ISOTROPY OF 8-DIMENSIONAL QUADRATIC FORMS OVER FUNCTION FIELDS OF QUADRICS

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ABSTRACT. Let F be a field of characteristic different from 2 and ϕ be an anisotropic 8-dimensional quadratic form over F with trivial determinant. We study the last open cases in the problem of describing the quadratic forms ψ such that ϕ becomes isotropic over the function field $F(\psi)$.

0. INTRODUCTION

Let F be a field of characteristic different from 2 and let ϕ and ψ be two anisotropic quadratic forms over F . An important problem in the algebraic theory of quadratic forms is to find conditions on ϕ and ψ so that $\phi_{F(\psi)}$ is isotropic. More precisely, one studies the question whether the isotropy of ϕ over $F(\psi)$ is standard in a sense.

In this paper we consider the case where ϕ is an 8-dimensional anisotropic quadratic form with trivial determinant. Necessity of certain sufficient conditions for isotropy of ϕ over $F(\psi)$ was studied by A. Laghribi; we call the isotropy, caused by one of these conditions, *L-standard*:

Definition. Let ϕ and ψ be anisotropic quadratic forms such that $\phi_{F(\psi)}$ is isotropic. Besides we suppose that $\dim \phi = 8$ and $\det \phi = 1$. We say that the isotropy of $\phi_{F(\psi)}$ is *L-standard*, if at least one of the following conditions holds:

- there exists a half-neighbor ϕ^* of ϕ such that $\psi \subset \phi^*$;
- there exists a 5-dimensional subform $\phi_0 \subset \phi$ with the following two properties:
 - the form ϕ_0 is a Pfister neighbor,
 - the form $(\phi_0)_{F(\psi)}$ is isotropic.

Otherwise, we say that the isotropy is *non-L-standard*.

In the case when $\dim \psi \geq 5$, the isotropy of $\phi_{F(\psi)}$ is always L-standard ([18], [17], see also [8]). The main result of our paper is the following

Theorem. *Let ϕ be an anisotropic 8-dimensional quadratic form with $\det \phi = 1$ and ψ be a 4-dimensional quadratic form with $\det \psi \neq 1$. Suppose that $\phi_{F(\psi)}$ is isotropic. Then the isotropy of $\phi_{F(\psi)}$ is L-standard except (possibly) the case $\text{ind } C(\phi) = \text{ind}(C(\phi) \otimes_F C_0(\psi)) = 4$.*

For the case where $\dim \psi = 3$ or where $\dim \psi = 4$ and $\det \psi = 1$ see §8.

1. TERMINOLOGY AND NOTATION

1.1. Quadratic forms. Mainly, we use notation of [19] and [27]. However there is certain slight difference: we denote by $\langle\langle a_1, \dots, a_n \rangle\rangle$ the n -fold Pfister form $\langle 1, -a_1 \rangle \otimes \dots \otimes \langle 1, -a_n \rangle$. We denote by $P_n(F)$ the set of all n -fold Pfister forms; $GP_n(F)$ is the set of the forms similar to a form from $P_n(F)$. For any field extension L/F , we denote by $P_n(L/F)$ the set of forms from $P_n(F)$ hyperbolic over L ; $GP_n(L/F)$ is the set of the forms similar to a form from $P_n(L/F)$.

We recall that a quadratic form ψ is called a (*Pfister*) *neighbor* (of a Pfister form π), if it is similar to a subform in π and $\dim \phi > \frac{1}{2} \dim \pi$. Two quadratic forms ϕ and ϕ^* are *half-neighbors*, if $\dim \phi = \dim \phi^*$ and there exists $s \in F^*$ such that the sum $\phi \perp s\phi^*$ is similar to a Pfister form.

For a quadratic form ϕ of dimension ≥ 3 , we denote by X_ϕ the projective variety given by the equation $\phi = 0$. We set $F(\phi) = F(X_\phi)$ if $\dim \phi \geq 3$; $F(\phi) = F(\sqrt{d})$ if $\dim \phi = 2$ and $d = d_\pm \phi \neq 1$; and $F(\phi) = F$ otherwise.

1.2. Algebras. We consider only finite-dimensional F -algebras.

For a central simple F -algebra D , we denote by $\deg(D)$, $[D]$, and $\exp(D)$ respectively the degree of D , the class of D in the Brauer group $\text{Br}(F)$, and the exponent of D , i.e., the order of $[D]$ in the Brauer group.

For a simple F -algebra A , we denote by $\text{ind}(A)$ the Schur index of A . For an algebra B of the form $B = A \times \dots \times A$ with simple A , we set $\text{ind } B = \text{ind } A$.

Let ϕ be a quadratic form. We denote by $C(\phi)$ the Clifford algebra of ϕ . By $C_0(\phi)$ we denote the even part of $C(\phi)$. Note that for any quadratic F -form ψ and any central simple F -algebra D , the index of $C_0(\psi) \otimes_F D$ is well-defined.

If $\phi \in I^2(F)$ then $C(\phi)$ is a central simple algebra. Its class $[C(\phi)]$ in the Brauer group $\text{Br}_2(F)$ is denoted by $c(\phi)$.

Let D be a central simple algebra. We denote by X_D the Severi-Brauer variety of D and by $F(D)$ the function field $F(X_D)$. For another central simple F -algebra D' and for a quadratic F -form ψ of dimension ≥ 3 , we set $F(D', D) \stackrel{\text{def}}{=} F(X_{D'} \times X_D)$ and $F(\psi, D) \stackrel{\text{def}}{=} F(X_\psi \times X_D)$.

1.3. Cohomology groups. By $H^*(F)$ we denote the graded ring of Galois cohomology $H^*(F, \mathbb{Z}/2\mathbb{Z}) \stackrel{\text{def}}{=} H^*(\text{Gal}(F_{\text{sep}}/F), \mathbb{Z}/2\mathbb{Z})$.

For $n = 0, 1, 2, 3$, there is a homomorphism $e^n : I^n(F) \rightarrow H^n(F)$ which is uniquely determined by the condition $e^n(\langle\langle a_1, \dots, a_n \rangle\rangle) = (a_1, \dots, a_n)$ (see [1]). The homomorphism e^n is surjective and $\ker e^n = I^{n+1}(F)$ for $n = 0, 1, 2, 3$ (see [20], [21], and [25]).

We also work with the cohomology groups $H^n(F, \mathbb{Q}/\mathbb{Z}(i))$, ($i = 0, 1, 2$), defined by B. Kahn (see [11]). For $n = 1, 2, 3$, the group $H^n(F)$ is naturally identified with $\text{Tors}_2 H^n(F, \mathbb{Q}/\mathbb{Z}(n-1))$.

1.4. K -theory and Chow groups. For a smooth algebraic F -variety X , its Grothendieck ring is denoted by $K(X)$. This ring is supplied with the filtration by codimension of support (also called the topological filtration).

We fix an algebraic closure $\bar{F}$ of the base field F and denote by $\bar{X}$ the $\bar{F}$ -variety $X_{\bar{F}}$. If the variety X is projective homogeneous, we identify $K(X)$ with a subring of $K(\bar{X})$ via the restriction homomorphism which is injective by [22].

For a ring (or a group) with filtration A , we denote by G^*A the adjoint graded ring (resp., the adjoint graded group).

There is a canonical surjective homomorphism of the graded Chow ring $\text{CH}^*(X)$ onto $G^*K(X)$, its kernel consists only of torsion elements and is trivial in the 0-th, 1-st, and 2-nd graded components ([28, §9]).

Let X_1 and X_2 be two smooth F -varieties. For any $x_1 \in K(X_1)$ and $x_2 \in K(X_2)$, we denote by $x_1 \boxtimes x_2$ the product $pr_1^*(x_1) \cdot pr_2^*(x_2) \in K(X_1 \times X_2)$ where pr_1^* and pr_2^* are the pull-backs with respect to the projections pr_1 and pr_2 of $X_1 \times X_2$ on X_1 and X_2 respectively. For an $\mathcal{O}_{X_1}$ -module $\mathcal{F}_1$ and an $\mathcal{O}_{X_2}$ -module $\mathcal{F}_2$, we denote by $\mathcal{F}_1 \boxtimes \mathcal{F}_2$ the tensor product $pr_1^*(\mathcal{F}_1) \otimes_{\mathcal{O}_X} pr_2^*(\mathcal{F}_2)$.

1.5. Relative groups. Let Φ be an arbitrary functor on the category of fields (of characteristic $\neq 2$) with values in the category of abelian groups. For a field extension L/F we use the notation $\Phi(L/F)$ for $\ker(\Phi(F) \rightarrow \Phi(L))$. Here is a list of examples that we need in this paper: $W(L/F)$, $I^n(L/F)$, $H^n(L/F)$, and $H^n(L/F, \mathbb{Q}/\mathbb{Z}(i))$.

2. THE GROUPS $H^3(F(\psi, D)/F)$ AND $I^3(F(\psi, D)/F)$

Proposition 2.1. *Let D be a central simple F -algebra of exponent 2 and ψ be a quadratic form of dimension ≥ 3 . Then there exists a natural isomorphism*

$$\frac{H^3(F(\psi, D)/F)}{H^3(F(\psi)/F) + H^3(F(D)/F)} \simeq \frac{\text{Tors CH}^2(X_\psi \times X_D)}{pr_\psi^* \text{Tors CH}^2(X_\psi) + pr_D^* \text{Tors CH}^2(X_D)}$$

where pr_ψ^* and pr_D^* are the pull-backs with respect to the projection pr_ψ and pr_D of $X_\psi \times X_D$ to X_ψ and X_D .

Proof. By [7, Prop. 2.2] the factor group

$$\frac{H^3(F(\psi, D)/F, \mathbb{Q}/\mathbb{Z}(2))}{H^3(F(\psi)/F, \mathbb{Q}/\mathbb{Z}(2)) + H^3(F(D)/F, \mathbb{Q}/\mathbb{Z}(2))}$$

is isomorphic to

$$\frac{\text{Tors CH}^2(X_\psi \times X_D)}{pr_\psi^* \text{Tors CH}^2(X_\psi) + pr_D^* \text{Tors CH}^2(X_D)}.$$

Now it is sufficient to apply isomorphisms

$$\begin{aligned} H^3(F(\psi)/F, \mathbb{Q}/\mathbb{Z}(2)) &= H^3(F(\psi)/F), \\ H^3(F(D)/F, \mathbb{Q}/\mathbb{Z}(2)) &= H^3(F(D)/F), \\ H^3(F(\psi, D)/F, \mathbb{Q}/\mathbb{Z}(2)) &= H^3(F(\psi, D)/F). \end{aligned}$$

(see e.g. [7, Lemma 2.8], [4, Lemma A.8], [8]). □

Corollary 2.2. *Let D be a biquaternion algebra and ψ be a 4-dimensional quadratic form. Then there exists a natural isomorphism*

$$\frac{H^3(F(\psi, D)/F)}{H^3(F(\psi)/F) + H^3(F(D)/F)} \simeq \text{Tors CH}^2(X_\psi \times X_D).$$

In particular, $2 \cdot \text{Tors CH}^2(X_\psi \times X_D) = 0$.

Proof. Since $\dim \psi = 4$, we have $\text{Tors CH}^2(X_\psi) = 0$ (see e.g. [12, Theorem 6.1] or [7, Lemma 2.4]). By [14, Proposition 5.3], we have $\text{Tors CH}^2(X_D) = 0$. To complete the proof it is sufficient to apply Proposition 2.1. $\square$

Lemma 2.3. *Let ψ be a quadratic form of dimension ≥ 3 . Then*

1. *the map $e^3 : P_3(F(\psi)/F) \rightarrow H^3(F(\psi)/F)$ is surjective;*
2. *$I^3(F(\psi)/F) + I^4(F) = P_3(F(\psi)/F) + I^4(F)$.*

Proof. 1. Really, we have to verify that the set $H^3(F(\psi)/F)$ consists of symbols. This fact is proved in [1, Satz 5.6].

2. Follows from Item 1 and from injectivity of $e^3 : I^3(F)/I^4(F) \rightarrow H^3(F)$. $\square$

Lemma 2.4. *Let D be a biquaternion algebra and q be an Albert form of D . Then*

1. *$H^3(F(D)/F) = [D] \cup H^1(F)$;*
2. *$I^3(F(D)/F) + I^4(F) = \{q \langle\langle s \rangle\rangle \mid s \in F^*\} + I^4(F)$;*
3. *the map $e^3 : I^3(F(D)/F) \rightarrow H^3(F(D)/F)$ is surjective.*

Proof. 1. See [23, Corollary 4.5].

2. Obviously $e^3(q \langle\langle s \rangle\rangle) = [D] \cup (s)$. Now, it is sufficient to apply Item 1 and injectivity of $e^3 : I^3(F)/I^4(F) \rightarrow H^3(F)$.

3. Follows from Items 1 and 2. $\square$

Lemma 2.5. *Let D be a biquaternion algebra and ψ be a quadratic form of dimension ≥ 3 . Then the natural homomorphism*

$$\frac{I^3(F(\psi, D)/F) + I^4(F)}{I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)} \rightarrow \frac{H^3(F(\psi, D)/F)}{H^3(F(\psi)/F) + H^3(F(D)/F)}$$

is injective.

In particular, the condition $H^3(F(\psi, D)/F) = H^3(F(\psi)/F) + H^3(F(D)/F)$ implies that $I^3(F(\psi, D)/F) \subset I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$.

Proof. Obvious consequence of the following facts:

- a) $I^3(F)/I^4(F) \rightarrow H^3(F)$ is injective;
- b) $I^3(F(\psi)/F) \rightarrow H^3(F(\psi)/F)$ is surjective (Lemma 2.3);
- c) $I^3(F(D)/F) \rightarrow H^3(F(D)/F)$ is surjective (Lemma 2.4). $\square$

Corollary 2.6. *Let D be a biquaternion algebra and ψ be a quadratic form of dimension ≥ 3 such that $\text{Tors CH}^2(X_\psi \times X_D) = 0$. Then $I^3(F(\psi, D)/F) \subset I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$.*

Proof. Follows from Lemma 2.5 and Proposition 2.1. $\square$

3. THE GROUP $K(X_\psi \times X_D)$

In this section, ψ is a quadratic F -form of dimension 4, D is a biquaternion F -algebra.

Lemma 3.1. *Consider the tensor product $K(X_\psi) \otimes_{\mathbb{Z}} K(X_D)$ together with the filtration induced by the topological filtrations on $K(X_\psi)$ and $K(X_D)$. The adjoint graded group $G^*(K(X_\psi) \otimes K(X_D))$ is torsion-free.*

Proof. The adjoint graded group $G^*K(X_\psi)$ is torsion-free (see e.g [12]). The adjoint graded group $G^*K(X_D)$ is torsion-free as well (see [13, Example]). We have a surjection

$$G^*K(X_\psi) \otimes G^*K(X_D) \twoheadrightarrow G^*(K(X_\psi) \otimes K(X_D)) .$$

The left-hand side term is a finitely generated torsion-free abelian group, i.e. a free abelian group of finite rank. This rank coincides with the rank of the right-hand side term. Therefore, the map is an isomorphism. $\square$

We consider the subgroup $K(X_\psi) \boxtimes K(X_D)$ of $K(X_\psi \times X_D)$ together with the filtration induced by the topological filtration on $K(X_\psi \times X_D)$.

Lemma 3.2. *The homomorphism $K(X_\psi) \otimes K(X_D) \rightarrow K(X_\psi) \boxtimes K(X_D)$ is an isomorphism of groups with filtrations.*

Proof. It is a homomorphism of groups respecting the filtrations. First of all let us check that it is an isomorphism of groups, regardless the filtrations. It is evidently an epimorphism. So, we only have to check the injectivity. Since for any extension of the base field F , the restriction homomorphism on the product $K(X_\psi) \otimes K(X_D)$ is injective, it suffices to check the injectivity in a split situation. However, if D splits, then X_D is isomorphic to a projective space; therefore the map $K(X_\psi) \otimes K(X_D) \rightarrow K(X_\psi) \boxtimes K(X_D)$ is an isomorphism.

To finish the proof, it suffices to show that the homomorphism of the adjoint graded groups is injective. Consider the commutative diagram

$$\begin{array}{ccc} G^*(K(\bar{X}_\psi) \otimes K(\bar{X}_D)) & \longrightarrow & G^*(K(\bar{X}_\psi) \boxtimes K(\bar{X}_D)) \\ \uparrow & & \uparrow \\ G^*(K(X_\psi) \otimes K(X_D)) & \longrightarrow & G^*(K(X_\psi) \boxtimes K(X_D)) \end{array}$$

The left hand side arrow is injective since the group $G^*(K(X_\psi) \otimes K(X_D))$ is torsion-free (Lemma 3.1) The upper arrow is injective because $\bar{X}_D$ is isomorphic to a projective space. Therefore the bottom arrow is injective too. $\square$

Corollary 3.3. *The group $G^*(K(X_\psi) \boxtimes K(X_D))$ is torsion-free.* $\square$

We denote by C the even Clifford algebra $C_0(\psi)$. Let $\mathcal{U}(2)$ be the Swan's sheaf $\mathcal{U}$ on the quadric X_ψ ([29, §6]), twisted twice. It has a structure of C -module. Let $\mathcal{J}$ be the canonical sheaf on the Severi-Brauer variety X_D ([24, §8.4]). It has a structure of D -module.

Put $\mathcal{F} \stackrel{\text{def}}{=} \mathcal{U}(2) \boxtimes \mathcal{J}$. It is a sheaf on $X_\psi \times X_D$ with a structure of $C \otimes_F D$ -module. Denote by f the homomorphism $K(C \otimes D) \rightarrow K(X_\psi \times X_D)$ given by the functor of tensor multiplication by $\mathcal{F}$ over $C \otimes D$.

Put $\mathcal{G} \stackrel{\text{def}}{=} \mathcal{U}(2) \boxtimes \mathcal{J}^{\otimes 3}$. It is a sheaf on $X_\psi \times X_D$ with a structure of $C \otimes_F D^{\otimes 3}$ -module. Consider the homomorphism $K(C \otimes D^{\otimes 3}) \rightarrow K(X_\psi \times X_D)$ given by the functor of tensor multiplication by $\mathcal{G}$ over $C \otimes D^{\otimes 3}$. Since the algebra $D^{\otimes 2}$ is split, the group $K(C \otimes D^{\otimes 3})$ is canonically isomorphic (via Morita equivalence) to $K(C \otimes D)$. Denote by g the composition $K(C \otimes D) \xrightarrow{\sim} K(C \otimes D^{\otimes 3}) \rightarrow K(X_\psi \times X_D)$.

Lemma 3.4. 1. *The homomorphism*

$$K(C \otimes D)^{\oplus 2} \rightarrow K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D)),$$

induced by the homomorphisms $f + g$, is surjective.

2. *If $C \otimes D$ is a skewfield, then $K(X_\psi \times X_D) = K(X_\psi) \boxtimes K(X_D)$.*

Proof. 1. Using the Swan's computation of the K-theory for quadrics [29, Theorem 9.1] (with $\mathcal{U}(2)$ instead of $\mathcal{U}$) and a generalized Peyre's version [23, Proposition 3.1] of the Quillen's computation of K-theory for Severi-Brauer schemes [24, Theorem 4.1 of §8], we get an isomorphism

$$K(F)^{\oplus 4} \oplus K(C)^{\oplus 2} \oplus K(D)^{\oplus 4} \oplus K(C \otimes D)^{\oplus 2} \simeq K(X_\psi \times X_D)$$

such that the image of $K(F)^{\oplus 4} \oplus K(C)^{\oplus 2} \oplus K(D)^{\oplus 4}$ is contained in $K(X_\psi) \boxtimes K(X_D)$ and the summand $K(C \otimes D)^{\oplus 2}$ is mapped into $K(X_\psi \times X_D)$ via $f + g$. Therefore, $K(C \otimes D)^{\oplus 2} \rightarrow K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D))$ is an epimorphism.

2. If the algebra $C \otimes D$ is a skewfield, then its class generates the group $K(C \otimes D)$. The images of this class under f, g are $\mathcal{F}, \mathcal{G} \in K(X_\psi) \boxtimes K(X_D)$. $\square$

Corollary 3.5. *If $C \otimes D$ is a skewfield, then the group $G^*K(X_\psi \times X_D)$ is torsion-free.*

Proof. By Lemma 3.4, $K(X_\psi \times X_D) = K(X_\psi) \boxtimes K(X_D)$. By Corollary 3.3, the group $G^*(K(X_\psi) \boxtimes K(X_D))$ is torsion-free. $\square$

4. THE GROUP $\text{Tors CH}^2(X_\psi \times X_D)$

Theorem 4.1. *Let D be a biquaternion algebra and ψ be an anisotropic 4-dimensional quadratic form with $\det \psi \neq 1$. Then the group $\text{CH}^2(X_\psi \times X_D)$ is torsion-free except (possibly) the following two cases:*

- (1) $\text{ind } C_0(\psi) \otimes D = \text{ind } D = 4$,
- (2) $\text{ind } C_0(\psi) \otimes D = 2$.

Proof. Set $C \stackrel{\text{def}}{=} C_0(\psi)$ and $s \stackrel{\text{def}}{=} \text{ind}(C \otimes D)$. The possible values of s are 1, 2, 4, 8.

Assume that $s = 8$. Since $\det \psi \neq 1$, it follows that $C \otimes D$ is a skewfield. Therefore, the group $\text{CH}^2(X_\psi \times X_D) \simeq G^2K(X_\psi \times X_D)$ is torsion-free by Corollary 3.5.

Now assume $s = 1$. Then $\text{ind}(C_0(\psi_{F(D)})) = 1$. Therefore the quadratic form $\psi_{F(D)}$ is isotropic. Hence the extension $F(\psi, D)/F(D)$ is purely transcendental and $H^3(F(\psi, D)/F) = H^3(F(D)/F)$. Now, Corollary 2.2 shows that $\text{Tors CH}^2(X_\psi \times X_D) = 0$.

Finally, assume $s = 4$ and $\text{ind } D \neq 4$. Then the biquaternion algebra D is Brauer equivalent to a quaternion F -algebra D' . Since $\text{Tors CH}^2(X_\psi \times X_D)$ depends only on the Brauer class of D (see e.g. [8]), we may replace D by D' . Let ψ' be a 3-dimensional quadratic F -form such that $C_0(\psi') \simeq D'$. The Severi-Brauer variety $X_{D'}$ is isomorphic to the conic $X_{\psi'}$. Since the tensor product $C_0(\psi) \otimes C_0(\psi')$ has index 4, it is a division algebra. Therefore, by [7, Corollary 4.4], the group $\text{CH}^2(X_\psi \times X_{\psi'})$ is torsion-free. $\square$

Remark 4.2. The assumption ψ is anisotropic is not essential: if ψ is isotropic, then $H^3(F(\psi, D)/F) = H^3(F(D)/F)$ and therefore the group $\text{CH}^2(X_\psi \times X_D)$ is torsion-free as well (by Corollary 2.2).

Proposition 4.3. *Let D be a biquaternion algebra and ψ be a 4-dimensional quadratic form with $\det \psi \neq 1$. Then the group $\text{Tors CH}^2(X_\psi \times X_D)$ is equal to zero or isomorphic to $\mathbb{Z}/2$.*

Proof. Since we already know that the torsion in the group $\text{CH}^2(X_\psi \times X_D)$ is annihilated by 2 (Corollary 2.2), it suffices to show that the torsion is cyclic.

Once again we set $C \stackrel{\text{def}}{=} C_0(\psi)$. According to Theorem 4.1, it suffices to consider the case where $\text{ind}(C \otimes D)$ equals 2 or 4.

Consider the quotient $K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D))$ with the filtration induced by the topological filtration on $K(X_\psi \times X_D)$. Since in the exact sequence of the adjoint graded groups

$$\begin{aligned} 0 \rightarrow G^*(K(X_\psi) \boxtimes K(X_D)) &\rightarrow G^*K(X_\psi \times X_D) \rightarrow \\ &\rightarrow G^*(K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D))) \rightarrow 0 \end{aligned}$$

the left-hand side term is torsion-free (Corollary 3.3), we have an injection

$$\text{Tors } G^*K(X_\psi \times X_D) \hookrightarrow G^*(K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D))).$$

Since $\text{CH}^2(X_\psi \times X_D) \simeq G^2K(X_\psi \times X_D)$, it suffices to show that the group $G^2(K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D)))$ is cyclic.

Denote by $h \in K(X_\psi)$ the class of a general hyperplane section of X_ψ . Let $\xi \in K(\bar{X}_D)$ be the class of the tautological linear bundle on the projective space $\bar{X}_D$. Note that for any $i \geq 0$, the multiple $(\text{ind } D^{\otimes i}) \cdot \xi^i$ of ξ^i belongs to $K(X_D)$. Thus $\xi^i \in K(X_D)$ for i even and $4\xi^i \in K(X_D)$ for i odd.

The algebra $C \otimes D$ is simple. Therefore, its Grothendieck group is cyclic. By Lemma 3.4, it follows that the quotient $K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D))$ is generated by two elements, namely by $(s/2)x$ and $(s/2)y$, where $x \stackrel{\text{def}}{=} (4 + 2h + h^2) \boxtimes \xi$, $y \stackrel{\text{def}}{=} (4 + 2h + h^2) \boxtimes \xi^3$, and $s \stackrel{\text{def}}{=} \text{ind } C \otimes D$ (we use here the equality $[\mathcal{U}(2)] = 4 + 2h + h^2 \in K(X_\psi)$, [12, Lemma 3.6]).

We have a congruence $x \equiv h^2 \boxtimes (\xi - 1) - h \boxtimes (\xi - 1)^2 \pmod{K(X_\psi) \boxtimes K(X_D)}$. The right side hand element belongs to $K(X_\psi \times X_D)^{(2)}$, because it is in $K(\bar{X}_\psi \times$

$\bar{X}_D)^{(2)}$ and $K(X_\psi \times X_D)^{(2)} = K(\bar{X}_\psi \times \bar{X}_D)^{(2)} \cap K(X_\psi \times X_D)$ (see [26, Lemme 6.3, (i)]). Therefore, $(s/2)x \in (K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D)))^{(2)}$ (take into account that the coefficient $s/2$ is an integer).

We also have another congruence modulo $K(X_\psi) \boxtimes K(X_D)$:

$$y - x \equiv (h^2 \boxtimes (\xi - 1) - h \boxtimes (\xi - 1)^2) \cdot (1 \boxtimes (\xi^2 - 1)) .$$

The right side hand element is in $K(X_\psi \times X_D)^3$ as a product of an element in $K(X_\psi \times X_D)^{(2)}$ and the element $1 \boxtimes (\xi^2 - 1) \in K(X_\psi \times X_D)^{(1)}$. Therefore, $(s/2)(y - x) \in (K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D)))^{(3)}$ and it follows that the group $G^2(K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D)))$ is generated by $(s/2)x$. So, in particular, this group is cyclic. $\square$

Remark 4.4. The assumption $\det \psi \neq 1$ is not essential: if $\det \psi = 1$, then $H^3(F(\psi, D)/F) = H^3(F(\psi', D)/F)$ and $H^3(F(\psi)/F) = H^3(F(\psi')/F)$, where ψ' is an arbitrary 3-dimensional subform of ψ ([7, Lemma 5.2]). Therefore,

$$\mathrm{CH}^2(X_\psi \times X_D) \simeq \mathrm{CH}^2(X_{\psi'} \times X_D) \simeq \mathrm{CH}^2(X_{D'} \times X_D)$$

where D' is the even Clifford algebra of ψ' . The group $\mathrm{Tors} \mathrm{CH}^2(X_{D'} \times X_D)$ is zero or $\mathbb{Z}/2$ according to [15, Théorème 6.1].

Theorem 4.5. Suppose that $\psi = \langle -x, -y, xy, d \rangle$ (with $d \notin F^{*2}$) and $D = (x, y) \otimes (u, v)$ where $x, y, d, u, v \in F^*$. Then $\mathrm{Tors} \mathrm{CH}^2(X_\psi \times X_D) = 0$.

Proof. The even Clifford algebra of the quadratic form ψ is isomorphic to the quaternion algebra $(x, y)_{F(\sqrt{d})}$. Therefore, the tensor product $C_0(\psi) \otimes D$ is Brauer equivalent to the quaternion algebra $(u, v)_{F(\sqrt{d})}$ and in particular has index 2 or 1. In the case, where the index is 1, we are done by Theorem 4.1. Let us assume the index equals to 2. It suffices to show that the element $x \stackrel{\mathrm{def}}{=} h^2 \boxtimes (\xi - 1) - h \boxtimes (\xi - 1)^2$ is in $K(X_\psi \times X_D)^{(3)}$ (see the proof of Proposition 4.3).

By definition, the element $h \in K(X_\psi)$ is the class of a hyperplane section of the quadric X_ψ . This hyperplane section is the quadric $X_{\psi'}$ determined by a 3-dimensional subform ψ' of ψ . Clearly, x is equal to the image of $x' \stackrel{\mathrm{def}}{=} h' \boxtimes (\xi - 1) - 1 \boxtimes (\xi - 1)^2 \in K(X_{\psi'} \times X_D)$ under the push-forward homomorphism $K(X_{\psi'} \times X_D) \rightarrow K(X_\psi \times X_D)$. Moreover, $x' \in K(X_{\psi'} \times X_D)^{(2)}$, because $x' \in K(\bar{X}_{\psi'} \times \bar{X}_D)^{(2)}$ and $K(X_{\psi'} \times K(X_D))^{(2)} = K(\bar{X}_{\psi'} \times \bar{X}_D)^{(2)} \cap K(X_{\psi'} \times K(X_D))$. Since the codimension of $X_{\psi'} \times X_D$ in $X_\psi \times X_D$ equals to 1, it follows that $x \in K(X_\psi \times X_D)^{(3)}$. $\square$

Remark 4.6. The assumption $d \notin F^{*2}$ is not essential: if $d \in F^{*2}$, then $\mathrm{Tors} \mathrm{CH}^2(X_\psi \times X_D) \simeq \mathrm{Tors} \mathrm{CH}^2(X_{C_0(\psi')} \times X_D)$, where $\psi' \stackrel{\mathrm{def}}{=} \langle -x, -y, xy \rangle$; since $C_0(\psi') \simeq (x, y)$, we have $\mathrm{ind} C_0(\psi') \otimes D \leq 2$; therefore, by [15, Théorème 6.1], the latter group is zero.

5. STANDARD ISOTROPY IN THE CASE $\text{Tors CH}^2(X_\psi \times X_D) = 0$

Definition 5.1. We say that (ϕ, D, q) is a special triple if the following conditions hold:

- 1) ϕ is an 8-dimensional anisotropic form with $\det \phi = 1$,
- 2) D is a biquaternion algebra,
- 3) q is an Albert form,
- 4) $[D] = c(\phi) = c(q) \in \text{Br}_2(F)$.

In this section we need the following

Theorem 5.2 ([18, Théorème 4] and [2, Corollary 9.3], see also [8]). *Let ϕ be an anisotropic 8-dimensional quadratic form with $\det \phi = 1$ and let D be an algebra such that $c(\phi) = [D]$. Then $\phi_{F(D)}$ is anisotropic.*

Our study of isotropy of 8-dimensional forms over function field of quadrics is based on the following assertion.

Proposition 5.3. *Let (ϕ, D, q) be a special triple and ψ be a quadratic form. Then*

1. *The following two conditions are equivalent:*
 - (i) $\phi + q \in I^3(F(\psi, D)/F)$;
 - (ii) $\phi_{F(\psi)}$ is isotropic.
2. *The following two conditions are equivalent:*
 - (i) $\phi + q \in I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$;
 - (ii) *there exists a 5-dimensional Pfister neighbor ϕ_0 such that $\phi_0 \subset \phi$ and $(\phi_0)_{F(\psi)}$ is isotropic.*

Proof. (1i) $\Rightarrow$ (1ii). Condition (1i) implies that the quadratic form $(\phi \perp q)_{F(\psi, D)}$ is hyperbolic. Since $q_{F(D)}$ is hyperbolic, it follows that $\phi_{F(\psi, D)}$ is hyperbolic.

Let $E \stackrel{\text{def}}{=} F(\psi)$. We see that $\phi_{E(D)}$ is hyperbolic. Theorem 5.2 implies that ϕ_E is isotropic, i.e., condition (1ii) holds.

(1ii) $\Rightarrow$ (1i). Suppose that $\phi_{F(\psi)}$ is isotropic. Since $c(\phi) = c(q)$, it follows that $\phi + q \in I^3(F)$. Hence it is sufficient to prove that $\phi_{F(\psi, D)}$ and $q_{F(\psi, D)}$ are hyperbolic. The form $q_{F(\psi, D)}$ is hyperbolic because $q_{F(D)}$ is. Since $c(\phi) = [D]$, we have $\phi_{F(\psi, D)} \in I^3(F(\psi, D))$. Since $\phi_{F(\psi)}$ is isotropic, we have $\dim(\phi_{F(\psi, D)})_{an} < 8$. The Arason-Pfister Hauptsatz shows that $\phi_{F(\psi, D)}$ is hyperbolic.

(2i) $\Rightarrow$ (2ii). By Lemmas 2.3 and 2.4, there exist $\pi \in P_3(F(\psi)/F)$ and $s \in F^*$ such that

$$\phi + q \equiv \pi + q \langle\langle s \rangle\rangle \pmod{I^4(F)}.$$

We have $\phi + sq \equiv \pi \pmod{I^4(F)}$. Since $\pi \in P_3(F)$, $\pi_{F(\pi)}$ is hyperbolic. Hence $(\phi + sq)_{F(\pi)} \equiv \pi_{F(\pi)} \equiv 0 \pmod{I^4(F)}$. Since $\dim(\phi + sq) \leq 8 + 4 < 16$, the Arason-Pfister Hauptsatz shows that $(\phi + sq)_{F(\pi)}$ is hyperbolic. Hence there exists a form γ such that $(\phi \perp sq)_{an} = \pi\gamma$. Clearly $0 < 8 - 6 \leq \dim(\phi \perp sq)_{an} \leq 8 + 6 < 16$. This implies that $\dim \gamma = 1$, i.e., there exists $k \in F^*$ such that $\gamma = \langle k \rangle$. Thus $\phi + sq = k\pi$. Therefore $(\phi \perp -k\pi) = -sq$. Hence ϕ and

$k\pi$ contain a common subform of dimension

$$\frac{\dim \phi + \dim \pi - \dim q}{2} = \frac{8 + 8 - 6}{2} = 5.$$

Let us denote such a form by ϕ_0 . Since $\dim \phi_0 = 5$ and $\phi_0 \subset k\pi$, it follows that ϕ_0 is a Pfister neighbor of π . Since $\pi \in P_3(F(\psi)/F)$, it follows that $(\phi_0)_{F(\psi)}$ is isotropic.

(2ii) $\Rightarrow$ (2i). Let ϕ_0 be a 5-dimensional quadratic form such as in (2ii). By the assumption, there exists $\pi \in GP_3(F)$ such that $\phi_0 \subset \pi$. Since $(\phi_0)_{F(\psi)}$ is isotropic, it follows that $\pi \in GP_3(F(\psi)/F)$.

Since $\phi_0 \subset \phi$ and $\phi_0 \subset \pi$, there exist 3-dimensional quadratic forms ρ', ρ'' such that $\phi = \phi_0 \perp \rho'$ and $\pi = \phi_0 \perp \rho''$. We set $\rho = \rho'' \perp -\rho'$. Clearly $\dim \rho = 6$. In the Witt ring $W(F)$ we have $\rho = \rho'' - \rho' = \pi - \phi$. In particular, $\rho \in I^2(F)$. Hence ρ is an Albert form.

We have $c(\rho) = c(\pi) + c(\phi) = 0 + c(\phi) = c(q)$. Hence ρ is similar to q ([10, Theorem 3.12]). Let $s \in F^*$ be such that $\rho = sq$. We have $\pi - \phi = \rho = sq$. Hence $\phi = \pi - sq$. Therefore

$$\begin{aligned} \phi + q &= \pi + q \langle\langle s \rangle\rangle \in GP_3(F(\psi)/F) + [q] \cdot I(F) \subset \\ &\subset I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F). \end{aligned}$$

□

Corollary 5.4. *Let (ϕ, D, q) be a special triple and ψ be a quadratic form of dimension ≥ 3 . Suppose that $\text{Tors CH}^2(X_\psi \times X_D) = 0$. Then the following conditions are equivalent:*

- (1) $\phi_{F(\psi)}$ is isotropic,
- (2) there exists a 5-dimensional Pfister neighbor ϕ_0 such that $\phi_0 \subset \phi$ and $(\phi_0)_{F(\psi)}$ is isotropic.

Proof. Obvious in view of Proposition 5.3 and Corollary 2.6. □

6. THE GROUP $H^3(F(\psi, D)/F)$ IN THE CASE $\text{ind}(C_0(\psi) \otimes_F D) = 2$

In this section we study the group $H^3(F(\psi, D)/F)$ in the case where ψ is a 4-dimensional quadratic form with a non-trivial discriminant, D is a biquaternion division F -algebra, and $\text{ind}(C_0(\psi) \otimes_F D) = 2$.

Let $d \stackrel{\text{def}}{=} \det \psi$. By our assumption, $d \notin F^{*2}$. Replacing ψ by a similar form, we can suppose $\psi = \langle -a, -b, ab, d \rangle$ with $a, b \in F^*$. Let $L \stackrel{\text{def}}{=} F(\sqrt{d})$. By our assumption, we have $\text{ind}((a, b) \otimes_F D)_L = 2$. Hence there exists a quaternion F -algebra Q such that $(a, b)_L + [D_L] = [Q_L]$ in $\text{Br}_2(L)$ (see [16, Proposition 16.2]). Let us write Q in the form $Q = (r, s)$ with $r, s \in F^*$. We set $\psi' = \langle -r, -s, rs, d \rangle$.

Let q be an Albert form corresponding to D .

Lemma 6.1. *There exist $k, k' \in F^*$ such that $k\psi + k'\psi' + q \in I^3(F)$.*

Proof. Since $(a, b)_L + [D_L] = [Q_L] = (r, s)_L$, it follows that $(a, b) + (r, s) + [D] \in \text{Br}_2(L/F)$. Hence there exist $k \in F^*$ such that $(a, b) + (r, s) + [D] = (d, k)$. Let $k' \stackrel{\text{def}}{=} -1$ and $\phi \stackrel{\text{def}}{=} k\psi + k'\psi' + q$. We claim that $\phi \in I^3(F)$. To prove this, it is sufficient to verify that $\phi \in I^2(F)$ and $c(\phi) = 0$. We have

$$\begin{aligned} \phi &= k\psi + k'\psi' + q = k \langle -a, -b, ab, d \rangle - \langle -r, -s, rs, d \rangle + q = \\ &= k(\langle\langle a, b \rangle\rangle - \langle\langle d \rangle\rangle) - (\langle\langle r, s \rangle\rangle - \langle\langle d \rangle\rangle) + q = \\ &= k \langle\langle a, b \rangle\rangle + \langle\langle d, k \rangle\rangle - \langle\langle r, s \rangle\rangle + q. \end{aligned}$$

Hence $\phi \in I^2(F)$ and $c(\phi) = (a, b) + (d, k) + (r, s) + c(q) = (a, b) + (d, k) + (r, s) + [D] = 0$. $\square$

Definition 6.2. Let D be a biquaternion algebra and ψ be a 4-dimensional quadratic form such that $\det \psi \neq 1$ and $\text{ind}(C_0(\psi) \otimes_F D) = 2$. We denote by $\Gamma(\psi, D)$ the set defined as follows

$$\{\gamma \in I^3(F) \mid \text{there exist } k, k', l \in F^* \text{ such that } \gamma = k\psi + k'\psi' + lq\},$$

where q is an Albert form corresponding to D and ψ' is a 4-dimensional quadratic form satisfying the following two properties: $\det \psi' = \det \psi$ and $C_0(\psi')$ is Brauer-equivalent to $C_0(\psi) \otimes_F D$.

Remark 6.3. 1. The set $\Gamma(\psi, D)$ does not depend on the choice of q and ψ' : indeed, the condition on q and ψ' determines them uniquely up to similarity.

2. Lemma 6.1 shows the set $\Gamma(\psi, D)$ is not empty.

Lemma 6.4. $\Gamma(\psi, D) \subset I^3(F(\psi, D)/F)$.

Proof. Let $\gamma = k\psi + k'\psi' + lq \in \Gamma(\psi, D)$. By the definition of $\Gamma(\psi, D)$, we have $\gamma \in I^3(F)$. Thus it is sufficient to prove that $\gamma_{F(\psi, D)}$ is hyperbolic. We have $\dim(\psi_{F(\psi)})_{an} \leq 2$ and $\dim(q_{F(D)})_{an} = 0$. Therefore $\dim(\gamma_{F(\psi, D)})_{an} = \dim((k\psi \perp k'\psi' \perp lq)_{F(\psi, D)})_{an} \leq 2 + 4 + 0 = 6 < 8$. Since $\gamma \in I^3(F)$, the Arason-Pfister Hauptsatz shows that $\gamma_{F(\psi, D)}$ is hyperbolic. $\square$

Lemma 6.5. Let $\gamma \in \Gamma(\psi, D)$, $\pi \in P_3(F(\psi)/F)$, and $s \in F^*$. Then there exists $\gamma' \in \Gamma(\psi, D)$ such that $\gamma' \equiv \gamma + \pi + q \langle\langle s \rangle\rangle \pmod{I^4(F)}$.

Proof. Let us write γ in the form $\gamma = k\psi + k'\psi' + lq$. Since $\pi \in P_3(F(\psi)/F)$, there exists $r \in F^*$ such that $\pi = \psi \langle\langle r \rangle\rangle$ ([6, Lemmas 6.1 and 6.2]).

We have $k\psi + \pi \equiv k\psi - k\pi = k\psi - k\psi \langle\langle r \rangle\rangle = rk\psi \pmod{I^4(F)}$ and $lq + q \langle\langle s \rangle\rangle \equiv lq - lq \langle\langle s \rangle\rangle = lsq \pmod{I^4(F)}$. Therefore

$$\gamma + \pi + q \langle\langle s \rangle\rangle = k\psi + k'\psi' + lq + \pi + q \langle\langle s \rangle\rangle \equiv rk\psi + k'\psi' + lsq \pmod{I^4(F)}.$$

Now it is sufficient to set $\gamma' = rk\psi + k'\psi' + lsq$. $\square$

Corollary 6.6. $\Gamma(\psi, D) + I^4(F) = \Gamma(\psi, D) + I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$.

Proof. Obvious in view of Lemmas 6.5, 2.3, and 2.4. $\square$

Lemma 6.7. *The following conditions are equivalent:*

- (1) $I^3(F(\psi, D)/F) \subset I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$;
- (2) $\Gamma(\psi, D) \subset I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$;
- (3) *there exists $\gamma \in \Gamma(\psi, D)$ such that $\gamma \in I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$;*
- (4) $\Gamma(\psi, D)$ *contains a hyperbolic form, i.e. $0 \in \Gamma(\psi, D)$;*
- (5) *there exist $x, y, u, v, d \in F^*$ such that $\psi \sim \langle -x, -y, xy, d \rangle$ and $D \cong (x, y) \otimes_F (u, v)$;*
- (6) $\text{Tors CH}^2(X_\psi \times X_D) = 0$;
- (7) $H^3(F(\psi, D)/F) = H^3(F(\psi)/F) + H^3(F(D)/F)$.

Proof. (1) $\Rightarrow$ (2). Obvious in view of Lemma 6.4.

(2) $\Rightarrow$ (3). Obvious, because $\Gamma(\psi, D)$ is not empty.

(3) $\Rightarrow$ (4). Condition (3) implies that $0 \in \Gamma(\psi, D) + I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$. It follows from Corollary 6.6 that $0 \in \Gamma(\psi, D) + I^4(F)$. Hence there exists $\gamma = k\psi + k'\psi' + lq \in \Gamma(\psi, D)$ such that $\gamma \in I^4(F)$. Since $\dim \gamma = 4 + 4 + 6 = 14 < 16$, the Arason-Pfister Hauptsatz shows that $\gamma = 0$.

(4) $\Rightarrow$ (5). Let $\gamma = k\psi + k'\psi' + lq$ be a hyperbolic form. We have $(k\psi \perp lq)_{an} = -k'\psi'_{an}$. Therefore $k\psi$ and $-lq$ contain a common subform of dimension

$$\frac{1}{2}(\dim \psi + \dim q - \dim \psi') = \frac{1}{2}(4 + 6 - 4) = 3.$$

Let us denote such a 3-dimensional form by τ . Let $x, y \in F^*$ be such that $\tau \sim \langle -x, -y, xy \rangle$. Thus $\langle -x, -y, xy \rangle$ is similar to a subform of ψ and similar to a subform of q . Let $d = \det \psi$. Since $\langle -x, -y, xy \rangle$ is similar to a subform of ψ it follows that $\langle -x, -y, xy, d \rangle$ is similar to ψ . Since $\langle -x, -y, xy \rangle$ is similar to a subform of the Albert form q , it follows that there exist $u, v \in F^*$ such that q is similar to $\langle -x, -y, xy, u, v, -uv \rangle$. Then $[D] = c(q) = (x, y) + (u, v)$. Therefore $D \cong (x, y) \otimes_F (u, v)$.

(5) $\Rightarrow$ (6). See Theorem 4.5.

(6) $\Rightarrow$ (7). See Proposition 2.1.

(7) $\Rightarrow$ (1). See Lemma 2.5. □

Proposition 6.8. *Let D be a biquaternion algebra and ψ be a 4-dimensional quadratic form such that $\det \psi \neq 1$ and $\text{ind}(D \otimes_F C_0(\psi)) = 2$. Then for any $\gamma \in \Gamma(\psi, D)$ one has*

$$H^3(F(\psi, D)/F) = H^3(F(\psi)/F) + H^3(F(D)/F) + e^3(\gamma)H^0(F).$$

Proof. By Lemma 6.4, the element $e^3(\gamma)$ belongs to $H^3(F(\psi, D)/F)$. If the group $\text{CH}^2(X_\psi \times X_D)$ is torsion-free, then, by Proposition 2.1, we have

$$H^3(F(\psi, D)/F) = H^3(F(\psi)/F) + H^3(F(D)/F)$$

and the proof is complete. If $\text{Tors CH}^2(X_\psi \times X_D) \neq 0$, Lemma 6.7 shows that $\gamma \notin I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$. It follows from Lemma 2.5 that $e^3(\gamma) \notin H^3(F(\psi)/F) + H^3(F(D)/F)$. To complete the proof it is sufficient to apply Proposition 4.3 and Proposition 2.1. □

Corollary 6.9. $I^3(F(\psi, D)/F) \subset I^3(F(\psi)/F) + I^3(F(D)/F) + \Gamma'(\psi, D) + I^4(F)$, where $\Gamma'(\psi, D) = \Gamma(\psi, D) \cup \{0\}$.

Proof. Let $\tau \in I^3(F(\psi, D)/F)$. Choose an element $\gamma \in \Gamma(\psi, D)$. By Proposition 6.8, either the element $e^3(\tau)$ or the element $e^3(\tau - \gamma)$ is in $H^3(F(\psi)/F) + H^3(F(D)/F)$. It remains to apply Lemma 2.5. $\square$

Proposition 6.10. Let $\lambda \in I^3(F(\psi, D)/F)$. Then at least one of the following conditions holds

- 1) $\lambda \in I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$;
- 2) $\lambda \in \Gamma(\psi, D) + I^4(F)$.

Proof. Obvious in view of Corollaries 6.9 and 6.6. $\square$

7. MAIN THEOREM

Theorem 7.1. Let ϕ be an anisotropic 8-dimensional quadratic form with $\det \phi = 1$ and let ψ be a 4-dimensional quadratic form with $\det \psi \neq 1$. Suppose that $\phi_{F(\psi)}$ is isotropic. Then the isotropy of $\phi_{F(\psi)}$ is L-standard except possibly the case $\text{ind } C(\phi) = \text{ind}(C(\phi) \otimes_F C_0(\psi)) = 4$.

Proof. In the case where $\text{ind } C(\phi) = 8$, the theorem is proved in [6]. Thus we can suppose that $\text{ind } C(\phi) \leq 4$. Then there exists a biquaternion algebra D such that $c(\phi) = [D]$. Let q be an Albert form corresponding to D . Clearly (ϕ, D, q) is a special triple.

By Corollary 5.4, we can suppose that $\text{Tors } \text{CH}^2(X_\psi \times X_D) \neq 0$. Theorem 4.1 asserts that at least one of the following conditions holds:

- 1) $\text{ind}(D \otimes C_0(\psi)) = \text{ind } D = 4$,
- 2) $\text{ind}(D \otimes C_0(\psi)) = 2$.

If $\text{ind}(D \otimes C_0(\psi)) = 2$, then all the conditions of Definition 6.2 hold. Thus we have a well-defined set $\Gamma(\psi, D)$.

By Proposition 5.3, we have $\phi + q \in I^3(F(\psi, D)/F)$. By Proposition 6.10, we see that at least one of the following condition holds:

- 1) $\phi + q \in I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$;
- 2) $\phi + q \in \Gamma(\psi, D) + I^4(F)$.

In the first case, Proposition 5.3 shows that there exists a 5-dimensional Pfister neighbor ϕ_0 such that $\phi_0 \subset \phi$ and $(\phi_0)_{F(\psi)}$ is isotropic. This implies that the isotropy of $\phi_{F(\psi)}$ is L-standard.

Therefore, we may assume that $\phi + q \in \Gamma(\psi, D) + I^4(F)$. Thus there exist $k, k', l \in F^*$ (and a 4-dimensional quadratic form ψ') such that

$$\phi + q \equiv k\psi + k'\psi' + lq \pmod{I^4(F)}.$$

Since $k\psi + k'\psi' + lq \in I^3(F)$ it follows that

$$k\psi + k'\psi' + lq \equiv l(k\psi + k'\psi' + lq) \pmod{I^4(F)}.$$

Therefore

$$\phi + q \equiv lk\psi + lk'\psi' + q \pmod{I^4(F)}.$$

Thus $\phi \equiv lk\psi + lk'\psi' \pmod{I^4(F)}$. Let $\phi^* \stackrel{\text{def}}{=} \psi \perp kk'\psi'$. Since $\phi \equiv lk\phi^* \pmod{I^4(F)}$ and $\dim \phi^* = 8$, it follows that ϕ and ϕ^* are half-neighbors. Since $\psi \subset \phi^*$, the isotropy is L-standard and the proof is complete. $\square$

Corollary 7.2. *Let ϕ be an anisotropic 8-dimensional quadratic form with $\det \phi = 1$ and let ψ be a 4-dimensional quadratic form with $\det \psi \neq 1$. Suppose that $\phi_{F(\psi)}$ is isotropic but the isotropy is not L-standard. Then ϕ can be written in the form $\phi = \pi_1 \perp \pi_2$ with $\pi_1, \pi_2 \in GP_2(F)$.*

Proof. Since $\det \phi = 1$, it is sufficient to verify that ϕ contains a 4-dimensional quadratic form with trivial determinant. Suppose at the moment that ϕ contains no 4-dimensional quadratic form with trivial determinant. Then [5, Theorem 6.1] implies that there exists a homomorphism of F -algebras, $C_0(\psi) \rightarrow C_0(\phi)$. Since $\det \phi = 1$ and $\dim \phi = 8$, there exists a 3-quaternion algebra A such that $C_0(\phi)$ has the form $A \times A$ and $C(\phi) \simeq M_2(A)$. Thus we get a homomorphism $C_0(\psi) \rightarrow A$ which is injective because $C_0(\psi)$ is a simple algebra. Then $\text{ind}(C_0(\psi) \otimes_F A) = 2$. Since $M_2(A) \simeq C(\phi)$, we have $\text{ind}(C_0(\psi) \otimes_F C(\phi)) = 2$. Theorem 7.1 implies that isotropy of $\phi_{F(\psi)}$ is L-standard. The contradiction obtained completes the proof. $\square$

It is a natural question if there exists an example of non-L-standard isotropy. Some way to construct non L-standard isotropy is based on the following

Lemma 7.3. *Let q be a 6-dimensional quadratic form and let ψ be a 4-dimensional quadratic form over a field k . Suppose that q is a $k(\psi)$ -minimal form (see definition in [3]). Let $F \stackrel{\text{def}}{=} k((t))$, $\phi \stackrel{\text{def}}{=} q \perp t \langle 1, \det(q) \rangle$. Then the form $\phi_{F(\psi)}$ is isotropic, but the isotropy is not L-standard.*

Proof. The proof of this lemma is absolutely analogous to the proof of [6, Theorem 5.1] and we omit it. $\square$

Corollary 7.4. *There exist a field F , an 8-dimensional quadratic form $\phi \in I^2(F)$, and a 4-dimensional F -form ψ with nontrivial determinant such that $\phi_{F(\psi)}$ is isotropic, but the isotropy is not L-standard.*

Proof. By [9] there exists a field k , a 6-dimensional quadratic form q and a 4-dimensional quadratic form ψ over a field k such that q is a $k(\psi)$ -minimal form. Now the required result follows from Lemma 7.3. $\square$

8. ISOTROPY OVER THE FUNCTION FIELD OF A CONIC

In this section we still assume that ϕ is an anisotropic 8-dimensional quadratic form of trivial determinant. We are interested in the question when ϕ is isotropic over the function field of a quadratic form ψ .

For the forms ψ of dimension ≥ 5 , the question was studied in [17] and [18]. The case $\dim \psi = 4$, $d_{\pm}\psi \neq 1$ is done in the previous section. Thus it suffices to consider only two cases: $\dim \psi = 3$ or $\psi \in GP_2(F)$. Since the function field of a form $\psi \in GP_2(F)$ is stably birational equivalent to the function field of

an arbitrary 3-dimensional subform of ψ (see e.g. [7, Lemma 5.2]), it suffices to handle only one of these cases.

Let us consider the case where $\psi \in GP_2(F)$.

Theorem 8.1. *Let ϕ be an anisotropic 8-dimensional quadratic form (we do not assume $\det \phi = 1$) and let $\psi \in GP_2(F)$. Then the form $\phi_{F(\psi)}$ is isotropic if and only if at least one of the following conditions holds:*

- a) *there exists a 10-dimensional form ϕ^* such that $\psi \subset \phi^*$ and $s\phi \equiv \phi^* \pmod{I^4(F)}$ for suitable $s \in F^*$;*
- b) *there exists a 5-dimensional subform $\phi_0 \subset \phi$ with the following two properties:*
 - *the form ϕ_0 is a Pfister neighbor,*
 - *the form $(\phi_0)_{F(\psi)}$ is isotropic.*

Proof. First, suppose that $\phi_{F(\psi)}$ is isotropic. The excellence property of the field extension $F(\psi)/F$ implies that there exist a 6-dimensional form τ and a form γ such that $\phi - \tau = \gamma\psi$ (see [3, Proof of Proposition 1.1]). Comparing dimensions, one can see that $1 \leq \dim \gamma \leq 3$. First, consider the case where $\dim \gamma$ is odd (i.e., $\dim \gamma = 1$ or 3). We set $s \stackrel{\text{def}}{=} d_{\pm} \gamma$. Clearly, $\gamma \equiv \langle s \rangle \pmod{I^2(F)}$. Hence $\gamma\psi \equiv s\psi \pmod{I^4(F)}$. Since $\phi - \tau = \gamma\psi$, it follows that $\phi = \tau + \gamma\psi \equiv \tau + s\psi \equiv s(\tau + \psi) \pmod{I^4(F)}$. Setting $\phi^* \stackrel{\text{def}}{=} s\tau \perp \psi$, one can see that condition a) holds. Now, suppose that $\dim \gamma$ is even, i.e., $\dim \gamma = 2$. Then $\gamma\psi \in GP_3(F)$. We have $\phi - \gamma\psi = \tau$. Therefore, the quadratic forms ϕ and $\gamma\psi$ contain a common subform of dimension $\frac{1}{2}(\dim \phi + \dim \gamma\psi - \dim \tau) = \frac{1}{2}(8 + 8 - 6) = 5$. Let us denote that 5-dimensional form by ϕ_0 . Clearly, condition b) holds.

Now, suppose that at least one of conditions a) and b) holds. We have to verify that $\phi_{F(\psi)}$ is isotropic. It is obvious in the case where condition b) holds. Suppose now that condition a) holds. Since $\psi \subset \phi^*$ and $\dim \phi^* = 10$, one has $\dim(\phi_{F(\psi)})_{an} \leq 6$. Therefore $\dim((\phi \perp -s\phi^*)_{F(\psi)})_{an} \leq 8 + 6 = 14 < 16$. Since $\phi \perp -s\phi^* \in I^4(F)$, the Arason-Pfister Hauptsatz implies that $\dim((\phi \perp -s\phi^*)_{F(\psi)})_{an}$ is hyperbolic. Hence $(\phi_{F(\psi)})_{an} = (s\phi_{F(\psi)}^*)_{an}$. Therefore $\dim(\phi_{F(\psi)})_{an} \leq 6$. Hence $\phi_{F(\psi)}$ is isotropic. $\square$

Corollary 8.2. *Let ϕ be an anisotropic 8-dimensional quadratic form with $\det \phi = 1$ and let ψ be a 4-dimensional quadratic form with $\det \psi = 1$ (or ψ is a 3-dimensional form). Suppose that $\phi_{F(\psi)}$ is isotropic but the isotropy is not L -standard. Then $\text{ind}(C(\phi) \otimes C_0(\psi)) = 4$.*

Proof. We can assume that $\psi \in GP_2(F)$ (if $\dim \psi = 3$, then we replace ψ by $\psi \perp \langle \det \psi \rangle$). Let us suppose that $\text{ind}(C(\phi) \otimes C_0(\psi)) \neq 4$. By Theorem 8.1, there exist a 10-dimensional form ϕ^* and an coefficient $s \in F^*$ such that $\psi \subset \phi^*$ and $s\phi \equiv \phi^* \pmod{I^4(F)}$. Then ϕ^* can be written in the form $\phi^* = \psi \perp q$. Clearly, q is an Albert form and $c(q) = c(\phi^*) + c(\psi)$. Therefore $\text{ind } C(q) = \text{ind}(C(\phi^*) \otimes C(\psi)) = \text{ind}(C(\phi) \otimes C_0(\psi)) \neq 4$. Hence, q is isotropic. Thus there exists a 4-dimensional form $\tilde{q}$ such that $\tilde{q}_{an} = q_{an}$. Set $\tilde{\phi}^* \stackrel{\text{def}}{=} \psi \perp \tilde{q}$.

Clearly, $\dim \tilde{\phi}^* = 8$, $\psi \subset \tilde{\phi}^*$ and $s\phi \equiv \phi^* \equiv \tilde{\phi}^* \pmod{I^4(F)}$. Therefore, the isotropy $\phi_{F(\psi)}$ is L-standard, a contradiction. $\square$

Remark 8.3. There are many examples of ϕ and ψ with $\psi \in GP_2(F)$ such that the isotropy of $\phi_{F(\psi)}$ is not L-standard. The condition of Theorem 8.1 can be regarded as a modification of the notion of the L-standard isotropy for the case $\psi \in GP_2(F)$.

We do not know the answer to the following

Question 8.4. *Let ϕ be a 8-dimensional quadratic form with trivial determinant and ψ be a 3-dimensional quadratic form such that $\phi_{F(\psi)}$ is isotropic. Is it always true that the isotropy is L-standard?*

The answer is evidently positive in the case where $\text{ind } C(\phi) = 1$. It is not difficult to show that the answer is also positive in the case where $\text{ind } C(\phi) = 2$.

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