varepsilon_{-1} = id$). We will additionally assume Γ' admits an extra simplicial degeneracy which realizes the contraction:

$$\{s'_{n+1} : \Gamma'_n \to \Gamma'_{n+1}\}_{n\geq -1} \ . \tag{1.4}$$

Such a degeneracy map exists for Γ' when Γ is a free simplicial resolution (or more generally a cartesian product of free resolutions).

Definition 1.5 . $(\Gamma, \mathbb{X}, L)$ is type $P(m)$ resp. $B(m)$ if Γ' admits an extra degeneracy $s'. = \{s'_{n+1}\}_{n\geq -1}$ with s'_{n+1} p-bounded resp. 1-bounded (with respect to the word-length function of Γ restricted to Γ') for all $n \leq m$.

Clearly, Γ' may be contractible without Γ being type $B(m)$ or even $P(m)$ for any m . We say that $(\Gamma, \mathbb{X}, L)$ is type P resp. type B if it is type $P(m)$ resp. type $B(m)$ for all m .

Suppose $(\pi, \mathbb{X})$ is a group with a countable set of generators $\mathbb{X}$ in which every element g has a unique reduced word representing it

$$g = x_1 x_2 \dots x_q \, , \quad x_i \in \mathbb{X} \tag{1.6}$$

and L is a word-length function on π . Then L is called semi-standard if it is bounded below by the standard word-length function and

$$L(g) = \sum_{i=1}^{p} L(x_i) \tag{1.7}$$

where g and the x_i are as in (1.6); thus, a semi-standard word-length function differs from the standard one only in how it is weighted on generators. Note that if L is semi-standard, then it is proper if and only if $L|_{\mathbb{X}} : \mathbb{X} \to \mathbb{N}$ is a proper function. A simplicial word-length function is semi-standard if it is semi-standard in each degree.

We will often used a fixed choice of section

$$s(0) : \pi \rightarrowtail \Gamma_0$$

of the augmentation map, on the level of sets. We will call such a section of sets admissable if it is minimal with respect to word-length and $s(1) = 1$.

§2 Constructing type B resolutions

Theorem 2.1. *Let π be a countably generated discrete group. Then there exists a free resolution $(\widetilde{\Gamma}, \widetilde{\mathbb{X}}, \widetilde{L})$ of π which is type B.*

Proof. Let $\mathbb{X}_0$ be a countable generating set for π, Γ_0 the free group on $\mathbb{X}_0$, and $\varepsilon : \Gamma_0 \twoheadrightarrow \Gamma_{-1} = \pi$ the obvious surjection. Choose a proper function $f_0 : \mathbb{X}_0 \to \mathbb{Z}^+$, and let L_0 be the unique semi-standard word-length function on Γ_0 whose restriction to $\mathbb{X}_0$ is f_0 . Define L_{-1} on Γ_{-1} by $L_{-1}(g) = \min_{h\in\varepsilon^{-1}(g)}\{L_0(h)\}$, and set $\mathbb{X}_{-1} = \mathbb{X}_0$. Then $(\Gamma(0), \mathbb{X}(0), L(0))$ is $(\Gamma_{-1}, \mathbb{X}_{-1}, L_{-1})$ in dimension -1, $(\Gamma_0, \mathbb{X}_0, L_0)$ in each non-negative degree, with all face and degeneracy maps above dimension 0 equal to the identity. Clearly $(\Gamma(0), \mathbb{X}(0), L(0))$ is 1-bounded.

By induction, we may assume that a 1-bounded free simplicial group $(\Gamma(m-1), \mathbb{X}(m-1), L(m-1))$ has been constructed such that the kernel of the augmentation map $\Gamma(m-1)' = \ker(\Gamma(m-1) \to \pi)$ is $(m-2)$-connected, and admits a contracting degeneracy $\{s'_{p+1}\}_{0\le p\le m-2}$ through dimension $(m-2)$ which is 1-bounded. Note that $\Gamma(m-1)'_{-1} = \{1\}$, $s'_0 : \Gamma(m-1)'_{-1} \to \Gamma(m-1)'_0$ is the inclusion of the trivial group, and $\partial_0 = \varepsilon : \Gamma(m-1)'_0 \to \Gamma(m-1)'_{-1}$ is the projection to the trivial group.

Fix a set $\mathbb{X}(m-1)'_{(m-1)}$ of generators for $\Gamma(m-1)'_{(m-1)}$. Let

$$\begin{aligned} \mathbb{X}(m)_j &= \mathbb{X}(m-1)_j \ \text{ for } \ j \le (m-1)\ , \\ \mathbb{X}(m)_m &= \mathbb{X}(m-1)_m \coprod \{y_{m,i}\}\ , \\ \mathbb{X}(m)_n &= \mathbb{X}(m-1)_n \coprod \{s(y_{m,i})\} \ \text{ for } \ n > m \end{aligned} \tag{2.2}$$

where the last coproduct is over all iterated degeneracies s from dimension m to dimension n. Face maps are given by

$$\begin{aligned} \partial_j(y_{m,i}) &= s'_{m-1}\partial_j(x_{m-1,i}) \ \text{ for } \ 0 \le j < m \\ \partial_m(y_{m,i}) &= x_{m-1,i} \end{aligned} \tag{2.3}$$

where $x_{m-1,i} \in \mathbb{X}(m-1)'_{m-1}$, and ∂_m induces an isomorphism of sets

$$\partial_m : \{y_{m,i}\} \stackrel{\cong}{\rightarrow} \mathbb{X}(m-1)'_{m-1}$$

Proceeding as before, we define $\Gamma(m)$ to be the free simplicial group on $\mathbb{X}(m)$. $L(m)$ is uniquely defined by the following four properties:

(2.4) i) It equals $L(m-1)$ on $\mathbb{X}(m-1)$

(2.4) ii) Given a collection of elements $\{y_k \in \{y_{m,i}\}\}$ with

$$\partial_m(y_k) = x_k \in \mathbb{X}(m-1)'_{m-1}$$

for each k, we have

$$L(m)_m(y_1^{n_1} y_2^{n_2} \ldots y_m^{n_m}) = L(m-1)_{m-1}(x_1^{n_1} x_2^{n_2} \ldots x_m^{n_m})$$

(2.4) iii) If $x \in \Gamma(m)_m$ is written as $x = w_1 w_2 \ldots w_p$ with w_{2i-1} in $\Gamma(m-1)_m$ and w_{2i} a product of powers of the elements $\{y_{m,i}\}$ then

$$L(m)_m(x) = \sum_{i=1}^{p} L(m)_m(w_i)$$

(2.4) iv) If $s : \Gamma(m)_m \rightarrow \Gamma(m)_n$ is an iterated degeneracy, then $L(m)_n(s(x)) = L(m)_m(x)$. If $w = w_1 w_2 \ldots w_q \in \Gamma(m)_n$ is a product of degenerate elements w_i then $L(m)_n(w) = \sum_{i=1}^{q} L(m)_n(w_i)$.

Then $L(m-1)$ proper implies $L(m)$ proper, and the contracting degeneracy s' for $\Gamma(m)'$ is now extended through dimension $(m-1)$ by setting

(2.5) $$s'_{m-1}(x_{m-1,i}) = y_{m,i} \, .$$

(2.3) and (2.5) guarantee that s' satisfies the required simplicial identities through dimension $(m-1)$. (2.3) – (2.5) and induction imply that all of the degeneracy maps (including s') through dimension $(m-1)$ and all of the face maps through dimension m are 1-bounded. Let

(2.6) $$(\widetilde{\Gamma}, \widetilde{\mathbb{X}}, \widetilde{L}) = \varinjlim_{m} \{(\Gamma(m), \mathbb{X}(m), L(m))\} \, .$$

Then $(\widetilde{\Gamma}, \widetilde{\mathbb{X}}, \widetilde{L})$ is a type B resolution of π . //

We remark that in the above construction, all of the degeneracies, including the extra contracting degeneracy, preserve word-length.

§3 Simplicial group algebras and resolutions

For a ring R, the correspondence $\pi \mapsto R[\pi]$ which associates to a discrete group π its group algebra with coefficients in R defines a functor from (groups) to (rings). This extends naturally to an augmented simplicial context, so that if Γ is an augmented simplicial group, $R[\Gamma]$ is the augmented simplicial ring

$$R[\Gamma] = \{[n] \mapsto R[\Gamma_n]\}_{n \geq -1}$$

with face and degeneracy maps induced by those of Γ.

Viewing $R[\Gamma]$ as an augmented simplicial abelian group by forgetting the ring structure, we may compute its combinatorial homotopy groups according to the definition given in section 1. On the other hand, by Moore's theorem there is an isomorphism

$$\pi_*(R[\Gamma]) \cong H_*(\widetilde{\Gamma}; R) \tag{3.1}$$

where the right-hand side is the homology of the simplicial set $\widetilde{\Gamma}$ with coefficients in R. Thus when Γ is a resolution we have

$$\pi_*(R[\Gamma]) = \begin{cases} R[\Gamma_{-1}] & * = 0 \\ 0 & * > 0 \end{cases} \tag{3.2}$$

For the remainder of the section we will assume Γ is a free simplicial resolution, with $s'_{.} = \{s'_{n+1} : \Gamma'_n \to \Gamma'_{n+1}\}_{n \geq -1}$ a fixed choice of extra degeneracy associated with the simplicially contractible subgroup Γ' as defined in (1.3). We also assume given an admissable section $s(0) : \Gamma_{-1} \rightarrowtail \Gamma_0$, $\varepsilon_0 \circ s(0) =$ identity, and define $s(n) = s_0^{(n)} \circ s(0) : \pi \rightarrowtail \Gamma_n$. Note that

$$\begin{aligned} \varepsilon_n \circ s(n) &= \text{ identity} && \forall\, n \geq 0\,, \\ \partial_i \circ s(n) &= s(n-1) && \forall n \geq 1, 0 \leq i \leq n\,, \\ s_i \circ s(n-1) &= s(n) && \forall n \geq 1, 0 \leq i \leq n-1\,. \end{aligned} \tag{3.3}$$

For notational convenience we will set $s(-1) = id$, and will write $C^a_*(\Gamma)$ for the associated chain complex of $\mathbb{C}[\Gamma]$. For $g \in \Gamma_n$, set $g' = g(s(n)(\overline{g}))^{-1}$ where $\overline{g}$ is the image of g in π under the epimorphism $\Gamma_n \overset{\varepsilon}{\twoheadrightarrow} \pi$. Set

$$\widetilde{s}_{n+1}(g) = s'_{n+1}(g')s(n+1)(\overline{g}) \tag{3.4}$$

This defines a map of sets $\widetilde{s}_{n+1} : \Gamma_n \to \Gamma_{n+1}$, with $\widetilde{s}_0 = s(0)$. By the simplicial identities we have

$$d_{n+1}\widetilde{s}_{n+1} = (-1)^{n+1}(id) + \widetilde{s}_n d_n \tag{3.5}$$

Thus $\widetilde{s}_* = \{(-1)^{n+1}\widetilde{s}_{n+1}\}_{n\geq -1}$ provides a chain contraction of the complex $C^a_*(\Gamma)$, hence an alternative proof of acyclicity above dimension 0. It also shows that $H_0(C^a_*(\Gamma)) = \mathbb{C}[\Gamma_{-1}]$.

§4 Simplicial rapid decay algebras

As usual, $C_r^*(\pi)$ will denote the reduced C^* algebra of π. For a discrete group π equipped with a word-length function L and a real number s, $H_L^s(\pi)$ is the Hilbert space of functions $f : \pi \to \mathbb{C}$ satisfying the inequality $\|f\|_{2,s,L} < \infty$, where the Sobolev norm $\|f\|_{2,s,L}$ is given by

(4.1)

$$\|f\|_{2,s,L} = (< f, f >_{2,s,L})^{1/2}, \qquad < f, g >_{2,s,L} = \sum_{h\in\Gamma} f(h)\overline{g}(h)(1 + L(h))^{2s}$$

For each s and L , $H_L^s(\pi)$ is the completion of the group algebra $\mathbb{C}[\pi]$ with respect to the norm induced by (4.1). Then

(4.2)
$$H_L^\infty(\pi) = \bigcap_s H_L^s(\pi)$$

The collection of norms $\{\| \quad \|_{2,n,L}\}_{n\in\mathbb{N}}$ give $H_L^\infty(\pi)$ the structure of a Fréchet space, as $H_L^t(\pi) \subset H_L^s(\pi)$ for $0 < s < t$. If there exist numbers $C > 1, s$ and word-length function L such that

(4.3)
$$\|f\| \leq C \cdot \|f\|_{2,s,L}$$

for all $f \in \mathbb{C}[\pi]$, where $\|f\|$ denotes the reduced C^*-norm of f, then $H_L^t(\pi)$ occurs as a subspace of $C^*\pi$ for $t > s$, and $H_L^\infty(\pi)$ is a subalgebra of $C^*(\pi)$. This property was first shown to hold for finitely-generated free groups by Haagarup [4], and later for finitely-generated hyperbolic groups by Jolissaint [5], [6] and P. de la Harpe [3]. $(F, \mathbb{X}, L)$ has property RD if $H_L^\infty(F)$ is a subalgebra of C^*F.

The following proposition provides the necessary extension of [4] to countably generated free groups. As in section 1, we start with a triple $(F, \mathbb{X}, L)$ where F is a free group and $\mathbb{X}$ is countable. $\mathbb{X}_m = \{x \in \mathbb{X} \mid L(x) \leq m\}$, F_m is the subgroup of F on the set $\mathbb{X}_m$, and $L_m = L|_{F_m}$.

Proposition 4.4. *$H_L^\infty(F)$ is a dense, holomorphically closed involutive subalgebra of $C_r^*(F)$ containing $\mathbb{C}[F]$.*

Proof. It will suffice to show that $H_L^\infty(F)$ is contained in $C_r^*(F)$. Let $x = \sum_g \lambda(g)g \in H_L^\infty(F)$, and for each m let $x_p = \sum_{g\in F_p} \lambda(g)g$. It follows from [4] that for all $N > m, n$

$$\|x_m - x_n\|_N < C\|x_m - x_n\|_{2,2,L_N} \tag{4.5}$$

where the norm on the left is the C^* norm in $C_r^*(F_N)$, C is a constant independent of all terms, and the norm on the right is in $H_{L_N}^\infty(F_N)$. Denote the norm in $C_r^*(F)$ by $\| \; \|$. Then (4.5) implies

$$\|x_m - x_n\| < C\|x_m - x_n\|_{2,2,L} \tag{4.6}$$

Thus the Cauchy sequence $\{x_p\}$ converges in $C_r^*(F)$. Since it converges to x in $\ell^2(F)$, $x \in C_r^*(F)$. //

It follows from (4.6) that $\|x\| < C\|x\|_{2,2,L}$ for all $x \in H_L^\infty(F)$, as in the case of finitely generated free groups. Not all groups have property RD. We will give a modified version of the constructions in [2] and [6] which applies to all countable groups. For $g \in \pi$, write $\delta_g \in \ell^2(\pi)$ for the characteristic function of g. Define χ_N to be the projection of $\ell^2(\pi)$ onto the subspace $\oplus_{L(g)=N}\mathbb{C}\delta_g$, and $p_N = \oplus_{m\leq N}\chi_m$ the projection onto the ball $B_N = \oplus_{L(g)\leq N}\mathbb{C}\delta_g$. We adopt the convention that $p_{-1} = 0$. For $a = \sum a(g)g \in \ell^2(\pi)$, $|a| = \sum |a(g)|g$, and $|a|(t) = \sum |a(g)|(1 + L(g))^t$. Also, we set $Q_m(a) = a|_{\overline{B_m}} = \sum_{L(g)>m} a(g)g$. If $a \in C_r^*(\pi)$, $Q_m(a)$ is computed via the standard embedding $C_r^*(\pi) \rightarrowtail \ell^2(\pi)$. Now following [2], we define for each $k \geq 0$ an unbounded operator $(D')^k : \ell^2(\pi) \to \ell^2(\pi)$ by

$$(D')^k(\delta_g) = (1 + L(g))^k\delta_g \tag{4.7}$$

and write $D^k = ad((D')^k)$ for the corresponding derivation on bounded operators. For $a \in C_r^*(\pi)$, $k \geq 0$, $\alpha \in (0, 1)$ we set

$$\begin{aligned} \widetilde{\gamma}_{N,\alpha,k}(a) &= D^k(Q_{N^\alpha}(|a|))p_{N-N^\alpha} \qquad\qquad N \geq 0 \\ \widetilde{\gamma}_{-1,\alpha,k}(a) &= D^k(|a|)p_0 \\ \widetilde{\beta}_{N,\alpha,k}(a) &= \|\widetilde{\gamma}_{N,\alpha,k}(a)\| + \|\widetilde{\gamma}_{N,\alpha,k}(a^*)\| \end{aligned} \tag{4.8}$$

It is easy to see that $\xi \in H_L^\infty(\pi)$ if and only if $|\xi| \in H_L^\infty(\pi)$; this is true for any countable group.

Definition 4.9. $\widetilde{T}_L^\infty(\pi)$ is the set consisting of all $a \in C_r^*(\pi)$ with the property that for all $\alpha \in (0,1)$ and $k, q \in \mathbb{N}$

$$\widetilde{\beta}_{N,\alpha,k}(a) = O(N^{-q})$$

as N tends to infinity.

Theorem 4.10. *For all* (π, L), $\widetilde{T}_L^\infty(\pi)$ *is a dense involutive subalgebra of* $C_r^*(\pi)$ *which contains* $\mathbb{C}[\pi]$. *Moreover, the inclusion* $\widetilde{T}_L^\infty(\pi) \rightarrowtail C_r^*(\pi)$ *induces an isomorphism of topological K-groups above dimension* 0.

Proof. That $\widetilde{T}_L^\infty(\pi)$ is an involutive algebra containing $\mathbb{C}[\pi]$ is proved in Proposition 4.17 below, and the same argument shows $\overline{T}_L^\infty(\pi)$ is an algebra, where $\overline{T}_L^\infty(\pi)$ is the subset of $C_r^*(\pi)$ consisting of those a for which $\widetilde{\beta}_{N,\alpha,0}(a) = O(N^{-q})$ for all $\alpha \in (0,1)$ and $q \geq 0$. By the argument of [6, Th. 1.4], $\overline{T}_L^\infty(\pi)$ is inverse-closed in $C_r^*(\pi)$; that is, $GL_n(\overline{T}_L^\infty(\pi)) = GL_n(C_r^*(\pi)) \cap M_n(\overline{T}_L^\infty(\pi))$ for all $n \geq 1$. The method of proof used in [2, Lemma 6.4] shows that $\widetilde{T}_L^\infty(\pi)$ is inverse-closed in $\overline{T}_L^\infty(\pi)$. Thus $\widetilde{T}_L^\infty(\pi)$ is inverse-closed in $C_r^*(\pi)$, implying by standard arguments that $GL(\widetilde{T}_L^\infty(\pi)) \xrightarrow{\cong} GL(C_r^*(\pi))$. //

We define semi-norms for $\widetilde{T}_L^\infty(\pi)$ by

$$\widetilde{\nu}_{q,\alpha,k}(a) = \sup_{N \geq -1} \{(N+2)^q \widetilde{\beta}_{N,\alpha,k}(a), \|D^k(|a|)\|\} \tag{4.11}$$

(note that $|a^*| = |a|^*$).

We now assume given a homomorphism $(F_1, L_1) \xrightarrow{\varphi} (F_2, L_2)$ of countable groups equipped with word-length functions. For $a \in \mathbb{C}[F_1]$ define $\widetilde{\gamma}^\varphi_{N,\alpha,k}(a) : \ell^2(F_2) \to \ell^2(F_2)$ by

$$\begin{aligned} \widetilde{\gamma}^\varphi_{N,\alpha,k}(a)(\psi) &= \varphi(D^k(Q_{N^\alpha}(|a|)))p_{N-N^\alpha}(\psi) \qquad N \geq 0 \\ \widetilde{\gamma}^\varphi_{-1,\alpha,k}(a)(\psi) &= \varphi(D^k(|a|))p_0(\psi) \end{aligned} \tag{4.12}$$

and set

$$\widetilde{\beta}^{\varphi}_{N,\alpha,k}(a) = \|\widetilde{\gamma}^{\varphi}_{N,\alpha,k}(a)\| + \|\widetilde{\gamma}^{\varphi}_{N,\alpha,k}(a^*)\| \tag{4.13}$$

Notice that when $\varphi = id$, $\widetilde{\beta}^{\varphi}_{N,\alpha,k}(x) = \widetilde{\beta}_{N,\alpha,k}(x)$.

Denote the semi-norms for $\widetilde{T}^{\infty}_{L_1}(F_1)$ resp. $\widetilde{T}^{\infty}_{L_2}(F_2)$ by $\widetilde{\nu}_{q,\alpha}$ resp. $\widetilde{\nu}'_{q,\alpha}$.

Definition 4.14. An element of $\widetilde{T}^{\infty}(F;\varphi)$ is represented by a sequence $x = \{x_m\}$ of elements of $\mathbb{C}[F]$ satisfying the following conditions

i) $x = \{x_m\}$ converges in $\widetilde{T}^{\infty}_{L_1}(F_1)$.

ii) $\varphi(x) = \{\varphi(x_m)\}$ converges in $\widetilde{T}^{\infty}_{L_2}(F_2)$.

iii) For all $\alpha \in (0,1)$ and $q, k \geq 0$, $\sup_m\{\widetilde{\beta}^{\varphi}_{N,\alpha,k}(x_m)\} \in O(N^{-q})$.

Two sequences $\{x_m\}$, $\{x'_m\}$ represent the same element in $\widetilde{T}^{\infty}(F;\varphi)$ iff $\varinjlim x_m = \varinjlim x'_m$ in $\widetilde{T}^{\infty}_{L_1}(F_1)$ and $\varinjlim \varphi(x_m) = \varinjlim \varphi(x'_m)$ in $\widetilde{T}^{\infty}_{L_2}(F_2)$

It is natural to define additional semi-norms for $\widetilde{T}^{\infty}(F;\varphi)$ by

$$\widetilde{\nu}^{\varphi}_{q,\alpha,k}(a) = \sup_{N\geq -1}\{(N+2)^q \widetilde{\beta}^{\varphi}_{N,\alpha,k}(a), \|\varphi(D^k(|a|))\|\} \tag{4.15}$$

Suppose that $a, b \in \mathbb{C}[F_1]$. Then for $N \geq 0$

$$\begin{aligned}
\|\widetilde{\gamma}^{\varphi}_{N,\alpha,k}(ab)\| &= \|\varphi(D^k(Q_{N^\alpha}(|ab|)))p_{N-N^\alpha}\| \\
&\leq \sum_{j=1}^{N^\alpha} \|\varphi(D^k(Q_j(|a|)Q_{N^\alpha-j}(|b|)))p_{N-N^\alpha}\|
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j=1}^{N^\alpha} \|\varphi(D^k(Q_j(|a|)))\varphi(Q_{N^\alpha-j}(|b|))p_{N-N^\alpha}\| + \sum_{j=1}^{N^\alpha} \|\varphi(Q_j(|a|))\varphi(D^k(Q_{N^\alpha-j}(|b|)))p_{N-N^\alpha}\| \\
&\leq \frac{N^\alpha}{2}\left(\|\varphi(D^k(|a|))\|\|\varphi(Q_{\frac{N^\alpha}{2}}(|b|))p_{N-N^\alpha}\| + \|\varphi(D^k(|b|))\|\|\varphi(Q_{\frac{N^\alpha}{2}}(|a|))p_{N-N^\alpha}\|\right) \\
&+\frac{N^\alpha}{2}\left(\|\varphi(|a|)\|\|\varphi(D^k(Q_{\frac{N^\alpha}{2}}(|b|)))p_{N-N^\alpha}\| + \|\varphi(|b|)\|\|\varphi(D^k(Q_{\frac{N^\alpha}{2}}(|a|)))p_{N-N^\alpha}\|\right)
\end{aligned}$$

This yields an inequality

$$\widetilde{\nu}^{\varphi}_{q,\alpha,k}(ab) \leq 2\left(\widetilde{\nu}^{\varphi}_{q+1,\frac{\alpha}{2},k}(a)\widetilde{\nu}^{\varphi}_{q+1,\frac{\alpha}{2},0}(b) + \widetilde{\nu}^{\varphi}_{q+1,\frac{\alpha}{2},k}(b)\widetilde{\nu}^{\varphi}_{q+1,\frac{\alpha}{2},0}(a)\right) \tag{4.16}$$

Proposition 4.17. $\widetilde{T}^\infty(F;\varphi)$ *is an involutive algebra containing* $\mathbb{C}[F_1]$.

Proof. It is clear that $\widetilde{T}^\infty(F;\varphi)$ is closed under addition, contains $\mathbb{C}[F_1]$ as the subset of eventually constant sequences, and is closed under the standard involution. Let $\{x_i\}$, $\{y_i\}$ be two elements of $\widetilde{T}^\infty(F;\varphi)$. Define the product to be the sequence $\{x_i y_i\}$. By (4.16) this sequence is again an element of $\widetilde{T}^\infty(F;\varphi)$. It is not hard to see that the equivalence class of $\{x_i y_i\}$ only depends on the equivalence class of $\{x_i\}$ and $\{y_i\}$. //

Suppose that $\{x_m\}$, $\{y_m\}$ are two sequences in $\widetilde{T}^\infty(F;\varphi)$ converging to the same element in $\widetilde{T}^\infty_{L_1}(F_1)$. By hypothesis, $\{x_m - y_m\}$ converges to zero in $\widetilde{T}^\infty_{L_1}(F_1)$. Having fixed ϵ we may choose M and p (which may depend on M) so that for all $m > p$

$$a = \|\varphi((x_m - y_m)|_{B_M})\| < \frac{\epsilon}{2}$$
$$b = \|\varphi((x_m - y_m)|_{\overline{B_M}})\| < \frac{\epsilon}{2}$$

Thus

$$\|\varphi(x_m - y_m)\| = \|\varphi(x_m) - \varphi(y_m)\| \leq a + b \leq \epsilon \tag{4.18}$$

where the C^*-norm is that of $C^*_r(F_2)$. Hence

Lemma 4.19. *If* $\{x_m\}, \{y_m\} \in \widetilde{T}^\infty(F;\varphi)$ *and* $\{x_m - y_m\}$ *converges to zero in* $\widetilde{T}^\infty_{L_1}(F_1)$ *, then* $\{\varphi(x_m) - \varphi(y_m)\}$ *converges to zero in* $\widetilde{T}^\infty_{L_2}(F_2)$ *.* //

Corollary 4.20. *If* $\{x_m\} \in \widetilde{T}^\infty(F;\varphi)$ *with* $x = \varinjlim x_m$ *in* $T^\infty_{L_1}(F_1)$ *, then the map* $\{x_m\} \mapsto x$ *determines an embedding of* $\widetilde{T}^\infty(F;\varphi)$ *in* $T^\infty_{L_1}(F_1)$ *. In particular, every element of* $\widetilde{T}^\infty(F;\varphi)$ *may be faithfully represented as an infinite series* $\{x_m\} = \sum \lambda_i g_i$ *where* $\lambda_i = \varinjlim x_m(g_i)$ *for each* i *.* //

This is not saying that if $\{x_m\}$ is an element of $\widetilde{T}^\infty(F;\varphi)$, $\{y_m\}$ a convergent sequence of group algebra elements in $T^\infty_{L_1}(F_1)$, and $\{x_m - y_m\}$ converges to zero in $T^\infty_{L_1}(F_1)$, then $\{y_m\}$ is also an element of $\widetilde{T}^\infty(F;\varphi)$.

The construction of the algebra $\widetilde{T}^\infty(F;\varphi)$ extends to the case where one has a collection $\{\varphi_\alpha : F \to F_\alpha\}$ of homomorphisms countable groups. We denote the resulting algebra by $T^\infty(F;\{\varphi_\alpha\})$. It is not hard to see that

$$T^\infty(F;\{\varphi_\alpha\}) = \bigcap_\alpha \widetilde{T}^\infty_L(F;\varphi_\alpha)\ .$$

Thus $T^\infty(F;\{\varphi_\alpha\})$ also admits a faithful representation as a subalgebra of $\widetilde{T}^\infty_L(F)$ in the manner described by the above corollary.

For the remainder of the section we will assume that $(\Gamma, \mathbb{X}, L)$ is a type B resolution of $\pi = \Gamma_{-1}$ with $(\Gamma_n, \mathbb{X}_n, L_n)$ free and countable for all $n \geq 0$, and with degeneracy maps (including the extra contracting degeneracy for Γ') which preserve word-length (from §2, we know that all countable groups $\pi = \Gamma_{-1}$ admit such a resolution). Viewing Γ as a contravariant functor $\Gamma : \Delta_+ \to (groups)$, we set

$$\mathcal{T}_n(\Gamma) = \Gamma(\Delta_+[n]) \qquad n \geq -1$$

where $\Delta_+[n] = \{Hom_{\Delta_+}([m],[n])\}$ is the collecton of morphisms in Δ_+ terminating at $[n]$. Thus $\mathcal{T}_n(\Gamma)$ contains all homomorphisms $\Gamma_n \to \Gamma_m$ which occur as iterated compositions of face and degeneracy maps (including the identity map). We define $\widetilde{T}^\infty_L(\Gamma)$ inductively by

$$\begin{gathered}\widetilde{T}^\infty_L(\Gamma)_{-1} = \widetilde{T}^\infty_{L_{-1}}(\Gamma_{-1}) \\ \widetilde{T}^\infty_L(\Gamma)_n = \{x \in \widetilde{T}^\infty(\Gamma_n;\mathcal{T}_n(\Gamma)) \mid \partial_i(x) \in \widetilde{T}^\infty_L(\Gamma)_{n-1}\, \forall 0 \leq i \leq n\}\end{gathered} \tag{4.21}$$

Note that this definition is dependent on both $\mathbb{X}$ and L ; for each n there is an embedding $\widetilde{T}^\infty_L(\Gamma)_n \rightarrowtail \widetilde{T}^\infty_{L_n}(\Gamma_n)$. When $n = -1$, this is the identity map.

Suppose that $x \in \mathbb{C}[\Gamma_n]$. Let $\phi \in \mathcal{T}_{n+1}$. As s_j preserves word-length we have

$$\begin{aligned}&\|\phi(D^k(Q_{N^\alpha}(s_j(|x|))))p_{N-N^\alpha}\| \\ =&\|\phi(s_j(D^k(Q_{N^\alpha}(|x|))))p_{N-N^\alpha}\| \\ =&\|(\phi\circ s_j)(D^k(Q_{N^\alpha}(|x|)))p_{N-N^\alpha}\|\end{aligned} \tag{4.22}$$

implying the simplicial structure on $\mathbb{C}[\Gamma]$ extends to a simplicial structure on $\widetilde{T}_L^\infty(\Gamma)$. Finally consider the map $\widetilde{s}_{n+1} : \Gamma_n \to \Gamma_{n+1}$ defined in (3.4). Since both s'_{n+1} and $s(n+1)$ preserve word-length, we have $L_{n+1}(\widetilde{s}_{n+1}(g)) \leq 3L_n(g)$. Again, let $x \in \mathbb{C}[\Gamma_n]$ and $\phi \in \mathcal{T}_{n+1}$. $\phi \circ \widetilde{s}_{n+1}$ may be factored as a composition $s_I \partial_J$, where s_I is a sequence of degeneracies followed by at most one factor of the form $\widetilde{s}_{m+1}$, and $\partial_J : \Gamma_{n+1} \to \Gamma_m$ is a sequence of boundary maps. Then

$$
\begin{aligned}
&\|\phi(D^k(Q_{N^\alpha}(\widetilde{s}_{n+1}(|x|))))p_{N-N^\alpha}\| \\
&\leq \|\phi(\widetilde{s}_{n+1}(D^k(Q_{\frac{N^\alpha}{3}}(|x|))))p_{N-N^\alpha}\| \\
&\leq \|(\phi \circ \widetilde{s}_{n+1})(Q_{\frac{N^\alpha}{3}}(|x|(k)))p_{N-N^\alpha}\| \\
&\leq \|s_I \partial_J (Q_{\frac{N^\alpha}{3}}(|x|(k)))\| \\
&\leq C\|s_I \partial_J (Q_{\frac{N^\alpha}{3}}(|x|(k)))\|_{2,s,L_{n+1}} \\
&\leq (3^s C)\|\partial_J (Q_{\frac{N^\alpha}{3}}(|x|(k)))\|_{2,s,L_m} \\
&\leq (3^s C)\|\partial_J (Q_{\frac{N^\alpha}{3}}(|x|(k+s)))\|_{2,0,L_m} \\
&\leq (3^s C)\|\partial_J (Q_{\frac{N^\alpha}{3}}(D^{k+s}(|x|)))p_{N-N^\alpha}\| \\
&= (3^s C)\|\partial_J (D^{k+s}(Q_{\frac{N^\alpha}{3}}(|x|)))p_{N-N^\alpha}\|
\end{aligned}
\tag{4.23}
$$

where C and s are as in (4.3).

In fact, we may derive an explicit bound for $\widetilde{s}_{n+1}$ with respect to the semi-norm $\widetilde{\nu}^\varphi_{q,\alpha,k}$. For considering the two cases $N^{\frac{\alpha}{2}} \geq 3$ and $N^{\frac{\alpha}{2}} < 3$, the series of inequalities in (4.23) above yield

$$
\widetilde{\nu}^\varphi_{q,\alpha,k}(\widetilde{s}_{n+1}(a)) \leq C(3^s)(9^{\frac{1}{\alpha}})\widetilde{\nu}^{\partial_J}_{q,\frac{\alpha}{2},k}(a) \tag{4.24}
$$

We conclude with a description of the degreewise topology on $\widetilde{T}_L^\infty(\Gamma)$. As in [6], $\widetilde{\nu}^\varphi_{q,\alpha,k} \leq \widetilde{\nu}^\varphi_{q',\alpha',k}$ if $q' \geq q$ and $\alpha' \leq \alpha$. Hence the topology on $\widetilde{T}_L^\infty(\Gamma)_n$ induced by the collection of semi-norms $\{\widetilde{\nu}^\varphi_{q,\alpha,k} \mid q,k \geq 0, \alpha \in (0,1), \varphi \in \mathcal{T}_n\}$ is equivalent to the topology induced by the countable sub-collection $\{\widetilde{\nu}^\varphi_{q,\frac{1}{q},k} \mid q \geq 1, k \geq 0, \varphi \in \mathcal{T}_n\}$. As we are applying Definition 4.14 for homomorphisms φ that bounded in word-length, condition (4.14) iii) implies (4.14) ii), and so $\widetilde{T}_L^\infty(\Gamma)_n$ may alternatively be defined as the completion of $\mathbb{C}[\Gamma_n]$ in the semi-norms $\{\widetilde{\nu}^{\varphi}_{q,\frac{1}{q},k} \mid q \geq 1, k \geq 0, \varphi \in \mathcal{T}_n\}$ (note that the set of maps $\mathcal{T}_n$ always contains the identity). Let $C^t_*(\Gamma)$ denote the associated chain complex of $\widetilde{T}^\infty_L(\Gamma)$. We may summararize the properties of $\widetilde{T}^\infty_L(\Gamma)$ in the following theorem. Write $\overline{L}$ for the word-length function L_{-1} on $\Gamma_{-1} = \pi$.

Theorem 4.25. $\widetilde{T}^\infty_L(\Gamma) = \{n \mapsto \widetilde{T}^\infty_L(\Gamma)_n\}_{n\geq -1}$ *is an augmented, simplicial involutive Fréchet algebra, containing* $\mathbb{C}[\Gamma]$, *with* $\widetilde{T}^\infty_L(\Gamma)_{-1} = \widetilde{T}^\infty_{\overline{L}}(\pi)$ *a dense, inverse-closed subalgebra of* $C^*_r(\pi)$. *In addition, the contraction* $\widetilde{s}_*$ *of* $C^a_*(\Gamma)$ *defined in (3.4) extends to a chain contraction of* $C^t_*(\Gamma)$, *bounded in semi-norms in the manner described by (4.24).* //

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