Excision in Cyclic Homology Theories

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Introduction

In this paper we describe a new approach to excision in periodic cyclic homology. It applies also to cyclic homology theories for topological algebras and establishes excision in entire, asymptotic and local bivariant cyclic cohomology. Among other things we obtain sharp upper bounds for the dimension shift of periodic cyclic cohomology under the boundary map associated to an extension of algebras.

Ever since the invention of cyclic homology by Connes and independently by Tsygan the problem of excision played a central role in the theory. It is concerned with the question of to what extent an extension

$$0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$$

of algebras induces natural long exact sequences on cyclic homology groups.

To put our approach into perspective we recall the previous work on excision in cyclic homology.

The first results on this problem were obtained by Wodzicki [Wo]. In 1989 he introduced the notion of H -unitality (homological unitality) for algebras and showed that excision in cyclic homology holds for all (linearly split) extensions,

$$0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$$

with fixed kernel I , if and only if I is H -unital as an algebra. This condition is however rarely satisfied in practice. For example excision in cyclic homology does not hold for nilpotent extensions because nilpotent algebras are never H -unital (unless they are zero).

If one considers periodic cyclic homology instead, the situation becomes different. A theorem of Goodwillie says that epimorphisms with nilpotent kernels are periodic cyclic homology equivalences and therefore nilpotent extensions satisfy excision in the periodic theory. One can thus hope that stronger excision results hold in the periodic setup and in fact Cuntz and Quillen [CQ3] succeeded in 1997 in proving that arbitrary linearly split extensions of k -algebras ($k \supset \mathbb{Q}$ a field) satisfy excision in (bivariant) periodic cyclic cohomology. Their proof uses their approach to cyclic homology via quasifree resolutions and the X -complex which they apply in the category of k -pro-algebras. It is based on the fact that in the category of pro-algebras every extension is HP-equivalent to an extension of pro-algebras with asymptotically H -unital kernel.

This result immediately raises the question whether excision holds for other $\mathbb{Z}/2$ -graded versions of cyclic homology as the entire, asymptotic or local cyclic theories [Co], [CM], [Pu1], [Pu2]. A quick inspection shows however that the Cuntz-Quillen proof does not carry over to the topological framework.

In this paper we develop a new approach to excision in $\mathbb{Z}/2$ -graded cyclic homology theories, based on the notion of a free extension. It is inspired by the work of Cuntz and Quillen but follows rather closely Wodzicki's original approach. Our proof has the advantage to carry over to entire, asymptotic and local cyclic cohomology so that excision holds for continuous linearly split extensions of Ind-(Fréchet)-algebras in all these theories.

As a consequence it turns out that entire cyclic cohomology, which is very hard to calculate directly, coincides with continuous periodic cyclic cohomology on a large class of nuclear, admissible Fréchet algebras, containing for example algebras of smooth functions on compact, smooth manifolds. Another consequence is the existence of a bivariant Chern-Connes-character on Kasparov's bivariant KK -theory with values in bivariant local cyclic cohomology (extending those of [Pu1], [Cu]) which is multiplicative up to a factor $2\pi i$.

Before explaining our proof, we want to recall Wodzicki's work.
Let

$$0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$$

be a linearly split extensions of k -algebras and let

$$0 \rightarrow \widehat{CC}_*(A, B) \rightarrow \widehat{CC}_*(A) \rightarrow \widehat{CC}_*(B) \rightarrow 0$$

be the induced linearly split extension of periodic cyclic bicomplexes. Excision is then equivalent to the statement that the natural inclusion

$$\widehat{CC}_*(I) \rightarrow \widehat{CC}_*(A, B)$$

is a continuous chain homotopy equivalence, respectively to the contractibility of the quotient complex $\widehat{CC}_*(A, B)/\widehat{CC}_*(I)$. Wodzicki introduces an increasing filtration of the associated quotient $C_*(A, B)/C_*(I)$ of Hochschild complexes such that the associated graded complex is isomorphic (via the Wodzicki isomorphism) to a direct sum of (shifted) tensor powers of the bar complex

$$C_n^{\text{bar}}(I) := I^{\otimes(n+1)} \text{ with } n \geq 0,$$

$$\partial(j_0 \otimes \dots \otimes j_{n+1}) := \sum_{i=0}^n (-1)^i j_0 \otimes \dots \otimes j_i j_{i+1} \otimes \dots \otimes j_{n+1}$$

He calls I H -unital if $C_*^{\text{bar}}(I)$ is acyclic. H -unitality of I clearly implies that $C_*(A, B)/C_*(I)$ and its cyclic analogue are acyclic which establishes excision.

Our approach to excision is based on the notion of free ideal extensions. A linearly split extension of k -algebras

$$0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$$

is called a free ideal extension if the ideal J is free as A left-(or right) module. We establish excision for free ideal extensions first and derive the general case from the fact that every extension is equivalent in periodic cyclic homology to a free ideal extension. The central part of our proof consists of a detailed analysis of the bar complex $C_*^{\text{bar}}(J)$ of a free ideal.

The bar homology of an ideal (or algebra) can be described independently of the complex used to define it as the Tor-group $\text{Tor}_{**+1}^{\tilde{J}}(k, k)$. Calculating these Tor groups via the standard bar resolution of k over $\tilde{J}$ one recovers the bar complex of J again. For a free ideal J , the $\tilde{J}$ -module k possesses however a different and smaller free resolution, depending naturally on the choice of an A -basis of J , which leads to a new complex $C_*^{\text{small}}(J)$ calculating the Tor-groups in question.

The crucial properties of this small complex are

- the differentials of $C_*^{\text{small}}(J)$ are zero
- the J -adic filtration of $C_*^{\text{small}}(J)$ satisfies

$$\text{Fil}_J^{*+2} C_*^{\text{small}}(J) = 0$$

For the original bar complex this implies that there exists a natural harmonic decomposition

$$C_*^{\text{bar}}(J) \simeq h(C_{*-1}^{\text{bar}}(J)) \oplus \Delta_* \oplus \partial(C_{**+1}^{\text{bar}}(J))$$

where

- the harmonic subspaces Δ_* are naturally isomorphic to the bar homology $H_*^{\text{bar}}(J)$.
- $\ker \partial = \Delta_* \oplus \partial(C_{**+1}^{\text{bar}}(J))$
- the operator $h: C_*^{\text{bar}}(J) \rightarrow C_{*-1}^{\text{bar}}(J)$ preserves J -adic filtrations and satisfies
 - $\ker h = h(C_{*-1}^{\text{bar}}(J)) \oplus \Delta_*$
 - $h: \partial(C_*^{\text{bar}}(J)) \xrightarrow{\simeq} h(C_{*-1}^{\text{bar}}(J))$
- $P := \text{Id} - (\partial h + h \partial)$ equals the canonical projection of $C_*^{\text{bar}}(J)$ into its harmonic subspace Δ_*
- P vanishes on $C_{*,\text{bar}}^>(J) := \text{Fil}_J^{*+2} C_*^{\text{bar}}(J)$

This is analogous to the Hodge decomposition in differential geometry where the operator h equals the adjoint d^* of the differential d .

From this we deduce that the identity of $C_*^{\text{bar}}(J)$ is naturally chain homotopic to a chain map annihilating $C_{*,\text{bar}}^>(J)$. All this is treated in section one of this paper.

In section two we define for a free extension the intermediate Hochschild complex $C_*^>(A)_J$ of chains whose J -adic valuation is strictly larger than their degree. One has $C_*(J) \subset C_*^>(A)_J \subset C_*(A, B)$ and these inclusions possess natural linear sections. Studying the Wodzicki-filtration on these inclusions, the Wodzicki isomorphism and the results of section one allow us to deduce the existence of a chain map

$$\phi : C_*(A, B)/C_*(J) \rightarrow C_*(A, B)/C_*(J)$$

which is naturally chain homotopic to the identity and

- annihilates the subcomplex $C_*^>(A)_J/C_*(J)$
- preserves Hodge filtrations
- preserves J -adic filtrations
- is continuous in the I -adic topology with respect to any ideal $I \subset J$ of A .

In section three we apply these results to the cyclic bicomplex.

The map ϕ strictly commutes with Connes differential B so that it defines a map of cyclic bicomplexes. The results of the previous section imply then that the composition

$$\widehat{CC}_*(A, B)/\widehat{CC}_*(J) \rightarrow \widehat{CC}_*(A, B)/\widehat{CC}_*^>(A, B)_J \xrightarrow{\phi} \widehat{CC}_*(A, B)/\widehat{CC}_*(J)$$

is chain homotopic to the identity. By Goodwillie's Theorem on excision for nilpotent extensions the complex $\widehat{CC}_*(A, B)/\widehat{CC}_*^>(A, B)_J$ is contractible. Therefore the complex $\widehat{CC}_*(A, B)/\widehat{CC}_*(J)$ is contractible, too, which establishes excision for free extensions and more generally for I -adic completions

$$0 \rightarrow \widehat{J} \rightarrow \widehat{A} \rightarrow B \rightarrow 0$$

of free extensions with respect to some ideal $I \subset J$ of A . From this one easily deduces excision in periodic cyclic homology for arbitrary extensions.

In section four we analyze the behaviour of the boundary map in periodic cyclic (co)homology with respect to the dimension filtration, which is induced by the Hodge-filtration on the cyclic bicomplex. Note that these dimension shifts are independent of the particular construction of the boundary map: excision means that the natural inclusion $\widehat{CC}_*(I) \hookrightarrow \widehat{CC}_*(A, B)$ is a chain homotopy equivalence and we are asking how the induced isomorphism on homology shifts dimensions. If an extension of algebras satisfies excision in ordinary cyclic homology, the boundary map in periodic cyclic (co)homology will shift dimensions only by one, examples show however that in general higher dimension shifts occur [Co1].

For arbitrary extensions

$$0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$$

with cohomological boundary map

$$\delta: HP^*(I) \rightarrow HP^{*+1}(B)$$

we establish the estimate

$$\dim(\delta(\alpha)) \leq 3 \dim(\alpha) + 3$$

and show that it is actually sharp.

A similar sharp estimate for the iterated boundary map in higher extensions is also given. A typical extension where a maximal dimension shift occurs is the universal extension

$$0 \rightarrow JA_n \rightarrow TA_n \rightarrow A_n \rightarrow 0$$

of [Cu] for $A_n = \Sigma^{3n+3}\mathbb{C}$, the $(3n+3)$ -fold algebraic suspension of $\mathbb{C}$ [Wo].

Given an extension $0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$ and a homology class $\alpha \in HP_n(B)$ of dimension n , the dimension of $\delta(\alpha)$ depends usually both on the extension and the choice of α .

We exhibit however a class of extensions for which the boundary map shows a remarkable rigidity:

If $0 \rightarrow I \rightarrow R \rightarrow S \rightarrow 0$ is an extension such that R is quasifree [CQ1] and such that $\delta: HP_*(S) \rightarrow HP_{*-1}(I)$ is injective, then for any $\alpha \in HP_*(B)$ the dimension of $\delta(\alpha)$ only depends on the dimension of α . The precise dimension formula (essentially $\dim \alpha = 3 \dim \delta(\alpha)$) is given in theorem 4.5. This result determines the behaviour of the bivariate Chern-character with respect to the natural filtrations on kk -theory and bivariate periodic cyclic cohomology. This was posed as a problem by Cuntz in [Cu].

Similar results hold for multiplicatively split extensions. This will be proved elsewhere. For a multiplicatively split extension

$$0 \rightarrow I \xrightarrow{i} A \xrightleftharpoons[f]{g} B \rightarrow 0$$

excision implies that i induces an isomorphism

$$i^*: HP^*(A) \supset \ker g^* \xrightarrow{\cong} HP^*(I)$$

There exist sharp estimates

$$\dim(i^*)^{-1}(\alpha) \leq \begin{cases} 2 \dim(\alpha) + 2 & \text{if } \deg(\alpha) = 0 \\ 2 \dim(\alpha) + 1 & \text{if } \deg(\alpha) = 1 \end{cases}$$

for the dimension shift of i^* . Again there exists a class of rigid split extensions where an exact dimension formula for the effect of $(i_*)^{-1}$ on periodic cyclic homology holds. It is given by the class of Cuntz extensions

$$0 \rightarrow qA \rightarrow QA \rightarrow A \rightarrow 0$$

for arbitrary algebras A . This class of extensions realizes the maximal dimension shifts occurring in multiplicatively split extensions.

The proof of these estimates is possible because all constructions of the first two sections preserve Hodge-filtrations. Possible dimension shifts are only caused by the use of Goodwillie's Theorem in constructing a homotopy inverse to the inclusion $\widehat{CC}_*^>(A)_J \rightarrow \widehat{CC}_*(A, B)$ and these can be controlled.

In section five we verify that our proof of excision applies to entire, asymptotic and more general local cyclic theories of Ind-algebras equipped with systems of supports. The necessary calculations are carried out in an appendix and are rather elaborate for one reason: the explicit homotopy operators of section one give rise to a contracting homotopy for $\mathrm{Gr}_W(CC_*^>(A)_J/CC_*(J))$ which is easily seen to be continuous. For the complex $CC_*^>(A)_J/CC_*(J)$ this implies only the existence of a selfmap homotopic to the identity which decreases Wodzicki degrees by one. The true contracting homotopy is obtained by infinite repetition and its continuity is not clear a priori but necessitates a careful inspection of the operators involved.

In section six we present two consequences of the general excision results of section five. By comparing the functorial behaviour of periodic and entire cyclic cohomology we answer an old question of Connes: the entire cyclic cohomology of the algebra of smooth functions on a smooth compact manifold equals its continuous periodic cyclic cohomology and is thus isomorphic to the $\mathbb{Z}/2$ -graded singular homology of the manifold.

The arguments of [Pu1] and [Cu] together with our excision results lead finally to a Grothendieck-Riemann-Roch Theorem in Noncommutative Geometry: there exists a bivariant Chern-Connes character

$$\mathrm{ch}_{\mathrm{biv}} : KK^*(-, -) \rightarrow HC_{lc}^*(-, -)$$

on Kasparov's bivariant KK -theory with values in bivariant local cyclic cohomology which is multiplicative up to $2\pi i$, extends the ordinary Chern character and is compatible with boundary maps in six term exact sequences. From [Pu1] it is known already that

$$\mathrm{ch}_{\mathrm{biv}} : KK^*(-, -) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow HC_{lc}^*(-, -)$$

is an isomorphism on a large class of C^* -algebras.

Remark: We work throughout this paper in the category of nonunital algebras.

This paper is the extended and revised version of the preprint [Pu2]. I want to thank Joachim Cuntz for fruitful discussions. I understand that Ralf Meyer has obtained another proof of excision in cyclic homology theory which is based on quasifree resolutions and the X -complex and thus in spirit closer to the work of Cuntz and Quillen.

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1 Bar Homology of Free Ideals

In this section we will analyze in detail the bar homology of a certain class of ideals. These ideals are far from being H -unital but their bar homology and therefore the obstruction to excision in cyclic homology for such ideals is of a particularly simple form.

The bar homology of ideals is known to play a crucial role in excision problems for cyclic homology. In fact for a k -algebra I Wodzicki [Wo] has established the equivalence of the following two assertions:

- Every extension $0 \rightarrow I \rightarrow R \rightarrow S \rightarrow 0$ of k -algebras with kernel I satisfies excision in cyclic homology, i.e. induces long exact sequences of cyclic homology groups.
- I is H -unital, i.e. the bar homology of I vanishes.

The bar homology $H_*^{\text{bar}}(A)$ of an algebra A has been introduced as the homology of the chain complex

$$C_n^{\text{bar}}(A) := A^{\otimes(n+1)} \text{ with } n \geq 0,$$

$$\partial_n(a_0 \otimes \dots \otimes a_{n+1}) := \sum_{i=0}^n (-1)^i a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_{n+1}$$

A more conceptual way to define it is the following. Let A be an algebra and let $\tilde{A}$ be the algebra obtained from A by adjoining a unit. The natural extension of algebras $0 \rightarrow A \rightarrow \tilde{A} \rightarrow k \rightarrow 0$ turns k and A into $\tilde{A}$ -(left)-modules.

There is a natural isomorphism

$$H_*^{\text{bar}}(A) \simeq \text{Tor}_*^{\tilde{A}}(k, A) \simeq \text{Tor}_{*+1}^{\tilde{A}}(k, k).$$

Free ideals

We define now the class of ideals investigated in this paper.

Definition 1.1 *An ideal J of an algebra A is called **free** iff it is free as A left- (or right-) module.*

This means that there exists a k -linear subspace $V \subset J$ such that the module multiplication defines an isomorphism

$$\tilde{A} \otimes_k V \xrightarrow{\simeq} J$$

of $\tilde{A}$ -modules. Such a linear subspace V is called a **basis** of J .

Remark 1.2 *In the sequel we will suppose that all free ideals J are free as A left-modules and that some basis $V \subset J$ has been fixed.*

The distinguished class of extensions for which an explicit proof of excision in periodic cyclic homology will be established is the following:

Definition 1.3 *A free ideal extension of k -algebras is a linear split extension*

$$0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$$

of algebras with kernel a free ideal J in A .

Thus a free ideal extension consists of the following data:

- An extension of k -algebras $0 \rightarrow J \rightarrow A \xrightarrow{f} B \rightarrow 0$
- A k -linear section $s : B \rightarrow A$ of f .
- A k -linear basis V of J as A left module.

Lemma 1.4 *Let $0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$ be a free ideal extension. Then*

- a) J^n is a free ideal of A with basis V^n
- b) $J^n = J^k \cdot V^{n-k} \simeq J^k \otimes V^{\otimes(n-k)}$ for $0 \leq k \leq n$
- c) *There exists a natural k -linear decomposition*

$$J \simeq V \oplus s(B)V \oplus J^2 \simeq \tilde{J} \otimes V \oplus s(B)V$$

Here naturality means naturality with respect to maps of free ideal extensions.

Proof: Fix $k, 0 \leq k \leq n$. Because J is free the iterated multiplication defines an isomorphism

$$J^k \otimes V^{\otimes(n-k)} \xrightarrow{\simeq} J^k V^{n-k} \subset J^n$$

On the other hand $J^n = J^{n-1} \cdot J = J^{n-1}(\tilde{A}V) = (J^{n-1}\tilde{A})V = J^{n-1}V$ and iteration shows that $J^n = J^k V^{n-k}$. This proves b) and a) is just the special case $k = 0$.

The k -linear decomposition of $\tilde{A}$: $\tilde{A} \simeq k \oplus s(B) \oplus J$ provides a decomposition $\tilde{A} \otimes_k V \simeq V \oplus s(B) \otimes_k V \oplus J \otimes_k V$. The module multiplication maps this isomorphically onto

$$J = \tilde{A} \cdot V \simeq V \oplus s(B)V \oplus JV = V \oplus s(B)V \oplus J^2 = \tilde{J} \otimes V \oplus s(B)V$$

□

After having established the basic notions of free ideals and extensions, we begin our analysis of the bar complex of a free ideal.

The small resolution of a free ideal

We will construct a natural resolution of a free ideal J by free $\tilde{J}$ -modules.

Lemma 1.5 *Let $0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$ be a free ideal extension of k -algebras.*

a) *The map*

$$\begin{aligned} \partial_0 : \tilde{J} \otimes_k V \oplus \tilde{J} \otimes_k s(B)V &\rightarrow J \\ (a \otimes v, a' \otimes v') &\mapsto av + a'v' \end{aligned}$$

is an epimorphism of $\tilde{J}$ -modules.

b) *The linear decomposition of (1.4) c) defines a k -linear section s_0 of ∂_0 with image*

$$\tilde{J} \otimes_k V \oplus 1 \otimes s(B)V \subset \tilde{J} \otimes_k V \oplus \tilde{J} \otimes s(B)V.$$

c) *The natural projection*

$$\tilde{J} \otimes_k V \oplus \tilde{J} \otimes_k s(B)V \rightarrow \tilde{J} \otimes_k s(B)V$$

of $\tilde{J}$ -modules maps $\text{Ker } \partial_0$ isomorphically onto the module $J \otimes_k s(B)V$.

We denote this isomorphism by

$$\alpha : \text{Ker } \partial_0 \xrightarrow{\cong} J \otimes_k s(B)V.$$

Its inverse is given on elementary tensors by

$$\alpha^{-1} : j \otimes s(b)v \rightarrow j \otimes s(b)v - js(b) \otimes v$$

Proof: Clearly ∂_0 is a map of $\tilde{J}$ -modules. Its surjectivity follows from b) which is a consequence of 1.4 c). By construction

$$(\text{Id} - s_0 \partial_0)(\tilde{J} \otimes_k (V \oplus s(B)V)) \subset J \otimes_k (V \oplus s(B)V)$$

so that the projection considered in c) maps $\text{Ker } \partial_0$ to $J \otimes_k s(B)V$. Moreover $\tilde{J} \otimes_k V \cap \text{Ker } \partial_0 = 0$ because the multiplication map is injective on $\tilde{J} \otimes_k V$. Therefore the projection defines a monomorphism $\alpha : \text{Ker } \partial_0 \rightarrow J \otimes_k s(B)V$.

Define a map of $\tilde{J}$ -modules by

$$\begin{aligned} \alpha^{-1} : J \otimes_k s(B)V &\rightarrow \text{Ker } \partial_0 \\ j \otimes s(b)v &\mapsto j \otimes s(b)v - js(b) \otimes v. \end{aligned}$$

It is well defined because $s(B) \otimes_k V \xrightarrow{\cong} s(B)V$ under the multiplication. Clearly α^{-1} is a right inverse to α and because α is injective α^{-1} is also a left inverse of α . Thus α and α^{-1} are isomorphisms inverse to each other. $\square$

Denote the module $\tilde{J} \otimes_k V \oplus \tilde{J} \otimes_k s(B)V$ by $P_0^{\text{small}}(J)$. It is a free $\tilde{J}$ -module with basis $V \oplus s(B)V$. The epimorphism $\partial_0 : P_0^{\text{small}} \rightarrow J \rightarrow 0$ therefore can

be considered as first step of a $\tilde{J}$ -free resolution of J . By 1.5 c) the kernel of ∂_0 is isomorphic as $\tilde{J}$ -module to a direct sum of copies of J . Thus one obtains by iteration an infinite $\tilde{J}$ -free resolution of J .

The construction is summarized in

Definition and Proposition 1.6 (Small Resolution)

Let

$$0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$$

be a free ideal extension of k -algebras.

a) For $n \geq 0$ define free $\tilde{J}$ -modules $P_n^{\text{small}}(J)$ by

$$P_n^{\text{small}}(J) := \tilde{J} \otimes_k (V \oplus s(B)V) \otimes_k s(B)V^{\otimes n}$$

A canonical basis of $P_n^{\text{small}}(J)$ is given by $(V \oplus s(B)V) \otimes s(B)V^{\otimes n}$

b) For $n > 0$ define homomorphisms of $\tilde{J}$ -modules

$$\partial_n : P_n^{\text{small}}(J) \rightarrow P_{n-1}^{\text{small}}(J) \quad \text{by} \quad \partial_n := \left(\alpha^{-1} \otimes \text{id}_{s(B)V}^{\otimes(n-1)} \right) \circ \left(\partial_0 \otimes \text{id}_{s(B)V}^{\otimes n} \right)$$

where ∂_0 and α are defined in lemma 1.5.

c) The complex

$$0 \leftarrow J \xleftarrow{\partial_0} P_0^{\text{small}}(J) \xleftarrow{\partial_1} \dots \xleftarrow{\partial_n} P_n^{\text{small}}(J) \xleftarrow{\partial_{n+1}} \dots$$

is acyclic and defines a resolution $P_{\bullet}^{\text{small}}(J)$ of J by free $\tilde{J}$ -modules. It is called the **small resolution** of J . It is natural with respect to maps of free ideal extensions.

d) The small resolution is k -linearly contractible. A natural contracting homotopy is provided by the k -linear maps $s_n, n \geq 0$ where s_0 is defined in lemma 1.5 b) and for $n > 0$

$$s_n : P_{n-1}^{\text{small}}(J) \rightarrow P_n^{\text{small}}(J) \quad \text{equals}$$

$$s_n := \left(s_0 \otimes \text{id}_{s(B)V}^{\otimes n} \right) \circ \left(p \otimes \text{id}_{s(B)V}^{\otimes(n-1)} \right)$$

where $p : \tilde{J} \otimes V \oplus \tilde{J} \otimes s(B)V \rightarrow J \otimes s(B)V$ is the natural projection.

e)

$$\partial_n(P_n^{\text{small}}(J)) \subset J \cdot P_{n-1}^{\text{small}}(J).$$

Proof: Assertions a) and b) are clear. The fact that $P_{\bullet}^{\text{small}}(J)$ is a complex follows from the identity $\partial_0 \circ \alpha^{-1} = 0$ of lemma 1.5 c).

It is clearly natural with respect to maps of free extensions. Its acyclicity follows from d). To check d) note that

$$\begin{aligned}\partial_{n+1} \circ s_{n+1} &= (\alpha^{-1} \circ p) \otimes \text{id}_{s(B)V}^{\otimes n} \quad \text{and} \\ s_n \circ \partial_n &= (s_0 \circ \partial_0) \otimes \text{id}_{s(B)V}^{\otimes n}.\end{aligned}$$

Thus it suffices to show that

$$\partial_1 \circ s_1 + s_0 \circ \partial_0 = \text{Id}_{P_0^{\text{small}}}$$

Now on the subspace $\text{Ker } \partial_0$ of P_0^{small} holds

$$\partial_1 s_1 + s_0 \partial_0 = \alpha^{-1} \circ p + s_0 \circ \partial_0 = \alpha^{-1} \circ p = \text{id}$$

by lemma 1.5 c). On the complementary subspace $s_0(J) \subset P_0^{\text{small}}(J)$ holds

$$\partial_1 s_1 + s_0 \partial_0 = \alpha^{-1} \circ p + s_0 \circ \partial_0 = s_0 \partial_0 = \text{id}.$$

Finally e) is clear from the definitions. □

Adic filtrations of the small resolution

The small resolution carries natural adic filtrations.

Definition 1.7

a) Define a decreasing filtration Fil_J , the J -adic filtration of $P_{\bullet}^{\text{small}}(J)$ by

$$\text{Fil}_J^m(P_n^{\text{small}}(J)) := J^{m-n-1} \cdot P_n^{\text{small}}(J)$$

where the notation $J^l := \tilde{A}$ for $l \leq 0$ is understood.

b) If $I \subset J \subset A$ is an auxiliary ideal of A , $I \neq J$, define the I -adic filtration of $P_{\bullet}^{\text{small}}(J)$ by

$$\text{Fil}_I^m(P_n^{\text{small}}(J)) := I^m \cdot P_n^{\text{small}}(J).$$

Lemma 1.8

a) The J -adic filtration of the small resolution is preserved by the differentials ∂ and the linear contracting homotopies s of $P_{\bullet}^{\text{small}}(J)$.

b) For $I \subset J$, $I \neq J$ the I -adic filtration of $P_{\bullet}^{\text{small}}$ is preserved by the differentials ∂ . For the contracting homotopy one has

$$s(\text{Fil}_I^m P_{\bullet}^{\text{small}}(J)) \subset \text{Fil}_I^{m-1} P_{\bullet}^{\text{small}}(J).$$

We will use now the small resolution to investigate the structure of the bar complex of a free ideal.

Comparison with the standard resolution

Recall that there is a standard resolution of a k -algebra J by free $\tilde{J}$ -modules, the **Bar-resolution** $P_{\bullet}^{\text{bar}}(J)$. It is defined as $P_n^{\text{bar}}(J) := \tilde{J} \otimes J^{\otimes(n+1)}$, which is a free $\tilde{J}$ -module with canonical basis $J^{\otimes(n+1)}$. The differentials are given by

$$\begin{aligned} \partial_n = b' : P_n^{\text{bar}}(J) &\rightarrow P_{n-1}^{\text{bar}}(J) \\ j_0 \otimes \dots \otimes j_{n+1} &\mapsto \sum_{i=0}^n (-1)^i j_0 \otimes \dots \otimes j_i j_{i+1} \otimes \dots \otimes j_{n+1} \end{aligned}$$

A k -linear contracting homotopy is provided by

$$\begin{aligned} s_n = s : P_{n-1}^{\text{bar}}(J) &\rightarrow P_n^{\text{bar}}(J) \\ j_0 \otimes \dots \otimes j_{n-1} &\mapsto 1 \otimes j_0 \otimes \dots \otimes j_{n-1}. \end{aligned}$$

It satisfies $s_{n+1} \circ s_n = 0$, $n \geq 0$. If $I \subset J$ is an auxiliary ideal (here the case $I = J$ is allowed), the I -adic filtration of $P_{\bullet}^{\text{bar}}$ is defined by

$$\text{Fil}_I^m(P_n^{\text{bar}}(J)) := \sum_{\substack{m_0 + \dots + m_{n+1} = m \\ m_i \geq 0}} I^{m_0} \otimes I^{m_1} \otimes \dots \otimes I^{m_{n+1}}$$

where the convention $I^0 = \tilde{J}$ holds. The I -adic filtration is preserved by the differentials and the contracting linear homotopy of $P_{\bullet}^{\text{bar}}$.

Let now A be a k -algebra and let $M \leftarrow P_{\bullet}, N \leftarrow Q_{\bullet}$ be resolutions of A -modules. Suppose that $P_{\bullet}, Q_{\bullet}$ are A -free resolutions with fixed basis and that linear contracting homotopies of both resolutions have been fixed.

Under these assumptions every A -module map $f : M \rightarrow N$ possesses a canonical extension to a map $f_{\bullet} : P_{\bullet} \rightarrow Q_{\bullet}$ of resolutions: if $\hat{f}_{\bullet}$ has been constructed inductively for $\bullet < n$ define f_n on the fixed basis V_n of P_n by $f_n|_{V_n} := s_n^Q \circ f_{n-1} \circ \partial_n^P$ and extend it to an A -module map. Similarly if $\rho : P_{\bullet} \rightarrow P_{\bullet}$ is a map of resolutions covering the zero map of M then ρ is canonically chain homotopic to zero.

Applied to the bar resolution and the small resolution of free ideals the discussion can be summarized as follows:

Definition and Proposition 1.9 *Let $0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$ be a free ideal extension of k -algebras and let $P_{\bullet}^{\text{bar}}(J)$ and $P_{\bullet}^{\text{small}}(J)$ be the bar-resolution respectively the small resolution of J by free $\tilde{J}$ -modules.*

a) *There exists a canonical and natural map of resolutions*

$$\varphi : P_{\bullet}^{\text{bar}}(J) \rightarrow P_{\bullet}^{\text{small}}(J)$$

covering the identity of J .

b) *There exists a canonical and natural map of resolutions*

$$\psi : P_{\bullet}^{\text{small}}(J) \rightarrow P_{\bullet}^{\text{bar}}(J)$$

covering the identity of J .

c)

$$\varphi \circ \psi = \text{Id}_{P_{\bullet}^{\text{small}}(J)}$$

d) *There exists a natural and canonical J -linear operator*

$$h : P_{\bullet}^{\text{bar}}(J) \rightarrow P_{\bullet+1}^{\text{bar}}(J)$$

defining a chain homotopy between $\psi \circ \varphi$ and the identity:

$$\text{Id} - \psi \circ \varphi = b' \circ h + h \circ b'$$

e) *The A -module maps φ, ψ and h preserve the J -adic filtrations of $P_{\bullet}^{\text{bar}}(J)$ and $P_{\bullet}^{\text{small}}(J)$.*

f) *Let $I \subset J, I \neq J$ be an auxiliary ideal of A . The map ψ preserves I -adic filtrations. For the other maps the inclusions*

$$\begin{aligned} \varphi(\text{Fil}_I^m(P_n^{\text{bar}}(J))) &\subset \text{Fil}_I^{m-n-1}P_n^{\text{small}}(J) \\ h(\text{Fil}_I^m(P_n^{\text{bar}}(J))) &\subset \text{Fil}_I^{m-n-1}(P_{n+1}^{\text{bar}}(J)) \end{aligned}$$

hold.

Proof: Assertions a) b) and d) follow from the preceeding discussion. As assertion c) will not be needed in the sequel we leave its proof to the reader. Finally e) and f) follow from lemma 1.8 and the explicit construction of the maps φ, ψ and h . $\square$

For later use we need more precise information about the maps involved in the previous proposition. This is provided by

Lemma 1.10 *Let $S \subset TJ$ be the left ideal generated by the subspaces*

$$J \otimes_k V \otimes_k s(B)V^{\otimes j}, j \geq 0$$

and let $S' \subset TJ$ be the twosided ideal generated by $J \otimes J^2$.

- a) $\varphi(1 \otimes (V \oplus s(B)V)^{\otimes n}) \subset 1 \otimes (V \oplus s(B)V) \otimes s(B)V^{\otimes(n-1)}$ and
 $\psi(1 \otimes (V \oplus s(B)V) \otimes s(B)V^{\otimes(n-1)}) \subset 1 \otimes (V \oplus s(B)) \otimes s(B)V^{\otimes(n-1)} + 1 \otimes S$
- b) $\varphi(1 \otimes S) = 0$ and $\varphi(1 \otimes S') = 0$
- c) $h(1 \otimes T(V \oplus s(B)V)) = 0$
- d) $h(1 \otimes S) = 0$
- e) $h(1 \otimes TJ) \subset \tilde{J} \otimes_k S$
- f) $h^2 = 0$

Proof:

a) The assertions follow by induction on the length of the considered tensors.

b) The linear contraction s' of $P_{\bullet}^{\text{small}}$ satisfies $s'^2 = 0$.

Thus by definition of the differential of $P_{\bullet}^{\text{bar}}$ follows

$$\varphi_n(1 \otimes j^1 \otimes \dots \otimes j^n) = s'_{n-1}(j^1 \varphi_{n-1}(1 \otimes j^2 \otimes \dots \otimes j^n)).$$

This formula implies that $\text{Ker } \varphi \cap 1 \otimes TJ$ is of the form $1 \otimes K$ for some left ideal $K \subset TJ$.

Suppose now $j^2 \in J^2$.

$$\text{Then } \varphi_n(j^0 \otimes \dots \otimes j^n) = j^0 s'_{n-1}(j^1 s'_{n-2}(j^2 \varphi_{n-2}(1 \otimes j^3 \otimes \dots \otimes j^n)))$$

$$\text{and } s'_{n-2}(j^2 \varphi_{n-2}(\dots)) \in J \otimes V \otimes s(B)V^{\otimes(n-1)}$$

$$\text{so that } j^1 s'_{n-2}(j^2 \varphi_{n-2}(\dots)) \in \text{Ker } s'_{n-1}.$$

If $j^2 \in V$ and $j^3 \otimes \dots \otimes j^n \in T(V \oplus s(B)V)$

$$\text{then } \varphi_{n-2}(1 \otimes j^3 \otimes \dots \otimes j^n) \in 1 \otimes (V \oplus s(B)V) \otimes s(B)V^{\otimes(n-2)} \quad \text{by a)}$$

so that again $s'_{n-2}(j^2 \varphi_{n-2}(1 \otimes j^3 \otimes \dots \otimes j^n)) \in J \otimes V \otimes s(B)V^{\otimes(n-1)}$ and the reasoning of the previous case applies.

c) As the linear contraction s of $P_{\bullet}^{\text{bar}}$ satisfies $s^2 = 0$ one has

$$\begin{aligned} h_n(1 \otimes j^1 \otimes \dots \otimes j^n) \\ = s_n((\text{Id} - \psi\varphi)(1 \otimes j^1 \otimes \dots \otimes j^n)) - s_n(j^1 h_{n-1}(1 \otimes j^2 \otimes \dots \otimes j^n)). \end{aligned}$$

The claim follows then from a) and induction over n .

d) The formulas in parts b) and c) of the proof show that

$\text{Ker } h \cap \text{Ker } \varphi \cap 1 \otimes TJ$ is of the form $1 \otimes K'$ for some left ideal $K' \subset TV$.

Let $\alpha = 1 \otimes j^1 \otimes \dots \otimes j^n \in 1 \otimes J \otimes V \otimes s(B)V^{\otimes(n-2)}$ be a generator of S .

Then the first summand of the formula for h_n in c) vanishes by b) and the second summand vanishes by c) and the induction hypothesis.

e) Claim e) has already been verified in a special case in c) so that it suffices to check the case of tensors with at least one factor in J^2 . Suppose that the assertion has already been verified for tensors for which at least one of its k last factors belongs to J^2 .

$$\text{Let } \alpha = 1 \otimes j^1 \otimes \dots \otimes j^n \in 1 \otimes J^{\otimes(n-k-1)} \otimes J^2 \otimes (s(B)V \oplus V)^{\otimes k}$$

$$\text{with } j^{n-k} = j'v' \in J^2 = JV$$

$$\text{and put } \alpha' := 1 \otimes j^1 \otimes \dots \otimes j' \otimes v' \otimes \dots \otimes j^n \in P_{n+1}^{\text{bar}}(J).$$

Then $\alpha' \in 1 \otimes S$ so that the identity $\text{Id} - \psi\varphi = b'h + hb'$ reduces to $\alpha' = h(b'(\alpha'))$ by b) and d).

Especially $h(b'(\alpha')) \in 1 \otimes S$. Now

$$b'(\alpha') = b'(1 \otimes \dots \otimes j') \otimes v' \otimes \dots \otimes j^n \pm \alpha + \sum \alpha'$$

where the first sum belongs to $\tilde{J} \otimes S$ and one of the k last factors of the summands α' is in J^2 .

Thus $h(\alpha) \in \tilde{J} \otimes S$ and the induction step is established.

f) The identity $h^2 = 0$ follows from e) and d).

□

The bar homology of free ideals

Lemma 1.11 *Let $0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$ be a free ideal extension of k -algebras. Let $P_{\bullet}^{\text{small}}(J)$ be the small resolution of J by free $\tilde{J}$ -modules. Put*

$$C_{*}^{\text{small}}(J) := k \otimes_{\tilde{J}} P_{*}^{\text{small}}(J)$$

a) *The complex $C_{*}^{\text{small}}(J)$ is given in degree n by*

$$C_n^{\text{small}}(J) \simeq (V \oplus s(B)V) \otimes s(B)V^{\otimes n}$$

and the differentials of $C_{}^{\text{small}}(J)$ vanish.*

b) *The J -adic filtration of $P_{*}^{\text{small}}(J)$ induces a J -adic filtration of $C_{*}^{\text{small}}(J)$ satisfying*

$$\text{Fil}_J^m(C_n^{\text{small}}(J)) = 0 \quad \text{for } m > n + 1.$$

c) *Let*

$$\varphi' := \text{id}_k \otimes_{\tilde{J}} \varphi : C_{*}^{\text{bar}}(J) \rightarrow C_{*}^{\text{small}}(J)$$

$$\psi' := \text{id}_k \otimes_{\tilde{J}} \psi : C_{*}^{\text{small}}(J) \rightarrow C_{*}^{\text{bar}}(J)$$

$$h' := \text{id}_k \otimes_{\tilde{J}} h : C_{*}^{\text{bar}}(J) \rightarrow C_{*+1}^{\text{bar}}(J)$$

be the chain maps respectively the homotopy operator induced from the maps of resolutions in 1.9.

They satisfy the identities

$$\varphi' \circ \psi' = \text{Id}, \quad \psi' \circ \varphi' = \text{Id} - (h' \circ \partial + \partial \circ h')$$

All these maps preserve J -adic filtrations.

Proof: The vanishing of the differentials of $C_{*}^{\text{small}}(J)$ follows from 1.6 e) because J is the kernel of the augmentation map $\tilde{J} \rightarrow k$. Similarly b) holds by definition of the J -adic filtration (see definition 1.7 a). $\square$

Note that $k \otimes_{\tilde{J}} P_{*}^{\text{bar}}(J) = C_{*}^{\text{bar}}(J)$ is the bar complex of J .

We are now ready to calculate the bar homology of free ideals. The crucial fact is that cycles of high J -adic filtration do not contribute to the bar homology. This is the central point in our approach to excision because it implies that on subcomplexes of high J -adic valuation free ideals behave as if they were H -unital. The bar homology of a free ideal is rather pathological as it consists only of the “obvious classes”, i.e. those cocycles which cannot be boundaries by comparison of J -adic valuations. This implies also that except in very degenerate cases free ideals are of infinite cohomological dimension.

Theorem 1.12 *Let $0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$ be a free ideal extension of k -algebras.*

a) *For all $n \geq 0$ there is a natural isomorphism*

$$H_n^{\text{bar}}(J) \simeq (V \oplus s(B)V) \otimes_k s(B)V^{\otimes n}$$

b) *The J -adic filtration of the bar complex $C_*^{\text{bar}}(J)$ induces a filtration of the bar homology $H_*^{\text{bar}}(J)$.*

It satisfies $\text{Fil}_J^m H_n^{\text{bar}}(J) = 0$ for $m > n + 1$.

Proof: As the bar resolution of a free ideal is naturally chain homotopy equivalent to the small resolution one deduces that the complexes $C_*^{\text{bar}}(J)$ and $C_*^{\text{small}}(J)$ are naturally chain homotopy equivalent. Thus

$$\begin{aligned} H_n^{\text{bar}}(J) &= H_n(C_*^{\text{bar}}(J)) \simeq H_n(C_*^{\text{small}}(J)) \\ &= C_n^{\text{small}}(J) \simeq (V \oplus s(B)V) \otimes s(B)V^{\otimes n} \end{aligned}$$

because the differentials of $C_*^{\text{small}}(J)$ vanish.

As moreover the natural chain homotopy equivalences between $C_*^{\text{bar}}(J)$ and $C_*^{\text{small}}(J)$ preserve J -adic filtrations one finds that

$$\text{Fil}_J^m H_n^{\text{bar}}(J) \simeq \text{Fil}_J^m H_n(C_*^{\text{small}}(J)) \simeq \text{Fil}_J^m (C_n^{\text{small}}(J)) = 0$$

for $m > n + 1$ by lemma 1.10 b). □

The harmonic decomposition of the bar complex

The comparison of the bar complex with the small complex does not only allow to determine the bar homology of J .

The fact that the differentials of the small complex vanish implies the existence of a natural Hodge-type decomposition of the bar-complex which, together with the result about the J -adic filtration, provides much more profound information about the bar complex of free ideals than the bare calculation of its homology.

Theorem 1.13 *Let $0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$ be a free ideal extension of k -algebras. Then there exists a natural decomposition of the bar complex $C_*^{\text{bar}}(J)$:*

$$C_n^{\text{bar}}(J) \simeq \text{Im}(h'_{n-1}) \oplus \Delta_n \oplus \text{Im}(\partial_{n+1}) \text{ for } n \geq 0$$

satisfying

a) *The image Δ of the natural idempotent chain map $P' := \psi' \circ \varphi'$ is isomorphic to the bar homology of J and called the “harmonic subspace” of $C_*^{\text{bar}}(J)$.*

b) The natural idempotent linear map $\partial h'$ defines a projection of $C_*^{\text{bar}}(J)$ onto $\text{Im}(\partial)$

c) The natural idempotent linear map $h'\partial$ defines a projection of $C_*^{\text{bar}}(J)$ onto $\text{Im}(h')$. Moreover $\text{Im}(h') \subset C_*^{\text{bar}}(J) \simeq TJ$ coincides with the left ideal S of TV generated by the spaces $J \otimes V \otimes s(B)V^{\otimes j}, j \geq 0$. Thus $\text{Im}(h') \subset C_*^{\text{bar}}(J)$ is an A -left-module and a left ideal in TJ .

d) The projectors $P', \partial h'$ and $h'\partial$ are orthogonal and complementary:

$$C_*^{\text{bar}}(J) = h'\partial(C_*^{\text{bar}}(J)) \oplus P'(C_*^{\text{bar}}(J)) \oplus \partial h'(C_*^{\text{bar}}(J)).$$

$$e) \quad \text{Ker } \partial = \Delta \oplus \partial h'(C_*^{\text{bar}}(J))$$

$$\text{Ker } h' = h'\partial(C_*^{\text{bar}}(J)) \oplus \Delta$$

$$\text{Ker } \partial \cap \text{Ker } h' = \Delta$$

$$h' \text{ maps } \partial h'(C_*^{\text{bar}}(J)) \text{ isomorphically onto } h'\partial(C_{*+1}^{\text{bar}}(J))$$

Proof: The chain map $\varphi' \circ \psi' : C_*^{\text{small}}(J) \rightarrow C_*^{\text{small}}(J)$ is chain homotopic to the identity by construction and equals thus the identity because the differential of the complex $C_*^{\text{small}}(J)$ is zero. Therefore $P' = \psi' \circ \varphi'$ is an idempotent chain map of $C_*^{\text{bar}}(J)$ whose image Δ is contained in $\text{Ker } \partial$.

The space Δ maps isomorphically onto $H_*^{\text{bar}}(J)$. This and the identity $\text{Id} - \psi'\varphi' = \partial h' + h'\partial$ imply moreover that $\partial h'$ and $h'\partial$ are idempotent linear maps and that the sum of the three idempotents equals one.

It remains to verify that the considered idempotents are orthogonal to each other which is a consequence of the identities $h'^2 = 0$ derived from lemma 1.10 f) and $h'\psi' = 0$ derived from lemma 1.10 a), c) and d). $\square$

The results obtained in section one will be used to study the excision problem in periodic cyclic homology. Following Wodzicki we begin with an analysis of the relative Hochschild complex associated to an extension.

2 Maps of Hochschild complexes associated to an extension

Let $0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$ be a free extension of k -algebras. The relative Hochschild complex $C_*(A, B)$ carries a natural filtration, the Wodzicki-filtration. The associated graded chain complex is a direct sum of the Hochschild-complex $C_*(J)$ of J and shifted powers of the bar complex of the ideal J .

This shows the important role played by the bar homology of ideals in the excision problem. We will introduce an intermediate relative Hochschild complex

$$C_*(J) \subset C_*^>(A)_J \subset C_*(A, B).$$

It consists of the Hochschild chains whose J -adic valuation is larger than their Hochschild degree. The results on the bar homology of free ideals of section one, especially about its J -adic filtration, imply that there is no cohomological obstruction to the contractibility of

$$\mathrm{Gr}_W(C_*^>(A)_J/C_*(J)) \text{ inside } \mathrm{Gr}_W(C_*(A, B)/C_*(J))$$

Here Gr_W denotes the associated graded complexes with respect to the Wodzicki-filtration. Inductively we will derive that the complex $C_*^>(A)_J/C_*(J)$ is contractible inside the relative Hochschild complex $C_*(A, B)/C_*(J)$.

This has to be done with some care as the intermediate Hochschild complex $C_*^>(A)_J$ tends to be a rather unstable object.

Let A be a k -algebra and let $C_*(A)$ be its Hochschild complex. The linear decomposition

$$C_n(A) \simeq \tilde{A} \otimes A^{\otimes n} \simeq A^{\otimes n+1} \oplus A^{\otimes n}$$

induces a natural linear decomposition

$$C_*(A) \simeq C'_*(A) \oplus C''_*(A)$$

The spaces $C'_*(A)$ form a subcomplex of $C_*(A)$, the quotient complex $C_*(A)/C'_*(A)$ is isomorphic to the bar complex $C_*^{\mathrm{bar}}(A)$.

Let $0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$ be an extension of k -algebras with k -linear split s . Then the linear decomposition $A \simeq J \oplus s(B)$ induces a linear decomposition of the Hochschild complex

$$C_*(A) : C_n(A) \simeq \tilde{A} \otimes A^{\otimes n} \simeq (k \oplus J \oplus s(B)) \otimes (J \oplus s(B))^{\otimes n}.$$

In the rest of this section it is understood that elements of $C_*(A)$ are decomposed according to this splitting so that each factor of a tensor monomial $\alpha = a^0 \otimes \dots \otimes a^n$ is supposed either to belong to J or to $s(B)$. The maximal strings $a^i \otimes \dots \otimes a^j$ of factors of α , counted in cyclic order, with $a^i \in J, \dots, a^j \in J$ are called the blocks of α .

Let $C_*(A)$ be the Hochschild complex of A and let Fil_J be its J -adic filtration.

We begin with the definition of the intermediate relative Hochschild complex.

Definition and Lemma 2.1

a) Let $\nu : \mathbb{N} \rightarrow \mathbb{N}$ be a monotone increasing function. Then the spaces

$$C_n^{>\nu}(A)_J := \mathrm{Fil}_J^{\nu(n+1)+1} C'_n(A) \oplus \mathrm{Fil}_J^{\nu(n)+1} C''_n(A)$$

form a subcomplex $C_*^{>\nu}(A)_J$ of $C_*(A)$.

b) One has $C_*^{>\nu}(A)_J \subset C_*^{>\nu'}(A)_J$ for $\nu \geq \nu'$.

The complex $C_*^{>\nu}(A)_J$ associated to the choices $\nu(n) = n$ respectively $\nu(n) = n - 1$ will be denoted by $C_*^>(A)_J$ respectively $C_*^{\geq}(A)_J$.

c) The quotient complex $C_*(A)/C_*^{>\nu}(A)_J$ is denoted by $C_*^{\leq\nu}(A)_J$.

If the given extension of algebras is free then the extension of complexes

$$0 \rightarrow C_*^{>\nu}(A)_J \rightarrow C_*(A) \rightarrow C_*^{\leq\nu}(A)_J \rightarrow 0$$

possesses a natural linear section.

We recall now the Wodzicki-filtration of the relative Hochschild complex.

The Wodzicki-Filtration [Wo]

Let $0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$ be an extension of k -algebras with k -linear split s .

Definition 2.2 *There exists an increasing filtration Fil_W of the Hochschild complex $C_*(A)$ which is given as follows:*

For every $l \in \mathbb{N}$ put

$$\text{Fil}_W^{4l} C_n(A) := \langle \tilde{a}^0 \otimes a^1 \otimes \dots \otimes a^n \mid \text{at most } l \text{ factors } a^i \text{ are in } s(B) \rangle$$

$$\text{Fil}_W^{4l+1} C_n(A) := (s(B) \otimes A^{\otimes n}) \cap \text{Fil}_W^{4l+4} C_n(A) + \text{Fil}_W^{4l} C_n(A)$$

$$\text{Fil}_W^{4l+2} C_n(A) := (\tilde{A} \otimes A^{\otimes(n-1)} \otimes B) \cap \text{Fil}_W^{4l+4} C_n(A) + \text{Fil}_W^{4l+1} C_n(A)$$

$$\begin{aligned} \text{Fil}_W^{4l+3} C_n(A) &:= (J \otimes A^{\otimes(n-1)} \otimes J) \cap \text{Fil}_W^{4l+4} C_n(A) \\ &+ (1 \otimes J \otimes A^{\otimes(n-2)} \otimes J) \cap \text{Fil}_W^{4l+4} C_n(A) + \text{Fil}_W^{4l+2} C_n(A) \end{aligned}$$

The Wodzicki-filtration satisfies

- $\text{Fil}_W^0 C_*(A) = C_*(J)$
- $\text{Fil}_W^l C_n(A) = C_n(A)$ if $l \geq 4n + 4$

Following Wodzicki we identify the associated graded complex of the Hochschild complex with its Wodzicki-filtration as a direct sum of tensor powers of the bar complex of J .

Lemma 2.3

a) *There exists a natural, filtration preserving linear isomorphism*

$$\text{Gr}_W C_*(A) \simeq C_*(A).$$

Especially the subquotients of the Wodzicki-filtration can be identified linearly with subspaces of $C_(A)$.*

b) The complex $\text{Fil}_W^{4l+1}C_*(A)/\text{Fil}_W^{4l}C_*(A)$ is naturally isomorphic to the direct sum of $(s(B)^{\otimes(l+1)}, 0)$ and the complexes

$$C^{(k_0, \dots, k_j)} := s(B)^{\otimes k_0}[k_0] \otimes C_*^{\text{bar}}(J) \otimes s(B)^{\otimes k_1}[k_1 + 1] \otimes C_*^{\text{bar}}(J) \otimes \dots \\ \dots \otimes s(B)^{\otimes k_{j-1}}[k_{j-1} + 1] \otimes C_*^{\text{bar}}(J) \otimes s(B)^{\otimes k_j}[k_j]$$

where the direct sum runs over all tuples $(k_0, \dots, k_j)$ of integers satisfying $k_0 > 0, \dots, k_{j-1} > 0, k_j \geq 0, k_0 + \dots + k_j = l + 1$.

c) The complex $\text{Fil}_W^{4l+2}C_*(A)/\text{Fil}_W^{4l+1}C_*(A)$ is naturally isomorphic to the direct sum of the complexes

$$C_*^{(k_1, \dots, k_j)} := C_*^{\text{bar}}(J) \otimes s(B)^{\otimes k_1}[k_1 + 1] \otimes \dots \\ \dots \otimes s(B)^{\otimes k_{j-1}}[k_{j-1} + 1] \otimes C_*^{\text{bar}}(J) \otimes s(B)^{\otimes k_j}[k_j]$$

and

$$C_*'^{(i_0, \dots, i_j)} := 1 \otimes s(B)^{\otimes i_0}[i_0 + 1] \otimes C_*^{\text{bar}}(J) \otimes \dots \\ \dots \otimes C_*^{\text{bar}}(J) \otimes s(B)^{\otimes i_j}[i_j]$$

where the first resp. second direct sum runs over all tuples $(k_1, \dots, k_j)$ resp. $(i_0, \dots, i_j)$ of integers satisfying $k_1 + \dots + k_j = i_0 + \dots + i_j = l + 1, k_1 > 0, \dots, k_j > 0, i_0 \geq 0, i_1 > 0, \dots, i_j > 0$.

d) The quotient complexes $\text{Fil}_W^{4l+3}C_*(A)/\text{Fil}_W^{4l+2}C_*$ are naturally contractible.

e) The quotient complexes $\text{Fil}_W^{4l+4}C_*(A)/\text{Fil}_W^{4l+3}C_*$ are naturally isomorphic to the direct sum of complexes

$$C_*''^{(k_0, \dots, k_{j-1})} \\ = 1 \otimes s(B)^{\otimes k_0}[k_0 + 1] \otimes C_*^{\text{bar}}(J) \otimes \dots \otimes s(B)^{\otimes k_{j-1}}[k_{j-1} + 1] \otimes C_*^{\text{bar}}(J)$$

where the direct sum runs over all tuples of integers satisfying $k_0 > 0, \dots, k_{j-1} > 0, k_0 + \dots + k_{j-1} = l + 1$.

Proof: All parts except d) are evident.

The isomorphism of part a) identifies $\text{Fil}_W^{4l+3}C_*(A)/\text{Fil}_W^{4l+2}C_*(A)$ with the subspace

$$(J \otimes A^{\otimes(n-1)} \otimes J \oplus 1 \otimes J \otimes A^{\otimes n-2} \otimes J) \cap (\text{Fil}_W^{4l+4}C_n(A)/\text{Fil}_W^{4l}C_n(A))$$

of $C_*(A)$.

Define a homotopy operator by $h(a^0 \otimes \dots \otimes a^n) := 1 \otimes a^0 \otimes \dots \otimes a^n$.

Then $\kappa := \text{Id} - (h\partial + \partial h)$ satisfies

$$\kappa(a^0 da^1 \dots da^n) = \begin{cases} (-1)^{n-1} da^n a^0 da^1 \dots da^{n-1} & , \quad a^{n-1} \in J \\ 0 & , \quad a^{n-1} \in s(B) \end{cases}$$

so that $H_\kappa := \sum_{i=0}^{\infty} h \circ \kappa^i$ converges and defines a contracting homotopy of the considered complex. $\square$

It will be necessary to refine the Wodzicki-filtration by taking into account the harmonic decomposition of the bar complex.

Definition 2.4 Let $0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$ be a free ideal extension of k -algebras. Recall the harmonic decomposition of the bar complex $C_*^{\text{bar}}(J)$.

Define $\text{Fil}_\Delta^r C_*(A) \subset C_*(A)$, $r \geq 0$, as the smallest linear subspace of the Hochschild complex $C_*(A)$ such that

- $\text{Fil}_\Delta^r C_*(A) \cap C'_*(A)$ contains all tensor monomials in $C'_*(A)$ with at least $r+1$ blocks in $\text{Im}(h')$ and at least one factor in $s(B)$.
- $\text{Fil}_\Delta^r C_*(A) \cap C''_*(A)$ contains all tensor monomials in $C''_*(A)$ with at least r blocks in $\text{Im}(h')$ and at least one factor in $s(B)$.

We put $\text{Fil}_\Delta^{-1} C_*(A) := C_*(A)$.

We note

Lemma 2.5 The Hochschild differential ∂ maps $\text{Fil}_\Delta^r C_*(A)$ into $\text{Fil}_\Delta^{r-1} C_*(A)$ for $r \neq 1$ and $\partial(\text{Fil}_\Delta^1 C_*(A)) \subset \text{Fil}_\Delta^0 C_*(A) + C_*(J)$

Proof: This follows from the fact that $\text{Im}(h')$ is an A -left-module and a left ideal in TJ (1.13 c) which implies that the number of blocks in $\text{Im}(h')$ of a tensor monomial can decrease by at most one under the differential b . $\square$

We can construct now a chain map which is naturally homotopic to the identity of the Hochschild-complex and such that the induced map of associated graded complexes (with respect to the Wodzicki-filtration) annihilates the image of $C_*^>(A)_J$ in Wodzicki-degree l . This is possible because by our analysis of the bar complex of J the intermediate Hochschild-complex $C_*^>(A)_J$ does not meet the obstruction against excision encoded in the homology of $\text{Gr}_W(C_*(A))$. The proposition is a crucial step towards excision in periodic cyclic homology.

Proposition 2.6 Let $0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$ be a free ideal extension of k -algebras. For every $l > 0$ there exists a natural chain map

$$\phi_l : C_*(A) \rightarrow C_*(A)$$

satisfying

- i) ϕ_l is naturally chain homotopic to the identity, i.e. there exists a natural operator $\chi_l : C_*(A) \rightarrow C_{*+1}(A)$ such that

$$\phi_l = \text{Id}_{C_*(A)} - (\partial \circ \chi_l + \chi_l \circ \partial)$$

- ii) The maps ϕ_l and χ_l preserve J -adic filtrations and Wodzicki-filtrations

- iii) ϕ_l equals the identity on $\text{Fil}_W^{l-1} C_*(A)$ and χ_l vanishes on $\text{Fil}_W^{l-1} C_*(A)$.

- iv) ϕ_l preserves the subcomplex

$$\tilde{C}_*^>(A)_J := C_*^>(A)_J + C_*^>(A)_J \cap C'_*(A) \cap \text{Fil}_\Delta^0 C_*(A)$$

and maps $\text{Fil}_W^l \tilde{C}_*^>(A)_J$ into $\text{Fil}_W^{l-1} \tilde{C}_*^>(A)_J$.

v) If $I \subset J$ is an auxiliary ideal of A then

$$\begin{aligned}\phi_l(\text{Fil}_I^k C_n(A)) &\subset \text{Fil}_I^{k-2(n+1)} C_n(A) \\ \chi_l(\text{Fil}_I^k C_n(A)) &\subset \text{Fil}_I^{k-2(n+1)} C_{n+1}(A)\end{aligned}$$

Proof: Identify according to 2.3a) $C_*(A)$ with the associated graded space $\text{Gr}_W C_*(A)$. Let for the moment

$$\chi_l : \text{Fil}_W^l C_*(A) / \text{Fil}_W^{l-1} C_*(A) \rightarrow \text{Fil}_W^l C_{*+1}(A) / \text{Fil}_W^{l-1} C_{*+1}(A)$$

be any linear operator preserving J -adic filtrations and extend it to all of $\text{Gr}_W C_*(A) \simeq C_*(A)$ by setting $\chi_l = 0$ in Wodzicki degrees different from l . Then the pair $(\chi_l, \phi_l := \text{Id} - (\chi_l \partial + \partial \chi_l))$ satisfies assertions i) ii) iii). We distinguish several cases. If l is of the form $l = 4m + 1$ decompose the complex $\text{Fil}_W^l C_*(A) / \text{Fil}_W^{l-1} C_*(A)$ according to lemma 2.3 b). The operator χ_l will be constructed as a similar direct sum

$$\chi_l = \bigoplus \chi^{(k_0, \dots, k_j)} \text{ on the space } \bigoplus C_*^{(k_0, \dots, k_j)}$$

where the operator $\chi^{(k_0, \dots, k_j)}$ are defined on the space

$$s(B)^{\otimes k_0} \otimes C_{n_1}^{\text{bar}}(J) \otimes s(B)^{\otimes k_1} \otimes \dots \otimes s(B)^{\otimes k_{j-1}} \otimes C_{n_j}^{\text{bar}}(J) \otimes s(B)^{\otimes k_j}$$

by the sum

$$\chi^{(k_0, \dots, k_j)} := \sum_{i=1}^j (-1)^{\kappa(i)} \text{Id}^{\otimes k_0} \otimes P' \otimes \text{Id}^{\otimes k_1} \otimes \dots \otimes \text{Id}^{\otimes k_{i-1}} \otimes h' \otimes \text{Id}^{\otimes (k_i + \dots + n_j + k_j)}$$

with $\kappa(i) := k_0 + n_1 + k_1 + \dots + k_{i-1}$.

Here P' denotes the canonical projection of $C_*^{\text{bar}}(J)$ onto its harmonic subspace (1.13) and h' is the associated homotopy operator (1.13). It is easily verified that on $C^{(k_0, \dots, k_j)}$

$$\begin{aligned}\phi^{(k_0, \dots, k_j)} &:= \text{Id} - (\partial \circ \chi^{(k_0, \dots, k_j)} + \chi^{(k_0, \dots, k_j)} \circ \partial) \\ &= \text{Id}^{\otimes k_0} \otimes P' \otimes \text{Id}^{\otimes k_1} \otimes \dots \otimes \text{Id}^{\otimes k_{j-1}} \otimes P' \otimes \text{Id}^{\otimes k_j}\end{aligned}$$

With this the definition in the case $l = 4m + 1$ is complete. The definition in the cases $l = 4m + 2, l = 4m$ is similar. In the remaining case $l = 4m + 3$ one takes as operator χ_l on the subquotient $\text{Fil}_W^{4m+3} C_*(A) / \text{Fil}_W^{4m+2} C_*(A)$ the contracting homotopy operator of lemma 2.3 d). With this the homotopy operator χ_l is well defined and assertions i), ii), and iii) hold.

The fact that $\tilde{C}_*^>(A)_J$ is stable under ϕ_l is a consequence of the definitions and is most easily proved by verifying the following claims:

- $\phi_l(C_*^>(A)_J \cap C'_*(A)) \subset C_*^>(A)_J$
- $\phi_l(C_*^>(A)_J \cap C''_*(A)) \subset C_*^>(A)_J \cap C''_*(A) + C_*^>(A)_J \cap C'_*(A) \cap \text{Fil}_\Delta^0 C_*(A)$

- $\phi_l(C_*^{\geq}(A)_J \cap C'_*(A) \cap \text{Fil}_{\Delta}^0 C_*(A))$
 $\subset C_*^{\geq}(A)_J \cap C''_*(A) + C_*^{\geq}(A)_J \cap C'_*(A) \cap \text{Fil}_{\Delta}^0 C_*(A)$

The map ϕ_l preserves the Wodzicki-filtration and in order to show that it maps $\text{Fil}_W^l \tilde{C}_*^{\geq}(A)_J$ into $\text{Fil}_W^{l-1} \tilde{C}_*^{\geq}(A)$ it suffices to verify that the induced map on $\text{Fil}_W^l / \text{Fil}_W^{l-1}$ annihilates the image of $\tilde{C}_*^{\geq}(A)_J$. This image is spanned linearly by tensor monomials satisfying one of the following conditions:

- at least one block is contained in $\text{Fil}_J^{*+2} C_*^{\text{bar}}(J) \subset C_*^{\text{bar}} \subset TJ$
- at least one block is contained in $\text{Im}(h') \subset C_*^{\text{bar}}(J) \subset TJ$

The map induced by ϕ_l on the associated graded complex equals zero if l is of the form $l=4m+3$ and is otherwise a direct sum of maps

$$\text{Id}_{s(B) \otimes k_0} \otimes P' \otimes \text{Id}_{s(B) \otimes k_1} \dots \otimes P' \otimes \text{Id}_{s(B) \otimes k_j}.$$

By the harmonic decomposition theorem 1.13 both $\text{Fil}_J^{*+2} C_*^{\text{bar}}(J)$ and $\text{Im}(h')$ are contained in the kernel of the harmonic projection P' . So all tensor monomials containing at least one block of the form described above are annihilated by the map induced by ϕ_l . This establishes our claim.

Assertion v) follows finally from the fact that the isomorphism of lemma 2.3 a) and the decomposition lemma 2.3 b) strictly preserve I -adic and J -adic filtrations and from lemma 1.11 c). $\square$

Iterating the construction one obtains the contractibility of the intermediate Hochschild complex $C_*^{\geq}(A)_J / C_*(J)$ inside $C_*(A, B) / C_*(J)$.

Theorem 2.7 *Let*

$$0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$$

be a free ideal extension of k -algebras. There exists a natural chain map

$$\phi : C_*(A) \rightarrow C_*(A)$$

satisfying

- i) ϕ is naturally chain homotopic to the identity, i.e. there exists a natural linear operator

$$\chi : C_*(A) \rightarrow C_{*+1}(A)$$

such that

$$\phi = \text{Id} - (\partial \circ \chi + \chi \circ \partial)$$

- ii) ϕ equals the identity on the subcomplex $C_*(J)$

- iii) ϕ maps the subcomplex $C_*^{\geq}(A)_J$ into $C_*(J)$.

iv) The maps ϕ and χ preserve J -adic and Wodzicki-filtrations.

v) If $I \subset J$ is an auxiliary ideal of A then

$$\phi(\mathrm{Fil}_I^k C_n(A)) \subset \mathrm{Fil}_I^{k-8(n+1)^2} C_n(A)$$

$$\chi(\mathrm{Fil}_I^k C_n(A)) \subset \mathrm{Fil}_I^{k-8(n+1)^2} C_{n+1}(A)$$

Proof: Define ϕ and χ on $C_n(A)$ by the formulas

$$\phi := \phi_1 \circ \dots \circ \phi_l$$

$$\chi := \sum_{i=1}^l \chi_i \circ \phi_{i+1} \circ \dots \circ \phi_l$$

where $l \geq 4n + 4$ is any integer.

This definition is independent of the choice of l by proposition 2.6 iii) and defines maps

$$\phi : C_*(A) \rightarrow C_*(A), \quad \chi : C_*(A) \rightarrow C_{*+1}(A)$$

satisfying

$$\phi = \mathrm{Id} - (\partial \circ \chi + \chi \circ \partial).$$

Moreover ϕ equals the identity on $\mathrm{Fil}_W^0 C_*(A) = C_*(J)$ and maps $C_*^>(A)_J$ into $\mathrm{Fil}_W^0 C_*(A)_J = C_*(J)$ by 2.6 iv).

Finally ϕ and χ preserve Wodzicki and J -adic filtrations because the maps ϕ_l and χ_l do so and v) follows from proposition 2.6 v) by noting that $C_n(A) \subset \mathrm{Fil}_W^{4n+4} C_*(A)$. $\square$

Now we pass from the Hochschild complex to the cyclic bicomplex.

3 Excision in Periodic Cyclic Homology

In this section we extend the results of the previous section to the relative cyclic bicomplex associated to a free extension. We introduce the intermediate cyclic bicomplex

$$CC_*(J) \subset CC_*^>(A)_J \subset CC_*(A, B)$$

and show that quotient complex $CC_*^>(A)_J / CC_*(J)$ is contractible.

This is done as follows. In section two the relative contractibility of the complex $C_*(A)_J^> / C_*(J)$ was realized by a chain map

$$\phi : C_*(A, B) / C_*(J) \rightarrow C_*(A, B) / C_*(J)$$

which is chain homotopic to the identity and annihilates $C_*^>(A)_J$. It turns out that this map commutes with Connes differential B and is compatible with the Hodge- and adic filtrations. From this we derive a diagram of chain maps

$$\widehat{CC}_*(A, B)/\widehat{CC}_*(J) \xrightarrow{p} \widehat{CC}_*(A, B)/\widehat{CC}_*^>(A)_J \xrightarrow{\phi} \widehat{CC}_*(A, B)/\widehat{CC}_*(J)$$

of complete cyclic bicomplexes. The composition $\phi \circ p$ is chain homotopic to the identity. Contrary to Hochschild- and cyclic homology the bivariant cyclic theory satisfies excision for nilpotent extensions due to a well known theorem of Goodwillie [Go]. This result applied in our context says that the chain complex $\widehat{CC}_*(A, B)/\widehat{CC}_*^>(A)_J$ is naturally contractible.

Thus the identity of the relative cyclic complex $\widehat{CC}_*(A, B)/\widehat{CC}_*(J)$ factors up homotopy through a contractible complex. Therefore the relative cyclic bicomplex of a free extension is contractible itself.

This establishes excision for free ideal extensions and the general case is deduced at the end of this section by a reduction argument based again on Goodwillie's Theorem.

We begin by recalling some basic notions about cyclic homology

Definition 3.1 *Let A be a k -algebra.*

a) *The $\mathbb{Z}/2$ -graded chain complex*

$$(CC_*(A), \partial) := \left(\bigoplus_{n=0}^{\infty} C_{*+2n}(A), b + B \right) \simeq \left(\bigoplus_{n=0}^{\infty} \Omega^{*+2n}(A), b + B \right)$$

is the (noncomplete) cyclic bicomplex of A . It is an acyclic chain complex.

b) *The Hodge-filtration of $CC_*(A)$ is the decreasing filtration defined by the subcomplexes*

$$\text{Fil}_{\text{Hodge}}^m CC_*(A) := b\Omega^m(A) \oplus \bigoplus_{k=m}^{\infty} \Omega^k(A)$$

The completion with respect to the Hodge-filtration is denoted by $\widehat{CC}_(A)$. This complex of adically complete vector spaces calculates the periodic cyclic homology $HP_*(A)$ of A .*

c) *Any ideal $I \subset A$ defines an I -adic filtration of the cyclic bicomplex $CC_*(A)$. The completion of $CC_*(A)$ with respect to the filtration generated by Hodge- and I -adic filtration is the complete cyclic bicomplex $\widehat{CC}_*(\widehat{A})$ of the I -adic completion $\widehat{A}$ of A . It calculates the periodic cyclic homology $HP_*(\widehat{A})$ of $\widehat{A}$.*

Definition and Lemma 3.2 *Let $J \subset A$ be an ideal and let $\nu : \mathbb{N} \rightarrow \mathbb{N}$ be a monotone increasing function.*

a) *The space*

$$CC_*^{>\nu}(A)_J := \bigoplus_{n=0}^{\infty} C_{*+2n}^{>\nu}(A)$$

defines a subcomplex of $CC_(A)$. Denote the quotient complex by $CC_*^{\leq\nu}(A)$.*

b) *Suppose that J is a free ideal. Then the extension*

$$0 \rightarrow CC_*^{>\nu}(A)_J \rightarrow CC_*(A) \rightarrow CC_*^{\leq\nu}(A)_J \rightarrow 0$$

possesses a natural k -linear splitting.

c) *The splitting of b) is continuous with respect to the Hodge-filtration and with respect to the I -adic filtration associated to any ideal $I \subset J$ of A . Therefore the extensions*

$$0 \rightarrow \widehat{CC}_*^{>\nu}(A)_J \rightarrow \widehat{CC}_*(A) \rightarrow \widehat{CC}_*^{\leq\nu}(A)_J \rightarrow 0$$

and

$$0 \rightarrow \widehat{CC}_*^{>\nu}(\widehat{A})_J \rightarrow \widehat{CC}_*(\widehat{A}) \rightarrow \widehat{CC}_*^{\leq\nu}(\widehat{A})_J \rightarrow 0$$

possess natural, continuous k -linear splittings.

We want to translate theorem 2.7 about the relative Hochschild-complex associated to a free ideal extension into a similar theorem about cyclic bicomplexes. In fact we will show that the chain map ϕ of theorem 2.7 commutes with Connes differential B . Due to the construction of ϕ it is difficult to verify this directly. Instead we construct an operator χ_{cyc} of degree one on the cyclic bicomplex and verify the identity

$$\phi = \text{Id} - \partial_{\text{cyc}} \cdot \chi_{\text{cyc}} + \chi_{\text{cyc}} \cdot \partial_{\text{cyc}}$$

which shows at the same time that ϕ commutes with B and that it is chain homotopic to the identity. The equality above is not verified directly but follows from studying two auxiliary filtrations of the cyclic bicomplex $CC_*(A)$.

Lemma 3.3 *Recall the filtration of the Hochschild complex defined in definition 2.4*

- a) χ_l maps $\text{Fil}_{\Delta}^r C_*(A)$ into $\text{Fil}_{\Delta}^{r+1} C_*(A)$ for all r .
- b) The spaces $\text{Fil}_{\Delta}^r C_*(A)$ are stable under ϕ_l for $r > 0$.
- c) The Connes-operator B maps $\text{Fil}_{\Delta}^r C_*(A)$ into $\text{Fil}_{\Delta}^{r+1} C_*(A)$.
- d) $\text{Fil}_{\Delta}^r C_*(A) \subset \text{Ker } \phi$ for $r > 0$.

Proof:

- a) follows from the definition of χ_l and the harmonic decomposition
- b) is obvious from 2.5 and a) and the fact that χ_l vanishes on $C_*(J)$.
- c) is obvious from the definitions.
- d) As $\text{Im}(h') \subset \text{Ker } P'$ by the harmonic decomposition one has that ϕ_l maps $\text{Fil}_W^l \text{Fil}_\Delta^r C_*(A)$ into $\text{Fil}_W^{l-1} \text{Fil}_\Delta^r C_*(A)$ for $r > 0$ by b) and by the same argument as in the proof of 2.6 iv).
As moreover $\text{Fil}_\Delta^r C_*(A) \cap \text{Fil}_W^0 C_*(A) = 0$ for $r > 0$ we are done.

□

Corollary 3.4 *In the notations of lemma 2.5*

- a) $\phi \circ B \circ \chi = 0$
- b) $(\chi \circ B)^k \circ \chi$ vanishes on $C_n(A)$ for $2k > n + 1$

Proof:

- a) By lemma 3.3
 $B \circ \chi(C_*(A)) \subset B(\text{Fil}_\Delta^0 C_*(A)) \subset \text{Fil}_\Delta^1 C_*(A) \subset \text{Ker } \phi$.
- b) $(-\chi \circ B)^k \circ \chi$ maps $C_n(A)$ into $\text{Fil}_\Delta^{2k} C_{n+2k+1}(A)$ by lemma 3.3.
The space $\text{Fil}_\Delta^r C_m(A)$ is however zero for $2r > m$.

□

The second auxiliary filtration is given by

Definition 3.5 *Let $0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$ be a free ideal extension of k -algebras and recall the harmonic decomposition (1.13). Define $C_*^\Delta(A) \subset C_*(A)$ as the smallest subspace such that under the decomposition of lemma 2.3 $C_*^\Delta(A)$ contains all the spaces*

- $s(B)^{\otimes k_0} \otimes \text{Ker } h' \otimes s(B)^{\otimes k_1} \otimes \dots \otimes \text{Ker } h' \otimes s(B)^{\otimes k_j}$
for $k_0 \geq 0, k_1 > 0, \dots, k_{j-1} > 0, k_j \geq 0, k_0 + k_j > 0, j > 0$
- $1 \otimes s(B)^{\otimes k_0} \otimes \text{Ker } h' \otimes \dots \otimes \text{Ker } h' \otimes s(B)^{\otimes k_j}$
for $k_0 \geq 0, k_1 > 0, \dots, k_{j-1} > 0, k_j \geq 0, k_0 + k_j > 0, j > 0$
- All tensor monomials in $1 \otimes J \otimes TA \otimes J$
for which all blocks except at most one are in $\text{Ker}(h')$

Lemma 3.6 *In the notations of lemma 2.5 and 3.5*

- a) $C_*^\Delta(A)$ is stable under ϕ_l for all l .

- b) χ_l and χ vanish on $C_*^\Delta(A)$.
- c) ϕ_l maps $\text{Fil}_W^l C_*(A)$ to $\text{Fil}_W^{l-1} C_*(A) + C_*^\Delta(A)$
- d) $\text{Im } \phi \subset C_*(J) + C_*^\Delta(A)$
- e) $C_*^\Delta(A)$ is stable under Connes operator B .

Proof: a) and b) follow from the definition of χ_l and the argument in the proof of 2.5, c) follows from the inclusion $\text{Im } P' \subset \text{Ker } h'$ (1.13), and d) is a consequence of c) and a). e) is obvious from the definitions. $\square$

Corollary 3.7 *In the notations of theorem 2.7*

$$\chi \circ B \circ \phi = 0$$

Proof:

By lemma 3.6

$$B(\phi(C_*(A))) \subset B(C_*(J) + C_*^\Delta(A)) \subset C_*(J) + C_*^\Delta(A) \subset \text{Ker } \chi$$

$\square$

We can now derive the cyclic version of theorem 2.7.

Theorem 3.8 *Let $0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$ be a free ideal extension of k -algebras.*

Let $\phi : CC_(A) \rightarrow CC_*(A)$ the linear map induced from the corresponding map of Hochschild complexes in lemma 2.5. Then*

- i) ϕ is a chain map of cyclic bicomplexes.
- ii) ϕ is chain homotopic to the identity via a natural homotopy operator χ_{cyc} .
- iii) ϕ equals the identity on $CC_*(J)$ and maps $CC_*^>(A)_J$ onto $CC_*(J)$
- iv) ϕ and χ_{cyc} preserve J -adic and Hodge-filtrations.
- v) The maps ϕ and the homotopy operator χ_{cyc} connecting ϕ and the identity are both continuous in any topology generated by the Hodge-filtration and the I -adic filtration associated to an auxiliary ideal $I \subset J$ of A .

Proof: Recall the homotopy operator χ of theorem 2.7 and view it as linear operator on $CC_*(A)$ via the identification in definition 3.1 a). Put

$$\chi_{cyc} := \sum_{k=0}^{\infty} (-\chi \circ B)^k \circ \chi$$

where B is Connes operator (see [Kh], [Pu]). The map χ_{cyc} is well defined by definition 3.5 b). A straightforward calculation using 3.5 a) and theorem 3.8 shows that

$$\phi = \text{Id} - ((b + B) \circ \chi_{cyc} + \chi_{cyc} \circ (b + B))$$

From this it follows that ϕ is a chain map and that ϕ is chain homotopic to the identity.

Assertions iii), iv) and v) are then a consequence of 2.7

□

With theorem 3.8 we are ready to establish excision in periodic cyclic homology for free ideal extensions.

Theorem 3.9 *Let*

$$0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$$

be a free ideal extension of k -algebras and let $I \subset J$ be an auxiliary ideal. Denote by $\hat{J}$ and $\hat{A}$ the I -adic completions of J respectively A .

Then

i) The natural inclusions

$$\widehat{CC}_*(J) \rightarrow \widehat{CC}_*(A, B)$$

$$\widehat{CC}_*(\hat{J}) \rightarrow \widehat{CC}_*(\hat{A}, B)$$

are natural chain homotopy equivalences. Consequently free ideal extensions satisfy excision in bivariate periodic cyclic cohomology.

ii) The chain map ϕ of theorem 3.8 defines deformation retractions

$$\phi : \widehat{CC}_*^>(A)_J \rightarrow \widehat{CC}_*(J)$$

$$\hat{\phi} : \widehat{CC}_*^>(\hat{A})_{\hat{J}} \rightarrow \widehat{CC}_*(\hat{J})$$

of complexes.

Proof: Let $\widehat{CC}_*(A, B)$ be the relative cyclic bicomplex of the pair (A, B) . It is stable under the chain map ϕ and the homotopy operator χ_{cyc} , because $CC_*(A, B) = \text{Fil}_J^1(CC_*(A))$ and ϕ and χ_{cyc} preserve the J -adic filtration of $CC_*(A)$. The results of theorem 3.8 imply that ϕ fits into a diagram of complexes

$$\widehat{CC}_*(A, B)/\widehat{CC}_*(J) \xrightarrow{p} \widehat{CC}_*(A, B)/\widehat{CC}_*^>(A)_J \xrightarrow{\phi} \widehat{CC}_*(A, B)/\widehat{CC}_*(J)$$

where the first chain map is the natural projection. Moreover they show that the composition $\phi \circ p$ is chain homotopic to the identity. By Goodwillie's Theorem [Go],[Lo] the complex $\widehat{CC}_*(A, B)/\widehat{CC}_*(A)_J$ is however naturally contractible. The identity of $\widehat{CC}_*(A, B)/\widehat{CC}_*(J)$ factors therefore up to natural homotopy through a naturally contractible complex and is thus naturally null-homotopic itself. Consequently the inclusion $\widehat{CC}_*(J) \subset \widehat{CC}_*(A, B)$ is a natural chain homotopy equivalence.

As all involved maps are continuous with respect to I -adic topologies the same result holds for the I -adic completions of J and A .

Finally ii) follows from the fact that the map induced by ϕ in the diagram above is a chain homotopy equivalence (both complexes are contractible) and the five lemma. $\square$

Excision in periodic cyclic homology for free ideal extensions finally implies excision for arbitrary extensions.

Theorem 3.10 (Excision in Periodic Cyclic Homology)

Let

$$0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$$

be a linearly split extension of k -algebras.

i) The natural inclusion $\widehat{CC}_*(I) \rightarrow \widehat{CC}_*(A, B)$ is a natural chain homotopy-equivalence.

ii) The given extension induces natural six term exact sequences

$$\begin{array}{ccccc} HP^*(-, I) & \rightarrow & HP^*(-, A) & \rightarrow & HP^*(-, B) \\ \delta \uparrow & & & & \downarrow \delta \\ HP^{*+1}(-, B) & \leftarrow & HP^{*+1}(-, A) & \leftarrow & HP^{*+1}(-, I) \\ \\ HP^*(I, -) & \leftarrow & HP^*(A, -) & \leftarrow & HP^*(B, -) \\ \delta \downarrow & & & & \uparrow \delta \\ HP^{*+1}(B, -) & \rightarrow & HP^{*+1}(A, -) & \rightarrow & HP^{*+1}(I, -) \end{array}$$

in bivariate periodic cyclic cohomology.

iii) Both statements also hold for continuously linearly split extensions of adically complete algebras.

Proof: Let $\pi : TA \rightarrow A$ be the natural linearly split epimorphism [CQ1] and denote by IA its kernel. Consider the map of linear split extensions

$$\begin{array}{ccccccc} 0 & \rightarrow & I & \rightarrow & A & \xrightarrow{s} & B \rightarrow 0 \\ & & \uparrow & & \uparrow \pi & & \parallel \\ 0 & \rightarrow & J & \rightarrow & TA & \rightarrow & B \rightarrow 0 \end{array}$$

It is easily verified that J is a free ideal of TA with basis

$$\begin{array}{ccccc} V & := & I & \oplus & I \otimes B & \oplus & B \otimes B \\ & & \in & & \in & & \in \\ & & \rho(i), & & \rho(i) \otimes \rho(s(b)), & & \omega(s(b), s(b')) \end{array}$$

Moreover J contains the ideal $IA = \text{Ker } \pi$. Theorem 3.9 implies therefore that the linearly split extension

$$0 \rightarrow \widehat{J} \rightarrow \widehat{TA} \rightarrow B \rightarrow 0$$

of IA -adic completions satisfies excision in bivariant periodic cyclic cohomology. The vertical maps in the diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & I & \rightarrow & A & \rightarrow & B \rightarrow 0 \\ & & \uparrow & & \uparrow & & \parallel \\ 0 & \rightarrow & \widehat{J} & \rightarrow & \widehat{TA} & \rightarrow & B \rightarrow 0 \end{array}$$

are linear split epimorphisms with topologically nilpotent kernel and induce thus chain homotopy equivalences of cyclic bicomplexes by Goodwillie's Theorem [Go]. Therefore the original extension satisfies excision in bivariant periodic cyclic cohomology, too. The whole construction is continuous with respect to adic topologies induced from ideals and natural with respect to maps of linearly split extensions. The six term exact sequences of bivariant periodic cyclic cohomology groups will consequently also be natural with respect to maps of extensions and will hold for continuously linearly split extensions of adically complete algebras. $\square$

4 Dimension shifts under the boundary map in periodic cyclic homology

In this section we introduce the notion of dimension of a periodic cyclic (co)homology class and investigate its behaviour under the boundary map in periodic cyclic (co)homology. We present two results.

- A sharp upper bound for the dimension shift of the boundary map in periodic cyclic cohomology.
- An exact formula for the dimension shift of the boundary map in periodic cyclic homology for extensions $0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$, where A is quasifree and the boundary map $\delta: HP_*(B) \rightarrow HP_{*+1}(I)$ is injective.

It should be noted that these dimension shifts are independent of the concrete realization of the boundary map as explained in the introduction.

We start with the definition and a few comments on the notion of dimension.

The $\mathbb{Z}/2$ -graded periodic cyclic bicomplex can be obtained as the inverse limit of the integer graded cyclic complex under powers of the S -operator of degree two. This gives rise to the exact sequence

$$0 \rightarrow \varprojlim_S^1 HC_{*+2n-1}(A) \rightarrow HP_*(A) \rightarrow \varprojlim_S HC_{*2n}(A) \rightarrow 0$$

of homology groups respectively to the formula

$$HP^*(A) \simeq \varinjlim_S HC^{**+2n}(A)$$

for cohomology groups.

Definition 4.1

- a) *The dimension filtrations on periodic cyclic homology HP_* are the decreasing filtrations*

$$\text{Fil}_{\dim}^{2k} HP_0(A) := \ker (HP_0(A) \rightarrow HC_{2k-2}(A))$$

$$\text{Fil}_{\dim}^{2k+1} HP_1(A) := \ker (HP_1(A) \rightarrow HC_{2k-1}(A))$$

- b) *The dimension filtration on periodic cyclic cohomology HP^* is the increasing filtration*

$$\text{Fil}_{\dim}^k HP^*(A) := \text{Im} (HC^k(A) \rightarrow HP^*(A))$$

The associated grading is called the grading by dimensions, the degree of a class $\alpha \in HP(A)$ being denoted by $\dim(\alpha)$.

The dimension of a periodic homology class can be an integer or $+\infty$. The latter happens iff the class is an element of the subspace $\varprojlim_S^1 HC_{*+2n-1}(A)$ of HP_* . The dimension of a cohomology class is always an integer. The dimension of even (odd) classes in periodic cyclic (co)homology is even (odd) if it is finite.

We want to investigate the behaviour of the dimension under the boundary map in periodic cyclic cohomology. First we recall a few facts about this problem which are known.

- i) If $0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$ is an extension with H -unital kernel I then excision holds already in cyclic cohomology [Wo] and the boundary map in cyclic cohomology is also a boundary map in periodic cyclic cohomology. Therefore $\dim(\delta(\alpha)) \leq \dim(\alpha) + 1$ in this case.
- ii) Let M be a compact manifold of even dimension $2n$ and let $S^*M \subset T^*M$ be its cosphere bundle. Consider the pseudodifferential operator extension

$$0 \rightarrow \mathcal{P}^{<0}(M) \rightarrow \mathcal{P}^{\leq 0}(M) \xrightarrow{\sigma} \mathcal{C}^\infty(S^*M) \rightarrow 0$$

where σ is the principal symbol. This is an extension of Fréchet-algebras with continuous linear split.

If the considered pseudodifferential operators act on a suitable Hilbert space of L^2 -functions on M or L^2 -sections of some bundle over M , it follows from Weyl's Theorem on the eigenvalue distribution of differential operators that $\mathcal{P}^{<0}(M) \subset l^{2n+1}(\mathcal{H})$, i.e. the pseudodifferential operators of negative order belong to the Schatten ideal $l^{2n+1} \subset \mathcal{B}(\mathcal{H})$. The periodic cyclic cohomology $HP^0(l^{2n+1}(\mathcal{H})) \simeq \mathbb{C}$ has the canonical generator $[S^n \tau] \in HC^{2n}(l^{2n+1}(\mathcal{H}))$, τ the trace, which yields by restriction a class $\alpha \in HC^{2n}(\mathcal{P}^{<0}(M))$. The Atiyah-Singer index theorem and the compatibility of the boundary maps in K -theory and periodic cyclic cohomology imply that α maps under the boundary map associated to the pseudodifferential extension to the Poincaré dual of the total Todd class

$$\text{Todd}(T^*M) \cap [S^*M] \in H_*(S^*M, \mathbb{C}) \simeq HP^*(\mathcal{C}^\infty(S^*M))$$

The Todd class coming from the Todd genus it is of the form $\text{Todd}(T^*M) = 1 + \text{higher order classes}$.

So its Poincaré dual is of the form $\text{Todd}(T^*M) \cap [S^*M] = \text{fundamental class of } S^*M + \text{lower order classes}$.

Especially $\deg(\text{Todd}(T^*M) \cap [S^*M]) = \dim(\delta(\alpha)) = 4n - 1$.

Thus the dimension of the generator of $HP^0(l^{(2n+1)}(\mathcal{H}))$ is essentially multiplied by 2 in this example:

$$\dim(\delta(\alpha)) \geq 2 \dim(\alpha) - 1$$

The same holds more generally for extensions

$$0 \rightarrow l^p(\mathcal{H}) \rightarrow \mathcal{E} \rightarrow A \rightarrow 0$$

defining odd, finitely summable Fredholm modules [Co1].

- iii) The Cuntz-Quillen proof [CQ3] of excision in periodic cyclic cohomology establishes the estimate

$$\dim(\delta(\alpha)) \leq 4 \cdot 2^{\dim(\alpha)}$$

which holds for arbitrary extensions of algebras and arbitrary cohomology classes.

The optimal estimate for the dimension shift under the boundary map is provided by

Theorem 4.2 *Let φ be the shift function associated to the boundary map in periodic cyclic cohomology: it is given for $n \geq 0$ by*

$$\varphi(n) := \sup \dim(\delta(\alpha))$$

where the supremum is taken over all pairs (ν, α) where

$$\nu : 0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$$

is a linearly split extension of k -algebras and $\alpha \in HP_*(B)$ is a homology class of dimension n .

Then one has

$$\varphi(n) = 3n + 3$$

More general one has

Theorem 4.3 Denote by $\delta: HP^*(I) \rightarrow HP^{*+m}(B)$, $m \geq 1$ the boundary map associated to a linearly contractible m -extension

$$0 \rightarrow I \rightarrow \mathcal{E}_1 \rightarrow \dots \rightarrow \mathcal{E}_m \rightarrow B \rightarrow 0$$

of k -algebras. Let φ_m be the associated shift function defined similarly to the shift function for the ordinary boundary map in 4.2.

Then

$$\varphi_m(n) = 3^m \cdot n + \frac{1}{2}(3^{m+1} - 3)$$

Before we come to the proofs we investigate the dimension shift under the boundary map on the level of chain complexes.

Proposition 4.4 Let $0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$ be a linearly split extension of k -algebras. Then there exists a natural chain map

$$\delta: \widehat{CC}_*(B) \rightarrow \widehat{CC}_{*+1}(I)$$

of $\mathbb{Z}/2$ -graded cyclic bicomplexes inducing the boundary map in bivariate periodic cyclic cohomology and such that the subcomplex $\text{Fil}_{\text{Hodge}}^{3n-1}(\widehat{CC}_n(B))$ is mapped into $\text{Fil}_{\text{Hodge}}^n(\widehat{CC}_n(I))$ for all $n \geq 0$.

Proof: Consider the diagram of complexes

$$\begin{array}{ccccccc} \widehat{CC}_*(I) & \longrightarrow & \widehat{CC}_*(A) & \xrightarrow{s} & \widehat{CC}_*(B) \\ \uparrow \pi_* & & \uparrow \pi_* & & \uparrow \text{Id} \\ \widehat{CC}_*(\hat{J}) & \longrightarrow & \widehat{CC}_*(\hat{TA}) & \xrightarrow{s'} & \widehat{CC}_*(B) \\ \uparrow \phi & & \uparrow \text{Id} & & \uparrow \pi'_* \\ 0 \longrightarrow \widehat{CC}_*^>(\hat{TA})_{\hat{J}} & \longrightarrow & \widehat{CC}_*(\hat{TA}) & \xrightarrow{s''} & \widehat{CC}_*^<(\hat{TA})_{\hat{J}} \longrightarrow 0 \end{array}$$

where s', s'' denote the natural linear sections. It is commutative up to natural chain homotopy.

A natural chain map representing the boundary map in periodic cyclic (co)homology can be defined by

$$\delta := \pi_* \circ \phi \circ s' \circ \sigma,$$

where $\delta' := [\partial, s'']$ is the boundary map of the bottom line extension of complexes and where $\sigma : \widehat{CC}_*(B) \rightarrow \widehat{CC}_*^{\leq}(\widehat{TA})_{\widehat{J}}$ is a chain map homotopy inverse to π'_* .

There are several ways to construct a homotopy inverse of π'_* : one can either use the Cartan homotopy formula for the cyclic bicomplex as Goodwillie [Go], [Lo], or can work in the Cuntz-Quillen picture and use the functoriality of the complex $X_*(\widehat{TA})$ under almost multiplicative linear maps [Pu], [Pu1]. Both constructions provide chain maps $\sigma : CC_*(B) \rightarrow CC_*^{\leq}(TA)_J$ satisfying

$$\sigma : \Omega^n B \rightarrow \bigoplus_{k=0}^{\infty} \text{Fil}_J^k \Omega^{n-2k}(TA)$$

and such that the induced chain map $\sigma : \widehat{CC}_*(B) \rightarrow \widehat{CC}_*^{\leq}(\widehat{TA})_{\widehat{J}}$ of completions is homotopy inverse to π'_* . Now note that

$$\text{Fil}_J^k \Omega^{n-2k}(TA) \subset CC_n^>(TA)_J$$

for $k > n - 2k$ by definition of $CC_n^>(TA)_J$. A straightforward calculation shows that

$$\sigma : \bigoplus_{k=0}^{\infty} \Omega^{3n-1+2k}(B) \rightarrow \bigoplus_{j=0}^{\infty} \Omega^{n+1+2j}(TA).$$

and consequently

$$\sigma : \text{Fil}_{\text{Hodge}}^{3n-1}(\widehat{CC}(B)) \rightarrow \text{Fil}_{\text{Hodge}}^{n+1}(\widehat{CC}_*^{\leq}(\widehat{TA})_{\widehat{J}})$$

The boundary map δ' shifts Hodge-filtrations by one. As finally the chain map ϕ (and of course π_*) preserve Hodge-filtrations 3.8 we obtain

$$\delta : \text{Fil}_{\text{Hodge}}^{3n-1}(\widehat{CC}(B)) \rightarrow \text{Fil}_{\text{Hodge}}^n(\widehat{CC}_*(I))$$

□

Proposition 4.4 answers the question about the dimension shift of the boundary map on the level of chain complexes. In order to see that these results hold also on a cohomological level and provide optimal estimates for the dimension shift we will study a special class of extensions. For these extensions one does not only get estimates but in fact exact formulas for the dimension shift under the boundary map. These extensions also turn out to be extremal in the sense that they provide examples of a maximal dimension shift.

Theorem 4.5 *Let $0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$ be a linearly split extension of k -algebras such that*

- *A is quasifree [CQ1]*
- *the induced map $HP_*(A) \rightarrow HP_*(B)$ is zero.*

Let $\mu: \mathbb{N} \cup \{\infty\} \rightarrow \mathbb{N} \cup \{\infty\}$ be the function defined by

$$\mu^{-1}(k) = \begin{cases} \{1, 3\} & k = 0 \\ \{0, 2, 4, 6\} & k = 1 \\ \{3n - 1, 3n + 1, 3n + 3\} & k = n > 1 \\ \infty & k = \infty \end{cases}$$

Then the dimension shift of the boundary map

$$\delta: HP_*(B) \rightarrow HP_{*+1}(I)$$

in periodic cyclic homology is given by the formula

$$\dim(\delta(\alpha)) = \mu(\dim(\alpha))$$

for all $\alpha \in HP_*(B)$.

Proof: Let $\alpha \in HP_*(B)$ be a homology class with

$$\dim(\alpha) \in \{3n - 1, 3n + 1, 3n + 3\} = \mu^{-1}(n), n > 1.$$

Then α can be realized by a cycle in $\text{Fil}_{\text{Hodge}}^{3n-1}(\widehat{CC}_{n+1}(B))$ and Proposition 4.4 implies $\dim(\delta(\alpha)) \geq n = \mu(\dim(\alpha))$.

Suppose now that $\dim(\delta(\alpha)) > n$.

Then $\dim(\delta(\alpha)) \geq n + 2$ because $\dim \delta(\alpha) \equiv \dim \alpha - 1 \pmod{2}$.

Let z be a cycle in $\text{Fil}_{\text{Hodge}}^{n+2}(\widehat{CC}_n(I))$ representing $\delta(\alpha)$ and identify it with its image in

$$\text{Fil}_{\text{Hodge}}^{n+2}(\widehat{CC}_n(A)) \subset \text{Fil}_{\text{Hodge}}^2(\widehat{CC}_n(A))$$

The algebra A being quasifree the complex $\text{Fil}_{\text{Hodge}}^2(\widehat{CC}_*(A))$ is contractible. If $\nabla: \Omega^1 A \rightarrow \Omega^2 A$ is a connection in the sense of Cuntz-Quillen [CQ1], then a contracting homotopy for $\text{Fil}_{\text{Hodge}}^2(\widehat{CC}_*(A))$ is provided by the operator

$$\nu := \sum_{k=0}^{\infty} (-\nabla B)^k \nabla,$$

where B is Connes differential (see [Kh], [Pu1]).

By the very definition of a connection $\nabla(dI^m) \subset \text{Fil}_I^{m-1}(\Omega A)$ for $n > 0$.

Thus one finds $\nu(x) \in \text{Fil}_I^1(\widehat{\Omega A}) + \prod_{j=0}^{\infty} \Omega^{3m-1+2j} A$ for $x \in \text{Fil}_I^m \Omega^m A$.

For the considered cycle $z \in \text{Fil}_{\text{Hodge}}^{n+2}(\widehat{CC}_n(I))$ this means that $z = \partial(\nu(z))$ where $\nu(z) \in \text{Fil}_I^1(\widehat{CC}_{n+1}(A)) + \text{Fil}_{\text{Hodge}}^{3n+5}(\widehat{CC}_{n+1}(A))$.

The image z' of $\nu(z)$ in $\widehat{CC}_{n+1}(B)$ is then a cycle in $\text{Fil}_{\text{Hodge}}^{3n+5}(\widehat{CC}_{n+1}(B))$. The homology class $\beta = [z'] \in HP_{n+1}(B)$ satisfies by construction

$$\delta(\beta) = [z] = \delta(\alpha) \text{ and } \dim(\beta) \geq 3n + 5.$$

If the boundary map $\delta: HP_{*+1}(B) \rightarrow HP_*(I)$ is injective, i.e. if the natural map $HP_*(A) \rightarrow HP_*(B)$ is zero, then it follows that

$$\alpha = \beta \text{ and } \dim(\alpha) \geq 3n + 5$$

which contradicts our assumption about $\dim(\alpha)$. The proof in the remaining cases is similar. $\square$

Now we can come back to the proofs of the previous theorems.

Proof of Theorem 4.2:

Proposition 4.4 implies the estimate

$$\varphi(n) \leq 3n + 3$$

for the shift function of the boundary map in periodic cyclic cohomology. We have to provide now sufficiently many examples of extensions of the type considered in theorem 4.5 to show that this estimate is in fact sharp.

Recall the notions of the algebraic cone and suspension functors [Wo]: If A is a unital algebra denote by CA the unital algebra of infinite matrices over A satisfying

- the set of entries is finite
- in every line and column there are only finitely many nonzero entries

it is called the cone of A .

The stable matrix algebra $M_\infty(A) = \varinjlim_n M_n(A)$ forms a twosided ideal in the cone CA and its quotient ΣA is called the algebraic suspension of A .

If A is unital, its suspension is unital, too, and the stable matrix algebra $M_\infty(A)$ is H -unital so that the extension

$$0 \rightarrow M_\infty(A) \rightarrow CA \rightarrow \Sigma A \rightarrow 0$$

satisfies excision in Hochschild- and cyclic homology [Wo].

As the cone is contractible in the sense that $HH_*(CA, CA) = 0$ [Wo] one deduces that $HH_*(\Sigma A, \Sigma A) \simeq HH_{*-1}(A, A)$.

Let now k be the ground field and let $\Sigma^{3n+3}k$ be its $(3n+3)$ -fold algebraic suspension. Consider the universal linear split extension [Cu]

$$0 \rightarrow J(\Sigma^{3n+3}k) \rightarrow T(\Sigma^{3n+3}k) \rightarrow \Sigma^{3n+3}k \rightarrow 0.$$

The Hochschild homology of $\Sigma^{3n+3}k$ vanishes except in degree $3n+3$ where it is of dimension one. The SBI-sequence implies then that $HP_*(\Sigma^{3n+3}k) \simeq k$ and that the dimension of a generator is $3n+3$. The tensor algebra $T(\Sigma^{3n+3}k)$ being both quasifree and contractible one deduces from theorem 4.5 that

$HP_*(J(\Sigma^{3n+3}k)) \simeq k$ and that the dimension of a generator equals $\mu(3n+3) = n$. This implies that the composition

$$HP_{n+1}(\Sigma^{3n+3}k) \xrightarrow{\delta} HP_n(J(\Sigma^{3n+3}k)) \rightarrow HC_n(J(\Sigma^{3n+3}k))$$

is injective.

Especially there exists a cyclic cohomology class $\gamma \in HC^n(J(\Sigma^{3n+3}k))$ such that $\langle \delta(x), \gamma \rangle = \langle x, \delta(\gamma) \rangle \neq 0$ for $0 \neq x \in HP_{n+1}(\Sigma^{3n+3}k)$.

Consequently $\dim(\gamma) \leq n$ and $\dim(\delta(\gamma)) = 3n + 3$ which establishes the equality $\varphi(n) = 3n + 3$. $\square$

Proof of Theorem 4.3:

The estimate $\dim(\delta(\alpha)) \leq 3 \dim(\alpha) + 3$ of 4.2 yields inductively the inequality $\varphi_m(n) \leq 3^m n + \frac{1}{2}(3^{m+1} - 3)$. The same argument as in the proof of theorem 4.2 applied to the universal, linearly contractible m -extension

$$0 \rightarrow J^m B \rightarrow T(J^{m-1} B) \rightarrow \dots \rightarrow TB \rightarrow B \rightarrow 0$$

for the algebra $B = \Sigma^{\varphi_m(n)} k$ shows that the estimate is actually sharp. $\square$

5 Excision in cyclic homology theories for topological algebras

In this section we discuss the excision problem in the context of topological algebras and topological cyclic theories. Our results imply that excision holds for Connes entire cyclic cohomology [Co2] and the asymptotic cyclic cohomology of [CM], [Pu].

Most interestingly, excision holds for local cyclic cohomology [Pu1], which is the best behaved and most powerful cyclic theory for topological algebras.

We treat the different topological cyclic theories case by case.

Excision in continuous cyclic cohomology

Recall the definition of continuous cyclic (co)homology as introduced and studied in [Co1].

Definition 5.1 *Let A be a Fréchet-algebra.*

- a) *The A -bimodule $\Omega^n A_{\text{top}}$ of differential n -forms over A is the Fréchet space and continuous A -bimodule*

$$\begin{aligned} \Omega^n A_{\text{top}} &:= \tilde{A} \otimes_{\pi} A \otimes_{\pi} \dots \otimes_{\pi} A \\ a^0 da^1 \dots da^n &\leftrightarrow a^0 \otimes a^1 \otimes \dots \otimes a_n \end{aligned}$$

- b) The Fréchet algebra ΩA_{top} is the completion of the space ΩA of algebraic differential forms over A in the coarsest locally convex topology making the projections $\Omega A \rightarrow \Omega^n A_{\text{top}}, n \geq 0$ continuous.
- c) The continuous cyclic bicomplex $CC_*^{\text{top}}(A)$ is the completion of the cyclic bicomplex $CC_*(A)$ in the coarsest locally convex topology making the projections

$$CC_*(A) \rightarrow CC_*(A)/\text{Fil}_{\text{Hodge}}^n CC_*(A) \simeq \bigoplus_{k=0}^{n-1} \Omega^k A_{\text{top}} \oplus \Omega^n A_{\text{top}} / \overline{b[\Omega A_{\text{top}}, A]}$$

for $n \geq 0$, continuous.

- d) The continuous cyclic homology of A is by definition

$$HP_*^{\text{cont}}(A) := H_*(CC_*^{\text{top}}(A))$$

the continuous cyclic cohomology of A equals

$$HP_{\text{top}}^*(A) := H^*(\text{Hom}_{\text{cont}}^*(CC_*^{\text{top}}(A), \mathbb{C}))$$

and in general the bivariant continuous cyclic cohomology of a pair of Fréchet-algebras (A, B) is given by

$$HP_{\text{cont}}^*(A, B) := H^*(\text{Hom}_{\text{cont}}^*(CC_*^{\text{top}}(A), CC_*^{\text{top}}(B)))$$

To extend our proof of excision to the continuous case we make the

Definition and Lemma 5.2

- a) Let A be a Fréchet algebra and let $J \subset A$ be a closed ideal of A . Then J is called a free ideal of A if there exists a closed linear subspace $V \subset J$ such that the multiplication defines a topological isomorphism

$$J \xleftarrow{\simeq} \tilde{A} \otimes_{\pi} V$$

A continuously linearly split extension $0 \rightarrow J \rightarrow A' \rightarrow B \rightarrow 0$ of Fréchet algebras is called a free ideal extension if $J \subset A'$ is a free ideal.

- b) Let $0 \rightarrow I \rightarrow A \xrightarrow{f} B \rightarrow 0$ be a continuously linearly split extension of Fréchet-algebras. Equip the tensor algebra $\mathcal{T}A$ with the Fréchet topology for which the linear isomorphism $\mathcal{T}A \simeq \Omega^{\text{ev}} A$ of [CQ1]-1.2 extends to an isomorphism of Fréchet-spaces $\mathcal{T}A \simeq \Omega^{\text{ev}} A_{\text{top}}$.

Then $0 \rightarrow J \rightarrow \mathcal{T}A \xrightarrow{f \circ \pi} B \rightarrow 0$ of Fréchet-algebras is a free ideal extension of Fréchet algebras.

It is easily verified that the constructions and results of sections one, two and three carry over to the context of continuous Hochschild- and cyclic complexes associated to free ideal extensions of Fréchet algebras.

Especially it follows that excision holds in continuous periodic cyclic (co)homology for the free ideal extension $0 \rightarrow J \rightarrow \mathcal{T}A \rightarrow B \rightarrow 0$ of 5.2.

As moreover Goodwillie's Theorem holds in the sense that the projections $\mathcal{T}A \rightarrow A$ and $J \rightarrow I$ are continuous HP -equivalences we obtain

Theorem 5.3 (Excision in Continuous Cyclic Homology)

i) Let $0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$ be a continuously linearly split extension of Fréchet algebras. Then the natural inclusions

$$CC_*^{\text{top}}(I) \subset CC_*^{\text{top}}(A, B) \subset \text{Cone}(CC_*^{\text{top}}(A), CC_*^{\text{top}}(B))[-1]$$

are natural continuous chain homotopy equivalences.

ii) Every continuously linearly split extension of Fréchet-algebras induces six-term exact sequences in bivariant continuous periodic cyclic cohomology. These six-term exact sequences are natural with respect to maps of extensions and compatible with the natural forgetful transformation from continuous to algebraic periodic cyclic cohomology.

The notion of dimension of classes in periodic cyclic (co)homology carries immediately over from the algebraic to the continuous context. Concerning the dimension shift of the boundary map we have

Theorem 5.4 *All results of section four, i.e. theorems 4.2, 4.3, 4.4 and 4.5 are also valid for continuous cyclic (co)homology.*

Proof: The demonstration of 4.4 carries over directly to the continuous context. Theorem 4.5 still holds if one supposes that the algebra A in the middle of the considered extension is continuously quasifree which means that A satisfies any of the equivalent conditions

- A has the lifting property with respect to nilpotent extensions of Fréchet-algebras with continuous linear splittings.
- there exists a continuous connection $\nabla: \Omega_{\text{top}}^1 A \rightarrow \Omega_{\text{top}}^2 A$.

The only part which has to be modified in the continuous context are the examples of extensions with maximal dimension shift. Once this is done, the continuous versions of 4.2 and 4.3 follow immediately.

Let us consider the Fréchet algebras $A_{3n+3} := C^\infty(S^{3n+3})$ of smooth functions on the $(3n+3)$ -sphere and the continuously linearly split extension

$$0 \rightarrow J^{\text{top}} A_{3n+3} \rightarrow T^{\text{top}} A_{3n+3} \rightarrow A_{3n+3} \rightarrow 0$$

where $T^{\text{top}} A$ is the completion of the tensor algebra TA in the coarsest Fréchet topology making the projections $TA \rightarrow A^{\otimes_n^2}$ for $n \geq 1$ continuous. It is well known that both $HP_*^{\text{cont}}(A_{3n+3})$ and $HP_{\text{cont}}^*(A_{3n+3})$ are twodimensional with one generator of dimension zero and the other, the fundamental class, of dimension $3n+3$. Moreover, the homological fundamental class α is detected by the cohomological fundamental class α' under the pairing between homology and cohomology.

By 4.4 the class $\delta(\alpha) \in HP_n^{\text{cont}}(J^{\text{top}}A_{3n+3})$ can be represented by a cycle z in $\Omega_{\text{top}}^{\geq n}(J^{\text{top}}A_{3n+3})$. Suppose now that the image $\pi(z)$ of z in

$$CC_*^{\text{top}}(J^{\text{top}}A_{3n+3})/\text{Fil}_{\text{Hodge}}^{n+2}CC_*^{\text{top}}(J^{\text{top}}A_{3n+3})$$

is contained in the closure of the subspace of boundaries. Then there exists a sequence of cycles $z_k \in CC_n^{\text{top}}(J^{\text{top}}A_{3n+3})$ representing the class $\delta(\alpha)$ such that

$$\lim_{k \rightarrow \infty} \pi(z_k) = 0 \quad \text{in} \quad CC_n^{\text{top}}(J^{\text{top}}A_{3n+3})/\text{Fil}_{\text{Hodge}}^{n+2}CC_n^{\text{top}}(J^{\text{top}}A_{3n+3})$$

The complex $CC_*^{\text{top}}(T^{\text{top}}A_{3n+3})$ fits into a continuous linear split extension

$$0 \rightarrow \text{Fil}_{\text{Hodge}}^2 CC_*^{\text{top}}(T^{\text{top}}A_{3n+3}) \rightarrow CC_*^{\text{top}}(T^{\text{top}}A_{3n+3}) \rightarrow X_*^{\text{top}}(T^{\text{top}}A_{3n+3}) \rightarrow 0$$

(A linear splitting on “one forms” is given by $s_1 := \text{Id} - b\nabla$). The tensor algebra being contractible and continuously quasifree, the X -complex $X_*^{\text{top}}(T^{\text{top}}A_{3n+3})$ is naturally contractible by the Cartan homotopy formula.

The tensor algebra being quasifree, the complex $\text{Fil}_{\text{Hodge}}^2 CC_*^{\text{top}}(T^{\text{top}}A_{3n+3})$ is contractible, a contracting homotopy being described in the proof of 4.5.

Both contracting homotopies together give rise to a continuous contracting homotopy h' of $CC_*^{\text{top}}(T^{\text{top}}A_{3n+3})$.

The cycles z_k map under transgression in the diagram

$$CC_*^{\text{top}}(J^{\text{top}}A_{3n+3}) \xrightarrow{i} CC_*^{\text{top}}(T^{\text{top}}A_{3n+3}) \xrightarrow{p} CC_*^{\text{top}}(A_{3n+3})$$

to cycles $v_k := p \circ h' \circ i(z_k)$ representing the fundamental class $\alpha \in HP_{n+1}^{\text{cont}}(A_{3n+3})$ and satisfying

$$\lim_{k \rightarrow \infty} \pi(v_k) = 0 \quad \text{in} \quad CC_{n+1}^{\text{top}}(A_{3n+3})/\text{Fil}_{\text{Hodge}}^{3n+5}CC_{n+1}^{\text{top}}(A_{3n+3})$$

This would however contradict the fact that the fundamental class α is detected by a continuous cocycle α' of dimension $3n + 3$.

Thus the image of the cycle z in

$$CC_n^{\text{top}}(J^{\text{top}}A_{3n+3})/\text{Fil}_{\text{Hodge}}^{n+2}CC_n^{\text{top}}(J^{\text{top}}A_{3n+3})$$

is not contained in the closure of the subspace of boundaries.

The Hahn-Banach Theorem guarantees then the existence of a continuous cocycle $\beta' \in CC_{\text{cont}}^n(J^{\text{top}}A_{3n+3})$ of dimension n satisfying

$$\langle \beta', \delta(\alpha) \rangle = \langle \delta(\beta'), \alpha \rangle \neq 0.$$

Thus $\dim[\beta'] \leq n$, $\dim \delta([\beta']) = \dim \alpha' = 3n + 3$ and the continuous versions of 4.2 and 4.3 are established. $\square$

Excision in cyclic and local cyclic cohomology of Ind-algebras with supports

We now study the excision problem for cyclic and local cyclic (co)homology of Ind-algebras equipped with a system of supports. These have been introduced and studied in [Pu1] and generalize (and contain as special case) the

entire cyclic cohomology of Connes [Co2] and the asymptotic cyclic cohomology introduced by Connes and Moscovici [CM] and developed in [Pu]. For the definition and the basic properties of Ind-algebras with supports we refer the reader to [Pu1].

First of all we have to clarify the notion of an extension of Ind-Algebras with supports.

Definition 5.5

a) An **extension** of Ind-(Fréchet)-algebras with supports is a diagram

$$0 \rightarrow (\mathcal{I}, \mathcal{S}') \rightarrow (\mathcal{A}, \mathcal{S}) \rightarrow (\mathcal{B}, \mathcal{S}'') \rightarrow 0$$

in the category of Ind-(Fréchet)-algebras and support preserving homomorphisms such that

$$\begin{array}{ccc} (\mathcal{I}, \mathcal{S}') & \rightarrow & 0 \\ \downarrow & & \downarrow \\ (\mathcal{A}, \mathcal{S}) & \rightarrow & (\mathcal{B}, \mathcal{S}'') \end{array}$$

is a pullback square and

$$\begin{array}{ccc} 0 & \rightarrow & (\mathcal{B}, \mathcal{S}'') \\ \uparrow & & \uparrow \\ (\mathcal{I}, \mathcal{S}') & \rightarrow & (\mathcal{A}, \mathcal{S}) \end{array}$$

is a pushout square.

b) A **linearly split extension** of Ind-(Fréchet)-algebras with supports is an extension

$$0 \rightarrow (\mathcal{I}, \mathcal{S}') \rightarrow (\mathcal{A}, \mathcal{S}) \xrightarrow[p]{s} (\mathcal{B}, \mathcal{S}'') \rightarrow 0$$

together with a linear, support preserving map of Ind-algebras

$s : (\mathcal{B}, \mathcal{S}'') \rightarrow (\mathcal{A}, \mathcal{S})$ satisfying $p \circ s = \text{Id}_{(\mathcal{B}, \mathcal{S}'')}$.

c) A linearly split extension of Ind-(Fréchet)-algebras with supports is called **strict** if it is isomorphic to a formal inductive limit

$$\varinjlim_{\alpha} \quad 0 \rightarrow (I_{\alpha}, \mathcal{S}'_{\alpha}) \rightarrow (A_{\alpha}, \mathcal{S}_{\alpha}) \xrightarrow[p_{\alpha}]{s_{\alpha}} (B_{\alpha}, \mathcal{S}''_{\alpha}) \rightarrow 0$$

of linearly split extensions.

d) An extension of Ind-(Fréchet)-algebras with supports is called **locally strict** if it is isomorphic to a formal inductive limit

$$\varinjlim_{\alpha} \quad 0 \rightarrow (I_{\alpha}, \mathcal{S}'_{\alpha}) \rightarrow (A_{\alpha}, \mathcal{S}_{\alpha}) \rightarrow (B_{\alpha}, \mathcal{S}''_{\alpha}) \rightarrow 0$$

where for each α the epimorphism $p_{\alpha} : A_{\alpha} \rightarrow B_{\alpha}$ admits a linear section s_{α} such that the composition $B_{\alpha} \xrightarrow{s_{\alpha}} A_{\alpha} \xrightarrow{j} A$ satisfies $(j \circ s_{\alpha})(\mathcal{S}''_{\alpha}) \prec \mathcal{S}$.

Remark 5.6 *If in a diagram*

$$0 \rightarrow (\mathcal{I}, \mathcal{S}') \xrightarrow{i} (\mathcal{A}, \mathcal{S}) \xrightarrow{p} (\mathcal{B}, \mathcal{S}'') \rightarrow 0$$

of Ind-(Fréchet)-algebras with supports the map i induces an isomorphism $i : (\mathcal{I}, \mathcal{S}') \xrightarrow{\cong} \ker p$ and if any of the conditions of b), c) and d) hold, then the diagram is automatically an extension, i.e. $(\mathcal{B}, \mathcal{S}'')$ is a pushout as asked for in a).

After this short discussion of the notion of extension of Ind-algebras with support we come to the central excision theorem of this paper. It states that excision holds for well behaved extensions of Ind-Fréchet algebras with supports.

As a special case of this we obtain that excision holds in Connes (bivariant) entire cyclic cohomology of Banach algebras.

Theorem 5.7 (Excision for cyclic cohomology of Ind algebras with supports)

Let

$$0 \rightarrow (\mathcal{I}, \mathcal{S}') \xrightarrow{i} (\mathcal{A}, \mathcal{S}) \xrightleftharpoons[p]{s} (\mathcal{B}, \mathcal{S}'') \rightarrow 0$$

be a strictly linearly split extension of Ind-algebras (Ind-Fréchet-algebras) with strictly convex (and complete) supports.

- a) *Denote by $(\mathcal{TA}, \mathcal{TS})$ the universal linear deformation of $(\mathcal{A}, \mathcal{S})$ [Pu1]. The natural morphism*

$$X_*(\mathcal{TI}, \mathcal{TS}') \rightarrow \text{Cone} (X_*(\mathcal{TA}, \mathcal{TS}), X_*(\mathcal{TB}, \mathcal{TS}''))[-1]$$

of X -complexes of universal linear deformations is an isomorphism in the homotopy category of Ind-complexes of complete, locally convex vector spaces.

- b) *The given extension induces natural six term exact sequences*

$$\begin{array}{ccccc} HC^*(-, (\mathcal{I}, \mathcal{S}')) & \xrightarrow{i_*} & HC^*(-, (\mathcal{A}, \mathcal{S})) & \xrightarrow{p_*} & HC^*(-, (\mathcal{B}, \mathcal{S}'')) \\ \delta \uparrow & & & & \downarrow \delta \\ HC^{*+1}(-, (\mathcal{B}, \mathcal{S}'')) & \xrightarrow{p_*} & HC^{*+1}(-, (\mathcal{A}, \mathcal{S})) & \xleftarrow{i_*} & HC^{*+1}(-, (\mathcal{I}, \mathcal{S}')) \end{array}$$

$$\begin{array}{ccccc} HC^*((\mathcal{I}, \mathcal{S}'), -) & \xleftarrow{i^*} & HC^*((\mathcal{A}, \mathcal{S}), -) & \xleftarrow{p^*} & HC^*((\mathcal{B}, \mathcal{S}''), -) \\ \delta \downarrow & & & & \uparrow \delta \\ HC^{*+1}((\mathcal{B}, \mathcal{S}''), -) & \xrightarrow{p^*} & HC^{*+1}((\mathcal{A}, \mathcal{S}), -) & \xrightarrow{i^*} & HC^{*+1}((\mathcal{I}, \mathcal{S}'), -) \end{array}$$

in bivariant cyclic cohomology of Ind-(Fréchet)-algebras with supports.

The six term exact sequences are moreover functorial with respect to maps of extensions (not necessarily preserving the linear splitting).

Proof: Denote by $(\mathcal{TA}, \mathcal{TS})$ the universal linear deformation of $(\mathcal{A}, \mathcal{S})$. It suffices to show that the natural morphism

$$X_*(\mathcal{TI}, \mathcal{TS}') \rightarrow \text{Cone}(X_*(\mathcal{TA}, \mathcal{TS}), X_*(\mathcal{TB}, \mathcal{TS}''))[-1]$$

of X -complexes of universal linear deformations is an isomorphism in the homotopy category of Ind-complexes of complete locally convex vector spaces. We will proceed in several steps.

Step 1:

The Cuntz-Quillen Theorem [CQ2] allows to identify the formal inductive limit of X -complexes above with a similar formal inductive limit of cyclic bicomplexes. Thus it has to be verified that the natural morphism

$$\mathcal{CC}_*(\mathcal{I}, \mathcal{S}') \rightarrow \text{Cone}(\mathcal{CC}_*(\mathcal{A}, \mathcal{S}), \mathcal{CC}_*(\mathcal{B}, \mathcal{S}''))[-1]$$

is a chain homotopy equivalence.

Step 2:

For each α , the linearly split extension of algebras

$$0 \rightarrow I_\alpha \rightarrow A_\alpha \xrightarrow{s_\alpha} B_\alpha \rightarrow 0$$

gives rise to the free ideal extension

$$0 \rightarrow J_\alpha \rightarrow TA_\alpha \rightarrow B_\alpha \rightarrow 0$$

and the associated linearly split extension of complexes

$$0 \rightarrow \mathcal{CC}_*^>(TA_\alpha)_{J_\alpha} \rightarrow \mathcal{CC}_*(TA_\alpha) \rightarrow \mathcal{CC}_*^<(TA_\alpha)_{J_\alpha} \rightarrow 0$$

These form an inductive system as α varies because the linear sections s_α are supposed to fit together to a strict linear section of the original extension of Ind-algebras.

Step 3:

Recall [Pu1] that the construction of the universal linear deformation of Ind-algebras can be iterated and yields canonically homotopy equivalent Ind-algebras

$$(\mathcal{T}(\mathcal{TA}), \mathcal{T}(\mathcal{TS})) \xleftarrow[\mathcal{T}_\rho]{\pi} (\mathcal{TA}, \mathcal{TS})$$

Taking X -complexes and applying the Cuntz-Quillen Theorem as in step one, one obtains natural morphisms of Ind-complexes

$$\mathcal{CC}_*(\mathcal{TA}, \mathcal{TS}) \xleftarrow[\mathcal{T}_{\rho*}]{\pi_*} \mathcal{CC}_*(\mathcal{A}, \mathcal{S})$$

which are isomorphisms inverse to each other in the homotopy category of Ind-complexes. Moreover $\pi_* \circ \mathcal{T}_{\rho*} = \text{Id}$ on the level of Ind-complexes.

Step 4:

The Ind-complex $\mathcal{CC}_*(\mathcal{TA}, \mathcal{TS})$ is a formal inductive limit of completions of subcomplexes of $CC_*(TA_\alpha)$. The extensions of complexes in Step two give rise after completion to an extension

$$0 \rightarrow \mathcal{CC}_*^>(\mathcal{TA}, \mathcal{TS})_{\mathcal{J}} \rightarrow \mathcal{CC}_*(\mathcal{TA}, \mathcal{TS}) \rightarrow \mathcal{CC}_*^{\leq}(\mathcal{TA}, \mathcal{TS})_{\mathcal{J}} \rightarrow 0$$

of Ind-complexes with continuous, linear section.

The morphisms in this extensions fit into the natural commutative diagrams

$$\begin{array}{ccc} \mathcal{CC}_*(\mathcal{I}, \mathcal{S}') & \rightarrow & \mathcal{CC}_*(\mathcal{A}, \mathcal{S}) \\ \mathcal{T}\rho_* \downarrow & & \downarrow \mathcal{T}\rho_* \\ \mathcal{CC}_*^>(\mathcal{TA}, \mathcal{TS})_{\mathcal{J}} & \rightarrow & \mathcal{CC}_*(\mathcal{TA}, \mathcal{TS}) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{CC}_*(\mathcal{TA}, \mathcal{TS}) & \rightarrow & \mathcal{CC}_*^{\leq}(\mathcal{TA}, \mathcal{TS})_{\mathcal{J}} \\ \pi_* \downarrow & & \downarrow \pi_* \\ \mathcal{CC}_*(\mathcal{A}, \mathcal{S}) & \rightarrow & \mathcal{CC}_*(\mathcal{B}, \mathcal{S}'') \end{array}$$

where the vertical morphisms in the second diagram are chain homotopy equivalences of Ind-complexes by the topological version of Goodwillie's Theorem [Pu1].

Step 5:

The natural morphism

$$\mathcal{CC}_*(\mathcal{I}, \mathcal{S}') \rightarrow \text{Cone}(\mathcal{CC}_*(\mathcal{A}, \mathcal{S}), \mathcal{CC}_*(\mathcal{B}, \mathcal{S}''))[-1]$$

factorizes by step three and four as

$$\begin{array}{ccc} & \mathcal{CC}_*(\mathcal{I}, \mathcal{S}') & \\ \mathcal{T}\rho_* \searrow & \mathcal{CC}_*^>(\mathcal{TA}, \mathcal{TS})_{\mathcal{J}} & \\ j \searrow & \text{Cone}(\mathcal{CC}_*(\mathcal{TA}, \mathcal{TS}), \mathcal{CC}_*^{\leq}(\mathcal{TA}, \mathcal{TS})_{\mathcal{J}})[-1] & \\ \text{Cone}(\pi_*) \searrow & \text{Cone}(\mathcal{CC}_*(\mathcal{A}, \mathcal{S}), \mathcal{CC}_*(\mathcal{B}, \mathcal{S}''))[-1] & \end{array}$$

The morphisms j and $\text{cone}(\pi_*)$ are isomorphisms in the homotopy category of Ind-complexes; the morphism j because the extension of complexes introduced in step four possesses a linear splitting and the morphism $\text{cone}(\pi_*)$ by the second diagram in step four. Thus it remains to verify that

$$\mathcal{T}\rho_*: \mathcal{CC}_*(\mathcal{I}, \mathcal{S}') \rightarrow \mathcal{CC}_*^>(\mathcal{TA}, \mathcal{TS})_{\mathcal{J}}$$

is an isomorphism, too.

Step 6:

Recall that we constructed in section three for every linearly split extension $0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$ of k -algebras an operator

$\chi_{cyc}: CC_*^>(TA)_J \rightarrow CC_{*+1}(TA, B)$ such that $\phi := \text{Id} - (\partial \circ \chi_{cyc} + \chi_{cyc} \circ \partial)$

maps $CC_*^>(TA)_J$ into $CC_*(J)$. The naturality of those maps implies that they give rise to an operator

$$\chi' := \pi_* \circ \chi_{\text{cyc}} : \varinjlim_{\alpha} CC_*^>(TA_{\alpha})_{J_{\alpha}} \rightarrow \varinjlim_{\alpha} CC_{*+1}(A_{\alpha}, B_{\alpha})$$

and a chain map

$$\phi' := \pi_* \circ \phi : \varinjlim_{\alpha} CC_*^>(TA_{\alpha})_{J_{\alpha}} \rightarrow \varinjlim_{\alpha} CC_*(I_{\alpha})$$

of Ind-complexes. The content of the Appendix, summarized in proposition 7.9, is equivalent to the assertion that these maps are continuous in the sense that they extend to an operator

$$\chi' : CC_*^>(\mathcal{TA}, \mathcal{TS})_{\mathcal{J}} \rightarrow CC_{*+1}((\mathcal{A}, \mathcal{S}), (\mathcal{B}, \mathcal{S}''))$$

and a chain map

$$\phi' : CC_*^>(\mathcal{TA}, \mathcal{TS})_{\mathcal{J}} \rightarrow CC_*(\mathcal{I}, \mathcal{S}')$$

We claim that the chain map ϕ' is homotopy inverse to

$$\mathcal{T}\rho_* : CC_*(\mathcal{I}, \mathcal{S}') \rightarrow CC_*^>(\mathcal{TA}, \mathcal{TS})_{\mathcal{J}}$$

which would finish the proof. The composition $\phi' \circ \mathcal{T}\rho_*$ equals the identity.

By arguments similar to those in step four, using the topological version of Goodwillie's Theorem [Pu1], the inclusion of Ind-complexes

$$j' : CC_*^>(\mathcal{TA}, \mathcal{TS})_{\mathcal{J}} \rightarrow CC_*((\mathcal{TA}, \mathcal{TS}), (\mathcal{TB}, \mathcal{TS}''))$$

is a chain homotopy equivalence. In order to verify that $\mathcal{T}\rho_* \circ \phi'$ is chain homotopic to the identity, it is therefore sufficient to show that $j' \circ \mathcal{T}\rho_* \circ \phi'$ is chain homotopic to j' .

The homotopy operator

$$H' : CC_*^>(\mathcal{TA}, \mathcal{TS})_{\mathcal{J}} \xrightarrow{\chi'} CC_{*+1}((\mathcal{A}, \mathcal{S}), (\mathcal{B}, \mathcal{S}'')) \xrightarrow{\mathcal{T}\rho_*} CC_{*+1}((\mathcal{TA}, \mathcal{TS}), (\mathcal{B}, \mathcal{S}''))$$

defines by construction a chain homotopy between $j' \circ \mathcal{T}\rho_* \circ \phi'$ and

$$\mathcal{T}\rho_* \circ \pi_* : CC_*^>(\mathcal{TA}, \mathcal{TS})_{\mathcal{J}} \rightarrow CC_*((\mathcal{TA}, \mathcal{TS}), (\mathcal{B}, \mathcal{S}''))$$

and as the latter map is chain homotopic to the inclusion j' by step four we are done. $\square$

Finally we come to excision in local bivariant cyclic cohomology of Ind-Fréchet algebras with supports. Due to the flexibility of the local cyclic theory excision holds here for a considerably larger class of extensions. We do not aim for maximal generality and note that the excision holds for a wider class of extensions than the one mentioned here. Details will be given elsewhere.

Theorem 5.8 (Excision for local cyclic cohomology)

Let

$$0 \rightarrow (\mathcal{I}, \mathcal{S}') \rightarrow (\mathcal{A}, \mathcal{S}) \rightarrow (\mathcal{B}, \mathcal{S}'') \rightarrow 0$$

be a locally strict extension of Ind-algebras (Ind-Fréchet-algebras) with strictly convex (and complete) supports.

- i) Denote by $(\mathcal{TA}, \mathcal{TS})$ the universal linear deformation of $(\mathcal{A}, \mathcal{S})$.
The natural morphism

$$X_*(\mathcal{TI}, \mathcal{TS}') \rightarrow \text{Cone}(X_*(\mathcal{TA}, \mathcal{TS}), X_*(\mathcal{TB}, \mathcal{TS}''))[-1]$$

of X -complexes of universal linear deformations is an isomorphism in the derived Ind-category of complexes of complete locally convex vector spaces.

- ii) The given extension induces six term exact sequences in local cyclic cohomology of Ind-(Fréchet)-algebras with supports similar to those in 5.7. The six-term exact sequences are natural with respect to maps of extensions and for a strictly linearly split extension they are compatible with the forgetful transformation from cyclic to local cyclic (co)homology of Ind-(Fréchet)-algebras with supports.

Proof: Repeating the procedure of the proof of theorem 5.7 one sees that the local strictness of the considered extension is sufficient to be able to apply criterion 2.11 of [Pu1] to conclude that the natural morphism

$$\mathcal{CC}_*(\mathcal{I}, \mathcal{S}') \rightarrow \text{Cone}(\mathcal{CC}_*(\mathcal{A}, \mathcal{S}), \mathcal{CC}_*(\mathcal{B}, \mathcal{S}''))[-1]$$

is an isomorphism in the derived Ind-category of complexes of complete, locally convex vector spaces. The compatibilities claimed in the statement are obvious. $\square$

6 Consequences

We present now two applications of the general excision theorems of the previous section.

The first concerns Connes entire cyclic cohomology of Banach algebras. This theory was introduced by Connes [Co2] as target of a Chern character for theta-summable Fredholm-modules. Connes also showed that the entire cyclic and periodic cyclic cohomology of the algebra of complex numbers coincide. It turned out however that entire cyclic cohomology is very difficult to calculate and essentially no calculations of entire cyclic cohomology groups except for those implied by smooth homotopy invariance and Morita invariance have been carried out successfully. (The only exception might be the calculation of the entire cyclic cohomology of the algebra of Laurent polynomials by Connes in

[Co] and the calculation of the entire cyclic cohomology of amenable Banach algebras by Khalkhali in [Kh].)

The results of the previous sections imply that entire cyclic cohomology satisfies excision for continuously linearly split extensions of Banach algebras. An elementary comparison argument shows then that entire and continuous periodic cyclic cohomology groups coincide on a large class of Banach algebras. This allows to determine the entire cyclic cohomology of Banach algebras in a large number of (hitherto unknown) cases.

Comparison of entire and continuous periodic cyclic cohomology

Let A be a Banach algebra or a nuclear, admissible Fréchet algebra. Then the family of bounded subsets of an open unit ball defines the system of bounded supports. It was shown in [Pu1], that the cyclic cohomology of a Banach- or nuclear admissible Fréchet algebra equipped with the system of bounded supports equals A.Connes entire cyclic cohomology. Thus the excision theorem 5.7 implies

Theorem 6.1 *Let*

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

be an extension of Banach- or nuclear admissible Fréchet algebras possessing a bounded, linear section. Then this extension induces a six-term exact sequence of entire cyclic cohomology groups

$$\begin{array}{ccccc} HC_{\varepsilon}^*(I) & \longleftarrow & HC_{\varepsilon}^*(A) & \longleftarrow & HC_{\varepsilon}^*(B) \\ \delta \downarrow & & & & \uparrow \delta \\ HC_{\varepsilon}^{*+1}(B) & \longrightarrow & HC_{\varepsilon}^{*+1}(A) & \longrightarrow & HC_{\varepsilon}^{*+1}(I) \end{array}$$

The same holds for entire cyclic homology and bivariant entire cyclic cohomology.

The forgetful transformation $HP_{\text{cont}}^* \rightarrow HC_{\varepsilon}^*$ from periodic to entire cyclic cohomology carries the long exact cohomology sequence of periodic cyclic cohomology groups associated to an extension to the corresponding long exact sequence of entire cyclic cohomology groups. This is evident from the construction of the long exact cohomology sequences. From this one can deduce the following consequence about the natural transformation from periodic to entire cyclic cohomology.

Theorem 6.2 *Let $\mathcal{C}$ be the smallest class of Banach- and nuclear admissible Fréchet algebras satisfying*

- $\mathbb{C} \in \mathcal{C}$
- $\mathcal{C}$ is closed under smooth homotopy equivalence
- *If in an extension with bounded linear splitting two algebras belong to $\mathcal{C}$ then also the third one.*

Then for every algebra A in $\mathcal{C}$ the natural forgetful transformation from periodic to entire cyclic cohomology induces an isomorphism

$$HP_{\text{cont}}^*(A) \xrightarrow{\simeq} HC_{\varepsilon}^*(A)$$

Proof: Consider the class $\mathcal{C}'$ of Banach- or nuclear admissible Fréchet-algebras for which the transformation from periodic to entire cyclic cohomology is an isomorphism. Then by Connes paper [Co2], $\mathcal{C}'$ contains the algebra of complex numbers. As both periodic and entire cyclic cohomology are smooth homotopy functors [Co] the class $\mathcal{C}'$ is closed under smooth homotopy equivalence. If finally an extension is given for which two of the involved algebras belong to $\mathcal{C}'$ then the excision theorem (3.10),(5.7), and the five lemma imply that also the third algebra in the extension belongs to $\mathcal{C}'$. Thus $\mathcal{C} \subset \mathcal{C}'$ and the theorem is proved. $\square$

As application of this we present the solution of a rather long standing problem in entire cyclic cohomology, the structure of the entire cyclic cohomology of $\mathcal{C}^\infty(S^1)$ or more general of the algebra of smooth functions on a smooth compact manifold.

Theorem 6.3 *Let M be a smooth, closed manifold and let $\mathcal{C}^\infty(M)$ be the nuclear admissible Fréchet- algebra of smooth functions on M . Then*

$$HC_{\varepsilon}^*(\mathcal{C}^\infty(M)) \simeq HP^*(\mathcal{C}^\infty(M)) \simeq H_*(M, \mathbb{C})$$

where on the right hand side $\mathbb{Z}/2$ - graded singular homology is understood.

First of all we need the well known

Lemma 6.4 *Let M be a smooth, closed manifold and let $N \subset M$ be a codimension one submanifold. Let $C^\infty(M, N)$ respectively $C_0^\infty(M, N)$ be the algebras of smooth functions on M vanishing along N , respectively vanishing of infinite order along N .*

Then the inclusion

$$C_0^\infty(M, N) \hookrightarrow C^\infty(M, N)$$

is a smooth homotopy equivalence.

Proof: Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be a strictly monotone increasing, smooth homeomorphism of the real line which is a diffeomorphism outside the origin, equals the identity outside $[-1, 1]$, satisfies $\varphi(-x) = -\varphi(x)$ and has vanishing Taylor series at the origin.

Let $N \subset M$ be the given codimension one submanifold. Then a tubular neighbourhood W of N in M can be identified with the total space of the normal bundle of N in M . We obtain

$$\tilde{N} \times_{\mathbb{Z}/2} \mathbb{R} \simeq \nu(N, M) \simeq W \subset M$$

where $\tilde{N}$ is the covering (of degree one or two) of N on which the normal bundle $\nu(N, M)$ becomes trivial.

One can now extend the smooth homeomorphism $\text{Id} \times \varphi$ of $\tilde{N} \times_{\mathbb{Z}/2} \mathbb{R}$ to a smooth homeomorphism ψ of M by putting it equal to the identity outside W .

The algebra homomorphism $\psi^*: C^\infty(M) \rightarrow C^\infty(M)$ maps $C^\infty(M, N)$ to $C_0^\infty(M, N)$ and is obviously an inverse to the inclusion $C_0^\infty(M, N) \hookrightarrow C^\infty(M, N)$ up to smooth homotopy. $\square$

Proof of Theorem 6.3: Let M be a smooth closed manifold. Let N be a smooth, closed submanifold $N \subset M$ of codimension one which decomposes M into two pieces $M_1, M_2 : M = M_1 \cup M_2, N = M_1 \cap M_2$, and (M_1, N) and (M_2, N) are smooth, compact manifolds with boundary. We show that if $\mathcal{C}^\infty(N), \mathcal{C}^\infty(M_1)$ and $\mathcal{C}^\infty(M_2)$ belong to the class $\mathcal{C}$ of theorem 6.2 then also $\mathcal{C}^\infty(M)$.

Consider the linear split extensions

$$\begin{aligned} 0 &\rightarrow \mathcal{C}^\infty(M_1, N) \rightarrow \mathcal{C}^\infty(M_1) \rightarrow \mathcal{C}^\infty(N) \rightarrow 0 \\ 0 &\rightarrow \mathcal{C}^\infty(M_2, N) \rightarrow \mathcal{C}^\infty(M_2) \rightarrow \mathcal{C}^\infty(N) \rightarrow 0 \end{aligned}$$

By assumption $\mathcal{C}^\infty(M_1), \mathcal{C}^\infty(M_2), \mathcal{C}^\infty(N) \in \mathcal{C}$ and therefore we know that $\mathcal{C}^\infty(M_1, N), \mathcal{C}^\infty(M_2, N) \in \mathcal{C}$. But then also $\mathcal{C}_0^\infty(M_1, N), \mathcal{C}_0^\infty(M_2, N) \in \mathcal{C}$ by Lemma (6.4). Consequently $\mathcal{C}_0^\infty(M, N) \simeq \mathcal{C}_0^\infty(M_1, N) \oplus \mathcal{C}_0^\infty(M_2, N) \in \mathcal{C}$ and running the argument backwards $\mathcal{C}^\infty(M, N) \in \mathcal{C}$ and $\mathcal{C}^\infty(M) \in \mathcal{C}$ by considering the linear split extension

$$0 \rightarrow \mathcal{C}^\infty(M, N) \rightarrow \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(N) \rightarrow 0.$$

The theorem follows then inductively by using a handle decomposition of M which can be obtained from some Morse function. $\square$

The bivariant Chern-Connes-character

Another consequence of excision in cyclic homology theories is the existence of a bivariant Chern-Connes-character on bivariant K -theory with values in a suitable bivariant cyclic cohomology theory.

Partial results in this direction have been obtained by Nistor [Ni] who constructed a bivariant Chern-Connes-character for finitely summable Fredholm modules. Constructions of a full bivariant Chern-Connes-character, i.e. a natural transformation of bifunctors have been achieved in [Pu], where a stable Chern-Connes-character on Kasparov KK -theory with values in bivariant asymptotic cyclic cohomology was invented and in [Cu], where a bivariant Chern-Connes-character on Cuntz kk -theory with values in bivariant periodic continuous cyclic cohomology was defined. The following theorem contains and generalizes both mentioned results. Especially it provides an unstable Chern-Connes-character on Kasparov KK -theory which is much more useful than the stable character of [Pu].

We consider the category of separable C^* -algebras. The family of compact subsets of the open unit ball of a C^* -algebra defines the system of compact supports. Let $HC_{lc}^*(-, -)$ be the bivariant local cyclic cohomology of C^* -algebras equipped with the system of compact supports [Pu2]. Then the local cyclic cohomology $HC_{lc}^*(-, -)$ defines a continuous homotopy bifunctor [Pu2], and is stable under tensorisation by the algebra of compact operators. Moreover, by theorem 5.8 local cyclic cohomology satisfies excision for extensions of C^* -algebras with bounded linear splitting. Now, by a well known argument of J.Cuntz, every stable and excisive homotopy bifunctor on the category of separable C^* -algebras factorizes through the category KK associated to Kasparov's bivariant K -Theory. Therefore we obtain along the lines of [Pu] the

Theorem 6.5 (Grothendieck-Riemann-Roch-Theorem in Noncommutative Geometry)

- a) *There exists a natural transformation of bifunctors on the category of separable C^* -algebras*

$$\text{ch}_{\text{biv}} : KK^*(-, -) \rightarrow HC_{lc}^*(-, -)$$

*from Kasparov's bivariant KK -theory to bivariant local cyclic cohomology called the **bivariant Chern-Connes-character**.*

- b) *The bivariant Chern-Connes-character is **multiplicative** up to a period factor $2\pi i$.*

For any separable C^ -algebras A, B, C the diagram*

$$\begin{array}{ccc} KK^j(A, B) \otimes KK^l(B, C) & \xrightarrow{\circ} & KK^{j+l}(A, C) \\ \downarrow \text{ch}_{\text{biv}} \otimes \text{ch}_{\text{biv}} & & \downarrow \text{ch}_{\text{biv}} \\ HC_{lc}^j(A, B) \otimes HC_{lc}^l(B, C) & \xrightarrow{\frac{1}{(2\pi i)^j} \circ} & HC_{lc}^{j+l}(A, C) \end{array}$$

commutes, where the upper horizontal map is the Kasparov product and the lower horizontal map is given by $\frac{1}{(2\pi i)^{jl}}$ times the composition product. (See [Pu] for an explanation of the factor $2\pi i$).

c) **(Grothendieck-Riemann-Roch Theorem)**

Let $\text{ch}' := \frac{1}{(2\pi i)^j} \text{ch} : K_j \rightarrow HC_j^{lc}$ be the normalized Chern character on K -theory, and let $\alpha \in KK^l(A, B)$. Then the diagram

$$\begin{array}{ccc} K_j(A) & \xrightarrow{-\otimes \alpha} & K_{j+l}(B) \\ \downarrow \text{ch}' & & \downarrow \text{ch}' \\ HC_j^{lc}(A) & \xrightarrow{-\circ \frac{1}{(2\pi i)^{jl}} \text{ch}_{\text{biv}}(\alpha)} & HC_{j+l}^{lc}(B) \end{array}$$

commutes.

d) Let $0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$ be an extension of separable C^* -algebras with completely positive splitting. Then the bivariant Chern-Connes-character is compatible with long exact sequences, i.e. the diagrams

$$\begin{array}{ccccccc} \longrightarrow & KK^j(-, B) & \xrightarrow{\delta} & KK^{j-1}(-, I) & \longrightarrow \\ & \downarrow \text{ch}_{\text{biv}} & & \downarrow \text{ch}_{\text{biv}} & \\ \longrightarrow & HC_{lc}^j(-, B) & \xrightarrow{(2\pi i)^j \delta} & HC_{lc}^{j-1}(-, I) & \longrightarrow \end{array}$$

and

$$\begin{array}{ccccccc} \longleftarrow & KK^{j+1}(B, -) & \xleftarrow{\delta} & KK^j(I, -) & \longleftarrow \\ & \downarrow \text{ch}_{\text{biv}} & & \downarrow \text{ch}_{\text{biv}} & \\ \longleftarrow & HC_{lc}^{j+1}(B, -) & \xleftarrow{(2\pi i)^j \delta} & HC_{lc}^j(I, -) & \longleftarrow \end{array}$$

commute.

Similar results hold for the different topological versions of bivariant K -theory in the sense of Cuntz.

The question in section 8 of [Cu] concerning the compatibility of the natural filtrations on bivariant K -theory and bivariant continuous periodic cyclic cohomology is affirmatively answered by theorems 4.5 and 5.4.

7 Appendix

In the appendix we want to make the homotopy operator χ_{cyc} associated to the free ideal extension

$$0 \rightarrow J \rightarrow TA \rightarrow B \rightarrow 0$$

sufficiently explicit to be able to verify its continuity properties.

First of all we need explicit formulas for the harmonic decomposition of $C_*^{\text{bar}}(J)$.

Lemma 7.1 *Let $0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$ be a free ideal extension and let $P' := \psi' \circ \varphi'$ and h' be the operators on $C_*^{\text{bar}}(J)$ defined in 1.11. Then the following formulas are valid:*

- $P'|_{s(\tilde{B})V} = \text{Id}$ and $P'|_{J^2} = 0$
- $P'(s(\tilde{B})V \otimes s(B)V \otimes \dots \otimes s(B)V)$
 $= s(\tilde{B})V \otimes P'(s(B)V \otimes \dots \otimes s(B)V) - s(\tilde{B})V s(B) \otimes P'(V \otimes \dots \otimes s(B)V)$
- $h'(J \otimes \dots \otimes JV \otimes s(B)V \otimes \dots \otimes s(B)V)$
 $= \pm J \otimes \dots \otimes J \otimes P'(V \otimes s(B)V \otimes \dots \otimes s(B)V)$
- $h'(J \otimes \dots \otimes J \otimes V \otimes s(B)V \otimes \dots \otimes s(B)V) = 0$

The complexity of the formulas for the operators ϕ and χ_{cyc} associated to a free ideal extension depends essentially on the structure of J as A -right-module. This structure is determined in the cases of interest in

Definition and Lemma 7.2

Let $0 \rightarrow I \rightarrow A \xrightarrow{s} B \rightarrow 0$ be a linearly split extension of k -algebras and let $0 \rightarrow J \rightarrow TA \rightarrow B \rightarrow 0$ be the associated free ideal extension. 3.10

Recall the basis of J is given in 3.10

a) *An element of TA is said to be of standard form if it corresponds to a tensor monomial in $TI * \Omega^{\text{ev}} B$ 3.10 under the linear isomorphism $TA \simeq \text{Gr}_J(TA) \simeq TI * \Omega^{\text{ev}} B$. An element of $C_*(TA)$ or $CC_*(TA)$ is said to be of standard form if it corresponds to a homogeneous differential form in ΩTA all of whose factors are elements of standard form of TA .*

b) *The natural basis V of J 3.10 decomposes as $V \simeq V' \oplus V''$:*

$$\begin{aligned} V' &\simeq I \oplus I \otimes B && \hookrightarrow J \\ (i, i' \otimes b) &&& \rightarrow \rho(i) + \rho(i')\rho(s(b)) \end{aligned}$$

$$\begin{aligned} V'' &\simeq B \otimes B && \hookrightarrow J \\ b \otimes b' &&& \rightarrow \omega(s(b), s(b')) \end{aligned}$$

c) *The following relations hold*

$$\rho(s(b))\rho(s(b')) = \rho(s(bb')) - \rho(j(b, b')) - \omega(s(b), s(b'))$$

and

$$\begin{aligned} &\omega(s(b), s(b'))\rho(s(b'')) \\ &= \rho(s(b))\omega(s(b'), s(b'')) + \rho(s(b))\rho(j(b', b'')) \\ &\quad + \omega(s(b), s(b'b'')) - \omega(s(bb'), s(b'')) \\ &\quad - \rho(s(b)j(b', b'')) + \rho(j(b, b')s(b')) - \rho(j(b, b'))\rho(s(b'')) \end{aligned}$$

where $j(b, b') := s(bb') - s(b)s(b') \in I$

These formulas express the elements $\rho(s(b)) \cdot \rho(s(b'))$ and $\omega(s(b), s(b'))\rho(s(b''))$ in standard form.

We define certain linear subspaces of $C_*^{\text{bar}}(J) \subset TJ$. It will turn out that Hochschild chains with blocks in these subspaces are annihilated by the operator χ_{cyc} . So terms of this kind can be neglected in all computations.

Definition and Lemma 7.3

a) Let $S' \subset TJ$ be the twosided ideal generated by the TA -bimodule spanned by

- $IA \otimes J \otimes \dots \otimes J \otimes V' \tilde{T}A$
- $\tilde{T}AV'' \otimes J \otimes \dots \otimes J \otimes \tilde{T}A \cdot V' \cdot \tilde{T}A$
- $\tilde{T}AV''V' \tilde{T}A$
- $J \otimes V'' \otimes J$
- $J \otimes V' \otimes s(B)V \otimes \dots \otimes s(B)V \otimes s(B)V'' \otimes J$

If TJ is identified with $C_*^{\text{bar}}(J)$, then

$$S' \subset \text{Fil}_{IA}^1 C_*^{\text{bar}}(J), h'(S') \subset S \cap \text{Fil}_{IA}^1 C_*^{\text{bar}}(J)$$

and S' is stable under the maps

$$m_b \circ h' : C_*^{\text{bar}}(J) \rightarrow C_{*+1}^{\text{bar}}(J)$$

given by the composition of h' and the right module multiplication with $\rho(s(b))$ for all $b \in B$.

b) Let $S'' \subset TJ$ be the twosided ideal generated by the TA -left module spanned by

- $J \otimes VV''$
- $J \otimes V' \otimes s(B)V \otimes \dots \otimes s(B)V \otimes s(B)V'V''$

If TJ is identified with $C_*^{\text{bar}}(J)$, then

$$S'' \subset \text{Fil}_{IA}^1 C_*^{\text{bar}}(J), h'(S'') \subset S \cap \text{Fil}_{IA}^1 C_*^{\text{bar}}(J)$$

and the subspace $S' + S'' + S \cap \text{Fil}_{IA}^1 C_*^{\text{bar}}(J)$ is stable under the maps $m_b \circ h'$ for all $b \in B$.

c) Denote by $\bar{S} \subset TJ$ the twosided ideal generated by $S' + S'' + h'(S' + S'')$.

The operators ϕ_l, χ_l map a tensor monomial into a sum of tensor monomials in standard form. The principal contribution to this sum is given by

Lemma 7.4 Consider the map

$$\begin{array}{ccc} m_b \circ h' : & C_*^{\text{bar}}(J) & \rightarrow C_{*+1}^{\text{bar}}(J) \\ & \alpha & \mapsto h'(\alpha) \cdot \rho(s(b)) \end{array}$$

given by the composition of h' and the right module multiplication with $\rho(s(b))$ for $b \in B$. Then the following formulas hold modulo the subspace $S + S' + S''$:

- $$\begin{aligned} & (m_b \circ h')(T\tilde{A}V''\omega(b'_0, b''_0) \otimes \rho(b_1)\omega(b'_1, b''_1) \otimes \dots \otimes \rho(b'_k)\omega(b'_k, b''_k)) \\ &= \pm T\tilde{A}V'' \otimes \rho(b'_0)\omega(b''_0, b_1) \otimes \rho(b'_1)\omega(b''_1, b_2) \otimes \dots \otimes \rho(b'_k)\omega(b''_k, b) \end{aligned}$$
- $$\begin{aligned} & (m_b \circ h')(T\tilde{A}V'\omega(b'_0, b''_0) \otimes \rho(b_1)\omega(b'_1, b''_1) \otimes \dots \otimes \rho(b_k)\omega(b'_k, b''_k)) \\ &= \sum_{i=0}^{k+1} \pm T\tilde{A}V' \otimes \rho(b'_0)\rho(j(b''_0, b_1)) \otimes \dots \otimes \rho(b'_{i-1})\rho(j(b''_{i-1}, b_i)) \otimes \\ & \quad \dots \otimes \rho(b'_i)\omega(b''_i, b_{i+1}) \otimes \dots \otimes \rho(b'_k)\omega(b'_k, b) \end{aligned}$$
- $$\begin{aligned} & (m_b \circ h')(J\rho(j) \otimes \rho(b_1)\omega(b'_1, b''_1) \otimes \dots \otimes \rho(b_k)\omega(b'_k, b''_k)) \\ &= \sum_{i=1}^k \pm J \otimes \rho(j) \otimes \rho(j(b_1, b'_1))\rho(j(b''_1, b_2)) \otimes \rho(b'_2)\rho(j(b''_2, b_3)) \otimes \\ & \quad \dots \otimes \rho(b'_i)\rho(j(b''_i, b_{i+1})) \otimes \rho(b'_{i+1})\omega(b''_{i+1}, b_{i+2}) \otimes \dots \otimes \rho(b'_k)\omega(b'_k, b) \end{aligned}$$
- $$\begin{aligned} & (m_b \circ h')(J\rho(j)\rho(b_0) \otimes \rho(b_1)\omega(b'_1, b''_1) \otimes \dots \otimes \rho(b_k)\omega(b'_k, b''_k)) \\ &= \sum_{i=1}^{k+1} \pm J \otimes \rho(j)\rho(j(b_0, b_1)) \otimes \rho(b'_1)\rho(j(b''_1, b_2)) \otimes \\ & \quad \dots \otimes \rho(b'_{i-1})\rho(j(b''_{i-1}, b_i)) \otimes \rho(b'_i)\omega(b''_i, b_{i+1}) \otimes \dots \otimes \rho(b'_k)\omega(b'_k, b) \\ & \quad + \sum_{i=1}^k \pm J \otimes \rho(j)\rho(b_0) \otimes \rho(j(b_1, b'_1))\rho(j(b''_1, b_2)) \otimes \rho(b'_2)\rho(j(b''_2, b_3)) \otimes \\ & \quad \dots \otimes \rho(b'_i)\rho(j(b''_i, b_{i+1})) \otimes \rho(b'_{i+1})\omega(b''_{i+1}, b_{i+2}) \otimes \dots \otimes \rho(b'_k)\omega(b'_k, b) \end{aligned}$$
- $$\begin{aligned} & (m_b \circ h')(JV' \otimes s(B)V' \otimes \dots \otimes s(B)\rho(j) \otimes \rho(b_1)\omega(b'_1, b''_1) \otimes \\ & \quad \dots \otimes \rho(b_k)\omega(b'_k, b''_k)) \\ &= \sum_{i=2}^{k+1} J \otimes P'(V' \otimes s(B)V' \otimes \dots \otimes s(B)\rho(j)) \otimes \rho(j(b_1, b'_1))\rho(j(b''_1, b_2)) \otimes \\ & \quad \dots \otimes \rho(b'_2)\rho(j(b''_2, b_3)) \otimes \dots \otimes \rho(b'_{i-1})\rho(j(b''_{i-1}, b_i)) \otimes \rho(b'_i)\omega(b''_i, b_{i+1}) \otimes \\ & \quad \dots \otimes \rho(b'_k)\omega(b'_k, b) \end{aligned}$$

- $$\begin{aligned}
& (m_b \circ h')(JV' \otimes s(B)V' \otimes \dots \otimes s(B)\rho(j)\rho(b_0) \otimes \rho(b_1)\omega(b'_1, b''_1) \otimes \\
& \quad \dots \otimes \rho(b_k)\omega(b'_k, b''_k)) \\
&= \sum_{i=1}^{k+1} \pm J \otimes P'(V' \otimes s(B)V' \otimes \dots \otimes s(B)\rho(j))\rho(j(b_0, b_1)) \\
& \quad \otimes \rho(b'_1)\rho(j(b''_1, b_2)) \otimes \dots \otimes \rho(b'_{i-1})\rho(j(b''_{i-1}, b_i)) \otimes \rho(b'_i)\omega(b''_i, b_{i+1}) \otimes \\
& \quad \dots \otimes \rho(b'_k)\omega(b''_k, b) \\
& \quad + \sum_{i=1}^k \pm J \otimes P'(V' \otimes s(B)V' \otimes \dots \otimes s(B)\rho(j)\rho(b_0)) \\
& \quad \otimes \rho(j(b_1, b'_1))\rho(j(b''_1, b_2)) \otimes \rho(b'_2)\rho(j(b''_2, b_3)) \otimes \\
& \quad \dots \otimes \rho(b'_i)\rho(j(b''_i, b_{i+1})) \otimes \rho(b'_{i+1})\omega(b''_{i+1}, b_{i+2}) \otimes \dots \otimes \rho(b'_k)\omega(b''_k, b)
\end{aligned}$$
- $$(m_b \circ h')(JV' \otimes s(B)V' \otimes \dots \otimes s(B)\rho(j)) = 0$$
- $$\begin{aligned}
& (m_b \circ h')(JV' \otimes s(B)V' \otimes \dots \otimes s(B)\rho(j)\rho(b')) \\
&= \pm J \otimes P'(V' \otimes s(B)V' \otimes \dots \otimes s(B)\rho(j)) \cdot \rho(j(b', b))
\end{aligned}$$

(Here we write $\rho(b), \omega(b', b'')$ instead of $\rho(s(b)), \omega(s(b'), s(b''))$, etc. and denote elements of I by j).

The explicit formulas of lemma 7.4 allow to give the following estimate for the complexity of the principal part of the operators χ and ϕ .

Lemma 7.5 *There exists a constant $C_0 > 0$ with the following property.*

Let $0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$ be a linearly split extension of k -algebras with associated free ideal extension $0 \rightarrow J \rightarrow TA \rightarrow B \rightarrow 0$. For an element

$$\alpha = \alpha_0 \otimes \dots \otimes \alpha_k \in \rho(s(B)) \otimes J \otimes \dots \otimes J \in C_k^{\text{bar}}(TA)$$

denote by

$$m_\alpha: C_*^{\text{bar}}(J) \rightarrow C_{*+k}^{\text{bar}}(J)$$

the right multiplication

$$m_\alpha(j_0 \otimes \dots \otimes j_l) := j_0 \otimes \dots \otimes j_l \alpha_0 \otimes \alpha_1 \otimes \dots \otimes \alpha_k.$$

Let now $\alpha^1, \dots, \alpha^j \in C_^{\text{bar}}(TA)$ be elements as above and $\beta, \beta' \in C_*^{\text{bar}}(J)$ be such that*

$$\beta' := (m_{\alpha^1} \circ h') \circ \dots \circ (m_{\alpha^j} \circ h')(\beta) \in C_n^{\text{bar}}(J)$$

Then, if $\alpha^1, \dots, \alpha^j$ and β are of standard form, the element β' can be written modulo $S + S' + S''$ as a sum of at most C_0^{n+1} elements of $C_n^{\text{bar}}(J)$ in standard form.

This follows by a careful counting from the explicit formulas of 7.4.

We describe now explicitly a subspace of $\text{Ker } \chi'_{cyc}$ which contains the majority of the terms occuring in the explicit formulas of the previous lemmata.

Definition and Lemma 7.6

Let the linear identification $C_(TA) \simeq \text{Gr}_W C_*(TA)$ be understood 2.3*

Denote by T the linear subspace of $C_(TA)$ spanned by*

- $\beta_0 \otimes \alpha_0 \otimes \beta_1 \otimes \dots \otimes \alpha_{k-1} \otimes \beta_k \otimes (S + \bar{S}) \otimes \beta_{k+1} \otimes \dots, \quad k > 0$
- $\beta_0 \otimes \alpha_0 \otimes \beta_1 \otimes \dots \otimes \alpha_{k-1} \otimes \beta_k \otimes (S + \bar{S}), \quad k > 0$
- $\alpha_0 \otimes \beta_1 \otimes \dots \otimes \alpha_{k-1} \otimes \beta_k \otimes (S + \bar{S}) \otimes \beta_{k+1} \otimes \dots \otimes \beta, \quad k > 0$
- $\dots \otimes \beta \otimes h'(\bar{S}) \otimes \beta'' \otimes \dots$
- $\alpha' \otimes \beta' \otimes \dots \otimes \beta'' \otimes \alpha'', \quad \alpha'' \otimes \alpha' \in \bar{S}$
- $(S \otimes \beta \otimes \dots \otimes \alpha') \cap \text{Fil}_{IA}^1 C_*(TA)$
- $1 \otimes \alpha' \otimes \dots \otimes \alpha''$

where $\alpha_i, \alpha', \alpha'' \in TJ \simeq C_^{\text{bar}}(J)$ and all factors of $\alpha_0, \dots, \alpha_{k-1}$ are of J -adic degree one, $\beta_i, \beta', \beta'' \in TB \simeq T(\rho(s(B)))$ and $P', S, \bar{S}$ are as in 1.13, 7.3.*

Then T is stable under the maps ϕ_l 2.6 and $B \circ \chi_l$ (B Connes differential) and is mapped under χ_l into $\text{Fil}_{IA}^1 C_(TA)$ for all l .*

This follows from explicit calculations.

The last assertion shows that $T \subset \text{Ker } (\chi'_{cyc})$.

We can now estimate the complexity of the map ϕ of Hochschild complexes modulo elements of the subspace T .

Lemma 7.7 *There exists a constant $C_1 > 0$ such that the following holds:*

If $0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$ is a linearly split extension of k -algebras and if $0 \rightarrow J \rightarrow TA \rightarrow B \rightarrow 0$ is the associated free ideal extension, then for every element (2.6)

$$\alpha \in \tilde{C}_n(TA)_J \cap \text{Fil}_W^l C_n(TA)$$

of standard form

$$\phi_{l-m} \circ \dots \circ \phi_{l-1} \circ \phi_l(\alpha) \in \tilde{C}_n(TA)_J \cap \text{Fil}_W^{l-m-1}(C_n(TA))$$

can be written modulo T as a sum of at most C_1^{n+1} elements of

$$\tilde{C}_n(TA)_J \cap \text{Fil}_W^{l-m-1} C_n(TA)$$

of standard form.

Proof:

Let $\alpha \in \tilde{C}_n(TA)_J$ be an element of Wodzicki degree l in standard form. Then α can be written as product of tensor monomials $\beta_i \in T(\rho(s(B)))$ and elements $\alpha_i \in C_*^{\text{bar}}(J)$ in standard form. The elements α_i are called the blocks of α . Their number is bounded by $\frac{n+1}{2}$. A block α_i is called special if $\alpha_i \in S$.

If $\alpha \in T$ then there is nothing to show. Otherwise there exists a smallest integer $k \geq 0$ such that all factors of the blocks $\alpha_0, \dots, \alpha_{k-1}$ are of J -adic degree one and such that α_k contains a factor of J -adic degree at least two.

If l is of the form $l = 4r + 3$ then $\phi_l(\alpha)$ equals zero or a cyclic translate of α modulo T .

If l is not of this form write α as

$$\alpha = \dots \otimes \alpha_1 \otimes \beta_1 \otimes \alpha_2 \otimes \dots \otimes \alpha_{k-1} \otimes \beta_{k-1} \otimes \alpha_k \otimes \rho(s(b)) \otimes a' \otimes \dots \otimes a^n$$

with $\alpha_k = j_{k_0} \otimes \dots \otimes j_{k_1}, a', \dots, a^n \in TA$.

Then $\chi_l(\alpha) =$

$$\dots \otimes P'(\alpha_1) \otimes \beta_1 \otimes \dots \otimes P'(\alpha_{k-1}) \otimes \beta_{k-1} \otimes h'(\alpha_k) \otimes \rho(s(b)) \otimes a' \otimes \dots \otimes a^n$$

and one finds

$$\begin{aligned} \phi_l(\alpha) &= \sum_{i=0}^{k_0} (-1)^i \pi_{l-1}(\partial_i(\chi_l(\alpha))) \pm \pi_{l-1}(\partial_{k_1+1}(\chi_l(\alpha))) \bar{+} \pi_{l-1}(\partial_{k_1+2}(\chi_l(\alpha))) \\ &\quad + \sum_{j=k_1+2}^{n-1} (-1)^j \pi_{l-1}(\partial_j(\chi_l(\alpha))) \pm \pi_{l-1}(\partial_n(\chi_l(\alpha))) \bar{+} \pi_{l-1}(\partial_{n+1}(\chi_l(\alpha))) \end{aligned}$$

where $\partial_i(a^0 \otimes \dots \otimes a^n) = a^0 \otimes \dots \otimes a^i a^{i+1} \otimes \dots \otimes a^n$ and where π_{l-1} is the linear projection onto the part of Wodzicki degree $< l$ (2.3). The structure modulo T of the summands is as follows:

i) The summands $\pi_{l-1}(\partial_{k+1}(\chi_l(\alpha))) =$

$$\pm \dots \otimes P(\alpha_1) \otimes \dots \otimes \beta_{k-1} \otimes (m_b \circ h')(\alpha_k) \otimes a' \otimes \dots \otimes a^n$$

and $\pi_{l-1}(\partial_{k_1+2}(\chi_l(\alpha)))$ are elements of the form considered in 7.4 and 7.5.

ii) The summands $\pi_{l-1}(\partial_j(\chi_l(\alpha)))$, $k_1 + 1 < j < n$ belong to T and can be neglected

iii) the summands $\pi_{l-1}(\partial_i(\chi_l(\alpha)))$, $0 \leq i < k_0$ possess a special block, the image of α_k under $\pi_{l-1} \circ \partial_i \circ \chi_l$. The element $\pi_{l-1}(\partial_i(\chi_l(\alpha)))$ is of the form

$$\dots \otimes \alpha_{k'} \otimes \dots \otimes h'(\alpha_{k''}) \otimes \dots$$

with $\alpha_{k'} = a_{k'_0} \otimes \dots \otimes a_{k'_1}$, $\alpha_{k''} = a_{k''_0} \otimes \dots \otimes a_{k''_1}$ where all factors of the blocks $\alpha_0, \dots, \alpha_{k'-1}, \alpha_{k'+1}, \dots, \alpha_{k''-1}$ are of J -adic degree one, $\alpha_{k'}$ contains a factor of J -adic degree at least two, and $h'(\alpha_{k''}) \in S$ is special.

iv) the summands $\pi_{l-1}(\partial_n(\chi_l(\alpha))), \pi_{l-1}(\partial_{n+1}(\chi_l(\alpha)))$ are (after possibly applying a cyclic translation $\phi_{4l'+3}$) of the form $\alpha_0 \otimes \beta_0 \otimes \dots \otimes \alpha_j \otimes \dots$ where either

- there exists a special block α_j , $j > 0$ such that the factors of all blocks $\alpha_1, \dots, \alpha_{j-1}$ are of J -adic degree one or
- the factors of all blocks except α_0 are of J -adic degree one and α_0 contains a factor of J -adic degree two.

Iterating the maps ϕ_l and neglecting all terms which belong to T one derives from this characterization of the summands of $\phi_l(\alpha)$ that

$$\phi_{l-m} \circ \dots \circ \phi_l(\alpha) = \sum_{i_0, \dots, i_w} \pm (\pi_{l-m-1} \circ \partial_{i_w} \circ \chi_l) \circ \dots \circ (\pi_{l-1} \circ \partial_{i_0} \circ \chi_l)(\alpha)$$

where the sum runs over a set of sequences of integers $(i_0, \dots, i_w)$ in $[0, n]$ satisfying

- the length of the sequence is at most $4n + 4$
- the values $n - 1$ and n occur at most twice
- after removing the elements of value $n - 1$ and n in the sequence it decomposes into a union of at most three subsequences $(i_\alpha, \dots, i_\beta)$ satisfying:

i) there exist integers $j_0, \dots, j_s, j_0 > i_\alpha$ such that the subsequences

$$(i_\alpha, \dots, i_{j_0}), (i_{j_0+1}, \dots, i_{j_1}), \dots, (i_{j_s}, \dots, i_\beta)$$

are strictly monotone increasing

ii) $i_{j_t} > i_v$ for $j_t < v \leq i_\beta$ and $t = 0, \dots, s$.

(In this sum we have actually neglected operators of the form $\phi_{4l'+3}$. They act for $l' > 2$ however only by a cyclic translation modulo T and therefore do not increase the complexity of the maps and the number of the created terms. We have not taken them into account just to keep notations short.)

It is easily seen that there is an at most exponential number (in n) sequences of this type. As moreover the results of lemma 7.4 and 7.5 apply to each of the summands for $\phi_{l-m} \circ \dots \circ \phi_l(\alpha)$ above we are done. $\square$

We pass now from the Hochschild complex to the cyclic bicomplex

Lemma 7.8 *There exists a constant $C_2 > 0$ such that the following holds:*

Let $0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$ be a linearly split extension of k -algebras and let $0 \rightarrow J \rightarrow TA \rightarrow B \rightarrow 0$ be the associated free ideal extension.

Define linear maps

$$\nu_l: C_*(TA) \rightarrow C_{*+1}(TA) \text{ by}$$

$$\nu_l|_{C_n(TA)} := \chi_l \circ \phi_{l+1} \circ \dots \circ \phi_{4n+4} \quad (1)$$

Let $(l_1, \dots, l_{j+1})$ be a $(j+1)$ -tuple of integers and denote by B Connes differential.

Then, if $\alpha \in C_n^>(TA)_J$ is an element in standard form, the element

$$\alpha' := \nu_{l_1} \circ B \circ \nu_{l_2} \circ \dots \circ \nu_{l_j} \circ B \circ \nu_{l_{j+1}}(\alpha)$$

vanishes for $j > n+1$ and can otherwise be written modulo $\text{Fil}_{IA}^1 C_*(TA)$ as a sum of most $(n+2)(n+4)\dots(n+2j)C_2^{(n+1)}$ elements of $C_{n+2l+1}(TA)$ in standard form.

Proof: As noted in 7.7 applying ϕ_l to an element of $C_*(TA)$ in standard form may increase but never can decrease the number of special blocks whereas $B \circ \chi_l$ increases the number of special blocks by one. As the Wodzicki-filtration is stable under ϕ_l and essentially stable under B and an element of $\text{Fil}_w^{4n+4} C_*(TA) (\supset C_n(TA))$ in standard form possesses at most $n+1$ blocks it follows that

$$\nu_{l_1} \circ B \circ \dots \circ B \circ \nu_{l_{j+1}}$$

vanishes on $C_n^>(TA)_J$ for $j > n+1$ (the homotopy operators χ_l vanish on elements all of whose blocks are special).

If $j \leq n+1$ a careful counting similar to that in the proof of 7.7 shows that an element $\alpha \in C_n^>(TA)_J$ of standard form is mapped under

$$\phi_{l_1+1} \circ \dots \circ \phi_m \circ B \circ \nu_{l_2} \circ \dots \circ B \circ \nu_{l_{j+1}}$$

into a sum of at most $(n+2)\dots(n+2j)C_1^{n+1}$ standard terms of $C_*(TA)$ modulo T . Lemma 7.7 and 7.4 imply then that

$$\alpha' = \nu_{l_1} \circ B \circ \nu_{l_2} \circ \dots \circ \nu_{l_j} \circ B \circ \nu_{l_{j+1}}(\alpha)$$

is given modulo $\text{Fil}_{IA}^1 C_*(TA)$ by a sum of at most $(n+2)\dots(n+2j)C_2^{n+1}$ elements of $C_{n+2j+1}(TA)$ in standard form. $\square$

We collect the results of our calculations in the following

Proposition 7.9 *Let $0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$ be a linearly split extension of k -algebras and let $0 \rightarrow J \rightarrow TA \rightarrow B \rightarrow 0$ be the associated free ideal extension.*

Let $\chi' := \pi_ \circ \chi_{cyc} : CC_*^>(TA)_J \rightarrow CC_{*+1}(I)$ be the homotopy operator of 3.8.*

Then the following holds:

a) χ' vanishes on $\text{Fil}_{IA}^k(\Omega^n TA) \cap CC_n^>(TA)_J$ for $k > n$

b) $\chi'(\Omega^n TA \cap CC_n^>(TA)_J) \subset \bigoplus_{k=0}^{\frac{n+1}{2}} \Omega^{n+2k+1} I$

c) *There exists a constant $C > 0$ (independent of the given extension) with the following property:*

If

$$\alpha = \alpha_0 d\alpha_1 \dots d\alpha_n \in \Omega^n TA \cap CC_n^>(TA)_J$$

where

$$\alpha_i = 1 \text{ or } \alpha_i = \rho(a_i^0)\omega(a_i^1, a_i^2) \dots \omega(a_i^{2k_i-1}, a_i^{2k_i}), k_i \geq 0$$

with $a_i^j \in F_I \cup s(F_B) \subset I \cup s(B) \subset A$ for some subsets $F_I \subset I$, $F_B \subset B$, then

$$\chi'(\alpha) = \sum_{k=0}^{\frac{n+1}{2}} \gamma_k$$

with $\gamma_k \in \Omega^{n+2k+1}I$ such that

- Each γ_k is a sum of at most $(n+2) \dots (n+2k) \cdot C^{n+1}$ homogeneous differential forms*

$$x_0 dx_1 \dots dx_{n+2k+1} \in \Omega I$$

- The factors x_i of the summands above are elements of the multiplicative closure of the set*

$$F_I^\infty \cup s(F_B^\infty) \cup \{s(bb') - s(b)s(b') | b, b' \in F_B^\infty\}$$

(Here F^∞ denotes the multiplicative closure of a subset F of an algebra).

Proof: By definition

$$\chi_{cyc} = \sum_{j=0}^{\infty} \sum_{l_1, \dots, l_{j+1}} \nu_{l_1} \circ B \circ \nu_{l_2} \circ \dots \circ B \circ \nu_{l_j} \circ B \circ \nu_{l_{j+1}}$$

Lemma 7.8 and the definition of χ_l show that the restriction of χ_{cyc} to $\Omega^n TA \cap CC_n^>(TA)$ equals

$$\sum_{j=0}^{\frac{n+1}{2}} \sum_{l_1, \dots, l_{j+1}} \nu_{l_1} \circ B \circ \nu_{l_2} \circ \dots \circ B \circ \nu_{l_{j+1}}$$

where the sum runs over all $(j+1)$ -tuples $(l_1, \dots, l_{j+1})$ satisfying

$$l_1 \leq l_2 + 4, \dots, l_j \leq l_{j+1} + 4, l_{j+1} \leq 4n + 4.$$

This implies on the one hand that

$$\chi'(\Omega^n TA \cap CC_n^>(TA)_J) \subset \bigoplus_{k=0}^{\frac{n+1}{2}} \Omega^{n+2k+1}I$$

and on the other hand that $\chi'(\alpha)$ is a sum of an exponential (in n) number of terms of the form studied in 7.8. Part c) follows then from the explicit formulas in 7.7, 7.2, 7.4 and 7.6 and from the estimate of 7.8. The explicit formulas of 7.1, 7.2, 7.4 and 7.7 allow finally also to verify part a) of the proposition. $\square$

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