Weight one motivic cohomology and K -theory

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Abstract

We study a filtration of the algebraic K -theory spectrum of a smooth variety whose “layers” ought to give the motivic cohomology groups. The central result is a computation of the weight one layer of the filtration. The computations show that the homotopy groups of the weight one piece coincides with the weight one motivic cohomology groups, thus providing evidence that the filtration is correct. Additionally, a novel filtration of $K_0(X)$ is studied, where X is any quasi-projective variety.

1 Introduction

The recent flurry of activity in the theory of motivic cohomology, notably the work of Friedlander, Suslin and Voevodsky culminating with Voevodsky’s proof of the Milnor conjecture [21], has yet to establish the precise relationship between algebraic K -theory and motivic cohomology. The guide for what this relationship should be comes from algebraic topology, where one has the classical Atiyah-Hirzebruch spectral sequence relating the topological K -theory of a space to its integral cohomology groups,

$$E_2^{p,q} = H^p(X, \mathbb{Z}_{\text{top}}(q)) \implies K_{\text{top}}^{p+q}(X),$$

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where we define $\mathbb{Z}_{\text{top}}(q)$ to be $\mathbb{Z}$ if q is even and zero otherwise. It has been conjectured for many years that an analogous “motivic” spectral sequence should exist in the world of algebraic varieties, relating the algebraic K -groups of a smooth variety to its motivic cohomology groups. Indeed, this conjecture predates many proposed definitions of the motivic cohomology groups and often has served as a hunting license for finding the “correct” definition of the motivic groups.

Following Lichtenbaum [12], the motivic cohomology groups of a smooth variety should be defined as the hypercohomology groups of members of an appropriately defined family of chain complexes of Zariski sheaves, the so-called “motivic complexes”. These chain complexes are indexed by the nonnegative integers, with the t^{th} one, referred to as the “weight t ” motivic complex, written as $\mathbb{Z}(t)$. Heuristic arguments tell us that $\mathbb{Z}(0)$ should be the complex with the constant sheaf $\mathbb{Z}$ concentrated in degree zero and that $\mathbb{Z}(1)$ should be the chain complex with the sheaf $\mathcal{O}^*$ concentrated in degree one. (We note here our convention that all chain complexes are indexed cohomologically.) In general, the chain complex $\mathbb{Z}(t)$ will be more complicated than these examples indicate, and in particular is not defined by concentrating a Zariski sheaf in one degree. The motivic cohomology groups of a variety X would then be defined by the formula

$$H_{\mathcal{M}}^n(X, \mathbb{Z}(t)) := \mathbb{H}^n(X, \mathbb{Z}(t)),$$

where the right hand side represents the Zariski hypercohomology groups of the weight t chain complex. Under this notation, the hoped-for motivic Atiyah-Hirzebruch spectral sequence becomes

$$H_{\mathcal{M}}^n(X, \mathbb{Z}(t)) \implies K_{2t-n}(X). \quad (1)$$

The first likely definition for the chain complexes $\mathbb{Z}(t)$ was given by Bloch [2], when he constructed chain complexes of cycle groups to define the “higher Chow groups”. The cohomology groups of Bloch’s complexes for a given variety X are written $\text{CH}^t(X, n)$. The indexing becomes a little confusing in that we should define

$$H_{\mathcal{M}}^n(X, \mathbb{Z}(t)) = \text{CH}^t(X, 2t - n).$$

More recently, the dramatic and important results of Voevodsky and Suslin-Voevodsky has brought another candidate for the motivic complexes to the

forefront (see [17] and [20] in particular). In addition to the wide range of desirable properties these complexes are known to satisfy, they have proved useful, playing the central role in Voevodsky's recently announced proof of the Milnor conjecture [21]. Additionally, Suslin and Voevodsky show that the definitions of motivic cohomology arising from these complexes and Bloch's complexes coincide for smooth varieties (see [15] and [20, Proposition 4.2.9]). We will therefore refer to the chain complexes of Suslin and Voevodsky as the motivic complexes and use the notation $\mathbb{Z}(t)$ to refer to these complexes from now on. We refer the reader to [16, Definition 2.1] for a precise definition of $\mathbb{Z}(t)$.

Recently, using the higher Chow groups as a definition for the motivic cohomology groups, Bloch and Lichtenbaum [3] established a spectral sequence having the form of (1) in the special case where X is the spectrum of a field. While being a very significant and ground-breaking result (in particular, it is used in an essential manner in Voevodsky's proof of the Quillen-Lichtenbaum conjecture with $\mathbb{Z}/2$ coefficients), their spectral sequence is not completely satisfactory. Due to the complicated nature of the construction, generalizing it to an arbitrary variety is a daunting task. Additionally, the desired multiplicative structure of the spectral sequence is still not known to exist.

In this paper, we explore an attempt to define the motivic spectral sequence which is based on the following idea, expounded by Grayson in [8]. Let us assume X is nonsingular. Suppose we are given a functorially defined filtration of the K -theory spectrum $K(X)$. By a filtration, we mean a sequence of maps

$$\dots \longrightarrow W^{t+1}(X) \longrightarrow W^t(X) \longrightarrow \dots \longrightarrow W^0(X) = K(X)$$

of spectra which fit into fibration sequences of the form

$$W^{t+1}(X) \longrightarrow W^t(X) \longrightarrow W^{t/t+1}(X). \quad (2)$$

Let us suppose a few basic properties hold – for example, that there is a multiplicative structure compatible with the filtration and $W^{0/1}(X)$ is an Eilenberg-MacLane spectrum whose only nonvanishing homotopy group is $\mathbb{Z}$, located in degree 0. Then the spectrum $W^{t/t+1}(X)$ can be shown to be equivalent to the geometric realization of a chain complex of abelian groups. The long exact sequences of homotopy groups defined by the fibration sequences (2) will determine an exact couple and hence a spectral sequence

abutting to the algebraic K -theory of X having the form

$$E_2^{p,q} = \pi_{-p-q} W^{-q/-q+1}(X) \implies K_{-p-q}(X).$$

The groups $\pi_n W^{t/t+1}(X)$ would then become viable candidates for the motivic cohomology groups of X . In other words, one can hope that $W^{t/t+1}(X)$ is the geometric realization of the motivic chain complex $\mathbb{Z}(t)[2t]$. (The degree shift is the same as the degree shift needed in relating the higher Chow groups to motivic cohomology.)

The main advantage of this approach in defining the motivic spectral sequence over an approach like that of Bloch and Lichtenbaum concerns naturality. Such issues as the multiplicative structure and the existence of the spectral sequence with coefficients (the former of which is still unresolved for the Bloch-Lichtenbaum spectral sequence and the latter of which requires some effort to prove (see [14])) become transparent. To see another possible advantage, suppose, using techniques analogous to those of Bloch and Lichtenbaum, one were able to show $W^{t/t+1}(X)$ agrees with the geometric realization of $\mathbb{Z}(t)[2t]$ at X in the special case when X is the spectrum of a field. Then the techniques of Voevodsky [22] should enable one to show the two agree locally on a smooth scheme. After sheafification in the Zariski topology, one would then obtain the motivic spectral sequence in full generality.

Landsburg [11] essentially gives one proposed definition of W^t . In his work the spectrum $W^t(X)$ is the geometric realization of a simplicial spectrum whose degree n part is defined, roughly speaking, to be the space $K^t(X \times \mathbb{A}^n)$, where the superscript t refers to the filtration given by codimension of support. The simplicial structure is given by identifying $\mathbb{A}^n$ with $\Delta^n := \text{Spec } \mathbb{Z}[x_0, \dots, x_n]/(x_0 + \dots + x_n - 1)$ and observing that the collection of schemes Δ^n forms a cosimplicial scheme in which the coface and coboundary maps are given just as in topology. Landsburg ultimately concluded the resulting spectral sequence must not be correct, but the calculations which led him to this conclusion appear to be flawed. See [24] for details.

More recently, Grayson, following a suggestion by Goodwillie and Lichtenbaum, proposed a candidate for the spectrum $W^t(\text{Spec } R)$ which uses the K -theory of automorphisms – that is, the K -theory of the category of projective R -modules which are equipped with a t -tuple of pairwise commuting automorphisms. Grayson's definition was further studied by the author in

[25], in which it is shown that if $R = k$ is an algebraically closed field, then the spectrum $W^{t/t+1}(k) \otimes \mathbb{Z}/n$ is homotopy equivalent to the realization of the complex $\mathbb{Z}(t) \otimes \mathbb{Z}/n[2t]$ evaluated at $\text{Spec } k$, provided the characteristic of k does not divide n . This should be likened to the fact that the space $BU \times \mathbb{Z}$ representing topological K -theory has $\mathbb{Z}$ for its even dimensional homotopy groups and 0 for its odd ones, and thus the Posnikov tower of $BU \times \mathbb{Z}$ with “finite coefficients” resembles the filtration by automorphisms of $K(\text{Spec } k; \mathbb{Z}/n)$.

Using [25, Theorem 7.11], the author also shows that Grayson’s weight filtration is equivalent locally on X to a variation of it which was also proposed by Goodwillie and Lichtenbaum. It is this filtration, which we define explicitly in section 2, that is the central object of study in this paper, and so we reserve the symbols W^t and $W^{t/t+1}$ to refer to this filtration from this point on. In [26], the author shows that the filtration W^t admits a notion of the Adams operations and furthermore, when X is a smooth variety over a perfect field, the filtration of $K_n(X)$ induced by the spectral sequence

$$E_2^{p,q} = \pi_{-p-q} W^{-q/-q+1}(X) \implies K_{-p-q}(X)$$

agrees up to torsion with the gamma-filtration. This provides evidence that the spectral sequence is in fact the long-sought motivic spectral sequence. For it is widely expected that the motivic spectral sequence degenerates up to torsion and the E_2 -terms tensored with $\mathbb{Q}$ are the weight t parts of the rational K -groups.

One hopes then that the homotopy groups $\pi_n W^{t/t+1}(X)$ coincide with the motivic cohomology groups of X , as defined for example by Voevodsky, so that K -theory is built out of the Eilenberg-MacLane spectra associated to motivic cohomology just as topological K -theory is built out of Eilenberg-MacLane spectra of the integers. Towards this end, the main result of this paper is a computation of the weight one piece of the W^t filtration for an arbitrary smooth affine variety. Specifically, we establish the following theorem.

Theorem 1. *Let $X = \text{Spec } R$ be a smooth affine variety defined over an infinite field. Then*

$$\pi_n W^{1/2}(X) = \begin{cases} \text{Pic}(R) & \text{if } n = 0, \\ R^\times & \text{if } n = 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

The reader should notice that these groups are precisely the Zariski cohomology groups of the sheaf $\mathcal{O}^*$; i.e., we have

$$\pi_n W^{1/2}(X) \cong H_{\mathcal{M}}^{2-n}(X, \mathbb{Z}(1)),$$

as conjectured.

In the course of the proof of our main result, we study a new filtration of $K_0(X)$, where X is a possibly singular quasi-projective variety. This filtration might be of independent interest, as it represents a ring filtration of $K_0(X)$, which is contravariantly functorial, agrees with the topological filtration when X is smooth over a field of characteristic zero, and contains the gamma filtration when X is arbitrary.

2 The filtration of $K(X)$

In this section we define the filtration W^t of the K -theory spectrum associated to a regular scheme. We need to set up some notation. Given a map $\pi : X \rightarrow Y$ of schemes, we define $\mathcal{M}(X/Y)$ to be the full subcategory of the category of coherent sheaves on X consisting of those sheaves $\mathcal{F}$ such that $\pi|_{\text{supp } \mathcal{F}} : \text{supp } \mathcal{F} \rightarrow Y$ is a finite morphism. One easily verifies that $\mathcal{M}(X/Y)$ is an abelian category. We define $\mathcal{P}(X/Y)$ to be the exact subcategory consisting of those sheaves $\mathcal{F}$ in $\mathcal{M}(X/Y)$ for which $\pi_* \mathcal{F}$ is a locally free sheaf on Y . Finally, given schemes X and Y , we define $\mathcal{M}(X, Y) = \mathcal{M}(X \times Y/X)$ and $\mathcal{P}(X, Y) = \mathcal{P}(X \times Y/X)$. The K -theory spectra of these four categories will be written as $K'(X/Y)$, $K(X/Y)$, $K'(X, Y)$, and $K(X, Y)$ respectively, and their associated homotopy groups are written $K'_n(X/Y)$, etc.

The spectrum $K(X, Y)$, as a functor in two variables, is contravariantly functorial in the first variable via pullback of sheaves and covariantly functorial in the second via pushforward of sheaves. Furthermore, we have pairings of the form

$$K(X, Y) \wedge K(Y, Z) \rightarrow K(X, Z)$$

induced from the biexact functor

$$\mathcal{P}(X, Y) \times \mathcal{P}(Y, Z) \rightarrow \mathcal{P}(X, Z)$$

given by $(\mathcal{F}, \mathcal{G}) \mapsto \pi_{Y*}(\pi_Z^*(\mathcal{F}) \otimes \pi_X^*(\mathcal{G}))$, where $\pi_Y : X \times Y \times Z \longrightarrow X \times Z$ is the natural projection and the maps π_X and π_Z are defined similarly. In particular, we obtain pairings

$$K_0(X, Y) \otimes K_0(Y, Z) \longrightarrow K_0(X, Z), \quad (4)$$

for all X , Y , and Z .

Definition 2.1. *The category $K_0(\text{Sch})$ is the additive category whose objects are schemes and for which, given schemes X and Y , the abelian group of maps from X to Y in $K_0(\text{Sch})$ is $K_0(X, Y)$. Composition is defined using the associated pairing (4).*

Observe that there is a functor from the category of schemes to $K_0(\text{Sch})$ which sends a map of schemes to the class of the structure sheaf of its graph.

A t -dimensional cube (or a t -cube, for short) of schemes is a functor from the poset of subsets of $\{1, \dots, t\}$ to the category of schemes. If X is a t -cube and Y is an s -cube, we may form their exterior product $X \boxtimes Y$, which is a $(t + s)$ -cube defined as follows. If I is a subset of $\{1, \dots, t\}$ and J is a subset of $\{1, \dots, s\}$, then we may think of the pair (I, J) as defining a subset of $\{1, \dots, t + s\}$ by identifying an element j of J with $j + t$ and taking the union of I and J . The $(t + s)$ -cube $X \boxtimes Y$ sends (I, J) to $X(I) \times Y(J)$.

We define $\mathbb{P}^{\wedge 1}$ to be the 1-cube $\{1\} \longrightarrow \mathbb{P}^1$, where $\{1\}$ denotes the point with coordinates $[1 : 1]$ for some chosen coordinate system. We define $\mathbb{P}^{\wedge t}$ by the recursive formula $\mathbb{P}^{\wedge t} = \mathbb{P}^{\wedge t-1} \boxtimes \mathbb{P}^{\wedge 1}$. The “final” vertex of $\mathbb{P}^{\wedge t}$ – i.e., the scheme associated to the set $\{1, \dots, t\}$ – is the cartesian product of t copies of $\mathbb{P}^1$, which we will write $\mathbb{P}^{\times t}$. Given a scheme X , we define $X \times \mathbb{P}^{\wedge t}$ to be the t -cube of schemes formed by taking the cartesian product with X at each vertex.

Given a covariant functor from schemes to chain complexes of abelian groups or, more generally, spectra, we may extend it to the category of t -cubes of schemes as follows. First apply the functor degreewise to each scheme in the cube to form a t -cube of chain complexes or spectra. Then take the associated “total object”. In the case of chain complexes, this amounts to regarding the t -cube of chain complexes as a multi-complex with $(t + 1)$ dimensions, concentrated in degrees 0 and -1 for all but one dimension, and then taking the associated total chain complex in the usual fashion. In the case of spectra, this construction amounts to taking the homotopy colimit

of the cubical diagram of spectra. In particular, we define $K(X, \mathbb{P}^{\wedge t})$ in this manner. Observe that the 1-cube $\mathbb{P}^{\wedge 1}$ is a “split cube”, in the sense that the one map comprising it is a split monomorphism. This induces a compatible family of splittings on the t -cube $\mathbb{P}^{\wedge t}$ and therefore allows one to identify $K_n(X, -)(\mathbb{P}^{\wedge t})$ with a single abelian group. In fact, we have $\pi_n K(X, \mathbb{P}^{\wedge t}) \cong K_n(X, -)(\mathbb{P}^{\wedge t})$, so that the notation $K_n(X, \mathbb{P}^{\wedge t})$ is not ambiguous.

We also want to make sense of the notation $K(X \times \mathbb{P}^{\wedge t})$. Observe that each of the maps in the cube $\mathbb{P}^{\wedge t}$ is a regular closed immersion, to which one can associate a well-defined pushforward map in K -theory. This allows us to construct a cube of spectra by applying $K(-)$ degree-wise to the cube $X \times \mathbb{P}^{\wedge t}$ and then taking the total space as before. In fact, choosing a rational point of $\mathbb{P}^{\times t}$ defines an embedding $X \rightarrow X \times \mathbb{P}^{\times t}$ which induces a pullback map $K(X \times \mathbb{P}^{\times t}) \rightarrow K(X)$. Using the projective bundle formula for K -theory, we see that there is an induced map $K(X \times \mathbb{P}^{\wedge t}) \rightarrow K(X)$ and furthermore that this map is a weak equivalence.

The “standard cosimplicial scheme”, written $\Delta^\bullet$, is the cosimplicial scheme whose degree n part is $\text{Spec } \mathbb{Z}[x_0, \dots, x_n]/(x_0 + \dots + x_n - 1)$. The coface and codegeneracy maps are defined as one defines them in algebraic topology.

We can now write down the definition of the filtration of $K(X)$. Assume X is a regular scheme. Define $W^t(X)$ to be the realization of the simplicial spectrum $d \mapsto K(X \times \Delta^d, \mathbb{P}^{\wedge t})$. Since X is regular, the natural map $K(X) \rightarrow K(X \times \Delta^d)$ is a weak equivalence, and thus $W^0(X)$ is weakly equivalent to $K(X)$. There is a map $W^{t+1}(X) \rightarrow W^t(X)$ which we now describe. Given any schemes X and Y , we define a map $K(X, Y \times \mathbb{P}^1) \rightarrow K(X, Y)$ as the difference of two maps, each of which is induced by an exact functor. The two exact functors from $\mathcal{P}(X, Y \times \mathbb{P}^1)$ to $\mathcal{P}(X, Y)$, which we write as ρ and $\rho(-1)$, send a coherent sheaf $\mathcal{F}$ to $\pi_* \mathcal{F}$ and $\pi_*(\mathcal{F} \otimes \mathcal{O}(-1))$, respectively. The map $\pi : X \times Y \times \mathbb{P}^1 \rightarrow X \times Y$ is the evident projection and the line bundle $\mathcal{O}(1)$ on $X \times Y \times \mathbb{P}^1$ is obtained by pulling back the standard very ample line bundle on $\mathbb{P}^1$. Observe that the composition

$$K(X, Y \times \{1\}) \rightarrow K(X, Y \times \mathbb{P}^1)^{\rho_* - \rho(-1)_*} K(X, Y)$$

is naturally homotopic to the zero map, and thus we obtain an induced map

$$K(X, Y \times \mathbb{P}^{\wedge 1}) \rightarrow K(X, Y).$$

Replacing Y with $\mathbb{P}^{\wedge t}$, we obtain the map

$$K(X, \mathbb{P}^{\wedge t+1}) \rightarrow K(X, \mathbb{P}^{\wedge t}),$$

which is natural in X . Replacing X with $X \times \Delta^\bullet$ and taking geometric realizations we obtain the desired map

$$W^{t+1}(X) \longrightarrow W^t(X).$$

There is an alternative construction of this map which runs as follows. Let Y be a product of projective lines and consider the inclusion of categories $\mathcal{P}(X, Y \times \mathbb{P}^1) \subset \mathcal{M}(X \times \mathbb{P}^1, Y)$. In fact, the image of this inclusion lies in the full subcategory $\mathcal{M}'(X \times \mathbb{P}^1, Y)$ of $\mathcal{M}(X \times \mathbb{P}^1, Y)$ consisting of those sheaves admitting finite resolutions by objects of $\mathcal{P}(X \times \mathbb{P}^1, Y)$ (in fact, resolutions of length two suffice). By Quillen's resolution theorem, this category has the same K -theory as $\mathcal{P}(X \times \mathbb{P}^1, Y)$. We define a functor $\mathcal{P}(X \times \mathbb{P}^1, Y) \longrightarrow \mathcal{P}(X, Y)$ by choosing any rational point of $\mathbb{P}^1$ and pulling back. We thus obtain the diagram

$$\begin{array}{ccc} K(X, Y \times \mathbb{P}^1) & \longrightarrow & K(\mathcal{M}'(X \times \mathbb{P}^1, Y)) \xleftarrow{\sim} K(X \times \mathbb{P}^1, Y) \\ & \searrow & \downarrow \\ & & K(X, Y) \end{array}$$

which induces a commutative diagram in the stable homotopy category. In particular, the composition of maps that yields $K(X, \mathbb{P}^{\wedge t}) \longrightarrow K(X)$ can be described using the map induced by the inclusion of categories $\mathcal{P}(X, \mathbb{P}^{\times t}) \subset \mathcal{M}^t(X \times \mathbb{P}^{\times t})$, where the superscript t denotes the full subcategory of coherent sheaves admitting resolutions of length at most t by locally free sheaves, together with the functor $\mathcal{P}(X \times \mathbb{P}^{\times t}) \longrightarrow \mathcal{P}(X)$ defined by pulling back along a rational point of $\mathbb{P}^{\times t}$. In other words, there is a commutative triangle

$$\begin{array}{ccc} K(X, \mathbb{P}^{\wedge t}) & \longrightarrow & K(X \times \mathbb{P}^{\wedge t}) \\ & \searrow & \downarrow \sim \\ & & K(X). \end{array}$$

3 The spectrum $W^{0/t}(R)$

Throughout this section we assume X is a regular, quasi-projective scheme over a field. We wish to describe in a convenient manner the spectrum

$W^{0/t}(X)$ which fits into a fibration sequence

$$W^t(X) \longrightarrow K(X) \longrightarrow W^{0/t}(X).$$

In [26, Lemma 7.8], it is shown that the sequence

$$K(X, \mathbb{P}^{\times t}) \longrightarrow K(X \times \mathbb{P}^{\times t}) \longrightarrow \varinjlim_Z K(X \times \mathbb{P}^{\times t} - Z), \quad (5)$$

where Z ranges over all closed subschemes of $X \times \mathbb{P}^{\times t}$ which map finitely to X , is a fibration sequence of spectra. Further, this sequence is natural in X , and so we may form another fibration sequence of spectra by applying (5) degreeewise to $X \times \Delta^\bullet$ and then taking geometric realizations. In order to better understand the base term of (5), it is convenient to work in the category of “pro schemes”. That is, we need to consider filtered projective systems of schemes which may or may not have an inverse limit defined within the category of schemes.

Definition 3.1. *For any scheme X , we let $X^{[t]}$ denote the projective system consisting of schemes of the form $X \times \mathbb{P}^{\times t} - Z$, where Z is a closed subscheme finite over X . The transition maps in this system are given by the evident open immersions.*

If we choose a very ample line bundle $\mathcal{L}$ for X , the system $X^{[t]}$ admits a nice cofinal subsystem. Suppose Z is any closed subscheme of $X \times \mathbb{P}^{\times t}$ which maps finitely to X . Observe that under any one of the t projections $X \times \mathbb{P}^{\times t} \longrightarrow X \times \mathbb{P}^1$, Z is mapped to a closed subscheme of $X \times \mathbb{P}^1$ which is finite over X . Letting S, T be homogeneous coordinate functions on $\mathbb{P}^1$, we know from [26, §2] that a homogeneous polynomial in $\Gamma(X, \mathcal{L}^{\otimes m})[S, T]$, for some integer m , whose coefficients generate the line bundle $\mathcal{L}^{\otimes m}$ defines a locally principal closed subscheme of $X \times \mathbb{P}^{\times t}$ which is finite and flat over X . In fact, also from [26, §2] we know that any closed subscheme of $X \times \mathbb{P}^1$ which is finite over X is contained in closed subscheme of this type. Thus, we can replace $X^{[t]}$ with the system defined as follows. Let $\mathcal{D}$ denote the set of t -tuples $(D_1, \dots, D_t)$, where D_i is a closed subset of $X \times \mathbb{P}^1$ defined by a homogeneous polynomial as before. Observe that $\mathcal{D}$ admits a partial ordering by using componentwise inclusion. For such a t -tuple $D = (D_1, \dots, D_t)$, let $|D|$ denote the closed subset of $X \times \mathbb{P}^1 \times \dots \times \mathbb{P}^1$ given by $\pi_1^{-1}(D_1) \cap \dots \cap \pi_t^{-1}(D_t)$, where $\pi_i : X \times \mathbb{P}^{\times t} \longrightarrow X \times \mathbb{P}^1$ is projection onto the i^{th} factor.

Then the system $X \times \mathbb{P}^{\times t} - |D|$, where D ranges over elements of $\mathcal{D}$, is a proscheme equivalent to $X^{[t]}$.

If $t = 1$, then the projective system $X^{[1]}$ does have limit inside the category of schemes. If $X = \operatorname{Spec} R$ is affine, we can equate $X^{[1]}$ with $\operatorname{Spec} R(x)$, where $R(x)$ is the result of inverting all polynomials of unit content in $R[x]$ (that is, polynomials whose coefficients generate R as an ideal). The ring $R(x)$ satisfies the “primitive criterion”: given a polynomial $f(y) \in R(x)[y]$ whose coefficients generate $R(x)$ as an ideal, there is an element r in $R(x)$ (in fact, we may take r to be a large power of x) such that $f(r)$ is a unit. Rings satisfying the primitive criterion are particularly nice from the point of view of K -theory as indicated by the results in [23] and [19]. We state here the main results about such rings we will be using.

Proposition 3.2. *(see [13]) If R is a noetherian ring satisfying the primitive criterion, then $K_0(\operatorname{Spec} R)$ is the free abelian group on the connected components of $\operatorname{Spec} R$ and $K_1(R) = R^\times$.*

Proposition 3.3. *If X is any quasi-projective scheme over an affine base scheme $\operatorname{Spec} A$, then $X^{[1]}$ is an affine scheme whose ring of regular functions satisfies the primitive criterion.*

Proof. Let X be an open subscheme of $\overline{X}$, which is itself a closed subscheme of $\mathbb{P}_A^n$. Suppose $\overline{X}$ is defined by the graded A -algebra R , so that the set of points of $\overline{X}$ is the set $\operatorname{Proj} R$, the collection of homogeneous prime ideals not containing the irrelevant prime ideal of R . Let C be the scheme $\overline{X} - X$, regarded as a reduced, closed subscheme of $\overline{X}$ so that C is defined by a homogeneous radical ideal I of R . We claim then that $X^{[1]}$ is the affine scheme associated to the ring which is the degree zero part of $S^{-1}R[t]$, where t is assigned degree zero and S consists of homogeneous elements $F(t) = f_0 + \cdots + f_n t^n$ such that $\operatorname{Proj} R/(f_0, \dots, f_n) \subset \operatorname{Proj} R/I$ as closed subsets of $\operatorname{Proj} R$.

To see this, observe first of all that an element $F(t)$ of S defines an effective divisor of $\overline{X} \times \mathbb{P}^1$ by taking the subscheme cut out by the associated homogenized polynomial $F(T, U)$ in $R[T, U]$. (The polynomial $F(T, U)$ is bihomogeneous in the bigraded ring $R[T, U]$.) The fibers over points of X of a divisor defined in this manner are all finite. Indeed, such a fiber is the projective variety associated to the graded ring $\kappa(p)[T, U]/\overline{F}(T, U)$, for some

$p \in \text{Proj } R$ which does not contain I . The condition on F ensures us that $\overline{F}$ is nonzero, and thus the fiber in question is finite.

Next we show that a closed subscheme Z of $X \times \mathbb{P}^1$ which is finite over X (or, equivalently, which has finite fibers over points of X) is contained in another such subscheme $\tilde{Z}$ having the additional property that $\tilde{Z} = D \cap X \times \mathbb{P}^1$, where D is an effective principal divisor of $\overline{X} \times \mathbb{P}^1$ that contains $C \times \mathbb{P}^1$. For consider the closure of Z in $\overline{X} \times \mathbb{P}^1$, which we write as $\overline{Z}$, and suppose $\overline{Z}$ is defined by the bihomogeneous ideal J of the bigraded ring $R[T, U]$. For each $p \in \text{Proj } R$ that does not contain I , there is a homogeneous element f_p in J which is not contained in $p[T, U]$, for this is precisely what it means for $Z \rightarrow X$ to be finite. By quasicompactness, there exist homogeneous elements $f_0, \dots, f_m$ of J so that for any prime p as before, one of the f_i does not lie in $p[T, U]$. We may in fact take the f_i to all have the same bidegree; say the degree of each f_i with respect to the variables T, U is d . Now define F to be $f_0 T^{(d+1)m} + f_1 T^{(d+1)(m-1)} U^{(d+1)} + \dots + f_m U^{(d+1)m}$ and observe that F is not contained in $p[T, U]$ for any $p \in \text{Proj } R$ which does not contain I . Thus, F cuts out an effective divisor of $\overline{X} \times \mathbb{P}^1$ whose fibers over points of X are finite and which contains $\overline{Z}$. Finally, let $g_0, \dots, g_l$ be elements of I , each of which is homogeneous of the same degree, such that $\text{Proj } R/I$ is cut out set-theoretically by the g_i 's. Now define G to be the bihomogeneous element $g_0 T^m + g_1 T^{m-1} U + \dots + g_l U^m$ in $R[T, U]$. Then G is not contained in $p[T, U]$ for any $p \in \text{Proj } R$ not containing I . Thus G also cuts out an effective divisor of $\overline{X} \times \mathbb{P}^1$ whose fibers over points of X are finite. Further, this divisor clearly contains the closed subscheme $C \times \mathbb{P}^1$. The divisor defined by $F \cdot G$ therefore has the desired properties; that is, the coefficients of $F \cdot G$ cut out C as a closed subset of $\text{Proj } R$.

These observations tell us that the limit $\varprojlim X \times \mathbb{P}^1 - Z$, where Z ranges over closed subschemes which are finite over X , and the limit $\varprojlim \overline{X} \times \mathbb{P}^1 - D$, where D ranges over those closed subschemes cut out by bihomogeneous polynomials $H = h_0 T^d + \dots + h_d U^d$ in $R[T, U]$ such that coefficients h_i cut out C as a closed subset of $\text{Proj } R$. In fact, each of these limits coincides with the limit $\varprojlim \overline{X} \times \mathbb{P}^1 - E$, where E ranges over principal divisors cut out by polynomials H whose coefficients cut out some closed subset of C in $\text{Proj } R$. This is so because each such E is contained a divisor D of the previous type by taking D to be the union of E with the divisor defined by the polynomial G constructed above. Observing that the divisor defined

by U is among this collection, we may remove it and thus we arrive at the identification $X^{[1]} = \text{Spec}(S^{-1}R[t])_0$ that we sought.

Now we must show the degree zero piece $A := (S^{-1}R[t])_0$ satisfied the primitive criterion. Suppose $H \in A[x]$ has unit content (with respect to the variable x) and write $H = (G_0 + G_1x + \cdots + G_mx^m)/F$, where $G_i \in R[t]$, $F \in S$, and $\deg F = \deg G_i$, for all i . Since H has unit content, there is an equation

$$\frac{G_0}{F} \frac{G'_0}{F'_0} + \cdots + \frac{G_m}{F} \frac{G'_m}{F'_m} = 1, \quad (6)$$

with $F'_i \in S$ and $G'_i \in R[t]$. Suppose there were a $p \in \text{Proj } R$ with $p \supset I$ and such that $G_i \in p[t]$, for all i . Then by equation (6), the product $FF'_0 \cdots F'_m$ would belong to $p[t]$ as well. But by definition of the set S , neither F nor any F'_i can belong to $p[t]$ for any $p \in \text{Proj } R$ containing I . We conclude that $(G_0, \dots, G_m)$ is not contained in $p[t]$ for any such p . In other words, letting $\text{cont}_t G$ denote the content of an element G of $R[t]$, we have that the ideal $\text{cont}_t G_0 + \cdots + \text{cont}_t G_m$ cuts out the closed subset C of Proj_R . But now $\text{cont}_t G_0 + \cdots + \text{cont}_t G_m = \text{cont}_t(G_0 + G_1t^N + \cdots + G_mt^{mN})$, when N is sufficiently large. It follows that $H(t^N)$ is a unit in A for large N . $\square$

Given a presheaf $\mathcal{F}$ on the category of schemes, we extend $\mathcal{F}$ to $X^{[t]}$ by defining

$$\mathcal{F}(X^{[t]}) := \varinjlim_Z \mathcal{F}(X \times \mathbb{P}^{\times t} - Z). \quad (7)$$

We will only consider presheaves $\mathcal{F}$ for which there is no conflict in the case $t = 1$; i.e., presheaves satisfying

$$\varinjlim_f \mathcal{F}(R[t]_f) \cong \mathcal{F}(R(t)). \quad (8)$$

In particular, we use formula (7) to define the K -groups of $X^{[t]}$ and also to define the groups $K_0(X^{[t]}, \mathbb{P}^{\times n})$, observing that in both cases equation (8) holds. There is also an induced γ -filtration on $K_0(X^{[t]})$, since $F_\gamma^j K_0(-)$ is contravariantly functorial. We will return to this point later. We can even define the spectrum $K(X^{[t]})$ by taking the evident inductive limit.

The following lemma is merely a matter of notation.

Lemma 3.4. *The sequence*

$$K(X, \mathbb{P}^{\times t}) \longrightarrow K(X \times \mathbb{P}^{\times t}) \longrightarrow K(X^{[t]})$$

is a homotopy fibration sequence of spectra.

We now extend the preceding definitions to cubical objects. The collection of proschemes $X^{[i]}$, for $i \leq t$, can be assembled into a cube in the following manner. First, we must modify the definition of $X^{[i]}$ slightly. In the indexing set $\mathcal{D}$, we will only allow i -tuples $(D_1, \dots, D_i)$ such that each D_j contains the closed subscheme $X \times \{1\} \subset X \times \mathbb{P}^1$. This added condition results in a cofinal indexing set, and so the resulting proscheme is equivalent to $X^{[t]}$. For I a subset of $\{1, \dots, t\}$, let $\mathbb{P}^{\times I}$ denote the closed subscheme of $\mathbb{P}^{\times t}$ consisting of points of $\mathbb{P}^{\times t}$ whose k^{th} component is the point 1 if k is not in I . Define $X^{[I]}$ to be the closed sub-proscheme of $X^{[t]}$ obtained by intersecting $X^{[t]}$ with $\mathbb{P}^{\times I}$ so that $X^{[I]}$ is the system involving the schemes

$$(X \times \mathbb{P}^{\times t} - |D|) \cap \mathbb{P}^{\times I}.$$

Observe that $X^{[I]}$ is isomorphic to $X^{[n]}$, if I has order n . Given an inclusion $I \subset J$ of two such subsets, there is an evident inclusion $X^{[I]} \longrightarrow X^{[J]}$. We let $X^{[\wedge t]}$ denote this cube of proschemes. Observe that $X^{[0]}$ is the empty scheme and so in particular we may identify $X^{[\wedge 1]}$ with $X^{[1]}$.

The schemes comprising the system $X^{[t]}$ are not affine, even when X is affine, an annoyance which makes it difficult to describe the associated K -theory spaces in terms of the general linear group. Such a description turns out to be valuable for our purposes, and we therefore need to find a way to describe $K(X^{[\wedge t]})$ in a convenient manner in terms of the K -theory of affine schemes. This is not difficult since $X^{[t]}$ can be described as the union of t proschemes U_i , where U_i is defined to be the system $\mathbb{P}_X^1 \times \dots \times (\mathbb{P}_X^1 - Z) \times \dots \times \mathbb{P}_X^1$, where Z ranges over closed subschemes of $\mathbb{P}_X^1$ (the i^{th} factor) which are finite over X . If $X = \text{Spec } R$, then once we take into account the cubical degeneracies, the proscheme U_i looks from a K -theoretic viewpoint like the spectrum $K(R(x_i))$. If we define $R(y_1, \dots, y_k) = R(y_1) \otimes_R \dots \otimes_R R(y_k)$, then the intersection $U_{i_1} \cap \dots \cap U_{i_k}$ looks, for K -theoretic purposes, like the spectrum $K(R(x_{i_1}, \dots, x_{i_k}))$. Heuristically, we have achieved the goal of defining $K(X^{[\wedge t]})$ canonically in terms of the K -theory of affine pieces.

Let us formalize this idea. We let Ord denote the usual simplicial category with objects $[n] := \{0, 1, \dots, n\}$, for $n \geq 0$, and morphisms given by

nondecreasing set maps. Recall that a cosimplicial (resp., simplicial) object in a category $\mathcal{C}$ is a covariant (resp., contravariant) functor from Ord to $\mathcal{C}$. Let $\text{Ord}^{\leq n}$ denote the full subcategory of Ord consisting of the objects $[0], \dots, [n]$. We define an n -truncated cosimplicial (resp., simplicial) object in $\mathcal{C}$ to be a covariant (resp., contravariant) functor from $\text{Ord}^{\leq n}$ to $\mathcal{C}$. Given a ring R , we define a cosimplicial ring $\tilde{R}$ using the formula

$$[n] \mapsto \tilde{R}_n = R(x_0, \dots, x_n).$$

A map $\alpha : [n] \longrightarrow [m]$ in Ord induces a map $\tilde{R}_n \longrightarrow \tilde{R}_m$ by sending x_i to $x_{\alpha(i)}$. We also define t -truncated cosimplicial rings $\tilde{R}^{\leq t}$ by restricting the domain of $\tilde{R}$ to $\text{Ord}^{\leq t}$.

Definition 3.5. For a ring R , define $G^{0/t}(R)$ to be the homotopy inverse limit of the diagram of spectra defined by applying the K -theory functor degreewise to the $(t-1)$ -truncated cosimplicial ring $\tilde{R}^{\leq t-1}$. In other words, we define

$$G^{0/t}(R) := \text{holim}_{[i] \in \text{Ord}^{\leq t-1}} [i] \mapsto K(R(x_0, \dots, x_i)).$$

Lemma 3.6. If $X = \text{Spec } R$, with R regular, then the spectrum $G^{0/t}(R)$ is homotopy equivalent to $K(X^{[\wedge t]})$.

Proof. As indicated previously, we let $U_i^{[t]}$ denote the open sub-proscheme of $X^{[t]}$ defined to be the projective system of schemes $X \times \mathbb{P}^{\times t} - \pi_i^{-1}(D_i)$, where π_i is the i^{th} projection map and D_i ranges over closed subschemes of $X \times \mathbb{P}^1$ which are finite over X and contain the point 1. By intersecting $U_i^{[t]}$ with each closed sub-proscheme in the cube $X^{[\wedge t]}$, we form a cube of proschemes, which we write $U_i^{[\wedge t]}$. More generally, for any subset J of $\{1, \dots, t\}$, we define $U_J^{[\wedge t]}$ to be the open subcube of proschemes of $X^{[\wedge t]}$ defined as the intersection

$$\bigcap_{j \in J} U_j^{[\wedge t]}.$$

The maps in the cube $U_J^{[\wedge t]}$ are regular closed immersions and so we apply K -theory and use the covariant maps induced by pushforward to obtain a cube of spectra. Our first claim is that the total spectrum of the cube $K(U_J^{[\wedge t]})$, for $J = \{j_1, \dots, j_l\}$ not empty, is homotopy equivalent to $K(\text{Spec } R(x_J))$, where we define $R(x_J) := R(x_{j_1}, \dots, x_{j_l})$. To see this, fix

a nonempty set J with l elements and observe that the cubical proscheme $U_J^{[\wedge t]} : I \mapsto U_J^{[I]}$ has the property that $U_J^{[I]} = \emptyset$ unless $J \subset I$. This allows us to identify $U_J^{[\wedge t]}$ with a $(t-l)$ -cube indexed by the subsets of $S := \{1, \dots, t\} - J$ by identifying such a subset A with the subset $A \cup J$ of $\{1, \dots, t\}$. Further, we have $U_J^{[A]} \cong \operatorname{Spec} R(x_J) \times \mathbb{P}^{\times A}$ and in fact the $(t-l)$ -cube $U_J^{[\wedge t]}$ is equivalent to $\operatorname{Spec} R(x_J)^{[\wedge t-l]}$. By pulling back along a rational point, we have as before a weak equivalence

$$K(U_J^{[\wedge t]}) \xrightarrow{\sim} K(\operatorname{Spec} R(x_J)).$$

We next use that K -theory satisfies Zariski descent, which can be expressed as follows. Let $\mathcal{V} = \{V_i \mid i = 1, \dots, m\}$, where the V_i form an open cover of some regular scheme Y . Now consider the punctured m -cocube indexed by the nonempty subsets of $\{1, \dots, m\}$ which sends a subset I to $\bigcap_{i \in I} V_i$. (That is, this is a contravariant functor from the poset of *nonempty* subsets of $\{1, \dots, m\}$ to the category of open subset of Y .) Let $\mathcal{C}(\mathcal{V})$ denote this punctured cube. Applying K -theory to $\mathcal{C}(\mathcal{V})$ results in a punctured cocube of spectra and Zariski descent amounts to the statement that the natural map

$$K(Y) \longrightarrow \operatorname{holim} K(\mathcal{C}(\mathcal{V}))$$

is a weak equivalence.

The notion of Zariski descent extends to the open covering

$$X^{[\wedge t]} = U_1^{[\wedge t]} \cup \dots \cup U_t^{[\wedge t]}.$$

This results in the formula

$$K(X^{[\wedge t]}) \xrightarrow{\sim} \operatorname{holim} K(\mathcal{C}(\{U_i^{[\wedge t]}\})),$$

where the right hand side is the homotopy limit of a cocube of cubes of proschemes! Fortunately, we can simplify this monstrosity by recalling that we have identified $K(U_J^{[\wedge t]})$ with $K(R(x_J))$, if J is nonempty, and furthermore the open inclusion $U_J^{[\wedge t]} \subset U_L^{[\wedge t]}$ associated to an inclusion $L \subset J$ corresponds to the ring map $R(x_L) \longrightarrow R(x_J)$. Thus, we have

$$K(X^{[\wedge t]}) \xrightarrow{\sim} \operatorname{holim}(J \mapsto K(R(x_J)))$$

where J ranges over the poset of the nonempty subsets of $\{1, \dots, t\}$.

To further simplify our description of $K(X^{[\wedge t]})$, let us observe that the category $\text{Ord}^{\leq t-1}$ is equivalent to the category $\mathcal{T}$ whose objects are the nonempty subsets of $\{1, \dots, t\}$ and whose morphisms are order preserving set maps. The poset of nonempty subsets of $\{1, \dots, t\}$, which we regard as a category and write as $\mathcal{P}$, is a subcategory of $\mathcal{T}$. Furthermore, the inclusion functor $\mathcal{P} \subset \mathcal{T}$ is (left) cofinal in the sense of [4, XI.9.1]. To show this, we must establish that the “fibers” over any $I \in \mathcal{T}$ are contractible. Fix an I and let W denote the fiber over I ; that is, W is the category consisting of pairs $(J, J \longrightarrow I)$ where J is an object of $\mathcal{P}$ but the map is an arbitrary set map. A map in W from $(J, J \longrightarrow I)$ to $(J', J' \longrightarrow I)$ is a map from J to J' (that is, an inclusion) so that the evident triangle commutes. In fact, W is a poset. For any integers $1 \leq x, y \leq t$ such that $y \in I$, define a subposet $W_{x,y}$ of W consisting of pairs $(J, J \longrightarrow I)$ such that x is in J and is sent to y under the set map. As x and y range over all possibilities, we obtain a covering of W and furthermore, given m pairs $(x_1, y_1), \dots, (x_m, y_m)$, the poset $\cap_i W_{x_i, y_i}$ is contractible since it contains the initial element $(\{x_1, \dots, x_m\}, \{x_1, \dots, x_m\} \longrightarrow I)$, in which the map sends x_i to y_i . It follows that W itself is contractible. The functor $I \mapsto R(x_I)$ from $\mathcal{P}$ to rings introduced earlier extends to a functor from $\mathcal{T}$ to rings, and thus using [4, XI.9.2], we have

$$K(X^{[\wedge t]}) \xrightarrow{\sim} \text{holim}_{\mathcal{P}}(K(R(x_I))) \xrightarrow{\sim} \text{holim}_{\mathcal{T}}(K(R(x_I))) \cong \text{holim}_{\text{Ord}^{\leq t-1}} K(R(x_0, \dots, x_i)).$$

□

In light of the lemma, we may now redefine $W^{0/t}(R)$, when R is regular, to be realization of the simplicial spectrum $[d] \mapsto G^{0/t}(R\Delta^d)$. Since finite homotopy inverse limits commute with homotopy colimits, we can also describe $W^{0/t}(R)$ as the homotopy inverse limit of the $(t-1)$ -truncated cosimplicial spectrum

$$[n] \mapsto |[d] \mapsto K(R\Delta^d(x_0, \dots, x_n))|,$$

where $0 \leq n \leq t-1$.

Remark 3.7. For R not necessarily regular, it seems most natural to define $W^{0/t}(R)$ just as in the regular case but using instead the functor K^B , as defined by Thomason [18]. The spectrum defined as the realization of $d \mapsto K^B(R\Delta^d)$ was defined by Weibel [27] to be “homotopy algebraic K -theory”

and is written $KH(R)$. The fibration sequence of the lemma would then become

$$W^t(R) \longrightarrow KH(R) \longrightarrow W^{0/t}(R),$$

and now serves as the definition of $W^t(R)$. This may prove to be a useful perspective in that it allows for dimension shifting arguments, using the fact that $KH(S^n R) \cong KH(R) \times \Omega^n KH(R)$, where $S^n R$ is the “ n -sphere” defined over R ; i.e., $S^n R$ the affine scheme $\text{Spec } R[x_0, \dots, x_n]/(x_0 + \dots + x_n - 1, x_0 \cdot \dots \cdot x_n)$.

By Proposition 3.2, $K_0(R(x))$ is the free abelian group on the connected components of $\text{Spec } R$, when R is any noetherian ring. Assume now that R is regular. Then the same can be said for $K_0(R(x_0, \dots, x_n))$, since $R(x_0, \dots, x_n)$ is a localization of $R(x_0)[x_1, \dots, x_n]$. Thus, the space

$$|d \mapsto K(R\Delta^d(x_0, \dots, x_n))|$$

factors as

$$H^0(\text{Spec } R, \mathbb{Z}) \times |d \mapsto BGL^+(R\Delta^d(x_0, \dots, x_n))|,$$

where for any scheme Y we define $H^0(Y, \mathbb{Z})$ to be the free abelian group on the connected components of Y . Observe that $R\Delta^d(x_0, \dots, x_n)$ forms a connected (and hence contractible) simplicial ring. Using the argument outlined in [1, Theorem 2.2], one shows that, for any connected simplicial ring R , the space $|d \mapsto BGL^+(R_d)|$ is homotopy equivalent to $\Omega^{-1}|d \mapsto GL(R_d)|$, a delooping of the geometric realization of the simplicial group $GL(R)$. Let us define, for a connected simplicial ring R , the ring ∂R_n to be the boundary of the n -simplices. That is, if $\partial\Delta^n$ denotes the subsimplicial set of $\Delta^n := \text{Hom}_{\text{Ord}}(-, [n])$ defined as the union of the $(n-1)$ -dimensional faces, then $\partial R_n := \text{Hom}_{\text{simp. sets}}(\partial\Delta^n, R)$. For example, if $R = R\Delta^\bullet$, then $\partial R\Delta^n \cong R\Delta^n/(x_0 \cdot \dots \cdot x_n)$. In general, there is a natural map $R_n \longrightarrow \partial R_n$, which is a surjection if R is connected.

Lemma 3.8. *There is a natural isomorphism*

$$\partial(R\Delta^n(x_0, \dots, x_t)) \cong (\partial R\Delta^n)(x_0, \dots, x_t),$$

for all n .

Proof. For any ring A , the ring $A(x_0, \dots, x_t)$ is the result of inverting all polynomials in $A[x_0, \dots, x_t]$ which depend on just one variable and have unit content. The lemma is a special case of the fact that the functor $A \mapsto A(x_0, \dots, x_t)$ preserves surjections and limits of diagrams of surjective ring maps. The proofs of these facts are straightforward. $\square$

The following lemma is surely well-known, but the reader may consult [24] for a proof.

Lemma 3.9. *For any connected simplicial ring R , we have*

$$\pi_n |d \mapsto GL(R_d)| \cong \operatorname{coker} (K_1(R_{n+1}) \longrightarrow K_1(\partial R_{n+1})).$$

Similarly,

$$\pi_n |d \mapsto SL(R_d)| \cong \operatorname{coker} (SK_1(R_{n+1}) \longrightarrow SK_1(\partial R_{n+1})).$$

It is also shown in [24] that, as a direct consequence of this lemma, $GL(R\Delta^\bullet(x))$ is contractible.

Proposition 3.10. *If R is regular, we have*

$$\pi_n W^{1/2}(R) = \begin{cases} \operatorname{Pic}(R) & \text{if } n = 0, \\ R^\times \oplus SK_1(R\partial\Delta^2(x, y))/SK_1(R\Delta^2(x, y)) & \text{if } n = 1, \\ SK_1(R\partial\Delta^{n+1}(x, y))/SK_1(R\Delta^{n+1}(x, y)) & \text{for all other } n. \end{cases}$$

Proof. Since $G^{0/2}(R)$ is by definition the homotopy inverse limit of

$$K(R(x)) \rightrightarrows K(R(x, y)),$$

we have a homotopy cartesian square

$$\begin{array}{ccc} G^{0/2}(R) & \longrightarrow & K(R(x)) \\ \downarrow & & \downarrow \\ K(R(y)) & \longrightarrow & K(R(x, y)). \end{array} \tag{9}$$

Using Lemma 3.6, if we introduce the simplicial ring $R\Delta^\bullet$ and take geometric realizations, we obtain the homotopy cartesian square

$$\begin{array}{ccc} W^{0/2}(R) & \longrightarrow & K(R\Delta^\bullet(x)) \\ \downarrow & & \downarrow \\ K(R\Delta^\bullet(y)) & \longrightarrow & K(R\Delta^\bullet(x, y)). \end{array}$$

From the preceding discussion, we have

$$K(R\Delta^\bullet(x_0, \dots, x_i)) \sim H^0(\text{Spec } R, \mathbb{Z}) \times \Omega^{-1} |GL(R\Delta^\bullet(x_0, \dots, x_i))|.$$

As mentioned, from [24, Corollary 2.3] we know that $GL(R\Delta^\bullet(t))$ is contractible. Using the natural map $W^{0/2}(R) \rightarrow W^{0/1}(R)$, whose homotopy fiber is $W^{1/2}(R)$, we obtain the formula

$$W^{1/2}(R) \sim \text{holim} (K(R\Delta^\bullet(y)) \rightarrow K(R\Delta^\bullet(x, y))).$$

Factoring out the common factors of $H^0(\text{Spec } R, \mathbb{Z})$, we obtain

$$W^{1/2}(R) \sim \text{holim} (|BGL(R\Delta^\bullet(y))| \rightarrow |BGL(R\Delta^\bullet(x, y))|)$$

and finally

$$W^{1/2}(R) \sim |GL(R\Delta^\bullet(x, y))|.$$

Now consider the determinant map

$$GL(R\Delta^\bullet(x, y)) \rightarrow U(R\Delta^\bullet(x, y)),$$

where U denotes taking groups of units. This map is a split surjection whose kernel is $SL(R\Delta^\bullet(x, y))$. Using Lemma 3.9 we get that

$$\pi_n W^{1/2}(R) \cong SK_1(R\partial\Delta^{n+1}(x, y)) / SK_1(R\Delta^{n+1}(x, y)) \oplus \pi_n(U(R\Delta^\bullet(x, y))).$$

From [23, Theorem 4.3], there is an exact sequence

$$0 \rightarrow U(R) \rightarrow U(R(x)) \oplus U(R(y)) \rightarrow U(R(x, y)) \rightarrow \text{Pic}(R) \rightarrow 0.$$

We therefore obtain an exact sequence of simplicial abelian groups

$$\begin{aligned} 0 \rightarrow U(R\Delta^\bullet) \rightarrow U(R\Delta^\bullet(x)) \oplus U(R\Delta^\bullet(y)) &\rightarrow U(R\Delta^\bullet(x, y)) \\ &\rightarrow \text{Pic}(R\Delta^\bullet) \rightarrow 0. \end{aligned} \tag{10}$$

The simplicial abelian group $U(R\Delta^\bullet(x))$ is in fact contractible. For the same reasoning used in Lemma 3.9 shows that $\pi_n(R\Delta^\bullet(x))$ is isomorphic to $\text{coker}(U(R\Delta^{n+1}(x)) \rightarrow U(R\partial\Delta^{n+1}(x)))$, which is easily shown to be zero since $R\Delta^{n+1} \rightarrow R\partial\Delta^{n+1}$ is a surjective ring map. (See the proof of [24, Corollary 2.3] for a more explicit proof.) The result now follows from the exact sequence (10), since both $U(-)$ and $\text{Pic}(-)$ are homotopy invariant functors on the category of regular rings. $\square$

Remark 3.11. *The results of the next section will enable us to show*

$$SK_1(R\partial\Delta^{n+1}(x, y)) = 0,$$

for all n , thereby proving the main result of this paper.

4 The filtration of K_0

In this section, we study the filtration on $K_0(X)$ induced by the spectra $W^t(X)$. The study of this filtration will be used in the next section to compute $W^{1/2}$, but it may be of independent interest. We will need to explore the filtration of $K_0(X)$ in the greatest possible generality and in particular we do not wish to assume that X is regular. Since in this generality $K(X \times \Delta^\bullet)$ is not the same as $K(X)$, we need to remove the simplicial construction from the filtration W^t in order to arrive at a filtration of $K_0(X)$. Specifically, the sequence of maps

$$\dots \rightarrow K(X, \mathbb{P}^{\wedge t+1}) \rightarrow K(X, \mathbb{P}^{\wedge t}) \rightarrow \dots \rightarrow K(X)$$

described in section 2 induces a sequence of maps of abelian groups

$$\dots \rightarrow K_0(X, \mathbb{P}^{\wedge t+1}) \rightarrow K_0(X, \mathbb{P}^{\wedge t}) \rightarrow \dots \rightarrow K_0(X)$$

by taking degree zero homotopy groups. Observe that the map $K_0(X, \mathbb{P}^{\times t}) \rightarrow K_0(X)$ is defined by the formula

$$[\mathcal{F}] \mapsto \sum_{(\epsilon_1, \dots, \epsilon_t) \in \{0, -1\}^t} (-1)^{\epsilon_1 + \dots + \epsilon_t} [\pi_*(\mathcal{F} \otimes L_1^{\epsilon_1} \otimes \dots \otimes L_t^{\epsilon_t})],$$

where L_i is the line bundle induced from the bundle $\mathcal{O}(1)$ on the i^{th} copy of $\mathbb{P}^1$ and $\pi : X \times \mathbb{P}^{\times t} \rightarrow X$ is the projection map.

Definition 4.1. *The $\mathbb{P}^1$ -filtration of $K_0(X)$, for X quasi-projective over an affine scheme, is the decreasing filtration given by*

$$F_{\mathbb{P}^1}^t K_0(X) := \text{im}(K_0(X, \mathbb{P}^{\times t}) \longrightarrow K_0(X)).$$

The use of $\mathbb{P}^{\wedge t}$ instead of $\mathbb{P}^{\times t}$ does not affect the definition of the $\mathbb{P}^1$ -filtration and we have therefore used $\mathbb{P}^{\times t}$ for simplicity. We wish to compare the $\mathbb{P}^1$ -filtration with two other common filtrations on $K_0(X)$, the gamma filtration and the topological filtration (see [7]). We begin with a few basic properties.

Proposition 4.2. *The $\mathbb{P}^1$ -filtration is a functorial ring filtration. For any scheme X , we have $F_{\mathbb{P}^1}^1 K_0(X) = F_{\text{top}}^1 K_0(X) = F_{\gamma}^1 K_0(X)$.*

Proof. Concerning the first claim, the functoriality is immediate from the construction since each map $K(X, \mathbb{P}^{\times t}) \longrightarrow K(X)$ is contravariantly functorial in X . To establish that the $\mathbb{P}^1$ -filtration is a ring filtration – i.e., under the multiplication in $K_0(X)$, we have $F_{\mathbb{P}^1}^t K_0(X) \cdot F_{\mathbb{P}^1}^s K_0(X) \subset F_{\mathbb{P}^1}^{t+s} K_0(X)$ – we introduce a biexact functor

$$\mathcal{P}(X, \mathbb{P}^{\times t}) \times \mathcal{P}(X, \mathbb{P}^{\times s}) \longrightarrow \mathcal{P}(X, \mathbb{P}^{\times t+s})$$

which sends a pair $(\mathcal{F}, \mathcal{G})$ to $p^* \mathcal{F} \otimes q^* \mathcal{G}$, where $p : X \times \mathbb{P}^{\times t} \times \mathbb{P}^{\times s} \longrightarrow X \times \mathbb{P}^{\times t}$ and $q : X \times \mathbb{P}^{\times t} \times \mathbb{P}^{\times s} \longrightarrow X \times \mathbb{P}^{\times s}$ are the evident projections. One may check that the resulting diagram

$$\begin{array}{ccc} K_0(X, \mathbb{P}^{\times t}) \otimes K_0(X, \mathbb{P}^{\times s}) & \longrightarrow & K_0(X, \mathbb{P}^{\times t+s}) \\ \downarrow & & \downarrow \\ K_0(X) \otimes K_0(X) & \longrightarrow & K_0(X) \end{array}$$

commutes, which establishes the first claim.

As is known, we have

$$F_{\text{top}}^1 K_0(X) = F_{\gamma}^1 K_0(X) = \ker \left(\text{rank} : K_0(X) \longrightarrow H^0(X, \mathbb{Z}) \right).$$

From Lemma 3.4, we have a right exact sequence

$$K_0(X, \mathbb{P}^{\wedge 1}) \longrightarrow K_0(X) \longrightarrow K_0(X^{[1]}) \longrightarrow 0.$$

Since X is quasi-projective, we know from Proposition 3.3 that $X^{[1]}$ is the affine scheme associated to a ring satisfying the primitive criterion, and hence $K_0(X^{[1]}) = H^0(X, \mathbb{Z})$. $\square$

In case X is a regular scheme, the t^{th} stage of the topological filtration can be described at the subgroup of $K_0(X)$ generated by classes of coherent sheaves whose supports have codimension at least t . In fact, it suffices to consider the collection of classes $[\mathcal{O}_Z] \in K'_0(X) \cong K_0(X)$, where Z ranges over the collection of closed, integral subschemes of X of codimension at least t . The map $K_0(X, \mathbb{P}^{\times t}) \longrightarrow K_0(X)$ can be described as the composition of maps

$$K_0(X, \mathbb{P}^{\times t}) \longrightarrow K'_0(X \times \mathbb{P}^{\times t}) \cong K_0(X \times \mathbb{P}^{\times t}) \longrightarrow K_0(X). \quad (11)$$

The first map is induced by the evident inclusion of exact categories and the last map is induced by pulling back along the splitting $X \longrightarrow X \times \mathbb{P}^{\times t}$ of the map $X \times \mathbb{P}^{\times t} \longrightarrow X$ induced by choosing a rational point of $\mathbb{P}^{\times t}$. For later use, we observe here that we can choose a splitting from a slightly larger class.

Lemma 4.3. *Let $i : X \longrightarrow X \times \mathbb{P}^{\times t}$ be any splitting of $X \times \mathbb{P}^{\times t} \longrightarrow X$ such that, if $\mathcal{L}_j$ is the line bundle induced from the natural very ample bundle $\mathcal{O}(1)$ on the j^{th} copy of $\mathbb{P}^1$, we have that $i^*\mathcal{L}_j$ is trivial for all j . Then the map $i^*K_0(X \times \mathbb{P}^{\times t}) \longrightarrow K_0(X)$ coincides with the map induced by choosing a rational point of $\mathbb{P}^{\times t}$.*

Proof. This is an immediate consequence of the decomposition

$$K_0(X \times \mathbb{P}^{\times t}) \cong \bigoplus_{n_i \in \{0,1\}} K_0(X) \cdot [\mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_t^{n_t}].$$

□

Since the topological filtration is functorial on quasi-projective schemes and the map $K_0(X, \mathbb{P}^{\times t}) \longrightarrow K_0(X \times \mathbb{P}^{\times t})$ clearly lands in $F_{\text{top}}^t K_0(X \times \mathbb{P}^{\times t})$, the description (11) of the map $K_0(X, \mathbb{P}^{\times t}) \longrightarrow K_0(X)$ gives immediately the following result.

Lemma 4.4. *For X a regular scheme, quasi-projective over an affine scheme, we have*

$$F_{\mathbb{P}^1}^t K_0(X) \subseteq F_{\text{top}}^t K_0(X),$$

for all t .

Let us return to the general situation in which X might not be regular. In order to compare the γ -filtration and the $\mathbb{P}^1$ -filtration, we need something like a splitting principal for the $\mathbb{P}^1$ -filtration. The following lemma gives the desired statement.

Lemma 4.5. *Let X be a quasi-projective variety over an infinite field k , and let $\mathcal{E}$ be a vector bundle on X of rank $r + 1$. Let π denote the element of $K_0(\mathbb{P}\mathcal{E}, X)$ given by the projection map $\mathbb{P}\mathcal{E} \rightarrow X$. Then there exists an element f of $K_0(X, \mathbb{P}\mathcal{E})$ such that under the pairing*

$$K_0(X, \mathbb{P}\mathcal{E}) \otimes K_0(\mathbb{P}\mathcal{E}, X) \longrightarrow K_0(X, X),$$

the pair (f, π) is sent to the class of the structure sheaf of the diagonal subscheme; i.e., we have $\pi \circ f = \text{id}$ in $K_0(\text{Sch})$.

Before proving this lemma, we state and prove a simple preliminary lemma.

Lemma 4.6. *Let Y be a projective variety of dimension d over an infinite field k . Then the Hilbert function of Y , which we write as $H_Y(m)$, satisfies*

$$H_Y(m) \geq \binom{m+d}{d},$$

for all integers m .

Proof. Let Y be the subvariety of $\text{Proj } k[X_0, \dots, X_N]$ defined by the homogeneous prime ideal I . By the linear version of Noether normalization [5, Theorem 13.3], there exist linear forms $Y_0, \dots, Y_d$ such that the map

$$k[Y_0, \dots, Y_d] \longrightarrow k[X_0, \dots, X_N]/I$$

is injective (and finite). The result follows, since the dimension of the m^{th} homogeneous part of $k[Y_0, \dots, Y_d]$ is $\binom{m+d}{d}$. $\square$

Proof of Lemma 4.5. We may assume X is connected. By twisting with an ample line bundle on X , we may also assume $\mathcal{E}$ is generated by global sections.

Observe that the diagonal map $X \rightarrow X \times X$ induces an injective homomorphism $K_0(X) \rightarrow K_0(X, X)$ of $K_0(X)$ -modules, and we regard elements of $K_0(X)$ as elements of $K_0(X, X)$ throughout this proof. Suppose we can

find two closed subschemes of $X \times \mathbb{P}\mathcal{E}$, say Z_1 and Z_2 , which are finite and flat over X of degrees d_1 and d_2 , respectively, with d_1 and d_2 relatively prime. Then each $[\mathcal{O}_{Z_i}]$ is an element of $K_0(X, \mathbb{P}\mathcal{E})$ and $\pi \circ [\mathcal{O}_{Z_i}]$ is the element of $K_0(X, X)$ induced by $[\pi_*(\mathcal{O}_{Z_i})] \in K_0(X)$. But $[\pi_*(\mathcal{O}_{Z_i})]$ has rank d_i in $K_0(X)$ and so for some suitable integers e_i we have that $\pi \circ (e_1[\mathcal{O}_{Z_1}] + e_2[\mathcal{O}_{Z_2}])$ is defined by an element of $K_0(X)$, call it g , of rank one. Observe g is a unit in $K_0(X)$ and therefore by defining

$$f = g^{-1} \cdot (e_1[\mathcal{O}_{Z_1}] + e_2[\mathcal{O}_{Z_2}]) \in K_0(X, \mathbb{P}\mathcal{E}),$$

the lemma would be proven.

It remains to show such a pair of closed subschemes exist. In fact, we will show that any closed subscheme of $X \times \mathbb{P}\mathcal{E}$ defined by r forms of sufficiently large degree will be finite and flat over X . To be specific, since $\mathcal{E}$ is generated by global sections, we can choose a surjection

$$\mathcal{O}_X^N \twoheadrightarrow \mathcal{E}, \quad (12)$$

for some suitably large N . We can therefore embed $\mathbb{P}\mathcal{E}$ as a closed subscheme of $X \times_k \mathbb{P}_k^N$. We show that if $f_1, \dots, f_r$ are homogeneous forms on $\mathbb{P}_k^N$ of sufficiently large degree n , then $X \times \mathbb{P}\mathcal{E} \cap X \times V(f_1, \dots, f_r)$ is finite and flat over X of degree n^r . (The notation $V(f_1, \dots, f_r)$ refers to the closed subscheme cut out by these forms.) Assuming this claim, a pair of closed subschemes having the desired properties exists.

Let $V = \Gamma(\mathbb{P}_k^N, \mathcal{O}_{\mathbb{P}^N}(1))$, a k -vector space of dimension $N + 1$. Given a point x of X , we let $\mathcal{E}_x$ denote the fiber over x – that is, the vector bundle on $\text{Spec } k(x)$ induced by pullback. We let $S^n V$ denote the n^{th} symmetric power of V . Fix an integer n and consider the subvariety Z of $X \times_k \mathbb{P} S^n V \times \dots \times \mathbb{P} S^n V$ (with r copies of $\mathbb{P} S^n V$) defined on coordinates as

$$\{(x, f_1, \dots, f_r) \mid \mathbb{P}\mathcal{E}_x \cap V(f_1, \dots, f_r) \subseteq \mathbb{P}_{k(x)}^N \text{ has positive dimension}\}.$$

Loosely speaking, the fiber of Z over a point x of X consists of r -tuples of homogeneous forms which are “bad at x ”. We can rephrase the definition of Z as follows. If $W = \Gamma(\text{Spec } k(x), \mathcal{E}_x)$ and $V_{k(x)} := V \otimes_k k(x) \twoheadrightarrow W$ is the surjection induced by (12), then Z_x , the fiber of Z over x , is the inverse image under the induced rational map

$$\mathbb{P} S^n V_{k(x)} \times \dots \times \mathbb{P} S^n V_{k(x)} \xrightarrow{\Phi} \mathbb{P} S^n W \times \dots \times \mathbb{P} S^n W$$

of the closed subscheme Z^r defined on coordinates as

$$Z^r := \{(g_1, \dots, g_r) \mid \dim V(g_1, \dots, g_r) > 0\}.$$

(The inverse image of a closed subscheme under a rational map is taken to mean the union of the center of the rational map with the set of points in the domain of definition that are mapped to the closed subscheme.)

We claim that $\text{codim } Z^r \geq n + 1$, for n sufficiently large. From this the lemma will follow. For observe that in this case $Z_x = \text{Center}(\Phi) \cup \Phi^{-1}(Z^r)$ has codimension at least $\min\{\text{codim } Z^r, \text{codim } \text{Center } \Phi\} = \min\{n + 1, \dim S^n W - 1\}$ for large n . It follows that the codimension of Z in $X \times_k \mathbb{P}S^n V \times \dots \times \mathbb{P}S^n V$ is larger than n , for n sufficiently large, and in particular $\text{codim } Z > \dim X$ for such n . Consequently, image of Z under the map $X \times_k \mathbb{P}S^n V \times \dots \times \mathbb{P}S^n V \longrightarrow \mathbb{P}S^n V \times \dots \times \mathbb{P}S^n V$ is contained in a proper closed subscheme and there is, therefore, a rational point $(f_1, \dots, f_r)$ of $\mathbb{P}S^n V \times \dots \times \mathbb{P}S^n V$ not contained in the image of Z . Checking the definition of Z , this means that the closed subscheme of $X \times \mathbb{P}\mathcal{E}$ defined by the equations $f_1, \dots, f_r$ is quasi-finite, hence finite, over X . By the local criterion of flatness, this closed subscheme must also be flat of degree n^r .

It remains to show $\text{codim } Z^r \geq n + 1$. Let us define closed subschemes $Z^i \subset (\mathbb{P}S^n W)^{\times i}$ on coordinates as

$$Z^i := \{(g_1, \dots, g_i) \mid \dim V(g_1, \dots, g_i) > r - i\}.$$

We show by induction on i that

$$\text{codim } Z^{i+1} \geq \alpha(n, i) := \min\left\{\binom{n+r-i}{n}, \binom{n+r}{n}\right\}.$$

For $i = 0$, the result is obvious. For $1 \leq i < r$, consider the projection

$$\pi : Z^{i+1} \longrightarrow (\mathbb{P}S^n W)^{\times i}.$$

By induction it follows that $\pi^{-1}Z^i$ has large enough codimension and it therefore suffices to show $Z^{i+1} - \pi^{-1}Z^i$ has codimension at least $\alpha(n, i)$. But let $(f_1, \dots, f_i)$ be a point in $(\mathbb{P}S^n W)^{\times i} - Z^i$. Then $Y = V(f_1, \dots, f_i)$ is a codimension i subvariety of $\mathbb{P}_k^r$. Suppose $Y_1, \dots, Y_m$ are the reduced components of Y . Then $\dim Y_j = r - i$, since Y is a complete intersection. By Lemma 4.6, we have that $H_{Y_j}(n) \geq \binom{n+r-i}{r-i}$. Observe that given f_{i+1} ,

the point $(f_1, \dots, f_{i+1})$ lies in Z^{i+1} if and only if f_{i+1} vanishes on some Y_j . In other words, the fiber of π over $(f_1, \dots, f_t)$ is the union of the m linear subvarieties of $\mathbb{P}S^n W$ defined by n^{th} graded pieces of the homogeneous prime ideals defining the subschemes $Y_1, \dots, Y_m$. Thus the fiber $\pi^{-1}(f_1, \dots, f_i)$ has codimension at least $\binom{n+r-i}{r-i}$. The claim and lemma follow. $\square$

Lemma 4.5 allows us to relate the γ - and $\mathbb{P}^1$ -filtrations. For we can apply the lemma repeatedly to show, more generally, that if $X = X_0, \dots, X_n = X'$ is a sequence of schemes such that $X_{i+1} = \mathbb{P}\mathcal{E}_i$, where $\mathcal{E}_i$ is a vector bundle on X_i , then there exists an f in $K_0(X, X')$ such that $\pi \circ f$ is the identity, where $\pi : X' \rightarrow X$ is the natural map. Now let y be an element of $F_\gamma^t K_0(X)$. Then there exists $\pi : X' \rightarrow X$ defined as before such that $\pi^*(y)$ is a sum of elements of the form $(u_1 - v_1) \cdots (u_t - v_t)$, where each u_i and v_i is the class of a line bundle on X' (see, for example, the proof of [7, Theorem V.3.9]). By Proposition 4.2, the element $\pi^*(y)$ lies in $F_{\mathbb{P}^1}^t K_0(X')$, since it is a sum of t -fold products of elements lying in $F_{\mathbb{P}^1}^1 K_0(X)$. But then a diagram chase in the commutative diagram

$$= \begin{pmatrix} K_0(X, \mathbb{P}^{\times t}) & \longrightarrow & K_0(X) \\ \downarrow \pi^* & & \downarrow \pi^* \\ K_0(X', \mathbb{P}^{\times t}) & \longrightarrow & K_0(X') \\ \downarrow f^* & & \downarrow f^* \\ K_0(X, \mathbb{P}^{\times t}) & \longrightarrow & K_0(X) \end{pmatrix} =$$

shows that y is in $F_{\mathbb{P}^1}^t K_0(X)$.

Theorem 4.7. *Let X be a quasi-projective variety over an infinite field. Then*

$$F_\gamma^t K_0(X) \subseteq F_{\mathbb{P}^1}^t K_0(X),$$

for all t . Moreover, if in addition X is regular, we have that

$$F_\gamma^t K_0(X) \otimes \mathbb{Q} = F_{\mathbb{P}^1}^t K_0(X) \otimes \mathbb{Q}.$$

Proof. The first claim is the content of the preceding remarks. The second claim follows from Lemma 4.4 and the fact that the gamma and topological filtrations agree rationally when X is regular [7, Proposition VI.5.5]. $\square$

The following result will be needed later.

Proposition 4.8. *If X is a quasi-projective variety over an infinite field, then*

$$K_0(X)/F_{\mathbb{P}^1}^2 K_0(X) \cong H^0(X, \mathbb{Z}) \oplus \text{Pic}(X).$$

Proof. Let F^t denote $F_{\mathbb{P}^1}^t K_0(X)$ temporarily. By Theorem 4.7 and the fact that $K_0(X)/F_{\gamma}^2 K_0(X) \cong H^0(X, \mathbb{Z}) \oplus \text{Pic}(X)$, there is a natural surjection

$$H^0(X, \mathbb{Z}) \oplus \text{Pic}(X) \twoheadrightarrow K_0(X)/F^2.$$

Consider the natural map $(\text{rk}, \det) : K_0(X) \rightarrow H^0(X, \mathbb{Z}) \oplus \text{Pic}(X)$. We claim the map induced by composition $K_0(X, \mathbb{P}^{\times 2}) \rightarrow H^0(X, \mathbb{Z}) \oplus \text{Pic}(X)$ is zero. One may easily check that the map to the first summand is zero.

Let Z be an arbitrary closed subscheme of $X \times \mathbb{P}^{\times 2}$ such that the induced map $\pi : Z \rightarrow X$ is finite. It suffices to show the composite $K_0(Z/X) \rightarrow K_0(X) \rightarrow \text{Pic}(X)$ is zero. The first map in this composition sends a generator $[\mathcal{F}]$ to

$$[\pi_* \mathcal{F}] - [\pi_*(\mathcal{F} \otimes \mathcal{L}_1^{-1})] - [\pi_*(\mathcal{F} \otimes \mathcal{L}_2^{-1})] + [\pi_*(\mathcal{F} \otimes \mathcal{L}_1^{-1} \otimes \mathcal{L}_2^{-1})],$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are the pullbacks under the two induced maps from Z to $\mathbb{P}^1$ of the line bundle $\mathcal{O}(1)$. Therefore, the map in question sends a generator $[\mathcal{F}]$ to

$$\frac{\det(\pi_* \mathcal{F}) \bullet \det(\pi_*(\mathcal{F} \otimes \mathcal{L}_1^{-1} \otimes \mathcal{L}_2^{-1}))}{\det(\pi_*(\mathcal{F} \otimes \mathcal{L}_1^{-1})) \bullet \det(\pi_*(\mathcal{F} \otimes \mathcal{L}_2^{-1}))}.$$

We recall some constructions which are developed more systematically in [26]. Let $S^k(Z/X)$ denote the k^{th} symmetric power of Z relative to X ; that is, $S^k(Z/X) := (Z \times_X \cdots \times_X Z)/\Sigma_k$. Then there is a mapping

$$\det : \text{ob } \mathcal{P}(Z/X) \rightarrow \Pi_k \text{ob } \mathcal{P}(S^k(Z/X)/X)$$

that takes an object $\mathcal{F}$ of $\mathcal{P}(Z/X)$ to its n^{th} exterior power with respect to X , where n is the rank of $\pi_* \mathcal{F}$ as an $\mathcal{O}_X$ -module. That is, we may regard $\mathcal{F}$ as a locally free coherent sheaf of $\mathcal{O}_X$ -modules equipped with the added structure of a map $\mathcal{O}_Z \otimes_{\mathcal{O}_X} \mathcal{F} \rightarrow \mathcal{F}$. (We are equating $\mathcal{O}_Z$ with the coherent sheaf of $\mathcal{O}_X$ -algebras defining $Z \rightarrow X$.) Then the n^{th} exterior power of $\mathcal{F}$ with respect to $\mathcal{O}_X$ admits a map

$$(\mathcal{O}_Z \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathcal{O}_Z)^{\Sigma_n} \otimes_{\mathcal{O}_X} \Lambda_{\mathcal{O}_X}^n \mathcal{F} \rightarrow \Lambda_{\mathcal{O}_X}^n \mathcal{F},$$

where $(\mathcal{O}_Z \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathcal{O}_Z)^{\Sigma_n}$ is the coherent $\mathcal{O}_X$ -algebra defined by taking Σ_n invariants and defines the scheme $S^n(Z/X)$. We thus obtain an object of $\mathcal{P}(S^n(Z/X)/X)$ which we write simply as $\det \mathcal{F}$. In fact, the determinant map lands in the subclass of $\text{ob } \mathcal{P}(S^n(Z/X)/X)$ consisting of objects whose pushforwards to X are line bundles. In particular, this means that the support of $\det \mathcal{F}$, which we write as W , maps isomorphically to X .

Observe that $S^n(Z/X)$ admits two naturally defined maps to $S^n(\mathbb{P}^1) \cong \mathbb{P}^n$, arising from the two given maps from Z to $\mathbb{P}^1$. This induces two maps from W to $\mathbb{P}^n$. Let us write the pullback of $\mathcal{O}(1)$ to W along these two maps as $\mathcal{L}_1^{(n)}$ and $\mathcal{L}_2^{(n)}$. It is not difficult to verify that we have the formula

$$\det(\mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{L}_i) \cong \det(\mathcal{F}) \otimes_{\mathcal{O}_W} \mathcal{L}_i^{(n)}.$$

Furthermore, if $p : W \rightarrow X$ is the structure map (which, recall, is actually an isomorphism), we have the isomorphism

$$p_* \det(\mathcal{F}) \cong \det \pi_* \mathcal{F}$$

of line bundles on X .

Using the fact that p is an isomorphism, we obtain

$$\begin{aligned} & \det(\pi_* \mathcal{F}) \otimes \det(\pi_*(\mathcal{F} \otimes \mathcal{L}_1^{-1} \otimes \mathcal{L}_2^{-1})) \\ & \cong p_*(\det \mathcal{F}) \otimes p_* \left(\det(\mathcal{F}) \otimes \mathcal{L}_1^{(n)-1} \otimes \mathcal{L}_2^{(n)-1} \right) \\ & \cong p_*(\det \mathcal{F}) \otimes p_*(\det \mathcal{F}) \otimes p_*(\mathcal{L}_1^{(n)-1}) \otimes p_*(\mathcal{L}_2^{(n)-1}) \\ & \cong \det \pi_*(\mathcal{F} \otimes \mathcal{L}_1^{-1}) \otimes \det \pi_*(\mathcal{F} \otimes \mathcal{L}_2^{-1}). \end{aligned}$$

This shows that the map $K_0(Z/X) \rightarrow \text{Pic}(X)$ is zero.

We therefore have an induced surjection $K_0(X)/F^2 K_0(X) \rightarrow H(X, \mathbb{Z}) \oplus \text{Pic}(X)$, which is clearly inverse to the surjective map going the opposite direction. $\square$

Corollary 4.9. *If R is a regular ring containing an infinite field, then*

$$SK_1 R(x, y) = 0.$$

Proof. We know from Proposition 3.10 that

$$K_0(R)/F^2 \rightarrow \pi_0 W^{0/1}(R) \oplus \pi_0 W^{1/2}(R) \cong H^0(X, \mathbb{Z}) \oplus \text{Pic}(R) \oplus SK_1(R(x, y))$$

is a surjection, since R is regular. $\square$

As mentioned previously, the generalization of this corollary to rings which are not regular turns out to be the crucial fact needed to prove the main theorem of this paper. This will be shown in section 5.

Though not needed in this paper, the following result might be of interest.

Proposition 4.10. *If X is a smooth variety over an algebraically closed field k , then*

$$F_{\mathbb{P}^1}^t K_0(X) = F_{\text{top}}^t K_0(X),$$

for all t .

Proof. We may assume X is connected. We already know $F_{\mathbb{P}^1}^t K_0(X) \subset F_{\text{top}}^t K_0(X)$. Let Z be a closed, integral subscheme of X of codimension at least t . It suffices to show $[\mathcal{O}_Z] \in F_{\mathbb{P}^1}^t K_0(X)$ and in fact, by Noetherian induction, it suffices to show the class of $[\mathcal{O}_Z]$ in the quotient $K_0(X)/F_{\text{top}}^{t+1} K_0(X)$ lies in the image of the composition

$$K_0(X, \mathbb{P}^{\times t}) \longrightarrow K_0(X) \longrightarrow K_0(X)/F_{\text{top}}^{t+1} K_0(X).$$

Following the work of Voevodsky and Suslin, we define $C_0(X, \mathbb{P}^{\times t})$ to be the free abelian group of closed, integral subschemes W of $X \times \mathbb{P}^{\times t}$ such that the map $W \longrightarrow X$ is finite and surjective. Observe that there are natural maps

$$C_0(X, \mathbb{P}^{\times t}) \longrightarrow \text{CH}^t(X \times \mathbb{P}^{\times t}) \longrightarrow \text{CH}^t(X).$$

The first map sends a generator W to its class in the codimension t Chow group and the second map is given by pulling back along a rational point of $\mathbb{P}^{\times t}$.

We now wish to establish that the map

$$C_0(X, \mathbb{P}^{\times t}) \longrightarrow \text{CH}^t(X)$$

is a surjection. This is essentially a consequence of Suslin's result [15, Theorem 2.1], but we need a modification of it due to Friedlander and Voevodsky. Specifically, appealing to [20, Proposition 4.2.3 and Theorem 4.2.9] and [6, Theorem 1.8], we conclude that π_0 of the simplicial abelian group $C_0(X \times \Delta^\bullet, \mathbb{P}^{\times t})$ is isomorphic to $\text{CH}^t(X \times \mathbb{P}^{\times t})$ under the evident map. (It is here that the field must be assumed algebraically closed, for at the present moment resolution of singularities is needed to establish some of the results

in [6].) Pulling back along a rational point of $\mathbb{P}^{\times t}$ induces a surjection onto the summand $\mathrm{CH}^t(X)$ of $\mathrm{CH}^t(X \times \mathbb{P}^{\times t})$. In particular, the evident map $C_0(X, \mathbb{P}^{\times t}) \rightarrow \mathrm{CH}^t(X)$ is a surjection.

Furthermore, using [10, Corollaire 2 to Théorème 2.12] we see that there is a commutative diagram

$$\begin{array}{ccccc} C_0(X, \mathbb{P}^{\times t}) & \longrightarrow & \mathrm{CH}^t(X \times \mathbb{P}^{\times t}) & \longrightarrow & \mathrm{CH}^t(X) \\ \downarrow & & \downarrow & & \downarrow \\ K'_0(X, \mathbb{P}^{\times t}) & \longrightarrow & K'_0(X \times \mathbb{P}^{\times t})/F_{\mathrm{top}}^{t+1} & \longrightarrow & K'_0(X)/F_{\mathrm{top}}^{t+1}, \end{array} \quad (13)$$

in which the vertical maps are each defined by associating to a closed, integral subscheme the class of its structure sheaf in the appropriate Grothendieck group. The arrows along the bottom row of (13) are defined analogously to the maps of K_0 -groups. This shows

$$F_{\mathrm{top}}^t K_0(X) \subset \mathrm{im}(K'_0(X, \mathbb{P}^{\times t}) \rightarrow K_0(X)/F_{\mathrm{top}}^{t+1} K_0(X)).$$

Finally, we know from [26, Lemma 2.1] that $K_0(X, \mathbb{P}^{\times t}) \cong K'_0(X, \mathbb{P}^{\times t})$ under the evident map, from which the result follows. $\square$

The t^{th} stage of the $\mathbb{P}^1$ -filtration of $K_0(X)$ vanishes precisely when the map

$$K_0(X, \mathbb{P}^{\times t}) \rightarrow K_0(X) \quad (14)$$

is zero. Theorem 4.7 can therefore be interpreted as giving a new vanishing criterion for the γ -filtration. We conclude this section by exploring the consequences of this new criterion, and ultimately establish a vanishing result for the cubical proschemes $X^{[\wedge t]}$.

To better understand when the map (14) is zero, let us set up some notation. Given a scheme X , a splitting of the map $X \times \mathbb{P}^{\times t} \rightarrow X$ – that is, an X -point of $X \times \mathbb{P}^{\times t}$ – is specified by giving t maps from X to $\mathbb{P}^1$. Each such map is specified by a line bundle $\mathcal{L}$ on X together with a surjection $\mathcal{O}_X \oplus \mathcal{O}_X \rightarrow \mathcal{L}$. Given an X -point of $X \times \mathbb{P}^{\times t}$, if $\mathcal{L}_1, \dots, \mathcal{L}_t$ are the t line bundles on X associated to the t maps from X to $\mathbb{P}^1$, let us say that the X -point is of type $(\mathcal{L}_1, \dots, \mathcal{L}_t)$. If each $\mathcal{L}_i$ is in fact the trivial line bundle, let us call the X -point *simple*.

Recall from Lemma 4.3 that the map $K_0(X, \mathbb{P}^{\times t}) \longrightarrow K_0(X)$ factors through the map $K_0(X \times \mathbb{P}^{\times t}) \longrightarrow K_0(X)$ defined by pullback along any simple X -point. Therefore, one situation in which the map (14) vanishes is when the scheme X has the following property:

P_t := any subscheme of $X \times \mathbb{P}^{\times t}$ which is finite over X has empty intersection with a simple X -point of $X \times \mathbb{P}^{\times t}$.

Observe that a scheme with property P_t has property P_s for all $s \geq t$, since the projection $\mathbb{P}^{\times s} \longrightarrow \mathbb{P}^{\times t}$ onto the first t factors sends a subscheme of $X \times \mathbb{P}^{\times s}$ which is finite over X to a subscheme of $X \times \mathbb{P}^{\times t}$ which is also finite over X .

The following proposition follows straight from the definitions.

Proposition 4.11. *An affine scheme $\text{Spec } R$ has property P_1 if and only if R satisfies the primitive criterion.*

As a consequence of this proposition, we obtain another proof that $K_0(R) = H^0(\text{Spec } R, \mathbb{Z})$ if R satisfies the primitive criterion.

Proposition 4.12. *Assume X is a quasi-projective variety of dimension at most $t - 1$ defined over an infinite field. Then X has property P_t .*

Proof. If Z is a closed subscheme of $X \times \mathbb{P}^{\times t}$ which is finite over X , then Z also has dimension at most $t - 1$. But then the closure of the image of Z under the map $X \times \mathbb{P}^{\times t} \longrightarrow \mathbb{P}^{\times t}$ has dimension at most $t - 1$ and thus its complement contains a rational point of $\mathbb{P}^{\times t}$. This rational point defines the desired simple X -point. $\square$

As a consequence of this proposition, one can conclude that the t^{th} stage of the γ -filtration vanishes for X of dimension at most $t - 1$. Of course, this is already known [7, Corollary 3.10].

In order to obtain new vanishing results for the γ -filtration, we wish to generalize the following observation. Recall that for $X = \text{Spec } R$, the scheme $X^{[1]}$ is equal to $\text{Spec } R(x)$, and that $R(x)$ satisfies the primitive criterion; i.e., the scheme $X^{[\wedge 1]} = X^{[1]}$ satisfies P_1 . To generalize this, we consider the filtered projective system $X^{[t]}$ and the related cube of filtered projective systems of schemes $X^{[\wedge t]}$. We wish to prove $X^{[\wedge t]}$ satisfies property P_t , suitably interpreted. Recall that $K_0(X^{[t]})$ is the inverse limit of the system of abelian

groups obtained by applying K_0 degreeewise, and that $K_0(X^{[t]})$ has a γ filtration defined analogously. It can be shown that the cube of abelian groups splits in a suitable sense so that $K_0(X^{[\wedge t]})$ is a direct summand of $K_0(X^{[t]})$. But in any case, it will suffice for our purposes to just define $K_0(X^{[\wedge t]})$ to be the cokernel of the map

$$\bigoplus_{I=\{1,\dots,t\}-\{i\}} K_0(X^{[I]}) \longrightarrow K_0(X^{[t]}).$$

This is a map of $K_0(X^{[t]})$ -modules and thus $K_0(X^{[\wedge t]})$ is a quotient ring of $K_0(X^{[t]})$. We define the γ -filtration of $K_0(X^{[\wedge t]})$ to be the filtration induced by this surjective map from the γ -filtration of $K_0(X^{[t]})$.

On the scheme $X \times \mathbb{P}^{\times t}$, let $\mathcal{L}_i$ denote the line bundle induced from the natural very ample bundle $\mathcal{O}(1)$ on the i^{th} copy of $\mathbb{P}^1$. The precise statement we need is the following.

Theorem 4.13. *Let X be a quasi-projective variety over an infinite field k . Let Z be an arbitrary closed subset of $X \times \mathbb{P}^{\times t}$ which is finite over X . Let Y denote the scheme $X \times \mathbb{P}^{\times t} - Z$. Suppose we are given a closed subscheme W of $Y \times \mathbb{P}^{\times t}$ which is finite over Y . Then, after possibly passing to a larger Z (still finite over X), there exists a Y -point of $Y \times \mathbb{P}^{\times t}$ of type $(\mathcal{L}_1^{n_1}, \dots, \mathcal{L}_t^{n_t})$, for some integers n_i , which is disjoint from W . Consequently, the t^{th} stage of the γ -filtration of $K_0(X^{[\wedge t]})$ vanishes.*

Proof. Once we have proven the first claim, the second claim follows. For let $g : Y \longrightarrow Y \times \mathbb{P}^{\times t}$ be the splitting given by the first claim and let $i : Y \longrightarrow Y \times \mathbb{P}^{\times t}$ be a splitting obtained by choosing a rational point of $\mathbb{P}^{\times t}$. Each splitting produces a map $K_0(Y \times \mathbb{P}^{\times t}) \longrightarrow K_0(X^{[\wedge t]})$ by composition. Let us write these maps as θ_g and θ_i . Every element of the t^{th} stage of the $\mathbb{P}^1$ -filtration of $K_0(X^{[\wedge t]})$ arises from a class in $K_0(Y \times \mathbb{P}^{\times t})$ supported on a subscheme W , for some W and Y such as we are considering. The map θ_g sends such a class to zero and it therefore suffices to show $\theta_g = \theta_i$. We have

$$K_0(Y \times \mathbb{P}^{\times t}) \cong \bigoplus_{(e_1, \dots, e_t) \in \{0,1\}^t} K_0(Y) \cdot [\mathcal{E}_1^{e_1} \otimes \dots \otimes \mathcal{E}_t^{e_t}],$$

where $\mathcal{E}_i$ is the line bundle induced from the line bundle $\mathcal{O}(1)$ on the i^{th} copy of $\mathbb{P}^1$. Given a γ in $K_0(Y \times \mathbb{P}^{\times t})$, the elements $\theta_g(\gamma)$ and $\theta_i(\gamma)$ differ by a factor of $[\mathcal{L}_1^{n_1} \otimes \dots \otimes \mathcal{L}_t^{n_t}]$, for some integers n_i . But under the ring map

$K_0(X \times \mathbb{P}^{\times t}) \longrightarrow K_0(X^{[\wedge t]})$, the element $[\mathcal{L}_i]$ is sent to 1, since $1 - [\mathcal{L}_i]$ is represented by the closed subscheme $X \times \mathbb{P}^{\times i-1} \times \{1\} \times \mathbb{P}^{\times t-i}$.

It remains to check the first claim. We may assume $t > 1$. Another way of phrasing this claim is to say that, given Z and W as in the statement, there is splitting $i : Y \longrightarrow Y \times \mathbb{P}^{\times t}$ (of the appropriate type) such that $i(Y) \cap W$ is finite over X . For given this, we can enlarge Z by replacing it with $Z \cup i^{-1}(W)$.

In fact, letting $\overline{W}$ denote the closure of W in $X \times \mathbb{P}^{\times t} \times \mathbb{P}^{\times t}$, we will establish the slightly stronger statement that a suitable splitting $i : X \times \mathbb{P}^{\times t} \longrightarrow X \times \mathbb{P}^{\times t} \times \mathbb{P}^{\times t}$ has the property that $i^{-1}(\overline{W}) \longrightarrow X$ is finite. We can describe the kind of splitting that will do the job. Let $i_N : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be the map sending the point $[x : y]$ to $[x^N : y^N]$, where N is any positive integer. We claim that the desired splitting is the map

$$\text{id}_X \times i_{N_1, \dots, N_t} : X \times \mathbb{P}^{\times t} \longrightarrow X \times \mathbb{P}^{\times t} \times \mathbb{P}^{\times t}$$

defined by

$$(x, p_1, \dots, p_t) \mapsto (x, p_1, \dots, p_t, i_{N_1}(p_1), \dots, i_{N_t}(p_t))$$

for suitable integers N_i . If the required finiteness result holds, the lemma will be proven since this splitting has type $(\mathcal{L}^{N_1}, \dots, \mathcal{L}^{N_t})$.

Since our goal is to show a certain proper map is finite, it suffices to work locally on each member of a covering of X by finitely many open subschemes and show for N_i chosen sufficiently large, the desired property is satisfied. We may therefore assume $X = \text{Spec } R$ is affine. Let the i^{th} copy of $\mathbb{P}^1$ in $X \times \mathbb{P}^{\times t}$ have homogeneous coordinate functions S_i, T_i and let the i^{th} copy of $\mathbb{P}^1$ in $Y \times \mathbb{P}^{\times t}$ have homogeneous coordinate functions U_i, V_i .

We know from [26, page 6] that W is contained in $\pi_1^{-1}E_1 \cap \dots \pi_t^{-1}E_t$, where $E_i \subset Y \times \mathbb{P}^1$ is finite over Y and π_i is projection onto the i^{th} factor of $Y \times \mathbb{P}^{\times t}$. Let D_i denote the closure of E_i in $X \times \mathbb{P}^{\times t} \times \mathbb{P}^1$. We might as well assume $W = \pi_1^{-1}E_1 \cap \dots \cap \pi_n^{-1}E_n$ so that $\overline{W} = \pi_1^{-1}D_1 \cap \dots \cap \pi_n^{-1}D_n$. Checking the definitions, we see that we need to show that for $N_1, \dots, N_t$ sufficiently large, we have

$$i_{N_1}^{-1}(D_1) \cap \dots \cap i_{N_t}^{-1}(D_t) \longrightarrow X$$

is finite. The following lemma provides the crucial step.

Lemma 4.14. *Let $Q \longrightarrow \mathbb{P}_R^{\times t}$ be a finite, surjective map and suppose we are given a closed subscheme D of $Q \times \mathbb{P}^1$ which has finite fibers over points of $Q - Z$, where Z is a closed subscheme of Q which is finite over $\text{Spec } R$ under the natural map. Then for N sufficiently large, we have $i_N^{-1}(D) \longrightarrow \mathbb{P}_R^{\times t-1}$ is finite, where we define $i_N : Q \longrightarrow Q \times \mathbb{P}^1$ to be the map induced by the map $\mathbb{P}_R^{\times t} \longrightarrow \mathbb{P}_R^{\times t} \times_R \mathbb{P}_R^1$ sending $(p_1, \dots, p_t)$ to $(p_1, \dots, p_t, i_N(p_{t+1}))$.*

As an immediate consequence of Lemma 4.14, if we are given t subschemes $D_1, \dots, D_t$ of $Q \times \mathbb{P}^1$ which each have finite fibers over points of Q except for those points in Q in a closed subscheme which is itself finite over $\text{Spec } R$, then for $N_1, \dots, N_t$ sufficiently large, we have $i_{N_1}^{-1}(D_1) \cap \dots \cap i_{N_t}^{-1}(D_t) \longrightarrow \text{Spec } R$ is finite. For indeed we may apply Lemma 4.14 to show that for N_1 sufficiently large, $i_{N_1}^{-1}(D_1) \longrightarrow \mathbb{P}_R^{\times t-1}$ is finite, and then we may proceed by induction after pulling back $D_2, \dots, D_t$ to $i_{N_1}^{-1}(D_1) \times \mathbb{P}^1$. In particular, taking $Q = \mathbb{P}_R^{\times t}$ and the D_i as before, we have proven Theorem 4.13.

It remains to prove Lemma 4.14. Let f be the given finite, surjective map $Q \longrightarrow \mathbb{P}_R^{\times t}$. The map $i_N^{-1}(D) \longrightarrow \mathbb{P}_R^{\times t-1}$ is finite provided the map $i_N^{-1}(f(D)) \longrightarrow \mathbb{P}_R^{\times t-1}$ is finite. In other words, we may as well assume $Q = \mathbb{P}_R^{\times t}$.

If $t = 1$, then D is a closed subscheme of $\mathbb{P}_R^{\times 2}$. Let y be a point of $\text{Spec } R$ corresponding to some prime ideal p . The fiber of D over y cannot be all of $\mathbb{P}_y^{\times 2}$. For otherwise, would could pick a $x \in \mathbb{P}_R^1$ lying over y and not belonging to Z . The fiber of D over such an x could not be finite, contrary to our assumptions. Therefore, there must be a bihomogeneous polynomial $G[S, T; U, V]$ which vanishes on D and which does not belong to $p[S, T; U, V]$. By quasi-compactness, there are polynomials $G_0, \dots, G_t$ such that for any p , there exists an i such that G_i is not in $p[S, T; U, V]$. In fact, we can choose all the G_i 's to have the same bidegree. Therefore, a suitable element of the form

$$G(S, T; U, V) = G_0 U^{Mm} + G_1 U^{M(m-1)} V^M + \dots + G_m V^{Mm}$$

has unit content and vanishes on D . Finally, observe that $G(S, T; U^N, V^N)$ also has unit content for N sufficiently large. Since the subscheme cut out by G contains D , it follows $i_N^{-1}(D)$ is finite over $\text{Spec } R$, as desired.

Now assume $t > 1$. Once again by [26, page 6] we know that $D \cap ((\mathbb{P}_R^{\times t} - Z) \times \mathbb{P}^1)$ is contained in a closed subscheme having the same properties and which is defined by a homogeneous polynomial F of degree m in $\Gamma(\mathbb{P}_R^{\times t} -$

$Z, \mathcal{L}^{\otimes n}[U, V]$, where $\mathcal{L}$ is an ample line bundle on $\mathbb{P}_R^{\times t} - Z$ and m and n are sufficiently large. Observe Z itself is contained in the intersection of t divisors of the form $\pi_i^{-1}(E_i)$, with $E_i \subset \mathbb{P}_R^1$. In particular, any point z of Z has the property that its associated local ring in $\mathbb{P}_R^{\times t}$ has depth at least t . Therefore, taking $\mathcal{L}$ to be the restriction of the ample line bundle $\mathcal{O}(1, \dots, 1)$ on $\mathbb{P}_R^{\times t}$, we know from [9, 21.13.3 and 21.13.4] that

$$\Gamma(\mathbb{P}_R^{\times t} - Z, \mathcal{L}^{\otimes n}) = \Gamma(\mathbb{P}_R^{\times t}, \mathcal{O}(n, \dots, n)).$$

(Recall that $t > 1$.) Thus, $F = F(S_1, T_1, \dots, S_t, T_t, U, V)$ is an element of $\Gamma(\mathbb{P}_R^{\times t} \times \mathbb{P}^1, \mathcal{O}(n, \dots, n, m))$. We may as well assume D is the subscheme defined by F . We need to show that $G_N := F(S_1, T_1, \dots, S_t, T_t, S_t^N, T_t^N)$ defines a closed subscheme of $\mathbb{P}_R^{\times t}$ which is finite over $\mathbb{P}_R^{\times t-1}$ under projection onto the first $t-1$ factors, when N is sufficiently large.

We claim $N > n$ does the trick. For otherwise, there would be a map $R \rightarrow k$, with k a field, and elements $\alpha_i, \beta_i \in k$ for $i = 1, \dots, n-1$ so that $G_N(\alpha_1, \beta_1, \dots, \alpha_{t-1}, \beta_{t-1}, S_t, T_t) = 0$ in $k[S_t, T_t]$. But then since $N > n$ it would follow that $F(\alpha_1, \beta_1, \dots, \alpha_{t-1}, \beta_{t-1}, S_t, T_t, U, V) = 0$. In other words, there would be a point x of $\mathbb{P}_R^{\times t-1}$ so that D contains the subscheme $\{x\} \times \mathbb{P}^1 \times \mathbb{P}^1$. The fibers of D over points of $\{x\} \times \mathbb{P}^1 \subset \mathbb{P}_R^{\times t}$ would all have dimension one, which would entail Z to contain $\{x\} \times \mathbb{P}^1$. But this cannot happen since Z is finite over $\text{Spec } R$. This concludes the proof of Lemma 4.14 and thus Theorem 4.13 has been established. $\square$

5 The space $W^{1/2}(R)$

In this final section we prove the theorem announced in the introduction.

Theorem 5.1. *If $X = \text{Spec } R$ is a smooth affine variety over an infinite field k , then*

$$\pi_n W^{1/2}(X) = \begin{cases} \text{Pic } X & \text{if } n = 0, \\ R^\times & \text{if } n = 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Our calculations in Proposition 3.10 show that it suffices to establish the equation

$$SK_1(R\partial\Delta^{n+1}(x, y)) = 0,$$

for all n . We show, in fact, that $SK_1(A(x, y)) = 0$ for any ring A finitely generated over k . For any scheme Z , let $U(Z)$ denote the global sections of the sheaf of units $\mathcal{O}^*$ on Z and for simplicity we define $U(A) := U(\operatorname{Spec} A)$ for any ring A .

Recall that the proscheme $X^{[2]}$ admits an open covering by two copies of $X^{[1]} \times \mathbb{P}^1$, which are actually schemes, and the intersection of these two open subsystem is the (scheme) $\operatorname{Spec} R(x, y)$. We can there form a long exact sequence of Zariski cohomology groups with coefficients in $\mathcal{O}^*$, where we define, in general, the cohomology groups of a proscheme to be the direct limit of the cohomology groups of the schemes comprising the system. Using the facts that $U(Y \times \mathbb{P}^1) \cong U(Y)$ and $\operatorname{Pic}(Y \times \mathbb{P}^1) \cong \operatorname{Pic}(Y) \oplus \mathbb{Z}$, this long exact sequence becomes

$$\begin{aligned} 0 \longrightarrow U(X^{[2]}) \longrightarrow U(R(x)) \oplus U(R(y)) \longrightarrow U(R(x, y)) \longrightarrow \\ \operatorname{Pic}(X^{[2]}) \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0, \end{aligned}$$

for which exactness on the right is a consequence of the fact that the line bundles $\mathcal{O}(1, 0)$ and $\mathcal{O}(0, 1)$ on $\operatorname{Spec} R \times \mathbb{P}^{\times 2}$ restrict the trivial bundles on $\operatorname{Spec} R(x, y)$.

By the naturality of the Mayer-Vietoris sequences for low degree K -groups and the Zariski cohomology of $\mathcal{O}^*$, we have a diagram

$$\begin{array}{ccc} K_1(\operatorname{Spec} A^{[2]}) & \longrightarrow & U(\operatorname{Spec} A^{[2]}) \\ \downarrow & & \downarrow \\ K_1(\operatorname{Spec} A(x) \times \mathbb{P}^1) \oplus K_1(\operatorname{Spec} A(y) \times \mathbb{P}^1) & \longrightarrow & U(A(x)) \oplus U(A(y)) \\ \downarrow & & \downarrow \\ K_1(A(x, y)) & \longrightarrow & U(A(x, y)) \\ \downarrow & & \downarrow \\ \tilde{K}_0(\operatorname{Spec} A^{[2]}) & \longrightarrow & \operatorname{Pic}(\operatorname{Spec} A^{[2]}) \\ \downarrow & & \downarrow \\ \tilde{K}_0(\operatorname{Spec} A(x) \times \mathbb{P}^1) \oplus \tilde{K}_0(\operatorname{Spec} A(y) \times \mathbb{P}^1) & \xrightarrow{\cong} & \mathbb{Z} \oplus \mathbb{Z}, \end{array} \quad (15)$$

whose columns are exact. Given a regular, closed immersion $Z \hookrightarrow X$, the composition $K_1(Z) \longrightarrow K_1(X) \longrightarrow U(X)$ is the zero map. This enables us

to modify diagram (15) by modding out the cubical degeneracies in the K_1 groups to obtain the diagram

$$\begin{array}{ccc}
K_1(\mathrm{Spec} A^{[\wedge 2]}) & \longrightarrow & U(\mathrm{Spec} A^{[2]}) \\
\downarrow & & \downarrow \\
K_1(A(x)) \oplus K_1(A(y)) & \xrightarrow{\cong} & U(A(x)) \oplus U(A(y)) \\
\downarrow & & \downarrow \\
K_1(A(x, y)) & \longrightarrow & U(A(x, y)) \\
\downarrow & & \downarrow \\
\tilde{K}_0(\mathrm{Spec} A^{[2]}) & \longrightarrow & \mathrm{Pic}(\mathrm{Spec} A^{[2]}) \\
\downarrow & & \downarrow \\
\tilde{K}_0(\mathrm{Spec} A(x) \times \mathbb{P}^1) \oplus \tilde{K}_0(\mathrm{Spec} A(y) \times \mathbb{P}^1) & \xrightarrow{\cong} & \mathbb{Z} \oplus \mathbb{Z}.
\end{array} \tag{16}$$

The map

$$\tilde{K}_0(\mathrm{Spec} A^{[2]}) \longrightarrow \tilde{K}_0(\mathrm{Spec} A(x) \times \mathbb{P}^1) \oplus \tilde{K}_0(\mathrm{Spec} A(y) \times \mathbb{P}^1) \cong \mathbb{Z} \oplus \mathbb{Z}$$

is a splitting for the injective map

$$\mathbb{Z} \oplus \mathbb{Z} \cong K_0(\mathrm{Spec} A(y) \times \{1\}) \oplus K_0(\mathrm{Spec} A(x) \times \{1\}) \longrightarrow \tilde{K}_0(\mathrm{Spec} A^{[2]}).$$

Since the second stage of the γ filtration on $K_0(\mathrm{Spec} A^{[\wedge 2]})$ vanishes, the image of the split injection

$$K_0(\mathrm{Spec} A(y) \times \{1\}) \oplus K_0(\mathrm{Spec} A(x) \times \{1\}) \longrightarrow \tilde{K}_0(\mathrm{Spec} A^{[2]})$$

contains the second stage of the γ -filtration of $K_0(\mathrm{Spec} A^{[2]})$, which is the kernel of the map

$$\tilde{K}_0(\mathrm{Spec} A^{[2]}) \longrightarrow \mathrm{Pic}(\mathrm{Spec} A^{[2]}).$$

Using the fact that $K_1(\mathrm{Spec} A^{[\wedge 2]}) \longrightarrow U(\mathrm{Spec} A^{[\wedge 2]})$ is a surjection, a diagram chase shows that the map $K_1(A(x, y)) \longrightarrow U(A(x, y))$ must be an isomorphism. □

Remark 5.2. *It is shown in [23] that $U(A(x, y))/U(A(x)) \cdot U(A(y)) \cong \mathrm{Pic} A$, for any reduced ring. Using the proof of the Theorem 5.1, we see that $K_0(\mathrm{Spec} A^{[\wedge 2]}) \cong H^0(\mathrm{Spec} A, \mathbb{Z}) \oplus \mathrm{Pic}(A)$, for any reduced ring A which is finitely generated over an infinite field.*

References

- [1] D. W. Anderson. Relationship among K -theories. In *Algebraic K-theory I*, volume 341. Springer-Verlag, 1972.
- [2] Specer Bloch. Algebraic cycles and higher K -theory. *Advances in Mathematics*, 61:267–304, 1986.
- [3] Spencer Bloch and Steven Lichtenbaum. A spectral sequence for motivic cohomology. Preprint.
- [4] A. K. Bousfield and D. M. Kan. *Homotopy Limits, Completions and Localizations*, volume 304 of *Lecture Notes in Mathematics*. Springer-Verlag, 1972.
- [5] David Eisenbud. *Commutative Algebra with a View toward Algebraic Geometry*, volume 150 of *Lecture Notes in Mathematics*. Springer-Verlag, 1995.
- [6] Eric Friedlander and Vladimir Voevodsky. Bivariant cycle cohomology. Preprint.
- [7] William Fulton and Serge Lang. *Riemann-Roch Algebra*. Springer-Verlag, 1985.
- [8] Daniel R. Grayson. Weight filtrations in algebraic K -theory. In *Motives*, volume 55 of *Proceedings of Symposia in Pure Mathematics*, pages 207–244. American Mathematical Society, 1994.
- [9] A. Grothendieck. *Éléments de Géométrie Algébrique IV: Étude Locale des Schémas et des Morphismes de Schémas*, volume 20, 24, 28 and 32 of *Publications Mathématiques*. I. H. E. S., 1967.
- [10] Alexander Grothendieck. Classes de faisceaux et théorème de riemann-roch. In P. Berthelot, A. Grothendieck, and L. Illusie, editors, *SGA6: Théories des Intersections et Théorème de Riemann-Roch*, volume 225. Springer-Verlag, Berlin, Heidelberg, New York, 1971.
- [11] Steven E. Landsburg. Some filtrations on higher K -theory and related invariants. *K-Theory*, 6:431–455, 1992.

- [12] Stephen Lichtenbaum. Motivic complexes. In *Motives*, volume 55 of *Proceedings of Symposia in Pure Mathematics*, pages 303–313. American Mathematical Society, 1994.
- [13] B. R. MacDonald and William C. Waterhouse. Projective modules over rings with many units. *Proc. Amer. Math. Soc.*, 83(3):455–458, 1981.
- [14] John Rognes and Chuck Weibel. Two-primary algebraic K -theory of rings of integers in number fields. Preprint, August 1997.
- [15] A. Suslin. Higher chow groups and étale cohomology. Preprint.
- [16] A. Suslin and V. Voevodsky. Bloch-Kato conjecture and motivic cohomology with finite coefficients. Preprint, April 1995.
- [17] A. Suslin and V. Voevodsky. Singular homology of abstract algebraic varieties. *Invent. Math.*, 123:61–94, 1996.
- [18] R. W. Thomason and T. Trobaugh. Higher algebraic K -theory of schemes and of derived categories. In *The Grothendieck Festschrift, Volume III*, volume 88 of *Progress in Math.*, pages 247–436. Birkhäuser, Boston, Basel, Berlin, 1990.
- [19] Wilberd van der Kallen. The K_2 of rings with many units. *Ann. Sci. École Norm. Sup.*, 10:473–515, 1977.
- [20] V. Voevodsky. Triangulated categories of motives over a field. Preprint.
- [21] V. Voevodsky. The Milnor conjecture. Preprint, December 1996.
- [22] Vladimir Voevodsky. Cohomological theory of presheaves with transfers. Preprint.
- [23] Mark E. Walker. The primitive topology of a scheme. To appear in the *Journal of Algebra*.
- [24] Mark E. Walker. Weight zero motivic cohomology and the general linear group of simplicial rings. Preprint.
- [25] Mark E. Walker. *Motivic Complexes and the K -theory of Automorphisms*. PhD thesis, University of Illinois, Urbana–Champaign, 1996.

- [26] Mark E. Walker. Adams operations for bivariant K -theory and a filtration using projective lines. Preprint, 1997.
- [27] C. Weibel. Homotopy algebraic K -theory. In *Algebraic K-theory and Number Theory*, volume 83 of *Contemporary Mathematics*, pages 461–488. American Mathematical Society, 1989.