ISOTROPY OF VIRTUAL ALBERT FORMS OVER FUNCTION FIELDS OF QUADRICS

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ABSTRACT. Let F be a field of characteristic different from 2 and ϕ a virtual Albert form over F , i.e. an anisotropic 6-dimensional quadratic form over F which is still anisotropic over the field $F(\sqrt{d_{\pm}\phi})$. We give a complete description of the quadratic forms ψ such that ϕ becomes isotropic over the function field $F(\psi)$. This completes the series of works ([H6], [Lag6], [Lag], [Lee], [M2]) where the question was considered previously.

0. INTRODUCTION

Let F be a field of characteristic different from 2 and let ϕ and ψ be two anisotropic quadratic forms over F . An important problem in the algebraic theory of quadratic forms is to find conditions on ϕ and ψ so that $\phi_{F(\psi)}$ is isotropic. In the case when $\dim \phi \leq 5$ the problem was completely solved in [H5] and [Sc]. For 6-dimensional quadratic forms, the problem was studied by D. W. Hoffmann ([H6]), A. Laghribi ([Lag6], [Lag]), D. Leep ([Lee]), and A. S. Merkuriev ([M2]), and was solved fully except for the following two cases (see [Lag6] and [Lag]):

- $\dim \psi = 4$, $d_{\pm} \psi \neq 1$, and $\text{ind}(C_0(\phi)) = 2$;
- $\dim \psi = 4$, $d_{\pm} \psi \neq 1$, $\text{ind}(C_0(\phi)) = 4$, and $d_{\pm} \phi = d_{\pm} \psi$.

In this paper the second case is studied completely. Our result (Theorem 5.1) and results of Laghribi, Leep and Merkuriev give rise to the following

Theorem. *Let ϕ be a 6-dimensional quadratic form such that $\text{ind}(C_0(\phi)) = 4$. In the case when $\psi \notin GP_2(F)$, the quadratic form $\phi_{F(\psi)}$ is isotropic if and only if ψ is similar to a subform of ϕ . In the case when $\psi \in GP_2(F)$, the form $\phi_{F(\psi)}$ is isotropic if and only if a 3-dimensional subform of ψ is similar to a subform of ϕ .*

We deduce Theorem 5.1 from a result on 8-dimensional forms (Proposition 4.1) which also has an independent value: together with [Lag8], it gives rise to Theorem 4.4 answering the question about isotropy of an 8-dimensional quadratic form ϕ with $\det \phi = 1$ and $\text{ind } C(\phi) = 8$ over function fields of quadrics.

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1. TERMINOLOGY, NOTATION, AND BACKGROUNDS

Quadratic forms. By $\phi \perp \psi$, $\phi \simeq \psi$, and $[\phi]$ we denote orthogonal sum of forms, isometry of forms, and the class of ϕ in the Witt ring $W(F)$ of the field F respectively. To simplify notation we will write $\phi + \psi$ instead of $[\phi] + [\psi]$. For a quadratic form ϕ of dimension n , we set $d_{\pm} \phi = (-1)^{n(n-1)/2} \det \phi$. We consider $d_{\pm} \phi$ as an element of F^*/F^{*2} . The maximal ideal of $W(F)$ consisting of the classes of the even-dimensional forms is denoted by $I(F)$. The anisotropic part of ϕ is denoted by ϕ_{an} . We denote by $\langle\langle a_1, \dots, a_n \rangle\rangle$ the n -fold Pfister form

$$\langle 1, -a_1 \rangle \otimes \dots \otimes \langle 1, -a_n \rangle$$

and by $P_n(F)$ the set of all n -fold Pfister forms. The set of all forms similar to n -fold Pfister forms we denote by $GP_n(F)$. For any field extension L/F , we put $\phi_L = \phi \otimes L$, $W(L/F) = \text{Ker}(W(F) \rightarrow W(L))$, and $I^n(L/F) = \text{Ker}(I^n(F) \rightarrow I^n(L))$.

For a quadratic form ϕ of dimension ≥ 3 , we denote by X_{ϕ} the projective variety given by the equation $\phi = 0$. We set $F(\phi) = F(X_{\phi})$ if $\dim \phi \geq 3$; $F(\phi) = F(\sqrt{d})$ if $\dim \phi = 2$ and $d = d_{\pm} \phi \neq 1$; and $F(\phi) = F$ otherwise.

Let $\psi \in GP_2(F)$ and ψ_0 be a 3-dimensional subform of ψ . Then $\psi_{F(\psi_0)}$ and $(\psi_0)_{F(\psi)}$ are isotropic. Hence for any quadratic form ϕ , the isotropy of $\phi_{F(\psi)}$ is equivalent to isotropy of $\phi_{F(\psi_0)}$. Thus, to give a complete description of the quadratic forms ψ such that ϕ becomes isotropic over the function field $F(\psi)$, it is sufficient to consider the case where $\psi \notin GP_2(F)$.

Let γ be a quadratic form of dimension 2^n . We say that γ^* is a *half-neighbor* of γ , if $\dim \gamma^* = 2^n$ and there exists $k \in F^*$ such that $\gamma^* \equiv k\gamma \pmod{I^{n+1}(F)}$.

Algebras. Let A be a central simple algebra over F . By $\deg(A)$, $\text{ind}(A)$, $[A]$, and $\exp(A)$ we denote respectively the degree of A , the Schur index of A , the class of A in the Brauer group $\text{Br}(F)$, and the order of $[A]$ in the Brauer group. By $\text{SB}(A)$ we denote the Severi–Brauer variety of A . If an algebra B has the form $B = A \times A$, we set $\text{ind } B = \text{ind } A$.

Let ϕ be a quadratic form. We denote by $C(\phi)$ the Clifford algebra of ϕ . By $C_0(\phi)$ we denote the even part of $C(\phi)$. If $\phi \in I^2(F)$ then $C(\phi)$ is a central simple algebra. Hence we get a well defined element $[C(\phi)]$ of $\text{Br}_2(F)$ which we denote by $c(\phi)$.

Quadratic forms of dimension 6. Let ϕ be an anisotropic quadratic form of dimension 6 and let $d = d_{\pm} \phi$. If $d = 1$, then ϕ is an Albert form. In this case the problem of isotropy of ϕ over the function field of a quadratic form ψ is completely solved ([Lee], [M2]): in the case when $\psi \notin GP_2(\psi)$, the form $\phi_{F(\psi)}$ is isotropic iff ψ is similar to a subform in ϕ .

Suppose now that $d \neq 1$. Then $C_0(\phi)$ is a central simple algebra over $L = F(\sqrt{d})$. In this case we have the following classification of anisotropic 6-dimension forms:

Type 1 is defined by one of the following equivalent conditions:

- $\text{ind}(C_0(\phi)) = 1$;
- ϕ_L is hyperbolic;

- ϕ has the form $\langle\langle d \rangle\rangle \mu$ where $\dim \mu = 3$;
- ϕ is a Pfister neighbor.

Type 2 is defined by one of the following equivalent conditions:

- $\text{ind}(C_0(\phi)) = 2$;
- ϕ_L is isotropic but not hyperbolic;
- ϕ is similar to a form of the kind $\langle\langle a, b \rangle\rangle \perp c \langle\langle d \rangle\rangle$, where $\langle\langle a, b \rangle\rangle_L$ is not isotropic.

Type 3 is defined by one of the following equivalent conditions:

- $\text{ind}(C_0(\phi)) = 4$;
- ϕ_L is anisotropic;

The quadratic form of the type 3 is called a *virtual Albert form*.

For the quadratic forms ϕ of type 1 (i.e. for the Pfister neighbors), the problem of isotropy $\phi_{F(\psi)}$ is completely solved by Arason–Pfister subform theorem. The case of quadratic forms of type 2 was studied by D. Hoffmann in [H6]: he found the conditions on ϕ and ψ so that $\phi_{F(\psi)}$ is isotropic excepting the case $\dim \psi = 4$.¹

The case of the quadratic forms ϕ of type 3 (virtual Albert forms) was studied completely by A. Laghribi in [Lag6, Lag] except for the case where $\dim \psi = 4$ and $d_{\pm} \psi \neq 1$. In this paper we complete the investigation of isotropy of virtual Albert forms over the function field of a quadric.

Cohomology groups. By $H^*(F)$ we denote the graded ring of Galois cohomology $H^*(F, \mathbb{Z}/2\mathbb{Z}) = H^*(\text{Gal}(F_{\text{sep}}/F), \mathbb{Z}/2\mathbb{Z})$. For any field extension L/F we set $H^n(L/F) = \text{Ker}(H^n(F) \rightarrow H^n(L))$.

We use the standard canonical isomorphisms $H^0(F) = \mathbb{Z}/2\mathbb{Z}$, $H^1(F) = F^*/F^{*2}$, and $H^2(F) = \text{Br}_2(F)$. Thus any element $a \in F^*$ gives rise to an element of $H^1(F)$; it is denoted by (a) . The cup product $(a_1) \cup \dots \cup (a_n)$ is denoted by $(a_1, \dots, a_n)$.

For $n = 0, 1, 2$ there is a homomorphism $e^n : I^n(F) \rightarrow H^n(F)$ defined as follows: $e^0(\phi) = \dim \phi \pmod{2}$, $e^1(\phi) = d_{\pm} \phi$, and $e^2(\phi) = c(\phi)$. Moreover, there exists a homomorphism $e^3 : I^3(F) \rightarrow H^3(F)$ which is uniquely determined by the condition $e^3(\langle\langle a_1, a_2, a_3 \rangle\rangle) = (a_1, a_2, a_3)$ (see [Ara]). The homomorphism e^n is surjective and $\text{Ker } e^n = I^{n+1}(F)$ for $n = 0, 1, 2, 3$ (see [M1], [MS], and [R]).

K-theory and Chow groups. In §2 we use the following notation.

Let X be a smooth algebraic F -variety. The Grothendieck ring of X is denoted by $K(X)$. This ring is supplied with the filtration “by codimension of support” (which respects the multiplication); the adjoint graded ring is denoted by $G^*K(X)$. There is a canonical surjective homomorphism of the graded Chow ring $\text{CH}^*(X)$ onto $G^*K(X)$; its kernel consists only of torsion elements and is trivial in the 0-th, 1-st and 2-nd graded components ([Su, §9]).

We fix a separable closure $\overline{F}$ of the ground field F and denote by $\overline{X}$ the variety $X_{\overline{F}}$. The image of the restriction homomorphism $G^*K(X) \rightarrow G^*K(\overline{X})$ is denoted by $\overline{G}^*K(X)$.

¹The case where ϕ is of type 2 and $\dim \psi = 4$ is studied recently in [IK].

We denote by $|S|$ the order of a finite set S .

2. COMPUTATION OF $H^3(F(\text{SB}(A) \times \text{SB}(B))/F)$

Theorem 2.1. *Let A and B be biquaternion division F -algebras such that $\text{ind}(A \otimes B) = 8$. Suppose that there exists a quadratic extension L/F such that A_L and B_L are not division algebras. Then*

$$H^3(F(\text{SB}(A) \times \text{SB}(B))/F) = [A] \cdot H^1(F) + [B] \cdot H^1(F).$$

Proof. We put $X = \text{SB}(A) \times \text{SB}(B)$.

Lemma 2.2 ([K, Prop. 2]).

$$|\text{Tors } G^*K(X)| = \frac{|G^*K(\overline{X})/\bar{G}^*K(X)|}{|K(\overline{X})/K(X)|}.$$

Lemma 2.3. $|K(\overline{X})/K(X)| = 2^{28}$.

Proof. Applying [Q, §8, Th. 4.1], one gets an isomorphism

$$K(X) \simeq K(F)^{\oplus 4} \oplus K(A)^{\oplus 4} \oplus K(B)^{\oplus 4} \oplus K(A \otimes B)^{\oplus 4}$$

which shows that

$$|K(\overline{X})/K(X)| = (\text{ind } A)^4 \cdot (\text{ind } B)^4 \cdot (\text{ind } A \otimes B)^4 = 2^{28}.$$

□

The variety $\overline{\text{SB}(A)}$ is a projective space; denote by f the class of a hyperplane in $G^1K(\overline{\text{SB}(A)})$.

Lemma 2.4. *For any $i \geq 0$, the group $\bar{G}^iK(\text{SB}(A_L))$ contains*

- f^i , if i is even,
- $2f^i$, if i is odd.

Proof. By [K, Lemma 3], for any i , one has an inclusion

$$\bar{G}^iK(\text{SB}(A_L)) \ni \frac{\text{ind } A_L}{(i, \text{ind } A_L)} f^i$$

where $(\cdot, \cdot)$ denotes the greatest common divisor. Since $\text{ind } A_L = 2$, the statement follows. □

Lemma 2.5. $\bar{G}^1K(\text{SB}(A)) \ni 2f$.

Proof. By the computation [Art, §2] of the Picard group of a Severi-Brauer variety, one knows that

$$\bar{G}^1K(\text{SB}(A)) \ni (\exp A)f.$$

Since $\exp A = 2$, the statement follows. □

The variety $\overline{\text{SB}(B)}$ is a projective space; denote by g the class of a hyperplane in $G^1K(\overline{\text{SB}(B)})$.

Corollary 2.6. *For any $i, j \geq 0$, the group $\bar{G}^{i+j}K(X)$ contains*

- $f^i \times g^j$, if $i = j = 0$;
- $2(f^i \times g^j)$, if
 i and j are even or
 $i = 0, j = 1$ or
 $i = 1, j = 0$;
- $4(f^i \times g^j)$, if
 $i + j$ is odd or
 $i = j = 1$;
- $8(f^i \times g^j)$ for any i, j .

Proof. The case $i = j = 0$ is evident.

If i and j are even, then $f^i \in \bar{G}^i K(\text{SB}(A_L))$ and $g^j \in \bar{G}^j K(\text{SB}(B_L))$ by Lemma 2.4. Thus $f^i \times g^j \in \bar{G}^{i+j} K(X_L)$ and the transfer argument shows that $2(f^i \times g^j) \in \bar{G}^{i+j} K(X)$.

By Lemma 2.5, $\bar{G}^1 K(\text{SB}(A)) \ni 2f$ and $\bar{G}^1 K(\text{SB}(B)) \ni 2g$. Therefore $\bar{G}^1 K(X)$ contains $2(f \times 1)$ and $2(1 \times g)$; moreover, $\bar{G}^2 K(X) \ni 4(f \times g)$.

If $i + j$ is odd, then $2(f^i \times g^j) \in \bar{G}^{i+j} K(X_L)$ by Lemma 2.4 and the transfer argument shows that $4(f^i \times g^j) \in \bar{G}^{i+j} K(X)$.

Since there exists a field extension of degree 8 simultaneously splitting the algebras A and B , the inclusion $8(f^i \times g^j) \in \bar{G}^{i+j} K(X)$ holds for any i, j . $\square$

Corollary 2.7. $|G^* K(\bar{X}) / \bar{G}^* K(X)| \leq 2^{28}$.

Proof. Since $\overline{\text{SB}(A)}$ and $\overline{\text{SB}(B)}$ are projective spaces, $G^* K(\bar{X})$ is an abelian group freely generated by $f^i \times g^j$ with $i, j = 0, 1, 2, 3$. By Lemma 2.6, we know that the following multiples of these generators are in $\bar{G}^* K(X)$:

$$\begin{aligned} &2^0 \cdot (f^0 \times g^0), \quad 2^1 \cdot (f^0 \times g^1), \quad 2^1 \cdot (f^0 \times g^2), \quad 2^2 \cdot (f^0 \times g^3), \\ &2^1 \cdot (f^1 \times g^0), \quad 2^2 \cdot (f^1 \times g^1), \quad 2^2 \cdot (f^1 \times g^2), \quad 2^3 \cdot (f^1 \times g^3), \\ &2^1 \cdot (f^2 \times g^0), \quad 2^2 \cdot (f^2 \times g^1), \quad 2^1 \cdot (f^2 \times g^2), \quad 2^2 \cdot (f^2 \times g^3), \\ &2^2 \cdot (f^3 \times g^0), \quad 2^3 \cdot (f^3 \times g^1), \quad 2^2 \cdot (f^3 \times g^2), \quad 2^3 \cdot (f^3 \times g^3). \end{aligned}$$

Taking the product of the coefficients, we get 2^{28} . $\square$

Corollary 2.8. $\text{Tors } G^* K(X) = 0$.

Proof. Follows from Lemma 2.2, Lemma 2.3 and Corollary 2.7. $\square$

Since the Chow group $\text{CH}^2(X)$ is isomorphic to $G^2 K(X)$ ([Su, §9]), we also get

Corollary 2.9. $\text{Tors } \text{CH}^2(X) = 0$. $\square$

To complete the proof of Theorem 2.1, we apply [P, Th. 4.1 with Remark 4.1]. By that result, there is a monomorphism

$$\frac{H^3(F(X)/F)}{[A] \cdot H^1(F) + [B] \cdot H^1(F)} \hookrightarrow \text{Tors } \text{CH}^2(X)$$

and so, by Corollary 2.9, we are done. $\square$

3. COMPUTATION OF $H^3(F(X_\psi \times \text{SB}(D))/F)$

Theorem 3.1. *Let $\psi = \langle -a, -b, ab, d \rangle$ be an anisotropic quadratic form over F with $d \notin F^{*2}$. Let $D = (a, b) \otimes (u, v) \otimes (d, s)$ be a division 3-quaternion algebra over F . Then the group*

$$H^3(F(X_\psi \times \text{SB}(D))/F)$$

is equal to

$$\{e^3(\psi \langle\langle k \rangle\rangle) \mid k \in F^* \text{ is such that } \psi \langle\langle k \rangle\rangle \in \text{GP}_3(F)\} + [D]H^1(F).$$

Proof. Put $\widehat{F} = F((t))$. Consider two biquaternion algebras $A = (a, b) \otimes (d, t)$ and $B = (d, st) \otimes (u, v)$ over $\widehat{F}$.

Since D is a division algebra, it follows that $\text{ind}(D) = 8$. Therefore

$$\text{ind}((a, b) \otimes (u, v))_{F(\sqrt{d})} = \text{ind } D_{F(\sqrt{d})} = 4.$$

Hence $(a, b)_{F(\sqrt{d})}$ and $(u, v)_{F(\sqrt{d})}$ are division $F(\sqrt{d})$ -algebras. By Tignol's theorem [T, Prop. 2.4], A and B are division $\widehat{F}$ -algebras as well.

Since $[A \otimes B] = [D_{\widehat{F}}]$ in $\text{Br}(\widehat{F})$, we have $\text{ind}(A \otimes_{\widehat{F}} B) = \text{ind}(D) = 8$. Since $A_{\widehat{F}(\sqrt{d})}$ and $B_{\widehat{F}(\sqrt{d})}$ are not division algebras, the conditions of Theorem 2.1 hold for the field $\widehat{F}$ and algebras A and B over $\widehat{F}$. Therefore

$$H^3(\widehat{F}(\text{SB}(A) \times \text{SB}(B))/\widehat{F}) = [A]H^1(\widehat{F}) + [B]H^1(\widehat{F}).$$

Let $E = \widehat{F}(\text{SB}(A) \times \text{SB}(B))$. Clearly $[A_E] = [B_E] = 0$. Hence $[D_E] = [A_E] + [B_E] = 0$. Thus $\text{SB}(D)_E$ is a rational variety.

Since $[A_E] = 0$, the Albert form $\langle -a, -b, ab, d, t, -dt \rangle_E$ of the biquaternion algebra $A_E = ((a, b) \otimes (d, t))_E$ is hyperbolic. Hence $\langle -a, -b, ab, d \rangle_E = \langle dt, -d \rangle_E$ in the Witt ring $W(E)$. Therefore ψ_E is isotropic. Hence $(X_\psi)_E$ is a rational variety.

Let $Y = X_\psi \times \text{SB}(D)$. Since $(X_\psi)_E$ and $\text{SB}(D)_E$ are rational, it follows that Y_E is rational. Hence $E(Y)/E$ is a purely transcendental extension. Therefore $H^3(E(Y)/F) = H^3(E/F)$. We have $H^3(F(Y)/F) \subset H^3(E(Y)/F) = H^3(E/F)$.

Let $u \in H^3(F(X_\psi \times \text{SB}(D))/F) = H^3(F(Y)/F)$. To prove the theorem, it is enough to show that u can be written in the form

$$u = e^3(\psi \langle\langle k \rangle\rangle) + [D] \cup (r)$$

with some $k, r \in F^*$.

Since $H^3(F(Y)/F) \subset H^3(E/F)$, it follows that

$$u_{\widehat{F}} \in H^3(E/\widehat{F}) = H^3(\widehat{F}(\text{SB}(A) \times \text{SB}(B))/\widehat{F}) = [A]H^1(\widehat{F}) + [B]H^1(\widehat{F}).$$

Since $[A] + [B] = [D_{\widehat{F}}]$, we have $u_{\widehat{F}} \in [A]H^1(\widehat{F}) + [D_{\widehat{F}}]H^1(\widehat{F})$. Hence there are $\alpha, \beta \in \widehat{F}$ such that $u_{\widehat{F}} = [A] \cup (\alpha) + [D_{\widehat{F}}] \cup (\beta)$. Since $\widehat{F}^*/\widehat{F}^{*2} \simeq F^*/F^{*2} \times \{1, t\}$,

we can suppose that $\alpha = kt^i$ and $\beta = rt^j$, where $k, r \in F^*$ and $i, j \in \{0, 1\}$. We have

$$\begin{aligned} u_{\widehat{F}} &= [A] \cup (\alpha) + [D_{\widehat{F}}] \cup (\beta) = \\ &= ((a, b) + (d, t)) \cup (kt^i) + [D] \cup (rt^j) = \\ &= ((a, b, k) + [D] \cup (r)) + (t) \cup (i(a, b) + (d, k(-1)^i) + j[D]). \end{aligned}$$

Using well-known isomorphism $H^i(F((t))) = H^i(F) \oplus H^{i-1}(F)$, we have

$$u = (a, b, k) + [D] \cup (r)$$

and

$$i(a, b) + (d, k(-1)^i) + j[D] = 0.$$

We claim that $j = 0$. Indeed, if $j \neq 0$ then $j = 1$ and hence $[D] = i(a, b) + (d, k(-1)^i)$. Therefore $[D] = [(a, b^i) \otimes (d, k(-1)^i)]$. Thus $\text{ind}(D) \leq 4$, a contradiction.

So $j = 0$, and we have $i(a, b) + (d, k(-1)^i) = 0$. Thereby $i(a, b)_{F(\sqrt{d})} = 0$. Since $(a, b)_{F(\sqrt{d})} \neq 0$, it follows that $i = 0$.

Since $i(a, b) + (d, k(-1)^i) = 0$ and $i = 0$, we have $(d, k) = 0$. Hence $\langle\langle d, k \rangle\rangle = 0$ in $W(F)$. Since $\psi = \langle -a, -b, ab, d \rangle = \langle\langle a, b \rangle\rangle - \langle\langle d \rangle\rangle$, we have $\psi \langle\langle k \rangle\rangle = (\langle\langle a, b \rangle\rangle - \langle\langle d \rangle\rangle) \langle\langle k \rangle\rangle = \langle\langle a, b, k \rangle\rangle - \langle\langle d, k \rangle\rangle = \langle\langle a, b, k \rangle\rangle$. Therefore $\psi \langle\langle k \rangle\rangle \in GP_3(F)$ and $e^3(\psi \langle\langle k \rangle\rangle) = (a, b, k)$.

Hence the element $u = (a, b, k) + [D] \cup (r)$ belongs to the set

$$\{e^3(\psi \langle\langle k \rangle\rangle) \mid k \text{ is such that } \psi \langle\langle k \rangle\rangle \in GP_3(F)\} + [D]H^1(F).$$

□

Proposition 3.2. *In the notation of Theorem 3.1, let $\xi \in I^2(F)$ be a quadratic form such that $c(\xi) = [D]$. Then for an arbitrary element $\pi \in I^3(F(X_\psi \times \text{SB}(D))/F)$ there are $k_1, k_2 \in F^*$ such that*

$$\pi \equiv \psi \langle\langle k_1 \rangle\rangle + \xi \langle\langle k_2 \rangle\rangle \pmod{I^4(F)}$$

Proof. Obviously $e^3(\pi) \in H^3(F(X_\psi \times \text{SB}(D))/F)$. It follows from Theorem 3.1 that there are $k_1, k_2 \in F^*$ such that $e^3(\pi) = e^3(\psi \langle\langle k_1 \rangle\rangle) + [D] \cup (k_2)$. Clearly $[D] \cup (k_2) = e^2(\xi) \cup e^1(\langle\langle k_2 \rangle\rangle) = e^3(\xi \langle\langle k_2 \rangle\rangle)$. Hence $e^3(\pi) = e^3(\psi \langle\langle k_1 \rangle\rangle) + e^3(\xi \langle\langle k_2 \rangle\rangle)$. Since $\text{Ker}(e^3 : I^3(F) \rightarrow H^3(F)) = I^4(F)$, we have $\pi \equiv \psi \langle\langle k_1 \rangle\rangle + \xi \langle\langle k_2 \rangle\rangle \pmod{I^4(F)}$. □

4. 8-DIMENSIONAL QUADRATIC FORMS

Proposition 4.1. *Let ϕ be an 8-dimensional quadratic form with $d_\pm \phi = 1$ and $\text{ind } C(\phi) = 8$. Let ψ be a 4-dimensional quadratic form with $d_\pm \psi \neq 1$. Suppose that $\phi_{F(\psi)}$ is isotropic. Then there exists a half-neighbor ϕ^* of ϕ such that $\psi \subset \phi^*$.*

Proof. Changing ψ by a coefficient we can assume that $\psi = \langle -a, -b, ab, d \rangle$ with $d \notin F^{*2}$. Clearly $C_0(\psi) = (a, b)_{F(\sqrt{d})}$. Since $\text{ind } C(\phi) = 8$, there exists a 3-quaternion division algebra D such that $c(\phi) = [D]$.

Lemma 4.2. *There exist $u, v, s \in F^*$ such that $D = (a, b) \otimes (u, v) \otimes (d, s)$.*

Proof. Since $\phi_{F(\psi)}$ is isotropic, it follows that $D_{F(\psi)}$ is not a division algebra. The index reduction formula [M3] shows that $\text{ind}(C_0(\psi) \otimes_F D) = 2$. Therefore $\text{ind}((a, b) \otimes_F D)_{F(\sqrt{d})} = 2$. Hence there are $u, v, s \in F^*$ such that $[(a, b) \otimes_F D] = [(u, v) \otimes_F (d, s)]$ in $\text{Br}_2(F)$. Hence $[D] = [(a, b) \otimes_F (u, v) \otimes_F (d, s)]$. Since $\deg D = 8$, we have $D = (a, b) \otimes (u, v) \otimes (d, s)$. $\square$

Consider the quadratic form

$$\gamma = \langle -a, -b, ab, d \rangle \perp -s \langle -u, -v, uv, d \rangle.$$

One can verify that $d_{\pm} \gamma = 1$ and $c(\gamma) = [D]$. Hence $c(\phi) = [D] = c(\gamma)$. Therefore $\phi + \gamma \in I^3(F)$.

Lemma 4.3. $\phi + \gamma \in I^3(F(X_{\psi} \times SB(D))/F)$.

Proof. Let $E = F(X_{\psi} \times SB(D))$. Since $\phi + \gamma \in I^3(F)$, it is sufficient to verify that ϕ_E and γ_E are hyperbolic. Obviously $[D_E] = 0$, and the form ψ_E is isotropic. Since $c(\phi_E) = c(\gamma_E) = [D_E] = 0$ and $\dim \phi = \dim \gamma = 8$, we have $\phi_E, \gamma_E \in GP_3(E)$. Hence it is sufficient to prove that ϕ_E and γ_E are isotropic. Since $\phi_{F(\psi)}$ and ψ_E are isotropic, ϕ_E is isotropic as well. Since $\psi \subset \gamma$ and ψ_E is isotropic, we see that γ_E is isotropic. $\square$

Now we can complete the proof of Proposition 4.1. By Proposition 3.2 and Lemma 4.3, there exist $k_1, k_2 \in F^*$ such that

$$\phi + \gamma \equiv \psi \langle\langle k_1 \rangle\rangle + \phi \langle\langle k_2 \rangle\rangle \pmod{I^4(F)}.$$

Let $\rho = -s \langle -u, -v, uv, d \rangle$. We have $\gamma = \psi + \rho$. Hence

$$\phi + \psi + \rho \equiv \psi - k_1 \psi + \phi - k_2 \phi \pmod{I^4(F)}.$$

Thus $k_1 \psi + \rho \equiv -k_2 \phi \pmod{I^4(F)}$. Hence $\psi + k_1 \rho \equiv -k_1 k_2 \phi \pmod{I^4(F)}$. We finish the proof by setting $\phi^* = \psi \perp k_1 \rho$. $\square$

Theorem 4.4. *Let ϕ be an 8-dimensional quadratic form with $d_{\pm} \phi = 1$ and $\text{ind } C(\phi) = 8$. Let ψ be a quadratic form of dimension ≥ 4 such that $\psi \notin GP_2(F)$. The following conditions are equivalent:*

- $\phi_{F(\psi)}$ is isotropic;
- there exists a half-neighbor ϕ^* of ϕ such that $\psi \subset \phi^*$.

Proof. The case $\dim \psi = 4$ is Proposition 4.1. In the case $\dim \psi \neq 4$ the statement was proved by Laghribi in [Lag8] and [Lag]. $\square$

5. MAIN THEOREM

Theorem 5.1. *Let ϕ be a virtual Albert form (i.e. a 6-dimensional quadratic form with $d_{\pm} \phi \notin F^{*2}$ and $\text{ind}(C_0(\phi)) = 4$). Let ψ be a 4-dimensional quadratic form such that $d_{\pm} \psi \neq 1$. The following conditions are equivalent:*

- (1): $\phi_{F(\psi)}$ is isotropic;
- (2): ψ is similar to a subform in ϕ .

Proof. (1) $\Rightarrow$ (2). Let $d = d_{\pm} \phi$. Consider 8-dimensional quadratic form $\xi = \phi \perp t \langle\langle d \rangle\rangle$ over the field $\widehat{F} = F((t))$. Since $d_{\pm} \phi \notin F^{*2}$ and $\text{ind}(C_0(\phi)) = 4$, one has $\text{ind}(C(\xi)) = 8$. Clearly $\xi_{\widehat{F}(\psi)}$ is isotropic. It follows from Proposition 4.1 that there exists a quadratic form ξ^* over $\widehat{F}$ such that ξ and ξ^* are half-neighbors and $\psi_{\widehat{F}} \subset \xi^*$.

Lemma 5.2. ξ^* is similar to ξ .

Proof. Since ξ and ξ^* are half-neighbors, there exists $k \in \widehat{F}$ such that $\xi \equiv k\xi^* \pmod{I^4(\widehat{F})}$. By Springer's theorem one can write $k\xi^*$ in the form $k\xi^* = \mu_0 \perp t\mu_1$, where quadratic forms μ_0 and μ_1 are defined over F . We have

$$\phi \perp t \langle\langle d \rangle\rangle = \xi \equiv k\xi^* = \mu_0 \perp t\mu_1 \pmod{I^4(\widehat{F})}.$$

Hence $\phi \equiv \mu_0 \pmod{I^3(F)}$, $\langle\langle d \rangle\rangle \equiv \mu_1 \pmod{I^3(F)}$, and $\phi + \langle\langle d \rangle\rangle \equiv \mu_0 + \mu_1 \pmod{I^4(F)}$. Therefore $\text{ind } C_0(\mu_0) = \text{ind } C_0(\phi) \geq 4$. Hence $\dim \mu_0 \geq 6$. Therefore $\dim \mu_1 \leq 2$. By Arason–Pfister Hauptsatz the condition $\langle\langle d \rangle\rangle \equiv \mu_1 \pmod{I^3(F)}$ implies that $\mu_1 = \langle\langle d \rangle\rangle$. Hence $\phi \equiv \mu_0 \pmod{I^4(F)}$. Applying Arason–Pfister Hauptsatz once again, we have $\phi = \mu_0$. Therefore $\xi = k\xi^*$. $\square$

Now we return to the proof of Theorem 5.1. Since ψ is similar to a subform in ξ^* , and ξ^* is similar to ξ , it follows that ψ is similar to a subform in $\xi = \phi \perp t \langle\langle d \rangle\rangle$. Thus ψ is similar to a subform of ϕ by the following obvious observation.

Lemma 5.3. Let ψ , γ_0 and γ_1 be anisotropic quadratic forms over F . The following conditions are equivalent:

- a) $\psi_{F((t))}$ is similar to a subform in $\gamma_0 \perp t\gamma_1$,
- b) ψ is similar either to a subform in γ_0 or to a subform in γ_1 . $\square$

Thus we have proved that condition (1) of Theorem 5.1 implies condition (2). On the other hand, condition (2) obviously implies condition (1). The proof of Theorem 5.1 is complete. $\square$

Theorem 5.4. Let ϕ be a virtual Albert form and let $\psi \notin GP_2(F)$. The quadratic form $\phi_{F(\psi)}$ is isotropic if and only if ψ is similar to a subform in ϕ .

Proof. This theorem was proved by A. Laghribi in the following cases ([Lag6], [Lag]):

- $\dim \psi \neq 4$;
- $\dim \psi = 4$, $d_{\pm} \psi \neq d_{\pm} \phi$.

Thus we can suppose that $\dim \psi = 4$. To complete the proof it is sufficient to apply Theorem 5.1. $\square$

Together with results described in §1, Theorem 5.4 gives rise to the following

Corollary 5.5. Let ϕ be a 6-dimensional quadratic form with $\text{ind}(C_0(\phi)) = 4$. In the case when $\psi \notin GP_2(F)$, the quadratic form $\phi_{F(\psi)}$ is isotropic if and only if ψ is similar to a subform of ϕ . In the case when $\psi \in GP_2(F)$, the form $\phi_{F(\psi)}$ is isotropic if and only if a 3-dimensional subform of ψ is similar to a subform of ϕ . $\square$

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