ÉTALE DESCENT FOR TWO-PRIMARY ALGEBRAIC K-THEORY OF TOTALLY IMAGINARY NUMBER FIELDS

J. Rognes and C. Weibel

March 11, 1998

Abstract. We show that the natural map from 2-adic algebraic K-theory to 2-adic étale K-theory induces an isomorphism in positive degrees for rings of 2-integers in totally imaginary number fields. In this sense, 2-adic algebraic K-theory of totally imaginary number fields satisfies étale descent in positive degrees.

Introduction

We prove the following theorem.

Theorem 0.1. *Let $\ell = 2$, and let F be a totally imaginary number field with ring of ℓ-integers $R_F = \mathcal{O}_F[\frac{1}{\ell}]$. The Dwyer–Friedlander map*

$$\phi_F \colon K(R_F)^\wedge_\ell \longrightarrow K^{\text{ét}}(R_F)^\wedge_\ell$$

induces isomorphisms

$$\phi_{F*} \colon K_*(R_F; \mathbb{Z}_\ell) \longrightarrow K^{\text{ét}}_*(R_F; \mathbb{Z}_\ell)$$

for all $* \geq 1$. Hence ϕ_F induces a homotopy equivalence of connected components.

In this sense 2-adic algebraic K-theory for totally imaginary number fields satisfies étale descent, since étale K-theory does so, more or less by construction.

The Dwyer–Friedlander map in question was constructed in [DF], where it was shown that 2-adically ϕ_{F*} is surjective in positive degrees when F contains $\sqrt{-1}$. Keeping this hypothesis it was proven in [RW, 0.4] that ϕ_{F*} is an isomorphism in positive degrees. The present note uses a Tate spectral sequence to extend this result to arbitrary totally imaginary number fields.

The authors would like to thank B. Kahn and R. Thomason for discussions, dating from 1992, about the Tate spectral sequence.

1. Descent and codescent

Let F be a totally imaginary number field, and let ℓ be any prime, even or odd. We shall write R_F for the ring $\mathcal{O}_F[\frac{1}{\ell}]$ of ℓ-integers in F. We are interested in the étale cohomology of the sheaves $\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)$ and $\mathbb{Z}_\ell(i)$, always supposing $i \geq 2$.

We shall use the following facts, which may be found in [So] and [RW]. First, $H^n_{\text{ét}}(R_F; \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) = 0$ unless $n = 0, 1$, and $H^n_{\text{ét}}(R_F; \mathbb{Z}_\ell(i)) = 0$ unless $n = 1, 2$.

Next, the groups $H^0_{\text{ét}}(R_F;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) = \mathbb{Z}/w_i^{(\ell)}(F)$ and $H^2_{\text{ét}}(R_F;\mathbb{Z}_\ell(i))$ are finite. Also $H^1_{\text{ét}}(R_F;\mathbb{Z}_\ell(i))$ is a finitely generated $\mathbb{Z}_\ell$ -module, and $H^1_{\text{ét}}(R_F;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$ is Pontryagin dual to one such.

Now suppose E/F is a Galois extension of totally imaginary number fields, with group $G = \text{Gal}(E/F)$. For each étale coefficient sheaf M on R_F, there is a first quadrant Hochschild–Serre spectral sequence with E_2-term

$$E_2^{p,q} = H^p(G; H^q_{\text{ét}}(R_E; M))$$

converging to $H^{p+q}_{\text{ét}}(R_F;M)$. See e.g. [Mi, III.2.20]. Writing A^G for the invariants $H^0(G;A)$ of a G-module A, this yields natural isomorphisms

$$H^0_{\text{ét}}(R_F;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \xrightarrow{\cong} H^0_{\text{ét}}(R_E;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))^G$$
$$H^1_{\text{ét}}(R_F;\mathbb{Z}_\ell(i)) \xrightarrow{\cong} H^1_{\text{ét}}(R_E;\mathbb{Z}_\ell(i))^G$$

compatible with the restriction maps for E/F. The second isomorphism uses that $H^0_{\text{ét}}(R_E;\mathbb{Z}_\ell(i)) = 0$ for $i \neq 0$.

When M is discrete, there is also a second quadrant Tate spectral sequence with E_2-term

$$E_2^{-p,q} = H_p(G; H^q_{\text{ét}}(R_E; M))$$

converging to $H^{-p+q}_{\text{ét}}(R_F;M)$. See [Se, Ch.I, App.1] or [Ka, 3.1]. If we write A_G for the coinvariants $H_0(G;A)$ of a G-module A, the edge maps are the corestriction maps $H^q_{\text{ét}}(R_E;M)_G \to H^q_{\text{ét}}(R_F;M)$. When $M = \mathbb{Q}_\ell(i)/\mathbb{Z}_\ell(i)$ and $i \geq 2$ then $H^q_{\text{ét}}(R_E;M) = 0$ for $q \geq 2$, so the edge map

$$H^1_{\text{ét}}(R_E;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))_G \xrightarrow{\cong} H^1_{\text{ét}}(R_F;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$$

is a natural isomorphism for E/F. Since its construction assumes that the coefficient module M is discrete, the Tate spectral sequence does not apply directly with $M = \mathbb{Z}_\ell(i)$. Nonetheless:

Lemma 1.1. *Let E/F be a Galois extension of totally imaginary number fields. For $i \geq 2$, the corestriction map induces an isomorphism*

$$H^2_{\text{ét}}(R_E;\mathbb{Z}_\ell(i))_G \xrightarrow{\cong} H^2_{\text{ét}}(R_F;\mathbb{Z}_\ell(i)).$$

Proof. The extension $\mathbb{Z}_\ell \to \mathbb{Q}_\ell \to \mathbb{Q}_\ell/\mathbb{Z}_\ell$ induces a short exact sequence

$$0 \to H^1_{\text{ét}}(R_E;\mathbb{Z}_\ell(i)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to H^1_{\text{ét}}(R_E;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \to H^2_{\text{ét}}(R_E;\mathbb{Z}_\ell(i)) \to 0\,.$$

This uses $H^2_{\text{ét}}(R_E;\mathbb{Q}_\ell(i)) = 0$. Since this sequence is natural with respect to automorphisms of E over F, it is G-equivariant.

The corestriction map induces a map of exact sequences

$$\begin{array}{ccccc} H^1_{\text{ét}}(R_E;\mathbb{Z}_\ell(i))_G \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell & \longrightarrow & H^1_{\text{ét}}(R_E;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))_G & \longrightarrow & H^2_{\text{ét}}(R_E;\mathbb{Z}_\ell(i))_G \\ \downarrow & & \downarrow\cong & & \downarrow \\ H^1_{\text{ét}}(R_F;\mathbb{Z}_\ell(i)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell & \longrightarrow & H^1_{\text{ét}}(R_F;\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) & \longrightarrow & H^2_{\text{ét}}(R_F;\mathbb{Z}_\ell(i)) \end{array}$$

The upper and lower right horizontal maps are surjective, and the lower left horizontal map is injective. The middle vertical map is an isomorphism by the Tate spectral sequence. Hence the right hand vertical map is surjective. Its kernel is a finite group, and isomorphic to the cokernel of the left hand vertical map, which is divisible. Hence these groups are zero, and the right hand corestriction map must be an isomorphism. □

2. Proof of the Theorem

Let $K_*(R_F;\mathbb{Z}_\ell)$ and $K_*^{\text{ét}}(R_F;\mathbb{Z}_\ell)$ denote the algebraic and étale K-theory of R_F with ℓ-adic coefficients, respectively. We need to show that the universal maps

$$\phi_F\colon K_*(R_F;\mathbb{Z}_\ell) \to K_*^{\text{ét}}(R_F;\mathbb{Z}_\ell)$$

are isomorphisms.

The edge maps in the Dwyer–Friedlander spectral sequence [DF] induce natural isomorphisms

$$DF_{i,1}\colon K_{2i-1}^{\text{ét}}(R_F;\mathbb{Z}_\ell) \xrightarrow{\cong} H^1_{\text{ét}}(R_F;\mathbb{Z}_\ell(i))$$
$$DF_{i,2}\colon K_{2i-2}^{\text{ét}}(R_F;\mathbb{Z}_\ell) \xrightarrow{\cong} H^2_{\text{ét}}(R_F;\mathbb{Z}_\ell(i))$$

in total degrees $* \geq 1$. When $\ell = 2$, the Bloch–Lichtenbaum spectral sequence [BL] leads (via [RW]) to natural isomorphisms

$$BL_{i,1}\colon K_{2i-1}(R_F;\mathbb{Z}_\ell) \xrightarrow{\cong} H^1_{\text{ét}}(R_F;\mathbb{Z}_\ell(i))$$
$$BL_{i,2}\colon K_{2i-2}(R_F;\mathbb{Z}_\ell) \xrightarrow{\cong} H^2_{\text{ét}}(R_F;\mathbb{Z}_\ell(i))$$

in total degrees $* \geq 1$.

Now let F be a totally imaginary number field, not containing $\sqrt{-1}$, and let $E = F(\sqrt{-1})$ be the quadratic extension containing $\sqrt{-1}$. Then $G = \mathrm{Gal}(E/F)$ is cyclic of order 2. It is known that ϕ_E is surjective for all $* \geq 1$, and that it is an isomorphism for $\ell = 2$, by [DF] and [RW].

For $* = 2i-1$, we use the following commutative diagram:

$$\begin{array}{ccccccc}
H^1_{\text{ét}}(R_F;\mathbb{Z}_\ell(i)) & \xleftarrow[\cong]{BL_{i,1}} & K_{2i-1}(R_F;\mathbb{Z}_\ell) & \xrightarrow{\phi_F} & K_{2i-1}^{\text{ét}}(R_F;\mathbb{Z}_\ell) & \xrightarrow[\cong]{DF_{i,1}} & H^1_{\text{ét}}(R_F;\mathbb{Z}_\ell(i)) \\
\downarrow\cong & & \downarrow & & \downarrow & & \downarrow\cong \\
H^1_{\text{ét}}(R_E;\mathbb{Z}_\ell(i))^G & \xleftarrow[\cong]{BL_{i,1}^G} & K_{2i-1}(R_E;\mathbb{Z}_\ell)^G & \xrightarrow[\cong]{\phi_E^G} & K_{2i-1}^{\text{ét}}(R_E;\mathbb{Z}_\ell)^G & \xrightarrow[\cong]{DF_{i,1}^G} & H^1_{\text{ét}}(R_E;\mathbb{Z}_\ell(i))^G
\end{array}$$

Before taking G-invariants, the bottom row consists of isomorphisms, and is G-equivariant by naturality with respect to the Galois-automorphisms of E/F. Hence the row of G-invariant submodules also consists of isomorphisms.

Using the descent isomorphism at both ends and convergence of the Dwyer–Friedlander and Bloch–Lichtenbaum spectral sequences we deduce that ϕ_F is a natural isomorphism in degree $(2i-1)$.

For $* = 2i-2$, we use the following commutative diagram:

$$\begin{array}{ccccccc}
H^2_{\text{ét}}(R_E;\mathbb{Z}_\ell(i))_G & \xleftarrow[\cong]{(BL_{i,2})_G} & K_{2i-2}(R_E;\mathbb{Z}_\ell(i))_G & \xrightarrow[\cong]{(\phi_E)_G} & K_{2i-2}^{\text{ét}}(R_E;\mathbb{Z}_\ell(i))_G & \xrightarrow[\cong]{(DF_{i,2})_G} & H^2_{\text{ét}}(R_E;\mathbb{Z}_\ell(i))_G \\
\downarrow\cong & & \downarrow & & \downarrow & & \downarrow\cong \\
H^2_{\text{ét}}(R_F;\mathbb{Z}_\ell(i)) & \xleftarrow[\cong]{BL_{i,2}} & K_{2i-2}(R_F;\mathbb{Z}_\ell(i)) & \xrightarrow{\phi_F} & K_{2i-2}^{\text{ét}}(R_F;\mathbb{Z}_\ell(i)) & \xrightarrow[\cong]{DF_{i,2}} & H^2_{\text{ét}}(R_F;\mathbb{Z}_\ell(i))
\end{array}$$

Before taking G-coinvariants, the top row consists of isomorphisms and is G-equivariant as before. Hence also the row of G-coinvariant quotient modules consists of isomorphisms. Using Lemma 1.1 we deduce that ϕ_F is also a natural isomorphism in degree $(2i-2)$. This completes the proof of the theorem.

References

[BL] S. Bloch and S. Lichtenbaum, *A spectral sequence for motivic cohomology*, UIUC K-theory preprint server, no. 62 (1994).

[DF] W. Dwyer and E. Friedlander, *Algebraic and étale K-theory*, Trans. AMS **292** (1985), 247–280.

[Ka] B. Kahn, *Deux théorèmes de comparaison en cohomologie étale; applications*, Duke Math. J. **69** (1993), 137–165.

[Mi] J. S. Milne, *Étale Cohomology*, Princeton University Press, 1980.

[RW] J. Rognes and C. Weibel, *Two-primary algebraic K-theory of rings of integers in number fields*, UIUC K-theory preprint server, no. 220 (1997).

[Se] J.-P. Serre, *Cohomologie Galoisienne. 5eme ed., rev. et complete*, Lecture Notes in Mathematics, vol. 5, Springer Verlag, 1994.

[So] C. Soulé, *K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale*, Invent. Math. **55** (1979), 251–295.

Dept. of Math., University of Oslo, Norway
E-mail address: `rognes@math.uio.no`

Department of Mathematics, Rutgers University, U.S.A.
E-mail address: `weibel@math.rutgers.edu`