BLOCH-KATO CONJECTURE AND MOTIVIC COHOMOLOGY WITH FINITE COEFFICIENTS

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Contents

Introduction

§ 0. Notations, terminology and general remarks.

Throughout the paper we fix a field F and consider the category Sm/F of smooth separated schemes of finite type over F. We make Sm/F into a site using one of the following three topologies: Zariski topology, Nisnevich topology or etale topology. For any presheaf $\mathcal{F}$ on Sm/F we denote by $\mathcal{F}^{\sim}_{Zar}, \mathcal{F}^{\sim}_{Nis}$ and $\mathcal{F}^{\sim}_{et}$ the sheaf associated with $\mathcal{F}$ in Zariski, Nisnevich and etale topologies respectively.

For any site $\mathcal{C}$ we denote by $\mathcal{C}^{\sim}$ (resp. $\mathcal{C}^{\wedge}$) the category of abelian sheaves (resp. the category of abelian presheaves) on $\mathcal{C}$.

We denote by Sch/F the category of all separated schemes of finite type over F.

Let $\mathcal{F} : Sm/F \to \mathcal{A}$ be a (covariant) functor from Sm/F to an abelian category $\mathcal{A}$. Let further $X \in Sm/F$ be a smooth scheme and let $i : Y \hookrightarrow X$ be a smooth

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closed subscheme provided with a retraction $r : X \to Y$ (i.e. $ri = 1_Y$). In this case $\mathcal{F}(r)\mathcal{F}(i) = 1_{\mathcal{F}(Y)}$ and hence $\mathcal{F}(Y)$ is canonically a direct summand in $\mathcal{F}(X)$. We'll use the notation $\mathcal{F}(X/Y)$ for the complementary direct summand. Thus $\mathcal{F}(X/Y)$ may be identified either with the kernel of the homomorphism $\mathcal{F}(r) : \mathcal{F}(X) \to \mathcal{F}(Y)$ or with the cokernel of the homomorphism $\mathcal{F}(i) : \mathcal{F}(Y) \to \mathcal{F}(X)$. We'll use similar notations in case $\mathcal{F} : Sch/F \to \mathcal{A}$ is a functor from Sch/F to an abelian category $\mathcal{A}$ and $Y \subset X$ is a retract of $X \in Sch/F$.

More generally, assume we are given a scheme X and n closed subschemes $i_j : Y_j \hookrightarrow X \quad (1 \leq j \leq n)$ (the schemes X, Y_j should be smooth in case of the category Sm/F). Assume further that we are given retractions $r_j : X \to Y_j \quad (1 \leq j \leq n)$ such that the morphisms $\rho_j = i_j r_j : X \to X$ pairwise commute. In this case each of $\mathcal{F}(Y_j)$ may be identified with a direct summand in $\mathcal{F}(X)$. Moreover the sum $\sum_{j=1}^n \mathcal{F}(Y_j) \subset \mathcal{F}(X)$ is also a canonical direct summand: the corresponding projection is given by the formula

$$1_{\mathcal{F}(X)} - (1_{\mathcal{F}(X)} - \mathcal{F}(\rho_1)) \cdot \ldots \cdot (1_{\mathcal{F}(X)} - \mathcal{F}(\rho_n)) =$$

$$= \sum_{s=1}^n \sum_{1 \leq j_1 < \ldots < j_s \leq n} (-1)^{s-1} \mathcal{F}(\rho_{j_1} \cdot \ldots \cdot \rho_{j_s}) : \mathcal{F}(X) \to \sum_{j=1}^n \mathcal{F}(Y_j).$$

The above situation arises in particular in the following case. Let (Z, z_0) be a scheme provided with a distinguished rational point. Set $X = Z^{\times n}$. The point z_0 defines n embeddings of $Z^{\times(n-1)}$ in X . The corresponding subschemes $Y_j \subset X$ consist of all points with j-th coordinate equal to z_0, the corresponding morphisms $\rho_j : X \to X$ are given by the formula

$$\rho_j(z_1, \ldots, z_n) = (z_1, \ldots, \underset{j}{z_0}, \ldots, z_n)$$

and obviously pairwise commute. We'll use the notation $\mathcal{F}(Z^{\wedge n})$ for the direct summand of $\mathcal{F}(Z^{\times n})$ complementary to $\sum_{j=1}^n \mathcal{F}(Y_j)$. Throughout the paper this kind of notation is constantly used in the special case when $Z = \mathbb{G}_m = \mathbb{A}^1 \setminus 0$ and $z_0 = 1$ is the identity of the group scheme $\mathbb{G}_m$.

The previous construction has an obvious analogue for contravariant functors. In this case $\mathcal{F}(X/Y)$ may be identified either with the kernel of the homomorphism $\mathcal{F}(i) : \mathcal{F}(X) \to \mathcal{F}(Y)$ or with the cokernel of the homomorphism $\mathcal{F}(r) : \mathcal{F}(Y) \to \mathcal{F}(X)$. Furthermore the direct summand $\mathcal{F}(Z^{\wedge n}) \subset \mathcal{F}(Z^{\times n})$ may be identified with the intersection of kernels of restriction homomorphisms $\mathcal{F}(Z^{\times n}) \xrightarrow{\mathcal{F}(i_j)} \mathcal{F}(Z^{\times(n-1)})$.

The standart cosimplicial scheme $\Delta_F^\bullet$.

Recall that Δ_F^n is a closed subscheme in $\mathbb{A}_F^{n+1}$ defined by the equation $t_0 + \ldots + t_n = 1$. We'll refer to the points $v_i = (0, \ldots, \underset{i}{1}, \ldots, 0) \in \Delta^n$ $(0 \leq i \leq n)$ as vertices of Δ_n. Each nondecreasing map $\phi : [m] = \{0, 1, \ldots, m\} \to [n] = \{0, 1, \ldots, n\}$ defines the corresponding morphism of schemes

$$\Delta^m \to \Delta^n \ : \qquad (t_0, \ldots, t_m) \mapsto t_0 v_{\phi(0)} + \ldots + t_m v_{\phi(m)}.$$

There are $n+1$ coface morphisms $d^i : \Delta^{n-1} \to \Delta^n$ $(0 \le i \le n)$ (corresponding to strictly increasing maps $[n-1] \to [n]$). These coface morphisms are obviously closed embeddings, the corresponding closed subschemes of Δ^n are defined by equations $t_i = 0$ and are called the (codimension one) faces of Δ^n.

For any presheaf of abelian groups $\mathcal{F} : Sm/F \to Ab$ one can consider a simplicial presheaf $C_*(\mathcal{F})$ defined by the formula $C_n(\mathcal{F})(U) = \mathcal{F}(U \times \Delta^n)$. It's easy to see that homology presheaves $\mathcal{H}_i$ of the complex $C_*(\mathcal{F})$ are homotopy invariant - i.e. $\mathcal{H}_i(U \times \mathbb{A}^1) = \mathcal{H}_i(U)$ for any $U \in Sm/F$ (see [S-V § 7]). In the same way one can construct the bisimplicial presheaf $C_{*,*}(\mathcal{F})$, defined by the formula $C_{p,q}(\mathcal{F})(U) = \mathcal{F}(U \times \Delta^p \times \Delta^q)$. Homotopy invariance of the cohomology presheaves of $C_*(\mathcal{F})$ implies easily that the natural embedding $i : C_*(\mathcal{F}) \hookrightarrow Tot\ C_{*,*}(\mathcal{F})$ is a quasiisomorphism. The quasiisomorphism i has a canonical left inverse called the shuffle map (cf. [D] ch. 6, § 12), whose construction we are going to remind.

Every strictly increasing map $(\phi, \psi) : [p+q] \to [p] \times [q]$ defines a linear isomorphism of schemes

$$\Delta^{p+q} \to \Delta^p \times \Delta^q \quad (t_0, ..., t_{p+q}) \mapsto t_0 \cdot (v_{\phi(0)}, v_{\psi(0)}) + ... + t_{p+q} \cdot (v_{\phi(p+q)}, v_{\psi(p+q)})$$

and hence gives an isomorphism of presheaves $(\phi, \psi)^* : C_{p,q}(\mathcal{F}) \to C_{p+q}(\mathcal{F})$. Note further that strictly increasing maps $(\phi, \psi) : [p+q] \to [p] \times [q]$ are in one to one correspondence with (p,q) shuffles: each (p,q)-shuffle σ defines a map (ϕ, ψ) via the formula

$$\phi(x) = |\{1 \le i \le p : \sigma(i) \le x\}|$$
$$\psi(x) = |\{p+1 \le i \le p+q : \sigma(i) \le x\}|.$$

We'll use the notation $\sigma^* : C_{p,q}(\mathcal{F}) \xrightarrow{\sim} C_{p+q}(\mathcal{F})$ for the isomorphism of presheaves defined by the strictly increasing map (ϕ, ψ) corresponding to σ. The shuffle map $\nabla : Tot\ C_{*,*}(\mathcal{F}) \to C_*(\mathcal{F})$ is defined via the formula

$$\nabla_{p,q} = \sum_{\sigma - (p,q) - \text{shuffle}} \epsilon(\sigma)\sigma^* : C_{p,q}(\mathcal{F}) \to C_{p+q}(\mathcal{F}).$$

A well known and easy computation shows that ∇ is a homomorphism of complexes which is left inverse to i (i.e. $\nabla i = 1_{C_*(\mathcal{F})}$). It's not difficult to see that moreover the composition $i\nabla$ is homotopic to identity (so that i and ∇ are mutually unverse homotopy equivalences), but we won't need this fact.

Resolution of singularities

Definition 0.1. *We'll be saying that resolution of singularities holds over the field F, provided that the following two conditions are satisfied.*

1. *For any integral separated scheme of finite type X over F there exists a proper birational morphism $Y \to X$ with Y smooth (over F).*
2. *For any smooth integral scheme X over F and any birational proper morphism $Y \to X$ there exists a tower of morphisms $X_n \to X_{n-1} \to ... \to X_0 = X$, each stage of which is a blow up with a smooth center, and such that the composition morphism $X_n \to X$ may be factored through $Y \to X$.*

It's well known that resolution of singularities holds over fields of characteristic zero, whether it holds over (perfect) fields of positive characteristic remains one of the central problems of algebraic geometry. The following important theorem due to de Jong may be used sometimes as a replacement of the first of the above properties .

Theorem 0.2 [J]**.** *For any field F and any integral separated scheme of finite type X over F there exists a proper surjective morphism $Y \to X$ with Y smooth over F.*

It should be noted that this theorem implies that all results of [S-V] hold over an algebraically closed field of arbitrary characteristic.

Hypercohomology.

Motivic cohomology are defined as Zariski (or Nisnevich) hypercohomology with coefficients in a complex of sheaves which is not bounded below. We find it necessary to remind the definition and properties of hypercohomology in this generality.

Let $\mathcal{A}$ be an abelian category with enough injectives. Let further $\mathcal{F} : \mathcal{A} \to Ab$ be a left exact additive functor. Denote the right derived functors of $\mathcal{F}$ by $R^i\mathcal{F}$. Let finally $A^\bullet$ be a complex of degree $+1$ in $\mathcal{A}$ and let $A^\bullet \to I^{\bullet\bullet}$ be the Cartan-Eilenberg resolution of $A^\bullet$. The hypercohomology groups $R^*\mathcal{F}(A^\bullet)$ are defined as cohomology of the total complex corresponding to the bicomplex $\mathcal{F}(I^{\bullet\bullet})$

$$R^*\mathcal{F}(A^\bullet) = H^*(Tot(\mathcal{F}(I^{\bullet\bullet}))).$$

The following two results concerning hypercohomology constitute a minor generalization of well-known facts. For the sake of completness we sketch the proof below.

Theorem 0.3. *Assume that either the complex $A^\bullet$ is bounded below or the functor $\mathcal{F}$ is of finite cohomological dimension (i.e. $R^i(\mathcal{F}) = 0$ for $i >> 0$). Then both spectral sequences of the bicomplex $\mathcal{F}(I^{\bullet\bullet})$ are strongly convergent. Thus we have two hypercohomology spectrtal sequences*

$$\mathrm{I}_1^{pq} = R^p\mathcal{F}(A^q) \Rightarrow R^{p+q}\mathcal{F}(A^\bullet)$$
$$\mathrm{II}_2^{pq} = R^q\mathcal{F}(H^p(A^\bullet)) \Rightarrow R^{p+q}\mathcal{F}(A^\bullet).$$

Proof. The case when the complex $A^\bullet$ is bounded below is well-known so we'll consider only the case when the functor $\mathcal{F}$ is of finite cohomological dimension. Let $0 \to A^\bullet \xrightarrow{\epsilon} I^{\bullet 0} \xrightarrow{d'} I^{\bullet 1} \xrightarrow{d'} \ \ldots$ be the Cartan-Eilenberg resolution of $A^\bullet$. Define the complex $A^{\bullet n}$ as the kernel of the homomorphism $d' : I^{\bullet n} \to I^{\bullet n+1}$. The following properties of the complex $A^{\bullet n}$ are straightforward from definitions.

(0.3.1). *For each i we have exact sequences*

$$0 \to A^i \to I^{i,0} \to \ldots \to I^{i,n-1} \to A^{i,n} \to 0$$
$$0 \to Z^i(A^\bullet) \to Z^i(I^{\bullet 0}) \to \ldots \to Z^i(I^{\bullet n-1}) \to Z^i(A^{\bullet n}) \to 0$$
$$0 \to B^i(A^\bullet) \to B^i(I^{\bullet 0}) \to \ldots \to B^i(I^{\bullet n-1}) \to B^i(A^{\bullet n}) \to 0$$
$$0 \to H^i(A^\bullet) \to H^i(I^{\bullet 0}) \to \ldots \to H^i(I^{\bullet n-1}) \to H^i(A^{\bullet n}) \to 0$$

We conclude immediately from (0.3.1) the following further properties of the complex $A^{\bullet n}$.

(0.3.2). $0 \to A^{\bullet n} \to I^{\bullet n} \xrightarrow{d'} I^{\bullet n+1} \xrightarrow{d'} \ \ldots$ *is the Cartan-Eilenberg resolution of the complex* $A^{\bullet n}$.

(0.3.3). *Assume that* $n \geq cd\ \mathcal{F}$ *then all objects* $A^{i,n}, Z^i(A^{\bullet n}), B^i(A^{\bullet n}), H^i(A^{\bullet n})$ *are* $\mathcal{F}$*-acyclic.*

Lemma 0.3.4. *Let* $\mathcal{F}$ *be a functor of finite cohomological dimension* d. *Let further* $A^\bullet$ *be a complex such that all objects* $A^i, H^i(A^\bullet)$ *are* $\mathcal{F}$*-acyclic, and let* $0 \to A^\bullet \to I^{\bullet\bullet}$ *be the Cartan-Eilenberg resolution of* $A^\bullet$*, then*

1. *The objects* $Z^i(A^\bullet), B^i(A^\bullet)$ *are also* $\mathcal{F}$*-acyclic for all* i.
2. $Z^i(\mathcal{F}(A^\bullet)) = \mathcal{F}(Z^i(A^\bullet))$, $B^i(\mathcal{F}(A^\bullet)) = \mathcal{F}(B^i(A^\bullet))$, $H^i(\mathcal{F}(A^\bullet)) = \mathcal{F}(H^i(A^\bullet))$ *for all* i.
3. $\mathcal{F}(A^\bullet) \to Tot\ \mathcal{F}(I^{\bullet\bullet})$ *is a quasiisomorphism (*$\mathcal{F}$*-acyclicity of* A^i *(for all* i*) alone is enough for this conclusion).*

Proof. To prove (1) one shows by inverse induction on $n \geq 1$ that $R^n\mathcal{F}(Z^i(A^\bullet)) = R^n\mathcal{F}(B^i(A^\bullet)) = 0$ for all i. The point (2) is immediate from (1). To prove (3) one notes that the i-th coloumn of the bicomplex $\mathcal{F}(I^{\bullet\bullet})$ is a resolution of $\mathcal{F}(A^i)$ which is enough to conclude that $\mathcal{F}(A^\bullet) \to Tot\ \mathcal{F}(I^{\bullet\bullet})$ is a quasiisomorphism.

To conclude the proof of the Theorem 0.3 we denote by $\tau_{\leq n}I^{\bullet\bullet}$ the subbicomplex $0 \to I^{\bullet 0} \to I^{\bullet 1} \to \ \ldots \ \to I^{\bullet n-1} \to A^{\bullet n} \to 0$ of $I^{\bullet\bullet}$. Taking $n \geq cd\ \mathcal{F}$ we see immediately from the results proved above that the embedding $\tau_{\leq n}I^{\bullet\bullet} \hookrightarrow I^{\bullet\bullet}$ induces a quasiisomorphism of complexes $Tot\ \mathcal{F}(\tau_{\leq n}I^{\bullet\bullet}) \xrightarrow{\sim} Tot\ \mathcal{F}(I^{\bullet\bullet})$. Moreover the first (respectively the second) spectral sequences of the bicomplexes $\mathcal{F}(\tau_{\leq n}I^{\bullet\bullet})$ and $\mathcal{F}(I^{\bullet\bullet})$ coincide from E_1-term on (resp. from E_2-term on).

In case the functor $\mathcal{F}$ coincides with $\mathrm{Hom}_{\mathcal{A}}(X, -)$ the corresponding hypercohomology groups are ususlly denoted $\mathrm{RHom}^*_{\mathcal{A}}(X, A^\bullet)$. They may be interpreted as appropriate Hom-groups in the derived category $D(\mathcal{A})$.

Proposition 0.4. *Assume that either the complex* $A^\bullet$ *is bounded below or* $\mathrm{Ext}^i(X, -) = 0$ *for* $i >> 0$. *Then* $\mathrm{RHom}^i_{\mathcal{A}}(X, A^\bullet) = \mathrm{Hom}_{D(\mathcal{A})}(X, A^\bullet[i])$.

Proof. The case when the complex $A^\bullet$ is bounded below is well-known, so we'll consider only the case when the cohomological dimension of the functor $\mathrm{Hom}_{\mathcal{A}}(X, -) = \mathcal{F}$ is finite. Note first of all that according to definitions

$$\mathrm{Hom}_{D(\mathcal{A})}(X, A^\bullet[i]) = \varinjlim_{A^\bullet \xrightarrow{\sim} B^\bullet} \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(X, B^\bullet[i]) = \varinjlim_{A^\bullet \xrightarrow{\sim} B^\bullet} H^i(\mathrm{Hom}_{\mathcal{A}}(X, B^\bullet))$$

where direct limit is taken over all quasiisomorphisms $A^\bullet \xrightarrow{\sim} B^\bullet$, $\mathcal{K}(\mathcal{A})$ denotes the homotopy category of complexes and $\mathrm{Hom}_{\mathcal{A}}(X, B^\bullet)$ is the complex of abelian groups $\ \ldots \to \mathrm{Hom}_{\mathcal{A}}(X, B^i) \to \mathrm{Hom}_{\mathcal{A}}(X, B^{i+1}) \to \ldots$.

Lemma 0.4.1. *Let $\mathcal{F}$ be a left exact additive functor of finite cohomological dimension d, then*

(1) *For any complex $A^\bullet$ there exists a quasiisomorphism $A^\bullet \to B^\bullet$, where the terms B^i are $\mathcal{F}$-acyclic for all i.*
(2) *Let $B^\bullet \xrightarrow{f} C^\bullet$ be a quasiisomorphism of complexes. Assume that all terms B^i and C^i are $\mathcal{F}$-acyclic. Then the induced homomorphism of complexes of abelian groups $\mathcal{F}(f) : \mathcal{F}(B^\bullet) \to \mathcal{F}(C^\bullet)$ is also a quasiisomorphism.*

Proof. To prove the point (1) it suffices to note that $A^\bullet \to Tot\ \tau_{\leq n} I^{\bullet\bullet}$ is always a quasiisomorphism. Moreover the terms of the complex $Tot\ \tau_{\leq n} I^{\bullet\bullet}$ are $\mathcal{F}$-acyclic for any left exact additive functor $\mathcal{F}$ of cohomological dimension$\leq d$ provided that $n \geq d$.

To prove the second statement denote by $co(f)$ the cone of f. The complex $co(f)$ is acyclic (since f is a quasiisomorphism) and consists of $\mathcal{F}$-acyclic terms. Lemma 0.3.4 shows that the complex $co(\mathcal{F}(f)) = \mathcal{F}(co(f))$ is acyclic as well and hence $\mathcal{F}(f)$ is a quasiisomorphism.

Lemma 0.4.1 shows that

$$\mathrm{Hom}_{D(\mathcal{A})}(X, A^\bullet[i]) = H^i(\mathrm{Hom}_{\mathcal{A}}(X, B^\bullet))$$

for any quasiisomorphism $A^\bullet \to B^\bullet$ from $A^\bullet$ to a complex with $\mathrm{Hom}_{\mathcal{A}}(X, -)$-acyclic terms. Applying this to the complex $Tot\ \tau_{\leq n} I^{\bullet\bullet}$ (with $n >> 0$) we easily conclude the proof of Proposition 0.4.

Ext-groups

For any two complexes $A^\bullet, B^\bullet \in D(\mathcal{A})$ we set

$$\mathrm{Ext}^i_{\mathcal{A}}(A^\bullet, B^\bullet) = \mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet[i]).$$

It's well-known that in case $A^\bullet, B^\bullet \in D^-(\mathcal{A})$ (resp. $\in D^+(\mathcal{A}), D^b(\mathcal{A})$) the corresponding Ext-groups coincide with Hom-groups in the category $D^-(\mathcal{A})$ (resp. in $D^+(\mathcal{A}), D^b(\mathcal{A})$).

Infinite direct sums in the derived categories.

Let $\mathcal{A}$ be an abelian category satisfying the Grothendieck's axiom Ab-?. Below we'll discuss infinite direct sums in the category $D^-(\mathcal{A})$. A similar discussion applies with minimal modifications to $D^+(\mathcal{A})$ and $D^b(\mathcal{A})$ as well.

Lemma 0.5. *Let $\{A_i^\bullet\}_{i\in I}$ be a family of complexes. Any quasiisomorphism $C^\bullet \to \bigoplus_{i\in I} A_i^\bullet$ may be dominated (in the category $\mathcal{K}(\mathcal{A})$) by a quasiisomorphism of the form $\bigoplus_{i\in I} C_i^\bullet \xrightarrow{\oplus f_i} \bigoplus_{i\in I} A_i^\bullet$ (where each $f_i : C_i^\bullet \to A_i^\bullet$ is a quasiisimorphism).*

Proof. A well-known property of the homotopy category $\mathcal{K}(\mathcal{A})$ shows that for each i there exists a complex $C_i^\bullet$ a quasiisomorphism $f_i : C_i^\bullet \to A_i^\bullet$ and a morphism $g_i : C_i^\bullet \to C^\bullet$ which make the following diagram commute up to homotopy.

$$\begin{array}{ccc} C_i^\bullet & \xrightarrow{g_i} & C^\bullet \\ {\scriptstyle f_i}\downarrow & & \downarrow \\ A_i^\bullet & \xrightarrow{in_i} & \bigoplus_{i\in I} A_i^\bullet \end{array}$$

A family of morphisms $g_i : C_i^\bullet \to C^\bullet$ defines a morphism $g = (g_i)_{i\in I} : \bigoplus C_i^\bullet \to C^\bullet$ and according to the construction the following diagram commutes up to homotopy

$$\begin{array}{ccc} \bigoplus_{i\in I} C_i^\bullet & \xrightarrow{g} & C^\bullet \\ {\scriptstyle \oplus f_i}\downarrow & & \downarrow \\ \bigoplus_{i\in I} A_i^\bullet & \xrightarrow{=} & \bigoplus_{i\in I} A_i^\bullet \end{array}$$

Corollary 0.5.1. *For any complex $B^\bullet$ we have a natural isomorphism*

$$\mathrm{Hom}_{D(\mathcal{A})}(\bigoplus_{i\in I} A_i^\bullet, B^\bullet) = \prod_{i\in I} \mathrm{Hom}_{D(\mathcal{A})}(A_i^\bullet, B^\bullet).$$

In other words $\bigoplus_{i\in I} A_i^\bullet$ is a direct sum of complexes $A_i^\bullet$ in the category $D(\mathcal{A})$ as well.

Proof. In view of definitions and Lemma 0.5 we have the following identifications

$$\mathrm{Hom}_{D(\mathcal{A})}(\bigoplus_{i\in I} A_i^\bullet, B^\bullet) = \varinjlim_{C^\bullet \xrightarrow{\sim} \bigoplus_{i\in I} A_i^\bullet} \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(C^\bullet, B^\bullet) =$$

$$= \varinjlim_{\{C_i^\bullet \xrightarrow{\sim} A_i^\bullet\}_{i\in I}} \prod_{i\in I} \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(C_i^\bullet, B^\bullet) = \prod_{i\in I} \mathrm{Hom}_{D(\mathcal{A})}(A_i^\bullet, B^\bullet).$$

We'll be saying that the family of complexes $\{A_i^\bullet\}_{i\in I}$ is uniformly bounded above (resp. uniformly homologically bounded above) iff there exists an integer N such that $A_i^n = 0$ (resp. $H^n(A_i^\bullet) = 0$) for all $i \in I$ and all $n \geq N$.

Corollary 0.5.2. *A direct sum of a family of complexes $\{A_i^\bullet\}_{i\in I} \in D^-(\mathcal{A})$ exists in $D^-(\mathcal{A})$ if and only if this family is uniformly homologically bounded above.*

Let $\{A_i^\bullet\}_{i\in I} \in D^-(\mathcal{A})$ be a family of complexes. We'll denote by $< \{A_i^\bullet\} > \subset D^-(\mathcal{A})$ the minimal full triangulated subcategory closed with respect to taking direct sums of uniformly homologically bounded above families of complexes. We'll be saying that the family $\{A_i^\bullet\}_{i\in I}$ weakly generates the category $< \{A_i^\bullet\}_{i\in I} >$.

Lemma 0.6. *Let $A^{\bullet\bullet}$ be a bounded above bicomplex in $\mathcal{A}$. Then*

$$Tot(A^{\bullet\bullet}) \in < \{A^{\bullet i}\}_{i\in\mathbb{Z}} > \subset D^-(\mathcal{A}).$$

Proof. Denote by $\sigma_{\geq n}A^{\bullet\bullet} \subset A^{\bullet\bullet}$ the subbicomplex of $A^{\bullet\bullet}$ consisting of terms A^{ij} with $j \geq n$. Denote further by $i_n : \sigma_{\geq n}A^{\bullet\bullet} \to A^{\bullet\bullet}$ and $j_n : \sigma_{\geq n}A^{\bullet\bullet} \to \sigma_{\geq n-1}A^{\bullet\bullet}$ the obvious embeddings. The complexes $Tot(\sigma_{\geq n}A^{\bullet\bullet})$ obviously lie in the triangulated subcategory generated by all $A^{\bullet i}$. Furthermore we have a short exact sequence of bicomplexes

$$0 \to \bigoplus_n \sigma_{\geq n}A^{\bullet\bullet} \xrightarrow{(1_{\sigma_{\geq n}A^{\bullet\bullet}} - j_n)} \bigoplus_n \sigma_{\geq n}A^{\bullet\bullet} \xrightarrow{(i_n)} A^{\bullet\bullet} \to 0$$

which shows that $Tot(A^{\bullet\bullet})$ lies in the subcategory weakly generated by $A^{\bullet i}$.

§ 1. Homotopy invariant presheaves with transfers.

For any $X, Y \in Sm/F$ define $Cor(X,Y)$ to be the free abelian group generated by closed integral subschemes $Z \subset X \times_F Y$ which are finite and surjective over a component of X. Let $X, Y, W \in Sm/F$ be smooth schemes and let $Z \in Cor(X,Y), T \in Cor(Y,W)$ be cycles on $X \times Y$ and $Y \times W$ each component of which is finite and surjective over a component of X (respectively over a component of Y). One checks easily that the cycles $Z \times W$ and $X \times T$ intersect properly on $X \times Y \times W$ and each component of the intersection cycle $(Z \times W)_\bullet(X \times T)$ is finite and surjective over a component of X. Thus setting $T \circ Z = (pr_{1,3})_*((Z \times Y)_\bullet(X \times T))$ we get a bilinear composition map

$$Cor(Y,W) \times Cor(X,Y) \to Cor(X,W).$$

In this way we get a new category (denoted $SmCor/F$) whose objects are smooth schemes of finite type over F and $Hom_{SmCor/F}(X,Y) = Cor(X,Y)$ – see [V 1] for details.

The category $SmCor/F$ is clearly additive, the direct sum of two schemes being given by their disjoint union. A presheaf with transfers on the category Sm/F is defined as a contravariant additive functor $\mathcal{F} : SmCor/F \to Ab$. Note that associating to each morphism $f : X \to Y$ its graph $\Gamma_f \in Cor(X,Y)$ we get a canonical map $Hom_{Sm/F}(X,Y) \to Cor(X,Y)$. In this way we get a canonical functor $Sm/F \to SmCor/F$, which allows to view presheaves with transfers on Sm/F as presheaves in the usual sence equipped with appropriate additional data.

For any $X \in Sm/F$ we'll denote by $\mathbb{Z}_{tr}(X)$ the corresponding representable functor, i.e. $\mathbb{Z}_{tr}(X)$ is a presheaf with transfers defined via the formula

$$\mathbb{Z}_{tr}(X)(U) = Cor(U,X).$$

One checks easily that the pesheaf $\mathbb{Z}_{tr}(X)$ is actually a sheaf in the etale topology (and a fortiory in Zariski and Nisnevich topologies as well). Direct sums of such sheaves are sometimes called free presheaves with transfer or free Nissnevich sheaves with transfers. This terminology is not very consistent since sheaves $\mathbb{Z}_{tr}(X)$ are obviously projective objects in the category of presheaves with transfers, but clearly not in the category of Nisnevich sheaves with transfers.

In a similar way we may associate a presheaf with transfers $\mathbb{Z}_{tr}(X)$ to each scheme of finite type $X \in Sch/F$. To be more precise for each $U \in Sm/F$ we define $\mathbb{Z}_{tr}(X)(U)$ to be the free abelian group generated by closed integral subschemes $Z \subset U \times X$ which are finite and surjective over a component of U. The same construction as above defines (for any $U, V \in Sm/F$) the bilinear pairing

$$Cor(V,U) \times \mathbb{Z}_{tr}(X)(U) \to \mathbb{Z}_{tr}(X)(V)$$

which makes $\mathbb{Z}_{tr}(X)$ into a presheaf with transfers. Moreover the presheaf with transfers $\mathbb{Z}_{tr}(X)$ is an etale sheaf for any $X \in Sch/F$. Obviously $\mathbb{Z}_{tr}$ is a functor from Sch/F to the category of (etale) sheaves with transfer.

Consider the standart cosimplicial object $\Delta^\bullet = \Delta^\bullet_F$ in Sm/F. For any presheaf of abelian groups $\mathcal{F}$ on Sm/F we get a simplicial presheaf $C_\bullet(\mathcal{F})$, by setting $C_n(\mathcal{F})(U) = \mathcal{F}(U \times \Delta^n)$. We'll use the same notation $C_\bullet(\mathcal{F})$ for the corresponding complex (of degree -1) of abelian presheaves. Usually we'll be dealing with complexes of degree $+1$, in particular, we reindex the complex $C_\bullet(\mathcal{F})$ (in the standart way), by setting

$$C^i(\mathcal{F}) = C_{-i}(\mathcal{F}).$$

Recall that a presheaf $\mathcal{F} : Sm/F \to Ab$ is said to be homotopy invariant, provided that for any $U \in Sm/F$ the natural homomorphism $\mathcal{F}(U) \to \to \mathcal{F}(U \times \mathbb{A}^1)$ is an isomorphism. It's easy to see ([S-V 1] Corollary 7.5) that cohomology presheaves of the complex $C^\bullet(\mathcal{F})$ are homotopy invariant.

The following result sums up some of the basic properties of homotopy invariant presheaves with transfers.

Theorem 1.1 [V 1]**.** *Let $\mathcal{F}$ be a homotopy invariant presheaf with transfers on Sm/F, then*

1. *The sheaf $\mathcal{F}^\sim_{Zar}$ coincides with $\mathcal{F}^\sim_{Nis}$ and has a natural structure of a homotopy invariant sheaf with transfers.*
2. *For any $X \in Sm/F$ and any $i \geq 0$ $H^i_{Zar}(X, \mathcal{F}^\sim_{Zar}) = H^i_{Nis}(X, \mathcal{F}^\sim_{Nis})$*
3. *The presheaves $X \mapsto H^i_{Zar}(X, \mathcal{F}^\sim_{Zar}) = H^i_{Nis}(X, \mathcal{F}^\sim_{Nis})$ are homotopy invariant (and have a natural structure of presheaves with transfers).*

Corollary 1.1.1. *Let $C^\bullet$ be a complex (of degree $+1$) of Nisnevich sheaves with transfers with homotopy invariant cohomology presheaves. Then for any scheme $X \in Sm/F$ we have a natural isomorphism of hypercohomology groups $H^*_{Zar}(X, C^\bullet) = H^*_{Nis}(X, C^\bullet)$.*

Proof. Note that cohomological dimension of X with respect to both Zariski and Nisnevich topology is finite (and equal to $dim\ X$). This remark and Theorem 0.3 give us two convergent hypercohomology spectral sequences (in which $\mathcal{H}^q$ denotes the q-th cohomology presheaf of $C^\bullet$)

$$_{Zar}\mathrm{II}_2^{pq} = H^p_{Zar}(X, (\mathcal{H}^q)^\sim_{Zar}) \Rightarrow H^{p+q}_{Zar}(X, C^\bullet)$$
$$_{Nis}\mathrm{II}_2^{pq} = H^p_{Nis}(X, (\mathcal{H}^q)^\sim_{Nis}) \Rightarrow H^{p+q}_{Nis}(X, C^\bullet)$$

Moreover we have a natural homomorphism of spectral sequences $_{Zar}\mathrm{II} \to {}_{Nis}\mathrm{II}$. Theorem 1.1 shows that the map on E_2 terms is an isomorphism, hence the map on the limits is an isomorphism as well.

Nisnevich topology is much more convenient in dealing with presheaves with transfers as one sees from the following Lemma (which fails in case of the Zariski topology).

Lemma 1.2 [V 1]**.** *Let $\mathcal{F}$ be any presheaf with transfers on Sm/F. Then there exists the unique structure of a Nisnevich sheaf with transfers on $\mathcal{F}^\sim_{Nis}$ making the homomorphism $\mathcal{F} \to \mathcal{F}^\sim_{Nis}$ into a homomorphism of presheaves with transfers.*

For any Nisnevich sheaf with transfers $\mathcal{G}$ and any homomorphism of presheaves with transfers $\mathcal{F} \to \mathcal{G}$ the associated homomorphism $\mathcal{F}^{\sim}_{Nis} \to \mathcal{G}$ is compatible with transfers.

Lemma 1.2 implies easily that Nisnevich sheaves with transfers form an abelian category which we denote $NSwT/F$.

Lemma 1.3. *Homotopy invariant Nisnevich sheaves with transfers form a full abelian subcategory in $NSwT/F$, closed under taking kernels and cokernels of morphisms and under extensions.*

Proof. Let $f : \mathcal{F} \to \mathcal{G}$ be a homomorphism of homotopy invariant sheaves with transfers. Denote by $\mathcal{K}$ and $\mathcal{C}$ the kernel and cokernel of f in the category of presheaves. Then $\mathcal{K}$ and $\mathcal{C}$ obviously are homotopy invariant presheaves with transfers. Furthermore $\mathcal{K}$ is a sheaf and coincides with $Ker\ f$, whereas $Coker\ f = \mathcal{C}^{\sim}_{Nis}$. Theorem 1.1 (1) shows now that $Coker\ f$ is homotopy invariant. Finally let $0 \to \mathcal{F} \to \mathcal{H} \to \mathcal{G} \to 0$ be an extension of Nisnevich sheaves with transfers and assume that $\mathcal{F}$ and $\mathcal{G}$ are homotopy invariant. For any $U \in Sm/F$ we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \to \mathcal{F}(U) & \longrightarrow & \mathcal{H}(U) & \longrightarrow & \mathcal{G}(U) & \longrightarrow & H^1_{Nis}(U, \mathcal{F}) \\
\cong\downarrow & & \downarrow & & \cong\downarrow & & \cong\downarrow \\
0 \to \mathcal{F}(U \times \mathbb{A}^1) & \longrightarrow & \mathcal{H}(U \times \mathbb{A}^1) & \longrightarrow & \mathcal{G}(U \times \mathbb{A}^1) & \longrightarrow & H^1_{Nis}(U \times \mathbb{A}^1, \mathcal{F})
\end{array}$$

Theorem 1.1 (3) shows that the right hand side vertical map is an isomorphism so that homotopy invariance of $\mathcal{H}$ follows from the five lemma.

Homotopy invariance of the cohomology presheaves of the complexes $C^\bullet(\mathcal{F})$ implies immediately the following fact.

Lemma 1.4. *Let $A^\bullet$ be a bounded above complex of Nisnevich sheaves with transfers. All cohomology presheaves of the complex $Tot\ C^\bullet(A^\bullet)$ are homotopy invariant. Hence all cohomology sheaves of $Tot\ C^\bullet(A^\bullet)$ are homotopy invariant as well.*

A convenient framework for dealing with homotopy invariant sheaves with transfers is provided by the tensor triangulated category $DM^-(F)$. Recall (see [V 2]) that $DM^-(F)$ is defined as the full subcategory of the derived category $D^-(NSwT/F)$ of bounded above complexes of Nisnevich sheaves with transfers comprising the complexes with homotopy invariant cohomology sheaves. Lemma 1.4 shows that for any bounded above copmplex of Nisnevich sheaves with transfers $A^\bullet$ the complex $C^\bullet(A^\bullet)$ is in $DM^-(F)$. For any $X \in Sm/F$ we define its motive $M(X)$ as the element $C^\bullet(\mathbb{Z}_{tr}(X)) \in DM^-(F)$. The following theorem relates $DM^-(F)$ to Nisnevich cohomology.

Theorem 1.5[V 2]**.** *For any complex $A^\bullet \in DM^-(F)$ and any $X \in Sm/F$ we have natural isomorphisms*

$$H^i_{Nis}(X, A^\bullet) = \mathrm{Hom}_{DM^-(F)}(M(X), A^\bullet[i]).$$

The proof splits naturally into two parts. In the first part we compare Nisnevich cohomology with Hom-groups in the category $D^-(NSwT/F)$. We start with the following Lemma.

Lemma 1.6 [V 1]. *Let $f : Y \to X$ be a Nisnevich covering of a (not necessarily smooth) scheme X. Then the following sequence of Nisnevich sheaves with transfers is exact*

$$0 \leftarrow \mathbb{Z}_{tr}(X) \xleftarrow{f_*} \mathbb{Z}_{tr}(Y) \xleftarrow{(p_2)_*-(p_1)_*} \mathbb{Z}_{tr}(Y \times_X Y) \xleftarrow{(p_{23})_*-(p_{13})_*+(p_{12})_*} \dots$$

Proof. It suffices to show that if S is a smooth henselian scheme then the sequence of abelian groups

$$A_\bullet(S) = (0 \leftarrow \mathbb{Z}_{tr}(X)(S) \xleftarrow{f_*} \mathbb{Z}_{tr}(Y)(S) \xleftarrow{(p_2)_*-(p_1)_*} \mathbb{Z}_{tr}(Y \times_X Y)(S) \leftarrow \dots)$$

is exact. Fix a closed integral subscheme $Z \subset X \times S$ finite and surjective over S and denote by $A_n^Z(S)$ the subgroup of $A_n(S)$ generated by closed integral subschemes $T \subset Y \times_X ... \times_X Y \times S$ (finite and surjective over S) whose set theoretical image in $X \times S$ coincides with Z. Obviously $A_\bullet^Z(S)$ is a subcomplex of $A_\bullet(S)$ and moreover $A_\bullet(S) = \oplus_Z A_\bullet^Z(S)$. Thus it suffices to show that every complex $A_\bullet^Z(S)$ is contractible. The scheme Z being finite and surjective over a henselian scheme S is henselian itself, which implies that the projection $p_1 : Z \to X$ admits a lifting $f : Z \to Y$. Let $T \subset \underbrace{Y \times_X ... \times_X Y}_{n} \times S$ be a generator of $A_n^Z(S)$ and let $g : T \to Z$ be the corresponding morphism. The morphism $T \xrightarrow{pr_1 \times_X fg \times pr_2} \underbrace{Y \times_X ... \times_X Y}_{n} \times_X Y \times S$ is obviously a closed embedding and hence defines a closed integral subscheme $T' = u_n(T) \subset \underbrace{Y \times_X ... \times_X Y}_{n+1} \times S$ (finite and surjective over S). In this way a get a homomorphism $u_n : A_n^Z(S) \to A_{n+1}^Z(S)$, and one checks immediately that u is a contracting homotopy for the complex $A_\bullet^Z(S)$.

Corollary 1.7. *Let $I \in NSwT/F$ be an injective Nisnevich sheaf with transfers. Then for any $X \in Sm/F$ $\quad H^i_{Nis}(X, I) = 0$ for all $i > 0$.*

Proof. We first compute Chech cohomology of I with respect to a Nisnevich covering $f : Y \to X$. These cohomology are the cohomology of the complex

$$\begin{array}{ccccc} I(Y) & \longrightarrow & I(Y \times_X Y) & \longrightarrow & \dots \\ {\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & \\ \mathrm{Hom}_{NSwT}(\mathbb{Z}_{tr}(Y), I) & \longrightarrow & \mathrm{Hom}_{NSwT}(\mathbb{Z}_{tr}(Y \times_X Y), I) & \longrightarrow & \dots \end{array}$$

Injectivity of I in $NSwT/F$ and Lemma 1.6 show that $\check{H}^i(Y/X, I) = 0$ for $i > 0$. Now the standard argument involving the Cartan-Leray spectral sequence ends up the proof.

Corollary 1.7.1. *For any Nisnevich sheaf with transfers $\mathcal{F}$ we have natural identifications*

$$\mathrm{Ext}^i_{NSwT}(\mathbb{Z}_{tr}(X), \mathcal{F}) = H^i_{Nis}(X, \mathcal{F}).$$

In particular $\mathrm{Ext}^i_{NSwT}(\mathbb{Z}_{tr}(X), -) = 0$ *for* $i > dim X$.

Proposition 1.8. *Let $A^\bullet \in D^-(NSwT/F)$ be a bounded above complex of Nisnevich sheaves with transfers. Then for any $X \in Sm/F$ we have a natural isomorphism.*

$$H^i_{Nis}(X, A^\bullet) = \mathrm{Hom}_{D^-(NSwT/F)}(\mathbb{Z}_{tr}(X), A^\bullet[i]).$$

Proof. Let $A^\bullet \to J^{\bullet\bullet}$ and $A^\bullet \to I^{\bullet\bullet}$ be the Cartan-Eilenberg resolutions of $A^\bullet$ in the categories of Nisnevich sheaves with transfers and all Nisnevich sheaves respectively. It's easy to see from the defining property of the Cartan-Eilenberg resolution that there exists a unique (up to a homotopy of bidegree $(0, -1)$) homomorphism of resolutions $J^{\bullet\bullet} \to I^{\bullet\bullet}$. Furthermore Proposition 0.4 allows us to identify $\mathrm{Hom}_{D^-(NSwT/F)}(\mathbb{Z}_{tr}(X), A^\bullet[i])$ to $H^i(Tot(J^{\bullet\bullet}(X)))$ whereas $H^i_{Nis}(X, A^\bullet)$ identifies canonically to $H^i(Tot(I^{\bullet\bullet}(X)))$. What we have to verify is that the above homomorphism of resolutions induces an isomorphism $H^*(Tot(J^{\bullet\bullet}(X))) \xrightarrow{\sim} H^*(Tot(I^{\bullet\bullet}(X)))$. Theorem 0.3 provides us with two spectral sequences converging to cohomology in question and Corollary 1.7.1 shows that the induced map on the E_1-terms of the first spectral sequences is an isomorphism. Thus the map on limits is an isomorphism as well.

In the second part of the proof we compare $\mathrm{Hom}_{D^-(NSwT/F)}(\mathbb{Z}_{tr}(X), A^\bullet)$ and $\mathrm{Hom}_{DM^-(F)}(M(X), A^\bullet) = \mathrm{Hom}_{D^-(NSwT/F)}(C^\bullet(\mathbb{Z}_{tr}(X)), A^\bullet)$.

A presheaf (resp. a presheaf with transfers) $\mathcal{F}$ is said to be contractible provided that there exists a homomorphism of presheaves (resp. of presheaves with transfers) $\phi : \mathcal{F} \to C^{-1}(\mathcal{F}) = C_1(\mathcal{F})$ with the property that $\partial_0 \phi = 0, \partial_1 \phi = 1_{\mathcal{F}}$. Here $\partial_0, \partial_1 : C_1(\mathcal{F}) \to C_0(\mathcal{F}) = \mathcal{F}$ are the face operations of the simplicial presheaf $C_\bullet(\mathcal{F})$. In other words to each section $s \in \mathcal{F}(U)$ one can associate in a natural way a section $\phi(s) \in C_1(\mathcal{F})(U) = \mathcal{F}(U \times \mathbb{A}^1)$ such that $\phi(s)_{|U\times 0} = 0, \phi(s)_{|U\times 1} = s$. It follows easily from the definition that Zariski (Nisnevich, etale) sheaf associated to a contractible presheaf is again contractible. Typical examples of contractible presheavs are given by the following Lemma.

Lemma 1.9. *For any presheaf (resp. presheaf with transfers) $\mathcal{F}$ the kernel of any iterated face operation $\partial : C_n(\mathcal{F}) \to C_0(\mathcal{F}) = \mathcal{F}$ is contractible.*

Proof. We consider only the special case $\mathcal{K} = Ker(\partial_0 : C_1(\mathcal{F}) \to \mathcal{F})$. The proof in the general case is similar. For any $X \in Sm/F$ we have:

$$\mathcal{K}(X) = \{s \in \mathcal{F}(X \times \mathbb{A}^1) : s|_{X\times 0} = 0\}.$$

Denote by $m : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ the multiplication morphism ($m(a, b) = ab$) and finally set $\phi(s) = (1_X \times m)^*(s) \in \mathcal{F}(X \times \mathbb{A}^1 \times \mathbb{A}^1)$. One checks immediately that $\phi(s)|_{X\times\mathbb{A}^1\times 0} = 0$, i.e. $\phi(s) \in \mathcal{K}(X \times \mathbb{A}^1)$ and further

$$\partial_0\phi(s) = \phi(s)|_{X\times 0\times\mathbb{A}^1} = 0, \quad \partial_1\phi(s) = \phi(s)|_{X\times 1\times\mathbb{A}^1} = s.$$

A Nisnevich sheaf $\mathcal{F}$ is said to be strongly homotopy invariant provided that all cohomology presheaves $X \mapsto H^i_{Nis}(X, \mathcal{F})$ are homotopy invariant. According to Theorem 1.1 each homotopy invariant Nisnevich sheaf with transfers is strongly homotopy invariant.

Proposition 1.10. *Let $\mathcal{G}$ and $\mathcal{F}$ be Nisnevich sheaves (resp. Nisnevich sheaves with transfers). Assume that $\mathcal{G}$ is contractible and $\mathcal{F}$ is strongly homotopy invariant. Then $\mathrm{Ext}^i_{Nis}(\mathcal{G}, \mathcal{F}) = 0$ for all $i \geq 0$ (resp. $\mathrm{Ext}^i_{NSwT}(\mathcal{G}, \mathcal{F}) = 0$ for all $i \geq 0$).*

Proof. We'll give a proof for Nisnevich sheaves, the case of Nisnevich sheaves with transfers is treated similarly. Let $0 \to \mathcal{F} \to I^\bullet$ be an injective resolution of $\mathcal{F}$. The cohomology presheaves of the complex $C_1(I^\bullet)$ are given by the formula $X \mapsto H^i_{Nis}(X \times \mathbb{A}^1, \mathcal{F}) = H^i_{Nis}(X, \mathcal{F})$. Since the sheaf associated to the presheaf $X \mapsto H^i_{Nis}(X, \mathcal{F})$ is trivial for $i > 0$ we conclude that $C_1(I^\bullet)$ is a resolution of $C_1(\mathcal{F}) = \mathcal{F}$. The face operations $\partial_0, \partial_1 : C_1(I^\bullet) \to I^\bullet$ give two maps of resolutions over the identity endomorphism of $\mathcal{F}$ and hence are homotopic. Thus there exists a family of sheaf homomorphisms $s_n : C_1(I^n) \to I^{n-1}$ such that $ds_n + s_{n+1}C_1(d) = \partial_1 - \partial_0$. Let now $\phi : \mathcal{G} \to C_1(\mathcal{G})$ be the homomorphism from the definition of a contractible sheaf. One checks immediately now that associating to each homomorphism $f \in \mathrm{Hom}_{Nis}(\mathcal{G}, I^n)$ the homomorphism $u_n(f) = s_n C_1(f)\phi \in \mathrm{Hom}_{Nis}(\mathcal{G}, I^{n-1})$ we get a contracting homotopy for the complex $\mathrm{Hom}_{Nis}(\mathcal{G}, I^\bullet)$.

Corollary 1.10.1. *Let $\mathcal{G}$ and $\mathcal{F}$ be Nisnevich sheaves with transfers. Assume that $\mathcal{F}$ is homotopy invariant (and hence strongly homotopy invariant) and $\mathcal{G}$ admits a finite resolution*

$$0 \to \mathcal{G} \to \mathcal{G}^0 \to \ \ldots \ \to \mathcal{G}^n \to 0$$

in which all $\mathcal{G}^i$ are contractible Nisnevich sheaves with transfers. Then $\mathrm{Ext}^i_{NSwT/F}(\mathcal{G}, \mathcal{F}) = 0$ for all $i \geq 0$.

Corollary 1.10.2. *Let $A^\bullet$ be a bounded above complex of Nisnevich sheaves with transfers. Assume that all A^i are contractible and all the cohomology sheaves $\mathcal{H}^i = H^i(A^\bullet)$ are homotopy invariant. Then the complex $A^\bullet$ is acyclic.*

Proof. We prove that $\mathcal{H}^i = 0$ by inverse induction on i. For $i >> 0 \quad \mathcal{H}^i = 0$ since $A^\bullet$ is bounded above. Assume now that $\mathcal{H}^j = 0$ for all $j > i$ and denote by $\mathcal{Z}^i$ the kernel of $d : A^i \to A^{i+1}$. According to our inductive assumption the sheaf $\mathcal{Z}^i$ has a finite resolution

$$0 \to \mathcal{Z}^i \to A^i \to A^{i+1} \to \ \ldots$$

with contractible terms. Since the sheaf $\mathcal{H}^i$ is homotopy invariant we conclude from Corollary 1.10.1 that $\mathrm{Hom}_{NSwT}(\mathcal{Z}^i, \mathcal{H}^i) = 0$. Thus the natural epimorphism $\mathcal{Z}^i \twoheadrightarrow \mathcal{H}^i$ is zero and hence $\mathcal{H}^i = 0$.

For any presheaf $\mathcal{F}$ and any $n \geq 0$ we have a natural monomorphism $\mathcal{F} \to C_n(\mathcal{F})$ split by any of the iterated face maps $\partial : C_n(\mathcal{F}) \to C_0(\mathcal{F}) = \mathcal{F}$. Denoting by $C_{0\bullet}(\mathcal{F})$ the constant simplicial presheaf (which has $\mathcal{F}$ in all dimensions and all face and degeneracy maps are identities) we get a natural monomorphism (split in each dimension) of simplicial presheaves $C_{0\bullet}(\mathcal{F}) \to C_\bullet(\mathcal{F})$ and hence also the associated monomorphism of compexes (of degree +1) $C_0^\bullet(\mathcal{F}) \to C^\bullet(\mathcal{F})$. The cokernel of this monomorphism consists of contractible presheaves according to Lemma 1.9. This construction generalizes immediately to complexes of presheaves. In particular, for each bounded above complex $A^\bullet$ of Nisnevich sheaves with transfers we have a natural monomorphism of complexes $C_0^\bullet(A^\bullet) \to C^\bullet(A^\bullet)$ whose cokernel consists of contractible sheaves. Note further that we also have a natural embedding $A^\bullet \to C_0^\bullet(A^\bullet)$ whose cokernel is contractible (in the usual sence of homological algebra) and hence acyclic.

Proposition 1.11. *Let $A^\bullet$ be a bounded above complex of Nisnevich sheaves with transfers.*

1. *If each A^n is contractible then the complex $C^\bullet(A^\bullet)$ is acyclic.*
2. *If the cohomology sheaves $H^i(A^\bullet)$ are homotopy invariant then the natural embedding $A^\bullet \to C^\bullet(A^\bullet)$ is a quasiisomorphism.*

Proof. If each A^n is contractible then each $C^i(A^n)$ is contractible as well, so that all terms of the complex $C^\bullet(A^\bullet)$ are contractible. Since the cohomology sheaves of this complex are homotopy invariant we conclude from Corollary 1.10.2 that this complex is acyclic.

Assume now that the cohomology sheaves of $A^\bullet$ are homotopy invariant. Lemma 1.3 implies immediately that all the cohomology sheaves of the complex $C^\bullet(A^\bullet)/C_0^\bullet(A^\bullet)$ are homotopy invariant. Since the terms of this complex are contractible we conclude from Corollary 1.10.2 that this complex is acyclic. Thus the natural embedding $C_0^\bullet(A^\bullet) \to C^\bullet(A^\bullet)$ is a quasiisomorphism and hence the embedding $A^\bullet \to C_0^\bullet(A^\bullet) \to C^\bullet(A^\bullet)$ is a quasiisomorphism as well.

Corollary 1.11.1. *Let $B^\bullet, A^\bullet$ be bounded above complexes of Nisnevich sheaves with transfers. Assume that the sheaves B^n are contractible and the cohomology sheaves $H^n(A^\bullet)$ are homotopy invariant. Then $\mathrm{Hom}_{D^-(NSwT/F)}(B^\bullet, A^\bullet) = 0$.*

Proof. It suffices to show that each homomorphism of complexes $f : B^\bullet \to A^\bullet$ gives zero in $\mathrm{Hom}_{D^-(NSwT/F)}(B^\bullet, A^\bullet)$. This follows immediately from the commutative diagram

$$\begin{array}{ccc} B^\bullet & \xrightarrow{f} & A^\bullet \\ \downarrow & & \downarrow \\ C^\bullet(B^\bullet) & \xrightarrow{C^\bullet(f)} & C^\bullet(A^\bullet) \end{array}$$

in which the right vertical arrow is a qasiisomorphism and the complex $C^\bullet(B^\bullet)$ is acyclic.

Corollary 1.11.2. *Let $B^\bullet, A^\bullet$ be bounded above complexes of Nisnevich sheaves with transfers. Assume that the cohomology sheaves $H^n(A^\bullet)$ are homotopy invariant. Then the natural embedding $B^\bullet \to C^\bullet(B^\bullet)$ induces an isomorphism*

$$\mathrm{Hom}_{D^-(NSwT/F)}(C^\bullet(B^\bullet), A^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{D^-(NSwT/F)}(B^\bullet, A^\bullet).$$

Proof. Once again we decompose the embedding $B^\bullet \to C^\bullet(B^\bullet)$ into a composition $B^\bullet \to C_0^\bullet(B^\bullet) \to C^\bullet(B^\bullet)$. The first arrow is a quasiisomorphism and hence induces isomorphisms on $\mathrm{Hom}_{D^-(NSwT/F)}(-, A^\bullet)$. The second embedding has cokernel consisting of contractible sheaves and hence also induces an isomorphism on $\mathrm{Hom}_{D^-(NSwT/F)}(-, A^\bullet)$ according to corollary 1.11.1.

Corollary 1.11.2 shows that if $A^\bullet \in DM^-(F)$ is a bounded above complex of Nisnevich sheaves with transfers with homotopy invariant cohomology sheaves then for any scheme $X \in Sm/F$ the natural embedding $\mathbb{Z}_{tr}(X) \to C^\bullet(\mathbb{Z}_{tr}(X)) = M(X)$ induces an isomorphism

$$\mathrm{Hom}_{D^-(NSwT/F)}(M(X), A^\bullet[i]) \xrightarrow{\sim} \mathrm{Hom}_{D^-(NSwT/F)}(\mathbb{Z}_{tr}(X), A^\bullet[i])$$

which together with Corollary 1.8 ends up the proof of the Theorem 1.5.

Proposition 1.11 shows immediately that if $A^\bullet$ is an acyclic bounded above complex of Nisnevich sheaves with transfers then the complex $C^\bullet(A^\bullet)$ is acyclic as well. This implies further that if $f : A^\bullet \to B^\bullet$ is a quasiisomorphism of bounded above complexes of Nisnevich sheaves with transfers then $C^\bullet(f)$ is also a quasiisomorphism. Thus we get a functor

$$C^\bullet : D^-(NSwT/F) \to DM^-(F).$$

Denote by $\mathcal{A}$ the thick triangulated subcategory of $D^-(NSwT/F)$ comprising those complexes $A^\bullet$ for which the complex $C^\bullet(A^\bullet)$ is acyclic. The previous discussion proves immediately the following result.

Theorem 1.12 [V 2]**.** *(1) The functor $C^\bullet$ takes distinguished triangles to distinguished triangles and commutes with direct sums (of homologically bounded above families).*
(2) The functor $C^\bullet$ is left adjoint to the embedding functor $DM^-(F) \hookrightarrow$ $\hookrightarrow D^-(NSwT/F)$ and establishes an equivalence of $DM^-(F)$ with the localization of $D^-(NSwT/F)$ with respect to the thick triangulated subcategory $\mathcal{A}$.

We finish this section by a more detailed description of the category $\mathcal{A}$.

Lemma 1.13. *A complex $A^\bullet \in D^-(NSwT)$ is in $\mathcal{A}$ if and only if it is quasi-isomorphic to a (bounded above) complex of contractible Nisnevich sheaves with transfers.*

Proof. If all the entries of the complex $A^\bullet$ are contractible then the complex $C^\bullet(A^\bullet)$ is acyclic according to Proposition 1.11. Assume now that the complex $C^\bullet(A^\bullet)$ is acyclic. The distinguished triangle

$$C_0^\bullet(A^\bullet) \to C^\bullet(A^\bullet) \to C^\bullet(A^\bullet)/C_0^\bullet(A^\bullet) \to C_0^\bullet(A^\bullet)[1] \to \ldots$$

shows that the complex $C_0^\bullet(A^\bullet)$ (which is quasiisomorphic to $A^\bullet$) is quasiisomorphic to a complex $C^\bullet(A^\bullet)/C_0^\bullet(A^\bullet)[-1]$ with contractible entries.

Let $X \in Sch/F$ be a scheme with a distinguished rational point $x_0 \in X$. We'll be saying that the scheme X is (algebraically) contractible (to the point x_0) iff there exists a morphism $f : X \times \mathbb{A}^1 \to X$ such that $f|_{X\times 1} = 1_X, f|_{X\times 0} = x_0$ and $f|_{x_0\times\mathbb{A}^1} = x_0$.

Proposition 1.14. *(1) Assume that the scheme (X, x_0) is contractible. Then for any $Y \in Sch/F$ the sheaf $\mathbb{Z}_{tr}(Y \times X/Y \times x_0)$ is contractible.*
(2) The category $\mathcal{A}$ is weakly generated by the contractible sheaves $\mathbb{Z}_{tr}(Y \times \mathbb{A}^1/Y \times 0) \quad (Y \in Sm/F)$.

Proof. Let $f : X \times \mathbb{A}^1 \to X$ be the contraction of X. We define the homomorphism of sheaves $\phi : \mathbb{Z}_{tr}(Y \times X) \to C_1(\mathbb{Z}_{tr}(Y \times X))$ as the composition

$$\mathbb{Z}_{tr}(Y \times X) \xrightarrow{Z \mapsto Z \times \Delta_{\mathbb{A}^1}} C_1(\mathbb{Z}_{tr}(Y \times X \times \mathbb{A}^1)) \xrightarrow{C_1(1_Y \times f)} C_1(\mathbb{Z}_{tr}(Y \times X))$$

Here the first arrow sends a section of $\mathbb{Z}_{tr}(Y \times X)$ over a smooth scheme U (i.e. a cycle $Z \subset Y \times X \times U$ each component of which is finite and surjective over a component of U) to the section of $\mathbb{Z}_{tr}(Y \times X \times \mathbb{A}^1)$ over $U \times \mathbb{A}^1$ given by the cycle $Z \times \Delta_{\mathbb{A}^1} \subset Y \times X \times A^1 \times U \times \mathbb{A}^1$ (each component of which is obviously finite and surjective over a component of $U \times \mathbb{A}^1$). One checks immediately that ϕ takes $\mathbb{Z}_{tr}(Y \times x_0)$ to $C_1(\mathbb{Z}_{tr}(Y \times x_0))$ and hence defines a homomorphism $\overline{\phi}$: $\mathbb{Z}_{tr}(Y \times X/Y \times x_0) \to C_1(\mathbb{Z}_{tr}(Y \times X/Y \times x_0))$ and furthermore $\partial_1 \overline{\phi} = Id, \partial_0 \overline{\phi} = 0$.

To prove the second statement it suffices (in view of Lemmas 0.6 and 1.13) to show that every contractible sheaf $\mathcal{F} \in NSwT$ admits a left resolution by the direct sums of sheaves of the form $\mathbb{Z}_{tr}(Y \times \mathbb{A}^1/Y)$. We'll show more generally that such resolutions exist for all sheaves $\mathcal{F} \in NSwT$ for which the complex $C^\bullet(\mathcal{F})$ is acyclic (i.e. $\mathcal{F} \in \mathcal{A}$). Note first of all that to give a homomorphism of sheaves with transfers $\mathbb{Z}_{tr}(Y \times \mathbb{A}^1/Y) \to \mathcal{F}$ is the same as to give a section $s \in \mathcal{F}(Y \times \mathbb{A}^1)$ such that $s|_{Y \times 0} = 0$. Surjectivity of the homomorphism $C_1(\mathcal{F}) \xrightarrow{\partial_1 - \partial_0} \mathcal{F}$ implies that locally in the Nisnevich topology each section $t \in \mathcal{F}(Y)$ may be written in the form $t = s|_{Y \times 1}$, where $s \in \mathcal{F}(Y \times \mathbb{A}^1)$ is a section for which $s|_{Y \times 0} = 0$. This remark shows that the natural homomorphism

$$\bigoplus_{Y \in Sm/F} \bigoplus_{s \in \mathcal{F}(Y \times \mathbb{A}^1) : s|_{Y \times 0} = 0} \mathbb{Z}_{tr}(Y \times \mathbb{A}^1/Y) \to \mathcal{F}$$

is surjective. The kernel of this homomorphism is again in $\mathcal{A}$, so that the construction may be repeated.

§ 2 Tensor structure on the category $DM^-(F)$.

To define the tensor structure on the category $DM^-(F)$ we start with the definition of the tensor product of presheaves with transfers. The definition we are about to give is dictated by the following (expected) properties of the tensor product operation: it should commute with any direct sums and moreover should commute with tensoring (in the usual sence) with arbitrary abelian groups, be right exact and satisfy the property that (here the superscript *pr* stands for the tensor product in the category of presheaves) $\mathbb{Z}_{tr}(X) \otimes_{tr}^{pr} \mathbb{Z}_{tr}(Y) = \mathbb{Z}_{tr}(X \times Y)$. Note that for any presheaf with transfers $\mathcal{F}$ we have a natural exact sequence:

$$0 \leftarrow \mathcal{F} \leftarrow \oplus_{X \in Sm/F} \mathcal{F}(X) \otimes \mathbb{Z}_{tr}(X) \leftarrow \oplus_{f \in Cor(X,Y)} \mathcal{F}(Y) \otimes \mathbb{Z}_{tr}(X)$$

tensoring such resolutions for $\mathcal{F}$ and $\mathcal{G}$ we come immediately to the following formula for $\mathcal{F} \otimes_{tr}^{pr} \mathcal{G}$

$$\mathcal{F} \otimes_{tr}^{pr} \mathcal{G} = Coker(\bigoplus_{X \in Sm/F, g \in Cor(Y',Y)} \mathcal{F}(X) \otimes \mathcal{G}(Y) \otimes \mathbb{Z}_{tr}(X \times Y') \oplus$$
$$\oplus \bigoplus_{Y \in Sm/F, f \in Cor(X',X)} \mathcal{F}(X) \otimes \mathcal{G}(Y) \otimes \mathbb{Z}_{tr}(X' \times Y) \longrightarrow$$
$$\longrightarrow \bigoplus_{X,Y \in Sm/F} \mathcal{F}(X) \otimes \mathcal{G}(Y) \otimes \mathbb{Z}_{tr}(X \times Y)).$$

In other words the presheaf with transfers $\mathcal{F} \otimes_{tr}^{pr} \mathcal{G}$ is given by the formula:

$$(\mathcal{F} \otimes_{tr}^{pr} \mathcal{G})(Z) = \bigoplus_{X,Y \in Sm/F} \mathcal{F}(X) \otimes \mathcal{G}(Y) \otimes Cor(Z, X \times Y)/\Lambda$$

where Λ is the subgroup generated by elements of the following form

$$\phi \otimes \psi \otimes (f \times 1_Y) \cdot h - f^*(\phi) \otimes \psi \otimes h : \begin{array}{l} f \in Cor(X', X), \phi \in \mathcal{F}(X) \\ \psi \in \mathcal{G}(Y), h \in Cor(Z, X' \times Y) \end{array}$$
$$\phi \otimes \psi \otimes (1_X \times g) \cdot h - \phi \otimes g^*(\psi) \otimes h : \begin{array}{l} g \in Cor(Y', Y), \phi \in \mathcal{F}(X) \\ \psi \in \mathcal{G}(Y), h \in Cor(Z, X \times Y') \end{array}$$

With this definition one verifies easily that $\otimes_{tr}^{pr}$ has all the expected properties. In particular, the functor $\otimes_{tr}^{pr}$ is right exact in each variable, commutative and associative (up to a natural isomorphism) and $\mathbb{Z}_{tr}(X) \otimes_{tr}^{pr} \mathbb{Z}_{tr}(Y) = \mathbb{Z}_{tr}(X \times Y)$. Next one defines the tensor product of Nisnevich sheaves with transfers as the sheaf associated with their tensor product in the category of presheaves. We denote the last operation by $\otimes_{tr}$. The bifunctor $\otimes_{tr} : NSwT/F \times NSwT/F \to NSwT/F$ is still right exact, commutative and associative , commutes with arbitrary direct sums and satisfies the identity $\mathbb{Z}_{tr}(X) \otimes_{tr} \mathbb{Z}_{tr}(Y) = \mathbb{Z}_{tr}(X \times Y)$. The following universal mapping property of the tensor product of Nisnevich sheaves with transfers is obvious from the definition.

Lemma 2.1. *Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be Nisnevich sheaves with transfers. To give a homomorphism of sheaves with transfers $\mathcal{F} \otimes_{tr} \mathcal{G} \xrightarrow{p} \mathcal{H}$ is the same as to give bilinear maps $p_{X,Y} : \mathcal{F}(X) \times \mathcal{G}(Y) \to \mathcal{H}(X \times Y)$ ($X, Y \in Sm/F$) which satisfy the following properties*

(1) *For any $f \in Cor(X', X)$ the following diagram commutes*

$$\begin{array}{ccc} \mathcal{F}(X) \times \mathcal{G}(Y) & \xrightarrow{p_{X,Y}} & \mathcal{H}(X \times Y) \\ {\scriptstyle f^* \times 1_{\mathcal{G}(Y)}} \downarrow & & \downarrow {\scriptstyle (f \times 1_Y)^*} \\ \mathcal{F}(X') \times \mathcal{G}(Y) & \xrightarrow{p_{X',Y}} & \mathcal{H}(X' \times Y) \end{array}$$

(2) *For any $g \in Cor(Y', Y)$ the following diagram commutes*

$$\begin{array}{ccc} \mathcal{F}(X) \times \mathcal{G}(Y) & \xrightarrow{p_{X,Y}} & \mathcal{H}(X \times Y) \\ {\scriptstyle 1_{\mathcal{F}(X)} \times g^*} \downarrow & & {\scriptstyle (1_X \times g)^*} \downarrow \\ \mathcal{F}(X) \times \mathcal{G}(Y') & \xrightarrow{p_{X,Y'}} & \mathcal{H}(X \times Y') \end{array}$$

To extend this operation on the category $D^-(NSwT/F)$ we need a few auxiliary results. For any $X \in Sm/F$ denote by L_i^X the left derived functors of the right exact functor

$$\mathbb{Z}_{tr}(X) \otimes_{tr}^{pr} - : \text{Presheaves with transfers} \to \text{Presheaves with transfers}.$$

Note that the sheaves $\mathbb{Z}_{tr}(X)$ are projective in the category of presheaves with transfer, but we don't know whether they are flat (probably not).

Lemma 2.2. *Let $\mathcal{F}$ be a presheaf with transfers such that $\mathcal{F}_{Nis}^{\sim} = 0$. Then $L_i^X(\mathcal{F})_{Nis}^{\sim} = 0$ for all $i \geq 0$ and all $X \in Sm/F$.*

Proof. We first construct certain special presheaves with transfers $\mathcal{F}$ for which $\mathcal{F}_{Nis}^{\sim} = 0$. Let $f : Y' \to Y$ be a Nisnevich covering of $Y \in Sm/F$. Denote by $\mathcal{H}_i(Y'/Y)$ the homology presheaves of the complex

$$\check{C}_\bullet(Y'/Y): \qquad 0 \leftarrow \mathbb{Z}_{tr}(Y) \xleftarrow{f_*} \mathbb{Z}_{tr}(Y') \leftarrow \mathbb{Z}_{tr}(Y' \times_Y Y') \leftarrow \dots \quad .$$

Lemma 1.6 shows that $\mathcal{H}_i(Y'/Y)_{Nis}^{\sim} = 0$ for all i. Moreover it's clear from the definition that for each presheaf with transfers $\mathcal{F}$ for which $\mathcal{F}_{Nis}^{\sim} = 0$ there exists an epimorphism onto $\mathcal{F}$ from a direct sum of presheaves of the form $\mathcal{H}_0(Y'/Y)$. We now start proving our statement by induction on i. For $i < 0$ everything is obvious. Assume now that the statement holds for all $i < j$. The previous remark shows that to prove the vanishing of $L_j^X(\mathcal{F})_{Nis}^{\sim}$ for any presheaf $\mathcal{F}$ with $\mathcal{F}_{Nis}^{\sim} = 0$ it suffices to consider the special case $\mathcal{F} = \mathcal{H}_0(Y'/Y)$. Tensoring the complex $\check{C}_\bullet(Y'/Y)$ with $\mathbb{Z}_{tr}(X)$ we get a spectral sequence

$$E_{pq}^2 = L_p^X(\mathcal{H}_q(Y'/Y)) \Rightarrow H_{p+q}(\mathbb{Z}_{tr}(X) \otimes_{tr}^{pr} \check{C}_\bullet(Y'/Y)).$$

Note further that the complex $\mathbb{Z}_{tr}(X) \otimes_{tr}^{pr} \check{C}_\bullet(Y'/Y))$ is equal to $\check{C}_\bullet(X \times Y'/X \times Y)$, so that the limit of our spectral sequence coincides with $\mathcal{H}_*(X \times Y'/X \times Y)$. This shows that applying to the above spectral sequence the exact functor $\tilde{}_{Nis}$ we get a new spectral sequence which converges to zero:

$$E_{pq}^2 = L_p^X(\mathcal{H}_q(Y'/Y))_{Nis}^{\sim} \Rightarrow 0$$

Applying the induction hypothesis we get immediately the desired conclusion.

Corollary 2.3. *Let $A_\bullet$ be a bounded below complex (of degree -1) of free presheaves with transfers. Assume that $H_i(A_\bullet)^{\sim}_{Nis} = 0$ for all i. Then $H_i(\mathbb{Z}_{tr}(X) \otimes^{pr}_{tr} A_\bullet)^{\sim}_{Nis} = 0$ for all i and all $X \in Sm/F$.*

Proof. This statement follows immediately from Lemma 2.2 in view of the spectral sequence

$$E^2_{pq} = L^X_p(H_q(A_\bullet)) \Rightarrow H_{p+q}(\mathbb{Z}_{tr}(X) \otimes^{pr}_{tr} A_\bullet).$$

We return back to complexes of degree +1 and cohomological notations. Corollary 2.3 implies easily the following result

Corollary 2.4. *Let $A^\bullet$ and $B^\bullet$ be bounded above complexes (of degree +1) of free Nisnevich sheaves with transfers. Assume that either $A^\bullet$ or $B^\bullet$ is acyclic, then the complex $A^\bullet \otimes_{tr} B^\bullet$ is acyclic as well.*

Corollary 2.5. *Let $A^\bullet_1 \to A^\bullet_2$ be a quasiisomorphism of bounded above complexes of free Nisnevich sheaves with transfers. Then for any bounded above complex $B^\bullet$ of free Nisnevich sheaves with transfers the induced homomorphism $A^\bullet_1 \otimes_{tr} B^\bullet \to A^\bullet_2 \otimes_{tr} B^\bullet$ is a quasiisomorphism as well.*

For any Nisnevich sheaf with transfers $\mathcal{F}$ we have a natural epimorphism onto $\mathcal{F}$ from a free Nisnevich sheaf with transfers:$\bigoplus_{X \in Sm/F, \phi \in \mathcal{F}(X)} \mathbb{Z}_{tr}(X) \to \mathcal{F}$. Repeating this construction we get a canonical free resolution $\mathcal{X}_\bullet(\mathcal{F}) \to \mathcal{F}$. As always we reindex $\mathcal{X}_\bullet$ in cohomological terms, so that $\mathcal{X}^\bullet(\mathcal{F})$ is a complex of degree +1 concentrated in nonpositive degrees. Applying the functor $\mathcal{X}^\bullet$ to a bounded above complex $A^\bullet$ of Nisnevich sheaves with transfers we get a free bounded above complex $\mathcal{X}^\bullet(A^\bullet)$ and a natural quasiisomorphism $\mathcal{X}^\bullet(A^\bullet) \to A^\bullet$. We define the tensor product operation on the category $D^-(NSwT/F)$ via the formula

$$A^\bullet \otimes^L B^\bullet = \mathcal{X}^\bullet(A^\bullet) \otimes_{tr} \mathcal{X}^\bullet(B^\bullet).$$

Corollary 2.5 shows that this construction is well defined and moreover $A^\bullet \otimes^L B^\bullet = A^\bullet \otimes_{tr} B^\bullet$ for any complexes $A^\bullet, B^\bullet$ of free Nisnevich sheaves with transfers (more generally for any complexes whose entries are direct summands in appropriate free sheaves, i.e. are projective in the category of presheaves with transfers). The following properties of the bifunctor $\otimes^L$ are now straightforward.

Lemma 2.6. *(1) The functor $\otimes^L$ is commutative and associative (up to a natural isomorphism)*
(2) The functor $\otimes^L$ takes distinguished triangles in each variable to distinguished triangles.
(3) The functor $\otimes^L$ takes direct sums (of homologically bounded families) to direct sums.

To pass from the category $D^-(NSwT/F)$ to the category $DM^-(F)$ we have to verify that the tensor product $A^\bullet \otimes^L B^\bullet$ of two complexes, one of which is in the localizing subcategory $\mathcal{A}$ is again in $\mathcal{A}$.

Proposition 2.7. *Let $A^\bullet, B^\bullet$ be bounded above complexes of Nisnevich sheaves with transfers. Assume that $A^\bullet \in \mathcal{A}$, then $A^\bullet \otimes^L B^\bullet \in \mathcal{A}$.*

Proof. Since the category $\mathcal{A}$ is weakly generated by sheaves of the form $\mathbb{Z}_{tr}(Y \times \mathbb{A}^1/Y)$ $(Y \in Sm/F)$ and the category $D^-(NSwT)$ is weakly generated by sheaves $\mathbb{Z}_{tr}(X)$ $(X \in Sm/F)$ it suffices to note that

$$\mathbb{Z}_{tr}(Y \times \mathbb{A}^1/Y) \otimes^L \mathbb{Z}_{tr}(X) = \mathbb{Z}_{tr}(Y \times \mathbb{A}^1/Y) \otimes_{tr} \mathbb{Z}_{tr}(X) = \\ = \mathbb{Z}_{tr}(Y \times X \times \mathbb{A}^1/Y \times X) \in \mathcal{A}.$$

Proposition 2.7 together with Theorem 1.12 show that the tensor structure on the category $D^-(NSwT/F)$ induces (via the localizing functor $C^\bullet$) a tensor structure on the category $DM^-(F)$. Explicitly the tensor product operation on the category $DM^-(F)$ is given by the formula

$$A^\bullet \otimes B^\bullet = C^\bullet(A^\bullet \otimes^L B^\bullet).$$

The following result sums up the main properties of this tensor product operation.

Proposition 2.8. *(1) The functor $\otimes : DM^-(F) \times DM^-(F) \to DM^-(F)$ is commutative and associative (up to a natural isomorphism).*
(2) The functor $\otimes$ takes distinguished triangles in each variable to distinguished triangles.
(3) The functor $\otimes$ takes direct sums (of homologically bounded families) to direct sums.
(4) For any $A^\bullet, B^\bullet \in D^-(NSwT)$ we have a natural isomorphism $C^\bullet(A^\bullet \otimes^L B^\bullet) \xrightarrow{\sim} C^\bullet(A^\bullet) \otimes C^\bullet(B^\bullet)$. In particular for all schemes $X, Y \in Sm/F$ we have a natural isomorphism $M(X \times Y) \cong M(X) \otimes M(Y)$.

§ 3. Motivic Cohomology

The concept of motivic cohomology (as we understand it today) goes back to the beginning of the 80's when A. Beilinson conjectured that there should exist complexes of Zariski sheaves $\mathbb{Z}(n)$ on Sm/F which have (among others) the following properties:

- **Be1** (Normalization) The complex $\mathbb{Z}(0)$ is quasiisomorphic to the constant sheaf $\mathbb{Z}$ positioned in degree 0, the complex $\mathbb{Z}(1)$ is quasiisomorphic to the sheaf $\mathcal{O}^*$ of invertible functions positioned in degree 1.
- **Be2** (The Beilinson Soulé Vanishing Conjecture) For $n > 0$ the complex $\mathbb{Z}(n)$ is acyclic outside the interval $[1, n]$, the n-th cohomology sheaf $H^n(\mathbb{Z}(n))$ coincides with the sheaf $\mathcal{K}_n^M$ of Milnor K_n-groups.
- **Be3** (Relationship to K-theory) For any $X \in Sm/F$ there exists a natural spectral sequence

$$E_2^{p,q} = H^p_{Zar}(X, \mathbb{Z}(q)) \Rightarrow K_{2q-p}(X)$$

which rationally degenerates and defines isomorphisms of $H^*_{Zar}(X, \mathbb{Z}(*))\otimes\mathbb{Q}$ with subsequent factors of the γ filtration on $K_*(X) \otimes \mathbb{Q}$.

Be4 (The Beilinson-Lichtenbaum Conjecture) For any l prime to $char F$ the complex $\mathbb{Z}/l(n) = \mathbb{Z}(n) \otimes^L \mathbb{Z}/l$ is quasiisomorphic to $\tau_{\leq n} R\pi_*(\mu_l^{\otimes n})$. Here π is the obvious morphism of sites $(Sm/F)_{et} \to (Sm/F)_{Zar}$ and $\tau_{\leq n}$ denotes trancation at the level n.

In this section we define (following [V 2]) the motivic complexes $\mathbb{Z}(n)$ and establish some of there elementary properties.

Consider the sheaf $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})$. According to definitions (see §0) $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})$ coincides with $\mathbb{Z}_{tr}(\mathbb{G}_m^{\times n})/\mathcal{D}_n$ (here $\mathbb{G}_m$ stands for the multiplicative group scheme $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$) and $\mathcal{D}_n$ is the sum of images of homomorphisms

$$\mathbb{Z}_{tr}(\mathbb{G}_m^{\times n-1}) \to \mathbb{Z}_{tr}(\mathbb{G}_m^{\times n})$$

given by the embeddings of the form

$$(x_1, \dots, x_{n-1}) \mapsto (x_1, \dots, 1, \dots, x_{n-1}).$$

The subsheaf $\mathcal{D}_n$ is in fact a direct summand of $\mathbb{Z}_{tr}(\mathbb{G}_m^{\times n})$ (see §0). The corresponding projection $p : \mathbb{Z}_{tr}(\mathbb{G}_m^{\times n}) \to \mathcal{D}_n$ is given by the formula

$$p = \sum_I (-1)^{card(I)-1} (p_I)_*,$$

where I runs through all non empty subsets of $\{1, ..., n\}$ and $p_I : \mathbb{G}_m^{\times n} \to \mathbb{G}_m^{\times n}$ is the coordinate morphism, replacing all I-entries by $1 \in \mathbb{G}_m$. The sheaf $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})$ coincides with the complementary direct summand and may be also identified with the intersection of kernels of homomorphisms $(p_I)_*$ over all non empty subsets $I \subset \{1, ..., n\}$ (since $(p_I)_* p = (p_I)_*$ for any $I \neq \emptyset$). Furthermore, with these notations we have a canonical direct sum decomposition

$$\mathbb{Z}_{tr}(\mathbb{G}_m^{\times n}) = \bigoplus_{m=0}^{n} \bigoplus_{\substack{I \subset \{1,\dots,n\} \\ card\ I = m}} \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge m}) = \bigoplus_{m=0}^{n} \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge m})^{\binom{n}{m}} \tag{3.0}$$

Definition 3.1. *The motivic complex $\mathbb{Z}(n)$ of weight n on Sm/F is the complex $C^*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n}))[-n] = C^*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\times n})/\mathcal{D}_n)[-n]$. For a smooth scheme X over F we define its motivic cohomology groups $H^i_{\mathcal{M}}(X, \mathbb{Z}(n))$ as Zariski hypercohomology $H^i_{Zar}(X, \mathbb{Z}(n))$.*

Note that $\mathbb{Z}(n)$ is a complex of sheaves with transfers in Nisnevich topology (actually even in the etale topology) with homotopy invariant cohomology presheaves. Thus Corollary 1.2 shows that $H^*_{Zar}(X, \mathbb{Z}(n)) = H^*_{Nis}(X, \mathbb{Z}(n))$. Using further Theorem 1.5 we see that $H^i_{\mathcal{M}}(X, \mathbb{Z}(n)) = \mathrm{Hom}_{DM^-(F)}(M(X), \mathbb{Z}(n))$.

The motiv $\mathbb{Z}(1)$ is called the Tate motiv. For any $X \in DM^-(F)$ we denote by $X(n)$ the tensor product $X \otimes \mathbb{Z}(n)$. In particular for an abelian group A we denote by $A(n)$ the tensor product complex $A\otimes\mathbb{Z}(n)$. Proposition 2.8 implies easily, that $A(n)$ may be identified with the complex $C^*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})\otimes A)[-n]$. In particular $\mathbb{Z}/l(n)=C^*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})/l)[-n]$.

Proposition 2.8 implies that for each m,n we have a canonical isomorphism $\mathbb{Z}(m)\otimes\mathbb{Z}(n)\xrightarrow{\sim}\mathbb{Z}(m+n)$ and the following diagrams commute for all m,n,p

$$\begin{array}{ccc} \mathbb{Z}(m)\otimes\mathbb{Z}(n)\otimes\mathbb{Z}(p) & \xrightarrow{\sim} & \mathbb{Z}(m+n)\otimes\mathbb{Z}(p) \\ \cong\downarrow & & \cong\downarrow \\ \mathbb{Z}(m)\otimes\mathbb{Z}(n+p) & \xrightarrow{\sim} & \mathbb{Z}(m+n+p). \end{array}$$

Lemma 3.2.

(1) *The complex $\mathbb{Z}(0)$ is canonically quasisomorphic to the constant sheaf $\mathbb{Z}$, positioned in degree 0, the complex $\mathbb{Z}(1)$ is canonically quasiisomorphic to the sheaf $\mathcal{O}^*$ positioned in degree one.*

(2) *The complex $\mathbb{Z}(n)$ is acyclic in degrees $>n$.*

(3) *For all $n,m\geq 0$ we have a natural isomorphism $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})\otimes_{tr}\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge m})=\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge(n+m)})$ and hence a natural quasiisomorphism $\mathbb{Z}(n)\otimes\mathbb{Z}(m)\stackrel{\sim}{=}\mathbb{Z}(n+m)$.*

Proof. Identification of the complex $\mathbb{Z}(0)$ is straightforward. The complex $\mathbb{Z}(1)$ is a canonical direct summand in $C^*(\mathbb{Z}_{tr}(\mathbb{G}_m))[-1]$. To identify $\mathbb{Z}(1)$ we start by computing the cohomology presheaves of the complex $C^*(\mathbb{Z}_{tr}(\mathbb{G}_m))[-1]$. Let $U\in Sm/F$ be any smooth affine scheme. Denoting by $\mathcal{H}^i$ the i-th cohomology presheaf of $C^*(\mathbb{Z}_{tr}(\mathbb{G}_m))[-1]$ we have (using the notations introduced in [S-V]) the following identifications

$$\mathcal{H}^i(U)=H_{1-i}(C_*(\mathbb{Z}_{tr}(\mathbb{G}_m))(U))=H^{sin}_{1-i}(U\times\mathbb{G}_m/U).$$

Computation of singular homology of relative smooth curves - [S-V §3] shows that

$$\begin{aligned}\mathcal{H}^i(U)=&0 \qquad\qquad i\neq 1\\ \mathcal{H}^1(U)=&Pic(U\times\mathbb{P}^1,U\times 0\coprod U\times\infty).\end{aligned}$$

The exact sequence relating relative Picard group to the absolute ones shows immediately that $Pic(U\times\mathbb{P}^1,U\times 0\coprod U\times\infty)=\mathcal{O}^*(U)\oplus\mathbb{Z}$. This computation implies easily that all cohomology sheaves of $\mathbb{Z}(1)$ except for H^1 vanish and $H^1(\mathbb{Z}(1))=\mathcal{O}^*$.

Acyclicity of the complex $\mathbb{Z}(n)$ in degrees $>n$ is obvious from the construction. The last statement follows from the identification $\mathbb{Z}_{tr}(\mathbb{G}_m^{\times n})\otimes_{tr}\mathbb{Z}_{tr}(\mathbb{G}_m^{\times m})=\mathbb{Z}_{tr}(\mathbb{G}_m^{\times(n+m)})$ and right exactness of the functor $\otimes_{tr}$.

Remark 3.2.0 The proof of the Lemma 3.2 gives the following explicit way to identify the presheaf $\mathcal{H}^1(\mathbb{Z}(1))$ with $\mathcal{O}^*$. To each $f\in\mathcal{O}^*(U)=\mathrm{Hom}_{Sm/F}(U,\mathbb{G}_m)$ we associate the class of its graph $\Gamma_f\in\mathbb{Z}_{tr}(\mathbb{G}_m)(U)$ in $\mathcal{H}^1(\mathbb{Z}(1))(U)=\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge 1})(U)/$ $/\operatorname{Im}(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge 1})(U\times\mathbb{A}^1))$. Vice versa start with an irreducible cycle $Z\subset U\times\mathbb{G}_m$ finite and surjective over U. This cycle defines an invertible function $p_2\in\mathcal{O}^*(Z)$, taking the norm of this function from Z to U gives us an invertible function $f_Z=N_{Z/U}(p_2)\in\mathcal{O}^*(U)$. The two above maps define mutually inverse isomorphisms of $\mathcal{O}^*(U)$ and $\mathcal{H}^1(\mathbb{Z}(1))(U)$ (at least for affine U's).

Corollary 3.2.1. *For any $X \in Sm/F$ we have the following formulae:*

$$H^i_{\mathcal{M}}(X, \mathbb{Z}(0)) = \begin{cases} H^0(X, \mathbb{Z}) & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

$$H^i_{\mathcal{M}}(X, \mathbb{Z}(1)) = \begin{cases} H^0(X, \mathcal{O}^*) & \text{if } i = 1 \\ Pic(X) & \text{if } i = 2 \\ 0 & \text{if } i \neq 1, 2 \end{cases}$$

The formula (3.0) gives the following identification of the motiv of $\mathbb{G}_m^{\times n}$.

Lemma 3.3. *For any $n \geq 0$ we have the natural direct sum decomposition*

$$M(\mathbb{G}_m^{\times n}) = \bigoplus_{m=0}^{n} \mathbb{Z}(m)[m]^{\binom{n}{m}}.$$

For any $X, Y \in Sm/F$ we get a natural pairing

$$\begin{aligned} &H^i_{\mathcal{M}}(X, \mathbb{Z}(n)) \otimes H^j_{\mathcal{M}}(Y, \mathbb{Z}(m)) = \mathrm{Hom}_{DM^-(F)}(M(X), \mathbb{Z}(n)[i]) \otimes \\ &\otimes \mathrm{Hom}_{DM^-(F)}(M(Y), \mathbb{Z}(m)[j]) \to \mathrm{Hom}_{DM^-(F)}(M(X) \otimes M(Y), \\ &(\mathbb{Z}(n) \otimes \mathbb{Z}(m))[i+j]) = H^{i+j}_{\mathcal{M}}(X \times Y, \mathbb{Z}(n+m)). \end{aligned}$$

Using these pairings and the homomorphism in motivic cohomology induced by the diagonal embedding $\Delta : X \to X \times X$ one makes $H^{*,*}_{\mathcal{M}}(X) = \bigoplus_{i,n} H^i_{\mathcal{M}}(X, \mathbb{Z}(n))$ into a bigraded associative ring. Note further that $H^{1,1}_{\mathcal{M}}(SpecF) = F^*$. Thus we get a canonical homomorphism from the tensor algebra $T(F^*)$ of the multiplicative group F^* to the subring $\oplus_{n=0}^{\infty} H^{n,n}_{\mathcal{M}}(SpecF)$ of the motivic cohomology ring $H^{*,*}_{\mathcal{M}}(SpecF)$.

Theorem 3.4. *The natural homomorphism $T(F^*) \to \oplus_{n=0}^{\infty} H^{n,n}_{\mathcal{M}}(SpecF)$ defines an isomorphism $K^M_*(F) \xrightarrow{\sim} \oplus_{n=0}^{\infty} H^{n,n}_{\mathcal{M}}(SpecF)$ (here $K^M_*(F)$ is the Milnor ring of the field F - see* [B-T]*).*

Proof. We'll use the notation $H^{*,*}_{\mathcal{M}}(F)$ for $H^{*,*}_{\mathcal{M}}(SpecF)$. Note first of all that motivic cohomology $H^*_{\mathcal{M}}(F, \mathbb{Z}(n))$ coincides up to a shift of degrees with homology of the complex $C_*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n}))(F)$: $H^i_{\mathcal{M}}(F, \mathbb{Z}(n)) = H_{n-i}(C_*(\mathbb{G}_m^{\wedge n})(F))$. In particular

$$H^{n,n}_{\mathcal{M}}(F) = Coker\left(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})(\mathbb{A}^1_F) \xrightarrow{\partial_0 - \partial_1} \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})(SpecF)\right).$$

Moreover products in the motivic cohomology ring $H^{*,*}_{\mathcal{M}}(F)$ may be computed as follows. Every pair of cycles $Z \subset \mathbb{G}_m^{\times n} \times \Delta^p$, $T \subset \mathbb{G}_m^{\times m} \times \Delta^q$ defines a cycle $Z \times T \subset \mathbb{G}_m^{\times n+m} \times \Delta^p \times \Delta^q$. In this way we get a homomorphism of complexes $C_*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\times n}))(F) \otimes C_*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\times m}))(F) \to C_{*,*}(\mathbb{Z}_{tr}(\mathbb{G}_m^{\times n+m}))(F)$, which obviously factors to define a homomorphism $C_*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n}))(F) \otimes C_*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge m}))(F) \to C_{*,*}(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n+m}))(F)$. Composing this homomorphism with the shuffle map (see

§ 0) we finally get a homomorphism of complexes $\mathbb{Z}(n)(F) \otimes \mathbb{Z}(m)(F) \to \mathbb{Z}(n+m)(F)$ which defines the desired pairings in cohomology.

Every n-tuple $(a_1, ..., a_n)$ of elements of F^* defines an F-rational point $(a_1, ..., a_n) \in \mathbb{G}_m^{\times n}$. We'll denote by $[a_1, ..., a_n]$ the class of this rational point in $H^{n,n}_{\mathcal{M}}(F)$. According to definitions $[a_1, ..., a_n] = 0$ if at least one of a_i is equal to 1. Moreover the definition of products in motivic cohomology implies easily that $[a_1, ..., a_n] \cdot [b_1, ..., b_m] = [a_1, ..., a_n, b_1, ...b_m]$. This remark shows that the homomorphism $(F^*)^{\otimes n} \to H^{n,n}_{\mathcal{M}}(F)$ sends the tensor $a_1 \otimes ... \otimes a_n$ to $[a_1, ..., a_n]$). We first check that the homomorphism $(F^*)^{\otimes n} \to H^{n,n}(F)$ factors through $K_n^M(F)$. To do so it suffices to verify that that the class of the rational point $(a, 1-a) \in \mathbb{G}_m^{\times 2}$ dies in $H^{2,2}_{\mathcal{M}}(F)$. The proof of this fact requires the use of the transfer homomorphisms in motivic cohomology. We start by reminding the definition and properties of these homomorphisms.

Let $j : F \hookrightarrow E$ be a field extension. Extending scalars in cycles we get a canonical homomorphism of complexes $C_*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n}))(F) \to C_*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n}))(E)$ and the induced homomorphism in motivic cohomology $j_* : H^i_{\mathcal{M}}(F, \mathbb{Z}(n)) \to H^i_{\mathcal{M}}(E, \mathbb{Z}(n))$. We often use the notation x_E for the image $j_*(x) \in H^i_{\mathcal{M}}(E, \mathbb{Z}(n))$ of the element $x \in H^i_{\mathcal{M}}(F, \mathbb{Z}(n))$. The homomorphism $j_* : H^{*,*}_{\mathcal{M}}(F) \to H^{*,*}_{\mathcal{M}}(E)$ is obviously compatible with products in motivic cohomology.

Assume now that E/F is a finite field extension. Taking direct images of cycles we get a homomorphism of complexes $C_*(\mathbb{G}_m^{\wedge n})(E) \to C_*(\mathbb{G}_m^{\wedge n})(F)$ and hence the induced homomorphisms in homology groups

$$N_{E/F} : H_i(C_*(\mathbb{G}_m^{\wedge n})(E)) = H^{n-i,n}_{\mathcal{M}}(E) \to H_i(C_*(\mathbb{G}_m^{\wedge n})(F)) = H^{n-i,n}_{\mathcal{M}}(F).$$

Lemma 3.4.1. *Let E/F be a finite field extension and let further K/F be a finite normal field extension such that $\mathrm{Hom}_F(E, K) \neq \emptyset$. For any $x \in H^{n,n}_{\mathcal{M}}(E)$ and $y \in H^{m,m}_{\mathcal{M}}(F)$ we have:*

1. *$N_{E/F}(y_E \cdot x) = y \cdot N_{E/F}(x)$, $N_{E/F}(x \cdot y_E) = N_{E/F}(x) \cdot y$. In particular $N_{E/F}(y_E) = [E : F] \cdot y$.*
2. *$(N_{E/F}(x))_K = [E : F]_i \cdot \sum_{j \in \mathrm{Hom}_F(E,K)} j_*(x)$ (here $[E/F]_i$ stands for the inseparable degree of the field extension E/F).*
3. *The homomorphism $N_{E/F} : E^* = H^{1,1}_{\mathcal{M}}(E) \to F^* = H^{1,1}_{\mathcal{M}}(F)$ coincides with the usual norm homomorphism in field extensions.*

Proof. The first two properties follow immediately from the corresponding properties of the direct image homomorphism for cycles. The last property follows from (2), since the homomorphism $F^* = H^{1,1}_{\mathcal{M}}(F) \to K^* = H^{1,1}_{\mathcal{M}}(K)$ is injective.

Lemma 3.4.2. *Assume that there exists an integer $N > 0$ such that $N \cdot [a, 1-a] = 0$ for any field F and any element $a \in F^* \setminus \{1\}$ (resp. $N \cdot [a, -a] = 0$ for any F and any $a \in F^*$). Then $[a, 1-a] = 0 \quad \forall F \quad \forall a \in F^* \setminus \{1\}$ (resp. $[a, -a] = 0 \quad \forall F \quad \forall a \in F^*$).*

Proof. The proof is the same in both cases so we'll consider only the first one. It suffices to show that if all symbols of the form $[a, 1-a]$ are killed by an integer

$N = Mp$, where p is a prime then all such symbols are killed by M already. The case $a \in (F^*)^p$ is easy so we'll assume that a is not p-th power. Set $\alpha = a^{1/p}$, $E = F(\alpha)$. According to our assumptions $0 = Mp \cdot [\alpha, 1-\alpha] = M \cdot [a, 1-\alpha]$. Applying to this equation the homomorphism $N_{E/F}$ and using Lemma 3.4.1 we get:

$$0 = N_{E/F}(M \cdot [a, 1-\alpha]) = M \cdot [a] \cdot N_{E/F}([1-\alpha]) = M \cdot [a] \cdot [N_{E/F}(1-\alpha)] = M \cdot [a, 1-a].$$

Proposition 3.4.3. *For any field F and any $a \in F^* \setminus \{1\}$ the element $[a, 1-a] \in H^{2,2}_{\mathcal{M}}(F)$ is trivial.*

Proof. For any $a, b \in F^* \setminus \{1\}$ consider the closed subscheme $Y \subset \mathbb{A}^1 \times \mathbb{G}_m$, given by the equation

$$X^2 - (t \cdot (a+b) + (1-t) \cdot (1+ab))X + ab = 0$$

(here t is the coordinate in $\mathbb{A}^1$ and X is the coordinate in $\mathbb{G}_m$). One checks immediately that the projection $p_2 : Y \to \mathbb{G}_m$ is an isomorphism so that in particular Y is integral. Moreover it's clear that Y is finite and surjective over $\mathbb{A}^1$ and hence defines an element $y \in \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge 1})(\mathbb{A}^1)$. Note that $\partial_0(y) = [ab] + [1] = [ab]$ and $\partial_1(y) = [a] + [b]$ (this shows again that $[ab] = [a] + [b]$). Next consider the closed embedding $Y \hookrightarrow \mathbb{A}^1 \times \mathbb{G}_m^{\times 2}$ obtained by means of the diagonal embedding $\mathbb{G}_m \hookrightarrow \mathbb{G}_m^{\times 2}$ and denote by $y' \in \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge 2})(\mathbb{A}^1)$ the corresponding element. The faces of y' are given by the formulae $\partial_0(y') = [ab, ab] + [1, 1] = [ab, ab]$,$\partial_1(y') = [a, a] + [b, b]$. This gives us the relation $[ab, ab] = [a, a] + [b, b]$. Taking here $a = b$ and using bimultiplicativity of the symbol $[-, -]$ we see that $2 \cdot [a, -a] = 0 \quad \forall a \in F^*$. Using Lemma 3.4.2 we conclude that in fact $[a, -a] = 0 \quad \forall a \in F^*$. The relation $[a, -a] = 0$ implies immediately that $[a, 1-a] + [a^{-1}, 1-a^{-1}] = 0 \quad \forall a \in F^* \setminus \{1\}$. Assume now that $a \in F^*$ is an element such that $a^6 \neq 1$. Consider a closed subscheme $Z \subset \mathbb{A}^1 \times (\mathbb{G}_m \setminus \{1\})$ given by the equation

$$X^3 - t \cdot (a^3+1) \cdot X^2 + t \cdot (a^3+1) \cdot X - a^3 = 0.$$

Once again one checks immediately that the projection $p_2 : Z \to \mathbb{G}_m \setminus \{1\}$ is an isomorphism, so that Z is, in particular, integral. The fiber of Z over $0 \in \mathbb{A}^1$ consists of all cubical roots of a^3 and the fiber of Z over $1 \in \mathbb{A}^1$ consists of a^3 and two roots x_1, x_2 of the equation $X^2 - X + 1 = 0$. We embed Z into $\mathbb{A}^1 \times \mathbb{G}_m^{\times 2}$ using the embedding $\mathbb{G}_m \setminus \{1\} \hookrightarrow \mathbb{G}_m^{\times 2} \quad x \mapsto (x, 1-x)$ and denote by z' the corresponding element in $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge 2})(\mathbb{A}^1)$. Assume for the moment that F contains all roots (in the algebraic closure of F) of the equation $X^6 - 1 = 0$. In this case $\partial_0(z') = [a, 1-a] + [\xi a, 1-\xi a] + [\xi^2 a, 1-\xi^2 a] = [a, 1-a^3] + [\xi, (1-\xi a)(1-\xi^2 a)^2]$, where ξ is the generator of the group μ_3 ($\xi = 1$ if $char(F) = 3$). At the same time $\partial_1(z') = [a^3, 1-a^3] + [x_1, 1-x_1] + [x_2, 1-x_2] = [a^3, 1-a^3]$. Multiplying the relation $\partial_0(z') = \partial_1(z')$ by 3 and keeping in mind that $\xi^3 = 1$ we get the relation $2[a^3, 1-a^3] = 0$. Using once again the transfer argument we conclude further that $6[a, 1-a] = 0$ for any $a \in F^* \setminus \{1\}$. Lemma 3.4.2 shows finally that $[a, 1-a] = 0 \quad \forall a \in F^* \setminus \{1\}$.

Proposition 3.4.2 shows that the homomorphism $T(F^*) \to \oplus_{n=0}^{\infty} H^{n,n}_{\mathcal{M}}(F)$ factors through the Milnor ring $K^M_*(F)$. We denote by $\lambda = \lambda_F$ the resulting ring homomorphism $K^M_*(F) \to \oplus_{n=0}^{\infty} H^{n,n}_{\mathcal{M}}(F)$.

Lemma 3.4.4. *The homomorphism $\lambda : K^M_*(F) \to \oplus_{n=0}^{\infty} H^{n,n}_{\mathcal{M}}(F)$ is compatible with transfer maps in Milnor K-theory and motivic cohomology respectively. In other words for any finite field extension E/F the following diagram commutes*

$$\begin{array}{ccc} K^M_*(E) & \xrightarrow{\lambda_E} & \oplus_{n=0}^{\infty} H^{n,n}_{\mathcal{M}}(E) \\ {\scriptstyle N_{E/F}}\downarrow & & {\scriptstyle N_{E/F}}\downarrow \\ K^M_*(F) & \xrightarrow{\lambda_F} & \oplus_{n=0}^{\infty} H^{n,n}_{\mathcal{M}}(F). \end{array}$$

Proof. The general case is reduced easily to the case where the field F has no extensions of degree prime to p (for a certain prime integer p) and $[E : F] = p$. In this case the Milnor ring $K^M_*(E)$ is additively generated by symbols of the form $\{a_1, ..., a_{n-1}, b\}$ ($a_i \in F^*, b \in E^*$) - see [B-T]. Using Lemma 3.4.1 we easily conclude that both images of such a symbol in $H^{n,n}_{\mathcal{M}}(F)$ are equal to $\{a_1, ..., a_{n-1}, N_{E/F}(b)\}$.

Corollary 3.4.5. *The homomorphism $\lambda_F : K^M_n(F) \to H^{n,n}_{\mathcal{M}}(F)$ is surjective.*

Proof. The group $H^{n,n}_{\mathcal{M}}(F)$ is generated by classes of closed points in $\mathbb{G}_m^{\times n}$. Each closed point $x \in \mathbb{G}_m^{\times n}$ defines a finite field extension $E = F(x)$ of F and a rational point $x' = (x'_1, ..., x'_n) \in \mathbb{G}_m^{\times n}(E)$. Moreover according to the definition of the transfer maps in motivic cohomology we have : $[x] = N_{E/F}([x'])$. Lemma 3.4.4 shows now that $[x] = \lambda_F(N_{E/F}(\{x'_1, ..., x'_n\}))$.

End of the proof of the Theorem 3.4.
To finish the proof we define a homomorphism $H^{n,n}_{\mathcal{M}}(F) \to K^M_n(F)$ inverse to λ_F. Let $v \in \mathbb{G}_m^{\times n}$ be any closed point. The residue field $F(v)$ is a finite extension of F and we set $\theta(v) = N_{F(v)/F}(\{X_1(v), ..., X_n(v)\}) \in K^M_n(F)$. The usual argument involving the Weil reciprocity formula (see [N-S]) shows that the homomorphism $\theta : \mathbb{Z}_{tr}(\mathbb{G}_m^{\times n})(F) \to K^M_n(F)$ killes the image of $\partial_0 - \partial_1$ and thus defines a homomorphism $\theta : H^{n,n}_{\mathcal{M}}(F) \to K^M_n(F)$ inverse to λ_F.

§ 4. Fundamental distinguished triangles in the category $DM^-(F)$.

In this section we construct (following [V 2]) several fundamental distinguished triangles in the category $DM^-(F)$, which lead to various exact sequences of motivic cohomology. We also identify motives of affine and projective vector bundles and of blow ups (with smooth center).

Lemma 4.1. *Let $X = U \cup V$ be a (Zariski) open covering of a scheme $X \in Sm/F$. Then we have a natural distinguished triangle*

$$M(U \cap V) \to M(U) \oplus M(V) \to M(X) \to M(U \cap V)[1]$$

in the category $DM^-(F)$.

Proof. One checks easily that the following sequence of Nisnevich sheaves is exact (cf. Lemma 1.6)

$$0 \to \mathbb{Z}_{tr}(U \cap V) \to \mathbb{Z}_{tr}(U) \oplus \mathbb{Z}_{tr}(V) \to \mathbb{Z}_{tr}(X) \to 0.$$

This gives us a distinguished triangle

$$\mathbb{Z}_{tr}(U \cap V) \to \mathbb{Z}_{tr}(U) \oplus \mathbb{Z}_{tr}(V) \to \mathbb{Z}_{tr}(X) \to \mathbb{Z}_{tr}(U \cap V)[1]$$

in the category $D^-(NSwT/F)$. Applying to this triangle the functor C^* we get the triangle in question.

The proof of the following Proposition gives a typical example of the use of the above distinguished triangles.

Proposition 4.2. *Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module and let $p : Y = SpecS^*(\mathcal{E}^\#) \to X$ be the corresponding affine vector bundle. Then $M(p) : M(Y) \xrightarrow{\sim} M(X)$ is a quasiisomorphism.*

Proof. Let $X = U \cup V$ be an open covering of X. The morphism p induces a map from the distinguished triangle, corresponding to the covering $Y = p^{-1}(U) \cup p^{-1}(V)$ to the distinguished triangle, corresponding to the covering $X = U \cup V$. This shows immediately that if $M(p) : M(p^{-1}(U)) \to M(U)$, $M(p) : M(p^{-1}(V)) \to M(V)$ and $M(p) : M(p^{-1}(U \cap V)) \to M(U \cap V)$ are quasiisomorphisms, then $M(p) : M(Y) \to M(X)$ is a quasiisomorphism as well. This remark allows to reduce the general case of our statement to the situation when the locally free sheef $\mathcal{E}$ is trivial. In the latter case $Y = X \times \mathbb{A}^n$. Proceeding by induction on n it suffices to consider the case $n = 1$. Finally the natural morphism $M(i_0) : M(X) \to M(X \times \mathbb{A}^1)$ is a quasiisomorphism according to Proposition 1.14 and hence $M(p) : M(X \times \mathbb{A}^1) \to M(X)$ is also a quasiisomorphism (since $pi_0 = 1_X$).

As a next application of the distinguished triangle of Lemma 4.1 we compute the motives of the affine space without the origin and of the projective space.

Proposition 4.3. *There exists a canonical direct sum decomposition*

$$M(\mathbb{A}^n \setminus 0) = \mathbb{Z}(0) \oplus \mathbb{Z}(n)[2n-1].$$

Proof. We proceed by induction on n. The case $n = 1$ is covered by Lemma 3.3. In the general case we cover the scheme $X = \mathbb{A}^n \setminus 0$ by two open pieces $U = \{x \in \mathbb{A}^n : x_1 \neq 0\}$, $V = \{x \in \mathbb{A}^n : x_2 \neq 0$ or $x_3 \neq 0, \ldots,$ or $x_n \neq 0\}$. Obviously $V = \mathbb{A}^1 \times (\mathbb{A}^{n-1} \setminus 0)$, $U \cap V = \mathbb{G}_m \times (\mathbb{A}^{n-1} \setminus 0)$. Induction hypothesis and Proposition 4.2 show that $M(U) = \mathbb{Z}(0) \oplus \mathbb{Z}(1)[1]$, $M(V) = \mathbb{Z}(0) \oplus \mathbb{Z}(n-1)[2n-3]$, $M(U \cap V) = (\mathbb{Z}(0) \oplus \mathbb{Z}(1)[1]) \otimes (\mathbb{Z}(0) \oplus \mathbb{Z}(n-1)[2n-3]) = \mathbb{Z}(0) \oplus$

$\oplus\mathbb{Z}(n-1)[2n-3]\oplus\mathbb{Z}(1)[1]\oplus\mathbb{Z}(n)[2n-2]$. Furthermore it's clear from the construction that the homomorphism

$$M(U\cap V)=\mathbb{Z}(0)\oplus\mathbb{Z}(1)[1]\oplus\mathbb{Z}(n-1)[2n-3]\oplus\mathbb{Z}(n)[2n-2]\to$$
$$\to M(U)\oplus M(V)=\mathbb{Z}(0)\oplus\mathbb{Z}(1)[1]\oplus\mathbb{Z}(0)\oplus\mathbb{Z}(n-1)[2n-3]$$

sends the second and the third term on the left isomorphically onto the same summand on the right and sends $\mathbb{Z}(0)$ diagonally to $\mathbb{Z}(0)\oplus\mathbb{Z}(0)$. Thus the distinguished triangle, corresponding to the covering $X=U\cup V$ simplifies and takes the form

$$\mathbb{Z}(n)[2n-2]\to\mathbb{Z}(0)\to M(X)\to\mathbb{Z}(n)[2n-1].$$

Now it suffices to note that the morphism $M(SpecF)=\mathbb{Z}(0)\to M(X)$ is induced by the rational point $(1,...,0)\in X$ and is split by the obvious morphism $M(X)\to M(SpecF)=\mathbb{Z}(0)$ induced by the projection $X\to SpecF$.

The canonical line bundle $\mathcal{O}(1)$ defines an element $\tau\in Pic(\mathbb{P}^n)=H^2_{\mathcal{M}}(\mathbb{P}^n,\mathbb{Z}(1))=\mathrm{Hom}_{DM^-(F)}(M(\mathbb{P}^n),\mathbb{Z}(1)[2])$. Raising τ to powers $0,1,...,n$ we get the corresponding morphisms $\tau^k: M(\mathbb{P}^n)\to\mathbb{Z}(k)[2k]$.

Proposition 4.4. *The canonical morphism*

$$(1,\tau,...,\tau^n): M(\mathbb{P}^n)\to\mathbb{Z}(0)\oplus\mathbb{Z}(1)[2]\oplus...\oplus\mathbb{Z}(n)[2n]$$

is an isomorphism.

Proof. Consider an open covering of $\mathbb{P}^n$ by two open pieces $X=\mathbb{P}^n=U\cup V$, where $U=\mathbb{P}^n\setminus[0:...:0:1], V=(\mathbb{P}^n)_{T_n}$. It's well known that the projection with the center $[0:...:0:1]$ makes U into an affine line bundle over $\mathbb{P}^{n-1}$. Furthermore $V=\mathbb{A}^n, U\cap V=\mathbb{A}^n\setminus 0$. Using Propositions 4.2 and 4.3 we conclude that the distinguished triangle, corresponding to the above covering looks as follows:

$$\mathbb{Z}(0)\oplus\mathbb{Z}(n)[2n-1]\to\mathbb{Z}(0)\oplus M(\mathbb{P}^{n-1})\to M(\mathbb{P}^n)\to\mathbb{Z}(0)[1]\oplus\mathbb{Z}(n)[2n]$$

Neglecting the trivial summand $\mathbb{Z}(0)$, which maps diagonally into $\mathbb{Z}(0)\oplus M(\mathbb{P}^{n-1})$ and shifting to the right, this triangle is simplified to

$$M(\mathbb{P}^{n-1})\to M(\mathbb{P}^n)\to\mathbb{Z}(n)[2n]\to M(\mathbb{P}^{n-1})[1]$$

Induction hypothesis implies that the canonical morphism $M(\mathbb{P}^{n-1})\to M(\mathbb{P}^n)$ is split by the morphism $(1,\tau,...,\tau^{n-1})$. Thus we get a direct sum decomposition $M(\mathbb{P}^n)=M(\mathbb{P}^{n-1})\oplus\mathbb{Z}(n)[2n]$. This basically ends up the proof, the only thing still missing is the verification of the fact that the above morphism $M(\mathbb{P}^n)\to\mathbb{Z}(n)[2n]$ coincides (at least up to a sign) with τ^n. This is not hard to do, we refer the reader to [V 0] for technical details.

Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of constant rank $n+1$. Denote by $P=P(\mathcal{E})=Proj\ S^*(\mathcal{E}^\#)$ the corresponding projective bundle and by $p: P\to X$ the corresponding projection. The same as above the canonical line bundle $\mathcal{O}_P(1)$ defines a morphism $\tau: P\to\mathbb{Z}(1)[2]$.

Theorem 4.5. *The natural morphism*

$$M(P) \xrightarrow{M(\Delta)} M(P \times P) = M(P) \otimes M(P) \xrightarrow{M(p)\otimes(1,\tau,\dots,\tau^n)} M(X) \otimes (\mathbb{Z}(0)\oplus$$
$$\oplus\, \mathbb{Z}(1)[2] \oplus \dots \oplus \mathbb{Z}(n)[2n]) = M(X) \oplus M(X)(1)[2] \oplus \dots \oplus M(X)(n)[2n]$$

is an isomorphism.

Proof. Proceeding in the same way as in the proof of the Proposition 4.2 we easily reduce the general case to the case when the sheaf $\mathcal{E}$ is trivial. In the latter case $M(P(\mathcal{E})) = M(X) \otimes M(\mathbb{P}^n)$ and our statement follows from Proposition 4.4.

The following important theorem of Voevodsky is the basis for the construction of several distinguished triangles in the category $DM^-(F)$.

Theorem 4.6 [V 1]**.** *Assume that resolution of singularities holds over F. Let $\mathcal{F}$ be a presheaf with transfers such that for any connected scheme $X \in Sm/F$ and any section $s \in \mathcal{F}(X)$ there exists a proper birational morphism $f : Y \to X$ (with $Y \in Sm/F$) such that $f^*(s) = 0 \in \mathcal{F}(Y)$. Then the complex $C^*(\mathcal{F}^{\sim}_{Nis})$ is acyclic.*

Theorem 4.7. *Let $Z \subset X$ be a smooth subscheme in a smooth scheme X. Denote by $p : B_Z(X) \to X$ the blow up of X with center Z. Then we have a natural distinguished triangle*

$$M(p^{-1}(Z)) \to M(Z) \oplus M(B_Z(X)) \to M(X) \to M(p^{-1}(Z))[1].$$

Proof. It's easy to check that the sequence of Nisnevich sheaves with transfers

$$0 \to \mathbb{Z}_{tr}(p^{-1}(Z)) \to \mathbb{Z}_{tr}(Z) \oplus \mathbb{Z}_{tr}(B_Z(X)) \to \mathbb{Z}_{tr}(X)$$

is exact. Denote by $\mathcal{C}$ the cokernel of the last homomorphism above. The sheaf $\mathcal{C}$ is associated to the presheaf

$$U \mapsto \frac{\mathbb{Z}_{tr}(X)(U)}{Im(\mathbb{Z}_{tr}(Z)(U) \oplus \mathbb{Z}_{tr}(B_Z(X)(U))}.$$

Our statement would follow from the above exact sequence and Theorem 4.6 as soon as we show that the above presheaf satisfies the conditions of the Theorem 4.6. We start with a section $s \in \mathbb{Z}_{tr}(X)(U)$ (with $U \in Sm/F$ - smooth connected scheme) and want to show that it either comes from $\mathbb{Z}_{tr}(Z)(U)$ or otherwise can be lifted to $\mathbb{Z}_{tr}(B_Z(X))(V)$ after an appropriate base change $V \to U$ (with $V \to U$ proper and birational). We may clearly assume that s is represented by a closed integral subscheme $S \subset U \times X$, finite and surjective over U. The case when S is contained in $U \times Z$ is trivial. Thus we may assume that $S \not\subset U \times Z$. Denote by T the closure of $(1_U \times p)^{-1}(S \cap (U \times (X \setminus Z))$ in $U \times B_Z(X)$. The scheme T is clearly proper and surjective over U, but need not be finite over U. According to the Platification Theorem - see [R-G] there exists a blow up (not necessarily with smooth center) $U' \to U$ such that the proper inverse image T' of T in $U' \times B_Z(X)$

is flat over U'. Since $T' \to U'$ is proper and finite over the generic point of U' we conclude that $T' \to U'$ is finite. Finally there exists (according to the Definition 4.6) a proper birational morphism $V \to U'$ with V smooth over F. Thus we get a proper birational morphism $V \to U$ with V smooth, such that the proper inverse image T'' of T in $V \times B_Z(X)$ is finite over V. Obviously this T'' gives us a lifting to $\mathbb{Z}_{tr}(B_Z(X))(V)$ of the inverse image of S to $\mathbb{Z}_{tr}(X)(V)$.

Theorem 4.8. *In conditions and notations of Theorem 4.7 assume further that Z is everywhere of codimension c. Then we have a canonical direct sum decomposition*

$$M(B_Z(X)) = M(X) \oplus \bigoplus_{i=1}^{c-1} M(Z)(i)[2i].$$

Proof. Denote by $\mathcal{E}$ the conormal sheaf of Z in X. It's well known that the sheaf $\mathcal{E}$ is a locally free $\mathcal{O}_Z$-module of rank c and $p^{-1}(Z)$ may be identified with a projective bundle $P(\mathcal{E})$. Using Theorem 4.5 we see that the distinguished triangle of Theorem 4.7 may be rewritten in the following more simple form:

$$\bigoplus_{i=1}^{c-1} M(Z)(i)[2i] \to M(B_Z(X)) \to M(X) \to \bigoplus_{i=1}^{c-1} M(Z)(i)[2i+1]$$

It suffices now to show that the morphism $M(B_Z(X)) \xrightarrow{M(p)} M(X)$ admits a canonical section $\alpha_Z^X : M(X) \to M(B_Z(X))$. To do so we compare the above distinguished triangle to a similar distinguished triangle corresponding to the blow up $q : B_{Z\times\{0\}}(X \times \mathbb{A}^1) \to X \times \mathbb{A}^1$ of $Z \times \{0\} \subset X \times \mathbb{A}^1$. It's easy to see that the morphism $i_0 : X \to X \times \mathbb{A}^1$ induces a morphism (which we still denote i_0) $B_Z(X) \xrightarrow{i_0} B_{Z\times\{0\}}(X \times \mathbb{A}^1)$ (which takes $p^{-1}(Z)$ to $q^{-1}(Z \times \{0\})$). Moreover it's clear from the proof of the Theorem 4.7 that we get an induced homomorphism from the distinguished triangle corresponding to the blow up p to the one corresponding to the blow up q. Note further that $q^{-1}(Z \times \{0\}) = P(\mathcal{E} \oplus \mathcal{O}_Z)$ and the embedding $p^{-1}(Z) \xrightarrow{i_0} q^{-1}(Z \times \{0\})$ is the obvious one (corresponding to the embedding $\mathcal{E} \to \mathcal{E} \oplus \mathcal{O}_Z$). This implies immediately that the restriction of the canonical bundle $\mathcal{O}_{P(\mathcal{E}\oplus\mathcal{O}_Z)}(1)$ to $P(\mathcal{E})$ coincides with $\mathcal{O}_{P(\mathcal{E})}(1)$ and hence the following diagram commutes (for all $k \geq 0$)

$$\begin{array}{ccc} M(p^{-1}(Z)) = M(P(\mathcal{E})) & \xrightarrow{\tau^k} & \mathbb{Z}(k)[2k] \\ \downarrow{\scriptstyle M(i_0)} & & \downarrow{\scriptstyle =} \\ M(q^{-1}(Z \times \{0\})) = M(P(\mathcal{E} \oplus \mathcal{O}_Z)) & \xrightarrow{\tau^k} & \mathbb{Z}(k)[2k] \end{array}$$

Thus under the identification of the Theorem 4.5 the canonical morphism $M(i_0) : M(p^{-1}(Z) \to M(q^{-1}(Z \times \{0\})$ becomes an obvious embedding

$\oplus_{i=0}^{c-1} M(Z)(i)[2i] \xrightarrow{i_{c/c-1}} \oplus_{i=0}^{c} M(Z)(i)[2i]$. Denote by $pr_{c/c-1}$ the obvious projection of the direct sum on the right onto the direct sum on the left. Consider next the following commutative diagram with distinguished rows

$$\begin{array}{ccccccc} \bigoplus_{i=1}^{c-1} Z(i)[2i] & \longrightarrow & M(B_Z(X)) & \xrightarrow{M(p)} & M(X) & \longrightarrow & \dots \\ {\scriptstyle i_{c/c-1}}\downarrow & & {\scriptstyle M(i_0)}\downarrow & & {\scriptstyle M(i_0)}\downarrow & & \\ \bigoplus_{i=1}^{c} Z(i)[2i] & \longrightarrow & M(B_{Z\times\{0\}}(X\times\mathbb{A}^1)) & \xrightarrow{M(q)} & M(X\times\mathbb{A}^1) & \longrightarrow & \dots \end{array}$$

The morphism $M(i_0) : M(X) \to M(X \times \mathbb{A}^1)$ is an isomorphism with inverse $M(pr_1) : M(X \times \mathbb{A}^1) \to M(X)$ and hence coincides with $M(i_1)$. The morphism $i_1 : X \to X \times \mathbb{A}^1$ admits a canonical lifting to $B_{Z\times\{0\}}(X \times \mathbb{A}^1)$, which we still denote by i_1. Thus $M(i_1 \cdot pr_1)$ yields a section for $M(q)$ and hence the bottom triangle splits providing an isomorphism $M(B_{Z\times\{0\}}(X \times \mathbb{A}^1)) = M(X \times \mathbb{A}^1) \oplus \bigoplus_{i=1}^{c} M(Z)(i)[2i]$. Denote by $\tilde{\beta}$ the corresponding projection of $M(B_{Z\times\{0\}}(X \times \mathbb{A}^1))$ onto $\oplus_{i=1}^{c} M(Z)(i)[2i]$. We finally note that the homomorphism $\beta_Z^X = pr_{c/c-1} \cdot \tilde{\beta} \cdot M(i_0) : M(B_Z(X)) \to \oplus_{i=1}^{c-1} Z(i)[2i]$ is a left inverse for the canonical morphism $\oplus_{i=1}^{c-1} Z(i)[2i] \to M(B_Z(X))$ appearing in the top triangle. Thus the top triangle also splits canonically , which provides us with the section $\alpha_Z^X : M(X) \to M(B_Z(X))$ for $M(p)$.

To define the Gizin morphisms we need a slight generalization of Theorem 4.8. Let $U \hookrightarrow X$ be a (smooth) subscheme of $X \in Sm/F$ (not necessarily closed). We define the object $M(X/U) \in DM^-(F)$ as the cone of the homomorphism $M(U) \to M(X)$. Thus we always have a canonical distinguished triangle $M(U) \to M(X) \to M(X/U) \to M(U)[1]$. Whenever $f : X \to X'$ is a morphism of schemes, taking a subscheme $U \subset X$ to a subscheme $U' \subset X'$ we get an evident morphism $M(f) : M(X/U) \to M(X'/U')$. Let $Z \subset X$ be a smooth closed subscheme, and let $U \subset X$ be a smooth subscheme not intersecting Z. In this case U lifts canonically to a subscheme of $B_Z(X)$. The arguments used in the proof of the Theorem 4.7 apply without changes to this relative situation and provide us with a distinguished triangle

$$M(p^{-1}(Z)) \to M(B_Z(X)/U) \to M(X/U) \to M(p^{-1}(Z))[1].$$

Moreover the proof of the Theorem 4.8 also goes through nearly without changes. This time one has to compare the distinguished triangle in question to the distinguished triangle corresponding to q with motives taken relative to the subscheme $U \times \mathbb{A}^1$. We'll keep the notations introduced in the proof of Theorem 4.8 in this relative situation as well. Thus we get in particular canonical morphisms

$$M(X/U) \xrightarrow{\alpha_Z^X} M(B_Z(X)/U), \quad M(B_{Z\times\{0\}}(X \times \mathbb{A}^1)/U \times \mathbb{A}^1) \xrightarrow{\tilde{\beta}} \bigoplus_{i=1}^{c} M(Z)(i)[2i].$$

The absolute and relative morphisms thus obtained are compatible one with another, so that the following diagram (and a similar diagram involving $\widetilde{\beta}$) commutes:

$$\begin{array}{ccc} M(X) & \xrightarrow{\alpha_Z^X} & M(B_Z(X)) \\ \downarrow & & \downarrow \\ M(X/U) & \xrightarrow{\alpha_Z^X} & M(B_Z(X)/U). \end{array}$$

Let $Z \subset X$ be a closed subscheme of a scheme $X \in Sm/F$. Denoting the open subscheme $X \setminus Z$ by U we define the motive of X with supports in Z $M_Z(X)$ as the relative motive $M(X/U)$. Assume that Z is smooth everywhere of codimension c in X. In this case we can apply the previous discussion and get a canonical morphism

$$M_Z(X) \xrightarrow{\alpha_Z^X} M(B_Z(X)/U) \xrightarrow{M(i_0)} M(B_{Z\times\{0\}}(X\times\mathbb{A}^1)/U\times\mathbb{A}^1) \xrightarrow{\widetilde{\beta}} \bigoplus_{i=1}^{c} M(Z)(i)[2i]$$

It's easy to see that the composition of this morphism with the projection $pr_{c/c-1}$ onto the direct sum of the first $c-1$ summands is trivial. Thus the above morphism may be considered as a morphism from $M_Z(X)$ to $M(Z)(c)[2c]$, which is called the Gizin morphism and is denoted $G_Z^X : M_Z(X) \to M(Z)(c)[2c]$). The following naturality properties of the Gizin morphism are straightforward from the definition.

Lemma 4.9.

(1) *Let $f : X' \to X$ be a smooth morphism. Set $Z' = Z \times_X X'$, so that Z' is a smooth closed subscheme of X' everywhere of codimension c. Then the following diaggram commutes*

$$\begin{array}{ccc} M_{Z'}(X') & \xrightarrow{M(f)} & M_Z(X) \\ {\scriptstyle G_{Z'}^{X'}}\downarrow & & {\scriptstyle G_Z^X}\downarrow \\ M(Z')(c)[2c] & \xrightarrow{M(f|_{Z'})(c)[2c]} & M(Z)(c)[2c] \end{array}$$

(2) *Let $Y \in Sm/F$ be a smooth scheme. Then the following diagram commutes*

$$\begin{array}{ccc} M_{Z\times Y}(X\times Y) & \xrightarrow{G_{Z\times Y}^{X\times Y}} & M(Z\times Y)(c)[2c] \\ =\downarrow & & =\downarrow \\ M_Z(X)\otimes M(Y) & \xrightarrow{G_Z^X\otimes 1_{M(Y)}} & M(Z)(c)[2c]\otimes M(Y) \end{array}$$

(3) *Let $X = U \cup V$ be a Zariski open covering of a (smooth) scheme X. In this case we have a distinguished triangle*

$$M_{Z\cap(U\cap V)}(U\cap V) \to M_{Z\cap U}(U) \oplus M_{Z\cap V}(V) \to M_Z(X) \to M_{Z\cap(U\cap V)}(U\cap V)[1]$$

Moreover the Gizin morphisms

$$\begin{aligned} G^{U\cap V}_{Z\cap(U\cap V)} &: M_{Z\cap(U\cap V)}(U\cap V) \to M(Z\cap(U\cap V))(c)[2c], \\ G^{U}_{Z\cap U} &: M_{Z\cap U}(U) \to M(Z\cap U)(c)[2c], \\ G^{V}_{Z\cap V} &: M_{Z\cap V}(V) \to M(Z\cap V)(c)[2c], \\ G^{X}_{Z} &: M_{Z}(X) \to M(Z)(c)[2c] \end{aligned}$$

define a homomorphism from the above distinguished triangle to the distinguished triangle corresponding to the Zariski open covering $Z = (Z\cap U)\cup(Z\cap V)$ (twisted and shifted several times).

Theorem 4.10 (Cohomological purity in motivic cohomology)[V 2]. *Let $Z\subset X$ be a smooth closed subscheme everywhere of codimension c. Then the Gizin morphism $G^X_Z : M_Z(X)\to M(Z)(c)[2c]$ is an isomorphism.*

Proof. Lemma 4.9 (3) shows that the question is local with respect to the Zariski topology on X. In perticular we may assume that there exists an etale morphism $f : X \to \mathbb{A}^d$ such that $Z = f^{-1}(\mathbb{A}^{d-c}\times\{0\})$. Consider the cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{pr_2} & Z\times\mathbb{A}^c \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle f|_Z\times 1_{\mathbb{A}^c}} \\ X & \xrightarrow{f} & \mathbb{A}^d \end{array}$$

Since $f|_Z : Z \to \mathbb{A}^{d-c}$ is an etale morphism we conclude that the diagonal embedding $\Delta_Z : Z \to Z\times_{\mathbb{A}^{d-c}} Z$ is both open and closed. Thus we may find an open subscheme $X'\subset Y$ such that $X'\cap Z\times_{\mathbb{A}^{d-c}} Z = \Delta(Z)$. Denote by p_1, p_2 restrictions of projections pr_1, pr_2 to an open subscheme $X'\subset Y$. It's clear that the morphisms p_1, p_2 are etale. Moreover, identifying Z with its image in X' under Δ_Z we see easily, that $p_1^{-1}(Z) = Z$, $p_2^{-1}(Z\times\{0\}) = Z$ and the induced morphisms $(p_1)|_Z : Z\to Z, (p_2)|_Z : Z \to Z\times\{0\}$ are both equal to identity. Using Lemma 4.9 we get two commutative diagrams

$$\begin{array}{ccc} M_Z(X') & \xrightarrow{G^{X'}_Z} & M(Z)(c)[2c] \\ {\scriptstyle M(p_1)}\downarrow & & \downarrow{\scriptstyle =} \\ M_Z(X) & \xrightarrow{G^X_Z} & M(Z)(c)[2c] \end{array} \qquad \begin{array}{ccc} M_Z(X') & \xrightarrow{G^{X'}_Z} & M(Z)(c)[2c] \\ {\scriptstyle M(p_2)}\downarrow & & \downarrow{\scriptstyle =} \\ M_{Z\times\{0\}}(Z\times\mathbb{A}^c) & \xrightarrow{G^{Z\times\mathbb{A}^c}_{Z\times\{0\}}} & M(Z)(c)[2c] \end{array}$$

Lemma 4.11 below shows that the vertical arrows in the above diagrams are isomorphisms. Thus to show that G^X_Z is an isomorphism it suffices to show that $G^{Z\times\mathbb{A}^c}_{Z\times\{0\}} = 1_{M(Z)}\times G^{\mathbb{A}^c}_{\{0\}}$ is an isomorphism. Thus we are reduced to showing that the Gizin morphism $G^{\mathbb{A}^c}_{\{0\}} : M_{\{0\}}(\mathbb{A}^c) \to \mathbb{Z}(c)[2c]$ is an isomorphism. Proposition 4.3 implies easily that $M_{\{0\}}(\mathbb{A}^c) \cong \mathbb{Z}(c)[2c]$. Thus we only have to verify, that the isomorphism given by Proposition 4.3 coincides with the Gizin morphism. We omit the technical details of this verification - see [V 0].

Lemma 4.11 (Etale excision - [V 1]). *Let $f : X' \to X$ be an etale morphism of smooth schemes. Let further $Z \subset X$ be a smooth closed subscheme such that $Z' = f^{-1}(Z) \xrightarrow{f|_{Z'}} Z$ is an isomorphism. Then the induced morphism $M(f) : M_{Z'}(X') \to M_Z(X)$ is an izomorphism.*

Proof. The motiv $M_Z(X)$ (resp $M_{Z'}(X')$) may be identified with $C^*(\mathbb{Z}_{tr}(X)/\mathbb{Z}_{tr}(X \setminus Z))$ (resp. with $C^*(\mathbb{Z}_{tr}(X')/\mathbb{Z}_{tr}(X' \setminus Z'))$. Thus it suffices to show that the induced homomorphism of the quotient Nisnevich sheaves

$$f_* : \mathbb{Z}_{tr}(X')/\mathbb{Z}_{tr}(X' \setminus Z') \to \mathbb{Z}_{tr}(X)/\mathbb{Z}_{tr}(X \setminus Z)$$

is an isomorphism. To do so we have to compare the sections of these sheaves over a smooth henselian scheme S. Note that $(\mathbb{Z}_{tr}(X)/\mathbb{Z}_{tr}(X \setminus Z))(S)$ is a free abelian group with basis consisting of closed integral subschemes $T \subset X \times S$ finite and surjective over S, which are not contained in $(X \setminus Z) \times S$. Any such scheme T is again henselian and its (unique) closed point t_0 lies in $Z \times S$. Denote by T' the inverse image of T under the morphism $X' \times S \xrightarrow{f \times 1_S} X \times S$. The scheme T' is etale over the henselian scheme T and hence (see [M]) splits into a disjoint sum $T' = \widetilde{T}' \coprod T'_0 \coprod ... \coprod T'_n$, where the image of $\widetilde{T}'$ in T does not contain t_0 and the schemes T'_i are finite and etale over T and henselian. Since $t_0 \in Z \times S$ we conclude easily that the inverse image of t_0 in $X' \times S$ consists of only one point $t'_0 \in Z' \times S$ and moreover $F(t'_0) = F(t_0)$. This shows that $n = 0$ and the induced morphism $T'_0 \to T$ is an isomorphism. We define a map

$$\phi : (\mathbb{Z}_{tr}(X)/\mathbb{Z}_{tr}(X \setminus Z))(S) \to (\mathbb{Z}_{tr}(X')/\mathbb{Z}_{tr}(X' \setminus Z'))(S)$$

setting $\phi([T]) = [T'_0]$. The above discussion shows that $\phi f_* = Id, f_* \phi = Id$.

We refer the reader to [F-V] for the proof of the following important Theorem.

Theorem 4.12 (Quasiinvertibility of the Tate object). *Assume that resolution of singularities holds over F. Then for any complexes $A^\bullet, B^\bullet \in DM^-(F)$ tensoring with the identity endomorphism of $\mathbb{Z}(1)$ defines an isomorphism*

$$\mathrm{Hom}_{DM^-(F)}(A^\bullet, B^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{DM^-(F)}(A^\bullet(1), B^\bullet(1)).$$

Corollary 4.12.1. *Assume that resolution of singularities holds over F. Then for any scheme $X \in Sm/F$ multiplcation by $\tau \in H^2_{\mathcal{M}}(\mathbb{P}^1, \mathbb{Z}(1)) = H^2_{\mathcal{M}}(\mathbb{P}^1/SpecF, \mathbb{Z}(1))$ defines isomorphisms*

$$H^i_{\mathcal{M}}(X, \mathbb{Z}(n-1)) \xrightarrow{\sim} H^{i+2}_{\mathcal{M}}(X \times \mathbb{P}^1/X, \mathbb{Z}(n))$$
$$H^i_{\mathcal{M}}(X, \mathbb{Z}/l(n-1)) \xrightarrow{\sim} H^{i+2}_{\mathcal{M}}(X \times \mathbb{P}^1/X, \mathbb{Z}/l(n)).$$

Proof. This follows immediately from Theorem 4.12, since under identification $M(\mathbb{P}^1/SpecF) = \mathbb{Z}(1)[2]$ the element τ corresponds to the identity endomorphism of $\mathbb{Z}(1)$.

§ 5. MOTIVIC COHOMOLOGY OF NON SMOOTH SCHEMES AND THE CDH-TOPOLOGY.

Recall that to each scheme $X \in Sch/F$ we have associated in § 1 a Nisnevich sheaf with transfers $\mathbb{Z}_{tr}(X)$. This allows us to define a motiv $M(X)$ of an arbitrary scheme X as an object $C^*(\mathbb{Z}_{tr}(X))$ of the category $DM^-(F)$ and to define motivic cohomology of X with coefficients in an arbitrary complex $C^\bullet \in DM^-(F)$ via the formula

$$H^i_{\mathcal{M}}(X, C^\bullet) = \mathrm{Hom}_{DM^-(F)}(M(X), C^\bullet[i]).$$

Note that according to the Theorem 1.5 for smooth schemes the new definition coincides with the original one. Many of the properties of motivic cohomology proved in § 4 for smooth schemes ,in particular the Mayer-Vietoris distinguished triangle for open coverings, hold actually for arbitrary schemes. Moreover we also have the Mayer-Vietoris distinguished triangle for the closed coverings.

Lemma 5.1. *Let $X = U \cup V$ be a closed covering of a scheme $X \in Sch/F$. In this case we have the following distinguished triangle in the category $DM^-(F)$*

$$M(U \cap V) \to M(U) \oplus M(V) \to M(X) \to M(U \cap V)[1].$$

Proof. The result follows immediately since the following sequence of Nisnevich sheaves with transfers is exact (even as a sequence of presheaves).

$$0 \to \mathbb{Z}_{tr}(U \cap V) \to \mathbb{Z}_{tr}(U) \oplus \mathbb{Z}_{tr}(V) \to \mathbb{Z}_{tr}(X) \to 0.$$

The proof of the following result is identical to that of the Theorem 4.7.

Theorem 5.2. *Assume that resolution of singularities holds over F. Let $p : X' \to X$ be a proper morphism of schemes and let $T \subset X$ be a closed subscheme such that p is an isomorphism over $X \setminus T$. Then the bicomplex*

$$M(p^{-1}(T)) \longrightarrow M(T) \oplus M(X') \longrightarrow M(X)$$

is acyclic and hence we have a natural distinguished triangle in $DM^-(F)$

$$M(p^{-1}(T)) \longrightarrow M(T) \oplus M(X') \longrightarrow M(X) \longrightarrow M(p^{-1}(T))[1].$$

If $Z \subset X$ is a closed subscheme we define the motiv of X with supports in Z the same way as it was done for smooth schemes in § 4, i.e. $M_Z(X)$ is the cone of the homomorphism of complexes $M(X \setminus Z) \xrightarrow{M(i)} M(X)$.

Corollary 5.3. *In conditions and notations of the previous Theorem let Z be a closed subscheme of X. Then the bicomplex*

$$M_{p^{-1}(Z\cap T)}(p^{-1}(T)) \longrightarrow M_{Z\cap T}(T) \oplus M_{p^{-1}(Z)}(X') \longrightarrow M_Z(X)$$

is acyclic and hence we have a natural distinguished triangle in $DM^-(F)$

$$M_{p^{-1}(Z\cap T)}(p^{-1}(T)) \longrightarrow M_{Z\cap T}(T) \oplus M_{p^{-1}(Z)}(X') \longrightarrow M_Z(X) \longrightarrow$$
$$\longrightarrow M_{p^{-1}(Z\cap T)}(p^{-1}(T))[1].$$

The following Lemma provides some elementary properties of motives with supports.

Lemma 5.4. *(1) Assume that the closed subscheme* $Z \subset X$ *is a disjoint sum of two subschemes :* $Z = Z_1 \coprod Z_2$. *Then we have a natural isomorphism of motivs with supports :* $M_Z(X) = M_{Z_1}(X) \oplus M_{Z_2}(X)$.
(2) Let $Z' \subset Z \subset X$ *be a pair of closed subschemes. In this case we have a natural distinguished triangle in the category* $DM^-(F)$

$$M_{Z \setminus Z'}(X \setminus Z') \to M_Z(X) \to M_{Z'}(X) \to M_{Z \setminus Z'}(X \setminus Z')[1].$$

Proof. (1) Set $U_i = X \setminus Z_i, U = X \setminus Z$. Then $U = U_1 \cap U_2$ and $U_1 \cup U_2 = X$. The Mayer-Vietoris property for the open coverings implies the acyclicty of the following bicomplex

$$\begin{array}{ccccc} M(U) & \to & M(U_1) \oplus M(U_2) & \to & M(X) \\ \downarrow & & \downarrow & & \downarrow{=} \\ M(X) & \to & M(X) \oplus M(X) & \to & M(X). \end{array}$$

Thus the cone of the left vertical arrow is quasiisomorphic to the cone of the middle vertical arrow.

To prove (2) it suffices to note that the cone of the homomorphism $M_{Z \setminus Z'}(X \setminus Z') \to M_Z(X)$ coincides with the total complex corresponding to the bicomplex

$$\begin{array}{ccc} M(X \setminus Z) & \xrightarrow{=} & M(X \setminus Z) \\ \downarrow & & \downarrow \\ M(X \setminus Z') & \longrightarrow & M(X) \end{array}$$

and hence is canonically quasiisomorphic to $M_{Z'}(X)$.

Definition 5.5. *We define the notion of a strictly dense open subscheme* U *in the scheme* X *by induction on* $dim X$. *For schemes of dimension* 0 *the only strictly dense open subscheme of* X *is* X *itself. In general we'll be saying that* U *is strictly dense in* X *iff there exists a proper morphism* $p : X' \to X$ *with* $X' \in Sm/F$ *and a closed subscheme* $T \subset X$ *such that*

1. p *is an isomorphism over* $X \setminus T$
2. $dim T < dim X, dim\ p^{-1}(T) < dim X$
3. $p^{-1}(U)$ *is dense in* X', $U \cap T$ *is strictly dense in* T, $p^{-1}(U \cap T)$ *is strictly in* $p^{-1}(T)$.

Obviously for smooth schemes X strictly dense open subschemes are the same as dense open subschemes, in the general case this is however a more restricting condition. For one thing it's obvious that each strictly dense open subscheme should have a non-empty intersection with the singular locus of X. The above definition essentially was devised for the sake of the following result.

Theorem 5.6. *Assume that resolution of singularities holds over F. Let $C^\bullet \in DM^-(F)$ be a complex such that $H^*_Z(X, C^\bullet) = 0$ for any smooth irreducible scheme X and any irreducible closed smooth subscheme $Z \neq X$. Then $H^*_Z(X, C^\bullet) = 0$ for any $X \in Sch/F$ and any closed subscheme $Z \subset X$ such that $X \setminus Z$ is strictly dense in X.*

Proof. We first treat the case when X is smooth. The general case is reduced immediately to the case when X is smooth and irreducible. In case Z is smooth and irreducible there's nothing to prove. The case of an arbitrary smooth Z follows now from the additivity of $M_Z(X)$ with respect to Z:

$$M_Z(X) = M_{Z_1}(X) \oplus M_{Z_2}(X)$$

in case Z is a disjoint sum of two subschemes Z_1 and Z_2. In the general case we may assume Z reduced and proceed by induction on $dim\ Z$. Since over a perfect field each integral scheme is smooth at the generic point we can find a nonempty open $U \subset X$, which contains generic points of all components of Z and such that $Z \cap U$ is smooth. Denote by Z' the closed subscheme $Z \cap (X \setminus U)$. According to our construction $dim\ Z' < dim\ Z$. On the other hand the closed subscheme $Z \setminus Z' \subset X \setminus Z' = U$ is smooth, so that it suffices to use the existence of the distinguished triangle

$$M_{Z \setminus Z'}(X \setminus Z') \to M_Z(X) \to M_{Z'}(X) \to M_{Z \setminus Z'}(X \setminus Z')[1].$$

The case of an arbitrary scheme X follows now from Corollary 5.3 in view of the Definition 5.5.

Definition 5.7. *The cdh-topology (completely decomposed h-topology) on the category Sch/F is defined as the weakest Grothendieck topology for which the following two families of morphisms are coverings*

1. *Every Nisnevich covering is also a cdh-covering.*
2. *Let $p : X' \to X$ be a proper morphism of schemes and let $T \subset X$ be a closed subscheme such that p is an isomorphism over $X \setminus T$. Then the morphism $T \coprod X' \xrightarrow{(i,p)} X$ is a cdh-covering.*

Note that covering of a scheme X by its irreducible components (considered as closed reduced subschemes of X) is a cdh-covering. Furthermore it's easy to see from the definition than whenever $X' \to X$ is a cdh-covering each point $x \in X$ admits a lifting $x' \in X'$ with the same residue field: $F(x) = F(x')$. The last property characterizes the cdh-coverings in the case of proper morphisms.

Lemma 5.8. *Let $p : X' \to X$ be a proper morphism such that each point $x \in X$ admits a lifting $x' \in X'$ for which $F(x) = F(x')$. Then p is a cdh-covering.*

Proof. Proceeding by Noethirian induction we may assume that for any proper closed subscheme $T \subset X$ the induced morphism $X' \times_X T \to T$ is a cdh-covering. Now the statement for X is obvious unless X is integral. Assume that X is integral.

According to our assumptions there exists a closed integral subscheme $X'' \subset X'$ such that the morphism $p'' = p|_{X''} : X'' \to X$ is birational. Let $T \subset X$ be a proper closed subscheme for which p'' is an isomorphism over $X \setminus T$. According to definitions the family $X'' \xrightarrow{p''} X, T \hookrightarrow X$ is a cdh-covering. Furthermore $X' \times_X X'' \to X''$ is a cdh-covering (since it admits a section) and the morphism $X' \times_X T \to T$ is a cdh-covering according to the induction hypothesis. Thus $X' \to X$ is a cdh-covering locally in the cdh-topology and hence is a cdh-covering.

We'll call the cdh-coverings of Lemma 5.8 the proper cdh-coverings. A proper cdh-covering $p : X' \to X$ of an integral scheme X is called a proper birational cdh-covering in case the morphism p is an isomorphism over an appropriate neighbourhood of the generic point of X. Note that the proof of Lemma 5.8 shows that each proper cdh-covering of an integral scheme X admits a proper birational refinement.

The following Proposition gives a clearer view on the structure of cdh-coverings.

Proposition 5.9. *Every cdh-covering of a scheme $X \in Sch/F$ has a refinement of the following form*

$$\{U_i \to X' \to X\}_{i=1}^n$$

where $X' \to X$ is a proper cdh-covering and $\{U_i \to X'\}_{i=1}^n$ is a Nisnevich covering of the scheme X'.

Proof. To abbreviate terminology we call coverings of the above type special cdh-coverings. Note that to show that a given covering of a scheme X admits a special refinement is equivallent to showing that there exists a proper cdh-covering $X' \to X$ such that the induced covering of the scheme X' admits a Nisnevich refinement. Thus one may always replace the scheme X by any of its proper cdh-coverings (and the given covering of X by the induced covering of X'). Furthermore if $X = X_1 \coprod X_2$ is a disjoint sum of two schemes, then verifying the statement for X (and a given covering of X) is equivalent to verifying it for both X_i (and the induced coverings). The above remarks imply in particular that to verify our statement for a scheme X (and a given covering of X) is equivallent to veryfing it for all irreducible components of X, considered as closed integral subschemes of X (and the corresponding induced coverings of these components).

Note further that according to definitions each cdh-covering admits a refinement of the form $\{X_n^i \to X_n \to X_{n-1} \to \dots \to X_0 = X\}_{i=1}^m$, where $\{X_n^i \to X_n\}_{i=1}^m$ is a Nisnevich covering of the scheme X_n and each stage in the tower $X_n \to \dots \to X_0 = X$ is either a Nisnevich covering or a proper cdh-covering. Proceeding by induction on the length of the tower one sees easily that it suffices to verify that coverings of the form $T \xrightarrow{p} U \xrightarrow{q} X$ (where $T \xrightarrow{p} U$ is a proper covering and $U \xrightarrow{q} X$ is a Nisnevich covering) admit a special refinement.

Proceeding by the Noetherian induction we may assume that for any proper closed subscheme $Z \subset X$ the induced covering of Z admits a special refinement and the scheme X is integral. In this case the scheme U is a disjoint sum of its connected components $U = \coprod U_i$ and each U_i is integral. Denote the inverse image of U_i in T by T_i. Replacing $T \to U$ by a finer covering we may assume further that each $T_i \to U_i$ is a proper birational covering. Applying the Platification Theorem (see [R-G]) to the morphism $T \to X$ we find a proper closed subscheme $Z \subset X$ such that for each i the proper inverse image T_i' of T_i under the morphism $X' = B_Z(X) \to X$ is flat over X'. Thus for each i we get a commutative diagram

$$\begin{array}{ccc} T_i' & & \\ \downarrow & & \\ T_i \times_X X' & \longrightarrow & T_i \\ \downarrow & & \downarrow \\ U_i \times_X X' & \longrightarrow & U_i \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

Note that the scheme X' is integral and birationally isomorphic to X, from which one concludes easily that the scheme $U_i \times_X X'$ is integral as well. The morphism $T_i' \to U_i \times_X X'$ becomes flat being composed with the etale morphism $U_i \times_X X' \to X'$ and hence is flat itself. Moreover the scheme T_i' is integral and the morphism $T_i' \to U_i \times_X X'$ is proper and birational. Since every proper flat birational morphism of integral schemes is an isomorphism we conclude that $T_i \times_X X' \to U_i \times_X X'$ admits a section. Thus the induced covering of the scheme X' admits a Nisnevich refinement $U \times_X X' \to X'$. Finally according to our induction hypothesis there exists a proper covering $Z' \to Z$ such that the induced covering of Z' admits a Nisnevich refinement. The morphism $X' \coprod Z' \to X$ is a proper covering of X and the induced covering of the scheme $X' \coprod Z'$ admits a Nisnevich refinement, which ends up the proof.

¿From this point till the end of the § we assume that resolution of singularities holds over F. This assumption implies in particular that for smooth schemes Lemma 5.8 admits the following Improvement.

Lemma 5.10. *Let X be a smooth integral scheme. Then every proper birational morphism $X' \xrightarrow{p} X$ is a cdh-covering.*

Proof. According to definitions p admits a refinement which is a composition of blow up's with smooth centers. So it suffices to note that a blow up with a smooth center satisfies the conditions of the Lemma 5.8.

Consider an obvious morphism of sites $(Sch/F)_{cdh} \xrightarrow{\theta} (Sm/F)_{Nis}$ and the induced functors on the categories of abelian sheaves

$$(Sch/F)^{\sim}_{cdh} \underset{\theta^*}{\overset{\theta_*}{\rightleftarrows}} (Sm/F)^{\sim}_{Nis}$$

Proposition 5.11. *The functor $\theta^* : (Sm/F)^{\sim}_{Nis} \to (Sch/F)^{\sim}_{cdh}$ is exact.*

Proof. To prove the result we define a cdh-topology on the category Sm/F. This topology is not defined in terms of covering families (since there is not enough fiber products in the category Sm/F), but in terms of covering sieves. More precisely, every family of morphisms $\{X_i \to X\}_{i\in I}$ in Sch/F defines a sieve on Sm/F, consisting of those arrows $Y \to X$ in the category Sm/F, which may be factored through one of $X_i \to X$. The covering sieves of $X \in Sm/F$ for the cdh-topology on Sm/F are those which contain a sieve defined by an appropriate cdh-covering of X in Sch/F. One checks easily (using resolution of singularities) that restricting a cdh-sheaf from the category Sch/F to Sm/F we get again a cdh-sheaf, i.e. the inclusion functor $Sm/F \hookrightarrow Sch/F$ is continious (in the terminology of [SGA 4]). All in all we get a morphism of sites $\phi : (Sch/F)_{cdh} \to (Sm/F)_{cdh}$ and a pair of induced functors on the categories of sheaves

$$(Sch/F)^{\sim}_{cdh} \underset{\phi^*}{\overset{\phi_*}{\rightleftarrows}} (Sm/F)^{\sim}_{cdh}$$

Here ϕ_* is the restriction functor and the functor ϕ^* is left adjoint to ϕ_*. The functor ϕ_* has (in the present circumstances) also a right adjoint, which we denote by $e : (Sm/F)^{\sim}_{cdh} \to (Sch/F)^{\sim}_{cdh}$. In fact, the functor ϕ_* on the category of presheaves $\phi_* : (Sch/F)^{\wedge} \to (Sm/F)^{\wedge}$ has a right adjoint $e : (Sm/F)^{\wedge} \to (Sch/F)^{\wedge}$ given by the formula $e(\mathcal{F})(Y) = \mathrm{Hom}_{Funct(Sm/F,Sets)}(h_Y, \mathcal{F})$, where h_Y is the functor on Sm/F represented by the scheme $Y \in Sch/F$. One checks easily (once again using resolution of singularities) that the functor e actually takes cdh-sheaves to cdh-sheaves and moreover ϕ_* and e are mutually inverse equivalences of categories (cf. [S-V § 6]). This implies immediately that ϕ^* and e coincide and, in particular, the functor ϕ^* is exact. Finally the functor θ^* coincides with the composition of ϕ^* and the sheafification functor

$$(Sm/F)^{\sim}_{Nis} \xrightarrow{\mathcal{F}\mapsto\mathcal{F}^{\sim}_{cdh}} (Sm/F)^{\sim}_{cdh}$$

which is always exact - see [SGA 4].

In the sequell we'll usually identify the categories $(Sch/F)^{\sim}_{cdh}$ and $(Sm/F)^{\sim}_{cdh}$ via the quasiinverse equivalences ϕ_* and $\phi^* = e$. In particular we'll usually use the notation $(-)^{\sim}_{cdh}$ for the inverse image functor $\theta^* : (Sm/F)^{\sim}_{Nis} \to (Sch/F)^{\sim}_{cdh}$.

Proposition 5.12. *Let $\mathcal{F}$ be a homotopy invariant Nisnevich sheaf with transfers. Let further $\mathcal{G}$ be a Nisnevich sheaf such that $\mathcal{G}^{\sim}_{cdh} = 0$. Then $\mathrm{Ext}^*_{Nis}(\mathcal{G}, \mathcal{F}) = 0$.*

Proof. We first verify our statement for certain special sheaves $\mathcal{G}$. Let $Z \subset X$ be a closed smooth subscheme of a smooth scheme X. Denote by $\mathcal{G}(X, Z)$ the cokernel of the homomorphism $\mathbb{Z}_{Nis}(B_Z(X)) \to \mathbb{Z}_{Nis}(X)$.

(5.12.1). $\mathrm{Ext}^*_{Nis}(\mathcal{G}(X, Z), \mathcal{F}) = 0$.

Proof. One checks easily that the sheaf $\mathcal{G}(X, Z)$ fits into an exact sequence

$$0 \to \mathbb{Z}_{Nis}(p^{-1}(Z)) \to \mathbb{Z}_{Nis}(Z) \oplus \mathbb{Z}_{Nis}(B_Z(X)) \to \mathbb{Z}_{Nis}(X) \to \mathcal{G}(X, Z) \to 0$$

which gives us a spectral sequence converging to $\mathrm{Ext}^*_{Nis}(\mathcal{G}(X,Z),\mathcal{F})$ and whose E_1 term loks as follows:

$$H^*_{Nis}(X,\mathcal{F}) \to H^*_{Nis}(Z,\mathcal{F}) \oplus H^*_{Nis}(B_Z(X),\mathcal{F}) \to H^*_{Nis}(p^{-1}(Z),\mathcal{F}).$$

Since Nisnevich cohomology with coefficients in $\mathcal{F}$ satisfies descent for blow-up's - see Theorems 4.8 and 1.5 one concludes immediately that the E_2-term of the above spectral sequence is zero.

To conclude the proof we note that the condition $\mathcal{G}^{\sim}_{cdh} = 0$ is equivalent to the fact that for each integral scheme $X \in Sm/F$ and each section $s \in \mathcal{G}(X)$ there exists a cdh-covering sieve of X such that s dies being restricted to this sieve. Furthermore resolution of singularities together with Proposition 5.9 and Lemma 5.10 imply that every cdh-covering sieve of X admits a refinement defined by a covering family of the form $\{U_i \to X_n \to ... \to X_0 = X\}_{i=1}^n$ where $\{U_i \to X_n\}_{i=1}^n$ is a Nisnevich covering of X_n and $X_n \to ... \to X_0 = X$ is a tower of blow-up's with smooth centers. Thus for each $s \in \mathcal{G}(X)$ there exists a tower of blow-up's with smooth centers $X_n \to ... \to X_0 = X$ and a Nisnevich covering $\{U_i \to X_n\}_{i=1}^n$ of X_n such that $s|_{U_i} = 0 \quad \forall i$. Since $\mathcal{G}$ is a Nisnevich sheaf we conclude further that $s|_{X_n} = 0$. Denote by $\mathcal{G}_n$ the subsheaf of $\mathcal{G}$ generated by sections which can be killed by a tower of $\leq n$ blow-up's with smooth centers (in particular $\mathcal{G}_0 = 0$). According to the construction for each n the quotient sheaf $\mathcal{G}_n/\mathcal{G}_{n-1}$ is generated by sections which can be killed by just one blow-up with a smooth center. Thus for each n there exists an epimorphism onto $\mathcal{G}_n/\mathcal{G}_{n-1}$ from the direct sum of sheaves of the form $\mathcal{G}(X,Z)$. We now proceed to show that $\mathrm{Ext}^i_{Nis}(\mathcal{G},\mathcal{F}) = 0$ by induction on i. Induction hypothesis and (5.12.1) show that $\mathrm{Ext}^i_{Nis}(\mathcal{G}_n/\mathcal{G}_{n-1},\mathcal{F}) = 0$ for all n. This implies further that $\mathrm{Ext}^i_{Nis}(\mathcal{G}_n,\mathcal{F}) = 0$ for all n. Finally we use the exact sequence (cf. the proof of Lemma 0.6)

$$0 \to \bigoplus_{n=0}^{\infty} \mathcal{G}_n \to \bigoplus_{n=0}^{\infty} \mathcal{G}_n \to \mathcal{G} \to 0$$

to conclude that $\mathrm{Ext}^i_{Nis}(\mathcal{G},\mathcal{F}) = 0$.

Corollary 5.12.2. *In conditions and notations of Proposition 5.12 the Nisnevich sheaf $\mathcal{F}$ is a sheaf in the cdh-topology as well.*

Proof. One checks immediately that $\mathcal{F}$ is separated in the cdh-topology (to do so it suffices to note that the homomorphism $\mathcal{F}(X) \to \mathcal{F}(B_Z(X))$ is injective for a blow-up with a smooth center Z -see Theorem 4.8). This implies that the canonical homomorphism $\mathcal{F} \to \mathcal{F}^{\sim}_{cdh}$ is injective. Denote by $\mathcal{G}$ the quotient Nisnevich sheaf, so that $\mathcal{G}$ fits into an exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}^{\sim}_{cdh} \to \mathcal{G} \to 0$$

and, in particular $\mathcal{G}^{\sim}_{cdh} = 0$. Since $\mathrm{Ext}^1_{Nis}(\mathcal{G},\mathcal{F}) = 0$ we conclude that the above sequence of Nisnevich shieves splits, so that $\mathcal{G}$ is a direct summand in the cdh-sheaf $\mathcal{F}^{\sim}_{cdh}$ and hence is a cdh-sheaf itself. Thus $\mathcal{G} = 0$.

Denote by $\alpha : (Sm/F)_{cdh} \longrightarrow (Sm/F)_{Nis}$ the obvious morphism of cites. Note that $\theta = \alpha\phi$, moreover the functor $\alpha_* : (Sm/F)^{\sim}_{cdh} \longrightarrow (Sm/F)^{\sim}_{Nis}$ is just the forgetful functor, whereas the functor $\alpha^* : (Sm/F)^{\sim}_{NIs} \longrightarrow (Sm/F)^{\sim}_{cdh}$ coincides with the sheafification functor $(-)^{\sim}_{cdh}$.

Corollary 5.12.3. *Let $\mathcal{F}$ be a homotopy invariant Nisnevich sheaf with transfers. Then $R^i\alpha_*(\mathcal{F}) = 0$ for $i > 0$ and hence $H^*_{cdh}(X, \mathcal{F}) = H^*_{Nis}(X, \mathcal{F})$ for any $X \in Sm/F$.*

Proof. Let $0 \to \mathcal{F} \xrightarrow{\epsilon} I^0 \xrightarrow{d} I^1 \xrightarrow{d} \ \ldots$ be the cdh-injective resolution of the cdh-sheaf $\mathcal{F}$. Applying the functor α_* to the above resolution we get the same complex of sheaves, but considered now as a complex of Nisnevich sheaves. Thus we have to verify that Nisnevich cohomology sheaves $\mathcal{H}^i$ of the complex $I^\bullet$ are trivial for $i > 0$. We proceed by induction on i. Note first that $(\mathcal{H}^i)^{\sim}_{cdh} = 0$. Denote by $\mathcal{Z}^i$ the kernel of the homomorphism $I^i \xrightarrow{d} I^{i+1}$, so that we get an exact sequence of Nisnevich sheaves

$$0 \to \mathcal{F} \to I^0 \to \ldots \to I^{i-1} \xrightarrow{d} \mathcal{Z}^i \xrightarrow{p} \mathcal{H}^i \to 0$$

and hence an element in $\mathrm{Ext}^{i+1}_{Nis}(\mathcal{H}^i, \mathcal{F})$. The resulting extension class is trivial according to Proposition 5.12. Since the sheaves $I^0, ..., I^{i-1}$ are injective considered as Nisnevich sheaves as well this implies easily that the epimorphism $\mathcal{Z}^i \xrightarrow{p} \mathcal{H}^i$ splits and hence $\mathcal{H}^i$ is a direct summand of the cdh-sheaf $\mathcal{Z}^i$ and hence is a cdh-sheaf itself. Thus $\mathcal{H}^i = 0$.

We delay the proof of the following result till § 12.

Theorem 5.13. *The cdh-cohomological dimension of any scheme $X \in Sch/F$ is finite (and $\leq dimX$).*

Theorem 5.13 together with results of § 0 show that cdh-hypercohomology of a scheme $X \in Sch/F$ with coefficients in an arbitrary complex of cdh-sheaves makes perfect sence and has all the expected properties.

Theorem 5.14. *For any complex $C^\bullet \in DM^-(F)$ and any scheme $X \in Sch/F$ we have natural identifications*

$$H^*_{\mathcal{M}}(X, C^\bullet) = H^*_{cdh}(X, (C^\bullet)^{\sim}_{cdh}).$$

Proof. For any $X \in Sch/F$ let $\mathbb{Z}_{Nis}(X) : Sm/F \to Ab$ be the Nisnevich sheaf associated to the presheaf $U \mapsto \mathbb{Z}[Hom_{Sch/F}(U, X)]$. One checks easily that $\mathbb{Z}_{Nis}(X)$ is a subsheaf in $\mathbb{Z}_{tr}(X)$ and moreover $\mathbb{Z}_{Nis}(X)^{\sim}_{cdh} = \mathbb{Z}_{cdh}(X)$. This remark gives us a sequence of natural homomorphisms

$$H^i_{\mathcal{M}}(X, C^\bullet) = \mathrm{Ext}^i_{NSwT}(\mathbb{Z}_{tr}(X), C^\bullet) \to \mathrm{Ext}^i_{Nis}(\mathbb{Z}_{tr}(X), C^\bullet) \to$$
$$\to \mathrm{Ext}^i_{Nis}(\mathbb{Z}_{Nis}(X), C^\bullet) \to \mathrm{Ext}^i_{cdh}(\mathbb{Z}_{Nis}(X)^{\sim}_{cdh}, (C^\bullet)^{\sim}_{cdh}) = H^i_{cdh}(X, (C^\bullet)^{\sim}_{cdh}).$$

We have to show that the composition of these homomorphisms is an isomorphism. We first treat the case of a smooth X. In this case the composition of the first two homomorphisms is an isomorphism according to the Theorem 1.5. The proof of the fact that the last map is also an isomorphism is identical to the proof of Corollary 1.1.1 taking into account Theorem 5.13 and Corollary 5.12.3. In the general case we proceed by induction on $dim X$. Using additional induction on the number of irreducible components of X and Lemma 5.1 one reduces the general case first to the case of an integral scheme X. In the latter case resolution of singularities gives us a proper birational morphism $X' \xrightarrow{p} X$ with X' smooth and irreducible. Let $Z \subset X$ be a proper closed subscheme such that p is an isomorphism over $X \setminus Z$. In view of Theorem 5.2 (and Lemma 12.1) we get a commutative diagram with exact rows (in which all motivic cohomology are taken with coefficients in $C^\bullet$ and all cdh-cohomology are taken with coefficients in $(C^\bullet)^\sim_{cdh}$).

$$\begin{array}{ccccccccc}
\longrightarrow & H^{i-1}_{\mathcal{M}}(p^{-1}(Z)) & \longrightarrow & H^i_{\mathcal{M}}(X) & \longrightarrow & H^i_{\mathcal{M}}(X') \oplus H^i_{\mathcal{M}}(Z) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & H^{i-1}_{cdh}(p^{-1}(Z)) & \longrightarrow & H^i_{cdh}(X) & \longrightarrow & H^i_{cdh}(X') \oplus H^i_{cdh}(Z) & \longrightarrow
\end{array}$$

Induction hypothesis and the five-Lemma finish the proof.

§ 6. Truncated etale cohomology and the Beilinson-Lichtenbaum Conjecture.

The usual way to formulate the Beilinson-Lichtenbaum Conjecture (cf. § 3) is to say that a complex of Nisnevich sheaves $\mathbb{Z}/l(n)$ is canonically quasiisomorphic to $\tau_{\leq n} R(\pi_0)_*(\mu_l^{\otimes n})$, where

$$\pi_0 : (Sm/F)_{et} \to (Sm/F)_{Nis}$$

is the obvious morphism of sites and $\tau_{\leq n}$ denotes the level n truncation functor. We modify this conjecture slightly so that it would concern an isomorphism of two objects of $DM^-(F)$. Throughout this section F is a perfect field and l is an integer prime to $char F$.

We start by reminding a few elementary properties of h-sheaves to be used later. Recall that h-topology on the category Sch/F is a Grothendieck topology which is stronger than the etale topology and which is essentially characterized by the property that every proper surjective morphism is an h-covering (see [V 0] or [S-V] for the explicit definition). Denote by

$$(Sch/F)_h \xrightarrow{\pi} (Sm/F)_{Nis} \qquad (Sch/F)_h \xrightarrow{\rho} (Sm/F)_{et}$$

the obvious morphism of sites (so that $\pi = \pi_0 \rho$).

Lemma 6.1. *The inverse image functors*

$$(Sm/F)^\sim_{Nis} \xrightarrow{\pi^*} (Sch/F)^\sim_h \qquad (Sm/F)^\sim_{et} \xrightarrow{\rho^*} (Sch/F)^\sim_h$$

are exact.

Proof. The proof is identical to that of Proposition 5.11, using now Theorem 0.2 (which shows that every scheme $X \in Sch/F$ admits a smooth h-covering) instead of resolution of singularities.

The proof of Lemma 6.1 is based (the same as the proof of Proposition 5.11) on the fact that one can define h-topology on the category Sm/F and moreover the categories of sheaves $(Sch/F)^{\sim}_h$ and $(Sm/F)^{\sim}_h$ are canonically equivalent. In the sequel we'll usually identify these categories and often use the notation $(-)^{\sim}_h$ for both π^* and ρ^*.

Corollary 6.1.1. *The direct image functors*

$$\pi_* : (Sch/F)^{\sim}_h \to (Sm/F)^{\sim}_{Nis} \qquad \rho_* : (Sch/F)^{\sim}_h \to (Sm/F)^{\sim}_{et}$$

take injective sheaves to injective sheaves.

Lemma 6.2. $R^i\rho_*(\mu_l^{\otimes n}) = \begin{cases} \mu_l^{\otimes n} & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$

Proof. The morphism ρ may be factored as a composition

$$(Sch/F)_h \xrightarrow{\alpha} (Sch/F)_{et} \xrightarrow{\beta} (Sm/F)_{et}$$

Since the restriction functor $\beta_* : (Sch/F)^{\sim}_{et} \to (Sm/F)^{\sim}_{et}$ is obviously exact we conclude that $R^i\rho_* = \beta_* R^i\alpha_*$. Finally the vanishing of $R^i\alpha_*(\mu_l^{\otimes n})$ for $i > 0$ was proved in [S-V § 10].

Corollary 6.2.1. *For any etale sheaf $\mathcal{F} \in (Sm/F)^{\sim}_{et}$ we have natural isomorphisms*

$$\mathrm{Ext}^*_{et}(\mathcal{F}, \mu_l^{\otimes n}) = \mathrm{Ext}^*_h(\rho^*\mathcal{F}, \mu_l^{\otimes n})$$

Lemma 6.3. *For any $X \in Sm/F$ the natural embedding of etale sheaves $\mathbb{Z}_{et}(X) \hookrightarrow \mathbb{Z}_{tr}(X)$ induces an isomorphism of h-sheaves $\mathbb{Z}_h(X) = (\mathbb{Z}_{et}(X))^{\sim}_h \xrightarrow{\sim} (\mathbb{Z}_{tr}(X))^{\sim}_h$.*

Proof. Note that the etale sheaf $\mathbb{Z}_{et}(X)$ may be identified with a subsheaf of $\mathbb{Z}_{tr}(X)$, whose sections over $U \in Sm/F$ are integral linear combinations of closed integral subschemes $Z \subset X \times U$ which are finite surjective and etale over a component of U. In view of Lemma 6.1 it suffices to show that $(\mathbb{Z}_{tr}(X)/\mathbb{Z}_{et}(X))^{\sim}_h = 0$. To do so we'll show that for any $U \in Sm/F$ and any $Z \in \mathbb{Z}_{tr}(X)(U)$ there exists an h-covering $\widetilde{U} \to U$ with $\widetilde{U} \in Sm/F$ such that the inverse image of Z in $\mathbb{Z}_{tr}(X)(\widetilde{U})$ belongs to $\mathbb{Z}_{et}(X)(\widetilde{U})$. Obviously we may suppose that U is irreducible and Z is a closed integral subscheme in $X \times U$ finite and surjective over U. We proceed by induction on degree $d = [Z : U]$. Let $U' \to Z$ be a proper surjective morphism with U' smooth and irreducible (see Theorem 0.2). The inverse image of Z in $\mathbb{Z}_{tr}(X)(U')$ equals $Z' = cycle(U' \times_U Z)$. One of components of the above cycle is the graph of the section $U' \to U' \times_U Z$ (which belongs to $\mathbb{Z}_{et}(X)(U')$) and all the other components are of degree $< d$.

Corollary 6.3.1. *For any $X \in Sm/F$ we have natural isomorphisms*

$$\mathrm{Ext}^*_{et}(\mathbb{Z}_{tr}(X), \mu_l^{\otimes n}) = \mathrm{Ext}^*_{et}(\mathbb{Z}_{et}(X), \mu_l^{\otimes n}) = H^*_{et}(X, \mu_l^{\otimes n}) = H^*_h(X, \mu_l^{\otimes n}).$$

The following Lemma shows that the result of Lemma 6.3 is valid for non smooth schemes as well.

Lemma 6.3.2. *For any $X \in Sch/F$ we have a natural identification $\mathbb{Z}_{tr}(X)^\sim_h = \mathbb{Z}_h(X)$.*

Proof. Define a presheaf $\mathbb{Z}_{Sm/F}(X)$ on the category Sm/F via the formula $\mathbb{Z}_{Sm/F}(X)(U) = \mathbb{Z}[\mathrm{Hom}_{Sch/F}(U, X)]$, i.e. $\mathbb{Z}_{Sm/F}(X)$ is the restriction to Sm/F of the free presheaf on Sch/F generated by X. One checks easily (using as always Theorem 0.2), that for any h-sheaf $\mathcal{G}$ on the category Sch/F we have a canonical identification

$$\mathrm{Hom}_{(Sm/F)^\wedge}(\mathbb{Z}_{Sm/F}(X), \pi_*(\mathcal{G})) = \mathcal{G}(X).$$

The above formula shows that $(\mathbb{Z}_{Sm/F}(X))^\sim_h = \mathbb{Z}_h(X)$. Furthermore we have an obvious embedding $\mathbb{Z}_{Sm/F}(X) \hookrightarrow \mathbb{Z}_{tr}(X)$ and the arguement used in the proof of Lemma 6.3 shows that the h-sheafification of the quotient $\mathbb{Z}_{tr}(X)/\mathbb{Z}_{Sm/F}(X)$ is trivial. This shows that $\mathbb{Z}_h(X) = \mathbb{Z}_{Sm/F}(X)^\sim_h = \mathbb{Z}_{tr}(X)^\sim_h$.

The proof of the following Lemma is identical to that of Proposition 1.10.

Lemma 6.4. *Let $\mathcal{F}$ be a contractible etale sheaf and let $\mathcal{G}$ be a strongly homotopy invariant etale sheaf. Then $\mathrm{Ext}^*_{et}(\mathcal{F}, \mathcal{G}) = 0$.*

Corollary 6.4.1. *Let $C^\bullet$ be a bounded above complex of contractible etale sheaves and let $\mathcal{G}$ be a strongly homotopy invariant etale sheaf. Then $\mathrm{Ext}^*_{et}(C^\bullet, \mathcal{G}) \stackrel{def}{=} \mathrm{Hom}_{D^-((Sm/F)^\sim_{et})}(C^\bullet, \mathcal{G}[*]) = 0$.*

Proof. Let $\mathcal{B}$ be the full triangulated subcategory of $D^-((Sm/F)_{et})$ consisting of complexes with trivial Ext^* to $\mathcal{G}$. The category $\mathcal{B}$ is obviously closed with respect to taking arbitrary direct sums (of homologically bounded families) and contains all C^i according to Lemma 6.4. Lemma 0.6 shows now that $C^* \in \mathcal{B}$.

Corollary 6.4.2. *For any $X \in Sm/F$ we have natural identifications*

$$\mathrm{Ext}^*_{et}(C^*(\mathbb{Z}_{tr}(X)), \mu_l^{\otimes n}) = \mathrm{Ext}^*_{et}(\mathbb{Z}_{tr}(X), \mu_l^{\otimes n}) = H^*_{et}(X, \mu_l^{\otimes n}).$$

Proof. Recall that the complex $C^*(\mathbb{Z}_{tr}(X))$ contains a subcomplex $C^*_0(\mathbb{Z}_{tr}(X))$ which is canonically quasiisomorphic to $\mathbb{Z}_{tr}(X)$ and such that the factorcomplex C^*/C^*_0 consists of contractible sheaves. Our statement follows now from Corollary 6.4.1, taking into account that the etale sheaf $\mu_l^{\otimes n}$ is strongly homotopy invariant.

Lemma 6.5. *For any h-sheaf $\mathcal{G}$ the Nisnevich sheaf $\pi_*(\mathcal{G})$ has a natural structure of a Nisnevich sheaf with transfers. Moreover for any $\mathcal{F} \in NSwT/F$ we have a natural identification*

$$\mathrm{Hom}_{Nis}(\mathcal{F}, \pi_*(\mathcal{G})) = \mathrm{Hom}_{NSwT}(\mathcal{F}, \pi_*(\mathcal{G})).$$

In other words every homomorphism of Nisnevich sheaves $\mathcal{F} \to \pi_*(\mathcal{G})$ is compatible with transfers.

Proof. Recall that for any h-sheaf $\mathcal{G}$ and any finite surjective morphism $Z \to X$ from an integral scheme Z to an irreducible normal scheme X we defined in [S-V] the transfer homomorphism $Tr_{Z/X} : \mathcal{G}(Z) \to \mathcal{G}(X)$. Let now X, Y be smooth irreducible schemes and let $Z \subset X \times Y$ be a closed integral subscheme finite and surjective over X. We define the homomorphism $Z^* : \mathcal{G}(Y) \to \mathcal{G}(X)$ as the composition

$$\mathcal{G}(Y) \xrightarrow{(p_2)^*} \mathcal{G}(Z) \xrightarrow{Tr_{Z/X}} \mathcal{G}(X).$$

One checks easily, using the results of [S-V § 5], that this defines the structure of a Nisnevich sheaf with transfers on $\pi_*(\mathcal{G})$.

In proving the second statement we may assume that $\mathcal{F} = \mathbb{Z}_{tr}(X)$ for some $X \in Sm/F$. In this case

$$\mathrm{Hom}_{NSwT/F}(\mathbb{Z}_{tr}(X), \pi_*(\mathcal{G})) = \pi_*(\mathcal{G})(X) = \mathcal{G}(X)$$
$$\mathrm{Hom}_{Nis}(\mathbb{Z}_{tr}(X), \pi_*(\mathcal{G})) = \mathrm{Hom}_h(\pi^*(\mathbb{Z}_{tr}(X)), \mathcal{G}) = \mathrm{Hom}_h(\mathbb{Z}_h(X), \mathcal{G}) = \mathcal{G}(X).$$

The complex $R\pi_*(\mu_l^{\otimes n})$ may be defined as follows. Consider a resolution $J^\bullet(n)$

$$0 \to \mu_l^{\otimes n} \to J^0(n) \to J^1(n) \to \dots$$

of $\mu_l^{\otimes n}$ by injective h-sheaves. Then the complex $R\pi_*(\mu_l^{\otimes n})$ coincides with

$$\pi_*(J^\bullet(n)) = (\pi_*(J^0(n)) \to \pi_*(J^1(n)) \to \dots \quad).$$

According to Lemma 6.5 the complex $\pi_*(J^\bullet(n))$ is actually a complex of Nisnevich sheaves with transfers. Moreover the cohomology presheaves of $\pi_*(J^\bullet(n))$ are of the form

$$X \mapsto H^i_h(X, \mu_l^{\otimes n}) = H^i_{et}(X, \mu_l^{\otimes n})$$

and hence are homotopy invariant. This shows that the complex

$$B_l(n) = \tau_{\leq n}\pi_*(J^\bullet(n))$$

is a bounded above complex of Nisnevich sheaves with transfers with homotopy invariant cohomology sheaves, i.e. an object of $DM^-(F)$.

Sometimes it's preferable to work with resolutions of $\mu_l^{\otimes n}$ by injective objects in the category of h-sheaves of $\mathbb{Z}/l$-modules. For any such resolution $0 \to \mu_l^{\otimes n} \to I^\bullet(n)$ we have a unique (up to homotopy) homomorphism of resolutions $I^\bullet(n) \to J^\bullet(n)$ and the induced homomorphisms

$$\pi_*(I^\bullet(n)) \to \pi_*(J^\bullet(n)) \qquad \tau_{\leq n}\pi_*(I^\bullet(n)) \to \tau_{\leq n}\pi_*(J^\bullet(n))$$

are quasiisomorphism since injective sheaves of $\mathbb{Z}/l$-modules are acyclic (i.e. $H^*_h(X, I^k) = 0$ for any $X \in Sch/F$ any $k \geq 0$ and any $* > 0$). Thus we may equally define the complex $B_l(n)$ as $\tau_{\leq n}\pi_*(I^\bullet(n))$.

Lemma 6.6. *For any $n \geq 0$ we have natural identifications*

$$H^i_{et}(\mathbb{G}_m^{\wedge n}, \mu_l^{\otimes n}) = H^{i-n}_{et}(F, \mathbb{Z}/l).$$

Proof. We first compute etale cohomology of $\mathbb{G}^{\wedge n}_{m,\overline{F}}$. Note that according to the definition $H^*_{et}(\mathbb{G}^{\wedge n}_{m,\overline{F}}, \mu_l^{\otimes n})$ is a direct summand in $H^*_{et}(\mathbb{G}^{\times n}_{m,\overline{F}}, \mu_l^{\otimes n})$, consisting of cohomology classes which vanish being restricted to any copy of $\mathbb{G}^{\times(n-1)}_{m,\overline{F}}$ inside $\mathbb{G}^{\times n}_{m,\overline{F}}$. The well-known computation of the etale cohomology of $\mathbb{G}_{m,\overline{F}}$

$$H^*_{et}(\mathbb{G}_{m,\overline{F}}, \mu_l) = \begin{cases} \mu_l & \text{if } * = 0 \\ \mathbb{Z}/l & \text{if } * = 1 \\ 0 & \text{if } * > 1 \end{cases}$$

and the Künneth formula for etale cohomology imply easily that

$$H^*_{et}(\mathbb{G}^{\wedge n}_{m,\overline{F}}, \mu_l^{\otimes n}) = \begin{cases} \mathbb{Z}/l & \text{if } * = n \\ 0 & \text{if } * \neq n \end{cases}$$

We conclude the proof using the Hochschild-Serre spectral sequence

Remark 6.6.1. *We'll denote by α_n the canonical generator of $H^n_{et}(\mathbb{G}_m^{\wedge n}, \mu_l^{\otimes n})$. The proof of the Lemma shows that α_n coincides with the n-th power $\alpha_1 \in H^1_{et}(\mathbb{G}_m^{\wedge 1}, \mu_l)$ and, in particular $\alpha_n \cup \alpha_m = \alpha_{n+m}$ for all $n, m \geq 0$.*

Corollary 6.6.2. $\mathrm{Ext}^0_{et}(\mathbb{Z}/l(n), \mu_l^{\otimes n}) = \mathrm{Ext}^0_{et}(\mathbb{Z}(n), \mu_l^{\otimes n}) = H^n_{et}(\mathbb{G}_m^{\wedge n}, \mu_l^{\otimes n}) = \mathbb{Z}/l.$

Proof. This follows immediately from Corollary 6.4.2, Lemma 6.6 and an exact sequence of Ext-groups corresponding to the short exact sequence of complexes

$$0 \to \mathbb{Z}(n) \xrightarrow{l} \mathbb{Z}(n) \to \mathbb{Z}/l(n) \to 0$$

Proposition 6.7. *The complex of etale sheaves $\pi_0^*(\mathbb{Z}/l(n))$ is canonically quasiisomorphic to $\mu_l^{\otimes n}$ and hence the same is true for the complex of h-sheaves $\pi^*(\mathbb{Z}/l(n))$.*

Proof. Denote by $\mathcal{H}^i$ the i-th cohomology presheaf of $\mathbb{Z}/l(n)$. Our statement may be rephrased as follows. The etale sheaves $(\mathcal{H}^i)^\sim_{et}$ are trivial for $i \neq 0$ and the sheaf $(\mathcal{H}^0)^\sim_{et}$ is canonically isomorphic to $\mu_l^{\otimes n}$. We first show that $(\mathcal{H}^i)^\sim_{et} = 0$ for $i \neq 0$ and that the sheaf $(\mathcal{H}^0)^\sim_{et}$ is a locally constant sheaf of free $\mathbb{Z}/l$-modules of rank one. This amounts to showing that for any $X \in Sm/F$ and any closed point $x \in X$

$$\mathcal{H}^i(\mathcal{O}^{sh}_{X,x}) = \begin{cases} \tilde{=} \mathbb{Z}/l & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

Note first that $\mathcal{H}^i$ is a homotopy invariant presheaf of $\mathbb{Z}/l$-modules with transfers so that the rigidity theorem ([S-V] Theorem 4.4) shows that $\mathcal{H}^i(\mathcal{O}^{sh}_{X,x}) = \mathcal{H}^i(\overline{F})$. Futhermore (in notations of [S-V])

$$\mathcal{H}^i(\overline{F}) = H^{sin}_{n-i}(\mathbb{G}^{\wedge n}_{m,\overline{F}}), \mathbb{Z}/l)$$

The dual $\mathbb{Z}/l$-module $\mathcal{H}^i(\overline{F})^\#$ coincides with $H^{n-i}_{sin}(\mathbb{G}^{\wedge n}_{m,\overline{F}}, \mathbb{Z}/l) =$ $= H^{n-i}_{et}(\mathbb{G}^{\wedge n}_{m,\overline{F}}, \mathbb{Z}/l)$ - see [S-V]. This together with the computation of the etale cohomology of $\mathbb{G}^{\wedge n}_{m,\overline{F}}$ (see Lemma 6.6) proves our claim. To finish the proof of the Proposition we note that vanishing of $(\mathcal{H}^i)^\sim_{et}$ for $i \neq 0$ gives us canonical isomorphisms

$$\mathrm{Hom}_{et}((\mathcal{H}^0)^\sim_{et}, \mu_l^{\otimes n}) = \mathrm{Ext}^0_{et}(\mathbb{Z}/l(n), \mu_l^{\otimes n}) = \mathbb{Z}/l.$$

Proposition 6.7 provides us with canonical morphisms

$$\pi^*(\mathbb{Z}/l(n)) \to \tau_{\geq 0}\pi^*(\mathbb{Z}/l(n)) \to I^\bullet(n) \to J^\bullet(n)$$

Using further the adjunction relation between π^* and π_* we get a canonical morphism

$$\mathbb{Z}/l(n) \to \pi_*(I^\bullet(n)) \xrightarrow{\sim} \pi_*(J^\bullet(n))$$

which factors through $B_l(n) = \tau_{\leq n}(\pi_*(I^\bullet(n))$, since the complex $\mathbb{Z}/l(n)$ is bounded above at level n. We denote the resulting morphism $\mathbb{Z}/l(n) \to B_l(n)$ by α_n. Note that according to Lemma 6.5 the homomorphism α_n is compatible with transfers i.e. $\alpha_n \in \mathrm{Hom}_{DM^-(F)}(\mathbb{Z}/l(n), B_l(n))$.

Conjecture 6.8 (The Beilinson-Lichtenbaum Conjecture in weight n). *The canonical morphism $\alpha_n : \mathbb{Z}/l(n) \to B_l(n) = \tau_{\leq n}(\pi_*(I^\bullet(n)))$ is a quasiisomorphism.*

Remark 6.8.1 Lemma 6.2 and Corollary 6.1.1 show that the complex $\rho_*(J^\bullet(n))$ is an injective resolution of the etale sheaf $\mu_l^{\otimes n} \in (Sm/F)^\sim_{et}$. This implies immediately that the complex $\pi_*(J^\bullet(n)) = (\pi_0)_*(\rho_*(J^\bullet(n)))$ coincides (if we ignore the transfers) with $R(\pi_0)_*(\mu_l^{\otimes n})$ and $B_l(n)$ coincides with $\tau_{\leq n} R(\pi_0)_*(\mu_l^{\otimes n})$, so that the present formulation of the Beilinson-Lichtenbaum Conjecture is equivalent to the usual one.

Lemma 6.9. *To prove the Beilinson-Lichtenbaum Conjecture in the weight n it suffices to show that for any finitely generated field extension E/F the homomorphisms*

$$(\alpha_n)_* : H^i_{\mathcal{M}}(E, \mathbb{Z}/l(n)) \to H^i_{\mathcal{M}}(E, B_l(n))$$

are isomorphisms.

Proof. Let $\mathcal{H}^i$ (resp. $\widetilde{\mathcal{H}}^i$) denote the i-th cohomology sheaf of $\mathbb{Z}/l(n)$ (resp. of $B_l(n)$). The Beilinson-Lichtenbaum Conjecture asserts that the induced maps $(\alpha_n)_* : \mathcal{H}^i \to \widetilde{\mathcal{H}}^i$ are isomorphisms for all i. Note further that $\mathcal{H}^i$ and $\widetilde{\mathcal{H}}^i$ are homotopy invariant Nisnevich sheaves with transfers and according to [V 1] a homomorphism of Nisnevich sheaves with transfers is an isomorphism iff it induces isomorphisms on sections over an arbitrary finitely generated field extension of F.

The following Lemma allows to reduce the general case of the Beilinson-Lichtenbaum Conjecture to the case of prime numbers.

Lemma 6.10. *Let l, l' be two integers. If the Beilinson-Lichtenbaum Conjecture in the weight n holds modulo l and l' then it also holds modulo ll'. Thus to prove the Beilinson-Lichtenbaum Conjecture in the weight n modulo l it suffices to prove it modulo all prime divisors of l.*

Proof. Note that in $DM^-(F)$ we have a distinguished Bockstein triangle

$$\mathbb{Z}/l'(n) \to \mathbb{Z}/ll'(n) \to \mathbb{Z}/l(n) \to \mathbb{Z}/l'(n)[1].$$

Validity of the Beilinson-Lichtenbaum Conjecture modulo l implies that we have a similar distinguished triangle involving $B(n)$:

$$B_{l'}(n) \to B_{ll'}(n) \to B_l(n) \to B_{l'}(n)[1].$$

To check this one has only to verify that the induced homomorphism on the n-th cohomology sheaves $\mathcal{H}^n(B_{ll'}(n)) \to \mathcal{H}^n(B_l(n))$ is surjective. This follows however from the commutative diagram

$$\begin{array}{ccc} \mathcal{H}^n(\mathbb{Z}(n)) & \xrightarrow{=} & \mathcal{H}^n(\mathbb{Z}(n)) \\ \downarrow & & \downarrow \\ \mathcal{H}^n(B_{ll'}(n)) & \longrightarrow & \mathcal{H}^n(B_l(n)) \end{array}$$

in which the right vertical homomorphism is surjective. Finally one checks easily that the homomorphisms α_n give a map from the first distinguished triangle to the second one and our statement follows in view of the five-Lemma.

We end up this section with a few remarks concerning motivic cohomology with coefficients in $B_l(n)$.

Lemma 6.11. *For any $X \in Sch/F$ and any $i \leq n$ we have natural isomorphisms*

$$H^i_{\mathcal{M}}(X, B_l(n)) = H^i_h(X, \mu_l^{\otimes n}) = H^i_{et}(X, \mu_l^{\otimes n}).$$

Proof. According to definitions and Corollary 1.11.2 we have natural identifications

$$H^i_{\mathcal{M}}(X, B_l(n)) = \mathrm{Ext}^i_{NSwT}(\mathbb{Z}_{tr}(X), B_l(n)).$$

The short exact sequence of complexes

$$0 \to B_l(n) \to \pi_*(J^\bullet(n)) \to \tau_{\geq n+1}\pi_*(J^\bullet(n)) \to 0$$

gives rise to a long exact sequence of Ext-groups The Ext-groups from $\mathbb{Z}_{tr}(X)$ to $\tau_{\geq n+1}\pi_*(J^\bullet(n))$ are trivial in degrees $\leq n$ by obvious reasons. The complex $\pi_*(J^\bullet(n))$ consists of injective Nisnevich sheaves with transfers and hence

$$\mathrm{Ext}^*_{NSwT}(\mathbb{Z}_{tr}(X), \pi_*(J^\bullet(n))) = H^*(\mathrm{Hom}_{NSwT}(\mathbb{Z}_{tr}(X), \pi_*(J^\bullet(n)))) =$$
$$H^*(\mathrm{Hom}_{Nis}(\mathbb{Z}_{tr}(X), \pi_*(J^\bullet(n)))) = H^*(\mathrm{Hom}_h(\mathbb{Z}_h(X), J^\bullet(n))) = H^*_h(X, \mu_l^{\otimes n}).$$

Corollary 6.11.1. $\mathrm{Hom}_{DM^-(F)}(\mathbb{Z}/l(n), B_l(n)) = \mathrm{Hom}_{DM^-(F)}(\mathbb{Z}(n), B_l(n)) = = H^n_{et}(\mathbb{G}_m^{\wedge n}, \mu_l^{\otimes n}) = \mathbb{Z}/l$. *Under this identification the homomorphism* α_n *corresponds to the cohomology class denoted by* α_n *previously.*

Proof. The first statement follows from Lemma 6.11 and the long exact cohomology sequence, corresponding to the short exact sequence of complexes

$$0 \to \mathbb{Z}(n) \xrightarrow{l} \mathbb{Z}(n) \to \mathbb{Z}/l(n) \to 0$$

The second statement is obvious from the construction.

Remark 6.11.2 Corollary 6.11.1 shows that $\mathrm{Hom}_{DM^-(F)}(\mathbb{Z}/l(n), B_l(n))$ may be identified with $\mathrm{Hom}_{DM^-(F)}(\mathbb{Z}(n), B_l(n))$, we'll use the same notation α_n for the corresponding morphism $\mathbb{Z}(n) \to B_l(n)$.

§ 7. The Bloch-Kato Conjecture.

We keep notations and assumptions of the previous section.

Proposition 7.1. *For any* $n, m \geq 0$ *there are natural pairings* $B_l(n) \otimes B_l(m) \to \to B_l(n+m)$. *Furthermore for any* $X, Y \in Sch/F$ *and any integers* $i \leq n, j \leq m$ *the resulting pairing in cohomology*

$$H^i_{et}(X, \mu_l^{\otimes n}) \otimes H^j_{et}(Y, \mu_l^{\otimes m}) = H^i_{\mathcal{M}}(X, B_l(n)) \otimes H^j_{\mathcal{M}}(Y, B_l(m)) \to$$
$$\to H^{i+j}_{\mathcal{M}}(X \times Y, B_l(n+m)) = H^{i+j}_{et}(X \times Y, \mu_l^{\otimes(n+m)})$$

coincides with the usual product pairing in etale cohomology.

Proof. Note that for any injective h-sheaf of $\mathbb{Z}/l$-modules I all groups of sections $I(X)$ are injective $\mathbb{Z}/l$-modules and hence free $\mathbb{Z}/l$-modules. This remark implies immediately that the complex $I^\bullet(n) \otimes_h I^\bullet(m)$ is a resolution of $\mu_l^{\otimes n} \otimes_h \mu_l^{\otimes m} = \mu_l^{\otimes n+m}$, which gives us a natural homomorphism of complexes $I^\bullet(n) \otimes_h I^\bullet(m) \to I^\bullet(n+m)$. Furthermore one checks easily (using Lemma 2.1) that for any h-sheaves $\mathcal{F}, \mathcal{G}$ there exists a canonical homomorphism of Nisnevich sheaves with transfers $\pi_*(\mathcal{F}) \otimes_{tr} \pi_*(\mathcal{G}) \to \pi_*(\mathcal{F} \otimes_h \mathcal{G})$. This gives us a homomorphism of complexes

$$B_l(n) \otimes^L B_l(m) \to B_l(n) \otimes_{tr} B_l(m) \to \pi_*(I^\bullet(n)) \otimes_{tr} \pi_*(I^\bullet(m)) \to$$
$$\to \pi_*(I^\bullet(n) \otimes_h I^\bullet(m)) \to \pi_*(I^\bullet(n+m))$$

which factors through $B_l(n+m)$ since the complex $B_l(n) \otimes^L B_l(m)$ is bounded above at the level $n+m$. Applying finally the localising functor C^* to both sides we get the desired pairing

$$B_l(n) \otimes B_l(m) = C^*(B_l(n) \otimes^L B_l(m)) \to C^*(B_l(n+m)) \xleftarrow{\sim} B_l(n+m)$$

The second statement is obvious from the construction.

Corollary 7.1.1. *For any $n, m \geq 0$ the following diagram in $DM^-(F)$ commutes*

$$\begin{array}{ccc} \mathbb{Z}/l(n) \otimes \mathbb{Z}/l(m) & \xrightarrow{\sim} & \mathbb{Z}/l(n+m) \\ \downarrow{\scriptstyle \alpha_n \otimes \alpha_m} & & \downarrow{\scriptstyle \alpha_{n+m}} \\ B_l(n) \otimes B_l(m) & \longrightarrow & B_l(n+m) \end{array}$$

Proof. We have to show that two resulting maps in $DM^-(F)$ from $\mathbb{Z}/l(n) \otimes \mathbb{Z}/l(m) = \mathbb{Z}/l(n+m)$ to $B_l(n+m)$ coincide. To do so we note that $\mathrm{Hom}_{DM^-(F)}(\mathbb{Z}/l(n+m), B_l(n+m)) = H^{n+m}_{et}(\mathbb{G}_m^{\wedge(n+m)}, \mu_l^{\otimes(n+m)})$ $(= \mathbb{Z}/l)$ and the resulting maps correspond to the cohomology classes α_{n+m} and $\alpha_n \cup \alpha_m = \alpha_{n+m}$ respectively.

Lemma 7.2. *For any (finitely generated) field extension E/F we have natural identifications:*

$$H^i_{\mathcal{M}}(E, \mathbb{Z}/l(1)) = \begin{cases} H^0(E, \mu_l) & \text{if } i = 0 \\ E^*/E^{*l} & \text{if } i = 1 \\ 0 & \text{if } i \neq 0, 1 \end{cases}$$

The homomorphism

$$(\alpha_1)_* : H^i_{\mathcal{M}}(E, \mathbb{Z}/l(1)) \to H^i_{\mathcal{M}}(E, B_l(1)) = \begin{cases} H^i_{et}(E, \mu_l) & \text{if } i = 0, 1 \\ 0 & \text{if } i \neq 0, 1 \end{cases}$$

is identity in degree 0 and coincides with the Kümmer isomorphism $\chi : E^*/E^{*l} \xrightarrow{\sim} H^1_{et}(E, \mu_l)$ in degree one. In particular the Beilinson-Lichtenbaum Conjecture is true in weight 1.

Proof. As was noted in § 3 the complex $\mathbb{Z}(1)$ is naturally quasiisomorphic to $\mathcal{O}^*[-1]$ and hence the complex $\mathbb{Z}/l(1)$ is naturally quasiisomorphic to the complex

$$\mathcal{O}^* \xrightarrow{l} \mathcal{O}^*$$

which gives immediately the computation of motivic cohomology of any field with coefficients in $\mathbb{Z}/l(1)$. To compute the action of α_1 in cohomology we may proceed as follows. Let $0 \to \mathcal{O}^* \to \mathcal{L}^\bullet$ be an injective resolution of the etale sheaf $\mathcal{O}^* \in (Sm/F)^{\sim}_{et}$. The cone $J^\bullet = Cone(\mathcal{L}^\bullet \xrightarrow{l} \mathcal{L}^\bullet)$ gives us an injective resolution of the etale sheaf μ_l. According to definitions and Remark 5.8.1 the homomorphism α_1 corresponds (by adjunction) to the natural quasiisomorphism of complexes of etale sheaves

$$Cone(\mathcal{O}^* \xrightarrow{l} \mathcal{O}^*) \to Cone(\mathcal{L}^\bullet \xrightarrow{l} \mathcal{L}^\bullet).$$

Thus homomorphisms induced by α_1 in motivic cohomology of E may be computed using the homomorphism of complexes

$$Cone(E^* \xrightarrow{l} E^*) \to Cone(\mathcal{L}^\bullet(E) \xrightarrow{l} \mathcal{L}^\bullet(E)) = J^\bullet(E).$$

Since the cohomology of $\mathcal{L}^\bullet(E)$ coincides with $H^*_{et}(E, \mathcal{O}^*)$ we get (applying Hilbert's Theorem 90) the following commutative diagram with exact rows

$$\begin{array}{ccccccccccc} 0 & \to & \mu_l(E) & \to & E^* & \xrightarrow{l} & E^* & \to & E^*/E^{*l} & \to & 0 \\ & & \downarrow (\alpha_1)_* & & \downarrow = & & \downarrow = & & \downarrow (\alpha_1)_* & & \\ 0 & \to & H^0(J^\bullet(E)) & \to & H^0(\mathcal{L}^\bullet(E)) & \xrightarrow{l} & H^0(\mathcal{L}^\bullet(E)) & \xrightarrow{\delta} & H^1(J^\bullet(E)) & \to & 0 \end{array}$$

The diagram shows immediately that the homomorphism $(\alpha_1)_* : E^* \to H^1(J^\bullet(E)) = H^1_{et}(E, \mu_l)$ coincides with the connecting homomorphism $\delta : H^0_{et}(E, \mathcal{O}^*) \to H^1_{et}(E, \mu_l)$, corresponding to the short exact sequence of etale sheaves

$$0 \to \mu_l \to \mathcal{O}^* \xrightarrow{l} \mathcal{O}^* \to 0$$

i.e. $E^*/E^{*l} \xrightarrow{(\alpha_1)_*} H^1_{et}(E, \mu_l)$ coincides with the Kümmer isomorphism.

Corollary 7.2.1. *For any $n \geq 0$ and any finitely generated field extension E/F the homomorphism in motivic cohomology induced by α_n*

$$(\alpha_n)_* : K^M_n(E)/l = H^n_{\mathcal{M}}(E, \mathbb{Z}/l(n)) \to H^n_{\mathcal{M}}(E, B_l(n)) = H^n_{et}(E, \mu_l^{\otimes n})$$

coincides with the norm residue homomorphism.

Proof. Corollary 7.1.1 Lemma 7.2 and Proposition 7.1 show that $(\alpha_n)_*$ sends the symbol $\{a_1, ..., a_n\}$ to $\chi(a_1) \cup ... \cup \chi(a_n)$.

Corollary 7.2.1 shows that the Beilinson-Lichtenbaum Conjecture includes as a special case the following conjecture due to S. Bloch and K. Kato.

Conjecture 7.3 (Bloch-Kato Conjecture for a field E in weight n). *The norm-residue homomorphism $\chi_n : K^M_n(E)/l \to H^n_{et}(E, \mu_l^{\otimes n})$ is an isomorphism.*

We'll be also considering the following weaker form of the Bloch-Kato Conjecture.

Conjecture 7.3.1 (Weak Bloch-Kato Conjecture for a field E in weight n). *The norm-residue homomorphism $\chi_n : K^M_n(E)/l \to H^n_{et}(E, \mu_l^{\otimes n})$ is an epimorphism.*

Our main result shows that the validity of the Weak Bloch-Kato Conjecture alone implies the validity of the Beilinson-Lichtenbaum Conjecture.

Theorem 7.4. *Assume that resolution of singularities holds over F and the Weak Bloch-Kato Conjecture (modulo l) holds in weights $\leq n$ for all finitely generated field extensions E/F. Then the Beilinson-Lichtenbaum Conjecture (modulo any power of l) also holds over F in weights $\leq n$.*

The proof of this Theorem occupies the rest of this section and the next 3 sections. Proceeding by induction on n we may (and will) assume that the Beilinson-Lichtenbaum Conjecture is already known to be true in weights $< n$ and, in particular, the strong form of the Bloch-Kato Conjecture (for finitely generated field extensions E/F) is also true in weights $< n$. We start with the following elementary observation.

Lemma 7.5. *Assume that l is prime. Then for any finitely generated field extension E/F the induced homomorphisms in motivic cohomology*

$$(\alpha_n)_* : H^i_{\mathcal{M}}(E, \mathbb{Z}/l(n)) \to H^i_{\mathcal{M}}(E, B_l(n)) = H^i_{et}(E, \mu_l^{\otimes n})$$

are surjective for all $i \le n$.

Proof. We may obviously assume that $i \le n-1$. Furthermore since the degree of the field extension $E(\mu_l)/E$ is prime to l we see, using the standard trick with transfer maps, that it suffices to consider the case when E contains the primitive l-th root of unity. In the latter case our statement follows immediately from the commutativity of the following diagram

$$\begin{array}{ccc} H^0_{\mathcal{M}}(E, \mathbb{Z}/l(1)) \otimes H^i_{\mathcal{M}}(E, \mathbb{Z}/l(n-1)) & \longrightarrow & H^i_{\mathcal{M}}(E, \mathbb{Z}/l(n)) \\ \cong \downarrow & & \downarrow \\ H^0_{et}(E, \mu_l) \otimes H^i_{et}(E, \mu_l^{\otimes(n-1)}) & \xrightarrow{\sim} & H^i_{et}(E, \mu_l^{\otimes n}) \end{array}$$

We finish this section with the following important result, which gives the first step towards the proof of Theorem 7.4.

Proposition 7.6. *For any affine semilocal scheme S essentially of finite type over F the homomorphism in motivic cohomology induced by α_n*

$$(\alpha_n)_* : H^n_{\mathcal{M}}(S, \mathbb{Z}/l(n)) \to H^n_{\mathcal{M}}(S, B_l(n)) = H^n_{et}(S, \mu_l^{\otimes n})$$

is surjective.

Proof. Assume first that S is smooth (and irreducible). Denote by $\mathcal{K}_n$ (resp. $\mathcal{H}^n$) the n-th cohomology sheaf of the complex $\mathbb{Z}/l(n)$ (resp. of $B_l(n)$). Since all higher dimensional cohomology groups of a smooth semilocal scheme with coefficients in a homotopy invariant sheaf with transfers vanish (see [V 1]) we conclude that $H^n_{\mathcal{M}}(S, \mathbb{Z}/l(n)) = H^0_{Zar}(S, \mathcal{K}_n), H^n_{\mathcal{M}}(S, B_l(n)) = H^0_{Zar}(S, \mathcal{H}^n)$. The cokernel $\mathcal{L}$ of the homomorphism $(\alpha_n)_* : \mathcal{K}_n \to \mathcal{H}^n$ is a homotopy invariant Nisnevich sheaf with transfers which vanish on all finitely generated field extensions of F according to our assumptions and hence is zero ([V 1]). Moreover the sequence

$$H^0_{Zar}(S, \mathcal{K}_n) \to H^0_{Zar}(S, \mathcal{H}^n) \to H^0_{Zar}(S, \mathcal{L}) = 0$$

is exact ([V 1]), which ends up the proof in the smooth case.

For an arbitrary affine semilocal scheme we proceed as follows. Embed S as a closed subscheme into a smooth affine semilocal scheme T and denote by I the defining ideal of S. Let T^h_I be the henselization of T along I. The scheme S embeds canonically as a closed subscheme of T^h_I and according to a Theorem of O. Gabber [G] the restriction homomorphism in etale cohomology $H^n_{et}(T^h_I, \mu_l^{\otimes n}) \to H^n_{et}(S, \mu_l^{\otimes n})$ is an isomorphism. On the other hand the scheme T^h_I is a filtered inverse limit of smooth semilocal schemes essentially of finite type over F (each containing S as a closed subscheme). Since etale cohomology commutes with filtered limits we conclude that for every cohomology class $x \in H^n_{et}(S, \mu_l^{\otimes n})$ there exists a smooth affine semilocal scheme T' essentially of finite type over F, containing S as a closed subscheme such that x is the restriction to S of an appropriate cohomology class $x' \in H^n_{et}(T', \mu_l^{\otimes n})$. Now it suffices to use the commutativity of the following diagram

$$\begin{array}{ccc} H^n_{\mathcal{M}}(T', \mathbb{Z}/l(n)) & \xrightarrow{\twoheadrightarrow} & H^n_{et}(T', \mu_l^{\otimes n}) \\ \downarrow & & \downarrow \\ H^n_{\mathcal{M}}(S, \mathbb{Z}/l(n)) & \longrightarrow & H^n_{et}(S, \mu_l^{\otimes n}). \end{array}$$

§ 8. $B_l(n)$-cohomology with supports.

Lemma 8.1.

$$H^i_{\mathcal{M}}(\mathbb{P}^1_F, \mathbb{Z}/l(1)) = \begin{cases} H^i_{\mathcal{M}}(F, \mathbb{Z}/l(1)) & \textit{if } i = 0, 1 \\ \mathbb{Z}/l & \textit{if } i = 2 \\ 0 & \textit{if } i \neq 0, 1, 2 \end{cases}$$

Proof. This follows easily from the computation of motivic cohomology with coefficients in $\mathbb{Z}(1) = \mathcal{O}^*[-1]$ - see Corollary 3.2.1.

$$H^i_{\mathcal{M}}(\mathbb{P}^1_F, \mathbb{Z}(1)) = \begin{cases} F^* & \text{if } i = 1 \\ Pic(\mathbb{P}^1_F) = \mathbb{Z} & \text{if } i = 2 \\ 0 & \text{if } i \neq 1, 2 \end{cases}$$

Remark 8.1.1 The complex $\mathbb{Z}/l(1)$ is naturally qasiisomorphic to $B_l(1) = \tau_{\leq 1}\pi_*(I^\bullet(1))$. It has exactly two non-zero cohomology sheaves: $\mathcal{H}^0(1)$ and $\mathcal{H}^1(1)$. Moreover the sheaf $\mathcal{H}^0(1)$ coincides with μ_l and the sheaf $\mathcal{H}^1(1)$ is the sheaf associated with the preasheaf $U \mapsto H^1_{et}(U, \mu_l)$. Higher dimensional Zariski cohomology with coefficients in μ_l vanish, from which we conclude that $H^2_{\mathcal{M}}(\mathbb{P}^1_F, \mathbb{Z}/l(1)) = H^1_{Zar}(\mathbb{P}^1_F, \mathcal{H}^1(1))$. Furthermore Zariski cohomology of any smooth scheme with coefficients in $\mathcal{H}^1(1)$ may be computed using the Gersten complexes - see [B-O]. In particular $H^*_{Zar}(\mathbb{P}^1_F, \mathcal{H}^1(1))$ may be computed using the Gersten complex

$$H^1_{et}(F(\mathbb{P}^1), \mu_l) \to \bigoplus_{x \in (\mathbb{P}^1_F)^1} \mathbb{Z}/l$$

Here and in the sequell we denote by U^i the set of points of codimension i on a smooth scheme U. We'll be using the notation β for the canonical generator of $H^2_{\mathcal{M}}(\mathbb{P}^1_F, \mathbb{Z}/l(1)) = H^2_{\mathcal{M}}(\mathbb{P}^1_F, B_l(1)) = H^1_{Zar}(\mathbb{P}^1_F, \mathcal{H}^1(1))$. In terms of the above Gersten complex β may be represented by the class of any rational point of $\mathbb{P}^1_F$.

Usually we'll be thinking of β as represented by the class of the rational point $\infty \in \mathbb{P}^1_F$.

Remark 8.1.2 We use 0 as a distinguished rational point on $\mathbb{P}^1_F$. For any $X \in Sm/F$ this allows us to identify X with a closed subscheme $X \times 0 \subset X \times \mathbb{P}^1_F$. The motiv $M(X \times \mathbb{P}^1_F/X)$ coincides with $M(X)(1)[2]$ - see § 4. Motivic cohomology $H^*_{\mathcal{M}}(X \times \mathbb{P}^1_F/X, C^*)$ may be identified with the kernel of the natural split epimorphism $H^*_{\mathcal{M}}(X \times \mathbb{P}^1_F, C^*) \to H^*_{\mathcal{M}}(X, C^*)$ induced by the closed embedding $X = X \times 0 \hookrightarrow X \times \mathbb{P}^1_F$. Since motivic cohomology groups are homotopy invariant $H^*_{\mathcal{M}}(X \times \mathbb{P}^1_F/X, C^*)$ may be also identified with the kernel of the canonical homomorphism $H^*_{\mathcal{M}}(X \times \mathbb{P}^1_F, C^*) \to H^*_{\mathcal{M}}(X \times \mathbb{A}^1_F, C^*)$, where we identify $\mathbb{A}^1_F$ with the open subscheme $\mathbb{P}^1_F \setminus \infty$.

Proposition 8.2. *For any $X \in Sm/F$ multiplication by $\beta \in H^2_{\mathcal{M}}(\mathbb{P}^1_F/Spec\ F, B_l(1))$ defines isomorphisms*

$$H^{i-2}_{\mathcal{M}}(X, B_l(n-1)) \xrightarrow[\sim]{\beta} H^i_{\mathcal{M}}(X \times \mathbb{P}^1_F/X, B_l(n)).$$

Proof. The same construction as in § 7 defines natural pairings

$$\tau_{\leq k-1}\pi_* I^\bullet(n-1) \otimes \tau_{\leq 1}\pi_* I^\bullet(1) \longrightarrow \tau_{\leq k}\pi_* I^\bullet(n).$$

We'll show more generally that multiplication by β induces isomorphisms

$$H^{i-2}_{\mathcal{M}}(X, \tau_{\leq k-1}\pi_* I^\bullet(n-1)) \xrightarrow[\sim]{\beta} H^i_{\mathcal{M}}(X \times \mathbb{P}^1_F/X, \tau_{\leq k}\pi_* I^\bullet(n))$$

for all $k \geq 0$. Denote by $\mathcal{H}^k(n)$ the k-th cohomology sheaf of the complex $\pi_* I^\bullet(n)$. Thus $\mathcal{H}^k(n)$ is the Zariski sheaf associated to the presheaf $U \mapsto H^k_{et}(U, \mu_l^{\otimes n})$ ($\mathcal{H}^k(n) = 0$ for $k > n$). Proceeding by induction on k it suffices to show that for each k the multiplication by $\beta \in H^1_{Zar}(\mathbb{P}^1_F, \mathcal{H}^1(1))$ induces isomorphisms

$$H^{i-1}_{Zar}(X, \mathcal{H}^{k-1}(n-1)) \xrightarrow[\sim]{\beta} H^i_{Zar}(X \times \mathbb{P}^1_F/X, \mathcal{H}^k(n)).$$

Zariski cohomology of any smooth scheme Y with coefficients in $\mathcal{H}^k(n)$ may be computed using the Gersten complex

$$\bigoplus_{y \in Y^0} H^k_{et}(F(y), \mu_l^{\otimes n}) \to \bigoplus_{y \in Y^1} H^{k-1}_{et}(F(y), \mu_l^{\otimes n-1} \to \ \ldots$$

Remark 8.1.2 shows further that $H^i_{Zar}(X \times \mathbb{P}^1_F/X, \mathcal{H}^k(n))$ may be computed using the Gersten bicomplex

$$\begin{array}{ccccc}
\bigoplus_{y \in (X \times \mathbb{P}^1_F)^0} H^k_{et}(F(y), \mu_l^{\otimes n}) & \longrightarrow & \bigoplus_{y \in (X \times \mathbb{P}^1_F)^1} H^{k-1}_{et}(F(y), \mu_l^{\otimes (n-1)}) & \longrightarrow \\
\downarrow & & \downarrow & \\
\bigoplus_{y \in (X \times \mathbb{A}^1_F)^0} H^k_{et}(F(y), \mu_l^{\otimes n}) & \longrightarrow & \bigoplus_{y \in (X \times \mathbb{A}^1_F)^1} H^{k-1}_{et}(F(y), \mu_l^{\otimes (n-1)}) & \longrightarrow
\end{array}$$

since the vertical arrows of the above bicomplex are split surjective, this bicomplex is naturally quasiiisomorphic to the kernel of the vertical homomorphism of Gersten complexes, which coincides with the Gersten complex

$$\bigoplus_{x\in X^0} H^{k-1}_{et}(F(x),\mu_l^{\otimes(n-1)}) \to \bigoplus_{x\in X^1} H^{k-2}_{et}(F(x),\mu_l^{\otimes(n-2)})$$

shifted to the right by 1. This gives us the desired isomorphisms

$$H^i_{Zar}(X\times\mathbb{P}^1_F/X,\mathcal{H}^k(n)) = H^{i-1}_{Zar}(X,\mathcal{H}^{k-1}(n-1))$$

Moreover it's easy to see that this isomorphism coincides (up to a sign) with multiplication by β.

Corollary 8.3. *For any $Z \in Sm/F$ we have natural izomorphisms $H^*_{\mathcal{M}}(Z(d),B_l(n)) = H^*_{\mathcal{M}}(Z,B_l(n-d))$. Moreover (assuming that resolution of singularities holds over F) the following diagram commutes*

$$\begin{array}{ccc} H^*_{\mathcal{M}}(Z,\mathbb{Z}/l(n-d)) & \xrightarrow{\sim} & H^*_{\mathcal{M}}(Z(d),\mathbb{Z}/l(n)) \\ {\scriptstyle (\alpha_{n-d})_*}\downarrow & & {\scriptstyle (\alpha_n)_*}\downarrow \\ H^*_{\mathcal{M}}(Z,B_l(n-d)) & \xrightarrow{\sim} & H^*_{\mathcal{M}}(Z(d),B_l(n)) \end{array}$$

Proof. Proceeding by induction on d it suffices to consider the case $d=1$. The motiv $Z(1)$ may be identified with $Z\times\mathbb{P}^1_F/Z[-2]$. Proposition 8.2 shows that multiplication by β gives the desired isomorphism. Moreover Corollary 4.12.1 shows that multiplication by $\tau \in H^2_{\mathcal{M}}(\mathbb{P}^1,\mathbb{Z}(1))$ gives an isomorphism in motivic cohomology with coefficients in $\mathbb{Z}/l(-)$. Compatibility of products in motivic cohomology with $\mathbb{Z}/l(-)$ and $B_l(-)$ coefficients together with the fact that β coincides with the image of τ under the homomorphism $\mathbb{Z}(1)\to\mathbb{Z}/l(1)=B_l(1)$ proves the commutativity of the above diagram.

Corollary 8.4. *Let $X \in Sm/F$ be a smooth irreducible scheme and let $Z \neq X$ be a smooth irreducible closed subscheme. Then the map in motivic cohomology with supports in Z*

$$H^*_Z(X,\mathbb{Z}/l(n)) \to H^*_Z(X,B_l(n))$$

is an isomorphism.

Proof. Recall that motivic cohomology of X with supports in Z is defined as the motivic cihimilogy of the cone $M_Z(X) = (M(X)\to M(U))$ $(U = X\backslash Z)$. Recall also that in case Z is smooth everywhere of codimension d the motiv $M_Z(X)$ is naturally isomorphic to $M(Z)(d)[2d]$ - cf § 4. Thus our statement follows immediately from Corollary 8.3 (and the induction hypothesis).

Using now Theorem 5.6 we get finally the following result.

Theorem 8.5. *Let $X \in Sch/F$ be a scheme and let $Z \subset X$ be a closed subscheme such that $U = X\setminus Z$ is strictly dense in X. Then the natural map in motivic cohomology with supports in Z*

$$(\alpha_n)_* : H^*_Z(X,\mathbb{Z}/l(n)) \to H^*_Z(X,B_l(n))$$

is an isomorphism.

§ 9. Shift of degrees

Let $\partial\Delta^n$ denote the closed subscheme of Δ^n, given by the equation $t_0 \cdot \ldots \cdot t_n = 0$, in other words $\partial\Delta^n$ is the union of all codimension one faces of Δ^n. Usually we'll identify the scheme Δ^{n-1} with the face $t_n = 0$ of Δ^n and denote the union of all other faces by $\sigma\Delta^n$. Thus $\sigma\Delta^n$ is a closed subscheme of Δ^n given by the equation $t_0 \cdot \ldots \cdot t_{n-1} = 0$. Obviously

$$\partial\Delta^n = \Delta^{n-1} \cup \sigma\Delta^n \quad \text{and} \quad \Delta^{n-1} \cap \sigma\Delta^n = \partial\Delta^{n-1}.$$

Lemma 9.1. *The scheme $\sigma\Delta^n$ is algebraically contractible.*

Proof. The linear contraction of Δ^n to the vertex $v_n = (0, ..., 0, 1)$

$$\Delta^n \times \mathbb{A}^1 \to \Delta^n \qquad (t_0, ..., t_n) \times t \mapsto t \cdot (t_0, ..., t_n) + (1 - t) \cdot (0, ..., 0, 1)$$

obviously maps $\sigma\Delta^n \times \mathbb{A}^1$ to $\sigma\Delta^n$ and defines the desired contraction.

Let $X \in Sch/F$ be a scheme of finite type over F. Each of the vertices $v_i \in \partial\Delta^n$ defines a natural morphism $M(X) \xrightarrow{(1_X \times v_i)_*} M(X \times \partial\Delta^n)$ which is right inverse to the morphism defined by the projection $M(X \times \partial\Delta^n) \xrightarrow{(p_1)_*} M(X)$. Thus the choice of the vertex v_i allows us to identify $M(X)$ with a direct summand in $M(X \times \partial\Delta^n)$. In case $n > 1$ this identification is really independent of the choice of the vertex. In the case $n = 1$ we'll be using the vertex v_0 to make the above identification. We'll denote by $M(X \times \partial\Delta^n/X)$ the complimentary direct summand. The motivic cohomology groups $H^i_{\mathcal{M}}(X \times \partial\Delta^n/X, C^*)$ coincide with the kernel of the natural split epimorphism

$$H^i_{\mathcal{M}}(X \times \partial\Delta^n, C^*) \xrightarrow{(1_X \times v_0)^*} H^i_{\mathcal{M}}(X, C^*).$$

Lemma 9.2. *For any $X \in Sch/F$ any $n > 0$ and any $C^* \in DM^-(F)$ we have natural isomorphisms*

$$\delta^{n-1} : H^{i-(n-1)}_{\mathcal{M}}(X, C^*) \xrightarrow{\sim} H^i_{\mathcal{M}}(X \times \partial\Delta^n/X, C^*).$$

Proof. We proceed by induction on n. The case $n = 1$ is trivial, so assume that $n > 1$. The closed covering $X \times \partial\Delta^n = X \times \Delta^{n-1} \cup X \times \sigma\Delta^n$ gives us a long exact Mayer-Vietoris sequence, which (since the schemes Δ^{n-1} and $\sigma\Delta^n$ are contractible) looks as follows (all motivic cohomology groups are taken with coefficients in C^*):

$$H^{i-1}_{\mathcal{M}}(X \times \partial\Delta^n) \to H^{i-1}_{\mathcal{M}}(X) \oplus H^{i-1}_{\mathcal{M}}(X) \to H^{i-1}_{\mathcal{M}}(X \times \partial\Delta^{n-1}) \xrightarrow{\delta} H^i_{\mathcal{M}}(X \times \partial\Delta^n)$$

One checks easily that the image of the homomorphism

$$H^{i-1}_{\mathcal{M}}(X \times \partial\Delta^n) \to H^{i-1}_{\mathcal{M}}(X) \oplus H^{i-1}_{\mathcal{M}}(X)$$

consists of all elements of the form $y \oplus (-y) \quad y \in H^{i-1}_{\mathcal{M}}(X)$ and hence the above exact sequence gives rise to the following four term exact sequence

$$0 \to H^{i-1}_{\mathcal{M}}(X) \xrightarrow{(p_1)^*} H^{i-1}_{\mathcal{M}}(X \times \partial\Delta^{n-1}) \xrightarrow{\delta} H^i_{\mathcal{M}}(X \times \partial\Delta^n) \xrightarrow{(1_X \times v_0)^*} H^i_{\mathcal{M}}(X) \to 0$$

Thus the homomorphism δ defines an isomorphism

$$\delta : H^{i-1}_{\mathcal{M}}(X \times \partial\Delta^{n-1}/X) \xrightarrow{\sim} H^i_{\mathcal{M}}(X \times \partial\Delta^n/X)$$

which concludes the proof.

Proposition 9.3. *In conditions and notations of Theorem 7.4 the induced homomorphism in motivic cohomology*

$$(\alpha_n)_* : H^i_{\mathcal{M}}(E, \mathbb{Z}/l(n)) \to H^i_{\mathcal{M}}(E, B_l(n)) = H^i_{et}(E, \mu_l^{\otimes n})$$

is surjective for any finitely generated field extension E/F.

Proof. In the most important case when l is prime this is trivial - see Lemma 7.5. In the general case one has to use a more subtle arguement. Lemma 9.2 shows that motivic cohomology groups in question may be identified with $H^n_{\mathcal{M}}(\partial\Delta_E^{n-i+1}, \mathbb{Z}/l(n))$ and $H^n_{\mathcal{M}}(\partial\Delta_E^{n-i+1}, B_l(n))$ respectively. For any open subscheme $U \subset \partial\Delta_E^{n-i+1}$ containing all the vertices v_i we have an exact sequence of motivic cohomology groups (both with $\mathbb{Z}/l(n)$ and $B_l(n)$-coefficients)

$$H^n_T(\partial\Delta_E^{n-i+1}) \to H^n_{\mathcal{M}}(\partial\Delta_E^{n-i+1}) \to H^n_{\mathcal{M}}(U) \to H^{n+1}_T(\partial\Delta_E^{n-i+1})$$

where we denoted by T the closed subscheme $\partial\Delta_E^{n-i+1} \setminus U$. Taking direct limits of these sequences over all U's we get the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
H^n_{\mathbb{T}}(\partial\Delta_E^{n-i+1}) & \longrightarrow & H^n_{\mathcal{M}}(\partial\Delta_E^{n-i+1}) & \longrightarrow & H^n_{\mathcal{M}}(U) & \longrightarrow & H^{n+1}_{\mathbb{T}}(\partial\Delta_E^{n-i+1}) \\
\cong\big\downarrow & & \big\downarrow & & \text{epi}\big\downarrow & & \cong\big\downarrow \\
H^n_{\mathbb{T}}(\partial\Delta_E^{n-i+1}) & \longrightarrow & H^n_{\mathcal{M}}(\partial\Delta_E^{n-i+1}) & \longrightarrow & H^n_{\mathcal{M}}(U) & \longrightarrow & H^{n+1}_{\mathbb{T}}(\partial\Delta_E^{n-i+1})
\end{array}$$

Here the top row motivic cohomology are taken with $\mathbb{Z}/l(n)$-coefficients, bottom row motivic cohomology are taken with $B_l(n)$-coefficients, U denotes the semilocalization of the scheme $\partial\Delta_E^{n-i+1}$ at the points v_i and $H^n_{\mathbb{T}}(\partial\Delta_E^{n-i+1})$ stands for

$$\varinjlim_T H^n_T(\partial\Delta_E^{n-i+1})$$

where direct limit is taken over all closed reduced subschemes containing no vertices. Note that the first and the last vertical maps in the above diagram are isomorphisms according to Theorem 8.5 (one checks easily that every open subscheme of $\partial\Delta_E^{n-i+1}$ containing all vertices is strictly dense). Finally the second from the right vertical map is surjective according to Proposition 7.6. An easy diagram chase shows that the homomorphism $H^n_{\mathcal{M}}(\partial\Delta_E^{n-i+1}, \mathbb{Z}/l(n)) \to H^n_{\mathcal{M}}(\partial\Delta_E^{n-i+1}, B_l(n))$ is surjective as well.

Lemma 9.2 gives us, in particular, for all $i < n$ and all finitely generated field extensions E/F a natural isomorphism

$$\delta^{n-i} : H^i_{\mathcal{M}}(E, \mathbb{Z}/l(n)) \xrightarrow{\sim} H^n_{\mathcal{M}}(\partial\Delta_E^{n-i+1}/SpecE, \mathbb{Z}/l(n)) \subset H^n_{\mathcal{M}}(\partial\Delta_E^{n-i+1}, \mathbb{Z}/l(n)).$$

We'll need below an explicit formula for this isomorphism.

Note first of all that for any $X \in Sch/F$ and any complex $C^* \in DM^-(F)$ we have a natural homomorphism $H^i((C^*)^{\sim}_{cdh}(X)) \to H^i_{\mathcal{M}}(X, C^*)$ which in certain cases (in the case of fields for example) is an isomorphism. We'll show, in particular, that the map $\delta^{n-i} : H^i_{\mathcal{M}}(E, \mathbb{Z}/l(n)) \to H^n_{\mathcal{M}}(\partial\Delta^{n-i+1}_E, \mathbb{Z}/l(n))$ factors through

$$H^n(\mathbb{Z}/l(n)^{\sim}_{cdh}(\partial\Delta^{n-i+1}_E)) =$$
$$= \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})^{\sim}_{cdh}(\partial\Delta^{n-i+1}_E)/\operatorname{Im}(C_1(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})^{\sim}_{cdh}(\partial\Delta^{n-i+1}_E)).$$

Our reasoning applies more generally to any complex of the form $C^*(\mathcal{F})[-n]$. The motivic cohomology group $H^i_{\mathcal{M}}(E, C^*(\mathcal{F})[-n])$ coincides with the $(n-i)$-th homology group of the simplicial abelian group $\mathcal{F}(\Delta^{\bullet}_E)$ and may be computed using the corressponding Moore complex, i.e it may be identified with an appropriate quotient of $\mathcal{F}(\Delta^{n-i}_E)_0 = \{s \in \mathcal{F}(\Delta^{n-i}_E) : \partial_j(s) = 0 \quad (0 \le j \le n-i)\}$. Denote by $p : \Delta^{n-i} \xrightarrow{\sim} \Delta^{n-i}$ an automorphism of Δ^{n-i} given by the formula $(t_0, ..., t_{n-i}) \mapsto (t_{n-i}, ..., t_0)$. The section $p^*(s)$ (the same as s) vanishes on all faces of Δ^{n-i}_E and hence may be glued together with the zero section on $\sigma\Delta^{n-i+1}_E$ to give a section $\lambda(s)$ of $\mathcal{F}^{\sim}_{cdh}$ over $\partial\Delta^{n-i+1}_E$. In this way we get a homomorphism

$$\mathcal{F}(\Delta^{n-i}_E)_0 \xrightarrow{\lambda} \mathcal{F}^{\sim}_{cdh}(\partial\Delta^{n-i+1}_E) \xrightarrow{\text{can}} H^n((C^*(\mathcal{F})[-n])^{\sim}_{cdh}(\partial\Delta^{n-i+1}_E)) \xrightarrow{\text{can}}$$
$$\xrightarrow{\text{can}} H^n_{\mathcal{M}}(\partial\Delta^{n-i+1}_E, C^*(\mathcal{F})[-n]).$$

Lemma 9.4. *The natural isomorphism*

$$\delta^{n-i} : H^i_{\mathcal{M}}(E, C^*(\mathcal{F})[-n]) \xrightarrow{\sim} H^n_{\mathcal{M}}(\partial\Delta^{n-i+1}_E/SpecE, C^*(\mathcal{F})[-n])) \subset$$
$$\subset H^n_{\mathcal{M}}(\partial\Delta^{n-i+1}_E, C^*(\mathcal{F})[-n])$$

takes the homology class of $s \in \mathcal{F}(\Delta^{n-i}_E)_0$ to the canonical image of $\lambda(s) \in \mathcal{F}^{\sim}_{cdh}(\partial\Delta^{n-i+1}_E)$ in $H^n_{\mathcal{M}}(\partial\Delta^{n-i+1}_E, C^*(\mathcal{F})[-n])$ (possibly up to a sign).

Proof. For each $0 \le j \le n-i$ we define a $(j+i)$-dimentional cocycle s^j of the complex $(C^*(\mathcal{F})[-n])^{\sim}_{cdh}(\partial\Delta^{j+1}_E)$, i.e. a $n-(j+i)$ dimensional cycle of the complex $(C_*(\mathcal{F}))^{\sim}_{cdh}(\partial\Delta^{j+1}_E)$ so that the following properties hold

1. $s^0|_{v_0} = 0 \quad s^0|_{v_1} = s$
2. $\delta(s^j) = \pm s^{j+1}$
3. $s^{n-i} = \lambda(s)$

Define a morphism $q_j : \Delta^{n-i-j} \times \Delta^{j+1} \to \Delta^{n-i}$ via the formula

$$(t_0, ..., t_{n-i-j}) \times (t'_0, ..., t'_{j+1}) \mapsto t'_0(t_0, ..., t_{n-i-j}, 0, ..., 0) + t'_{j+1}v_{n-i-j} + ... + t'_1 v_{n-i}$$

and let $p_j : \Delta^{n-i-j} \times \Delta^j \to \Delta^{n-i}$ be the restriction of q_j. One checks easily that p_j maps all faces of $\Delta^{n-i-j} \times \Delta^j$ to faces of Δ^{n-i} . This implies, in particular, that $(p_j)^*(s) \in C_{n-i-j}(\mathcal{F})(\Delta^j_E)$ vanishes on all faces of Δ^j_E and hence may be glued together (along $\partial\Delta^j_E$) with the zero section to give a section s^j of $C_{n-i-j}(\mathcal{F})^{\sim}_{cdh}$

over $\partial\Delta_E^{j+1}$. Moreover all faces of s^j are equal to zero and hence s^j is a cycle. To compute $\delta(s^j)$ we note that s^j extends to a section $q_j^*(s)$ of $C_{n-i-j}(\mathcal{F})$ over Δ_E^{j+1} and hence $\delta(s^j)$ is represented by the cycle in $C_{n-i-j-1}(\mathcal{F})(\partial\Delta_E^{j+2})$ whose restriction to Δ_E^{j+1} is equal to $\sum_{k=0}^{n-i-j-1}(-1)^k\partial_k(q_j^*(s))$ and whose restriction to other faces of $\partial\Delta_E^{j+2}$ is zero. Finally an immediate verification shows that

$$\partial_k(q_j^*(s)) = \begin{cases} 0 & \text{if } k \neq n-i-j \\ p_{j+1}^*(s) & \text{if } k = n-i-j \end{cases}$$

For any presheaf $\mathcal{F}$ define a new presheaf $\widetilde{C}_1(\mathcal{F})$ via the formula

$$\widetilde{C}_1(\mathcal{F})(X) = \varinjlim_{X\times\{0,1\}\subset U\subset X\times\mathbb{A}^1} \mathcal{F}(U) \qquad (X \in Sm/F)$$

Here the direct limit is taken over all open subschemes $U \subset X \times \mathbb{A}^1$ which contain $X \times \{0,1\}$. The presheaf $\widetilde{C}_1(\mathcal{F})(X)$ is equipped with two obvious homomorphisms

$$\widetilde{C}_1(\mathcal{F}) \underset{i_1^*}{\overset{i_0^*}{\rightrightarrows}} \mathcal{F}$$

Moreover a straightforward verification shows that if $\mathcal{F}$ is a presheaf with transfers, then $\widetilde{C}_1(\mathcal{F})$ also has a natural structure of a presheaf with transfers and the homomorphisms i_0^*, i_1^* are compatible with transfers.

Definition 9.5. *We'll be saying that a presheaf with transfers $\mathcal{F}$ is rationally contractible iff there exists a homomorphism of presheaves with transfers $s : \mathcal{F} \to \widetilde{C}_1(\mathcal{F})$, such that $i_0^*s = 0, i_1^*s = Id$.*

Proposition 9.6. *The presheaves with transfers $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})$, $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})/l$ are rationally contractible.*

Proof. Obviously it suffices to establish rational contractibility of the sheaf $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})$. This follows easily from the fact that the scheme $\mathbb{G}_m^{\times n}$,being an open subscheme in a contractible scheme $\mathbb{A}^n$, is rationally contractible. To be more precise consider a morphism of schemes

$$\mathbb{G}_m^{\times n} \times \mathbb{A}^1 \xrightarrow{\psi} \mathbb{A}^n : \quad (t_1, ..., t_n) \times t \mapsto t \cdot (t_1, ..., t_n) + (1-t) \cdot (1, ..., 1)$$

and set $W = \psi^{-1}(\mathbb{G}_m^{\times n}), T = \mathbb{G}_m^{\times n} \times \mathbb{A}^1 \setminus W$. Thus W is an open subscheme of $\mathbb{G}_m^{\times n} \times \mathbb{A}^1$, containing $\mathbb{G}_m^{\times n} \times \{0,1\}$ and ψ defines a morphism $\psi : W \to \mathbb{G}_m^{\times n}$ such that

$$\psi|_{\mathbb{G}_m^{\times n}\times 1} = Id, \qquad \psi|_{\mathbb{G}_m^{\times n}\times 0} = (1, ..., 1)$$

Let $X \in Sm/F$ be a smooth irreducibla scheme and let $Z \in \mathbb{Z}_{tr}(\mathbb{G}_m^{\times n})(X)$ be a closed integral subscheme of $X \times \mathbb{G}_m^{\times n}$ finite and surjective over X. The closed integral subscheme

$$Z \times \mathbb{A}^1 = Z \times \Delta_{\mathbb{A}^1} \subset (X \times \mathbb{A}^1) \times (\mathbb{G}_m^{\times n} \times \mathbb{A}^1)$$

defines an element in $\mathbb{Z}_{tr}(\mathbb{G}_m^{\times n} \times \mathbb{A}^1)(X \times \mathbb{A}^1)$. Set

$$U_Z = X \times \mathbb{A}^1 \setminus pr_{X \times \mathbb{A}^1}(Z \times \mathbb{A}^1 \cap ((X \times \mathbb{A}^1) \times T)$$

The scheme U_Z is an open subscheme of $X \times \mathbb{A}^1$, containing $X \times \{0, 1\}$. Moreover

$$Z \times \mathbb{A}^1 \cap (U_Z \times (\mathbb{G}_m^{\times n} \times \mathbb{A}^1)) \subset U_Z \times W$$

and this closed integral subschem defines an element in $\mathbb{Z}_{tr}(W)(U_Z)$. Applying to this element the homomorphism $\mathbb{Z}_{tr}(\psi) : \mathbb{Z}_{tr}(W) \to \mathbb{Z}_{tr}(\mathbb{G}_m^{\times n})$ we get an element $s(Z) \in \mathbb{Z}_{tr}(\mathbb{G}_m^{\times n})(U_Z)$. A straightforward verification shows that in this way we get a homomorphism of presheaves with transfers $s : \mathbb{Z}_{tr}(\mathbb{G}_m^{\times n}) \to \widetilde{C}_1(\mathbb{Z}_{tr}(\mathbb{G}_m^{\times n}))$. Finally the composition $i_1^* s$ is the identity map, whereas the composition $i_0^* s : \mathbb{Z}_{tr}(\mathbb{G}_m^{\times n}) \to \mathbb{Z}_{tr}(\mathbb{G}_m^{\times n})$ is induced by the constant morphism $\mathbb{G}_m^{\times n} \to \mathbb{G}_m^{\times n} : (t_1, ..., t_n) \mapsto (1, ..., 1)$. This implies that the factor sheaf $\mathbb{Z}_{tr}(\mathbb{G}_m^{\times n})/\mathbb{Z}_{tr}((1, ..., 1))$ is rationally contractible and hence its direct summand $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})$ is also rationally contractible.

Corollary 9.7. *For any $z \in H^i_{\mathcal{M}}(E, \mathbb{Z}/l(n))$ there exists an open subscheme $U \subset \partial\Delta_E^{n-i+1} \times \mathbb{A}^1$ containing $\partial\Delta_E^{n-i+1} \times \{0, 1\}$ and an element $y \in H^n_{\mathcal{M}}(U, \mathbb{Z}/l(n))$ such that $y|_{\partial\Delta_E^{n-i+1} \times 0} = 0, y|_{\partial\Delta_E^{n-i+1} \times 1} = \delta^{n-i}(z)$.*

Proof. In view of Lemma 9.4 it suffices to show that for any element $z \in (\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})/l)(\Delta^{n-i})_0$ there exists an open subscheme U as above and a section $y \in (\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})/l)^{\sim}_{cdh}(U)$ such that

$$y|\partial\Delta_E^{n-i+1} \times 0 = 0 \quad y|\Delta_E^{n-i} \times 1 = z \quad y|\sigma\Delta^{n-i+1} \times 1 = 0$$

According to Proposition 9.6 there exists an open subscheme $\Delta_E^{n-i} \times \{0, 1\} \subset V \subset \Delta_E^{n-i} \times \mathbb{A}^1$ and a section $s(z) \in (\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})/l)(V)$ such that

$$s(z)|_{\Delta_E^{n-i} \times 0} = 0 \quad s(z)|_{\Delta_E^{n-i} \times 1} = z$$

Moreover for any $0 \leq j \leq n - i$ the restriction of $s(z)$ to $V \cap \partial_j\Delta_E^{n-i}$ dies in the direct limit

$$\varinjlim_{\Delta_E^{n-i-1} \times \{0,1\} \subset W \subset \Delta_E^{n-i-1} \times \mathbb{A}^1} \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})/l(W)$$

Hence decreasing V if necessary we may assume that this restriction is trivial for all j. This shows immediately that ,denoting $\Delta_E^{n-i} \times \mathbb{A}^1 \setminus V$ by T, one may extend the section $s(z)$ to a section y of $(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})/l)^{\sim}_{cdh}$ over $\partial\Delta_E^{n-i+1} \times \mathbb{A}^1 \setminus T$ which is zero over $\sigma\Delta_E^{n-i+1} \times \mathbb{A}^1 \setminus T$.

§ 10. Proof of the main Theorem.

Denote by S the scheme $\mathbb{A}^1$ with points 0 and 1 identied. We denote by $pt \in S$ the unique singular point of S.

Lemma 10.1. *Let $X \in Sch/F$ be a scheme of finite type over F and let $U \subset X \times \mathbb{A}^1$ be an open subscheme containing $X \times \{0,1\}$. Denote by $\widetilde{U}$ the image of U in $X \times S$ under the canonical projection $X \times \mathbb{A}^1 \xrightarrow{p} X \times S$. Then $\widetilde{U}$ is an open subscheme in $X \times S$ and for any complex $C^* \in DM^-(F)$ we have a natural long exact sequence of motivic cohomology groups*

$$\ldots \longrightarrow H^n_{\mathcal{M}}(\widetilde{U}, C^*) \xrightarrow{p^*} H^n_{\mathcal{M}}(U, C^*) \xrightarrow{i_0^* - i_1^*} H^n_{\mathcal{M}}(X, C^*) \xrightarrow{\widetilde{\delta}} H^{n+1}_{\mathcal{M}}(\widetilde{U}, C^*) \longrightarrow \ldots$$

Proof. The first statement is trivial. To prove the second it suffices to note that the sequence of Nisnevich sheaves with transfers

$$0 \longrightarrow \mathbb{Z}_{tr}(X) \xrightarrow{(i_0)_* - (i_1)_*} \mathbb{Z}_{tr}(U) \xrightarrow{p_*} \mathbb{Z}_{tr}(\widetilde{U}) \longrightarrow 0$$

is exact (even as a sequence of presheaves).

Corollary 10.2. *For any $X \in Sch/F$ we have a natural isomorphism*

$$\widetilde{\delta} : H^n_{\mathcal{M}}(X, C^*) \xrightarrow{\sim} H^{n+1}_{\mathcal{M}}(X \times S/X, C^*) \subset H^{n+1}_{\mathcal{M}}(X \times S, C^*)$$

Lemma 8.2 and Corollary 10.2 give us (for any $i < n$ and any finitely generated field extension E/F) a sequence of split monomorphisms of motivic cohomology groups with coefficients in an arbitrary complex $C^* \in DM^-(F)$

$$H^i_{\mathcal{M}}(E) \xrightarrow{\delta^{n-i}} H^n_{\mathcal{M}}(\partial\Delta_E^{n-i+1}) \xrightarrow{\widetilde{\delta}} H^{n+1}_{\mathcal{M}}(\partial\Delta_E^{n-i+1} \times S)$$

We denote the composition $\widetilde{\delta}\delta^{n-i}$ of the above split monomorphisms by ξ.

Proposition 10.3. *For any $z \in H^i_{\mathcal{M}}(E, \mathbb{Z}/l(n))$ there exists an open subscheme $\widetilde{U} \subset \partial\Delta_E^{n-i+1} \times S$, containing $\partial\Delta_E^{n-i-1} \times pt$ such that the restriction of $\xi(z) \in H^{n+1}_{\mathcal{M}}(\partial\Delta_E^{n-i+1} \times S, \mathbb{Z}/l(n))$ to $\widetilde{U}$ is zero.*

Proof. Let $X \times \{0,1\} \subset U \subset X \times \mathbb{A}^1$, $y \in H^n_{\mathcal{M}}(U, \mathbb{Z}/l(n))$ be an open subscheme and a motivic cohomology class such that $i_0^*(y) = 0$, $i_1^*(y) = \delta^{n-i}(z)$ - see Corollary 8.6. Let further $\widetilde{U}$ be the image of U in $\partial\Delta_E^{n-i+1} \times S$. According to Lemma 10.1 we have a commutative diagram with exact rows (all motivic cohomology groups are taken with $\mathbb{Z}/l(n)$ coefficients)

$$\begin{array}{ccccc}
0 & \longrightarrow & H^n_{\mathcal{M}}(\partial\Delta_E^{n-i+1}) & \xrightarrow{\widetilde{\delta}} & H^{n+1}_{\mathcal{M}}(\partial\Delta_E^{n-i+1} \times S) \\
\downarrow & & \downarrow{=} & & \downarrow \\
H^n_{\mathcal{M}}(U) & \xrightarrow{i_0^* - i_1^*} & H^n_{\mathcal{M}}(\partial\Delta_E^{n-i+1}) & \xrightarrow{\widetilde{\delta}} & H^{n+1}_{\mathcal{M}}(\widetilde{U})
\end{array}$$

The above diagram shows that the image of $\xi(z) = \widetilde{\delta}\delta^{n-i}(z)$ in $H^{n+1}_{\mathcal{M}}(\widetilde{U})$ is zero.

We lieve the proof of the following Lemma as an easy exercise to the reader.

Lemma 10.4. *Every open subscheme of* $\partial\Delta^{n-i+1}\times S$*, which contains all vertices* $v_i \times pt \quad (0 \le i \le n)$ *is strictly dense.*

Proof of the Theorem 7.4

According to Lemma 6.9 and Proposition 9.3 it suffices to show that for any $i \le n$ and any finitely generated field extension E/F the map in motivic cohomology

$$(\alpha_n)_* : H^i_{\mathcal{M}}(E, \mathbb{Z}/l(n)) \to H^i_{\mathcal{M}}(E, B_l(n))$$

is injective. Note further that according to Lemma 9.2 and Corollary 10.2 the motivic cohomology groups in question inject canonically to $H^{n+1}_{\mathcal{M}}(\partial\Delta^{n-i+1}_E \times S, \mathbb{Z}/l(n))$ and $H^{n+1}_{\mathcal{M}}(\partial\Delta^{n-i+1}_E \times S, B_l(n))$ respectively. For any open subscheme $U \subset \partial\Delta^{n-i+1}_E \times S$ containing all the vertices $v_i \times pt$ we have an exact sequence of motivic cohomology groups (both with $\mathbb{Z}/l(n)$ and $B_l(n)$-coefficients)

$$H^n_{\mathcal{M}}(U) \to H^{n+1}_T(\partial\Delta^{n-i+1}_E \times S) \to H^{n+1}(\partial\Delta^{n-i+1}_E \times S) \to H^{n+1}_{\mathcal{M}}(U)$$

where we denoted by T the closed subscheme $\partial\Delta^{n-i+1}_E \times S \setminus U$. Taking direct limits of these sequences over all U's we get the following commutative diagram with exact rows

$$\begin{array}{ccccc}
H^n_{\mathcal{M}}(U, \mathbb{Z}/l(n)) & \longrightarrow & \varinjlim_T H^{n+1}_T(\partial\Delta^{n-i+1}_E \times S, \mathbb{Z}/l(n)) & \longrightarrow & \\
{\scriptstyle(\alpha_n)_*}\downarrow & & {\scriptstyle(\alpha_n)_*}\downarrow & & \\
H^n_{\mathcal{M}}(U, B_l(n)(n)) & \longrightarrow & \varinjlim_T H^{n+1}_T(\partial\Delta^{n-i+1}_E \times S, B_l(n)) & \longrightarrow &
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & H^{n+1}(\partial\Delta^{n-i+1}_E \times S, \mathbb{Z}/l(n)) & \longrightarrow & H^{n+1}_{\mathcal{M}}(U, \mathbb{Z}/l(n)) \\
& {\scriptstyle(\alpha_n)_*}\downarrow & & {\scriptstyle(\alpha_n)_*}\downarrow \\
\longrightarrow & H^{n+1}(\partial\Delta^{n-i+1}_E \times S, B_l(n)) & \longrightarrow & H^{n+1}_{\mathcal{M}}(U, B_l(n))
\end{array}$$

where the direct limit is taken over all closed subsets T of $\partial\Delta^{n-i+1}_E \times S$ containing no vertices and U this time denotes the semilocalization of $\partial\Delta^{n-i+1}_E \times S$ at the points $v_i \times pt$. Note finally that the first vertical arrow in the above diagram is surjective according to Proposition 7.6 and the second one is an isomorphism according to Theorem 8.5 and Lemma 10.4. An easy diagram chase shows that the intersection of the kernels of homomorphisms

$$H^{n+1}(\partial\Delta^{n-i+1}_E \times S, \mathbb{Z}/l(n)) \to H^{n+1}_{\mathcal{M}}(U, \mathbb{Z}/l(n))$$
$$H^{n+1}(\partial\Delta^{n-i+1}_E \times S, \mathbb{Z}/l(n)) \to H^{n+1}(\partial\Delta^{n-i+1}_E \times S, B_l(n))$$

is trivial. For any motivic cohomology class $z \in H^i_{\mathcal{M}}(E, \mathbb{Z}/l(n))$ its image $\xi(z) \in H^{n+1}(\partial\Delta^{n-i+1}_E \times S, \mathbb{Z}/l(n))$ dies in $H^{n+1}_{\mathcal{M}}(U, \mathbb{Z}/l(n))$ according to Proposition 10.3. Thus if z dies in $H^i_{\mathcal{M}}(E, B_l(n))$ (in which case $\xi(z)$ dies in $H^{n+1}(\partial\Delta^{n-i+1}_E \times S, B_l(n))$) then $\xi(z)$ is zero and hence z itself is zero.

§ 11. Bloch-Kato conjecture and vanishing of the Bockstein homomorphisms.

In this section we fix a prime integer l different from $charF$. Note that validity of the Weak Bloch-Kato Conjecture for a field F (modulo l) implies, in particular, that all homomorphisms $H^n_{et}(F, \mu_{l^{k+1}}^{\otimes n}) \to H^n_{et}(F, \mu_l^{\otimes n})$ are surjective and hence all the Bockstein homomorphisms $\beta_{1,k} : H^n_{et}(F, \mu_l^{\otimes n}) \to H^{n+1}_{et}(F, \mu_{l^k}^{\otimes n})$ corresponding to the short exact sequences of etale sheaves $0 \to \mu_{l^k}^{\otimes n} \to \mu_{l^{k+1}}^{\otimes n} \to \mu_l^{\otimes n} \to 0$ are trivial. The aime of this section is to show that vice versa vanishing of the above Bockstein homomorphisms implies the validity of the Bloch-Kato Conjecture (and hence of the Beilinson-Lichtenbaum Conjecture as well). In the special case $l = 2$ such kind of statement was proved previously by A. Merkurjev [Me] using very different technique. We start with the following elementary Lemma.

Lemma 11.1. *The following conditions are equivalent*

1. *All Bockstein homomorphisms* $\beta_{1,k} : H^n_{et}(F, \mu_l^{\otimes n}) \to H^{n+1}_{et}(F, \mu_{l^k}^{\otimes n})$ *are trivial.*
2. *All homomorphisms* $H^n_{et}(F, \mu_{l^{k+1}}^{\otimes n}) \to H^n_{et}(F, \mu_l^{\otimes n})$ *are surjective.*
3. *All Bockstein homomorphisms* $\beta_{k,1} : H^n_{et}(F, \mu_{l^k}^{\otimes n}) \to H^{n+1}_{et}(F, \mu_l^{\otimes n})$ *are trivial.*
4. *All homomorphisms* $H^n_{et}(F, \mu_{l^{k+1}}^{\otimes n}) \to H^n_{et}(F, \mu_{l^k}^{\otimes n})$ *are surjective.*
5. *All Bockstein homomorphisms* $\beta_{k,s} : H^n_{et}(F, \mu_{l^k}^{\otimes n}) \to H^{n+1}_{et}(F, \mu_{l^s}^{\otimes n})$ *are trivial.*
6. *The group* $H^n_{et}(F, \mathbb{Q}_l/\mathbb{Z}_l(n))$ *is* l*-divisible.*

Proof. The equivalence of (1) and (2), the same as equivalence of (3) and (4) is obvious. Furthermore (4) clearly implies (2) and (5) implies (3). To end up the proof we only have to prove the implications (2)⇒(6) and (6)⇒(5). To prove the first implication we note that the composition $H^n_{et}(F, \mu_{l^{k+1}}^{\otimes n}) \to H^n_{et}(F, \mu_l^{\otimes n}) \to H^n_{et}(F, \mu_{l^{k+1}}^{\otimes n})$ coincides with the multiplication by l^k. Thus (2) implies that the image of $H^n_{et}(F, \mu_l^{\otimes n})$ in $H^n_{et}(F, \mathbb{Q}_l/\mathbb{Z}_l(n))$ consists of infinitly l-divisible elements. Since this image is exactly the l-torsion subgroup of the l-primary group $H^n_{et}(F, \mathbb{Q}_l/\mathbb{Z}_l(n))$ we conclude easily that the latter group is l-divisible. Finally (6) implies vanishing of the Bockstein homomorphisms $\beta_{\infty,s}$ which implies the vanishing of all $\beta_{k,s}$ in view of the commutative diagram

$$\begin{array}{ccc} H^n_{et}(F, \mu_{l^k}^{\otimes n}) & \xrightarrow{\beta_{k,s}} & H^{n+1}_{et}(F, \mu_{l^s}^{\otimes n}) \\ \downarrow & & \downarrow = \\ H^n_{et}(F, \mathbb{Q}_l/\mathbb{Z}_l(n)) & \xrightarrow{\beta_{\infty,s}} & H^{n+1}_{et}(F, \mu_{l^s}^{\otimes n}) \end{array}$$

Lemma 11.2. *Assume that the field* F *is infinite. Let* $s \in H^n_{et}(F, \mathbb{Z}/l)$ $(n > 0)$ *be an etale cohomology class. Then there exists an integer* N*, an open subscheme* $U \subset \mathbb{A}^N$*, an etale Galois covering* $p : U \to V$*, an etale cohomology class* $s' \in H^n_{et}(V, \mathbb{Z}/l)$

and two rational points $v_0, v_1 \in V$ *such that* $v_0^*(s') = 0, v_1^*(s') = s$ *(and* v_0 *admits a rational lifting* u_0 *to* U*).*

Proof. Note first of all that for any Galois covering $p : U \to V$ with a Galois group G we have a natural homomorphism $\gamma_p : H^n(G, \mathbb{Z}/l) \to H^n(G, H^0_{et}(U, \mathbb{Z}/l)) \to H^n_{et}(V, \mathbb{Z}/l)$. We can always represent s as a cohomology class of the Galois group $G = Gal(E/F)$ of an appropriate Galois extension E/F, i.e. right s in the form $s = \gamma_{p_1}(s_0)$, where $p_1 : SpecE \to SpecF$ is a Galois covering with the group G and $s_0 \in H^n(G, \mathbb{Z}/l)$ is an appropriate cohomology class. Set $N = |G|$, let G act on $\mathbb{A}^N$ by permuting the coordinates and let $U \subset \mathbb{A}^N$ be the open subscheme consisting of points with all coordinates different. Thus the action of G on U is free and hence $p : U \to V = U/G$ is a Galois covering. Finally set $s' = \gamma_p(s_0)$. Let $u_0 \in U$ be any F-rational point and let $v_0 = p(u_0)$ be its image in V. The cohomology class $v_0^*(s')$ coincides with $\gamma_{p_0}(s_0)$, where $p_0 : U_0 = p^{-1}(v_0) \to SpecF$ is the induced Galois covering. To show that $v_0^*(s') = 0$ it suffices to note that the fiber $p^{-1}(v_0)$ consists of N different rational points hence $H^0_{et}(U_0, \mathbb{Z}/l)$ is the induced G module and $H^*(G, H^0_{et}(U_0, \mathbb{Z}/l)) = 0$ for $* > 0$. Finally choose a normal basis $e_g \quad (g \in G)$ for the extension E/F. The family $\{e_g\}_{g \in G}$ defines a point $u_1 \in U$ with the residue field $F(u_1) = E$, one checks immediately that this point is G-invariant (and the induced action of G on $F(u_1) = E$ is the original one). This shows that the image $p(u_1) = v_1$ is an F-rational point and the induced Galois covering $p_1 : U_1 = p^{-1}(v_1) \to SpecF$ coincides with what we denoted by p_1 above. Thus $v_1^*(s') = \gamma_{p_1}(s_0) = s$.

Proposition 11.3. *Assume that resolution of singularities holds over* F. *Assume further that the Beilinson-Lichtenbaum Conjecture (modulo* l*) holds over* F *in weights* $< n$. *Let finally* $U \subset \mathbb{A}^N$ *be an open subscheme and let* $u \in U$ *be an* F*-rational point. In this case the canonical homomorphisms*

$$(\alpha_{l^k})_* : H^*_{\mathcal{M}}(U/u, \mathbb{Z}/l^k(n)) \to H^*_{\mathcal{M}}(U/u, B_{l^k}(n))$$

are isomorphisms for all $k > 0$.

Proof. Set $Z = \mathbb{A}^N \setminus U$. One checks immediately that for any $C^\bullet \in DM^-(F)$ we have a natural isomorphism $H^*_{\mathcal{M}}(U/u, C^\bullet) = H^{*+1}_Z(\mathbb{A}^N, C^\bullet)$. Thus our statement follows from Theorem 8.5.

Theorem 11.4. *Assume that resolution of singularities holds over* F. *Assume further that the Bockstein homomorphisms* $\beta^i_{1,k} : H^i_{et}(E, \mu_l^{\otimes i}) \to H^{i+1}_{et}(E, \mu_{l^k}^{\otimes i})$ *are trivial for all* k, *all* $i \le n$ *and all finitely generated field extensions* E/F. *Then the Beilison-Lichtenbaum Conjecture holds over* F *in weights* $\le n$.

Proof. Proceeding by induction on n we may assume that Beilinson-Lichtenbaum Conjecture is known to be true over F in weights $< n$. Define the complex $B_{l^\infty}(n)$ the same way the complex $B_{l^k}(n)$ was defined in § 6. Thus $B_{l^\infty}(n) = \tau_{\le n}(\pi_*(J^\bullet))$ where this time $J^\bullet$ is the injective resolution of the h-sheaf $\mathbb{Q}_l/\mathbb{Z}_l(n) = \mu_{l^\infty}^{\otimes n} = \varinjlim_k \mu_{l^k}^{\otimes n}$. We first show that the homomorphism $\alpha_n : \mathbb{Q}_l/\mathbb{Z}_l(n) = \varinjlim_k \mathbb{Z}/l^k(n) = \mathbb{Z}(n) \otimes \mathbb{Q}_l/\mathbb{Z}_l \to B_{l^\infty}(n)$ is a quasiisomorphism. The proof repeats word by word that of the Theorem 7.4, provided we can show that for any finitely generated field extension E/F the homomorphism in motivic cohomology

$$H^n_{\mathcal{M}}(E, \mathbb{Q}_l/\mathbb{Z}_l(n)) \to H^n_{\mathcal{M}}(E, B_{l^\infty}(n)) = H^n_{et}(E, \mathbb{Q}_l/\mathbb{Z}_l(n))$$

is surjective. Note that both l-primary groups above are l-divisible: the one on the left coicides with $K^M_n(E) \otimes \mathbb{Q}_l/\mathbb{Z}_l$ and the one on the right is l-divisible according to our assumptions. Thus to prove the surjectivity of the above map it suffices to show that the elements of l-torsion in $H^n_{et}(E, \mathbb{Q}_l/\mathbb{Z}_l(n))$ admit a lifting to $H^n_{\mathcal{M}}(E, \mathbb{Q}_l/\mathbb{Z}_l(n))$. Note also that the l-torsion subgroup in $H^n_{et}(E, \mathbb{Q}_l/\mathbb{Z}_l(n))$ coincides with the image of $H^n_{et}(E, \mu_l^{\otimes n})$. As always we may assume that E contains a primitive l-th route of unity ξ (which allows to identify $H^n_{et}(E, \mu_l^{\otimes n})$ with $H^n_{et}(E, \mathbb{Z}/l))$ and obviously we may assume that E is infinite. Start with a cohomology class $s \in H^n_{et}(E, \mu_l^{\otimes n})$. Applying Lemma 12.2 we get an open subscheme $U \subset \mathbb{A}^N$ a Galois covering $p : U \to V$, a cohomology class $s' \in H^n_{et}(V, \mu_l^{\otimes n})$ and two rational points $v_0, v_1 \in V$ such that $v_0^*(s') = 0, v_1^*(s') = s$. Write N in the form $N = l^k M$ where M is prime to l. Surjectivity of the homomorphism $H^n_{et}(E, \mu_{l^{k+1}}^{\otimes n}) \to H^n_{et}(E, \mu_l^{\otimes n})$ for all field extensions E/F implies immediately that the same holds for any smooth semilocal scheme over F (this is another general property of homotopy invariant presheaves with transfers - see [V 1]). Thus decreasing if necessary V we may assume that s' may be lifted to a cohomology class $s'' \in H^n_{et}(V, \mu_{l^{k+1}}^{\otimes n})$. Let $u_0 \in U$ be any rational point over v_0. Applying Proposition 11.3 to the etale cohomology class $p^*(s'') - p^*(s'')(u_0) = p^*(s'') - s''(v_0) \in H^n_{et}(U/u_0, \mu_{l^{k+1}}^{\otimes n})$ we get a motivic cohomology class $t'' \in H^n_{\mathcal{M}}(U/u_0, \mathbb{Z}/l^{k+1}(n))$ whose image in $H^n_{et}(U, \mu_{l^{k+1}}^{\otimes n})$ equals $p^*(s'') - s''(u_0)$. Set $t' = p_*(t'') \in H^n_{\mathcal{M}}(V, \mathbb{Z}/l^{k+1}(n))$. The image of this motivic cohomology class in $H^n_{et}(V, \mu_{l^{k+1}}^{\otimes n})$ equals $N(s'' - s''(u_0))$ and hence equals M times the image of $s' - s'(u_0) = s' \in H^n(V, \mu_l^{\otimes n})$ in $H^n_{et}(V, \mu_{l^{k+1}}^{\otimes n})$. Setting finally $t = M' v_1^*(t')$, where $M'M \equiv 1 \mod l^{k+1}$ we get a motivic cohomology class $t \in H^n_{\mathcal{M}}(E, \mathbb{Z}/l^{k+1}(n))$ whose image in $H^n_{et}(E, \mu_{l^{k+1}}^{\otimes n})$ equals the image of s in this group. Thus the image of $H^n(E, \mu_l^{\otimes n})$ in $H^n_{et}(E, Q_l/\mathbb{Z}_l(n))$ can be lifted to $H^n_{\mathcal{M}}(E, \mathbb{Q}_l/\mathbb{Z}_l(n))$ and hence the homomorphism

$$H^n_{\mathcal{M}}(E, \mathbb{Q}_l/\mathbb{Z}_l(n)) \to H^n_{\mathcal{M}}(E, B_{l^\infty}(n)) = H^n_{et}(E, \mathbb{Q}_l/\mathbb{Z}_l(n))$$

is surjective. Finally to prove the Beilinson-Lichtenbaum Conjecture modulo l we use the Bockstein distinguished triangles in $DM^-(F)$

$$\begin{array}{ccccccc}
\mathbb{Z}/l(n) & \longrightarrow & \mathbb{Q}_l/\mathbb{Z}_l(n) & \xrightarrow{l} & \mathbb{Q}_l/\mathbb{Z}_l(n) & \longrightarrow & \mathbb{Z}/l(n)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B_l(n) & \longrightarrow & B_{l^\infty}(n) & \xrightarrow{l} & B_{l^\infty}(n) & \longrightarrow & B_l(n)[1]
\end{array}$$

- cf. the proof of the Lemma 6.10.

§ 12. Appendix: cdh-cohomological dimension of Noetherian schemes.

For any Noetherian scheme S we denote by Sch/S the category of separated schemes of finite type over S. One defines the cdh-topology on Sch/S in the same way it was done in § 5 in case $S = SpecF$. The proofs of Lemma 5.8 and Proposition 5.9 were given in such a way that they work without any changes over an arbitrary Noetherian base scheme S.

Lemma 12.1. *Let $p : Y \to X$ be a proper morphism (of separated schemes of finite type over S) and let $Z \subset X$ be a closed subscheme such that p is an isomorphism over $X \setminus Z$. In this case we have an exact sequence of cdh-sheaves*

$$0 \to \mathbb{Z}_{cdh}(p^{-1}(Z)) \longrightarrow \mathbb{Z}_{cdh}(Z) \oplus \mathbb{Z}_{cdh}(Y) \longrightarrow \mathbb{Z}_{cdh}(X) \to 0$$

and hence for any cdh-sheaf $\mathcal{F} \in (Sch/S)^{\sim}_{cdh}$ we get an exact sequence of cdh-cohomology groups

$$H^i_{cdh}(X, \mathcal{F}) \to H^i_{cdh}(Z, \mathcal{F}) \oplus H^i_{cdh}(Y, \mathcal{F}) \to H^i_{cdh}(p^{-1}(Z), \mathcal{F}) \to H^{i+1}_{cdh}(X, \mathcal{F}) \to \dots$$

In particular for any $X \in Sch/S$ we have a natural isomorphism

$$H^*_{cdh}(X, \mathcal{F}) = H^*_{cdh}(X_{red}, \mathcal{F})$$

and for any closed covering $X = Y \cup Z$ we have a long exact Mayer-Vietoris sequence

$$H^i_{cdh}(X, \mathcal{F}) \to H^i_{cdh}(Z, \mathcal{F}) \oplus H^i_{cdh}(Y, \mathcal{F}) \to H^i_{cdh}(Y \cap Z, \mathcal{F}) \to H^{i+1}_{cdh}(X, \mathcal{F}) \to \dots$$

Proof. Recall that the sheaf $\mathbb{Z}_{cdh}(X)$ is defined as the cdh-sheaf associated to the presheaf $\mathbb{Z}(X) : \quad U \mapsto \mathbb{Z}[Hom_S(U, X)]$. An immediate verification shows the exactness of the following sequence of presheaves

$$0 \to \mathbb{Z}(p^{-1}(Z)) \longrightarrow \mathbb{Z}(Z) \oplus \mathbb{Z}(Y) \longrightarrow \mathbb{Z}(X).$$

Since the sheafification functor is exact this proves the exactness of the above sequence of sheaves without zero on the right. Finally the homomorphism $\mathbb{Z}_{cdh}(Z) \oplus \mathbb{Z}_{cdh}(Y) \longrightarrow \mathbb{Z}_{cdh}(X)$ is surjective since $Y \coprod Z \to X$ is a cdh-covering of X.

Ue used implicitly the second statement of the following Lemma several times in § 5.

Lemma 12.2. *1) Let $f : Y \to X$ be a morphism of finite type. Assume that the point $x \in X$ admits a lifting $y \in Y$ such that the residue fields at x and y coincide: $k(x) = k(y)$. Then there exists an open neighbourhood of x in $\overline{x}$ each point of which has the same property.*
2) Every Nisnevich covering $\{U_i \to X\}_{i \in I}$ of a Noetherian scheme X admits a finite subcovering.

Proof. 1) Consider $\overline{y}$ and $\overline{x}$ as closed integral subschemes in Y and X respectively. The induced morphism $f : \overline{y} \to \overline{x}$ is still of finite type and is an isomorphism over

the generic point x of $\overline{x}$, hence it is an isomorphism over some open neighbourhood of x in $\overline{x}$.
2)Proceeding by Noetherian induction we may assume that for any proper closed subscheme $Z \subset X$ the induced Nisnevich covering $\{U_i \times_X Z \to Z\}_{i \in I}$ has a finite subcovering. Let $x_1, ..., x_n \in X$ be generic points of components of X and let $U_{i_1}, ..., U_{i_n}$ be members of our family such that x_k admits a lifting to U_{i_k} with the same residue field. According to part 1) there exist open neighbourhoods $x_k \in V_k$ such that each point of V_k admits a lifting to U_{i_k} with the same residue field. Set $Z = X \setminus \cup_{k=1}^n V_k$ and consider Z as a (proper) closed reduced subscheme of X. According to our induction hypothesis there exists a finite subset $J \subset I$ such that each point $z \in Z$ admits a lifting with the same residue field to one of $U_j \quad j \in J$. Obviously the subfamily defined by the finite set $J \cup \{i_1, ..., i_n\}$ is a finite Nisnevich subcovering of the original covering.

Corollary 12.2.1. *Let I be a filtering partially ordered set and let $i \mapsto X_i$ be a functor from I to the category of Noetherian schemes such that all the transition morphisms $p_{ij} : X_i \to X_j \quad (i \geq j)$ are affine. Set $X_\infty = \varprojlim_{i \in I} X_i$, denote by $p_i : X_\infty \to X_i$ the structure morphisms and assume that the scheme X_∞ is again Noetherian. Assume further that we are given an index $i \in I$ and a morphism of finite type $Y_i \to X_i$ such that each point of X_∞ admits a lifting with the same residue field to $Y_\infty = Y_i \times_{X_i} X_\infty$. Then there exists an index $j \geq i$ for which the morphism $Y_j = Y_i \times_{X_i} X_j \to X_j$ has the same property.*

Proof. For any $j \geq i$ let $U_j \subset X_j$ denote the subset of X_j, consisting of points which admit a lifting to Y_j with the same residue field. Lemma 12.2 shows that the set U_j is ind-constructible in X_j - [EGA-4 (1.9.10)]. Note further that a point $x_j \in X_j$ with the image $x_i = p_{ji}(x_j)$ in X_i belongs to U_j if and only if the scheme $Y_i \times_{X_i} Speck(x_i)$ of finite type over the field $k(x_i)$ has a rational point over the extension field $k(x_i) \subset k(x_j)$. This remark shows easily that for $j' \geq j$ we have an inclusion $p_{j'j}^{-1}(U_j) \subset U_{j'}$ and moreover

$$X_\infty = U_\infty = \bigcup_{j \geq i} p_j^{-1}(U_j).$$

Using finally [EGA-4 (8.3.4)] we conclude that there exists an index $j \geq i$ such that $U_j = X_j$.

Corollary 12.2.2. *In conditions and notations of (12.2.1) assume that $Y_\infty \to X_\infty$ is a Nisnevich covering (resp. cdh-covering) then there exists an index $j \geq i$ such that $Y_j \to X_j$ has the same property.*

Proof. For Nisnevich coverings our statement follows immediately from [EGA-4 (17.7.8)] and Corollary 12.2.1. Assume now that $Y_\infty \to X_\infty$ is a cdh-covering. According to Proposition 5.9 there exists a refinefent of this covering of the form $U_\infty \to X'_\infty \to X_\infty$ where $U_\infty \to X'_\infty$ is a Nisnevich covering and $X'_\infty \to X_\infty$ is a proper cdh-covering. Furthermore, according to [EGA-4 (8.8.2)] we can find an index $j \geq i$ and a tower of morphisms $U_j \to X'_j \to X_j$ of finite type such that $U_\infty = U_j \times_{X_j} X_\infty, X'_\infty = X'_j \times_{X_j} X_\infty$. Using once again [EGA-4 (8.8.2)] we see, that increasing j, we may assume also that there exists a morphism $U_j \to Y_j$ over X_j. Finally we conclude, using Corollary (12.2.1) above and [EGA-4 (8.10.5), (17.7.8)] that increasing j we may assume that $U_j \to X'_j$ is a Nisnevich covering and $X'_j \to X_j$ is a proper cdh-covering. This shows that $U_j \to X_j$ is a cdh-covering and hence $Y_j \to X_j$ is a cdh-covering as well.

The main goal of this Appendix is to show that the cdh-cohomological dimension of a scheme $X \in Sch/S$ is finite provided that $dimX$ is finite and moreover $cd_{cdh}X \leq dimX$. The proof we are about to give was suggested to us by O. Gabber. We start with a few standard remarks.

For any morphism of (Noetherian) schemes $f : S' \to S$ we denote by the same letter f the associated morphism of sites

$$(Sch/S')_{cdh} \xrightarrow{f} (Sch/S)_{cdh} : \quad X/S \mapsto X_{S'}/S' = X \times_S S'/S'$$

and denote by f_* and f^* the corresponding functors on the categories of abelian sheaves

$$(Sch/S')^{\sim}_{cdh} \underset{f^*}{\overset{f_*}{\rightleftarrows}} (Sch/S)^{\sim}_{cdh}.$$

Thus the direct image functor f_* is given by the formula $(f_*\mathcal{F})(X/S) = \mathcal{F}(X_{S'}/S')$. This functor is left exact and preserves injectives. The inverse image functor f^* is left adjoint to f_*. This functor is exact (since the sites in question have arbitrary fiber products) and satisfies the following property: $f^*(\mathbb{Z}_{cdh}(X/S)) = \mathbb{Z}_{cdh}(X_{S'}/S')$.

Assume that the morphism $f : S' \to S$ is a separated morphism of finite type (so that $S' \in Sch/S$). In this case one checks easily that the inverse image functor f^* is given by the formula $(f^*\mathcal{F})(X'/S') = \mathcal{F}(X'/S)$. Furthermoer the functor f^* has an exact left adjoint , which is denoted $f_!$ and is called the extension by zero functor. In the above situation the functor f^* takes injectives to injectives, which gives immediately for any scheme $X \in Sch/S'$ the following natural isomorphism in cohomology

$$H^*_{cdh}(X/S', f^*\mathcal{F}) = H^*_{cdh}(X/S, \mathcal{F}).$$

The above formula shows that cdh-cohomology essentially does not depend on the base scheme S: for any separated scheme $f : X \to S$ of finite type over S and any cdh-sheaf $\mathcal{F} \in (Sch/S)^{\sim}_{cdh}$ we have a natural isomorphism $H^*_{cdh}(X/S, \mathcal{F}) = H^*_{cdh}(X/X, f^*\mathcal{F})$.

Assume now that I is a filtering partially ordered set (i.e. $I \neq \emptyset$ and for any $i_1, i_2 \in I$ there exists $i \in I$ such that $i \geq i_1, i_2$). Assume further that we are given a functor $i \mapsto S_i$ from I to Sch/S, such that all transition morphisms $S_i \to S_j \quad (i \geq j)$ are affine. Set $S_\infty = \varprojlim_{i \in I} S_i$ and assume that the scheme S_∞ is again Noetherian. Denote by $f_\infty : S_\infty \to S$ the corresponding structure morphism.

Proposition 12.3. *For any abelian sheaf $\mathcal{F} \in (Sch/S)^{\sim}_{cdh}$ we have a natural isomorphism $\varinjlim_{i \in I} H^*_{cdh}(S_i, \mathcal{F}) = H^*_{cdh}(S_\infty, f^*(\mathcal{F}))$.*

Proof. Note that in the above situation the category Sch/S_∞ is a direct limit of categories Sch/S_i in the sence that for any $i \in I$ and any schemes $X, Y \in Sch/S_i$ we have a natural bijection

$$\mathrm{Hom}_{S_\infty}(X \times_{S_i} S_\infty, Y \times_{S_i} S_\infty) = \varinjlim_{j \le i} \mathrm{Hom}_{S_j}(X \times_{S_i} S_j, Y \times_{S_i} S_j)$$

and furthermore for each $X_\infty \in Sch/S_\infty$ there exists an index $i \in I$ and a scheme $X_i \in Sch/S_i$ such that $X_\infty = X_i \times_{S_i} S_\infty$ - see [EGA-4]. Moreover Corollary 12.2.2 shows that given a finite family of morphisms $\{X_i^k \to X_i\}_{k=1}^K$ over S_i, the corresponding family $\{X_i^k \times_{S_i} S_\infty \to X_i \times_{S_i} S_\infty\}_{k=1}^K$ is a cdh-covering iff there exists $j \ge i$ such that $\{X_i^k \times_{S_i} S_j \to X_i \times_{S_i} S_j\}_{k=1}^K$ is a cdh-covering. Using these remarks one checks easily that the sheaf $f^*(\mathcal{F})$ is given by the formula

$$f^*(\mathcal{F})(X_i \times_{S_i} S_\infty) = \varinjlim_{j \ge i} \mathcal{F}(X_i \times_{S_i} S_j).$$

Next one checks that the inverse image of an injective sheaf is at least acyclic.

(12.3.1). *Assume that the sheaf $\mathcal{F}$ is injective. Then for any $X \in Sch/S_\infty$ the cohomology groups $H^*_{cdh}(X, f^*(\mathcal{F}))$ vanish for $* > 0$.*

Proof. Let $p : Y \to X$ be a cdh-covering of X. According to what was said above there exists an index $i \in I$ and a cdh-covering $p_i : Y_i \to X_i$ in the category Sch/S_i such that $p_i \times_{S_i} S_\infty = p$. The above explicit formula for the sheaf $f^*(\mathcal{F})$ shows immediately that the Čech complex of the sheaf $f^*(\mathcal{F})$ corresponding to the covering p coincides with the direct limit over $j \ge i$ of Čech complexes of $\mathcal{F}$, corresponding to the coverings $p_i \times_{S_i} S_j : Y_i \times_{S_i} S_j \to X_i \times_{S_i} S_j$ and hence is acyclic. Vanishing of all Čech cohomology groups of the sheaf $f^*(\mathcal{F})$ implies vanishing of all cohomology groups $H^*_{cdh}(X, f^*(\mathcal{F}))$ in view of the Cartan-Leray spectral sequence.

In the general case pick up an injective resolution $0 \to \mathcal{F} \to J^\bullet$ of the sheaf $\mathcal{F}$. Applying to this resolution the functor f^* we get a resolution $f^*(J^\bullet)$ of $f^*(\mathcal{F})$ consisting of acyclic sheaves. Hence the cohomology groups $H^*_{cdh}(S_\infty, f^*(\mathcal{F}))$ coincide with cohomology of the complex

$$f^*(J^\bullet)(S_\infty) = \varinjlim_{i \in I} J^\bullet(S_i).$$

So it remains to use again that cohomology commutes with filtering direct limits.

In the situation of Proposition 12.3 we usually use notation $H^*_{cdh}(S_\infty, \mathcal{F})$ instead of $H^*_{cdh}(S_\infty, f^*(\mathcal{F}))$.

Lemma 12.4. *Let $p : S' \to S$ be a birational morphism of finite type of integral Noetherian schemes of finite Krull dimension. Then $dimS' \leq dimS$.*

Proof. Since $dimS'$ coincides with the maximum of dimensions of local rings $\mathcal{O}_{S',s'}$ ($s' \in S'$) it suffices to note that for any point $s' \in S'$ we have, according to the "Dimensions Formula" - [EGA-4 (5.5.8)], the following inequality (in which we denote $p(s') \in S$ by s and denote by $k(S)$ (resp. $k(S')$)the field of rational functions on S (resp. S'))

$$dim\mathcal{O}_{S',s'} \leq dim\mathcal{O}_{S',s'} + tr.deg_{k(s)}k(s') \leq dim\mathcal{O}_{S,s} + tr.deg_{k(S)}k(S') = dim\mathcal{O}_{S,s}.$$

Theorem 12.5. *Let S be a Noetherian scheme of finite Krull dimension. Then cohomological dimension of S with respect to the cdh-topology is finite and $\leq dimS$.*

Proof. We proceed by induction on $dimS$. The case $dimS = 0$ being trivial, assume that $dimS = d > 0$ and the result is known to be true for schemes of dimension $< d$. Using the Mayer-Vietoris exact sequences in cdh-cohomology, corresponding to the closed coverings, one reduces easily the general case to the case when S is integral. Denote by α the obvious morphism of sites

$$\alpha : (Sch/S)_{cdh} \to (Sch/S)_{Nis}$$

and consider the corresponding Leray spectral sequence

$$E_2^{pq} = H^p_{Nis}(S, R^q\alpha_*(\mathcal{F})) \Rightarrow H^{p+q}_{cdh}(S, \mathcal{F}).$$

Note that the stalk of the sheaf $R^q\alpha_*(\mathcal{F})$ at a point $x \in S$ is the direct limit of cohomology groups $H^q_{cdh}(U, \mathcal{F})$ over Nisnevich neighbourhoods $(U, u) \to (S, x)$ of the point x. Moreover we may assume these neighbourhoods to be affine and connected. Such neighbourhoods form a filtering partially ordered set and the resulting functor from this poset to Sch/S satisfies the conditions of the Proposition 12.3. Thus we conclude that for any $x \in S$ $R^q\alpha_*(\mathcal{F})_x = H^q_{cdh}(Spec\mathcal{O}^h_x, \mathcal{F})$. Induction hypothesis gives us now the following conclusion

$$R^q\alpha_*(\mathcal{F})_x = 0 \text{ in case } \min(q, d) > dim\mathcal{O}^h_x = codim_S x.$$

Applying the same reasoning to an arbitrary scheme etale over S we see that for $q < d$ the Nisnevich sheaf $R^q\alpha_*(\mathcal{F})$ is supported in dimension $\leq d - q$ and hence $H^p_{Nis}(S, R^q\alpha_*(\mathcal{F})) = 0$ for $p > d - q$ -see [Nis]. Furthermore for $q \geq d$ the Nisnevich sheaf $R^q\alpha_*(\mathcal{F})$ is supported in dimension 0 and hence $H^p_{Nis}(S, R^q\alpha_*(\mathcal{F})) = 0$ for $p > 0$. The Leray spectral sequence shows now that for $n > d$ we have an isomorphism $H^n_{cdh}(S, \mathcal{F}) = H^0_{Nis}(S, R^n\alpha_*(\mathcal{F}))$ and ,in particular, each cohomology class $h \in H^n_{cdh}(S, \mathcal{F})$ for which there exists a Nisnevich covering $\{U_i \to S\}_{i=1}^n$ such that $h|_{U_i}$ is trivial for all i is trivial itself.

To finish the proof start with an arbitrary cohomology class $h \in H^n_{cdh}(S, \mathcal{F})$ ($n > d$). There exists a cdh-covering $\{X_i \to S\}$ such that h dies being restricted to each X_i. Furthermore Proposition 5.9 shows that this covering has a refinement of the form $\{U_i' \to S' \to S\}_{i=1}^n$, where $S' \to S$ is a proper birational cdh-covering and $\{U_i' \to S'\}_{i=1}^n$ is a Nisnevich covering of the scheme S'. Let $S'' \subset S'$ be the closure of the inverse image of the generic point of S and let $\{U_i'' \to S''\}$ be the induced Nisnevich covering of the scheme S''. Thus S'' is an integral scheme of dimension $\leq d$ and the morphism $S'' \to S$ is proper and birational. The cohomology class $h|_{S''}$ dies being restricted to all U_i'' and hence is trivial. Let finally $Z \subset S$ be a proper closed subscheme such that the morphism $p : S'' \to S$ is an isomorphism outside Z. The long exact cohomology sequence

$$0 = H^{n-1}_{cdh}(p^{-1}(Z), \mathcal{F}) \to H^n_{cdh}(S, \mathcal{F}) \to H^n_{cdh}(S'', \mathcal{F}) \oplus H^n_{cdh}(Z, \mathcal{F}) = H^n_{cdh}(S'', \mathcal{F})$$

shows that the restriction homomorphism $H^n_{cdh}(S, \mathcal{F}) \to H^n_{cdh}(S'', \mathcal{F})$ is injective. Thus $h = 0$.

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