# SEMI-STABLE CONJECTURE FOR VERTICAL LOG-SMOOTH FAMILIES

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## 1. Introduction

The purpose of this paper is to generalize our results from [16] to certain log-smooth families. More precisely, we would like to show that, for odd primes, the Semi-stable Conjecture of Jannsen and Fontaine [6] (proved before by Tsuji [20] and Faltings [4]) is true for projective vertical fine and saturated log-smooth families with reduction of Cartier type. We will derive it from Thomason's comparison theorem [18] between algebraic and étale K-theories.

Recall the formulation of this conjecture. Let $K$ be a complete discrete valuation field of mixed characteristic $(0,p)$ with ring of integers $V$ and a perfect residue field $k$. Let $X^{\times}$ be a fine log-smooth projective $V^{\times}$-scheme, where $V$ is equipped with the log-structure associated to the closed point, such that the generic fiber $X_K$ is smooth over $K$ and the special fiber $X_0^{\times}$ is of Cartier type. Then the conjecture postulates an existence of a natural period isomorphism

$$\alpha : H^*(X_{\overline{K}},\mathbf{Q}_p)\otimes_{\mathbf{Q}_p} B_{\mathrm{st}} \simeq H^*_{\mathrm{cr}}(X_0^{\times}/W(k)^0)\otimes_{W(k)} B_{\mathrm{st}},$$

where $\overline{K}$ is an algebraic closure of $K$, $W(k)^0$ is the ring $W(k)$ equipped with the log-structure associated to $(\mathbf{N}\to W(k), 1\mapsto 0)$, and $B_{\mathrm{st}}$ is a certain ring of periods introduced by Fontaine [6]. The ring $B_{\mathrm{st}}$ is equipped with Galois action, Frobenius and monodromy operators. The log-crystalline cohomology groups $H^*_{\mathrm{cr}}(X^{\times}/W(k)^0)[1/p]$ are also equipped with Frobenius and monodromy operators, and the period isomorphism is expected to preserve these structures. Moreover, the ring $B_{\mathrm{st}}$ maps naturally into another ring of periods $B_{dR}$, which is equipped with a decreasing filtration. There is also a canonical isomorphism of $K\otimes_{W(k)} H^*_{\mathrm{cr}}(X^{\times}/W(k)^0)$ with the de Rham cohomology groups $H^*_{dR}(X_K/K)$ via which Hodge filtration induces a descending filtration on the log-crystalline groups. The base change of the period isomorphism to $B_{dR}$ is expected to yield an isomorphism on the filtrations. As a corollary, one gets that étale cohomology can be recovered from de Rham cohomology (with all the extra structures) and vice versa.

To prove the conjecture, by a standard argument (see, [7], [3]), it suffices to construct a map

$$\alpha : H^*(X_{\overline{K}},\mathbf{Q}_p) \longrightarrow H^*_{\mathrm{cr}}(X_0^{\times}/W(k)^0)\otimes_{W(k)} B_{\mathrm{st}} \tag{1.1}$$

compatible with all the above structures, and, in addition, with Poincaré duality and the trace map.

Our idea was to construct the map $\alpha$ by passing via the higher K-theory groups. In the first step of the construction one passes from the right hand side of (1.1) to a cohomology group of the scheme $X^{\times}_{\overline{V}}$, where $\overline{V}$ is the integral closure of $V$ in $\overline{K}$. Namely, by an argument of Kato [13],

$$\begin{aligned}&\mathbf{Q}\otimes\operatorname{proj}\lim_{n} H^{*}_{\mathrm{cr}}(X^{\times}_{\overline{V},n}/W_n(k))\\ &\qquad\simeq \ker(\mathrm{monodromy}: H^{*}_{\mathrm{cr}}(X^{\times}_{0}/W(k)^0)\otimes B^{+}_{\mathrm{st}}\to H^{*}_{\mathrm{cr}}(X^{\times}_{0}/W(k)^0)\otimes B^{+}_{\mathrm{st}}),\end{aligned}$$

where $B^{+}_{\mathrm{st}}$ is a subring of $B_{\mathrm{st}}$. Hence, we need to construct a well-behaved map

$$\alpha^{n}_{ai}: H^{a}(X_{\overline{K}}, \mathbf{Z}/p^n(i)) \to H^{a}_{\mathrm{cr}}(X^{\times}_{\overline{V},n}/W_n(k), \mathcal{O}_{X^{\times}_{\overline{V},n}/W_n(k)}(i)),$$

at least for large enough $i$. Here, the twist in the crystalline cohomology refers to twisting the filtration and the Frobenius.

Our construction is based on the following diagram

$$\begin{tikzcd}
``F^{i}_{\gamma}/F^{i+1}_{\gamma}K_j(X_{\overline{V}};\mathbf{Z}/p^n)" \arrow[r, "\sim", "j^{*}"'] \arrow[d, "\overline{c}^{\mathrm{cr}}_{ij}"] & F^{i}_{\gamma}/F^{i+1}_{\gamma}K_j(X_{\overline{K}};\mathbf{Z}/p^n) \arrow[d, "\overline{c}^{\text{ét}}_{ij}"] \\
H^{2i-j}_{\mathrm{cr}}(X^{\times}_{\overline{V},n}/W_n(k), \mathcal{O}_{X^{\times}_{\overline{V},n}/W_n(k)}(i)) & H^{2i-j}(X_{\overline{K}}, \mathbf{Z}/p^n(i)), \arrow[l, "\alpha^{n}_{2i-j,i}"']
\end{tikzcd}$$

where $K_j(\cdot;\mathbf{Z}/p^n)$ is the K-theory with coefficients and $F^{i}_{\gamma}K_j(\cdot;\mathbf{Z}/p^n)$ is the $\gamma$-filtration. The term in the left upper corner stands, loosely speaking, for a limit of K-theory groups of (global) resolutions of the schemes $X^{\times}_{V'}$, for $V'$ a finite extension of $V$. The maps

$$\begin{aligned}\overline{c}^{\text{ét}}_{ij} &: K_j(X_{\overline{K}};\mathbf{Z}/p^n)\to H^{2i-j}(X_{\overline{K}}, \mathbf{Z}/p^n(i)),\\ \overline{c}^{\mathrm{cr}}_{ij} &: K_j(X_{\overline{V}};\mathbf{Z}/p^n)\to H^{2i-j}_{\mathrm{cr}}(X^{\times}_{\overline{V},n}/W_n(k), \mathcal{O}_{X^{\times}_{\overline{V},n}/W_n(k)}(i))\end{aligned}$$

are the étale [8] and the log-crystalline Chern class maps respectively. The log-crystalline Chern class map is the composition of the standard crystalline Chern class map $\overline{c}^{\mathrm{cr}}_{ij}$ : $K_j(X_{\overline{V}};\mathbf{Z}/p^n)\to H^{2i-j}_{\mathrm{cr}}(X_{\overline{V},n}/W_n(k), \mathcal{O}_{X_{\overline{V},n}/W_n(k)}(i))$ with the natural map

$$H^{2i-j}_{\mathrm{cr}}(X_{\overline{V},n}/W_n(k), \mathcal{O}_{X_{\overline{V},n}/W_n(k)})\to H^{2i-j}_{\mathrm{cr}}(X^{\times}_{\overline{V},n}/W_n(k), \mathcal{O}_{X_{\overline{V},n}/W_n(k)}).$$

A priori, because of the nature of the term "$F^{i}_{\gamma}/F^{i+1}_{\gamma}K_j(X_{\overline{V}};\mathbf{Z}/p^n)$" its domain is really the K-theory of the resolutions of the schemes $X^{\times}_{V'}$ and it lands in the log-crystalline cohomology groups of these resolutions. We show though (Corollary 2.1) that the cohomology of these resolutions is isomorphic to the cohomology of the original schemes $X^{\times}_{V'}$.

The arrow $\omega_{ij}$ is to be constructed below. Our map $\alpha^{n}_{2i-j,i}$ will make the above diagram commute. First, we prove that the restriction $j^{*}$ is an isomorphism. Again, it is in a very loose sense, but good enough to allow us to lift K-theory classes from the generic fiber to the model. Namely, by [16, Lemma 3.1], we know that we can lift these classes to $K'$-theory of the schemes $X^{\times}_{V'}$. Then, we show that there is a well-behaved map from the $K'$-theory of the schemes $X^{\times}_{V'}$ to the $K'$-theory of their resolutions (since the resolutions are regular, it is just K-theory). Our construction of this map hinges on the fact that the resolution maps are log-étale hence have the unique lifting property over exact nilpotent immersions. This gives us the required lifting. Finally, we show that this lifting is unambiguous modulo a field extension and a change of resolution (Lemma 3.1).

Next, using the fact that [16], for large $j$, the étale Chern class map $\overline{c}_{ij}^{\text{ét}}$ is surjective, modulo a constant depending only on the dimension of $X_K$ and $i, j$, and the elements in its kernel are annihilated, modulo the same constant, by a power of the Bott element $\beta_n \in K_2(X_{\overline{K}}; \mathbf{Z}/p^n)$, we construct the dotted arrow $\omega_{ij}$ in the above diagram: a map defined only modulo powers of the Bott element $\beta_n$ and the above constant, which makes the diagram commute. We set $\alpha^n_{2i-j,i} = \overline{c}_{ij}^{\text{cr}}\,\omega_{ij}$. Twisting the maps $\alpha^n_{2i-j,i}$ by some power of the Bott element and a constant, we get a well-defined map. We take the projective limit, over $n$, of these maps and after tensoring it with $\mathbf{Q}$ and appropriately untwisting, we get our map $\alpha_{2i-j,i}$.

Our construction of the map $\alpha_{2i-j,i}$ makes it now very easy to check its compatibility with Poincaré duality and trace maps.

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Throughout the paper, let $p$ be a fixed odd prime, let $\overline{K}$ denote a chosen algebraic closure of a field $K$, and, for a scheme $X$, let $X_n = X \otimes \mathbf{Z}/p^n$.

## 2. Preliminaries

Let $V$ be a complete discrete valuation ring with fraction field $K$ of characteristic 0 and with perfect residue field $k$ of characteristic $p$. Let $W(k)$ be the ring of Witt vectors with coefficients in $k$ with fraction field $K_0$. Set $G_K = \mathrm{Gal}(\overline{K}/K)$, and let $\sigma$ be the absolute Frobenius on $W(\overline{k})$. For a $V$-scheme $X$, let $X_0$ denote the special fiber of $X$. We will denote by $V$, $V^\times$, and $V^0$ the scheme $\mathrm{Spec}(V)$ with the trivial, canonical (i.e., associated to the closed point), and $(\mathbf{N} \to V, 1 \mapsto 0)$ log-structure respectively, and, for a log-scheme $X^\times$, we will denote by $X$ the underlying scheme.

2.1. **The rings of periods.** Let's recall the definitions of the rings $B_{\text{cr}}$, $B_{dR}$, $B_{\text{st}}$ of Fontaine [5], [6], [7]. We have

$$B^+_{\text{cr},n} = H^0_{\text{cr}}(\mathrm{Spec}(\overline{V}_n/W_n(k)), \quad B^+_{\text{cr}} = \operatorname{proj\,lim}_n B^+_{\text{cr},n}[1/p], \quad B_{\text{cr}} = B^+_{\text{cr}}[t^{-1}],$$

where $\overline{V}$ is the integral closure of $V$ in $\overline{K}$ and $t$ is a certain element of $B^+_{\text{cr}}$ (see [5] for a precise definition of $t$). The ring $B^+_{\text{cr}}$ is a topological $K_0$-module equipped with a Frobenius $\phi$ coming from the crystalline cohomology and a natural $G_K$-action. We have that $\phi(t) = pt$ and that $G_K$ acts on $t$ via the cyclotomic character.

The canonical morphism $B^+_{\text{cr},n} \to \overline{V}/p^n$ is surjective. Let $J_{\text{cr},n}$ denote its kernel. Let

$$B^+_{dR} = \operatorname{proj\,lim}_r (\mathbf{Q} \otimes \operatorname{proj\,lim}_n B^+_{\text{cr},n}/J^{[r]}_{\text{cr},n}), \quad B_{dR} = B^+_{dR}[t^{-1}].$$

The ring $B^+_{dR}$ has a discrete valuation given by powers of $t$. Its quotient field is $B_{dR}$. We will denote by $F^n B_{dR}$ the filtration induced on $B_{dR}$ by powers of $t$.

Let $\pi$ be a uniformizer of $V$. Fontaine associates to it an element $u_\pi$ of $B^+_{dR}$. Let $B^+_{\text{st}}$ denote the subring of $B_{dR}$ generated by $B^+_{\text{cr}}$ and $u_\pi$. Fontaine shows that $u_\pi$ is transcendental over $B^+_{\text{cr}}$. Hence $B^+_{\text{st}}$ is a polynomial algebra in one variable over $B^+_{\text{cr}}$. This construction *does* depend on the choice of $\pi$. The action of $G_K$ on $B^+_{dR}$ restricts well to $B^+_{\text{st}}$. Frobenius $\phi$ extends to $B^+_{\text{st}}$ by $\phi(u_\pi) = pu_\pi$ and one defines a monodromy operator $N : B^+_{\text{st}} \to B^+_{\text{st}}$ as a unique $B^+_{\text{cr}}$-derivation such that $Nu_\pi = -1$. We have $N\phi = p\phi N$.

Let $B_{\mathrm{st}} = B_{\mathrm{cr}}[u_\pi]$. Different choices of the uniformizer $\pi$ yield isomorphic rings $B^+_{\mathrm{st}}$, so we can and we will identify them via these isomorphisms. The dependence on $\pi$ will then be encoded in the morphism $\iota_\pi : B^+_{\mathrm{st}} \hookrightarrow B^+_{dR}$.

We will need the following crystalline interpretation of the ring $B^+_{\mathrm{st}}$. Let $R^\times_n$ denote the PD-envelope of the ring $W_n[x]$ with respect to the closed immersion $W_n[x] \to V_n$, $x \to \pi$, equipped with the log-structure associated to $\mathbf{N} \to R_n$, $1 \to x$. Let

$$\widehat{B}^+_{\mathrm{st}} = \operatorname{proj}\lim_n H^0_{\mathrm{cr}}(\mathrm{Spec}(\mathcal{O}_{\overline{V},n})/R^\times_n)[1/p].$$

The ring $\widehat{B}^+_{\mathrm{st}}$ has a natural action of $G_K$, Frobenius $\phi$, and a monodromy operator $N$. Kato [13, 3.7] shows that the ring $B^+_{\mathrm{st}}$ is canonically (and compatibly with all the structures) isomorphic to the elements of $\widehat{B}^+_{\mathrm{st}}$ annihilated by a power of the monodromy operator $N$.

More generally, for any fine log-scheme $X^\times$, which is log-smooth and proper over $V^\times$, and whose special fiber is of Cartier type, set

$$H^i_{\mathrm{cr}}(X^\times_{\overline{V}}/W(k)) := \operatorname{proj}\lim_n H^i_{\mathrm{cr}}(X^\times_{\overline{V},n}/W_n(k)), \qquad H^i_{\mathrm{cr}}(X^\times) := \mathbf{Q} \otimes H^i_{\mathrm{cr}}(X^\times_0/W(k)^0).$$

Then Kato defines [13, 4.2] a canonical morphism

$$h_\pi : \mathbf{Q} \otimes H^i_{\mathrm{cr}}(X^\times_{\overline{V}}/W(k)) \longrightarrow \widehat{B}^+_{\mathrm{st}} \otimes_{K_0} H^i_{\mathrm{cr}}(X^\times)$$

and computes [13, 4.5] that we have morphisms

$$\begin{aligned} \mathbf{Q}\otimes H^i_{\mathrm{cr}}(X^\times_{\overline{V}}/W(k)) &\xrightarrow[h_\pi]{} \ker(N : \widehat{B}^+_{\mathrm{st}} \otimes_{K_0} H^i_{\mathrm{cr}}(X^\times) \longrightarrow \widehat{B}^+_{\mathrm{st}} \otimes_{K_0} H^i_{\mathrm{cr}}(X^\times)) \\ &\xleftarrow{\sim} \ker(N : B^+_{\mathrm{st}} \otimes_{K_0} H^i_{\mathrm{cr}}(X^\times) \longrightarrow B^+_{\mathrm{st}} \otimes_{K_0} H^i_{\mathrm{cr}}(X^\times)). \end{aligned}$$

It can be checked (cf. [20, 4.5.6-7]) that these morphisms are compatible with the product structure, Galois action, and the Frobenius.

Moreover, Hyodo and Kato [11, 5.1] have constructed a canonical $K$-isomorphism

$$\rho_\pi : K \otimes_{K_0} H^i_{\mathrm{cr}}(X^\times) \xrightarrow{\sim} H^i_{dR}(X^\times_K/K),$$

which is compatible with products [20, 4.4.13]. Hence the composition

$$\rho_\pi h_\pi : \mathbf{Q} \otimes H^i_{\mathrm{cr}}(X^\times_{\overline{V}}/W(k)) \longrightarrow B^+_{\mathrm{st}} \otimes_{K_0} H^i_{dR}(X^\times_K/K)$$

is functorial in $X^\times$ and compatible with products, Galois action, and the Frobenius and the monodromy operators.

Let $X^\times$ be any fine log-scheme, which is log-smooth and proper over $V^\times$ with saturated log-structure on the generic fiber. Then we have the following isomorphisms

$$B^+_{dR} \otimes_{K_0} H^i_{dR}(X^\times_K/K) \xrightarrow{\sim} \operatorname{proj}\lim_s (\mathbf{Q} \otimes H^i_{\mathrm{cr}}(X^\times_{\overline{V}}/V^\times, \mathcal{O}/J^{[s]})) \quad [20, 4.7.6], \tag{2.1}$$

$$F^r(B^+_{dR} \otimes_{K_0} H^i_{dR}(X^\times_K/K)) \xrightarrow{\sim} \operatorname{proj}\lim_{s \geq r} (\mathbf{Q} \otimes H^i_{\mathrm{cr}}(X^\times_{\overline{V}}/V^\times, J^{[r]}/J^{[s]})) \quad [20, 4.7.13]. \tag{2.2}$$

2.2. **Syntomic regulators.** Let $f_1 : T_1 \to T_2$ and $f_2 : T_1 \to T_2$ be two morphisms of topoi. Consider the induced topos $(T_1, T_2)$ with sheaves given by tuples $\mathcal{F} = (\mathcal{F}_{T_1}, \mathcal{F}_{T_2}, \omega_1, \omega_2)$, where $\mathcal{F}_{T_1}$, $\mathcal{F}_{T_2}$ are sheaves on $T_1$ and $T_2$ respectively, and $\omega_i$ is a morphism $f_i^*\mathcal{F}_{T_2} \to \mathcal{F}_{T_1}$, $i = 1, 2$, and morphisms given by pairs of maps $\mathcal{F}_{T_i} \to \mathcal{G}_{T_i}$ compatible with the maps $\omega_i$. For a complex of sheaves of abelian groups $\mathcal{F}^{\cdot}$ on $(T_1, T_2)$, we have

$$\Gamma(T_1, T_2, \mathcal{F}^{\cdot}) = \ker(\Gamma(T_2, \mathcal{F}^{\cdot}_{T_2}) \xrightarrow{\omega_1 - \omega_2} \Gamma(T_1, \mathcal{F}^{\cdot}_{T_1})).$$

It is a left exact functor and its right derived functor $R\Gamma(T_1, T_2, \cdot)$ is also the right derived functor of the functor

$$\Gamma'(T_1, T_2, \mathcal{F}^{\cdot}) := \mathrm{Cone}(\Gamma(T_2, \mathcal{F}^{\cdot}_{T_2}) \xrightarrow{\omega_1 - \omega_2} \Gamma(T_1, \mathcal{F}^{\cdot}_{T_1}))[-1].$$

In particular, we have a distinguished triangle of functors

$$R\Gamma(T_1, T_2, \mathcal{F}^{\cdot}) \longrightarrow R\Gamma(T_2, \mathcal{F}^{\cdot}_{T_2}) \xrightarrow{\omega_1 - \omega_2} R\Gamma(T_1, \mathcal{F}^{\cdot}_{T_1}) \longrightarrow$$

For two complexes of sheaves $\mathcal{F}^{\cdot} = (\mathcal{F}^{\cdot}_{T_1}, \mathcal{F}^{\cdot}_{T_2}, \omega_{\mathcal{F}^{\cdot},1}, \omega_{\mathcal{F}^{\cdot},2})$ and $\mathcal{G}^{\cdot} = (\mathcal{G}^{\cdot}_{T_1}, \mathcal{G}^{\cdot}_{T_2}, \omega_{\mathcal{G}^{\cdot},1}, \omega_{\mathcal{G}^{\cdot},2})$ on $(T_1, T_2)$, we define the tensor product $\mathcal{F}^{\cdot} \otimes_f \mathcal{G}^{\cdot}$ to be the complex with

$$(\mathcal{F}^{\cdot} \otimes_f \mathcal{G}^{\cdot})_{T_2} := \mathcal{F}^{\cdot}_{T_2} \otimes \mathcal{G}^{\cdot}_{T_2},$$
$$(\mathcal{F}^{\cdot} \otimes_f \mathcal{G}^{\cdot})_{T_1} := \mathrm{Cone}((f_1^*\mathcal{F}^{\cdot}_{T_2} \oplus f_2^*\mathcal{F}^{\cdot}_{T_2}) \otimes \mathcal{G}^{\cdot}_{T_1} \oplus \mathcal{F}^{\cdot}_{T_1} \otimes (f_1^*\mathcal{G}^{\cdot}_{T_2} \oplus f_2^*\mathcal{G}^{\cdot}_{T_2}) \xrightarrow{\rho} \mathcal{F}^{\cdot}_{T_1} \otimes \mathcal{G}^{\cdot}_{T_1})[-1],$$

where $\rho = (\omega_{\mathcal{F}^{\cdot},1} + \omega_{\mathcal{F}^{\cdot},2}) \otimes \mathrm{Id} - \mathrm{Id} \otimes (\omega_{\mathcal{G}^{\cdot},1} + \omega_{\mathcal{G}^{\cdot},2})$. For the connecting morphisms

$$\omega_{\mathcal{F}^{\cdot} \otimes_f \mathcal{G}^{\cdot},i} : f_i^*(\mathcal{F}^{\cdot} \otimes_f \mathcal{G}^{\cdot})_{T_2} \longrightarrow (\mathcal{F}^{\cdot} \otimes_f \mathcal{G}^{\cdot})_{T_1},$$

we take $\omega_{\mathcal{F}^{\cdot} \otimes \mathcal{G}^{\cdot},i} = \omega_{\mathcal{F}^{\cdot},i} \otimes \mathrm{Id} + \mathrm{Id} \otimes \omega_{\mathcal{G}^{\cdot},i}$. There is a canonical map from $\mathcal{F}^{\cdot} \otimes^{\mathbf{L}} \mathcal{G}^{\cdot}$ to $\mathcal{F}^{\cdot} \otimes^{\mathbf{L}}_f \mathcal{G}^{\cdot}$ in the derived category $\mathbf{D}(T_1, T_2)$ of sheaves of abelian groups on $(T_1, T_2)$. Hence, for $\mathcal{F}^{\cdot}$, $\mathcal{G}^{\cdot}$ in $\mathbf{D}^b(T_1, T_2)$ such that $R\Gamma(T_1, T_2, \mathcal{F}^{\cdot})$, $R\Gamma(T_1, T_2, \mathcal{G}^{\cdot})$, and $\mathcal{F}^{\cdot} \otimes^{\mathbf{L}}_f \mathcal{G}^{\cdot}$ are in the corresponding derived categories of bounded complexes, the corresponding left derived functor gives a morphism $R\Gamma(T_1, T_2, \mathcal{F}^{\cdot}) \otimes^{\mathbf{L}} R\Gamma(T_1, T_2, \mathcal{G}^{\cdot}) \to R\Gamma(T_1, T_2, \mathcal{F}^{\cdot} \otimes^{\mathbf{L}}_f \mathcal{G}^{\cdot})$.

Let Ł be the category of integral log-schemes over $W(k)$. For $X^{\times} \in$ Ł, $n \geq 1$, define the topoi $T_1^n(X^{\times}) := (X_1^{\times}/W_n(k))_{\mathrm{cr}}$, $T_2^n(X^{\times}) := (X_n^{\times}/W_n(k))_{\mathrm{cr}}$, and the topos $T^n(X^{\times}) := (T_1^n(X^{\times}), T_2^n(X^{\times}))$ as above with the maps $f_1 := i$, $f_2 := i\phi$, where $\phi$ is the Frobenius on $(X_1^{\times}/W_n(k))_{\mathrm{cr}}$ and $i$ is the natural morphism $i : (X_1^{\times}/W_n(k))_{\mathrm{cr}} \to (X_n^{\times}/W_n(k))_{\mathrm{cr}}$. Define a complex $\mathcal{S}^{\cdot}_n(m)_X$ of sheaves on $T^n(X^{\times})$ by

$$\mathcal{S}^{\cdot}_n(m)_{T_1^n(X^{\times})} := \mathcal{O}_{X_1^{\times}/W_n(k)}, \qquad \mathcal{S}^{\cdot}_n(m)_{T_2^n(X^{\times})} := J^{[m]}_{X_n^{\times}/W_n(k)}.$$

The connecting morphism $\omega_1$ is given by the map $i^*J^{[m]}_{X_n^{\times}/W_n(k)} \xrightarrow{p^m} \mathcal{O}_{X_1^{\times}/W_n(k)}$. The other connecting morphism $\omega_2$ is given by the map $\phi^*i^*J^{[m]}_{X_n^{\times}/W_n(k)} \to \phi^*\mathcal{O}_{X_1^{\times}/W_n(k)} \xrightarrow{\mathrm{Id}} \mathcal{O}_{X_1^{\times}/W_n(k)}$.

We have a functorial in $X^{\times}$ pairing

$$\mathcal{S}^{\cdot}_n(m')_{X^{\times}} \otimes^{\mathbf{L}}_f \mathcal{S}^{\cdot}_n(m)_{X^{\times}} \longrightarrow \mathcal{S}^{\cdot}_n(m' + m)_{X^{\times}}.$$

It is defined by a family of maps

$$\cup_{\alpha} : \mathcal{S}^{\cdot}_n(m')_{X^{\times}} \otimes_f \mathcal{S}^{\cdot}_n(m)_{X^{\times}} \longrightarrow \mathcal{S}^{\cdot}_n(m' + m)_{X^{\times}},$$

for every $\alpha \in \mathbf{Z}_p$. To construct $\cup_\alpha$, we need to define two morphisms of complexes

$$\begin{aligned}\cup_{\alpha,T_2^n} :& J^{[m']}_{X_n^\times/W_n(k)} \otimes J^{[m]}_{X_n^\times/W_n(k)} \longrightarrow J^{[m'+m]}_{X_n^\times/W_n(k)},\\ \cup_{\alpha,T_1^n} :& \operatorname{Cone}\Big( (i^*J^{[m']}_{X_n^\times/W_n(k)} \oplus \phi^* i^* J^{[m']}_{X_n^\times/W_n(k)}) \otimes \mathcal{O}_{X_1^\times/W_n(k)}\\ &\qquad \oplus \mathcal{O}_{X_1^\times/W_n(k)} \otimes (i^*J^{[m]}_{X_n^\times/W_n(k)} \oplus \phi^* i^* J^{[m]}_{X_n^\times/W_n(k)})\\ &\qquad\qquad \xrightarrow{\rho} \mathcal{O}_{X_1^\times/W_n(k)} \otimes \mathcal{O}_{X_1^\times/W_n(k)}\Big)[-1] \longrightarrow \mathcal{O}_{X_1^\times/W_n(k)}\end{aligned}$$

compatible with the connecting morphisms. Set

$$\cup_{\alpha,T_2^n}(a_1 \otimes a_2) = a_1 a_2$$

and define $\cup_{\alpha,T_1^n}$ by the maps

$$\begin{aligned} i^*J^{[m']}_{X_n^\times/W_n(k)} \otimes \mathcal{O}_{X_1^\times/W_n(k)} &\longrightarrow \mathcal{O}_{X_1^\times/W_n(k)}, & x \otimes w &\longrightarrow (1-\alpha)p^{m'}xw\\ \phi^* i^*J^{[m']}_{X_n^\times/W_n(k)} \otimes \mathcal{O}_{X_1^\times/W_n(k)} &\longrightarrow \mathcal{O}_{X_1^\times/W_n(k)}, & x \otimes w &\longrightarrow \alpha\phi(x)w\\ \mathcal{O}_{X_1^\times/W_n(k)} \otimes i^*J^{[m]}_{X_n^\times/W_n(k)} &\longrightarrow \mathcal{O}_{X_1^\times/W_n(k)}, & x \otimes w &\longrightarrow \alpha p^m xw\\ \mathcal{O}_{X_1^\times/W_n(k)} \otimes \phi^* i^*J^{[m]}_{X_n^\times/W_n(k)} &\longrightarrow \mathcal{O}_{X_1^\times/W_n(k)}, & x \otimes w &\longrightarrow (1-\alpha)x\phi(w)\end{aligned}$$

Compatibility with the connecting morphism is easy to check. The maps $\cup_\alpha$ form a homotopic family, $\cup_0$ and $\cup_1$ are associative, and $\cup_\alpha$, $\cup_{1-\alpha}$ anticommute.

Let $\Pi$ be the category of topoi $T^n(X^\times)$ as above for $X^\times \in$ Ł, and let $\Pi^\sim$ be the category of sheaves in $\Pi$, i.e., collections of sheaves $\mathcal{F}_{T^n(X^\times)}$ for every $T^n(X^\times)$, and morphisms $f^*\mathcal{F}_{T^n(X^\times)} \to \mathcal{F}_{T^n(Y^\times)}$ for every $V$-morphism $f : Y^\times \to X^\times$ satisfying the usual compatibilities.

For any diagram $X^\times : I \to$ Ł denote by $T^n(X^\times)^\sim$ the category of sheaves in $T^n(X^\times)$. Denote by $\Gamma(T^n(X^\times), \cdot)$ the functor given by the composition

$$\Pi^\sim \xrightarrow{(\cdot)_{T^n(X^\times)}} T^n(X^\times)^\sim \xrightarrow{\Gamma} \mathrm{Sets}^I \xrightarrow{\operatorname{proj\,lim}} \mathrm{Sets}$$

The complexes $\mathcal{S}^{\cdot}_n(m)_{Y^\times}$, $Y^\times \in$ Ł, define a sheaf $\mathcal{S}^{\cdot}_n(m)$ in $\Pi$. Set

$$H_f(X^\times, \mathcal{O}_n(m)) := R\Gamma(T^n(X^\times, \mathcal{S}^{\cdot}_n(m)).$$

The above pairings define an anticommutative and associative product

$$H_f(X^\times, \mathcal{O}_n(m')) \otimes^{\mathbf{L}} H_f(X^\times, \mathcal{O}_n(m)) \to H_f(X^\times, \mathcal{O}_n(m'+m)).$$

We have a distinguished triangle of functors on Ł

$$H_f(X^\times, \mathcal{O}_n(m)) \longrightarrow H_{\mathrm{cr}}(X_n^\times/W_n(k), J^{[m]}_{X_n^\times/W_n(k)}) \xrightarrow{\beta} H_{\mathrm{cr}}(X_1^\times/W_n(k), \mathcal{O}_{X_1^\times/W_n(k)}) \longrightarrow$$

where $\beta(x,y) = (p^m x - \phi(x))$.

The functors $H_f$ are also cohomologies of certain complexes of sheaves in the Zariski topology. Consider the left-exact functor $\alpha_* : \Pi^\sim \to$ Ł$^\sim$, $\alpha_*(\mathcal{F})(X^\times) := \Gamma(T^n(X^\times), \mathcal{F})$. For $\mathcal{F} = ((\mathcal{F}_1, \mathcal{F}_2), \omega_1, \omega_2)$, we have a distinguished triangle

$$R\alpha_*(\mathcal{F})_{X^\times} \longrightarrow Ru_{X_n^\times/W_n(k)*}\mathcal{F}_2 \xrightarrow{\omega_1-\omega_2} Ru_{X_1^\times/W_n(k)*}\mathcal{F}_1 \longrightarrow,$$

where, for a $W_n(k)$-diagram $Y^\times$, $u_{Y^\times/W_n(k)} : (Y^\times/W_n(k))_{\mathrm{cr}} \to Y^\times_{\mathrm{Zar}}$ is the natural projection. Set

$$\mathcal{S}^{\cdot}_n(m)_{\mathrm{Zar}} := R\alpha_*\mathcal{S}^{\cdot}_n(m).$$

We get a canonical isomorphism $R\Gamma(X^\times, \mathcal{S}^{\cdot}_n(m)_{\mathrm{Zar}}) \simeq H_f(X^\times, \mathcal{O}_n(m))$, a distinguished triangle

$$\mathcal{S}^{\cdot}_n(m)_{\mathrm{Zar},X^\times} \longrightarrow Ru_{X^\times_n/W_n(k)*}J^{[m]}_{X^\times_n/W_n(k)} \xrightarrow{\beta} Ru_{X^\times_1/W_n(k)*}\mathcal{O}_{X^\times_n/W_n(k)} \longrightarrow,$$

and an anticommutative and associative multiplication

$$\mathcal{S}^{\cdot}_n(m')_{\mathrm{Zar}} \otimes^{\mathbf{L}} \mathcal{S}^{\cdot}_n(m)_{\mathrm{Zar}} \to \mathcal{S}^{\cdot}_n(m'+m)_{\mathrm{Zar}}$$

*Remark 1.* For any syntomic fine log-scheme $X^\times$ over $W(k)$, the cohomology $H^*_f(X^\times, \mathcal{O}_n(*))$ is isomorphic to the syntomic cohomology defined by Tsuji [20, 2.1].

For a scheme $X$, let $K(X)$ be the spectrum of Quillen K-theory. For a natural number $l \geq 2$, let $K/l(X)$ denote the corresponding spectrum mod $l$ [18, A.5]. Set $K_i(X) := \pi_i(K(X))$ and $K_i(X; \mathbf{Z}/l) := \pi_i(K/l(X))$.

Using Illusie's computation of the crystalline cohomology of $B.GL_m/W(k)$ [10, II], one can define universal classes $C_{i,m} \in H^{2i}_f(B.GL_m/W(k), \mathcal{O}_n(i))$. For any integral log-scheme $X^\times$ over $W(k)$, via the method of Gillet [8, 2.22], [16], they yield functorial and compatible families of Chern classes

$$\begin{aligned} c_{ij} &: K_j(X) \longrightarrow H^{2i-j}(X, \mathcal{S}^{\cdot}_n(i)_{\mathrm{Zar}}) \quad \text{for } j \geq 0,\\ \overline{c}_{ij} &: K_j(X; \mathbf{Z}/p^n) \longrightarrow H^{2i-j}(X, \mathcal{S}^{\cdot}_n(i)_{\mathrm{Zar}}) \quad \text{for } j \geq 2. \end{aligned}$$

These classes composed with the natural morphism $H^{2i-j}(X, \mathcal{S}^{\cdot}_n(i)_{\mathrm{Zar}}) \longrightarrow H^{2i-j}(X^\times, \mathcal{S}^{\cdot}_n(i)_{\mathrm{Zar}})$ give Chern classes

$$\begin{aligned} c^{\mathrm{syn}}_{ij} &: K_j(X) \longrightarrow H^{2i-j}_f(X^\times, \mathcal{O}_n(i)) \quad \text{for } j \geq 0,\\ \overline{c}^{\mathrm{syn}}_{ij} &: K_j(X; \mathbf{Z}/p^n) \longrightarrow H^{2i-j}_f(X^\times, \mathcal{O}_n(i)) \quad \text{for } j \geq 2. \end{aligned}$$

*Remark 2.* Interpreted appropriately, Gross' syntomic Chern classes [10] are equal to the above classes $c_{ij}$, $\overline{c}_{ij}$.

*Remark 3.* Much less general theory of syntomic cohomology than defined here would have sufficed for this paper. We will need though the general theory in the future to be able to work with syntomic cohomology of simplicial schemes.

**Lemma 2.1.** *The above Chern classes have the following properties*

- $c^{\mathrm{syn}}_{ij}$*, for* $j > 0$*, is a group homomorphism.*
- $\overline{c}^{\mathrm{syn}}_{ij}$*, for* $j \geq 2$ *and* $p \neq 2$*, is a group homomorphism.*
- *If* $\alpha \in K_l(X; \mathbf{Z}/p^n)$ *and* $\alpha' \in K_q(X; \mathbf{Z}/p^n)$*, then*

$$\overline{c}^{\mathrm{syn}}_{ij}(\alpha\alpha') = -\sum_{r+s=i} \frac{(i-1)!}{(r-1)!(s-1)!}\overline{c}^{\mathrm{syn}}_{rl}(\alpha)\overline{c}^{\mathrm{syn}}_{sq}(\alpha'),$$

*assuming that* $l, q \geq 2$, $l + q = j$, $2i \geq j$, $i \geq 0$, $p \neq 2$.

*Moreover, if the scheme* $X$ *is regular then*

- *If* $\alpha \in F^j_\gamma K_0(X)$, $j \neq 0$, *and* $\alpha' \in F^k_\gamma K_q(X; \mathbf{Z}/p^n)$, $q \geq 2$, $p \neq 2$, *then*

  $$\overline{c}^{\mathrm{syn}}_{j+k,q}(\alpha\alpha') = -\frac{(j+k-1)!}{(j-1)!(k-1)!} c^{\mathrm{syn}}_{j0}(\alpha)\overline{c}^{\mathrm{syn}}_{kq}(\alpha').$$

  *Similarly, for* $c^{\mathrm{syn}}_{j+k,q}(\alpha\alpha')$, *if* $\alpha' \in F^k_\gamma K_0(X)$, $k \neq 0$.
- *The integral Chern class maps* $c^{\mathrm{syn}}_{i0}$ *restrict to zero on* $F^{i+1}_\gamma K_0(X)$, $i \geq 1$.
- *For* $p \neq 2$, *the Chern class maps* $\overline{c}^{\mathrm{syn}}_{ij}$ *restrict to zero on* $F^{i+1}_\gamma K_j(X; \mathbf{Z}/p^n)$, $j \geq 2$.
- *The classes* $c_{ij}$ *and* $\overline{c}_{ij}$ *are compatible with the crystalline Chern classes*

  $$c^{\mathrm{cr}}_{ij} : K_j(X) \longrightarrow H^{2i-j}_{\mathrm{cr}}(X_n/W_n(k)), \qquad \overline{c}^{\mathrm{cr}}_{ij} : K_j(X; \mathbf{Z}/p^n) \longrightarrow H^{2i-j}_{\mathrm{cr}}(X_n/W_n(k)).$$

*Proof.* The proof is analogous to the proof of Lemma 2.1 in [16]. □

### 2.3. A pullback map in $K'$-theory.

We will construct here well-behaving maps from the $K'$ -theory of certain log-schemes to the $K'$ -theory of their resolutions. The resolution morphisms are rather ugly: they are composed of normalizations and blow-ups at not-necessarily regularly embedded centers. In particular, they will not, in general, have a finite Tor-dimension (examples are easy to generate using [1, III.3.4]). Since the maps of finite Tor-dimension form the most general class of maps (that I am aware of) through which we can pullback K'-theory classes [19, 3.14.1], we have to resort here to a very ad hoc construction. Troughout the rest of the paper, unless otherwise stated, we work in the category of fine and saturated log-schemes.

We call a morphism $g : U^\times \to Y^\times$ of log-schemes a *modification* if it is a sequence of (saturated) log-blow-ups [12, 1.6]. Hence $g$ is a universally surjective map that is log-étale and induces an identity on the open subset of triviality of $Y^\times$. Clearly modifications are stable under compositions; that they are also stable under base change was proved in [17, ?].

We say that a $V^\times$-log-scheme $Y^\times$ has a (generalized) semi-stable reduction if $Y$ is regular, the special fiber $Y_0$ is a divisor with normal crossings, and the log-structure on $Y^\times$ is defined by $Y_0$. Let $X^\times$ be a vertical (i.e., the open subset of triviality of log-structure of $X^\times$ is $X[1/p]$), $V^\times$-log-scheme. A morphism $f : U^\times \to X^\times$ is called a semi-stable model, if it is a modification and the log-scheme $U^\times$ has a semi-stable reduction.

**Lemma 2.2.** *Any log-scheme* $X^\times$ *that is vertical and log-smooth over* $V^\times$ *admits a semi-stable model.*

*Proof.* By Corollary 3.2 of [17], we can find a sequence of saturated log-blow-ups $f : U^\times \to X^\times$ from a regular scheme $U$, such that, for any $x \in U$, $(M/\mathcal{O}^*)_{U,x} \simeq \mathbf{N}^{r(x)}$, for some $r(x) \geq 0$, where $M$ is the sheaf of monoids on $U$. That $U$ has a semi-stable reduction was computed by Nakayama in [15, A2,2.4]. □

We will work now in the category of fine log-schemes. Assume that the residue field of $V$ is algebraically closed. Let $V'$ be a finite totally ramified extension of $V$. Let $X^\times$ be a vertical $V^\times$-log-scheme with the underlying scheme $X$ regular. Since the structure map $X^\times \to V^\times$ is integral, underlying scheme of $X^\times_{V'}$ is $X_{V'}$ (we take $X^\times_{V'}$ in the category of fine log-schemes). For any log-étale map $f : U^\times \longrightarrow X^\times_{V'}$ from a semi-stable $V'^\times$-log-scheme

$U^{\times}$, we will construct well-behaving homomorphisms

$$K'_j(X_{V'}) \xrightarrow{f^*} K_j(U); \quad K'_j(X_{V'};\mathbf{Z}/n) \xrightarrow{f^*} K_j(U;\mathbf{Z}/n).$$

We will describe the construction for the K-theory with no coefficients. The case with coefficients is similar.

Choose a uniformizer $\pi'$ of $V'$ and consider the associated exact immersion $i : X^{\heartsuit}_{V'} \hookrightarrow X[T]^{\times}, T \mapsto \pi'$. Here, $X[T]^{\times} = X^{\times} \times_V \operatorname{Spec}(V[T])^{\times}$, where $\operatorname{Spec}(V[T])^{\times}$ has log-structure induced by $\mathbf{N} \to V[T], 1 \mapsto T$. Since the morphism $\operatorname{Spec}(V[T])^{\times} \to V$ is integral, the underlying scheme of $X[T]^{\times}$ is $X[T] := X \times_{\operatorname{Spec}(V)} \operatorname{Spec}(V[T])$. In particular, it is regular. The log-scheme $X^{\heartsuit}_{V'}$ arises from the log-scheme $X^{\times}_{V'}$ by modifying the log-structure so as to make the closed immersion $i : X^{\times}_{V'} \hookrightarrow X[T]^{\times}, T \mapsto \pi'$ into an exact one. More explicitely, $V'^{\heartsuit} = V'^{\times} \times_V V^{\times}$ and $X^{\heartsuit}_{V'} = X^{\times} \times_{V^{\times}} V'^{\heartsuit} = X^{\times}_{V'} \times_V V^{\times} = X^{\times} \times_V V'^{\times}$. In particular, there is a natural morphism $V'^{\heartsuit} \to V'^{\times}$. As a base change of the integral morphism $V^{\times} \to V$, it is integral. Hence the natural morphisms $X^{\heartsuit}_{V'} \to X^{\times}_{V'}$ are integral as well.

Consider now the log-scheme $U^{\heartsuit} = U^{\times} \times_{X^{\times}_{V'}} X^{\heartsuit}_{V'} = U^{\times} \times_V V^{\times}$. Notice that the map $X^{\heartsuit}_{V'} \to X^{\times}_{V'}$ being integral, the underlying scheme of $U^{\heartsuit}$ is the same as the one of $U^{\times}$. Since the original map $f : U^{\times} \to X^{\times}_{V'}$ was log-étale, so is the induced map $f^{\heartsuit} : U^{\heartsuit} \to X^{\heartsuit}_{V'}$. Hence there exists a unique log-étale lifting $q : U^{\times}_T \to X[T]^{\times}_{X^{\heartsuit}_{V'}}$ of the scheme $U^{\heartsuit}$, where $X[T]^{\times}_{X^{\heartsuit}_{V'}}$ is the completion of the scheme $X[T]^{\times}$ along the closed subscheme $X^{\heartsuit}_{V'}$ (understood as the inductive limit of the exact closed immersions of successive neighborhoods of $X^{\heartsuit}_{V'}$ in $X[T]^{\times}$).

To construct $f^* : K'_j(X_{V'}) \to K_j(U)$, choose a finite affine Zariski covering $\mathcal{X} = \{X_i \hookrightarrow X | i \in I\}$ of $X$ and a finite affine Zariski covering $\mathcal{U} = \{U_j \hookrightarrow U | j \in J\}$ of $U$ refining the cover $f^{-1}(\mathcal{X}_{V'})$. For any set $L$ of $U_j \to U$ drawn from $\mathcal{U}$, and for the fiber product $U_L$ over $U$ of the elements of $L$, let $U_L = \operatorname{Spec}(B_L)$ and $U_{T,L} = \operatorname{Spf}(C_L)$. Consider the following maps of spectra

$$\begin{aligned}
f_1^* : K(X[T], X_{V'}) \to & \operatorname{holim}\Big(\prod_{i_0 \in I} K(X_{i_0}[T], X_{i_0 V'}) \to \prod_{(i_0,i_1)\in I^2} K(X_{i_0,i_1}[T], X_{i_0,i_1 V'}) \to\Big) \\
& \to \operatorname{holim}\Big(\prod_{j_0 \in J} \widetilde{K}(C_{j_0}, B_{j_0}) \to \prod_{(j_0,j_1)\in J^2} \widetilde{K}(C_{j_0,j_1}, B_{j_0,j_1}) \to\Big), \\
f_2^* : K(U) & \xrightarrow{\sim} \operatorname{holim}\Big(\prod_{j_0 \in J} K(B_{j_0}) \to \prod_{(j_0,j_1)\in J^2} K(B_{j_0,j_1}) \to\Big) \\
& \underset{i_*}{\xrightarrow{\sim}} \operatorname{holim}\Big(\prod_{j_0 \in J} \widetilde{K}(C_{j_0}, B_{j_0}) \to \prod_{(j_0,j_1)\in J^2} \widetilde{K}(C_{j_0,j_1}, B_{j_0,j_1}) \to\Big).
\end{aligned}$$

The next to the last map is a homotopy equivalence by [19, 8.3].

For the above definition to work we need a strictly functorial spectra $K(\,)$ (instead of functors up to homotopy). To achieve that, it suffices to rigidify the underlying category valued functors to make $f^*g^* = (gf)^*$ on $\mathcal{O}$-modules strictly (cf., [9, 5.1.2]).

For a closed immersion of noetherian schemes $Z \subset Y$, let $\mathcal{P}'(Y,Z)$ denote the complicial biWaldhausen category [19, 1.2.10] whose objects associate to every morphism $f : T \to Y$ from a noetherian scheme $T$ a bounded complex of locally free sheaves $\mathcal{F}^{\cdot}_f$ on $T$ acyclic on $T \setminus T_Z$ and to every morphism $g : (f_1 : T_1 \to Y) \to (f_2 : T_2 \to Y)$ an *isomorphism* $g^*(\mathcal{F}^{\cdot}_{f_2}) \xrightarrow{\sim} \mathcal{F}^{\cdot}_{f_1}$ satisfying the usual compatibility condition (note that, since $g^*$ is exact on elements of $\mathcal{P}(T_2, T_{2,Z})$ – the corresponding category on the small Zariski site of $T_2$, $g^*(\mathcal{F}^{\cdot}_{f_2}) \in \mathcal{P}(T_1, T_{1,Z})$).

Set $K(Y, Z)$ to be the K-theory spectrum of Thomason [19, 3.8.3] built from the category $\mathcal{P}'(Y, Z)$ (we choose the default setting here: cofibrations are the degree-wise split monomorphisms and weak equivalences are the quasi-isomorphisms). It is now clear that $K(Y, Z)$ is strictly functorial (on noetherian schemes). As the simple lemma below states this spectrum is homotopy equivalent to any of the standard models of $K(Y, Z)$ [19, 3.5,3.7].

**Lemma 2.3.** *The restriction functor* $r : \mathcal{P}'(Y, Z) \to \mathcal{P}(Y, Z)$ *is exact* [19, 1.2.3] *and induces an equivalence of categories. In particular, it yields a homotopy equivalence of K-theory spectra.*

*Proof.* Consider the functor $i : \mathcal{P}(Y, Z) \to \mathcal{P}'(Y, Z)$. Since the complexes in $\mathcal{P}(Y, Z)$ are bounded and built from sheaves acyclic for the pullbacks, this functor is well-defined and exact. We have that $ri = \mathrm{Id}$ and that $ir$ is equivalent to Id, hence the lemma. □

For the map $i_*$ in the above diagram to be defined at all we have to choose yet another set of models for the K-theory spectra. Recall [19, 3.18], that the cartesian diagram of maps of schemes on the left in the following yields the commutative, up to a canonical homotopy, diagram of spectra on the right assuming the immersions $i_Z$ and $i_{Z'}$ to be regular of codimension 1.

$$\begin{tikzcd}
Z'^{\times} \arrow[r, "i_{Z'}"] \arrow[d, "g"'] & Y'^{\times} \arrow[d, "h"'] & K(Z') \arrow[r, "i_{Z'*}"] & K(Y', Z') \\
Z^{\times} \arrow[r, "i_Z"] & Y^{\times} & K(Z) \arrow[r, "i_{Z*}"] \arrow[u, "g^*"] & K(Y, Z). \arrow[u, "h^*"']
\end{tikzcd} \tag{2.3}$$

Assume now that all the schemes in the above diagram have ample families of line bundles [19, 2.1.3]. In the case when $Y$ is regular, we would like the above diagram of spectra to commute on the nose. By the following lemma this would yield our map $i_*$ on holim's.

**Lemma 2.4.** *For every subset* $L \subset J$ *of the index set* $J$*, the ring* $C_L$ *is regular and the immersion* $i_{B_L} : \mathrm{Spec}(B_L) \hookrightarrow \mathrm{Spec}(C_L)$ *is a regular immersion of codimension* 1.

*Proof.* By assumption, étale locally $\mathrm{Spec}(B_L)$ has a generalized semi-stable reduction over $V'$, i.e., $\mathrm{Spec}(B_L)$ can be covered with étale opens $\mathrm{Spec}(B')$ that are strict étale over the fiber product $Z^{\times}$ of a diagram $V' \xrightarrow{1 \mapsto \pi'} \mathbf{Z}[\mathbf{N}] \xleftarrow{\mathbf{Z}h} \mathbf{Z}[\mathbf{N}^r]$ for some $r \geq 1$, where $h$ is an injection. Consider the fiber product $Z_1^{\times}$ of a diagram $V[T] \xrightarrow{1 \mapsto T} \mathbf{Z}[\mathbf{N}] \xleftarrow{\mathbf{Z}h} \mathbf{Z}[\mathbf{N}^r]$. The scheme $Z_1^{\times}$ lifts the scheme $Z^{\times}$ from $V'^{\times}$ to the log-scheme $\mathrm{Spec}(V[T])^{\times}$ via the exact immersion $V'^{\times} \hookrightarrow \mathrm{Spec}(V[T])^{\times}$, $T \mapsto \pi'$. Let $Z^{\heartsuit}$ and $Z_1^{\heartsuit}$ be the base change of the schemes $Z^{\times}$ and $Z_1^{\times}$ to $V[T]^{\times}$. Since this base change does not change the underlying schemes, both schemes are classically regular. Locally, we can now lift $\mathrm{Spec}(B')$ to an étale $Z_1$-scheme $\mathrm{Spec}(B_1)$. Since the completion of $\mathrm{Spec}(B_1)$ along $\mathrm{Spec}(B)$ admits a formally étale map to $\mathrm{Spf}(C_L)$, the above allows us to cover $\mathrm{Spec}(C_L)$ with flat regular schemes. Hence $C_L$ is regular, as wanted.

Also, the immersion $Z \hookrightarrow Z_1$ is clearly regular. Therefore, so is the immersion $\operatorname{Spec}(B_L) \hookrightarrow \operatorname{Spec}(B_1)$, and by descent the immersion $\operatorname{Spec}(B_L) \hookrightarrow \operatorname{Spec}(C_L)$, as claimed. □

Let $\mathcal{T}(Y,Z)$ be the category of morphisms $f : T \to Y$ from a noetherian scheme $T$ such that the immersion $T_Z \subset T$ is regular (note that $\mathcal{T}(Y,Z)$ includes open immersions). Let $\mathcal{S}(Y,Z)$ be the category of coherent sheaves on $Y$ such that $Lh^*\mathcal{F} = h^*\mathcal{F}$ for all maps in $\mathcal{T}(Y,Z)$.

To achieve the commutativity we want, take for $K(Y,Z)$ the K-theory spectrum of the complicial biWaldhausen category $\mathcal{M}'(Y,Z)$ with objects associating to every map $(f : T \to Y) \in \mathcal{T}(Y,Z)$ a bounded complex $\mathcal{F}^{\cdot}_f$ of coherent sheaves from the category $\mathcal{S}(Y,Z)$ acyclic on $T \setminus T_Z$ and to every morphism $g : (f_1 : T_1 \to Y) \to (f_2 : T_2 \to Y)$ an isomorphism $g^*(\mathcal{F}^{\cdot}_{f_2}) \xrightarrow{\sim} \mathcal{F}^{\cdot}_{f_1}$ satisfying the usual compatibility condition (as before, since $g^*$ is exact on complexes built from elements of $\mathcal{S}(T_2, T_{2,Z})$, $g^*(\mathcal{F}^{\cdot}_{f_2}) \in \mathcal{T}(T_1, T_{1,Z})$). To distinguish this spectrum from other models, we will denote it by $\widetilde{K}(Y,Z)$.

**Lemma 2.5.** *The restriction functor from $\mathcal{M}'(Y,Z)$ to $\mathcal{M}(Y,Z)$ is exact and an equivalence of categories. In particular, it induces a homotopy equivalence of the K-theory spectra.*

*Proof.* Identical to the proof of Lemma 2.3. □

**Lemma 2.6.**
1. *$\widetilde{K}(Y,Z)$ is a strict contravariant functor on the maps from $\mathcal{T}(Y,Z)$.*
2. *If $Y$ is regular, then the natural map*

$$K(Y,Z) \to \widetilde{K}(Y,Z)$$

*is a homotopy equivalence.*

3. *For the left diagram in (2.3), the following diagram of spectra strictly commutes*

$$\begin{tikzcd}
K(Z') \arrow[r, "i_{Z'*}"] & \widetilde{K}(Y',Z') \\
K(Z) \arrow[u, "g^*"] \arrow[r, "i_{Z*}"] & \widetilde{K}(Y,Z). \arrow[u, "h^*"']
\end{tikzcd}$$

*Here, the pushforwards $i_{Z*}$, $i_{Z'*}$ are induced by the corresponding pushforward functors on sheaves.*

*Proof.* The first statement is clear.

For the second, by Lemma 2.3 and 2.5, it suffices to work on the small Zariski site of $Y$. By [19, 1.9.8], it suffices now to show that every complex from $\mathcal{S}(Y,Z)$ is quasi-isomorphic to a bounded complex of locally free sheaves. Since $Y$ is regular, every bounded complex of coherent sheaves is perfect [19, 3.21], hence quasi-isomorphic to a bounded complex of locally free sheaves [19, 2.3.1.d].

For the third statement, notice that the pushforward $i_{Z*}$ is well-defined, i.e., that $i_{Z*}\mathcal{F}^{\cdot} \in \mathcal{M}'(Y,Z)$ for any $\mathcal{F}^{\cdot} \in \mathcal{P}(Z)$. First, let's check that the sheaves in the complex $(i_{Z*}\mathcal{F}^{\cdot})_g$ belong to $\mathcal{S}(T,T_Z)$ for any map $(g : T \to Y) \in \mathcal{T}(Y,Z)$. It suffices to check that $Lt^*\mathcal{E} = t^*\mathcal{E}$ for any vector bundle $\mathcal{E}$ on $T_Z$ and any map $(t : W \to T) \in \mathcal{T}(T,T_Z)$. Localizing, assume that $T = \operatorname{Spec}(D)$, $W = \operatorname{Spec}(D')$, $T_Z = \operatorname{Spec}(A)$, $W_Z = \operatorname{Spec}(A')$, and that a regular element $c \in D$ is a generator of the ideal of $\mathrm{Spec}(A)$ in $\mathrm{Spec}(D)$. It suffices to show that $\mathrm{Tor}_k^D(D', A) = 0$ for $k \geq 1$. But $h^*(c)$ generates the ideal of $\mathrm{Spec}(A')$ in $\mathrm{Spec}(D')$, hence, by assumption, is a regular element. Thus vanishing of the Tor-groups we want.

It remains now to check that for every morphism $g : (f_1 : T_1 \to Y) \to (f_2 : T_2 \to Y)$ the map $g^*((i_{Z*}\mathcal{F}^{\cdot})_{f_2}) \xrightarrow{\beta_g} (i_{Z*}\mathcal{F}^{\cdot})_{f_1}$ is an isomorphism and that it satisfies the usual compatibility condition. The map $\beta_g$ is the composition

$$g^*(i_{Z*}\mathcal{F}^{\cdot})_{f_2} = g^* i_{T_{2,Z}*}\mathcal{F}^{\cdot}_{f_{2,Z}} \xrightarrow{\gamma} i_{T_{1,Z}*} g_Z^* \mathcal{F}^{\cdot}_{f_{2,Z}} \to i_{T_{1,Z}*}\mathcal{F}^{\cdot}_{f_{1,Z}} = (i_{Z*}\mathcal{F}^{\cdot})_{f_1}.$$

Here the map $\gamma$ is a base change morphism. Since $i_Z$ is a closed immersion, $\gamma$ is an isomorphism. Hence so is $\beta_g$. The transitivity condition follows easily from the corresponding one for the base change morphism and the complex $\mathcal{F}^{\cdot} \in \mathcal{P}(Z)$. Similarly, we check that the pushforward $i_{Z'*}$ is well-defined.

Now, to check commutativity of the diagram in the lemma, we easily compute that

$$h^* i_{Z*} = i_{Z'*} h^* : ((g : T \to Z) \mapsto \mathcal{F}^{\cdot}_g)) \mapsto ((t : W \to Y') \mapsto i_{W_Z*}\mathcal{F}^{\cdot}_{(ht)_Z}),$$

as claimed. □

Having defined the map $i_*$ we now claim that it is a weak equivalence. It suffices to check, that for every subset $L$ of the index set $J$, the map $K(B_L) \xrightarrow{i_{L*}} \widetilde{K}(C_L, B_L)$ is a weak equivalence. To show this consider the following commutative diagram

$$\begin{tikzcd}
K(B_L) \arrow[r, "i_{L*}"] \arrow[d, "\wr"'] & \widetilde{K}(C_L, B_L) \arrow[d, "\wr"'] \\
G(B_L) \arrow[r, "i_G"] & G(C_L, B_L).
\end{tikzcd}$$

Here $G()$ is the $G$-theory spectrum defined by Thomason [19, 3.3]. Since $B_L$ and $C_L$ are regular, by the Poincaré duality [19, 3.21] the vertical maps are homotopy equivalences. Since the map $i_G$ is a weak equivalence, so is the map $i_{L*}$.

Define $f^* : K'_j(X_{V'}) \to K_j(U)$ as the composition

$$K'_j(X_{V'}) \to K'_j(X[T], X_{V'}) \xleftarrow{\sim} K_j(X[T], X_{V'}) \xrightarrow{(f_2^*)^{-1} f_1^*} K_j(U).$$

Using properties of log-étale morphisms, it is easy to check that this construction is independent of the choice of the lifting $U_T^{\times}$ and the affine coverings.

**Lemma 2.7.** *The map $f^* : K'_j(X_{V'}) \to K_j(U)$ has the following properties*

1. *It is independent of the choice of the uniformizer $\pi'$.*
2. *Let $V_1$ be a finite extension of $V'$ and let $f_1 : U_1^{\times} \to X_{V_1}^{\times}$ be a log-étale map from a semi-stable $V_1^{\times}$-log-scheme. Let $p : U_1^{\times} \to U^{\times}$ be a map over the natural map $g : X_{V_1}^{\times} \to X_{V'}^{\times}$. Then the following diagram commutes*

$$\begin{tikzcd}
K'_i(X_{V'}) \arrow[r, "g^*"] \arrow[d, "f^*"] & K'_i(X_{V_1}) \arrow[d, "f_1^*"] \\
K_i(U) \arrow[r, "p^*"] & K_i(U_1)
\end{tikzcd}$$

3. *The following diagrams commute*

$$\begin{tikzcd}
K'_i(X_{V'}) \arrow[r, "f^*"] & K_i(U) & K'_i(X_{V'}) \arrow[r, "f^*"] \arrow[d, "j_X^*"] & K_i(U) \arrow[d, "j_U^*"] \\
K_i(X_{V'}) \arrow[r, "f^*"] \arrow[u] & K_i(U) \arrow[u, "\mathrm{Id}"] & K'_i(X_{K'}) \arrow[r, "f^*_{K'}"] & K_i(U_{K'}),
\end{tikzcd}$$

*where* $j_X : X^\times_{K'} \hookrightarrow X^\times_{V'}$, $j_U : U^\times_{K'} \hookrightarrow U^\times$.

*Proof.* For the first property, notice that two different choices of the uniformizer differ by a unit $u \in V[T]^\times_{V'^\heartsuit}$. Sending $T$ to $uT$ induces a map on the formal log-scheme $X[T]^\times_{X^\heartsuit_{V'}}$ compatible with the identity on $X^\heartsuit_{V'}$. This map lifts to a map on the liftings $U^\times_T$ of $U^\heartsuit$ used in our construction, which is compatible with the identity map on $U^\heartsuit$. The lifted map yields a map of the simplicial schemes used in the construction. The first property follows now easily from functoriality of the K-theory spectrum, flatness of the change of the uniformizer maps, and Lemma 2.6.

For the second property, choose uniformizers $\pi_1 \in V_1$ and $\pi' \in V'$ such that $\pi_1^k = \pi'$ for some $k$. That allows us to choose compatible (via the map $T \mapsto T^k$) embeddings of the schemes $X^\heartsuit_{V'}$ and $X^\heartsuit_{V_1}$ into the scheme $X[T]^\times$. Using properties of log-étale maps, we can lift the map $T \mapsto T^k$ to a map between the liftings $U^\times_T$ and $U^\times_{1,T}$ used in our construction. The second property follows now from functoriality of the K-theory spectrum, the fact that the map $T \mapsto T^k$ is flat, and Lemma 2.6.

For the third property, pass to the spectra, use that the maps $j_X, j_U$ are flat, and evoke again Lemma 2.6. □

2.4. **A special case of log-étale descent.** The following proposition will be essential in our construction of the comparison morphism; it will allow us to descent the syntomic cohomology of the regular resolution to that of the original scheme. We are back in the category of fine and saturated log-schemes.

**Proposition 2.1.** *For any* $n \geq 1$, $r \geq 0$, *any log-smooth separated scheme of finite type* $X^\times \to V^\times$, *and any modification* $\pi : U^\times \to X^\times$, *there is a natural isomorphism*

$$H^*_{\mathrm{cr}}(X^\times_n/W_n(k), J^{[r]}_{X^\times_n/W_n(k)}) \underset{\sim}{\xrightarrow{\pi^*}} H^*_{\mathrm{cr}}(U^\times_n/W_n(k), J^{[r]}_{U^\times_n/W_n(k)}).$$

*Proof.* By Zariski descent for log-crystalline cohomology, we may assume $X$ to be affine. There is a commutative diagram of maps of topoi

$$\begin{tikzcd}
 & (U^\times_n/W_n(k))_{\mathrm{cr}} \arrow[dl, "\pi"] \arrow[dr, "f_{U^\times}"] & \\
(X^\times_n/W_n(k))_{\mathrm{cr}} \arrow[rr, "f_{X^\times}"] & & (V^\times_n/W_n(k))_{\mathrm{cr}}.
\end{tikzcd}$$

It suffices to show that the morphism $Rf_{X^\times *}J^{[r]}_{X^\times_n/W_n(k)} \xrightarrow{\pi^*} Rf_{U^\times *}J^{[r]}_{U^\times_n/W_n(k)}$ is a quasi-isomorphism.

Let $i_T : S^\times \hookrightarrow T^\times \to W_n(k), p : S^\times \to V^\times_n$, be a thickening. Notice that the log-scheme $T^\times$ is saturated: $S^\times$ is saturated ($p$ being étale), $i$ is a nilimmersion, and, since $i$ is an exact morphism, we have $i^{-1}(M_{T^\times}/\mathcal{O}^*_{T^\times}) \simeq M_{S^\times}/\mathcal{O}^*_{S^\times}$, where, for a log-scheme $Y^\times$, $M_{Y^\times}$

denotes its log-structure. We have canonically $(Rf_{X^\times *}J^{[r]}_{X_n^\times/W_n(k)})_{T^\times} \simeq Rf_{X_S^\times/T^\times *}(J^{[r]}_{X_S^\times/T^\times})$, where $f_{X_S^\times/T^\times}$ is the composition

$$f_{X_S^\times/T^\times} : (X_S^\times/T^\times)_{\mathrm{cr}} \longrightarrow (X_S^\times)_{\text{ét}} \longrightarrow T^\times_{\text{ét}}.$$

We may assume that $S^\times = V(k')^\times_n$, where $k'$ is a finite field extension of $k$ and $V(k')$ denotes the unramified extension of $V$ corresponding to $k'$. Let $Y^\times := W_n(k')[x]^\times$ be the scheme $\mathrm{Spec}(W_n(k')[x])$ equipped with the log-structure associated to the map $\mathbf{N} \to W_n(k')[x]$, $1 \mapsto x$. We have an exact closed immersion $i_W : V(k')^\times_n \longleftarrow W_n(k')[x]^\times$ given by sending $x$ to a uniformizer of $V(k')$. Let $I$ be the kernel of $i_W$. It is a principal ideal.

We may also assume that there exists a retraction $h : T^\times \to W_n(k')[x]^\times$ such that $hi_T = i_W$ ($W_n(k')[x]^\times$ being log-smooth over $W_n(k)$). Notice that, since $T^\times$ is equipped with divided powers, the retraction $h$ factors through a closed subscheme $Y_m^\times$ of $W_n(k')[x]^\times$ given by $I^m$.

Since $X_S^\times$ is affine and log-smooth over $S^\times$, it can be lifted to a (necessarily saturated) log-smooth scheme $X_S^\times \hookrightarrow X_{Y_m^\times} \to Y_m^\times$. Let $X_T^\times$ denotes the pullback of $X_{Y_m^\times}$ via $h$ to $T^\times$. The scheme $X_T^\times$ is flat over $T^\times$ (as it is log-smooth and integral over $T^\times$ – the monoid $M_{T^\times}/\mathcal{O}^*_{T^\times}$ being generated by one element) thus $D_{X_S^\times}(X_T^\times) = X_T^\times$ and canonically $Rf_{X_S^\times/T^\times}(J^{[r]}_{X_S^\times/T^\times}) \simeq Rg_{X*}\Omega^{\geq r}_{X_T^\times/T^\times}$, where $g_X : X_T^\times \to T^\times$ is the structure map.

Since $\pi : U^\times \to X^\times$ is log-étale, we have a following cartesian diagram of maps of log-schemes

$$\begin{tikzcd}
U_S^\times \arrow[r] \arrow[d, "\pi"] & U_T^\times \arrow[r] \arrow[d, "\pi_T"] & U_{Y_m}^\times \arrow[d, "\pi_{Y_m}"] \\
X_S^\times \arrow[r] \arrow[d] & X_T^\times \arrow[r] \arrow[d, "g_X"] & X_{Y_m}^\times \arrow[d] \\
S^\times \arrow[r, "i_T"] & T^\times \arrow[r, "h"] & Y_m^\times,
\end{tikzcd}$$

where $\pi_T$, $\pi_{Y_m}$ are log-étale liftings of $\pi$. Due to the fact that the morphisms $X_{Y_m}^\times \to Y_m^\times$ and $U_{Y_m}^\times \to Y_m^\times$ are integral it is a cartesian diagram of schemes as well. Since there is an isomorphism $(Rf_{U^\times *}J^{[r]}_{U_n^\times/W_n(k)})_{T^\times} \simeq Rg_{U*}\Omega^{\geq r}_{U_T^\times/T^\times}$, where $g_U = g_X\pi_T$, it suffices to show that the natural morphism

$$\Omega^{\geq r}_{X_T^\times/T^\times} \xrightarrow{\pi_T^*} R\pi_{T*}\Omega^{\geq r}_{U_T^\times/T^\times}$$

is a quasi-isomorphism. Since

$$R\pi_{T*}\Omega^{\geq r}_{U_T^\times/T^\times} = R\pi_{T*}\pi_T^*\Omega^{\geq r}_{X_T^\times/T^\times} = \Omega^{\geq r}_{X_T^\times/T^\times} \otimes R\pi_{T*}\mathcal{O}_{U_T},$$

it suffices to show that the natural morphism $\mathcal{O}_{X_T} \xrightarrow{\pi_T^*} R\pi_{T*}\mathcal{O}_{U_T}$ is a quasi-isomorphism, or, because the map $X_T \longrightarrow X_{Y_m}$ is a homeomorphism of topological spaces, that the natural morphism $\mathcal{O}_{X_T} \xrightarrow{\pi^*_{Y_m}} R\pi_{Y_m*}\mathcal{O}_{U_T} = R\pi_{Y_m*}\pi^*_{Y_m*}\mathcal{O}_{X_T} = \mathcal{O}_{X_T} \otimes^{\mathbf{L}} R\pi_{Y_m*}\mathcal{O}_{U_{Y,m}}$ is a quasi-isomorphism.

It suffices thus to show that $\mathcal{O}_{X_{Y,m}} \xrightarrow{\pi^*_{Y_m}} R\pi_{Y_m*}\mathcal{O}_{U_{Y,m}}$ is a quasi-isomorphism. We argue by induction on $m$. Assume that the statement is true for $m=1$. Since $U_{Y,m}$ is flat over $Y_m$, we have the following exact sequence

$$0 \longrightarrow \mathcal{O}_{U_{Y,m-1}} \longrightarrow \mathcal{O}_{U_{Y,m}} \longrightarrow \mathcal{O}_{U_{Y,1}} \longrightarrow 0.$$

This and the induction hypothesis yield that the sequence

$$0 \longrightarrow \Gamma(U_{Y,m-1}, \mathcal{O}_{U_{Y,m-1}}) \longrightarrow \Gamma(U_{Y,m}, \mathcal{O}_{U_{Y,m}}) \longrightarrow \Gamma(U_{Y,1}, \mathcal{O}_{U_{Y,1}}) \longrightarrow 0$$

is exact and that $H^i(U_{Y,m}, \mathcal{O}_{U_{Y,m}}) = 0$, for $i > 0$. Evoking once more the induction hypothesis, we get that $\Gamma(X_{Y,m}, \mathcal{O}_{X_{Y,m}}) \xrightarrow{\sim} \Gamma(U_{Y,m}, \mathcal{O}_{U_{Y,m}})$. Since $X_{Y,m}$ is affine, this gives us what we wanted.

It remains to show that $\mathcal{O}_{X_S} \xrightarrow{\pi^*_S} R\pi_{S*}\mathcal{O}_{U_S}$ is a quasi-isomorphism. Since $U^\times \to X^\times$ is a modification, this is just a $\mathbf{Z}/p^n$-version of Corollary 2.2 from [17]. $\square$

**Corollary 2.1.** *Let $X^\times \to V^\times$ be any log-smooth separated scheme of finite-type. Then for any $n \geq 1$, $r \geq 0$, and any modification $\pi : U^\times \to X^\times$, there is a natural isomorphism*

$$H^*_f(X^\times, \mathcal{O}_n(r)) \underset{\sim}{\xrightarrow{\pi^*}} H^*_f(U^\times, \mathcal{O}_n(r)).$$

*Proof.* Use the distinguished triangle

$$\longrightarrow H_f(X^\times, \mathcal{O}_n(r)) \longrightarrow H_{\mathrm{cr}}(X^\times_n/W_n(k), J^{[r]}_{X^\times_n/W_n(k)}) \xrightarrow{\beta} H_{\mathrm{cr}}(X^\times_1/W_n(k), \mathcal{O}_{X^\times_1/W_n(k)}) \longrightarrow,$$

where $\beta(x,y) = (p^r x - \phi(x))$ and Proposition 2.1. $\square$

## 3. COMPARISON THEOREM

For what follows choose a sequence of nontrivial $p$-roots of unity $\zeta_n \in \overline{\mathbf{Q}}_p$, $\zeta_n^{p^n} = 1$, $\zeta_{n+1}^p = \zeta_n$, and take for $t \in B_{\mathrm{cr}}$ and for the compatible sequence of Bott classes $\beta_n \in K_2(\overline{\mathbf{Q}}_p; \mathbf{Z}/p^n)$, $\tilde{\beta}_n \in K_2(\overline{\mathbf{Z}}_p; \mathbf{Z}/p^n)$, $n \geq 1$, the elements associated to the sequence $\zeta = (\zeta_n)$. Recall that we can invert the étale Chern class $\overline{c}^{\text{ét}}_{ij}$ modulo the Bott element, i.e., we have [16, Prop. 4.1.]

**Proposition 3.1.** *Let $Y$ be a smooth quasi-projective scheme over $\overline{K}$. Let $p \neq 2$. Let $n$ be such that $p^n \geq 5$. Let $i$ be any positive integer. There exists a constant $T = T(d,i,j)$ depending only on the dimension $d$ of $Y$ and $i, j$ such that, for $j \geq 2N(d)$, where $N(d) = (2/3)d(d+1)(d+2)$,*

1. *the cokernel of $\overline{c}^{\text{ét}}_{ij} : F^i_\gamma/F^{i+1}_\gamma K_j(Y; \mathbf{Z}/p^n) \longrightarrow H^{2i-j}(Y, \mathbf{Z}/p^n(i))$ is annihilated by $T$;*
2. *if $[x]$ is in the kernel of $\overline{c}^{\text{ét}}_{ij}$, then $T[x] = [z]$ for $z \in F^i_\gamma K_j(Y; \mathbf{Z}/p^n)$ annihilated by $\beta_n^{N(d)}$.*

Let $X^\times$ be any log-smooth vertical universally saturated quasi-projective scheme over $V^\times$ of pure relative dimension $d$. Assume that $b \geq d + N(d)$. Define a transformation

$$\alpha^n_{ab} : H^a(X_{\overline{K}}, \mathbf{Z}/p^n(b)) \longrightarrow H^a_{\mathrm{cr}}(X^\times_{\overline{V},n}/W_n(k), \mathcal{O}_{X^\times_{\overline{V},n}/W_n(k)})\{-b-N(d)\},$$

where $\{-b-N(d)\}$ denotes a twist of the filtration and the Frobenius, in the following way. By passing to an extension of $V$, assume that the residue field of $V$ is algebraically closed. For $x \in H^a(X_{\overline{K}}, \mathbf{Z}/p^n(b))$, take $(\overline{c}^{\text{ét}}_{b,2b-a})^{-1}(x) \in F^b_\gamma/F^{b+1}_\gamma K_{2b-a}(X_{\overline{K}}; \mathbf{Z}/p^n)$ to be any element in the preimage of $T(d,b,2b-a)x$ (Proposition 3.1 yields that $T(d,b,2b-a)x$ lies in the image of $\overline{c}^{\text{ét}}_{b,2b-a}$). Let $x_1\in F^b_{\gamma}K_{2b-a}(X_{\overline{K}};\mathbf{Z}/p^n)$ be a lifting of the element $(\overline{c}^{\text{ét}}_{b,2b-a})^{-1}(x)$.

Choose a semi-stable model $Y^{\times}\xrightarrow{\mu}X^{\times}$ of $X^{\times}$ (we need to modify $X^{\times}$ to be able to pullback $K'$-theory classes). Recall [16, Lemma 3.1] that the open immersion $j: Y_{\overline{V}}\hookrightarrow Y_{\overline{K}}=X_{\overline{K}}$ induces a natural isomorphism

$$K'_j(Y_{\overline{V}};\mathbf{Z}/p^n)\underset{\sim}{\xrightarrow{j^*}}K_j(X_{\overline{K}};\mathbf{Z}/p^n). \tag{3.1}$$

We claim that we can find a finite extension $V_1$ of $V$, a semi-stable model $U^{\times}$ of $Y^{\times}_{V_1}$ (hence of $X^{\times}_{V_1}$), and an element $x'_1\in F^b_{\gamma}K_{2b-a}(U;\mathbf{Z}/p^n)$ such that $j^*_U(x'_1)=x_1$. To see that, assume that for some finite extension $K_1$ of $K$, $x_1$ can be written as $a\gamma_{i_1}(y_1)\cup\ldots\cup\gamma_{i_n}(y_n)$ with $y_i\in K_{j_i}(X_{K_1};\mathbf{Z}/p^n)$, $y_i\geq 2$, $a\in F^{i_0}_{\gamma}K_0(X_{K_1})$, $i_0\geq 1$. After possibly extending the field $K_1$, there exist elements $y'_i\in K'_{j_i}(Y_{V_1};\mathbf{Z}/p^n)$ restricting to all $y_i$'s. Take any semi-stable model $U^{\times}\xrightarrow{\pi}Y^{\times}_{V_1}$. Note that, since the scheme $U$ is regular, the element $a\in F^{i_0}_{\gamma}K_0(X_{K_1})$ can be lifted to an element $a'\in F^{i_0}_{\gamma}K_0(U)$. We can now take for $x'_1$ the product of the elements $\gamma_{i_l}(\pi_1^*(y'_l))$ and the element $a'$. Here $\pi_1: U^{\times}\longrightarrow Y^{\times f}_{V_1}$, where $Y^{\times f}_{V_1}$ denotes the base change of $Y^{\times}$ in the category of *fine* log-schemes, is the composition of the map $\pi$ with the natural (log-étale) map $Y^{\times}_{V_1}\longrightarrow Y^{\times f}_{V_1}$. Notice that the scheme $Y^{\times}$ need not be universally saturated, hence the underlying scheme of $Y^{\times}_{V_1}$ may not be $Y_{V_1}$. On the other hand, the natural morphism $Y^{\times}\to V^{\times}$ being integral, the underlying scheme of $Y^{\times f}_{V_1}$ is always $Y_{V_1}$.

Since, by Lemma 2.7, there is a commutative diagram

$$\begin{tikzcd}
K_j(U;\mathbf{Z}/p^n) \arrow[r, "j_U^*"] & K_j(U_{K_1};\mathbf{Z}/p^n) \arrow[r, "\mathrm{Id}"] & K_j(U_{K_1};\mathbf{Z}/p^n) \\
K'_j(Y_{V_1};\mathbf{Z}/p^n) \arrow[u, "\pi_1^*"'] \arrow[r, "j_Y^*"] & K'_j(Y_{K_1};\mathbf{Z}/p^n) \arrow[u, "\pi_{K_1}^*"'] & K_j(X_{K_1};\mathbf{Z}/p^n), \arrow[l, "\sim"] \arrow[u, "\pi_{K_1}^*"', "\wr"]
\end{tikzcd}$$

$j^*_U(x'_1)=x_1$, as wanted.

Having found a lifting $x'_1$ of $x_1$, we can now show its uniqueness (modulo a base extension and a change of the semi-stable model).

**Lemma 3.1.** *Let $U^{\times}$ be a regular, vertical log-scheme log-smooth over $V^{\times}$. Let $x,y$ be two elements in $K_j(U;\mathbf{Z}/p^n)$ restricting to the same element $z\in K_j(U_K;\mathbf{Z}/p^n)$. Then there exists a finite extension $V_2$ of $V$ and a semi-stable model $U_2^{\times}\longrightarrow U^{\times}_{V_2}$ such that $p^*(x)=p^*(y)$ in $K_j(U_2;\mathbf{Z}/p^n)$, where the map $p$ is equal to the composition $p: U_2^{\times}\longrightarrow U^{\times}_{V_2}\longrightarrow U^{\times}$.*

*Proof.* By [16, Lemma 3.1], there exists a finite extension $V_2$ of $V$ such that $f^*(x)=f^*(y)$, where $f^*: K_j(U;\mathbf{Z}/p^n)\to K'_j(U_{V_2};\mathbf{Z}/p^n)$ is the natural map. Take a regular model $U_2^{\times}\to U^{\times}_{V_2}$. Let $U^{\times f}_{V_2}$ denote the base change of $U^{\times}$ in the category of fine log-schemes. Let $g: U_2^{\times}\to U^{\times f}_{V_2}$ and $f: U^{\times f}_{V_2}\to U^{\times}$ denote the induced maps. Since, by Lemma 2.7, the following diagram commutes

$$\begin{tikzcd}
K_j(U;\mathbf{Z}/p^n) \arrow[r, "\mathrm{Id}", "\sim"'] \arrow[d] & K_j(U;\mathbf{Z}/p^n) \arrow[r, "p^*"] \arrow[d, "f^*"'] & K_j(U_2;\mathbf{Z}/p^n) \arrow[d, "\wr"'] \\
K_j(U_{V_2};\mathbf{Z}/p^n) \arrow[r] & K'_j(U_{V_2};\mathbf{Z}/p^n) \arrow[r, "g^*"] & K_j(U_2;\mathbf{Z}/p^n),
\end{tikzcd}$$

we are done. □

Set

$$\alpha^n_{ab}(x) := l(b,d)t^{N(d)}\psi_n((\mu_{V_1}\pi)^*)^{-1}(\overline{c}^{\mathrm{syn}}_{b,2b-a}(T(d,b,2b-a)x'_1))$$

($\mu_{V_1}\pi$ induces an isomorphism on the syntomic cohomology groups by Corollary 2.1) where $\psi_n$ is the natural projection $H^a_f(X^{\times}_{\overline{V}},\mathcal{O}_n(b)) \to H^a_{\mathrm{cr}}(X^{\times}_{\overline{V},n}/W_n(k),\mathcal{O}_{X^{\times}_{\overline{V},n}/W_n(k)})\{-b\}$ and we set $l(b,d) = (-1)^{N(d)}(N(d)+b-1)!/(b-1)!$.

**Lemma 3.2.** *The transformation $\alpha^n_{ab}$ is a well-defined natural group homomorphism. It is also Galois equivariant provided that we twist the action of $G_K$ on the target by the $N(d)$'th power of the cyclotomic character.*

*Moreover,*

$$l(b,d)T(d,b,2b-a)^2\alpha^n_{a,b+1}(\zeta_n x) = l(b+1,d)T(d,b+1,2b+2-a)^2 t\alpha^n_{ab}(x).$$

*Proof.* We have made several choices in our construction of $\alpha^n_{ab}$.

That the choice of $V_1$ and the semi-stable model $U^{\times}$ (hence of the element $x'_1$ as well) is of no importance follows from Lemma 3.1 and functoriality of Chern classes with respect to pullbacks.

The ambiguity introduced by the choice of the lifting $x_1$ comes from an element of $F^{b+1}_{\gamma}K_{2b-a}(X_{\overline{K}};\mathbf{Z}/p^n)$. By Lemma 3.1, after perhaps passing to a finite extension of the base ring and changing the semi-stable model $U^{\times}$, we get that the ambiguity in the choice of the corresponding element $x'_1$ comes from an element of $F^{b+1}_{\gamma}K_{2b-a}(U;\mathbf{Z}/p^n)$. This disappears when we apply the syntomic Chern classes (Lemma 2.1).

Finally, to kill the ambiguity in the choice of $(\overline{c}^{\text{ét}}_{b,2b-a})^{-1}(x)$ we first multiply it by $T(d,b,2b-a)$. Then, by Proposition 3.1, the ambiguity comes from the class of an element of $F^b_{\gamma}K_{2b-a}(X_{\overline{K}};\mathbf{Z}/p^n)$ annihilated by $\beta^{N(d)}_n$. By Lemma 3.1, after perhaps passing to a finite extension of the base ring and changing the semi-stable model $U^{\times}$, we get that the ambiguity in the choice of the corresponding element $x'_1$ comes from an element $z \in F^b_{\gamma}K_{2b-a}(U;\mathbf{Z}/p^n)$ annihilated by $\tilde{\beta}^{N(d)}_n$ and an element of $F^{b+1}_{\gamma}K_{2b-a}(U;\mathbf{Z}/p^n)$. Since $\overline{c}^{\mathrm{syn}}_{1,2}(\tilde{\beta}_n) = t$ [16, Lemma 4.1], we have

$$\begin{aligned}
l(b,d)t^{N(d)}\psi_n((\mu_{V_1}\pi)^*)^{-1}(\overline{c}^{\mathrm{syn}}_{b,2b-a}(z)) &= l(b,d)\psi_n((\mu_{V_1}\pi)^*)^{-1}(\overline{c}^{\mathrm{syn}}_{1,2}(\tilde{\beta}^{N(d)}_n)\overline{c}^{\mathrm{syn}}_{b,2b-a}(z)) \\
&= \psi_n((\mu_{V_1}\pi)^*)^{-1}(\overline{c}^{\mathrm{syn}}_{b+N(d),2b+2N(d)-a}(\tilde{\beta}^{N(d)}_n z)) = 0
\end{aligned}$$

Hence this ambiguity disappears after we multiply by $l(b,d)t^{N(d)}$ and apply the syntomic Chern class morphisms.

For the independence of the choice of the semi-stable model $Y^\times$ and for functoriality, use functoriality of Chern classes and K-theory with respect to pullbacks, Proposition 3.1, Lemma 3.1, and Lemma 2.1.

Since all the genuine maps we used were group homomorphisms (Lemma 2.1), so is the map $\alpha^n_{ab}$.

That it is also Galois equivariant follows from functoriality of Chern classes and functoriality of K-theory with respect to pullbacks.

For the last statement of the lemma, let $x_1' \in F^b_\gamma K_{2b-a}(U;\mathbf{Z}/p^n)$ and $x_1 \in F^b_\gamma K_{2b-a}(X_{K_1};\mathbf{Z}/p^n)$ be the elements from the construction of $\alpha^n_{ab}(x)$. Since $\overline{c}^{\text{ét}}_{1,2}(\beta_n) = \zeta_n$ [16, Lemma 4.1], we have

$$\overline{c}^{\text{ét}}_{b+1,2b+2-a}(\beta_n x_1) = l(b,d)\overline{c}^{\text{ét}}_{1,2}(\beta_n)\overline{c}^{\text{ét}}_{b,2b-a}(x_1) = l(b,d)\zeta_n T(d,b,2b-a)x.$$

Hence as above

$$\begin{aligned} l(b,d)T(d,b,2b-a)^2\alpha^n_{a,b+1}(\zeta_n x) &= l(b+1,d)T(d,b,2b-a)t^{N(d)} \\ &\qquad \psi_n((\mu_{V_1}\pi)^*)^{-1}(\overline{c}^{\text{syn}}_{b+1,2b+2-a}(T(d,b+1,2b+2-a)^2\tilde{\beta}_n x_1')) \\ &= l(b,d)l(b+1,d)T(d,b+1,2b+2-a)^2 t^{N(d)} \\ &\qquad \psi_n((\mu_{V_1}\pi)^*)^{-1}(\overline{c}^{\text{syn}}_{1,2}(\tilde{\beta}_n)\overline{c}^{\text{syn}}_{b,2b-a}(T(d,b,2b-a)x_1')) \\ &= l(b+1,d)T(d,b+1,2b+2-a)^2 t\alpha^n_{a,b}(x), \end{aligned}$$

as wanted. □

Let $b \geq d + N(d)$ and assume $X$ to be projective over $V$. Define a morphism

$$\alpha_{ab} : H^a(X_{\overline{K}}, \mathbf{Q}_p(b)) \longrightarrow H^a_{\text{cr}}(X_0^\times/W(k)^0, \mathcal{O}_{X_0^\times/W(k)^0}) \otimes_{W(k)} B_{\text{st}}\{-b\}$$

as the composition of $\mathbf{Q} \otimes \operatorname{proj\,lim}_n \alpha^n_{ab}$ with the map (see section 2.1)

$$\mathbf{Q} \otimes \operatorname*{proj\,lim}_n H^a_{\text{cr}}(X^\times_{\overline{V},n}/W_n(k), \mathcal{O}_{X^\times_{\overline{V},n}/W_n(k)}) \xrightarrow{h_\pi} H^a_{\text{cr}}(X_0^\times/W(k)^0, \mathcal{O}_{X_0^\times/W(k)^0}) \otimes_{W(k)} B_{\text{st}}$$

and with the division by $l(b,d)T(d,b,2b-a)^2 t^{N(d)}$. The morphism $\alpha_{ab}$ is functorial in $X^\times$, preserves the Frobenius, the action of $\text{Gal}(\overline{K}/K)$ and the monodromy operator, and, after extension to $B_{dR}$, is compatible with filtrations (use the isomorphism (2.2)).

We would also like to know how the map $\alpha_{ab}$ behaves with respect to finite base changes. In what follows, we will add the subscript $\pi$ to $\alpha_{ab}$ to underscore the fact that in the definition of this map we made a choice of a uniformizer. Let $V_1$ be a finite extension of $V$ with fraction field $K_1$ and residue field $k_1$. Let $e$ be the ramification index of $K_1$ over $K$ and let $\pi_1$ be a uniformizer of $V_1$. Set $X_1^\times := X^\times_{V_1}$.

**Lemma 3.3.** *The following diagrams commute*

$$\begin{tikzcd} H^a(X_{\overline{K}}, \mathbf{Q}_p(b)) \otimes_{\mathbf{Q}_p} B_{\text{st}} \arrow[r, "\alpha_{ab,\pi}"] \arrow[d, equal] & H^a_{\text{cr}}(X^\times) \otimes_{W(k)} B_{\text{st}} \arrow[d, "\wr"'] \\ H^a(X_{1,\overline{K}_1}, \mathbf{Q}_p(b)) \otimes_{\mathbf{Q}_p} B_{\text{st}} \arrow[r, "\alpha_{ab,\pi_1}"] & H^a_{\text{cr}}(X_1^\times) \otimes_{W(k_1)} B_{\text{st}}, \end{tikzcd}$$

$$\begin{tikzcd}
H^a(X_{\overline{K}}, \mathbf{Q}_p(b)) \otimes_{\mathbf{Q}_p} B_{dR} \arrow[r, "\alpha^{dR}_{ab,\pi}"] \arrow[d, equal] & H^a_{dR}(X_K/K) \otimes_K B_{dR} \arrow[d, "\wr"'] \\
H^a(X_{1,\overline{K}_1}, \mathbf{Q}_p(b)) \otimes_{\mathbf{Q}_p} B_{dR} \arrow[r, "\alpha^{dR}_{ab,\pi_1}"] & H^a_{dR}(X_{K_1}/K_1) \otimes_{K_1} B_{dR}.
\end{tikzcd}$$

*In particular, the maps $\alpha_{ab}$ and $\alpha^{dR}_{ab}$ are independent of the choice of the uniformizer $\pi$.*

*Proof.* Arguing exactly like Tsuji in his proof of a similar statement [20, 4.10.4], we reduce to showing that the maps

$$\alpha^n_{ab} : H^a(X_{\overline{K}}, \mathbf{Z}/p^n(b)) \longrightarrow H^a_{\mathrm{cr}}(X^{\times}_{\overline{V},n}/W_n(k), \mathcal{O}_{X^{\times}_{\overline{V},n}/W_n(k)}),$$

$$\iota_\pi\rho_\pi h_\pi : H^a_{\mathrm{cr}}(X^{\times}_{\overline{V}}/W(k)) \longrightarrow H^a_{dR}(X_K/K) \otimes_K B^+_{dR}$$

are compatible with our base changes.

In the case of the map $\alpha^n_{ab}$ we use the same argument that we have used to prove its funtoriality (Lemma 3.2).

In the case of the map $\iota_\pi\rho_\pi h_\pi$, since from the definition [20, 4.7.3] of the isomorphism $B^+_{dR} \otimes_{K_0} H^i_{dR}(X_K/K) \xrightarrow{\sim} \operatorname{proj\,lim}_s(\mathbf{Q} \otimes H^{2j}_{\mathrm{cr}}(X^{\times}_{\overline{V}}/V^{\times}, \mathcal{O}/J^{[s]}))$ it is easy to see that it is compatible with our base change, it suffices to show that so is its composition with $\iota_\pi\rho_\pi h_\pi$. Since this composition is equal [20, 4.8.4] to the natural map

$$H^{2j}_{\mathrm{cr}}(X^{\times}_{\overline{V}}/W(k)) \longrightarrow \operatorname*{proj\,lim}_s(\mathbf{Q} \otimes H^{2j}_{\mathrm{cr}}(X^{\times}_{\overline{V}}/V^{\times}, \mathcal{O}/J^{[s]})),$$

this is clear. □

**Theorem 3.1.** *Let $X^{\times}$ be any projective log-smooth vertical $V^{\times}$-scheme with Cartier type reduction of pure relative dimension $d$. Then, assuming $b \geq 2d + N(2d)$, the natural morphism*

$$\alpha_{ab} : H^a(X_{\overline{K}}, \mathbf{Q}_p(b)) \otimes_{\mathbf{Q}_p} B_{\mathrm{st}} \longrightarrow H^a_{\mathrm{cr}}(X^{\times}_0/W(k)^0, \mathcal{O}_{X^{\times}_0/W(k)^0}) \otimes_{W(k)} B_{\mathrm{st}}\{-b\}$$

*is an isomorphism.*

*Moreover, the map $\alpha_{ab}$ preserves the Frobenius, the action of* $\mathrm{Gal}(\overline{K}/K)$ *and the monodromy operator. It is independent of the choice of $\pi$, compatible with base changes and Tate twists, and, after extension to $B_{dR}$, induces an isomorphism of filtrations.*

*Proof.* By passing to a finite extension of $K$ and using Lemma 3.3, we may assume that $X_K$ is geometrically irreducible. The line of the argument is standard [7]. Since both sides of $\alpha_{ab}$ have the same rank over $B_{\mathrm{st}}$, it suffices to show that the morphism $\alpha_{ab}$ is injective. For that, first, we have to check that the morphism $\alpha_{ab}$ is compatible with products. This follows from the fact that the morphism $h_\pi$ is compatible with products and from the following lemma

**Lemma 3.4.** *Let $x \in H^a(X_{\overline{K}}, \mathbf{Z}/p^n(b))$ and $y \in H^c(X_{\overline{K}}, \mathbf{Z}/p^n(e))$. Then*

$$\begin{aligned} &l(b,d)l(e,d)K(b,e)t^{N(d)}T(d,b,2b-a)^2T(d,e,2e-c)^2\alpha^n_{a+c,b+e}(x \cup y) \\ &\qquad = l(b+e,d)K(b,e)T(d,b+e,2b+2e-a-c)^2\alpha^n_{ab}(x) \cup \alpha^n_{ce}(y), \end{aligned}$$

*where $K(b,e) = (b+e-1)!/((b-1)!(e-1)!)$, assuming that all the indices are in the valid range.*

*Proof.* Use Lemma 3.1 and Lemma 2.1. ☐

Next, since the domain satisfies Poincaré duality

$$H^a(X_{\overline{K}},\mathbf{Q}_p(b))\otimes H^{2d-a}(X_{\overline{K}},\mathbf{Q}_p(b))\xrightarrow{\cup} H^{2d}(X_{\overline{K}},\mathbf{Q}_p(2b))$$

and there is a Poincaré-type pairing on the target

$$H^a_{\mathrm{cr}}(X_0^{\times}/W(k)^0,\mathcal{O}_{X_0^{\times}/W(k)^0})\otimes_{W(k)} B_{\mathrm{st}}\{-b\}\otimes H^{2d-a}_{\mathrm{cr}}(X_0^{\times}/W(k)^0,\mathcal{O}_{X_0^{\times}/W(k)^0})\otimes_{W(k)} B_{\mathrm{st}}\{-b\}$$
$$\xrightarrow{\cup} H^{2d}_{\mathrm{cr}}(X_0^{\times}/W(k)^0,\mathcal{O}_{X_0^{\times}/W(k)^0})\otimes_{W(k)} B_{\mathrm{st}}\{-2b\},$$

to show that $\alpha_{a,b}$ has a left inverse, it suffices to verify that the map $\alpha_{2d,2b}$ is an isomorphism. Since the morphisms

$$\rho_\pi : K\otimes_{K_0} H^{2d}_{\mathrm{cr}}(X_0^{\times}/W(k)^0,\mathcal{O}_{X_0^{\times}/W(k)^0})\to H^{2d}_{dR}(X_K/K),\qquad \iota_\pi : K\otimes_{K_0} B_{\mathrm{st}}\to B_{dR}$$

are isomorphisms, it suffices to show that so is the composition

$$H^{2d}(X_{\overline{K}},\mathbf{Q}_p(2b))\otimes_{\mathbf{Q}_p} B_{dR}\xrightarrow{\iota_\pi\rho_\pi\alpha_{2d,2b}} H^{2d}_{dR}(X_K/K)\otimes_{K_0} B_{dR}.$$

By functoriality, we may assume that the residue field of $K$ is algebraically closed. Let $P$ be a rational point of $X$ over $V$ (note that the special fiber of $X$ is reduced [20, 2.7.7]). Since $H^{2d}(X_{\overline{K}},\mathbf{Q}_p(2b))$ and $H^{2d}_{dR}(X_K/K)$ are generated by $\mathrm{cl}^{\text{ét}}(P_K)\zeta^{2b-d}$, respectively $\mathrm{cl}^{dR}(P_K)$, it suffices to show that

$$\iota_\pi\rho_\pi\alpha_{2d,2b}(\mathrm{cl}^{\text{ét}}(P_K)\zeta^{2b-d})=\mathrm{cl}^{dR}(P_K)t^{2b-d}.$$

Let $Y^{\times}\to X^{\times}$ be a semi-stable model and denote by $P'$ the unique $V$-point of $Y$ lying over $P$ (note that $Y_K\simeq X_K$ and $P'_K=P_K$). Let $[\mathcal{O}_{P'}]$ and $[\mathcal{O}_{P_K}]$ denote the class of $\mathcal{O}_{P'}$ and $\mathcal{O}_{P_K}$ in $K_0(Y)$ and $K_0(X_K)$, respectively (recall that $Y$ is regular). Just like in the proof of Lemma 4.2 in [16], multiplication formulas yield that

$$\overline{c}^{\text{ét}}_{2b,2(2b-d)}(s(d)[\mathcal{O}_{P_K}]\beta_n^{2b-d})=(-1)^{2b-1}(2b-1)!s(d)\,\mathrm{cl}^{\text{ét}}(P_K)\zeta_n^{2b-d}.$$

Here $s(d)$ is a constant (depending only on the dimension $d$), such that $s(d)[\mathcal{O}_{P'}]\in F^d_\gamma K_0(Y)$. Since we also have $c^{dR}_{d,0}([\mathcal{O}_{P_K}])=(-1)^{d-1}(d-1)!\,\mathrm{cl}^{dR}(P_K)$, by product formulas (see Lemma 2.1), it suffices to show that, for any $j\geq 0$, the following diagram commutes

$$\begin{tikzcd}
K_0(Y) \arrow[r, "c^{dR}_{j,0}"] \arrow[d, "c^{\mathrm{cr}}_{j,0}"'] & H^{2j}_{dR}(X_K/K) \arrow[r] & B^+_{dR}\otimes_{K_0} H^i_{dR}(X_K/K) \\
\mathbf{Q}\otimes H^{2j}_{\mathrm{cr}}(Y^{\times}/W(k)) & \mathbf{Q}\otimes H^{2j}_{\mathrm{cr}}(X^{\times}/W(k)) \arrow[l, "\sim"'] \arrow[r] & \mathbf{Q}\otimes H^{2j}_{\mathrm{cr}}(X^{\times}_{\overline{V}}/W(k)). \arrow[u, "\iota_\pi\rho_\pi h_\pi"']
\end{tikzcd}$$

Recall now [20, 4.7.6,4.8.4] that the composition

$$H^{2j}_{\mathrm{cr}}(X^{\times}_{\overline{V}}/W(k))\xrightarrow{\iota_\pi\rho_\pi h_\pi} B^+_{dR}\otimes_{K_0} H^i_{dR}(X_K/K)\xrightarrow{\sim}\operatorname{proj}\lim_s(\mathbf{Q}\otimes H^{2j}_{\mathrm{cr}}(X^{\times}_{\overline{V}}/V^{\times},\mathcal{O}/J^{[s]}))$$

is equal to the natural map

$$H^{2j}_{\mathrm{cr}}(X^{\times}_{\overline{V}}/W(k))\to\operatorname{proj}\lim_s(\mathbf{Q}\otimes H^{2j}_{\mathrm{cr}}(X^{\times}_{\overline{V}}/V^{\times},\mathcal{O}/J^{[s]})).$$

Hence, since all the morphisms in the above diagram are functorial and the isomorphism $H^{2j}_{dR}(Y_K/K)\xrightarrow{\sim}\operatorname{proj}\lim_s(\mathbf{Q}\otimes H^{2j}_{\mathrm{cr}}(Y^{\times}_{\overline{V}}/V^{\times},\mathcal{O}/J^{[s]}))$ (the base changes of $Y^{\times}$ are taken in the category of fine log-schemes ) is compatible (by definition [20, 4.7.3]) with the map

$H^{2j}_{\mathrm{cr}}(Y^\times/V^\times) \to \operatorname{proj\,lim}_s(\mathbf{Q}\otimes H^{2j}_{\mathrm{cr}}(Y^\times_{\overline{V}}/V^\times, \mathcal{O}/J^{[s]}))$, it suffices to check that the following diagram commutes

$$\begin{tikzcd}
K_0(Y) \arrow[r, "c^{dR}_{j,0}"] \arrow[d, "c^{\mathrm{cr}}_{j,0}"'] & H^{2j}_{dR}(Y_K/K) \\
\mathbf{Q}\otimes H^{2j}_{\mathrm{cr}}(Y^\times/W(k)) \arrow[r] & \mathbf{Q}\otimes H^{2j}_{\mathrm{cr}}(Y^\times/V^\times). \arrow[u, "\wr"']
\end{tikzcd}$$

By the universality of Chern classes and the splitting principle, it suffices to check that the above diagram commutes when we replace $Y^\times$ with any scheme $T^\times$ smooth and proper over $Y^\times$ and we restrict the maps $c^{dR}_{1,0}$ and $c^{\mathrm{cr}}_{1,0}$ to classes of line bundles on $T$.

Since the natural map $H^2_{\mathrm{cr}}(T_n/W_n(k)) \to H^2_{\mathrm{cr}}(T_n/V_n)$ is compatible with Chern classes, it suffices to show that the map

$$H^2_{\mathrm{cr}}(T/V) \to \mathbf{Q}\otimes H^2_{\mathrm{cr}}(T^\times/V^\times) \xrightarrow{\sim} H^2_{dR}(T_K/K)$$

is compatible with Chern classes of line bundles. By functoriality, it suffices to check it for $T^\times = B_{\cdot}\mathbf{G}^\times_{m,V}$ (the log-structure coming from the special fiber). Since the isomorphism $H^2_{\mathrm{cr}}(B_{\cdot}\mathbf{G}_{m,V_n}/V_n) \xrightarrow{\sim} H^2_{dR}(B_{\cdot}\mathbf{G}_{m,V_n}/V_n)$ is known to be compatible with Chern classes of line bundles [2, 3.4], it suffices to show that so is the natural morphism $H^2_{dR}(B_{\cdot}\mathbf{G}_{m,V_n}/V_n) \to H^2_{dR}(B_{\cdot}\mathbf{G}^\times_{m,V_n}/V^\times_n)$, what is clear.

To finish the proof of the theorem, it remains to show that $\alpha^{dR}_{ab}$ ($\alpha_{ab}$ extended to $B_{dR}$) induces an isomorphism on filtrations. Passing to the associated grading, one reduces to showing that the induced map

$$\overline{\alpha}^{dR,l}_{ab} : \mathbf{C}_p(l)\otimes_{\mathbf{Q}_p} H^a(X_{\overline{K}}, \mathbf{Q}_p(b)) \to \bigoplus_{j\in\mathbf{Z}} \mathbf{C}_p(b+l-j)\otimes_K H^{a-j}(X_K, \Omega^j_{X_K/K}), \quad l\in\mathbf{Z},$$

is injective. Since the domain of $\overline{\alpha}^{dR,l}_{ab}$ satisfies Poincaré duality and $\overline{\alpha}^{dR,l}_{ab}$ is compatible with products, for $\overline{\alpha}^{dR}_{ab}$ to have a left inverse, it suffices to show that

$$\overline{\alpha}^{dR,0}_{2d,2b} : \mathbf{C}_p\otimes_{\mathbf{Q}_p} H^{2d}(X_{\overline{K}}, \mathbf{Q}_p(2b)) \to \mathbf{C}_p(2b-d)\otimes_K H^d(X_K, \Omega^d_{X_K/K}) = \mathbf{C}_p(2b-d)\otimes_K H^{2d}_{dR}(X_K/K)$$

is an isomorphism. Since both the target and the domain are one-dimensional and, by the above, $\overline{\alpha}^{dR,0}_{2d,2b}(\mathrm{cl}^{\text{ét}}(P_K)\zeta^{2b-d}) = \mathrm{cl}^{dR}(P_K)t^{2b-d} \neq 0$, we are done. ☐

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