

CONGRUENCE SUBGROUPS AND TWISTED COHOMOLOGY OF $SL_n(F[t])$

KEVIN P. KNUDSON

In [7] we showed that if F is an infinite field, then the natural inclusion $SL_n(F) \rightarrow SL_n(F[t])$ induces an isomorphism

$$H_\bullet(SL_n(F), \mathbb{Z}) \longrightarrow H_\bullet(SL_n(F[t]), \mathbb{Z})$$

for all $n \geq 2$. Here, we study the extent to which this isomorphism holds when the trivial \mathbb{Z} coefficients are replaced by some rational representation of $SL_n(F)$. The group $SL_n(F[t])$ acts on such a representation via the map $SL_n(F[t]) \xrightarrow{t=0} SL_n(F)$.

There are two approaches one might take. The first is to use the spectral sequence associated to the action of $SL_n(F[t])$ on a certain Bruhat–Tits building \mathcal{X} . With this method, we obtain the following result. Suppose that F is a field of characteristic zero. Let V be an irreducible representation of $SL_n(F)$.

Theorem. (cf. Theorem 5.2) *The group $H^1(SL_n(F[t]), V)$ satisfies*

$$H^1(SL_n(F[t]), V) = \begin{cases} H^1(SL_n(F), V) & V \neq \text{Ad} \\ H^1(SL_n(F), V) \oplus F^\infty & V = \text{Ad}, n = 2 \\ H^1(SL_n(F), V) \oplus F & V = \text{Ad}, n \geq 3. \end{cases}$$

One expects that a similar result holds in positive characteristic as well. However, the author does not pretend to be an expert in representation theory, especially in positive characteristic; therefore, we restrict our attention to the characteristic zero case.

The homology of linear groups with twisted coefficients has been studied by Dwyer [2], van der Kallen [5], and others. Most known results concern what happens to these homology groups as n becomes large. In contrast, our results cover the unstable case.

The second approach to studying this question is to use the Hochschild–Serre spectral sequence associated to the extension

$$1 \longrightarrow K \longrightarrow SL_n(F[t]) \xrightarrow{t=0} SL_n(F) \longrightarrow 1$$

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where K denotes the subgroup of matrices which are congruent to the identity modulo t . A spectral sequence calculation shows that for any field F ,

$$H^1(SL_n(F[t]), V) = H^1(SL_n(F), V) \oplus \text{Hom}_{SL_n(F)}(H_1(K), V)$$

so that one need only compute the group $H_1(K)$ explicitly. This seems to be rather difficult in general. When $n = 2$, we have a free product decomposition

$$K = *_{s \in SL_2(F)/B} sCs^{-1}$$

where C is the upper triangular subgroup of K and B is the upper triangular subgroup of $SL_2(F)$. Hence we have

$$H_1(K) = \bigoplus_{s \in SL_2(F)/B} H_1(sCs^{-1})$$

and since C is an abelian group, this is easily calculated.

For $n \geq 3$, however, we have no such free product decomposition. In fact, the natural map $H_1(C) \rightarrow H_1(K)$ is no longer injective (consider the matrix $I + E_{12}(t^2) \in C$; this gives a nontrivial element of $H_1(C)$ which vanishes in $H_1(K)$). The group $H_1(K)$ appears to be rather complicated in general. For any field F , $H_1(K)$ surjects onto the adjoint representation $\mathfrak{sl}_n(F)$, and the kernel is nontrivial in general (see Section 7 below). We make the following conjecture.

Conjecture. *If $n \geq 3$, then $H_1(K) = \mathfrak{sl}_n(F)$ for any finite field F .*

One might prove this by finding a fundamental domain for the action of K on the Bruhat–Tits building \mathcal{X} and then utilizing the corresponding spectral sequence. We take the first steps toward this here. Unfortunately, the combinatorics involved are rather complicated. Still, we are able to prove the conjecture for $n = 3$, $F = \mathbb{F}_2, \mathbb{F}_3$.

The study of congruence subgroups has a long history, one which we will not try to reproduce here. The standard question one asks is whether a given normal subgroup of $SL_n(R)$ is a congruence subgroup. In particular, one could ask this question for the subgroups of K which form its lower central series. While these groups are close to being congruence subgroups in a certain sense (see Section 7), they actually are not.

This paper consists of two parts. The first part deals with the characteristic zero case and is organized as follows. In Section 1, we prove the above theorem in the case $n = 2$ by considering the long exact sequence associated to Nagao’s amalgamated free product decomposition of $SL_2(F[t])$. In Section 2, we describe the spectral sequence used in the proof of Theorem 5.2. In Section 3, we reprove the theorem in the case $n = 2$ using the spectral sequence of Section 2. This helps guide the way to the general result. In Section 4, we describe some equivariant homomorphisms. In Section 5, we prove Theorem 5.2.

The second part deals with the congruence subgroup K and its abelianization $H_1(K)$. In Section 6, we consider the maximal algebraic quotient of $H_1(K)$ in the characteristic zero case. In Section 7 we describe a descending

central series in K . In Section 8 we consider the action of K on the building \mathcal{X} and the resulting spectral sequence. In Section 9 we find a simpler subcomplex of \mathcal{X} which suffices to compute $H_1(K)$. Finally, in Section 10 we prove the above conjecture in certain cases.

This paper has its origins in discussions with Dick Hain, who wondered about the structure of $H_1(K)$ and also whether unstable homotopy invariance holds with nontrivial coefficients. Also, I should point out that this work owes a great deal to C. Soulé's paper [12]. I thank Eric Friedlander for many useful discussions. I am also grateful to the Mathematisches Forschungsinstitut Oberwolfach and l'Institut de Recherche Mathématique Avancée, Strasbourg (in particular to Jean-Louis Loday), for their hospitality during a visit in June 1996 when a preliminary version of this paper was written. Finally, I thank the referee who saved me from some embarrassing slips and pointed me to Krusemeyer's paper [8].

Notation. In Sections 1 through 6, F is assumed to be of characteristic zero; thereafter, F is allowed to be any field unless otherwise specified. If G is a group acting on a set X , we denote the invariants of the G -action by X^G . The symbol P typically denotes a parabolic subgroup of $SL_n(F)$ which contains the upper triangular subgroup B . We denote by Φ^+ the set of positive roots of $SL_n(F)$ determined by B ; this set consists of roots $\{\alpha_{ij} : i < j\}$. We denote by $E_{ij}(a)$ the matrix having i, j entry a and zeros elsewhere. For a polynomial $p(t)$, $p^{(k)}$ denotes the coefficient of t^k . If R is a ring, we denote by R^\times the group of units of R .

1. THE SL_2 CASE, PART I

We single out the case $n = 2$ because we may use Nagao's Theorem [10]

$$SL_2(F[t]) = SL_2(F) *_B B_t \quad (1)$$

to study $H^1(SL_2(F[t]), V)$ (here, B_t denotes the group of upper triangular matrices over $F[t]$). This amalgamated free product decomposition yields a long exact sequence

$$\cdots \rightarrow H^k(SL_2(F[t]), V) \rightarrow H^k(SL_2(F), V) \oplus H^k(B_t, V) \rightarrow H^k(B, V) \rightarrow \cdots$$

for computing the cohomology of $SL_2(F[t])$.

Proposition 1.1. *We have a short exact sequence*

$$0 \rightarrow H^1(SL_2(F[t]), V) \rightarrow H^1(SL_2(F), V) \oplus H^1(B_t, V) \rightarrow H^1(B, V) \rightarrow 0. \quad (2)$$

Proof. Since the map $B_t \xrightarrow{t=0} B$ is split by the natural inclusion $B \rightarrow B_t$, the induced map $H^k(B_t, V) \rightarrow H^k(B, V)$ is surjective. \square

We now study the relationship between $H^1(B_t, V)$ and $H^1(B, V)$. Consider the extension

$$1 \rightarrow C \rightarrow B_t \xrightarrow{t=0} B \rightarrow 1 \quad (3)$$

where

$$C = \left\{ \begin{pmatrix} 1 & tp(t) \\ 0 & 1 \end{pmatrix} : p(t) \in F[t] \right\}.$$

The Hochschild–Serre spectral sequence associated to this satisfies

$$E_2^{p,q} = H^p(B, H^q(C, V)) \implies H^{p+q}(B_t, V). \quad (4)$$

Since the extension (3) is split, the map $d_2^{0,1} : E_2^{0,1} \rightarrow E_2^{2,0}$ vanishes. It follows that we have a short exact sequence

$$0 \rightarrow H^1(B, H^0(C, V)) \rightarrow H^1(B_t, V) \rightarrow H^0(B, H^1(C, V)) \rightarrow 0. \quad (5)$$

Since C acts trivially on V , $H^0(C, V) = V$. Hence, the first term in (5) is simply $H^1(B, V)$. Observe that $H^0(B, H^1(C, V)) = \text{Hom}_B(H_1(C), V)$.

Proposition 1.2. *The group $\text{Hom}_B(H_1(C), V)$ satisfies*

$$\text{Hom}_B(H_1(C), V) = \begin{cases} \text{Hom}_F(tF[t], F) & V = \text{Ad} \\ 0 & V \neq \text{Ad}. \end{cases}$$

Proof. We may compute this group as follows. We have a split extension

$$1 \longrightarrow U \longrightarrow B \longrightarrow T \longrightarrow 1$$

where U is the subgroup of upper triangular unipotent matrices and T is the diagonal subgroup. The Hochschild–Serre spectral sequence implies that

$$H^0(B, H^1(C, V)) = H^0(T, H^0(U, H^1(C, V)));$$

that is,

$$\text{Hom}_B(H_1(C), V) = H^0(T, \text{Hom}_U(H_1(C), V)).$$

Observe that $H_1(C) = C \cong tF[t]$ and U acts trivially on C . It follows that

$$\text{Hom}_U(H_1(C), V) = \text{Hom}(C, V^U).$$

Now, $V = \text{Sym}^n S$, where S is the standard representation of $SL_2(F)$ and n is some nonnegative integer (see *e.g.* [3, Ch. 11]). It follows that

$$V^U = V_n$$

where V_n denotes the highest weight space of V . Observe that T acts on C with weight 2 and on V_n with weight n . Note also that

$$H^0(T, \text{Hom}(C, V_n)) = \text{Hom}_T(C, V_n).$$

We need the following result.

Lemma 1.3. *Suppose $f : C \rightarrow V^U$ is a T -equivariant group homomorphism. Then f is F -linear.*

Proof. We have $V = \text{Sym}^n S$ and $V^U = V_n$. If $a \in \mathbb{Z}$, denote by t_a the element of T with diagonal entries $a, \frac{1}{a}$. If $w \in C$, we have $t_a \cdot w = a^2 w$. Thus

$$a^n f(w) = t_a \cdot f(w) = f(t_a \cdot w) = f(a^2 w) = a^2 f(w)$$

(the last equality follows since f is \mathbb{Z} -linear). It follows that $f = 0$ unless $n = 2$.

Consider the case $n = 2$ (i.e., $V = \text{Ad}$). Let $\alpha \in F$ and consider the element $t_\alpha = \text{diag}(\alpha, \frac{1}{\alpha})$. Since f is T -equivariant, we have

$$\alpha^2 f(w) = t_\alpha \cdot f(w) = f(t_\alpha \cdot w) = f(\alpha^2 w)$$

for any $w \in C$. Note that since F has characteristic zero and f is \mathbb{Z} -linear, f is in fact \mathbb{Q} -linear. Note also that for any $\alpha \in F$, we have $\alpha = -\frac{1}{2} - \frac{1}{2}\alpha^2 + \frac{1}{2}(1 + \alpha)^2$. Thus,

$$\begin{aligned} f(\alpha w) &= f\left(\left(-\frac{1}{2} - \frac{1}{2}\alpha^2 + \frac{1}{2}(1 + \alpha)^2\right)w\right) \\ &= -\frac{1}{2}f(w) - \frac{1}{2}\alpha^2 f(w) + \frac{1}{2}(1 + \alpha)^2 f(w) \\ &= \alpha f(w). \end{aligned}$$

□

Returning to the proof of Proposition 1.2, the proof of Lemma 1.3 shows that $\text{Hom}_T(C, V_n) = 0$ unless $n = 2$. It follows that

$$\text{Hom}_B(C, V) = \begin{cases} 0 & V \neq \text{Ad} \\ \text{Hom}_F(tF[t], F) & V = \text{Ad}. \end{cases}$$

□

Corollary 1.4. *The group $H^1(B_t, V)$ satisfies*

$$H^1(B_t, V) = \begin{cases} H^1(B, V) & V \neq \text{Ad} \\ H^1(B, V) \oplus \text{Hom}_F(tF[t], F) & V = \text{Ad}. \end{cases}$$

Proof. This follows from the short exact sequence (5) and the fact that $H^1(B, V)$ is a direct summand of $H^1(B_t, V)$. □

Theorem 1.5. *The group $H^1(SL_2(F[t]), V)$ satisfies*

$$H^1(SL_2(F[t]), V) = \begin{cases} H^1(SL_2(F), V) & V \neq \text{Ad} \\ H^1(SL_2(F), V) \oplus \text{Hom}_F(tF[t], F) & V = \text{Ad}. \end{cases}$$

Proof. Consider the short exact sequence (2). By the corollary, the kernel of the map

$$H^1(SL_2(F), V) \oplus H^1(B_t, V) \longrightarrow H^1(B, V)$$

is isomorphic to

$$\begin{cases} H^1(SL_2(F), V) & V \neq \text{Ad} \\ H^1(SL_2(F), V) \oplus \text{Hom}_F(tF[t], F) & V = \text{Ad}. \end{cases}$$

The direct sum decomposition follows because $H^1(SL_2(F), V)$ is a direct summand of $H^1(SL_2(F[t]), V)$. □

We have seen that $H^1(SL_2(F[t]), \text{Ad})$ differs from $H^1(SL_2(F), \text{Ad})$ by the infinite dimensional F -vector space $\text{Hom}_B(H_1(C), \text{Ad})$. We describe an explicit basis for this space. For each $k \geq 1$, denote by φ_k the map $C \rightarrow \mathfrak{sl}_2(F)$ defined by

$$I + E_{12}(a_{12}^{(1)}t + a_{12}^{(2)}t^2 + \cdots + a_{12}^{(m)}t^m) \mapsto E_{12}(a_{12}^{(k)}).$$

Then the set $\{\varphi_k\}$ is a basis of $\text{Hom}_B(H_1(C), \text{Ad})$.

2. A SPECTRAL SEQUENCE

In general, we shall use the action of $SL_n(F[t])$ on a suitable simplicial complex to compute $H^1(SL_n(F[t]), V)$.

Denote by \mathcal{X} the Bruhat–Tits building associated to the vector space $F(t)^n$. Recall that the vertices of \mathcal{X} are equivalence classes of \mathcal{O} -lattices in $F(t)^n$ (here, \mathcal{O} consists of the set of a/b with $\deg b \geq \deg a$), where two lattices L and L' are equivalent if there is an $x \in F^\times$ with $L' = xL$. A collection of vertices $\Lambda_0, \Lambda_1, \dots, \Lambda_m$ forms an m -simplex if there are representatives L_i of the Λ_i with

$$t^{-1}L_0 \subset L_m \subset L_{m-1} \subset \cdots \subset L_0.$$

It is possible to put a metric on \mathcal{X} so that each edge in \mathcal{X} has length one. When we speak of the distance between vertices we implicitly use this metric. For a more complete description of \mathcal{X} , see, for example, [7].

The group $SL_n(F[t])$ acts on \mathcal{X} with fundamental domain an infinite wedge \mathcal{T} , which is the subcomplex of \mathcal{X} spanned by the vertices

$$[e_1t^{r_1}, e_2t^{r_2}, \dots, e_{n-1}t^{r_{n-1}}, e_n], \quad r_1 \geq r_2 \geq \cdots r_{n-1} \geq 0$$

where e_1, e_2, \dots, e_n denotes the standard basis of $F(t)^n$ (this is due to Serre [11] for $n = 2$ and Soulé [12] for $n \geq 3$). See Figure 1 for the case $n = 3$.

Denote by v_0 the vertex $[e_1, \dots, e_n]$ and by v_i the vertex

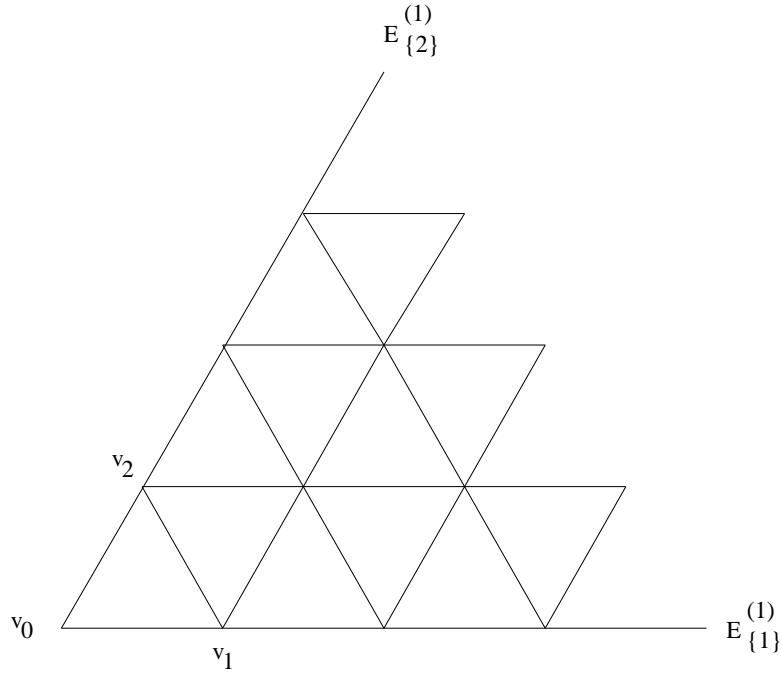
$$[e_1t, e_2t, \dots, e_it, e_{i+1}, \dots, e_n], \quad i = 1, 2, \dots, n-1.$$

For a k element subset $I = \{i_1, \dots, i_k\}$ of $\{1, 2, \dots, n-1\}$, define $E_I^{(k)}$ to be the subcomplex of \mathcal{T} which is the union of all rays with origin v_0 passing through the $(k-1)$ -simplex $\langle v_{i_1}, \dots, v_{i_k} \rangle$. There are $\binom{n-1}{k}$ such $E_I^{(k)}$. Observe that if $I = \{1, 2, \dots, n-1\}$, then $E_I^{(n-1)} = \mathcal{T}$. When we write $E_J^{(l)}$, the superscript l denotes the cardinality of the set J .

The structure of the various simplex stabilizers was described in [7]. If $(x_{ij}(t)) \in SL_n(F[t])$ stabilizes the vertex $[e_1t^{r_1}, \dots, e_{n-1}t^{r_{n-1}}, e_n]$, then we have

$$\deg x_{ij}(t) \leq r_i - r_j$$

(set $r_n = 0$). Note that since $r_1 \geq r_2 \geq \cdots \geq r_{n-1}$, some of the $x_{ij}(t)$ with $i > j$ may be 0. Denote the stabilizer of σ by Γ_σ . The group Γ_σ is the intersection of the stabilizers Γ_v where v ranges over the vertices of σ . In this case $\deg x_{ij}(t) \leq \min_{v \in \sigma} \{r_i^{(v)} - r_j^{(v)}\}$. Observe that in any case, the group Γ_σ has a block form where the blocks along the diagonal are matrices with


 FIGURE 1. The Fundamental Domain \mathcal{T} for $n = 3$

entries in F , blocks below are zero, and blocks above contain polynomials of bounded degree. In the case $n = 3$, we have the following block forms:

$$v_0 : \Gamma_{v_0} = SL_3(F)$$

$$\sigma \in E_{\{1\}}^{(1)} : \Gamma_\sigma = \left(\begin{array}{c|cc} * & * & * \\ 0 & * & * \\ 0 & * & * \end{array} \right)$$

$$\sigma \in E_{\{2\}}^{(1)} : \Gamma_\sigma = \left(\begin{array}{cc|c} * & * & * \\ * & * & * \\ 0 & 0 & * \end{array} \right)$$

$$\sigma \in \mathcal{T} - (E_{\{1\}}^{(1)} \cup E_{\{2\}}^{(1)}) : \Gamma_\sigma = \left(\begin{array}{ccc} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{array} \right)$$

We have a short exact sequence

$$1 \longrightarrow C_\sigma \longrightarrow \Gamma_\sigma \xrightarrow{t=0} P_\sigma \longrightarrow 1$$

where P_σ is a parabolic subgroup of $SL_n(F)$. From the above description of Γ_σ , we see that the group C_σ has a block form where blocks along the diagonal are identity matrices, blocks below are zero, and blocks above contain polynomials of bounded degree which are divisible by t .

Filter the complex \mathcal{T} by setting $W^{(0)} = v_0$ and

$$W^{(l)} = \bigcup_{|I|=l} E_I^{(l)}, \quad 1 \leq l \leq n-1$$

Observe that if σ and τ are simplices in the same component of $W^{(i)} - W^{(i-1)}$, then $P_\sigma = P_\tau$ since on such a component the relationships among the r_i defining the vertices do not vary from vertex to vertex (*i.e.*, if $r_i > r_{i+1}$ for one vertex in the component, then the same holds for every vertex in the component; since these relations determine which entries below the diagonal are zero, we see that the stabilizers of these vertices all have the same block form and hence so does the stabilizer of any simplex in the component).

The action of $SL_n(F[t])$ on \mathcal{X} gives rise to a spectral sequence converging to $H^\bullet(SL_n(F[t]), V)$ with E_1 -term

$$E_1^{p,q} = \prod_{\dim \sigma = p} H^q(\Gamma_\sigma, V) \quad (6)$$

where Γ_σ denotes the stabilizer of the simplex $\sigma \in \mathcal{T}$. We shall compute the terms $E_2^{p,0}$ and $E_2^{0,1}$ and thereby obtain the group $H^1(SL_n(F[t]), V)$.

Proposition 2.1. *The bottom row of the spectral sequence (6) satisfies*

$$E_2^{p,0} = 0, \quad p > 0.$$

Proof. We have a coefficient system \mathcal{H}^0 on \mathcal{T} given by

$$\mathcal{H}^0(\sigma) = H^0(\Gamma_\sigma, V).$$

Thus, the bottom row of (6) is simply the cochain complex $C^\bullet(\mathcal{T}, \mathcal{H}^0)$. The Hochschild–Serre spectral sequence associated to the extension $\Gamma_\sigma \rightarrow P_\sigma$ implies that

$$H^0(\Gamma_\sigma, V) = H^0(P_\sigma, V).$$

It follows that on each component of $W^{(i)} - W^{(i-1)}$, the system \mathcal{H}^0 is constant (see the remarks above). An easy modification of Lemma 5 of [13] shows that the inclusion $v_0 \rightarrow \mathcal{T}$ induces an isomorphism

$$H^\bullet(\mathcal{T}, \mathcal{H}^0) \xrightarrow{\cong} H^\bullet(v_0, \mathcal{H}^0)$$

(see [7] for the corresponding statement for homology). It follows that

$$E_2^{p,0} = \begin{cases} H^0(SL_n(F), V) & p = 0 \\ 0 & p > 0. \end{cases}$$

□

It remains to compute the term

$$E_2^{0,1} = \ker\{d_1 : E_1^{0,1} \rightarrow E_1^{1,1}\}$$

in the spectral sequence (6). That is, we must compute the group $H^0(\mathcal{T}, \mathcal{H}^1)$, where \mathcal{H}^1 is the system which assigns the group $H^1(\Gamma_\sigma, V)$ to σ .

3. THE SL_2 CASE, PART II

Before computing the group $H^0(\mathcal{T}, \mathcal{H}^1)$ in general, we first reconsider the case $n = 2$. This will help guide the way to the desired result. To this end, consider the fundamental domain \mathcal{T} . In this case, \mathcal{T} is an infinite path in the tree \mathcal{X} . Label the vertices v_0, v_1, \dots and the edges e_0, e_1, \dots . Then we have

$$\begin{aligned}\Gamma_{v_0} &= SL_2(F) \\ \Gamma_{v_i} &= \Gamma_{e_i} = \left\{ \begin{pmatrix} a & p(t) \\ 0 & 1/a \end{pmatrix} : \deg p(t) \leq i \right\}, \quad i \geq 1.\end{aligned}$$

For each $i \geq 1$ we have a split extension

$$1 \longrightarrow C_i \longrightarrow \Gamma_{v_i} \xrightarrow{t=0} B \longrightarrow 1.$$

Arguing as in the proof of Proposition 1.2 and again using Lemma 1.3 we see that

$$H^1(\Gamma_{v_i}, V) = \begin{cases} H^1(B, V) & V \neq \text{Ad} \\ H^1(B, V) \oplus \text{Hom}_B(C_i, V) & V = \text{Ad}. \end{cases}$$

Theorem 3.1. *The group $H^0(\mathcal{T}, \mathcal{H}^1)$ satisfies*

$$H^0(\mathcal{T}, \mathcal{H}^1) = \begin{cases} H^1(SL_2(F), V) & V \neq \text{Ad} \\ H^1(SL_2(F), V) \oplus \text{Hom}_B(C, V) & V = \text{Ad}. \end{cases}$$

Proof. If V is not the adjoint representation, we use the long exact sequence of the pair (\mathcal{T}, v_0) . Since $H^1(\Gamma_{v_i}, V) = H^1(B, V)$ for each $i \geq 1$, we see that $H^\bullet(\mathcal{T}, v_0; \mathcal{H}^1) = 0$. It follows that

$$H^0(\mathcal{T}, \mathcal{H}^1) = H^1(SL_2(F), V).$$

In the case $V = \text{Ad}$, we have

$$C^\bullet(\mathcal{T}, \mathcal{H}^1) = C^\bullet(\mathcal{T}, \mathcal{H}^1(P, V)) \oplus C^\bullet(\mathcal{T}, \mathcal{H}om(C, V))$$

where $\mathcal{H}^1(P, V)$ is the system

$$\sigma \mapsto H^1(P_\sigma, V)$$

and $\mathcal{H}om(C, V)$ is the system

$$\sigma \mapsto \text{Hom}_{P_\sigma}(C_\sigma, V).$$

Again, we see that $H^0(\mathcal{T}, \mathcal{H}^1(P, V)) = H^1(SL_2(F), V)$. It remains to compute $H^0(\mathcal{T}, \mathcal{H}om(C, V))$. For each i , we have

$$\begin{aligned}\text{Hom}_B(C_i, V) &= \text{Hom}_T(\{a_{12}^{(1)}t + \dots + a_{12}^{(i)}t^i\}, V) \\ &= F\{\varphi_j\}_{j=1}^i\end{aligned}$$

where $\varphi_j(a_{12}^{(1)}t + \dots + a_{12}^{(i)}t^i) = E_{12}(a_{12}^{(j)})$ (this follows from Lemma 1.3). Note that $C_{v_0} = C_{e_0} = 1$; *i.e.* $\text{Hom}(C_{v_0}, V) = 0$. It follows that

$$H^0(\mathcal{T}, \mathcal{H}om(C, V)) = H^0(\mathcal{T}', \mathcal{H}om(C, V)),$$

where \mathcal{T}' is the complex obtained from deleting v_0 and e_0 (but not v_1) from \mathcal{T} .

Consider the map $d : C^0(\mathcal{T}', \mathcal{H}om(C, V)) \rightarrow C^1(\mathcal{T}', \mathcal{H}om(C, V))$. This map is given by

$$(x_1, x_2, x_3, \dots) \mapsto (x_2|_{e_1} - x_1, x_3|_{e_2} - x_2, \dots)$$

where $x_k|_{e_{k-1}}$ denotes the image of x_k under the restriction map

$$\mathrm{Hom}_B(C_{v_k}, V) \rightarrow \mathrm{Hom}_B(C_{e_{k-1}}, V).$$

Note that d is F -linear since it is an alternating sum of restriction maps, each of which is clearly F -linear. The kernel of this map is spanned by the following elements:

$$\begin{aligned} &(\varphi_1, \varphi_1, \varphi_1, \dots), (0, \varphi_2, \varphi_2, \varphi_2, \dots), \\ &(0, 0, \varphi_3, \varphi_3, \varphi_3, \dots), \dots \end{aligned}$$

If we identify the k th element of this list with the map $\varphi_k : C \rightarrow V$, we see that

$$H^0(\mathcal{T}', \mathcal{H}om(C, V)) = \mathrm{Hom}_B(C, V).$$

It follows that

$$H^1(SL_2(F[t]), \mathrm{Ad}) = H^1(SL_2(F), \mathrm{Ad}) \oplus \mathrm{Hom}_B(C, \mathrm{Ad})$$

which is the desired result. \square

4. P_σ -EQUIVARIANT HOMOMORPHISMS

To compute the group $H^0(\mathcal{T}, \mathcal{H}^1)$ in general, we will need to consider the groups $\mathrm{Hom}_{P_\sigma}(H_1(C_\sigma), V)$, where the groups $H_1(C_\sigma)$ are more complicated than in the case $n = 2$. We begin by describing the structure of the $H_1(C_\sigma)$.

Recall that the elements $(x_{ij}(t))$ of the subgroup C_σ have a block form where the blocks on the diagonal are identity matrices, blocks below are zero, and blocks above contain polynomials of bounded degree which are divisible by t .

Using elementary row operations, we may write for any $(x_{ij}(t)) \in C_\sigma$,

$$(x_{ij}(t)) = \prod_{j=0}^{n-2} \prod_{i=1}^{n-j-1} (I + E_{i, n-j}(x_{i, n-j}(t))). \quad (7)$$

We also have

$$I + E_{ij}(x_{ij}(t)) = \prod_{l=1}^m (I + E_{ij}(x_{ij}^{(l)} t^l)). \quad (8)$$

Lemma 4.1. *If there exist distinct integers i, j, k and positive integers l, m with $l + m = r$ such that $I + E_{ik}(t^l), I + E_{kj}(t^m) \in C_\sigma$, then the element $I + E_{ij}(t^r)$ is zero in $H_1(C_\sigma)$.*

Proof. This follows from standard relations among commutators of elementary matrices. \square

We are now in a position to describe explicitly the structure of the groups $H_1(C_\sigma)$.

Proposition 4.2. *The group $H_1(C_\sigma)$ admits a decomposition*

$$H_1(C_\sigma) = \bigoplus_{\alpha \in \Phi^+} S_\alpha$$

where S_α is a weight space (possibly 0) for the positive root α . Moreover, each $S_{\alpha_{ij}}$ is graded by powers of t :

$$S_{\alpha_{ij}} = t^{l_1(\alpha_{ij})}W_{\alpha_{ij}} \oplus t^{l_2(\alpha_{ij})}W_{\alpha_{ij}} \oplus \dots \oplus t^{l_p(\alpha_{ij})}W_{\alpha_{ij}}$$

where $0 \leq l_1(\alpha_{ij}) < l_2(\alpha_{ij}) < \dots < l_p(\alpha_{ij}) \leq \min_{v \in \sigma} \{r_i^{(v)} - r_j^{(v)}\}$ and $W_{\alpha_{ij}}$ is a one-dimensional root space for α_{ij} . If $S_{\alpha_{ij}} \neq 0$, then $l_1(\alpha_{ij}) = 1$.

Proof. Observe that $H_1(C_\sigma)$ is a P_σ -module and as such admits a decomposition into weight spaces for the action of the diagonal subgroup T . In light of (7) and (8), we see that each element of $H_1(C_\sigma)$ may be written as a product

$$\prod_{i < j} I + E_{ij}(x_{ij}(t))$$

which implies the existence of the decomposition

$$H_1(C_\sigma) = \bigoplus_{\alpha \in \Phi^+} S_\alpha$$

(the only weights which can occur are positive roots since C_σ is upper triangular and T acts on $I + E_{ij}(x)$ with weight α_{ij}). The space $S_{\alpha_{ij}}$ is spanned by all elements of the form

$$I + E_{ij}(t^r)$$

where $1 \leq r \leq \min_{v \in \sigma} \{r_i^{(v)} - r_j^{(v)}\}$. If $\min_{v \in \sigma} \{r_i^{(v)} - r_j^{(v)}\} = 0$, then $S_{\alpha_{ij}} = 0$. By Lemma 4.1, if there is a k distinct from i and j with $I + E_{ik}(t^l), I + E_{kj}(t^m) \in C_\sigma$ ($l + m = r$), then $I + E_{ij}(t^r) = 0$ in $H_1(C_\sigma)$. Notice that the conditions of Lemma 4.1 cannot be met for $r = 1$ (since $l, m \geq 1$) so that if $S_{\alpha_{ij}} \neq 0$ (which can only occur if $\min_{v \in \sigma} \{r_i^{(v)} - r_j^{(v)}\} \geq 1$), then $I + E_{ij}(t) \in S_{\alpha_{ij}}$. It follows that

$$S_{\alpha_{ij}} = tW_{\alpha_{ij}} \oplus t^{l_2(\alpha_{ij})}W_{\alpha_{ij}} \oplus \dots \oplus t^{l_p(\alpha_{ij})}W_{\alpha_{ij}}$$

where $1 < l_2 < \dots < l_p \leq \min_{v \in \sigma} \{r_i^{(v)} - r_j^{(v)}\}$. \square

Note that it is possible for powers of t to get skipped. For example, let $n = 3$ and consider the following group:

$$C_\sigma = \begin{pmatrix} 1 & \deg \leq 1 & \deg \leq 3 \\ 0 & 1 & \deg \leq 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

This is the C_σ of some edge in \mathcal{T} . In $H_1(C_\sigma)$, the element

$$\begin{pmatrix} 1 & 0 & t^3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is nonzero, but the element

$$\begin{pmatrix} 1 & 0 & t^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is trivial. (In this case, $H_1(C_\sigma) = tW_{\alpha_{12}} \oplus tW_{\alpha_{23}} \oplus tW_{\alpha_{13}} \oplus t^3W_{\alpha_{13}}$.) This problem arises because for each i, j we are only allowed polynomials of degree at most $\min_{v \in \sigma} \{r_i^{(v)} - r_j^{(v)}\}$. This may allow certain elements to survive in H_1 .

However, if v is a vertex, this phenomenon does not occur for the following reason. For each i, j , we have $\deg x_{ij}(t) \leq r_i - r_j$. Consider the elements $I + E_{ij}(t^r)$, $1 \leq r \leq r_i - r_j$, in C_v . Suppose there exists a k such that C_v contains nonzero elements of the form $I + E_{ik}(p(t))$, $I + E_{kj}(q(t))$. Note that since C_v is upper triangular, such a k satisfies $i < k < j$. Then C_v contains all the $I + E_{ik}(t^l)$, $I + E_{kj}(t^m)$, $1 \leq l \leq r_i - r_k$, $1 \leq m \leq r_k - r_j$. Since $r_i \geq r_k \geq r_j$, it follows that it is always possible to satisfy the conditions of Lemma 4.1 for each $1 < r \leq r_i - r_j$; *i.e.*, $S_{\alpha_{ij}} = tW_{\alpha_{ij}}$. Note that such a pair i, j must satisfy $j > i + 1$ since each C_v is upper triangular.

We now determine the structure of the groups $\text{Hom}_{P_\sigma}(H_1(C_\sigma), V)$.

Lemma 4.3. *Suppose $n \geq 3$ and $f : H_1(C_\sigma) \rightarrow V$ is a P_σ -equivariant group homomorphism. Then $f(W_{\alpha_{ij}}) \subseteq V_{\alpha_{ij}}$.*

Proof. Write $V = \bigoplus_\lambda V_\lambda$ and choose a basis $\{v_\lambda\}_\lambda$ of V . Recall that each λ is a linear combination $\lambda = \sum_{p=1}^n m_p(\lambda)L_p$ where L_1, \dots, L_n are the weights of the standard representation. Recall also that $\sum_{i=1}^n L_i = 0$. Fix $i < j$. For $k \neq i$ denote by t_k the element

$$t_k = \text{diag}(1, \dots, 1, b, 1, \dots, 1, 1/b, 1, \dots, 1)$$

where b is an integer greater than 1, b appears in the i th place and $1/b$ appears in the k th place. Let $w \in W_{\alpha_{ij}}$ and write $f(w) = \sum_\lambda a_\lambda v_\lambda$. Note that if $k \neq j$, $t_k \cdot w = bw$ and $t_j \cdot w = b^2w$. Now if $k \neq j$,

$$\begin{aligned} t_k \cdot \sum_\lambda a_\lambda v_\lambda &= t_k \cdot f(w) \\ &= f(t_k \cdot w) \\ &= f(bw) \\ &= bf(w) \quad (f \text{ additive}) \\ &= \sum_\lambda ba_\lambda v_\lambda \end{aligned}$$

and similarly,

$$t_j \cdot \sum_{\lambda} a_{\lambda} v_{\lambda} = \sum_{\lambda} b^2 a_{\lambda} v_{\lambda}.$$

Now, if $\lambda = \sum_{p=1}^n m_p(\lambda) L_p$, then

$$t_k \cdot v_{\lambda} = b^{m_i(\lambda) - m_k(\lambda)} v_{\lambda}$$

for all $k \neq i$. Thus,

$$(b - b^{m_i(\lambda) - m_k(\lambda)}) a_{\lambda} = 0, \quad k \neq j$$

and

$$(b^2 - b^{m_i(\lambda) - m_j(\lambda)}) a_{\lambda} = 0.$$

So, if some $m_i(\lambda) - m_k(\lambda) \neq 1$ or $m_i(\lambda) - m_j(\lambda) \neq 2$, we see that $a_{\lambda} = 0$ (since F has characteristic zero).

Suppose that for $k \neq j$, $m_i(\lambda) - m_k(\lambda) = 1$ and $m_i(\lambda) - m_j(\lambda) = 2$. Then

$$\begin{aligned} \lambda &= (m_i(\lambda) - 1)L_1 + \cdots + m_i(\lambda)L_i + \cdots + (m_i(\lambda) - 2)L_j + \cdots \\ &\quad \cdots + (m_i(\lambda) - 1)L_n \\ &= m_i(\lambda) \sum_{p=1}^n L_p - \sum_{p \neq i} L_p - L_j \\ &= m_i(\lambda) \cdot 0 + L_i - \sum_{p=1}^n L_p - L_j \\ &= 0 + L_i - 0 - L_j \\ &= \alpha_{ij}. \end{aligned}$$

Thus, $a_{\lambda} = 0$ except for $\lambda = \alpha_{ij}$; *i.e.*, $f(w) = a_{\alpha_{ij}} v_{\alpha_{ij}}$. \square

Corollary 4.4. *If $n \geq 3$ and $f : H_1(C_{\sigma}) \rightarrow V$ is a P_{σ} -equivariant group homomorphism, then f is F -linear.*

Proof. Let $w \in W_{\alpha_{ij}}$ and let $\alpha \in F$. Let

$$t_{\alpha} = \text{diag}(1, \dots, 1, \alpha, 1, \dots, 1, 1/\alpha, 1, \dots, 1)$$

where α appears in the i th position and $1/\alpha$ appears in the k th position, where $k \neq j$. Then $t_{\alpha} \cdot w = \alpha w$. Hence,

$$t \cdot f(w) = f(t \cdot w) = f(\alpha w).$$

Since $f(W_{\alpha_{ij}}) \subseteq V_{\alpha_{ij}}$, $t \cdot f(w) = \alpha f(w)$. Thus, $f(\alpha w) = \alpha f(w)$ for all $\alpha \in F$. Since each element of $H_1(C_{\sigma})$ is an F -linear combination of elements of the various $W_{\alpha_{ij}}$, we see that f is F -linear. \square

The preceding result allows us to assume that the homomorphisms under consideration are F -linear.

Proposition 4.5. *If V is not the adjoint representation, then*

$$\text{Hom}_{P_{\sigma}}(H_1(C_{\sigma}), V) = 0.$$

Proof. Suppose f is a nonzero element in $\text{Hom}_{P_\sigma}(H_1(C_\sigma), V)$. Since V is irreducible, the image of f contains a highest weight vector whose weight γ must be a weight of $H_1(C_\sigma)$, hence a root. On the other hand, γ must be a highest weight of V , so we must have $V = \text{Ad}$. \square

This argument also shows that if $V = \text{Ad}$, then the restriction map $\text{Hom}_{P_\sigma}(H_1(C_\sigma), V) \rightarrow \text{Hom}_F(H_1(C_\sigma)_{\alpha_{1n}}, V_{\alpha_{1n}})$ is injective. Observe that each map $H_1(C_\sigma)_{\alpha_{1n}} \rightarrow V_{\alpha_{1n}}$ is a linear combination of maps of the form

$$I + E_{1n}(a_{1n}^{(1)}t + a_{1n}^{(2)}t^2 + \cdots + a_{1n}^{(l)}t^l) \mapsto E_{1n}(a_{1n}^{(k)}).$$

It follows that any f in $\text{Hom}_{P_\sigma}(H_1(C_\sigma), V)$ is a linear combination of maps $\varphi_k : H_1(C_\sigma) \rightarrow V$ defined by

$$\varphi_k : I + tX_1 + \cdots + t^l X_l \mapsto X_k.$$

Let us consider what the proposition says about $\text{Hom}_B(H_1(C), \text{Ad})$. If v is a vertex in $\mathcal{T} - W^{(n-2)}$ (i.e., v is in the interior of \mathcal{T}), then

$$H_1(C_v) = \bigoplus_i W_{\alpha_{i,i+1}}^{n_{\alpha_{i,i+1}}} \oplus \bigoplus_{j>i+1} tW_{\alpha_{ij}}$$

where $n_{\alpha_{i,i+1}} \geq 1$. In this case, $P_v = B$ and $\text{Hom}_{P_v}(H_1(C_v), \text{Ad})$ injects into $\text{Hom}_F(H_1(C_v)_{\alpha_{1n}}, \text{Ad}_{\alpha_{1n}}) = F$ (we assume that $n \geq 3$). Hence, up to scalars any nonzero map $f : H_1(C_v) \rightarrow \text{Ad}$ is of the form

$$f(I + tX_1 + t^2X_2 + \cdots + t^kX_k) = X_1.$$

Note that when $n = 2$ the situation is much different since $W_{\alpha_{12}}^{n_{\alpha_{12}}}$ is such that $n_{\alpha_{12}}$ can be greater than 1.

Thus we arrive at the following description of $\text{Hom}_B(H_1(C), \text{Ad})$.

Corollary 4.6. *The group $\text{Hom}_B(H_1(C), \text{Ad})$ satisfies*

$$\text{Hom}_B(H_1(C), \text{Ad}) = \begin{cases} F\{\varphi_k\}_{k=1}^\infty & n = 2 \\ F\{\varphi_1\} & n \geq 3 \end{cases}$$

where φ_k is the map $H_1(C) \rightarrow \text{Ad}$ defined by

$$\varphi_k(I + tX_1 + t^2X_2 + \cdots + t^lX_l) = X_k.$$

5. THE GROUP $H^1(SL_n(F[t]), V)$, $n \geq 3$

Having dispensed with the case $n = 2$, we now turn our attention to the groups $SL_n(F[t])$ for $n \geq 3$.

We begin by computing the groups $H^1(\Gamma_\sigma, V)$, where σ is a simplex in \mathcal{T} . We have a split short exact sequence

$$1 \longrightarrow C_\sigma \longrightarrow \Gamma_\sigma \xrightarrow{t=0} P_\sigma \longrightarrow 1.$$

By considering the Hochschild–Serre spectral sequence, we see that we have

$$H^1(\Gamma_\sigma, V) = H^1(P_\sigma, V) \oplus \text{Hom}_{P_\sigma}(H_1(C_\sigma), V).$$

In light of Proposition 4.5, we have the following result.

Proposition 5.1. *If V is not the adjoint representation, then*

$$H^1(SL_n(F[t]), V) = H^1(SL_n(F), V).$$

Proof. One checks easily in this case that $H^0(\mathcal{T}, \mathcal{H}^1) = H^1(SL_n(F), V)$ by considering the filtration W^\bullet of \mathcal{T} . \square

Now, if V is the adjoint representation we have already seen that the situation can be much different.

Theorem 5.2. *If V is the adjoint representation, then*

$$H^1(SL_n(F[t]), V) = H^1(SL_n(F), V) \oplus \text{Hom}_B(H_1(C), V).$$

Proof. As before, one checks that

$$H^0(\mathcal{T}, \mathcal{H}^1) = H^1(SL_n(F), V) \oplus H^0(\mathcal{T}, \mathcal{H}om(C, V)).$$

It remains to compute the kernel of the map

$$d : C^0(\mathcal{T}, \mathcal{H}om(C, V)) \longrightarrow C^1(\mathcal{T}, \mathcal{H}om(C, V)).$$

Suppose that v is a vertex in \mathcal{T} . The discussion following the proof of Proposition 4.5 shows that the group $\text{Hom}_{P_v}(H_1(C_v), V)$ has basis the maps φ_k where

$$\varphi_k(I + tX_1 + \cdots + t^l X_l) = X_k$$

and $k \leq \max\{r_i - r_j\}$ (here $v = [e_1 t^{r_1}, e_2 t^{r_2}, \dots, e_{n-1} t^{r_{n-1}}, e_n]$). We will show that only the map φ_1 corresponds to an element in the kernel of d . Note that d is F -linear since it is a linear combination of restriction maps, each of which is F -linear.

Suppose that $k \geq 2$ and consider the map φ_k . This map belongs to infinitely many $\text{Hom}_{P_v}(H_1(C_v), V)$. However, given a vertex v , the map φ_k can only belong to $\text{Hom}_{P_v}(H_1(C_v), V)$ if $r_i = r_{i+1}$ for some i (*i.e.*, v cannot lie in $\mathcal{T} - W^{(n-2)}$). In this case, there exists a vertex v' adjacent to v (where $r_i > r_{i+1}$) with $\varphi_k \notin \text{Hom}_{P_{v'}}(H_1(C_{v'}), V)$ and such that $C_e = C_v \cap C_{v'} = C_v$. The map $d : C^0(\mathcal{T}, \mathcal{H}om(C, V)) \longrightarrow C^1(\mathcal{T}, \mathcal{H}om(C, V))$ is then

$$d : (\varphi_k)_v \mapsto (\varphi_k)_e + \sum_{e'} (\varphi_k)_{e'}$$

where the sum is over the edges e' incident with v . However, since $\varphi_k \notin \text{Hom}_{P_{v'}}(H_1(C_{v'}), V)$, the e component of $d(\varphi_k)$ cannot be killed by a corresponding element of $\text{Hom}_{P_{v'}}(H_1(C_{v'}), V)$, *i.e.*, $d(\varphi_k) \neq 0$.

However, the map φ_1 does belong to each $\text{Hom}_{P_v}(H_1(C_v), V)$ and to each $\text{Hom}_{P_e}(H_1(C_e), V)$. Moreover, given an edge e with vertices v and v' , the restriction maps

$$\text{Hom}_{P_v}(H_1(C_v), V) \longrightarrow \text{Hom}_{P_e}(H_1(C_e), V)$$

and

$$\text{Hom}_{P_{v'}}(H_1(C_{v'}), V) \longrightarrow \text{Hom}_{P_e}(H_1(C_e), V)$$

map φ_1 to the same element (with opposite sign). It follows that the kernel of $d : C^0 \rightarrow C^1$ is spanned by the element $x = (x_v)_{v \in \mathcal{T}}$ defined by

$$x_v = \varphi_1 \in \text{Hom}_{P_v}(H_1(C_v), V).$$

If we identify this map with the map $\varphi_1 \in \text{Hom}_B(H_1(C), V)$, then we see that

$$H^0(\mathcal{T}, \mathcal{H}om(C, V)) \cong \text{Hom}_B(H_1(C), V).$$

This completes the proof of Theorem 5.2. \square

6. A SECOND SPECTRAL SEQUENCE

There is another spectral sequence which can be used to compute the group $H^1(SL_n(F[t]), V)$; namely, the Hochschild–Serre spectral sequence associated to the split extension

$$1 \longrightarrow K \longrightarrow SL_n(F[t]) \xrightarrow{t=0} SL_n(F) \longrightarrow 1$$

where K denotes the subgroup of $SL_n(F[t])$ consisting of those matrices which are congruent to the identity modulo t . Since K acts trivially on V and the extension splits, we see that the map

$$d_2^{0,1} : E_2^{0,1} \longrightarrow E_2^{2,0}$$

vanishes and hence,

$$\begin{aligned} H^1(SL_n(F[t]), V) &= E_2^{1,0} \oplus E_2^{0,1} \\ &= H^1(SL_n(F), V) \oplus H^0(SL_n(F), H^1(K, V)) \\ &= H^1(SL_n(F), V) \oplus \text{Hom}_{SL_n(F)}(H_1(K), V). \end{aligned}$$

The results of the preceding section now imply the following result.

Corollary 6.1. *If V is not the adjoint representation, then*

$$\text{Hom}_{SL_n(F)}(H_1(K), V) = 0;$$

that is, there are no $SL_n(F)$ -equivariant maps $H_1(K) \rightarrow V$. If V is the adjoint representation, then

$$\text{Hom}_{SL_n(F)}(H_1(K), V) = \text{Hom}_B(H_1(C), V). \quad \square$$

In other words, $SL_n(F)$ -equivariant maps $H_1(K) \rightarrow V$ are in one-to-one correspondence with B -equivariant maps $H_1(C) \rightarrow V$. Moreover, if we define $H_1^{\text{alg}}(K)$ to be the maximal algebraic quotient of $H_1(K)$ (*i.e.* the product of those $SL_n(F)$ representations which admit an equivariant homomorphism $H_1(K) \rightarrow V$) then we have

$$H_1^{\text{alg}}(K) = \text{Ad}.$$

The conclusion of Corollary 6.1 is clear in the case $n = 2$ since we have the free product decomposition (see *e.g.* [6])

$$K = *_{s \in \mathbb{P}^1(F)} s C s^{-1} \quad (\mathbb{P}^1(F) = SL_2(F)/B)$$

which implies that

$$H_1(K) = \text{Ind}_B^{SL_2(F)} H_1(C).$$

By Shapiro's Lemma, we have

$$H^0(B, H^1(C, V)) = H^0(SL_2(F), \text{Coind}_B^{SL_2(F)} H^1(C, V));$$

that is,

$$\begin{aligned} \text{Hom}_B(H_1(C), V) &= \left(\prod_{SL_2(F)/B} H^1(C, V) \right)^{SL_2(F)} \\ &= \text{Hom}_{SL_2(F)} \left(\bigoplus_{SL_2(F)/B} H_1(C), V \right) \\ &= \text{Hom}_{SL_2(F)}(H_1(K), V). \end{aligned}$$

Note that this argument works for any field F .

Proposition 6.2. *If F is any field, then*

$$H^1(SL_2(F[t], V) = H^1(SL_2(F), V) \oplus \text{Hom}_B(H_1(C), V). \quad \square$$

7. CENTRAL SERIES

In this section, we allow F to be any field of any characteristic.

For each $i \geq 1$, denote by K^i the subgroup of K consisting of matrices congruent to the identity modulo t^i . Each K^i is a normal subgroup of $SL_n(F[t])$ as it is the kernel of the map

$$SL_n(F[t]) \longrightarrow SL_n(F[t]/(t^i)).$$

Lemma 7.1. *For each i, j , $[K^i, K^j] \subseteq K^{i+j}$.*

Proof. If $X = I + t^i X_i + \dots + t^l X_l$ and $Y = I + t^j Y_j + \dots + t^m Y_m$, then

$$\begin{aligned} XYX^{-1}Y^{-1} &= (I + t^i X_i + \dots)(I + t^j Y_j + \dots) \\ &\quad (I - t^i X_i + \dots)(I - t^j Y_j + \dots) \\ &= (I + t^i X_i + t^j Y_j + \dots)(I - t^i X_i - t^j Y_j + \dots) \\ &= I + t^{i+j}(-X_i Y_j - Y_j X_i) + \dots \\ &\in K^{i+j}. \quad \square \end{aligned}$$

For each i define a map $\rho_i : K^i \longrightarrow \mathfrak{sl}_n(F)$ by

$$\rho_i(I + t^i X_i + \dots + t^l X_l) = X_i.$$

One checks easily that ρ_i is a surjective homomorphism (see Lee and Szczarba [9] for the \mathbb{Z} case) with kernel K^{i+1} . Hence for each $i \geq 1$, we have an isomorphism

$$K^i/K^{i+1} \cong \mathfrak{sl}_n(F).$$

For any group G , denote by Γ^\bullet the lower central series of G .

Lemma 7.2. *For each i , we have $\Gamma^i \subseteq K^i$.*

Proof. Since K^i/K^{i+1} is abelian for each i , K^\bullet is a central series in K and as such contains the lower central series. \square

Given a group G and a central series $G = G^1 \supseteq G^2 \supseteq \dots$ we have the associated graded Lie algebra

$$\mathrm{Gr}^\bullet G = \bigoplus_{i \geq 1} G^i/G^{i+1}.$$

Lemma 7.3. *The graded algebra $\mathrm{Gr}^\bullet G$ is generated by $\mathrm{Gr}^1 G$ if and only if $G^l = G^{l+1}\Gamma^l$ for each $l \geq 1$.*

Proof. Note that $\mathrm{Gr}_\Gamma^\bullet G = \bigoplus_{i \geq 1} \Gamma^i/\Gamma^{i+1}$ is generated by $\mathrm{Gr}_\Gamma^1 G = \Gamma^1/\Gamma^2$ and that we have a surjective map

$$\Gamma^1/\Gamma^2 \longrightarrow G^1/G^2.$$

For each l we have an exact sequence

$$0 \rightarrow (\Gamma^l \cap G^{l+1})/\Gamma^{l+1} \rightarrow \Gamma^l/\Gamma^{l+1} \rightarrow G^l/G^{l+1} \rightarrow G^l/G^{l+1}\Gamma^l \rightarrow 0.$$

We also have a commutative diagram

$$\begin{array}{ccc} (\Gamma^1/\Gamma^2)^{\otimes l} & \longrightarrow & (G^1/G^2)^{\otimes l} \\ \downarrow & & \downarrow \varphi \\ \Gamma^l/\Gamma^{l+1} & \xrightarrow{\psi} & G^l/G^{l+1} \end{array}$$

Suppose that $\mathrm{Gr}^\bullet G$ is generated by $\mathrm{Gr}^1 G$. Then φ is surjective. Commutativity of the diagram forces ψ to be surjective (since the top horizontal and left vertical maps are surjective). But then the exact sequence

$$\Gamma^l/\Gamma^{l+1} \xrightarrow{\psi} G^l/G^{l+1} \longrightarrow G^l/G^{l+1}\Gamma^l \longrightarrow 0$$

implies that $G^l/G^{l+1}\Gamma^l = 0$; that is, $G^l = G^{l+1}\Gamma^l$.

Conversely, if $G^l = G^{l+1}\Gamma^l$ for each l , then ψ is surjective. But then φ is also surjective; that is, $\mathrm{Gr}^\bullet G$ is generated by $\mathrm{Gr}^1 G$. \square

Now consider the algebra $\mathrm{Gr}^\bullet K = \bigoplus_{i \geq 1} K^i/K^{i+1}$. For each i , $\mathrm{Gr}^i K = \mathfrak{sl}_n(F)$. Since $\mathfrak{sl}_n(F) = [\mathfrak{sl}_n(F), \mathfrak{sl}_n(F)]$ (unless $n = 2$, $\mathrm{char} F = 2$), $\mathrm{Gr}^\bullet K$ is generated by $\mathrm{Gr}^1 K$. By the lemma, we have $K^l = K^{l+1}\Gamma^l$ for each l . Note that this implies that $K^l = K^{l+m}\Gamma^l$ for each $m \geq 1$. Since $\bigcap_m K^m = \{I\}$, we see that the filtrations K^\bullet and Γ^\bullet are close together in a certain sense. However, we will see that in general $K^l \neq \Gamma^l$.

Remark. The fact that $K^l = K^{l+m}\Gamma^l$ implies that in the completed group $\widehat{K} = \varprojlim K/K^i$, we have $\widehat{K}^l = \widehat{\Gamma}^l$ for each l .

Consider the short exact sequence

$$0 \longrightarrow K^2/\Gamma^2 \longrightarrow H_1(K) \xrightarrow{\rho_{1*}} \mathfrak{sl}_n(F) \longrightarrow 0.$$

The group K^2 is often denoted by $SL_n(F[t], (t^2))$ and we have

$$\Gamma^2 = [SL_n(F[t], (t)), SL_n(F[t], (t))].$$

Lemma 7.4. *If F is a field of characteristic not equal to 2 or 3 or if F is perfect, then $\Gamma^2 \subset E(F[t], (t^2))$.*

Proof. According to Vaserstein [14], the stable commutator subgroup

$$[GL(R, I), GL(R, J)]$$

is generated by elements $[E_{rs}(a), E_{sr}(b)]$ with $(a, b) \in (I \times J) \cup (R \times IJ)$ along with the $E_{rs}(x)$ with $x \in IJ$ (the latter elements are clearly in Γ^2 and in $E(F[t], (t^2))$). The corresponding Mennicke symbol is $\begin{bmatrix} a^2b \\ 1-ab \end{bmatrix}$. We show that these symbols vanish in $SK_1(F[t], (t^2))$.

To do this, we make use of computations in Krusemeyer [8, Lemma 12.3]. The hypotheses on F are needed to show the relations $\begin{bmatrix} t^2 \\ 1-t^2k \end{bmatrix} = 1$ and $\begin{bmatrix} t^3 \\ 1-t^2k \end{bmatrix} = 1$ for $k \in F[t]$. The hardest case is where $(a, b) \in (t) \times (t)$. Write $a = tf, b = tg, g = g_0 + g_1t + t^2h$, where $g_i \in F$ and $f, h \in F[t]$. Then

$$\begin{aligned} \begin{bmatrix} a^2b \\ 1-ab \end{bmatrix} &= \begin{bmatrix} t^3f^2g \\ 1-t^2fg \end{bmatrix} = \begin{bmatrix} t^3f^2g \\ 1-t^2fg \end{bmatrix} \begin{bmatrix} t^2 \\ 1-t^2fg \end{bmatrix} \\ &= \begin{bmatrix} t^5f^2g \\ 1-t^2fg \end{bmatrix} = \begin{bmatrix} t^3f \\ 1-t^2fg \end{bmatrix} \begin{bmatrix} fg \\ 1-t^2fg \end{bmatrix} \\ &= \begin{bmatrix} t^3f \\ 1-t^2fg \end{bmatrix} = \begin{bmatrix} t^5f \\ 1-t^2fg \end{bmatrix} = \begin{bmatrix} t^2f \\ 1-t^2fg \end{bmatrix} \\ &= \begin{bmatrix} t^2f \\ 1-t^3fg_1 \end{bmatrix} = \begin{bmatrix} t^3f \\ 1-t^3fg_1 \end{bmatrix} = 1. \end{aligned}$$

Similar computations show that $\begin{bmatrix} a^2b \\ 1-ab \end{bmatrix} = 1$ for $(a, b) \in F[t] \times (t^2)$. This completes the proof. \square

The lemma implies that we have a surjective map

$$K^2/\Gamma^2 \longrightarrow SK_1(F[t], (t^2)).$$

The latter group was computed by Krusemeyer [8]; it equals the module of differentials Ω_F^1 , if $\text{char } F \neq 2, 3$ or if F is perfect. Observe that this is nonzero in general.

Proposition 7.5. *Suppose $\text{char } F \neq 2, 3$ or F is perfect. If $\Omega_F^1 \neq 0$, then $K^l \neq \Gamma^l$ for all $l \geq 2$.*

Proof. Since K^2/Γ^2 surjects onto Ω_F^1 , we have $K^2 \neq \Gamma^2$. Suppose $K^l = \Gamma^l$ for some l . Then $K^2 = K^l\Gamma^2 = \Gamma^l\Gamma^2 \subseteq \Gamma^2$, a contradiction. \square

If $\Omega_F^1 = 0$, then we get no information about the structure of K^2/Γ^2 . Note that if F is a finite field, then $\Omega_F^1 = 0$.

8. THE ACTION OF K ON \mathcal{X}

We now turn our attention to the computation of the group $H_1(K)$ in the case where F is a finite field. While much of what we say below is true for any field, we restrict our attention to the finite case.

In [7, Prop. 4.1], we found a fundamental domain for the action of K on \mathcal{X} . Denote by S a set of coset representatives for $SL_n(F)/B$, where B is the upper triangular subgroup. Define a subcomplex \mathcal{Z} of \mathcal{X} by

$$\mathcal{Z} = \bigcup_{s \in S} s\mathcal{T}.$$

Then \mathcal{Z} is a fundamental domain for the action of K on \mathcal{X} .

If σ is a simplex in \mathcal{Z} , denote by K_σ the stabilizer of σ in K . Since \mathcal{X} is contractible, we have a spectral sequence converging to $H_\bullet(K)$ with E^1 -term

$$E_{p,q}^1 = \bigoplus_{\dim \sigma = p} H_q(K_\sigma) \quad (9)$$

where σ ranges over the simplices of \mathcal{Z} .

Observe that the bottom row $E_{*,0}^1$ is simply the simplicial chain complex $S_\bullet(\mathcal{Z})$. Note that \mathcal{Z} is contractible since the index set S is finite (F is a finite field) and each $s \in S$ fixes the initial vertex v_0 . Moreover, $s\mathcal{T} \cap s'\mathcal{T}$ is at most a codimension one face. It follows that the straight line contracting homotopies for each $s\mathcal{T}$ can be glued together to obtain a contracting homotopy of \mathcal{Z} . It follows that

$$E_{p,0}^2 = \begin{cases} \mathbb{Z} & p = 0 \\ 0 & p > 0. \end{cases}$$

Thus, to compute $H_1(K)$ we need only compute the group $E_{0,1}^2$.

9. REDUCTION

For each q define a coefficient system \mathcal{H}_q on \mathcal{Z} by setting $\mathcal{H}_q(\sigma) = H_q(K_\sigma)$. Then the q th row of the spectral sequence (9) is simply the chain complex $C_\bullet(\mathcal{Z}, \mathcal{H}_q)$, and the group $E_{p,q}^2$ is the group $H_p(\mathcal{Z}, \mathcal{H}_q)$.

In the previous section, we showed that $H_1(K) = E_{0,1}^2$. Thus we have

$$H_1(K) = H_0(\mathcal{Z}, \mathcal{H}_1).$$

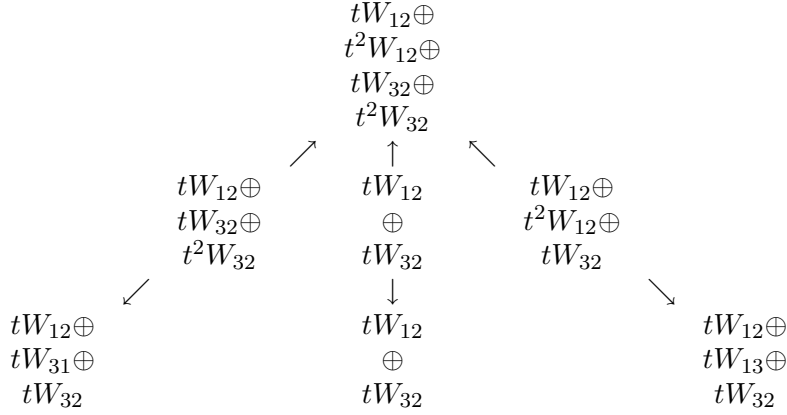
To compute the latter group, we first define a sequence of subcomplexes of \mathcal{Z} .

Denote by $\mathcal{Z}^{(1)}$ the 1-skeleton of \mathcal{Z} . Clearly, $H_0(\mathcal{Z}, \mathcal{H}_1) = H_0(\mathcal{Z}^{(1)}, \mathcal{H}_1)$. For each $i \geq 1$, let \mathcal{Z}_i be the subgraph of $\mathcal{Z}^{(1)}$ spanned by the vertices having distance at most i from v_0 . Then

$$\mathcal{Z}^{(1)} = \bigcup_{i \geq 1} \mathcal{Z}_i$$

and

$$H_0(\mathcal{Z}^{(1)}, \mathcal{H}_1) = \varinjlim H_0(\mathcal{Z}_i, \mathcal{H}_1).$$

FIGURE 2. The Vanishing of $H_0(\mathcal{Z}_{i+1}, \mathcal{Z}_i; \mathcal{H}_1)$

Proposition 9.1. *The inclusion $\mathcal{Z}_1 \longrightarrow \mathcal{Z}^{(1)}$ induces a surjection*

$$H_0(\mathcal{Z}_1, \mathcal{H}_1) \longrightarrow H_0(\mathcal{Z}^{(1)}, \mathcal{H}_1).$$

Proof. Consider the long exact sequence of the pair $(\mathcal{Z}_{i+1}, \mathcal{Z}_i)$ for $i \geq 1$:

$$H_1(\mathcal{Z}_{i+1}, \mathcal{Z}_i; \mathcal{H}_1) \rightarrow H_0(\mathcal{Z}_i, \mathcal{H}_1) \rightarrow H_0(\mathcal{Z}_{i+1}, \mathcal{H}_1) \rightarrow H_0(\mathcal{Z}_{i+1}, \mathcal{Z}_i; \mathcal{H}_1) \rightarrow 0.$$

Let v be a vertex in \mathcal{Z}_{i+1} . Then there exists an $s \in SL_n(F)/B$ and a vertex $v' \in \mathcal{T}$ such that $v = sv'$. It follows that $K_v = sK_{v'}s^{-1}$ and that $H_1(K_v)$ admits a decomposition

$$H_1(K_v) = tW_1 \oplus t^2W_2 \oplus \cdots \oplus t^{i+1}W_{i+1}$$

where each W_k is an F -vector space spanned by certain elements of $\mathfrak{sl}_n(F)$ (this follows from the decomposition of $H_1(K_{v'}) = H_1(C_{v'})$ described in Section 4).

Let $v \in \mathcal{Z}_{i+1} - \mathcal{Z}_i$. Then there exist $v_1, \dots, v_l \in \mathcal{Z}_i$ adjacent to v with $K_{v_j} \subset K_v$ for each j . If e_j denotes the edge connecting v_j with v , then $K_{e_j} = K_{v_j}$. Now, the boundary map

$$\partial : C_1(\mathcal{Z}_{i+1}, \mathcal{Z}_i; \mathcal{H}_1) \rightarrow C_0(\mathcal{Z}_{i+1}, \mathcal{Z}_i; \mathcal{H}_1)$$

maps $\bigoplus_{j=1}^l H_1(K_{e_j})$ surjectively onto the summand $tW_1 \oplus \cdots \oplus t^iW_i$ of

$$H_1(K_v) = tW_1 \oplus \cdots \oplus t^iW_i \oplus t^{i+1}W_{i+1}.$$

Figure 2 shows an example consisting of the vertices $v = [t^2e_1, e_2, t^2e_3]$, $v' = [t^2e_1, e_2, te_3]$, $v'' = [te_1, e_2, t^2e_3]$, $v_1 = [te_1, e_2, te_3]$, where the top group is $H_1(C_v)$, the three middle groups are $H_1(C_{v'',v})$, $H_1(C_{v_1,v})$, $H_1(C_{v',v})$, and the bottom groups are $H_1(C_{v''})$, $H_1(C_{v_1})$, $H_1(C_{v'})$. The space W_{ij} is a one-dimensional weight space for the root α_{ij} . The vertices v, v', v'' lie in \mathcal{Z}_2 and v_1 lies in \mathcal{Z}_1 .

It follows that the group $H_0(\mathcal{Z}_{i+1}, \mathcal{Z}_i; \mathcal{H}_1)$ can consist only of classes arising from the various $t^{i+1}W_{i+1}$. Moreover, such a class $t^{i+1}w_{i+1}$ can arise only from vertices of the form $v = s[e_1 t^{r_1}, \dots, e_{n-1} t^{r_{n-1}}, e_n] = sv'$ where $r_k = r_{k+1}$ for some $k = 1, 2, \dots, n-1$ (set $r_n = 0$). (Recall that those vertices for which the r_k are positive and distinct satisfy $H_1 = tW_1 \oplus \dots \oplus t^i W_i$ —see the discussion following Proposition 4.2.) However, v is adjacent to a vertex w in $\mathcal{Z}_{i+1} - \mathcal{Z}_i$ for which $r_k > r_{k+1}$ and hence the corresponding class $t^{i+1}w_{i+1}$ is trivial in $H_1(K_w)$. Also, the class $t^{i+1}w_{i+1}$ is nontrivial in $H_1(K_e)$ where e is the edge joining v and w . It follows that the boundary map hits the class $t^{i+1}w_{i+1}$ in $H_1(K_v)$; that is, $H_0(\mathcal{Z}_{i+1}, \mathcal{Z}_i; \mathcal{H}_1) = 0$. Figure 2 illustrates this for the classes $t^2 w_{12}, t^2 w_{32} \in H_1(C_v)$. The class $t^2 w_{12}$ is hit by the class $t^2 w_{12} \in H_1(C_{v'',v})$ (notice that this class is trivial in $H_1(C_{v''})$). Similarly, the class $t^2 w_{32}$ is hit by $t^2 w_{32} \in H_1(C_{v',v})$. Since $H_0(\mathcal{Z}^{(1)}, \mathcal{H}_1) = \varinjlim H_0(\mathcal{Z}_i, \mathcal{H}_1)$ and each map $H_0(\mathcal{Z}_i, \mathcal{H}_1) \rightarrow H_0(\mathcal{Z}_{i+1}, \mathcal{H}_1)$ is surjective, we have a surjection $H_0(\mathcal{Z}_1, \mathcal{H}_1) \rightarrow H_0(\mathcal{Z}^{(1)}, \mathcal{H}_1)$. \square

Remark. It is crucial that $n \geq 3$ in the above proof. When $n = 2$, each vertex v_{i+1} in $\mathcal{Z}_{i+1} - \mathcal{Z}_i$ is adjacent to a single vertex v_i in \mathcal{Z}_i . The stabilizer C_{e_i} of the edge joining v_i to v_{i+1} maps surjectively onto the part of $C_{v_{i+1}}$ consisting of polynomials of degree less than $i + 1$. However, v_{i+1} is not adjacent to any vertex in $\mathcal{Z}_{i+1} - \mathcal{Z}_i$. Hence,

$$H_0(\mathcal{Z}_{i+1}, \mathcal{Z}_i; \mathcal{H}_1) = \bigoplus_{s \in SL_2(F)/B} F$$

with basis the various $t^{i+1}w_{i+1}$.

Corollary 9.2. *The group $H_1(K)$ is a quotient of $H_0(\mathcal{Z}_1, \mathcal{H}_1)$.*

If the field F is finite, then we see immediately that the graph \mathcal{Z}_1 is finite and hence $H_1(K)$ is finite dimensional. This is in sharp contrast to the case $n = 2$.

10. COMPUTATION

We are now in a position to compute the group $H_1(K)$ in certain cases.

Theorem 10.1. *If $n = 3$ and $F = \mathbb{F}_2$, then $H_1(K) = \mathfrak{sl}_3(\mathbb{F}_2)$.*

Proof. By Proposition 9.1, we have a surjective map

$$H_0(\mathcal{Z}_1, \mathcal{H}_1) \longrightarrow H_1(K).$$

Since $H_1(K)$ surjects onto $\mathfrak{sl}_3(F)$, we need only check that $H_0(\mathcal{Z}_1, \mathcal{H}_1)$ is an 8-dimensional vector space.

The complex \mathcal{Z}_1 is the incidence geometry of \mathbb{F}_2^3 . It is a graph with 14 vertices and 21 edges. For each vertex v in \mathcal{Z}_1 , the group $H_1(K_v)$ is two-dimensional and for each edge e , the group $H_1(K_e)$ is one-dimensional. It follows that

$$\partial : C_1(\mathcal{Z}_1, \mathcal{H}_1) \longrightarrow C_0(\mathcal{Z}_1, \mathcal{H}_1)$$

is a map from a 21-dimensional vector space to a 28-dimensional vector space. Moreover, this map is clearly \mathbb{F}_2 -linear. Denote the generators of $C_0(\mathcal{Z}_1, \mathcal{H}_1)$ by f_1, f_2, \dots, f_{28} . The relations imposed by ∂ are as follows:

$$\begin{array}{lll} f_1 = f_3 & f_{15} = f_2 + f_4 & f_{14} = f_{20} \\ f_2 = f_{11} & f_{17} = f_5 + f_6 & f_{27} = f_{13} + f_{14} \\ f_{12} = f_{10} & f_{19} = f_7 + f_8 & f_{27} + f_{28} = f_{17} + f_{18} \\ f_9 = f_7 & f_{21} = f_9 + f_{10} & f_{28} = f_{21} + f_{22} \\ f_5 = f_8 & f_{23} = f_{11} + f_{12} & f_{25} + f_{26} = f_{19} + f_{20} \\ f_4 = f_6 & f_{18} = f_{24} & f_{26} = f_{23} + f_{24} \\ f_{13} = f_1 + f_2 & f_{16} = f_{22} & f_{25} = f_{15} + f_{16} \end{array}$$

By considering the 28×21 matrix associated to ∂ , we see that $H_0(\mathcal{Z}_1, \mathcal{H}_1)$ is an 8-dimensional F -vector space (the rank of ∂ is 20). \square

Remarks. 1. A similar argument can be used to compute $H_1(K) = \mathfrak{sl}_3(\mathbb{F}_3)$ for $F = \mathbb{F}_3$. In this case the complex \mathcal{Z}_1 has 25 vertices and 42 edges. The map ∂ is injective and hence $H_0(\mathcal{Z}_1, \mathcal{H}_1)$ is 8-dimensional.

2. To prove that $H_1(K) = \mathfrak{sl}_n(F)$ for all n , it suffices to show that the vector space $H_0(\mathcal{Z}_1, \mathcal{H}_1)$ has dimension $n^2 - 1$. Since \mathcal{Z}_1 is a finite complex it should be possible to carry this out. However, the combinatorics seem to be rather complicated in general. The complexity increases as n does and for a given n , the computation becomes more difficult as the cardinality of F increases.

3. As a corollary, we see that the subgroup K^2 of K consisting of matrices congruent to the identity modulo t^2 is equal to the commutator subgroup of K for $n = 3$, $F = \mathbb{F}_2, \mathbb{F}_3$.

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DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, IL 60208
E-mail address: `knudson@math.nwu.edu`