THE PLUS-CONSTRUCTION AS A LOCALIZATION

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Abstract

An initial survey contrasts two points of view in the historical development of the theory of localization. The first, starting with inversion of elements in a ring, leads to quotient categories and indirectly to the Q-construction. The second considers idempotent functors. This leads to the Berrick-Casacuberta description of the plus-construction on X as the idempotent functor that is nullification of X with respect to an acyclic space W. Focus on the case $X = B\,\mathrm{GL}\,R$ produces new results, including the classification of perfect normal subgroups of $\mathrm{GL}\,R$. When R is a group ring AG, links are obtained between these perfect normal subgroups and the A-representability of the group G. A final section studies the relationship between the plus-construction on $B\,\mathrm{GL}\,R$ and acyclicity of the space W, which prompts some general questions on the K-theory of rings.

0 Preface

This work derives from a talk with the same title, given to the Symposium at ICTP. I would like to thank the organizers, and especially Prof A Kuku, for arranging such a successful meeting. In view of the existence of the paper [10], whose results were a focus of the talk, I have chosen to concentrate here on new material, and minimize overlap with that paper.

Most algebraic K-theorists probably associate the term localization with a procedure that involves focusing on preferred elements of a ring or module by inverting others. In many cases this is an embedding. On the other hand, the plus-construction is designed to kill certain elements of the fundamental group of a space, resulting in a surjection. Here we consider the localization of rings and modules from two distinct viewpoints. Each has given rise to important developments. While the former is likely the more obvious to readers, it is the latter, emphasizing the functorial properties, that leads to the line of thought we explore later in this article.

Our short description of the historical development of localization theory starts with the "inversion of elements" approach. We note how various category-theoretic foundations of K-theory arise from this model. The second section introduces an alternative viewpoint. It concerns idempotent monads. Innovations in algebraic topology have enabled the plus-construction $X \to X^+$ to be accommodated within this framework as an example of W-nullification. As shown in [10], the key ingredient here is an understanding of the effect on fundamental groups of maps to X from an acyclic space W. In particular, the theory depends on the perfect normal subgroup structure of $\pi_1(X)$, and those perfect normal subgroups that are "swept" by W. This material is reviewed quickly in the third section.

In the fourth and fifth sections of the paper we concentrate on the case where $X = B\,\mathrm{GL}\,R$ for some ring R. First we characterize the perfect normal subgroups in general as the groups $E(R, I)$ with I an idempotent ideal of R. Then we look at those that may be swept by perfect groups and their classifying spaces. Group rings provide an important class of examples here.

The paper concludes with a discussion initiated in [10], concerning whether

W-nullification of a class of spaces being the same thing as the plus-construction forces acyclicity of W. The spaces of interest are again the classifying spaces of general linear groups of rings. It is shown that for finite-dimensional W, acyclicity does indeed follow.

The exposition of the first two sections claims no special "historical accuracy", a concept that most mathematicians anyway probably find unsatisfactory. As readers may be aware, sometimes the written record suggests that paper A preceded paper B, whereas in fact paper A was initiated by the private communication of the ideas of a draft of paper B. Since mathematical ideas are more important than egos, I have not stressed issues of priority in this account. Comments of the referee helpful to the exposition have been gratefully accepted.

1 A brief history of the philosophy of localization, I: the inversion approach

The prototypical localization is that of the integers at some prime; for instance $\mathbb{Z}_{(2)} = \mathbb{Z}[\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, ...]$. Evidently this is constructed by inverting elements, an idea that makes obvious the following generalization. For commutative rings R and their modules, one just chooses to invert the elements of a multiplicatively closed set Σ of non-zerodivisors, such as the complement of a prime ideal (the case that leads to a local ring, hence the name). The ring R_Σ is then defined by its traditional universal property, as follows.

Theorem 1.1 *R_Σ is the universal ring in which the elements of Σ become invertible, in that there is a canonical ring embedding $l : R \to R_\Sigma$ and if $f : R \to S$ is any other ring homomorphism that has the image of each element of Σ invertible, then there is a unique ring homomorphism $\overline{f} : R_\Sigma \to S$ with $l\overline{f} = f$. Moreover, every element of R_Σ has the form $l(r)l(\sigma)^{-1}$ for some $r \in R$, $\sigma \in \Sigma$.*

In particular, for a set of primes $P \subseteq \mathbb{Z}$ one can take Σ to comprise those integers coprime to all elements of P, to obtain the localization denoted $\mathbb{Z}_P$.

The game gets more interesting when the ring R is not commutative and Σ is not even central. There is clearly a problem with writing a product $r_1\sigma_1^{-1}r_2\sigma_2^{-1}$ in the required form $r\sigma^{-1}$ (suppressing the embedding l from the notation). This is overcome by imposing the (right) Ore condition: given $a \in R$, $\sigma \in \Sigma$, there exist $c \in R$, $\nu \in \Sigma$ with $a\nu = \sigma c$

$$\begin{tikzcd}
\bullet \arrow[r, dashed, "\exists c\cdot"] \arrow[d, dotted, no head, "\exists\nu\cdot"'] & \bullet \arrow[d, "\sigma\cdot"] \\
\bullet \arrow[r, "a\cdot"] & \bullet
\end{tikzcd}$$

This condition is both necessary and sufficient for a multiplicative subset Σ of non-zerodivisors to give rise to a homomorphism $l : R \to R_\Sigma$ with the properties of the above theorem; more generally, instead of consisting of non-zerodivisors Σ need only have the regularity property that, whenever $\sigma \in \Sigma$ has $a, b \in R$ with $\sigma a = \sigma b$, then for some $\sigma' \in \Sigma$, also $a\sigma' = b\sigma'$

$$\begin{tikzcd}
\bullet \arrow[r, dashed, "\exists\sigma'\cdot"] & \bullet \arrow[r, shift left, "a\cdot"] \arrow[r, shift right, "b\cdot"'] & \bullet \arrow[r, "\sigma\cdot"] & \bullet.
\end{tikzcd}$$

(See for example [31](2.1) for proofs.)

I have portrayed the above conditions diagramatically in order to suggest the generalization of [24]. If one generalizes from a ring as an additive category with a single object, to the case where Σ is a set of morphisms in a category $\mathcal{C}$ (under suitable smallness hypotheses), then Σ is said to admit a calculus of right fractions if, as well as being closed under composition and containing all identity morphisms, it satisfies the properties of the above diagrams. Note that this holds if Σ is a class of monomorphisms preserved by pullbacks (base change). Given this, one can form a category of fractions $\mathcal{C}[\Sigma^{-1}]$ having the same objects as $\mathcal{C}$. Its morphisms from Y to X are the diagrams of pairs (a, σ) with $\sigma \in \Sigma$

$$X \xleftarrow{a} Z \xrightarrow{\sigma} Y,$$

subject to the equivalence relation which is the symmetric closure of the order relation: $(a,\sigma) \leq (b,\tau)$ whenever there is a commuting diagram

$$\begin{tikzcd}
X \arrow[d, "\mathrm{id}"] & W \arrow[l, "b"'] \arrow[r, "\tau"] \arrow[d] & Y \arrow[d, "\mathrm{id}"] \\
X & Z \arrow[l, "a"'] \arrow[r, "\sigma"] & Y.
\end{tikzcd}$$

(In other words, the morphism set is obtained as a direct limit.) There results a localization functor $\mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ with the universal property generalizing that of the above theorem.

Theorem 1.2 [24] *The localization functor $L_\Sigma : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ makes the morphisms of Σ invertible and has the universal property that if $F : \mathcal{C} \to \mathcal{C}'$ is any functor that makes the morphisms of Σ invertible, then there exists a unique functor $\overline{F} : \mathcal{C}[\Sigma^{-1}] \to \mathcal{C}'$ with $L\overline{F} = F$.*

An important special case originated in algebraic topology, where Serre [41] wished to consider homotopy and homology groups localized at a particular prime, and also to discuss classes of abelian groups modulo a subclass, like the class of all finite abelian groups. This led to the concept of a Serre, or thick, subcategory $\mathcal{D}$ of an abelian category $\mathcal{C}$ [26]. Then Σ comprises those morphisms of $\mathcal{C}$ whose kernel and cokernel lie in $\mathcal{D}$. Inverting Σ consequently has the same effect as equating the objects of $\mathcal{D}$ to zero. The quotient category $\mathcal{C}/\mathcal{D}$ may be regarded as obtained either by inverting morphisms or by killing objects. (For recent expositions, see [45](10.3) and [11]ch.6.) Thus the significance of the inversion-of-elements viewpoint begins to recede, while the universal functorial property comes to the fore.

For a variant of the above that hugely influenced the development of algebraic K-theory, Quillen [39] took a class of "admissible" monomorphisms Σ_R, admitting a calculus of right fractions as above (given by cartesian squares), together with, dually, a class of admissible epimorphisms Σ_L admitting a calculus of left fractions (given by cocartesian squares). Working with $\mathcal{C}$ inside an ambient abelian category, he then performed the Q-construction by taking, as objects for $Q\mathcal{C}$ the objects of $\mathcal{C}$, and as morphisms from X to Y those pairs $(a,\sigma) \in \Sigma_L \times \Sigma_R$ subject to the equivalence relation given by $\leq$ itself ($\leq$ as above). This means that (a, σ) and (b, τ) define the same morphism if both $(a, \sigma) \leq (b, \tau)$ and $(b, \tau) \leq (a, \sigma)$, a definition easily checked to be equivalent to that of [39]. The following routine technical lemma shows that the procedure cannot be iterated to form anything else new.

Lemma 1.3 *Every morphism of $Q\mathcal{C}$ is a monomorphism, and every epimorphism of $Q\mathcal{C}$ is an isomorphism.*

In a departure from the situation of the previous theorem, Q is not a functor in the conventional sense, being covariant on Σ_R and contravariant on Σ_L.

Another development of Quillen [25] generalizes from inverting the module action, $(\sigma, m) \longmapsto \sigma.m$ of a submonoid Σ of regular elements of a ring R on a left R-module M, to inversion of the action of a monoidal category $\mathcal{S}$ on another category $\mathcal{M}$. This leads to a functor from $\mathcal{M}$ to a new category $\mathcal{S}^{-1}\mathcal{M}$. The motivating example is where $\mathcal{S} = \mathcal{M}$ is the category of all finitely projective modules over a given ring, with their isomorphisms as morphisms. Then the action of one isomorphism on another is given by taking their direct sum, regarded as an isomorphism on the direct sum of the underlying projective modules. Inversion now refers to inverting the effect on the homotopy type of the classifying space of the category. (See [39] for the relation between functors and homotopy of classifying spaces.) Again, a key result is that iteration of the procedure yields nothing new.

Theorem 1.4 [25]
(a) *$\mathcal{S}$ acts invertibly on $\mathcal{S}^{-1}\mathcal{M}$.*
(b) *The canonical functor $\mathcal{M} \mapsto \mathcal{S}^{-1}\mathcal{M}$ induces a homotopy equivalence on classifying spaces if and only if $\mathcal{S}$ acts invertibly on $\mathcal{M}$.*

2 A brief history of the philosophy of localization, II: the idempotence approach

The naive starting-point for the second approach is the localization of a $\mathbb{Z}$-module, in other words, an abelian group. Since in algebraic topology abelian groups commonly arise as invariants of a space, this prompted the challenge of finding a functorial map that could be applied to a space so as to induce localization of its homology and/or homotopy groups. (Thus Sullivan [42]p.0.1: *This compulsion to localize began with the author's work on invariants of combinatorial manifolds in 1965-67.*) The nonabelian nature of fundamental groups presented a major hurdle here. It was initially sidestepped by considering only simple (or abelian [37]) spaces. Such spaces mimic the algebraic situation of abelian groups; by means of Postnikov decompositions they are built out of Eilenberg-Mac Lane spaces $K(A, n)$, for which localization of the abelian group A induces topological localization. For a fixed set of primes $P \subseteq \mathbb{Z}$, an abelian group is said to be P-local if it is a $\mathbb{Z}_P$-module. A simple space is correspondingly called P-local if all its homotopy groups are $\mathbb{Z}_P$-modules.

Theorem 2.1 ([42] Theorem 2.1) *Let $l : X \to X'$ be a map of simple spaces. Then the following are equivalent.*

(i) *l is the initial map in the category of homotopy classes of maps under X into P-local spaces;*

(ii) *on integral homology groups l induces P-localization:*

$$\tilde{H}_*(X) \xrightarrow{l_*} \tilde{H}_*(X') \cong \tilde{H}_*(X) \otimes \mathbb{Z}_P;$$

(iii) *on homotopy groups l induces π-localization:*

$$\pi_*(X) \xrightarrow{l_*} \pi_*(X') \cong \pi_*(X) \otimes \mathbb{Z}_P.$$

Of course, simply-connected spaces afforded an especially tractable class of simple spaces for this theory [35].

The next step towards greater generality seized upon a further feature of localization: localization of modules is an exact functor. This suggested that in a situation where one had good control over exactness, one could expand the domain of localization beyond abelian groups. Thus, arguing algebraically via central extensions, or topologically via Postnikov systems of Eilenberg-Mac Lane spaces, several authors [17], [30], [37] provided a fruitful definition of localization for nilpotent groups and spaces. However, attempts to generalize to localization of all groups and spaces foundered, and relatively little progress was made in this direction for more than a decade. (Perhaps the search for an exactness property proved counterproductive.) Yet the seeds for ultimate success were sown at about this time. They lie in the following, routinely verified, alternative description of the universal property that has been noted several times above.

Lemma 2.2 *Let* $L : \mathcal{C} \to \mathcal{C}$ *be a functor and* $l : \mathrm{Id}_{\mathcal{C}} \to L$ *be a natural transformation. Define* $\mathcal{L}$ *to be the full subcategory of* $\mathcal{C}$ *whose objects are those* Y *isomorphic to some* LX *with* X *an object of* $\mathcal{C}$*. Then the following are equivalent.*

(i) *For each object* X *of* $\mathcal{C}$, $l_X{:}X \to LX$ *has the universal property with respect to each object* Y *of* $\mathcal{L}$:

$$\begin{tikzcd} X \arrow[rr, "l_X"] \arrow[dr, "f"] & & LX \arrow[dl, "\exists! f'"'] \\ & Y & \end{tikzcd}$$

(ii) *For each object* X *of* $\mathcal{C}$, $l_{LX} = Ll_X$ *and is an isomorphism.*

(iii) *For each object* X *of* $\mathcal{C}$, $l_{LX} = Ll_X$ *and has a left inverse.*

Adams [1] called $\mathcal{L}$ a reflective subcategory, whose objects are the local objects of the theory. From this viewpoint, a localizing functor L is just the same thing as an idempotent monad functor. (For, (ii) above is what a monad reduces to, under the further assumption of idempotence.) This is consistent with the theorems in the previous section that repetition of localization produces nothing new. In particular, the traditional localization of a ring or module is a localization in the above sense. In this light, here are three "new" examples of localizations.

Example 2.3

Abelianization of groups. As is well known, taking the quotient of a group G by its commutator subgroup $[G, G]$ produces a group G_{ab} which is the initial abelian group under G via the canonical homomorphism $l : G \to G/[G, G]$; moreover $(G_{\mathrm{ab}})_{\mathrm{ab}} = G_{\mathrm{ab}}$.

Hypoabelianization of groups. Here the local objects are the hypoabelian groups, namely those groups lacking a nontrivial perfect subgroup (a perfect subgroup being one generated by its commutators). Since every group G has a maximum perfect subgroup $\mathcal{P}G$, and always $\mathcal{P}(G/\mathcal{P}G) = 1$, the homomorphism $G \to G/\mathcal{P}G$ is initial among maps from G to hypoabelian groups. Evidently the construction is idempotent.

Plus-construction of spaces. For a connected space X, Quillen defined a map $q_X : X \to X^+$ that, as a universal construction, is initial among all maps from X to spaces whose fundamental group is hypoabelian. Viewed as an idempotent construction, q_X induces an isomorphism of homology groups (arbitrary abelian local coefficients) and hypoabelianization of fundamental groups.

An interesting point here, illustrated by the above examples, is that we are now as likely to be talking about killing elements (such as commutators) as about inverting elements. This highlights the contrast between the present approach and that of the previous section. Closely related concepts include factorization systems [16] and orthogonal pairs [21]. While some examples of idempotent monad functors on non-nilpotent groups [40], [15] and spaces [14] were defined in the mid-seventies, it was not until much later that their place in an overall scheme became apparent [20]. Thus, whereas for each prime p there is a canonical definition of the p-localization of a nilpotent group, for groups that need not be nilpotent a whole family of possible p-localizations offers itself, each a generalization of nilpotent localization. At one extreme is that of [40], while at the other is that of [13] and [19], with those of [15] sandwiched in between.

In the setting of the based homotopy category of CW-complexes, an important advance has been the notion of localization with respect to a map

$f : W \to W'$. Then a space X is said to be f-local (or f-periodic) if the induced map on based mapping spaces

$$f^* : \mathrm{map}_*(W', X) \to \mathrm{map}_*(W, X)$$

is a weak homotopy equivalence [16], [22]. This model seems to accommodate the examples of interest, for instance, Anderson's version of Sullivan's P-localization of simply connected spaces [3]. For further discussion, see [18], [23].

In fact, one can simplify further, and usefully consider W' to be a point. Thus one is left considering localization with respect to a space W. This concept, known as the W-nullification $P_W X$ of a space X, turns out to have illuminating application to the plus-construction. The challenge is to determine for which spaces W the plus-construction $q_X : X \to X^+$ coincides with the map $X \to P_W X$, where $\mathrm{map}_*(W, P_W X)$ is weakly contractible (that is, $P_W X$ is W-null), and $P_W X$ is initial in the homotopy category of W-null spaces under X.

3 The plus-construction as a localization or nullification

This section reviews some of the theory of [10], by which an acyclic space W yields the plus-construction as the W-nullification functor P_W. It involves the relative form of the plus-construction $X \to X^+_N$, characterized as follows.

Lemma 3.1 *Let X be any connected space. If N is a perfect normal subgroup of $\pi_1(X)$, then any map $h\colon X \to Y$ such that $h_*(N)$ is trivial factors through the Quillen map $q\colon X \to X^+_N$, uniquely up to homotopy under X.*

In our situation, a fixed connected, based space W, not necessarily acyclic but assumed to have perfect fundamental group, determines a choice of N in the following way. For every connected, based space X, the group $S(W, X)$ swept by W is the subgroup of $\pi_1(X)$ generated by the images of all homomorphisms $\pi_1(W) \to \pi_1(X)$ that are induced by based maps $W \to X$. It is a perfect normal subgroup of $\pi_1(X)$ with the following characterization.

Lemma 3.2 *For each X, the space $Y(1) = X^+_{S(W,X)}$ together with the corresponding Quillen map is initial in the homotopy category of spaces Y under X such that every composite map $W \to X \to Y$ sends $\pi_1(W)$ to 1.*

However, this is not quite the universal property that we are after. Although we seek the initial space Y under X such that $S(W,Y) = 1$, it may be that $S(W,Y(1)) \neq 1$, by virtue of the existence of a map from W to $Y(1)$ that fails to lift to X. To appreciate this point (and in the absence of a specific example!), it is simpler to consider the group-theoretic analogue.

For groups G and H, the normal subgroup $S(G,H)$ of H swept by G is that subgroup generated by the images of all homomorphisms $G \to H$. It is perfect when G is. Although $Q(H,1) = H/S(G,H)$ is initial in the category of groups Q under H such that every composite homomorphism $G \to H \to Q$ has a trivial image, there may exist a nontrivial homomorphism from G to $Q(H,1)$. The simplest example is when G, H and $Q(H,1)$ are cyclic of orders p, p^2 and p respectively. In order to construct a quotient of H that receives no nontrivial homomorphism from G, it is necessary to iterate transfinitely, as follows. Suppose that the quotient $Q(H,\alpha) = H/T(\alpha)$ is defined (where $T(1) = S(G,H)$). Then we define

$$H/T(\alpha+1) = Q(H,\alpha+1) = Q(Q(H,\alpha),1) = Q(H,\alpha)/S(G,Q(H,\alpha)).$$

For a limit ordinal α, define $T(\alpha)$ to be the union of the subgroups $T(\beta)$ with $\beta < \alpha$. Finally, set $T(G,H)$ equal to $T(\nu)$ for the smallest ordinal ν such that $T(\nu) = T(\nu+1)$. It is easy to check that for fixed G each $T(\alpha)$ is functorial in H, perfect when G is. Then $H/T(G,H)$ is the initial group under H that fails to receive a nontrivial homomorphism from G.

For the topological counterpart, instead of group quotients we proceed by taking relative plus-constructions. Put $N(1) = S(W,X)$ and, given the subgroup $N(\alpha)$ of $\pi_1(X)$ and space $Y(X,\alpha) = X^+_{N(\alpha)}$, define

$$Y(X,\alpha+1) = Y(Y(X,\alpha),1) = Y(X,\alpha)^+_{S(W,Y(X,\alpha))} = X^+_{N(\alpha+1)}.$$

Again, for a limit ordinal α, define $N(\alpha)$ to be the union of the subgroups $N(\beta)$ with $\beta < \alpha$. Again there is suitable functoriality, and the union of all $N(\gamma)$ is a functorial subgroup $T(W, X)$ of $\pi_1(X)$, such that the space $X^+_{T(W,X)}$ is the initial space in the homotopy category of spaces Y under X with $S(W, Y)$ trivial. Moreover, when $\pi_1(W)$ is perfect, so is $T(W, X)$ and we obtain a lattice of perfect normal subgroups of $\pi_1(X)$ (with all arrows inclusions):

$$\begin{tikzcd}
 & \mathcal{P}\pi_1(X) & \\
 & T(\pi_1(W), \pi_1(X)) \arrow[u] & \\
S(\pi_1(W), \pi_1(X)) \arrow[ur] & & T(W, X) \arrow[ul] \\
 & S(W, X) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Here is the result linking $T(W, X)$ to the W-nullification $P_W X$.

Theorem 3.3 *Let W be any connected space. Then the following statements are equivalent:*

(i) *W is acyclic.*

(ii) *The class of W-null spaces coincides with the class of spaces X such that $S(W, X)$ is trivial.*

(iii) *For every space X, the W-nullification map $l_X \colon X \to P_W X$ coincides, up to homotopy under X, with the plus-construction with respect to the perfect normal subgroup $T(W, X)$ of $\pi_1(X)$.*

(iv) *For every space X, the map $l_X \colon X \to P_W X$ is an integral homology equivalence.*

In the case of main interest to us here, the space X is the classifying space of a discrete group (indeed, a general linear group). Then the above lattice simplifies a little, by virtue of the adjunction between the fundamental group and classifying space functors.

Lemma 3.4 *When* $X = BH$, *and* $\pi_1(W)$ *is perfect,*

$$S(W,X) = S(\pi_1(W), \pi_1(X)) \leq T(W,X) \leq T(\pi_1(W), \pi_1(X)) \leq \mathcal{P}\pi_1(X).$$

One might imagine that since the inductive constructions for $T(W,X)$ and $T(\pi_1(W), \pi_1(X))$ have the same beginning, they would continue to be equal. We now introduce an example to show that that need not be the case.

Example 3.5
We take $W = BQ$ where Q is the simple alternating group

$$Q = \mathfrak{A}_5 = PSL(2,5) = \left\langle a, b, c \mid a^2 = b^3 = c^5 = abc = 1 \right\rangle,$$

and write L for its universal central extension, the binary icosahedral group

$$L = SL(2,5) = \left\langle x, y, z \mid x^2 = y^3 = z^5 = xyz \right\rangle,$$

having centre of order 2, generated by x^2. Let P be an amalgamated free product of these:

$$P = Q *_{a=x^2} L;$$

and put $X = BP$. Then the homology exact sequence for free products with amalgamation gives a short exact sequence

$$0 \to H_2(Q) \to H_2(P) \to H_1(\langle a \rangle) \to 0.$$

To see that this exact sequence is not split, we construct the Schur multiplier $H_2(P)$ of the perfect group P as the kernel of the projection from its universal central extension. The group U with presentation given by generators a, b, c, x, y, z, u and relations

$$a^2 = b^3 = c^5 = abc,\ x^2 = y^3 = z^5 = xyz,\ ax = xa,\ x^2 = au,\ u \in \mathcal{Z}(U)$$

is evidently perfect, and has the central element u of order 4 generating the kernel of the obvious homomorphism onto P. Thus U is the universal central extension and $H_2(P)$ is cyclic. So $H_2(Q)$ is not a direct summand of $H_2(P)$.

Now $S(W,X) = S(Q,P)$ is clearly the normal subgroup $\langle Q \rangle^P$ of P generated by its subgroup Q. Therefore $X^+_{S(W,X)}$ has fundamental group $P/\langle Q \rangle^P \cong$

Q. Since Q is simple, its only nontrivial endomorphisms are isomorphisms. Accordingly, any nontrivial map from W to BQ induces an isomorphism on homology. So, by our calculation above, it cannot factor through the space $X^+_{S(W,X)}$ that has the same homology as $X = BP$. Hence every map from W to $X^+_{S(W,X)}$ has nullhomotopic composite

$$W \to X^+_{S(W,X)} \to B(\pi_1(X^+_{S(W,X)})) = BQ$$

and must therefore be trivial on fundamental groups. The effect is that

$$S(W, X^+_{S(W,X)}) = 1,$$

while

$$S(W, B(\pi_1(X^+_{S(W,X)}))) = S(BQ, BQ) = Q.$$

Hence

$$T(BQ, BP) = S(Q, P) = \langle Q\rangle^P\,; \qquad T(Q, P) = P.$$

4 Normal subgroups of general linear groups

We examine the possible perfect normal subgroups of $\mathrm{GL}\,R$, in order to determine $T(W, B\,\mathrm{GL}\,R)$ in the next section. A key lemma is the Sandwich Theorem that constrains possible normal subgroups of $\mathrm{GL}\,R$, in terms of level ideals of R.

Lemma 4.1 [4](V.2.1). *A subgroup N of $\mathrm{GL}\,R$ is normal if and only if for some R-ideal $\mathfrak{a}$ (the* level *of N), which is necessarily unique,*

$$E(R, \mathfrak{a}) \leq N \leq \mathrm{GL}(R, \mathfrak{a});$$

moreover,

$$[E(R, \mathfrak{a}), ER] = E(R, \mathfrak{a}) = [\,\mathrm{GL}(R, \mathfrak{a}), \mathrm{GL}\,R].$$

We write $C_G(n)$ to denote the centralizer in G of an element n.

Lemma 4.2 *Let N be a normal subgroup of a subgroup H of a group G. If for each element n of N, $G = HC_G(n)$, then N is normal in G.*

PROOF. Given $g \in G$, write $g = hz$ where $h \in H$ and z commutes with n. Then $gng^{-1} = hnh^{-1} \in N$. $\square$

The following application seems remarkably to have been neglected in the literature (although (a) is implicit in [4](V.2.1)). For (b), we recall that the cone CR of a ring R is the ring comprising those matrices, indexed by the natural numbers, that have at most a finite number of nonzero entries in each row and column. Regard $\mathrm{GL}\, R$ as a normal subgroup of $\mathrm{GL}\, CR$ by identifying $\mathrm{GL}\, R$ with the group of all invertible matrices that differ from the identity matrix by a finite matrix (see [43] for details).

Proposition 4.3 (a) *Every normal subgroup of ER is also normal in* $\mathrm{GL}\, R$. (b) *Every normal subgroup of* $\mathrm{GL}\, R$ *is also normal in* $\mathrm{GL}\, CR$.

PROOF. It is shown in [5]pp.82,83 that the pairs $ER \leq \mathrm{GL}\, R$ and $\mathrm{GL}\, R \leq \mathrm{GL}\, CR$ satisfy the requirements of the lemma. Specifically, for (a), if $\alpha \in E_k R$, then any $\beta \in \mathrm{GL}_l\, R$ may be written as

$$\begin{aligned} \beta &= ((\beta \oplus I_k) \oplus (\beta \oplus I_k)^{-1}) \cdot (I_{l+k} \oplus (\beta \oplus I_k)) && (4.1) \\ &\in ER \cdot C_{\mathrm{GL}\, R}(\alpha). && (4.2) \end{aligned}$$

For (b), first regard an element α of $\mathrm{GL}\, R$ as lying in some $\mathrm{GL}(M_p R)$. Next, since $\mathrm{GL}\, CR = ECR$, we may write any element of $\mathrm{GL}\, CR$ as a product of elementary matrices $e_{st}(\gamma)$, where γ is an infinite matrix with only finitely many nonzero entries in each row and column. We can further write each $e_{st}(\gamma)$ as $e_{st}(\gamma_1)e_{st}(\gamma_2) = e_{st}(\gamma_1 + \gamma_2)$, where γ_1 lies in $M_p R$ and the nonzero (i, j)-entries of γ_2 have both i and j greater than p. So $e_{st}(\gamma_1) \in \mathrm{GL}\, R$, while $e_{st}(\gamma_2) \in C_{\mathrm{GL}\, CR}(\alpha)$. $\square$

The following basic trick appears in various guises in the literature. (For examples, see [36] (3.3.1), and the main theorem of [34].)

Lemma 4.4 *Let C be a ring with right ideals $\mathfrak{a}_i$ and groups $G_i \subseteq 1 + \mathfrak{a}_i$ $(i = 1, 2)$. Then the commutator subgroup $[G_1, G_2]$ satisfies*

$$[G_1, G_2] \subseteq 1 + \mathfrak{a}_1\mathfrak{a}_2 + \mathfrak{a}_2\mathfrak{a}_1.$$

Proof. For $x_i \in G_i$ we have

$$\begin{align}
x_1x_2x_1^{-1}x_2^{-1} - 1 &= (x_1x_2 - x_2x_1)x_1^{-1}x_2^{-1} \tag{4.3}\\
&= ((x_1-1)(x_2-1) - (x_2-1)(x_1-1))x_1^{-1}x_2^{-1} \tag{4.4}\\
&= (x_1-1)(x_2-1)x_1^{-1}x_2^{-1} - (x_2-1)(x_1-1)x_1^{-1}x_2^{-1} \tag{4.5}\\
&\in \mathfrak{a}_1\mathfrak{a}_2C + \mathfrak{a}_2\mathfrak{a}_1C, \tag{4.6}
\end{align}$$

as required. □

For an important application we take C to be the cone CR of a ring R. The subtler, unstable version of the next result forms the theorem of [33].

Lemma 4.5 *For $i = 1, 2$, let H_i be a normal subgroup of $\operatorname{GL} R$ of level $\mathfrak{i}_i$. Then the normal subgroup $[H_1, H_2]$ has level $\mathfrak{i}_1\mathfrak{i}_2 + \mathfrak{i}_2\mathfrak{i}_1$.*

Proof. We show that

$$E(R, \mathfrak{i}_1\mathfrak{i}_2 + \mathfrak{i}_2\mathfrak{i}_1) \leq [E(R, \mathfrak{i}_1), E(R, \mathfrak{i}_2)]$$

and

$$[\operatorname{GL}(R, \mathfrak{i}_1), \operatorname{GL}(R, \mathfrak{i}_2)] \leq \operatorname{GL}(R, \mathfrak{i}_1\mathfrak{i}_2 + \mathfrak{i}_2\mathfrak{i}_1),$$

forcing $[H_1, H_2]$ to have level $\mathfrak{i} = \mathfrak{i}_1\mathfrak{i}_2 + \mathfrak{i}_2\mathfrak{i}_1$.

To do this, first recall that $E(R, \mathfrak{i})$ has generators of the form $\alpha e_{ij}(a)\alpha^{-1}$ where $\alpha \in E(R)$ is a product of elementary matrices and $e_{ij}(a) \in E(\mathfrak{i})$ denotes the matrix that differs from the identity matrix in respect only of an entry $a \in \mathfrak{i}$ in the (i, j)-position. Our decomposition of $\mathfrak{i}$ means that we can write

$$a = \sum_p a'_p a''_p + \sum_q b''_q b'_q,$$

a finite sum with each $a'_p, b'_q \in \mathfrak{i}_1, a''_p, b''_q \in \mathfrak{i}_2$. So for any k distinct from both i and j,

$$\begin{align}
&\alpha e_{ij}(a)\alpha^{-1} \tag{4.7}\\
= &\prod_p [\alpha e_{ik}(a'_p)\alpha^{-1}, \alpha e_{kj}(a''_p)\alpha^{-1}] \prod_q [\alpha e_{kj}(b'_q)\alpha^{-1}, \alpha e_{ik}(b''_q)\alpha^{-1}]^{-1} \tag{4.8}\\
\in &[E(R, \mathfrak{i}_1), E(R, \mathfrak{i}_2)]. \tag{4.9}
\end{align}$$

Turning now to GL, we apply the lemma with $C = CR$ and $\mathfrak{a}_i$ the kernel of the canonical map $CR \to C(R/\mathfrak{i}_i)$. Thus $G_i = \mathrm{GL}(R, \mathfrak{i}_i) = \mathrm{GL}(R) \cap (1 + \mathfrak{a}_i)$. This gives the result. $\square$

Theorem 4.6 *For any ring R, the perfect normal subgroups of $\mathrm{GL}\,R$ are the groups $E(R, \mathfrak{i})$ with $\mathfrak{i}$ an idempotent ideal of R. Hence there is an isomorphism between the poset of perfect normal subgroups of $\mathrm{GL}\,R$ and the poset of idempotent ideals of R.*

PROOF. First recall that $E(R, \mathfrak{i})$ is necessarily normal in $\mathrm{GL}\,R$, and of level $\mathfrak{i}$, by Lemma 4.1. By the lemma above, the level of its commutator subgroup must be $\mathfrak{i}^2$. So when $\mathfrak{i}$ is an idempotent ideal of R the level is again $\mathfrak{i}$, making

$$E(R, \mathfrak{i}) \leq [E(R, \mathfrak{i}), E(R, \mathfrak{i})].$$

In other words, $E(R, \mathfrak{i})$ is perfect.

Conversely, let P be a perfect normal subgroup of $\mathrm{GL}\,R$, of level $\mathfrak{i}$, say. Then, since P is perfect,

$$P = [P, P] \leq [\mathrm{GL}\,R, P] = E(R, \mathfrak{i}),$$

the last equality by Lemma 4.1 again. Thus $P = E(R, \mathfrak{i})$. Again, the level of $[P, P]$ must be $\mathfrak{i}^2$, so P being perfect implies that $\mathfrak{i}$ is idempotent. $\square$

Theorem 4.7 *The normal subgroups N of $\mathrm{GL}\,R$ that have finitely generated abelianization N_{ab} are precisely those N of nonzero idempotent level $\mathfrak{a}$ with $N_{\mathrm{ab}} = N/E(R, \mathfrak{a})$ a finitely generated subgroup of $K_1(R, \mathfrak{a})$.*

PROOF. First observe that if N has idempotent level $\mathfrak{a}$, then $N' = [N, N]$ satisfies

$$E(R, \mathfrak{a}) = E(R, \mathfrak{a})' \leq N' \leq [\mathrm{GL}(R, \mathfrak{a}), \mathrm{GL}\,R] = E(R, \mathfrak{a}).$$

In the other direction, we show that the hypothesis of finite generation ensures that $N' = E(R, \mathfrak{a})$ (where $\mathfrak{a}$ is the level of N). Then since N' has level $\mathfrak{a}^2$ by Lemma 4.5, $\mathfrak{a}$ must be idempotent.

The key step is to show that the conjugation-induced action of $\mathrm{GL}(CR)$ on N_{ab} must be trivial. We do this by showing that $\mathrm{GL}(CR)$ must have trivial image in the group $\mathrm{Aut}(N_{\mathrm{ab}})$, which is residually finite by finite generation of N_{ab}. Here are two distinct proofs. First, the group $E(CR) = \mathrm{GL}(CR)$ is both torsion-generated [8] and acyclic [43]. So by [7], $\mathrm{GL}(CR)$ has no finite image. Alternatively, because $\mathrm{GL}(CR)$ is a binate group [6], it has no finite image [2].

We therefore have

$$[N, \mathrm{GL}(CR)] \leq N'.$$

Hence by Lemma 4.1

$$E(R,\mathfrak{a}) = [E(R,\mathfrak{a}), ER] \leq [N, \mathrm{GL}\, R] \leq N' \leq [\mathrm{GL}(R,\mathfrak{a}), \mathrm{GL}\, R] = E(R,\mathfrak{a}),$$

so that $N' = E(R,\mathfrak{a})$ as required. □

For rings whose idempotent ideals are easily described, the above theorems give a great deal of information. For example, Krull's Theorem implies that a commutative local ring (and hence any commutative domain) has no proper nonzero idempotent ideals.

Corollary 4.8 *Let R be a commutative domain. Then*

(a) *the only nontrivial perfect normal subgroup of $\mathrm{GL}\, R$ is $E(R)$; and*

(b) *a normal subgroup N of $\mathrm{GL}\, R$ has N_{ab} finitely generated if and only if N contains ER and N/ER is a finitely generated subgroup of K_1R.*

This leads to a surprising conclusion about the size of the abelian quotient group $E(R,\mathfrak{a})/\mathrm{GL}(R,\mathfrak{a})'$. In the two extreme cases $\mathfrak{a} = 0, R$, this group is trivial.

Corollary 4.9 *Let R be the ring of integers in a number field. For any proper nonzero ideal $\mathfrak{a}$, the abelian group $E(R,\mathfrak{a})/\mathrm{GL}(R,\mathfrak{a})'$ is not finitely generated.*

Proof. We consider the extension of abelian groups

$$E(R,\mathfrak{a})/\mathrm{GL}(R,\mathfrak{a})' \rightarrowtail \mathrm{GL}(R,\mathfrak{a})_{\mathrm{ab}} \twoheadrightarrow K_1(R,\mathfrak{a}).$$

Now by [4](X.3.6) $K_1(R,\mathfrak{a})$ is finitely generated. Since $\mathfrak{a}$ is not idempotent, and $\mathrm{GL}(R,\mathfrak{a})$ certainly has level $\mathfrak{a}$, by the previous corollary $\mathrm{GL}(R,\mathfrak{a})_{\mathrm{ab}}$ cannot be finitely generated. So neither can $E(R,\mathfrak{a})/\mathrm{GL}(R,\mathfrak{a})'$. □

5 The swept subgroups of a general linear group

By Corollary 4.8, the only nontrivial perfect normal subgroup of $\mathrm{GL}\,\mathbb{Z}$ is $E(\mathbb{Z})$, a fact we now exploit. We refer below to group homomorphisms $G \to \mathrm{GL}\,R$ as *R-representations* of a group G.

Theorem 5.1 *If a perfect group G affords a nontrivial integral representation, then, for all rings R and spaces W with $\pi_1(W) = G$,*

$$S(W, B\,\mathrm{GL}\,R) = S(G, \mathrm{GL}\,R) = T(W, B\,\mathrm{GL}\,R) = T(G, \mathrm{GL}\,R) = E(R).$$

PROOF. We have just confirmed this result for $R = \mathbb{Z}$. For the general case, consideration of the canonical map $\mathbb{Z} \to R$ shows that $S(G, \mathrm{GL}\,R)$ contains the normalizer of the image of $E(\mathbb{Z})$, and in particular, all even permutation matrices. However, such matrices normally generate $E(R)$ [28], [5](9.4). □

For G with $T(G, \mathrm{GL}\,R)$ smaller than $E(R)$, we must therefore consider groups with no nontrivial integral representation. These are necessarily large groups, having, for example, no subgroups of finite index. Evidently $T(G, \mathrm{GL}\,\mathbb{Z})$ is trivial. To obtain examples of nontrivial $T(G, \mathrm{GL}\,R)$, we take R to be the group ring $A[G]$ of G over some ring A, comprising formal finite sums $\sum a_i g_i = \sum g_i a_i$. The kernel of the homomorphism $A[G] \to A$ that sends each group element g_i to 1 is the augmentation ideal $\mathrm{Aug}_A(G)$; it is the ideal generated by all elements of the form $g_i - 1$. Note that g_i may be taken to be a generator of G because

$$g_i g - 1 = g_i(g-1) + (g_i - 1), \tag{5.10}$$

$$g_i^{-1} g - 1 = g_i^{-1}(g-1) - g_i^{-1}(g_i - 1). \tag{5.11}$$

The following definitions are suggested by Theorem 4.6. Given G, R as above, let $\mathfrak{s}, \mathfrak{t}$ (or $\mathfrak{s}_R, \mathfrak{t}_R$ when necessary) denote the idempotent R-ideals defined by

$$S(G, \mathrm{GL}\,R) = E(R, \mathfrak{s}), \qquad T(G, \mathrm{GL}\,R) = E(R, \mathfrak{t}).$$

By choosing $G_1 = G_2 = G$ and $\mathfrak{a}_1 = \mathfrak{a}_2 = \mathrm{Aug}_A(G)$ in Lemma 4.4, we have $[G, G] \subseteq 1 + (\mathrm{Aug}_A(G))^2$. Thus we obtain the following candidates for $\mathfrak{t}_{A[G]}$ and $T(G, \mathrm{GL}(A[G]))$.

Lemma 5.2 *If G is a perfect group, then for any ring A, $\operatorname{Aug}_A(G)$ is an idempotent ideal of $A[G]$ and $E(A[G], \operatorname{Aug}_A(G))$ is a perfect normal subgroup of* $\mathrm{GL}(A[G])$.

To describe the relationship between $\mathfrak{t}_A$ and $\mathfrak{t}_{A[G]}$, we recall that there is an inclusion of A in $A[G]$, split by the augmentation map. So any ideal $\mathfrak{i}$ of A determines its extended ideal $\mathfrak{i}[G]$ of $A[G]$, the image of the map $\mathfrak{i} \otimes_A A[G] \to A[G]$. Thus in particular, we have available for comparison three idempotent ideals of $A[G]$, namely $\mathfrak{t}_{A[G]}, \mathfrak{t}_A[G]$ and $\operatorname{Aug}_A(G)$. In order to compare them we require a general lemma on functors like $S(G, -)$, $T(G, -)$.

Lemma 5.3 *Suppose that P is a functor from groups to perfect groups such that if $K \leq H$ then $PK \leq PH \leq H$. If H is the semidirect product of N by Q, and $[N, H] \leq PN$, then PH is the semidirect product of PN by PQ.*

Proof. The five-term homology exact sequence for the extension

$$N \cap PH \rightarrowtail PH \overset{\curvearrowleft}{\twoheadrightarrow} PQ$$

(which is split by functoriality), reads

$$H_2(PH) \overset{\curvearrowleft}{\rightarrow} H_2(PQ) \to (N \cap PH)/[N \cap PH, PH] \to 0 \to 0.$$

So we have

$$PN \leq N \cap PH = [N \cap PH, PH] \leq [N, H] \leq PN,$$

and thus recognize the kernel of the extension as PN. □

Theorem 5.4 *For any perfect group G and ring A,*

$$\mathfrak{t}_{A[G]} = \mathfrak{t}_A[G] + \operatorname{Aug}_A(G), \tag{5.12}$$

$$\mathfrak{s}_{A[G]} = \mathfrak{s}_A[G] + \operatorname{Aug}_A(G). \tag{5.13}$$

Proof. Writing $\mathfrak{a}$ for $\operatorname{Aug}_A(G)$, we apply the lemma to the split extension

$$\mathrm{GL}(A[G], \mathfrak{a}) \rightarrowtail \mathrm{GL}(A[G]) \overset{\curvearrowleft}{\twoheadrightarrow} \mathrm{GL}\, A.$$

We first show that

$$T(G, \mathrm{GL}(A[G], \mathfrak{a})) = S(G, \mathrm{GL}(A[G], \mathfrak{a})) = E(A[G], \mathfrak{a}).$$

Since $\mathrm{GL}(A[G], \mathfrak{a})/E(A[G], \mathfrak{a}) = K_1(A[G], \mathfrak{a})$ is abelian, it suffices to show that

$$E(A[G], \mathfrak{a}) \leq S(G, \mathrm{GL}(A[G])).$$

Now $E(A[G], \mathfrak{a})$ is normally generated by matrices of the form $\begin{pmatrix} 1 & 1-g \\ 0 & 1 \end{pmatrix}$ with $g \in G$. (Finite matrices are as usual stabilized by the adjunction of 1 down the diagonal.) We use the canonical homomorphism

$$\kappa : G \to \mathrm{GL}(A[G], \mathfrak{a}), \qquad g \longmapsto \begin{pmatrix} g & & 0 \\ & 1 & \\ 0 & & \ddots \end{pmatrix}$$

and write

$$\begin{aligned} \begin{pmatrix} 1 & 1-g \\ 0 & 1 \end{pmatrix} &= [\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \kappa(g)] \qquad (5.14) \\ &\in S(G, \mathrm{GL}(A[G], \mathfrak{a})). \qquad (5.15) \end{aligned}$$

Next, we note that the condition

$$[\mathrm{GL}(A[G], \mathfrak{a}), \mathrm{GL}(A[G])] = E(A[G], \mathfrak{a})$$

is fulfilled by [4](V.2.1). So, with $i_* : \mathrm{GL}\, A \hookrightarrow \mathrm{GL}(A[G])$, the lemma gives

$$E(A[G], \mathfrak{t}_{A[G]}) = E(A[G], \mathfrak{a}) \cdot i_* E(A, \mathfrak{t}_A).$$

Since the right hand product is a normal subgroup of $E(A[G])$, it may be rewritten

$$\begin{aligned} &E(A[G], \mathfrak{a}) \cdot E(A[G], \mathfrak{t}_A[G]) \qquad (5.16) \\ &= E(A[G], \mathfrak{a} + \mathfrak{t}_A[G]). \qquad (5.17) \end{aligned}$$

Likewise, $\mathfrak{s}_{A[G]} = \mathfrak{a} + \mathfrak{s}_A[G]$. □

Corollary 5.5 *Let G be a perfect group.*

(a) *G has no nontrivial A-representation if and only if*

$$S(G, \mathrm{GL}(A[G])) = T(G, \mathrm{GL}(A[G])) = E(A[G], \mathrm{Aug}_A(G)).$$

(b) *When A is a commutative domain, G has a nontrivial A-representation if and only if*

$$S(G, \mathrm{GL}(A[G])) = T(G, \mathrm{GL}(A[G])) = E(A[G]).$$

Example 5.6
In the literature there are many perfect groups G and fields A (and its subdomains) for which there is no nontrivial A-representation. Although it is tempting simply to extract these from the listing of counter-A-linear groups supplied in [9], one should first note that counter-linearity there refers to finite-dimensional representations. A group may be counter-A-linear and yet admit an A-representation in our present sense. Nevertheless, available proofs of counter-A-linearity can in certain cases be readily extended to prove the absence of an A-representation; we suppress the details.

(a) When G is finitely generated, any A-representation is necessarily finite-dimensional. Then by [32] the image of G is residually finite, and so G admits an integral representation. It follows that if $T(G, \mathrm{GL}\,A)$ is nontrivial for some field A, then $T(G, \mathrm{GL}\,R) = \mathrm{GL}\,R$ for all rings R. On the other hand, Higman's four-generator, four-relator group [29] has $T(G, \mathrm{GL}\,A)$ trivial for all fields A.

(b) By adapting the proof of Theorem 3.2 of [2], one sees that any binate group [6] has no A-representation.

(c) Arguing as in [9](3.5), one concludes that the acyclic automorphism groups of de la Harpe & McDuff [27] also admit no A-representation.

(d) Suppose that G is a torsion-generated group with only finitely many nonzero $H_i(G; \mathbb{Z})$. Then by [12] G has no $\mathbb{C}$-representation.

6 Further relations with K-theory

The starting point for this discussion is the assertion in Theorem 3.3 above that if W is a space for which, for all spaces X, the map $X \to P_W X$ is acyclic, then W is acyclic. This is because in particular the acyclic map $W \to P_W W$ must be nullhomotopic. The observation suggests the question as to whether acyclicity of W can be deduced from the same property on a more restrictive class of spaces X. In particular, we are naturally interested here in considering classifying spaces of general linear groups. A first step in this direction is the following slight improvement on [10](5.5). (Unless otherwise stated, all homology and cohomology groups in this section have trivial integral coefficients.)

Lemma 6.1 *Let W be a space such that the map $B\mathrm{GL}\mathbb{C} \to P_W B\mathrm{GL}\mathbb{C}$ induces an injection on $H_1(-)$, where $\mathrm{GL}\mathbb{C}$ has the discrete topology. Then $H_1(W) = 0$.*

Proof. If the abelian group $H_1(W)$ is nonzero, then it admits a nontrivial homomorphism ψ to the group $\mathrm{GL}_1\mathbb{C}$, since that abelian group is divisible and contains elements of all finite orders. Now the composite map

$$W \to B\pi_1(W) \to BH_1(W) \xrightarrow{B\psi} B\mathrm{GL}_1\mathbb{C} \to B\mathrm{GL}\mathbb{C} \to P_W B\mathrm{GL}\mathbb{C}$$

is homotopically trivial. However, on the first homology group it coincides with an injection following ψ, since $H_1(B\mathrm{GL}_1\mathbb{C}) \to H_1(B\mathrm{GL}\mathbb{C})$ is inverse to the determinant isomorphism from $H_1(B\mathrm{GL}\mathbb{C}) = K_1(\mathbb{C})$ to $\mathrm{GL}_1\mathbb{C}$. This gives the desired contradiction. □

Here is a related result. Recall from Corollary 4.8 that $T(W, \mathrm{GL}\,\mathbb{C})$ is either trivial or $E\mathbb{C}$. Thus if $P_W B\,\mathrm{GL}\,\mathbb{C} = B\,\mathrm{GL}\,\mathbb{C}^+_{T(W,B\,\mathrm{GL}\,\mathbb{C})}$, then either $\mathrm{BGL}\,\mathbb{C}$ or $B\,\mathrm{GL}\,\mathbb{C}^+$ is W-null. Since W-nullity of X means that for all $n \geq 0$ $[W, \Omega^n X]$ is trivial, in the former case we have the complex topological K-theory $\tilde{KU}^*(W)$ zero. The latter case has a different outcome.

Proposition 6.2 *If $B\,\mathrm{GL}\,\mathbb{C}^+$ is W-null, then the Chern character*

$$\mathrm{ch} : \tilde{KU}^*(W) \to \tilde{H}^*(W; \mathbb{Q})$$

is an isomorphism.

Proof. We use the fibration $FU \to BU \to \prod K(\mathbb{Q}, 2n)$, the second map corresponding to the Chern character. Since by [44] FU is a retract of $B\,\mathrm{GL}\,\mathbb{C}^+$, we obtain the required isomorphism $[W, BU] \cong [W, \Omega^2 BU] \to [W, \Omega^2 \prod K(\mathbb{Q}, 2n)]$.

The same argument applies to ΣW to yield the odd-dimensional cohomology groups. □

In [10](5.6), Lemma 6.1 is used to show that it suffices to assume that P_W coincides with the plus-construction on classifying spaces of discrete groups, in order to conclude that W is acyclic. Note that this refers to the full plus-construction on BG, that is, with respect to $\mathcal{P}G$ rather than $T(W, BG)$.

In view of the historical association of the plus-construction with classifying spaces of the form $B\mathrm{GL}R$, Berrick & Casacuberta went on to ask whether Theorem 3.3 can be sharpened as follows.

Question 6.3 *Suppose that W is a space with the property that*

$$P_W B\mathrm{GL}R \simeq B\mathrm{GL}R^+$$

for every discrete associative ring R with unit. Is W then necessarily acyclic?

Variant 1. *If always*

$$P_W B\mathrm{GL}R \simeq B\mathrm{GL}R^+_T$$

where $T = T(W, B\mathrm{GL}R)$, is W then necessarily acyclic?

Variant 2. *Suppose that W has $[\Sigma^k W, B\,\mathrm{GL}\,R^+]$ trivial for all $k \geq 0$ and all rings R. Is W then necessarily acyclic?*

By Theorem 5.1, the first variant is equivalent to the original question when $\pi_1(W)$ has a nontrivial integral representation. Also, an affirmative answer to the second variant immediately implies an affirmative answer to the original question. We can now give an affirmative answer to these questions for finite-dimensional spaces W.

Theorem 6.4 *Let W be such that for all rings R the localization map $B\,\mathrm{GL}\,R \to P_W B\,\mathrm{GL}\,R$ induces a surjection on integral cohomology. If for all $q > n$ $H^q(W) = 0$, then $H^n(W) = 0$.*

PROOF. We suppose that $H^n(W) \neq 0$, and actually show that for any regular ring R with $K_0 R = \mathbb{Z}$, there exists $r \geq 1$ with $H^{n+r+1}(W; K_r R) \neq 0$. Thus for example, when R is the finite field $\mathbb{F}_q$ it follows that for some odd r, $H^{n+r+1}(W; \mathbb{Z}/(q^{(r+1)/2} - 1)) \neq 0$ [38]. This contradicts the vanishing of higher cohomology groups.

By Lemma 2 of [8] (applicable since R is regular), for $n \geq 2$ we may make the following identifications.

$$\begin{aligned}
[P_W B\,\mathrm{GL}\, S^n R, K(\mathbb{Z}, n)] &\twoheadrightarrow H^n(B\,\mathrm{GL}\, S^n R; \mathbb{Z}) && (6.18)\\
&= H^n(BES^n R; \mathbb{Z}) && (6.19)\\
&= \mathrm{Hom}(H_n(BES^n R), \mathbb{Z}) && (6.20)\\
&= \mathrm{Hom}(K_0 R, \mathbb{Z}) && (6.21)\\
&= \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}). && (6.22)
\end{aligned}$$

Under this correspondence, let f_n denote a representative of the homotopy class of maps from $P_W B\,\mathrm{GL}\, S^n R$ to $K(\mathbb{Z}, n)$ inducing the identity homomorphism on $\mathbb{Z}$, and let F_n be its homotopy fibre. Then the homotopy exact sequence of this fibration shows that F_n is n-connected, and for $r \geq 1$ $\pi_{n+r}(F_n) \cong K_r R$.

Now consider a map $g : W \to K(\mathbb{Z}, n)$ representing a nonzero element of $H^n(W)$. Then all the obstructions to factorizing g through f_n lie in the groups $H^{k+1}(W; \pi_k(F_n))$. Here the local coefficients are simple because $K(\mathbb{Z}, n)$ is simply-connected. So the obstruction groups with $k \leq n$ all vanish, by n-connectedness of F_n. However, by the hypothesis on g, it cannot factor through the W-null space $P_W B\,\mathrm{GL}\, S^n R$. Hence one of the remaining cohomology groups must be nonzero. □

The above proof exploits the existence of a class of rings that have vanishing K-groups below a certain dimension. In the other direction, if one wished to start with Lemma 6.1 and deduce the acyclicity of W by an inductive argument, it would be desirable to have to hand a class of rings whose K-groups vanish above a specified dimension.

Question 6.5 *Does there exist a ring R such that, for some n, $K_n(R)$ is nonzero, but for all $i > n$, $K_i(R) = 0$?*

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