ON THE HOMOLOGY OF GL_n AND HIGHER PRE-BLOCH GROUPS

SERGE YAGUNOV

INTRODUCTION

The purpose of the present work is to extend an approach used for investigation of the homology of GL_n by Suslin, Sah, and others. Studying of groups $H_*(GL_n)$ seems rather important, in particular, because of their close relation to algebraic K -theory. Unfortunately, these groups are much too big and complicated to be computed explicitly [10]. Therefore, all results allowing to compare groups $H_*(GL_n)$ for different values of n become quite important.

Almost all known methods of getting information about homology of GL_n (See, for example [3, 6, 7, 8, 9, 11, 12, 13, 15]) are based on the following approach. We choose some GL_n -resolution of the coefficient ring such that the corresponding hyperhomology spectral sequence has a desired form. This spectral sequence converges to $H_*(GL_n)$ and we usually want to build GL_n -resolution such that after passing to coinvariants it has first nontrivial homology in high enough dimension. This allows to analyze the relative homology group in corresponding dimension. In [8, 12] a complex constructed from points in general position of affine space is used. This proves the coincidence of the groups $H_n(GL_n(F), GL_{n-1}(F))$ and $K_n^M(F)$.

In [9, 13] we see a complex cooked up from distinct points of projective line $\mathbb{P}^1(F)$. After applying to this complex the functor of GL_n -coinvariants, we get a complex, whose first non-trivial homology group lies in dimension 3. This group $\wp(F)$ (we call it pre-Bloch group) coincides with $H_3(GL_2(F), GM_2(F))$.

The extension of the above method to n -dimensional projective space $\mathbb{P}^n$ would be quite powerful tool for the investigation of $H_{n+1}(GL_n, GM_n)$. Unfortunately differentials in corresponding spectral sequences are very hard to compute.

One possible approach to eliminate computational difficulties, at least in low dimensions, is presented in this paper. For any natural n we construct a complex of GL_n -modules such that the corresponding hyperhomology spectral sequence converges to $H_*(GL_n, GM_n)$. From these spectral sequences

we obtain a sequence of groups $\wp^n(F)$ which serve as a generalization of pre-Bloch group to higher dimensions. We expect the following conjectures to be true:

Conjecture 0.1. *Assume that the field F is infinite, $n > 1$ is an integer, and $n!$ is invertible in the coefficient ring $\mathbb{A}$. Then we have the following natural isomorphism:*

$$\wp^n(F) = \bigoplus_{i \geq 0} H_{n+1-2i}(GL_{n-2i}, \{GM_{n-2i}, GL_{n-2i-1}\})_{\text{ind}},$$

where “ $_{\text{ind}}$ ” means the indecomposable part of the homology.

Conjecture 0.2. *Let us assume in addition that the field F is algebraically closed. Then the groups $\wp^n(F)$ are divisible.*

In the present paper we prove the first conjecture for small dimensions ($n < 5$). For $n = 2$ Conjecture 0.2 is proven in [9, 13]. Nothing seems to be known about this conjecture for bigger n .

Acknowledgment. I’d like to thank a lot of people who read different versions of this text, gave me valuable comments, and whose interest to considered question was very stimulating. Among them are A.Bak, Ph.Elbaz-Vincent, E.Friedlander, J.F.Jardine, M.Kapranov, A.Suslin, V.Voevodsky.

Here I should also thank Universität Bielefeld and The University of Western Ontario for their hospitality, since the significant part of this text was written during my visits there.

This paper is mostly based on my Northwestern Ph.D. Thesis, so I’d like to thank Northwestern University for opportunity to work there for three years.

Remarks on notation. Everywhere in this paper F denotes an infinite field. We denote by e_i the i th standard basis n -column $(0, 0, \dots, 1, \dots, 0)^T$. Symbol E denotes n -column $(1, 1, \dots, 1)^T$. We use Σ_n for the group of n -permutations or a subgroup of GL_n naturally isomorphic to the group of n -permutations. GM_n denotes a group of nondegenerated monomial matrices of size n i.e. $GM_n(F) = (F^*)^n \rtimes \Sigma_n$.

Unless the opposite mentioned all the homology groups are assumed to have coefficients in some ring $\mathbb{A}$. This ring should satisfy the condition of invertibility of $n!$, where n is defined in each concrete case.

All the induced modules are implicitly over the ring $\mathbb{A}$. Namely, if H is a subgroup of a group G , then

$$\text{Ind}_H^G M \stackrel{\text{def}}{=} \mathbb{A}[G] \otimes_{\mathbb{A}[H]} M.$$

If M_* is a complex of H -modules and N_* is a G -complex, we denote by $\text{Cone}(M_* \rightarrow N_*)$ the *Cone* of morphism of complexes $\text{Ind}_H^G M_* \rightarrow N_*$ over G .

1. COMPLEXES RELATED TO GROUPS GM_n

We are beginning to construct the desired spectral sequence. Let n be fixed positive integer. We call an n -column of elements of F “monomial” if it has exactly one nonzero entry and “affine” if all its entries are nonzero.

Definition 1.1. We say that an $(n \times k)$ -matrix M has columns in general position (or satisfies GP-condition) if either $\text{rank } M = k$ for $k \leq n$, or any n -minor of M is nonzero.

Definition 1.2. We say that a certain matrix satisfies the strong general position condition (SGP-condition) if any minor of any size of this matrix is non-zero.

Denote by $\tilde{D}_m^n$ the set of all $n \times (m+1)$ matrices of the form

$$(1.1) \quad (v_0, v_1, \dots, v_j, w_{j+1}, \dots, w_m),$$

where v_i are monomial columns with nonzero entries in pairwise different positions, w_j are affine columns, and such that the submatrix $(w_{j+1}, \dots, w_m)$ satisfies the SGP-condition. (Both monomial and affine parts of this matrix allowed to be empty.) The positions of nonzero entries in the monomial part of a matrix from $\tilde{D}_*^n$ uniquely define an ordered subset of $\{1 \dots n\}$. Two matrices of $\tilde{D}_m^n$ are said to be D-equivalent if they have the same affine parts and the same corresponding ordered subsets. Denote by D_m^n a free $\mathbb{A}$ -module generated by all classes of equivalence formed by matrices of $\tilde{D}_m^n$. Let us define a differential operator $d: D_m^n \rightarrow D_{m-1}^n$ as an alternating sum of face operators d_i , which throw away the i -th column of matrix. One can easily see that the complex D_*^n is acyclic in positive dimensions. The natural action of the group GM_n on D_*^n supplies the complex D_*^n with the structure of GM_n -module. Let us now calculate the hyperhomology of GM_n with coefficients in D_*^n . First we compute the homology of the coinvariant complex $(D_*^n)_{GM_n}$.

Proposition 1.3.

$$H_i((D_*^n)_{GM_n}) = \begin{cases} \mathbb{A} & \text{for } i = 0 \\ 0 & \text{for } 0 < i < n+1 \end{cases}$$

and there exists a natural epimorphism $\mathbb{A} \otimes (F^*)^{[n/2]} \rightarrow H_{n+1}((D_*^n)_{GM_n})$.

Proof. Let's consider a filtration

$$(1.2) \quad 0 = F_{-1} \subset F_0 \subset \dots \subset F_n = D_*^n,$$

where $F_k D_*^n$ is a subcomplex of D_*^n generated by matrices having at most k monomial columns. One can easily check that this filtration is compatible with differential. Since the action of GM_n doesn't change the number of

monomial columns, the introduced filtration induces a filtration on the complex of coinvariants $(D_*^n)_{GM_n}$ for which we also use F . Let $E_{*,*}^*$ be the spectral sequence associated to the latter filtration. Its first term has the form $E_{p,q}^1 = H_{p+q}(F_{q/q-1}((D_*^n)_{GM_n}))$.

Let us note that for $p \geq 0$ any generator of $F_{q/q-1}((D_{p+q}^n)_{GM_n})$ can be presented uniquely by the following class of matrices:

$$(1.3) \quad \begin{pmatrix} & & & & & w_{11} & w_{12} & \cdots & w_{1,p} \\ e_1 & e_2 & \cdots & e_q & E & w_{21} & w_{22} & \cdots & w_{2,p} \\ & & & & & \vdots & \vdots & & \vdots \\ & & & & & w_{n1} & w_{n2} & \cdots & w_{n,p} \end{pmatrix}_{\Sigma_{n-q}}$$

where the group Σ_{n-q} acts on the matrix permuting its last $n-q$ rows. For $p = -1$ (i.e. in the lowest nontrivial dimension $q-1$) the corresponding term of this complex equals to $\mathbb{A}$. One can easily see that the differential in the factor-complex $F_{q/q-1}$ acts only in the affine part throwing away the columns successively. Therefore, we can forget about the monomial part of matrix.

Lemma 1.4. *Suppose that $n!$ is invertible in the coefficient ring $\mathbb{A}$. Then*

$$E_{p,q}^\infty = E_{p,q}^2 = \begin{cases} \text{Ker}(H_{p+n}F_{n/n-1} \xrightarrow{d_1} H_{p+n-1}F_{n-1/n-2}); & \text{if } q = n \\ 0 & \text{otherwise.} \end{cases}$$

Proof of this lemma is rather technical and tedious. Two different proofs (both computational) can be found in [16] and [17]. Let us now finish calculating of the E^2 -term.

Consider a complex of $\mathbb{A}$ -modules

$$(1.4) \quad \mathbb{A} \longleftarrow \mathbb{A}[\mathcal{F}_0] \longleftarrow \mathbb{A}[\mathcal{F}_1] \longleftarrow \cdots$$

where $\mathcal{F}_i$ is a set of all $n \times (i+1)$ -matrices satisfying the SGP-condition and the differential is defined in the standard way. This complex gives us a free $(F^*)^n$ -resolution of $\mathbb{A}$. One can easily verify that

$$(1.5) \quad F_{n-1/n-2}((D_*^n)_{GM_n})[n-1] \simeq F_{n/n-1}((D_*^n)_{GM_n})[n] \simeq \mathbb{A}[\mathcal{F}_*]_{(F^*)^n},$$

where the last term is the complex of $(F^*)^n$ -coinvariants of $\mathbb{A}[\mathcal{F}_*]$. These isomorphisms imply that

$$(1.6) \quad H_{p+n}F_{n/n-1} \simeq H_{p+n-1}F_{n-1/n-2} \simeq H_1(\mathbb{A}[\mathcal{F}_*]_{(F^*)^n}) \simeq \mathbb{A} \otimes (F^*)^n.$$

The map $d_1: \mathbb{A} \otimes (F^*)^n \rightarrow \mathbb{A} \otimes (F^*)^n$ looks now as follows

$$(1.7) \quad d_1(1 \otimes (a_1, a_2, \dots, a_n)) = 1 \otimes \left(\frac{a_2}{a_1}, 1, \frac{a_4}{a_3}, 1, \dots, \frac{a_n}{a_{n-1}}, 1 \right)$$

if n is even and

$$(1.8) \quad d_1(1 \otimes (a_1, a_2, \dots, a_n)) = 1 \otimes \left(a_2, a_2, a_4, a_4, \dots, a_{\frac{n-1}{2}}, a_{\frac{n-1}{2}}, \prod_{i=1}^n a_i^{(-1)^{i+1}} \right)$$

if n is odd.

Therefore, the desired kernel is generated by elements of the form

$$(1.9) \quad (x_1, x_1, x_2, x_2, \dots, x_{\frac{n}{2}}, x_{\frac{n}{2}})$$

for even n and

$$(1.10) \quad \left(x_1, 1, x_2, 1, \dots, x_{\frac{n-1}{2}}, 1, \prod_{i=1}^{(n-1)/2} x_i^{-1} \right)$$

for odd one.

In both cases the group $(F^*)^{[n/2]}$ covers the kernel that completely proves Proposition 1.3. □

Let's now find the homology of GM_n with coefficients in D_m^n .

Proposition 1.5. *Let $p > 0$ and let $n!$ be invertible in the coefficient ring. Then*

$$H_p(GM_n, D_q^n) = \begin{cases} H_p((F^*)^{q+1} \times GM_{n-q+1}) & 0 \leq q \leq n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The Shapiro lemma reduces the problem to computing the homology of stabilizers for each orbit of GM_n -action onto D_q^n .

The considered action splits the module D_q^n into three types of orbits. First of them is an orbit presented by a matrix without affine part. Such an orbit appears only if $q < n$ and its stabilizer is $(F^*)^{q+1} \times GM_{n-q+1}$.

The second type consists of matrices with only one affine column. There is only one orbit of this type, presented by the matrix $(e_1, \dots, e_q, E)$. Stabilizer of this matrix is Σ_{n-q} . This orbit is nonempty iff $q \leq n$. Since the order of Σ_{n-q} is invertible in the coefficient ring we have $H_p(\Sigma_{n-q}) = 0$ for $p > 0$. (See Corollary 10.2 [2])

The third type includes all the orbits consisting of matrices with more than one affine column. One can easily see that the stabilizer of such a matrix is trivial.

Therefore, only one summand, corresponding to the orbit of the first type really appears in homology. □

2. COMPLEXES RELATED TO GROUPS $H_*(GL_n, GM_n)$

In this section we are going to introduce a GL_n -complex C_*^n constructed using a mixture of points in general position of projective and affine spaces. For the induced complex $\text{Ind}_{GM_n}^{GL_n} D_*^n$ of GL_n -modules we construct a natural morphism of complexes $\text{Ind}_{GM_n}^{GL_n} D_*^n \rightarrow C_*^n$, which happens to be an epimorphism.

The kernel of this map K_*^n will be an object of our main interest. Hyperhomology spectral sequence of GL_n with coefficients in K_*^n converges to the relative homology of the pair (GL_n, GM_n) and is exactly the spectral sequence, mentioned in the introduction.

Let us start with the following construction. Set $\tilde{C}_0^n \leftarrow \tilde{C}_1^n \leftarrow \dots$ to be a complex which has the free $\mathbb{A}$ -module generated by $n \times (m+1)$ matrices with columns in general position in dimension m . Differential operator d is given by the formula

$$(2.1) \quad d(a_0, a_1, \dots, a_q) = \sum_{i=0}^q (-1)^i (a_0, a_1, \dots, \hat{a}_i, \dots, a_q)$$

Denote by $C_*^{n;p}$ a subcomplex of $\tilde{C}_*^n$ generated by elements

$$(2.2) \quad (x_0, x_1, \dots, x_p, x_{p+1}, \dots, x_m) - (\alpha_0 x_0, \alpha_1 x_1, \dots, \alpha_p x_p, x_{p+1}, \dots, x_m)$$

in dimension m if $m > p$ and by

$$(2.3) \quad (x_0, x_1, \dots, x_m) - (\alpha_0 x_0, \alpha_1 x_1, \dots, \alpha_m x_m)$$

otherwise. The coefficients α_i above are arbitrary elements of F^* .

Now, we set $C_*^n = \tilde{C}_*^n / C_*^{n;n-1}$. One can easily verify that Complex C_*^n has no nontrivial homology in positive dimensions.

Consider the hyperhomology spectral sequence of GL_n with coefficients in C_*^n . The group GL_n acts on C_m^n transitively for $m < n$ and freely for $m \geq n$. The canonical generator in dimension m has the form $(e_1, e_2, \dots, e_{m+1})$ if $m < n$ and $(e_1, e_2, \dots, e_n, E)$ in dimension $m = n$. Its stabilizer is the affine group

$$(2.4) \quad \text{Aff}_{n-m-1, m+1} = \begin{pmatrix} F^* & & 0 & \\ & \ddots & & * \\ 0 & & F^* & \\ 0 & \cdots & 0 & GL_{n-m-1} \end{pmatrix}$$

if $m < n$ and trivial if $m \geq n$.

Now we can compute the E^1 -term of the spectral sequence under consideration. The Shapiro lemma together with an easy modification of Theorem 1.8. [12] gives us the following

Lemma 2.1.

$$E_{p,q}^1 = H_p(GL_n, C_q^n) = \begin{cases} H_p((F^*)^{q+1} \times GL_{n-q-1}) & q < n \\ 0 & \text{otherwise.} \end{cases}$$

(We set GL_0 to be the trivial group.)

□

Let us compute also some entries in the 0th row of the E^2 -term. We have $E_{0,q}^2 = H_q((C_*^n)_{GL_n})$. Since GL_n acts transitively in low dimensions, corresponding coinvariant groups are $\mathbb{A}$ and differentials are alternatively trivial and identical. Therefore, we can see that

$$(2.5) \quad E_{0,q}^2 = \begin{cases} \mathbb{A} & q = 0 \\ 0 & 0 < q \leq n. \end{cases}$$

Consider a GL_n -complex obtained from D_*^n by the extension of scalars.

Lemma 2.2. *The map $\varphi: \text{Ind}_{GM_n}^{GL_n} D_*^n \rightarrow C_*^n$, sending $a \otimes g$ to ag is an epimorphism of complexes.*

Proof. For any $(n \times q)$ -matrix B satisfying GP-condition we can find such a matrix $G \in GL_n$ that $GB = (e_1, e_2, \dots, e_n, b_1, b_2, \dots, b_{q-n})$. (If $q < n$ the result just needs to be cut at the point e_q .) Let us note now that the submatrix $(b_1, b_2, \dots, b_{q-n})$ should satisfy SGP-condition i.e. the matrix GB belongs to $\tilde{D}_*^n$. Therefore, $B = \varphi(G^{-1} \otimes GB)$. □

Denote the kernel of the map φ by K_*^n . Now we want to compute the hyperhomology of GL_n with coefficients in K_*^n .

Proposition 2.3. $H_p(GL_n, K_*^n) = H_{p+1}(GL_n, GM_n)$.

Proof. Consider an epimorphism of GL_n -modules $\text{Ind}_{GM_n}^{GL_n} \mathbb{A} \rightarrow \mathbb{A}$, where GL_n acts trivially on $\mathbb{A}$. Denote its kernel by $\tilde{K}^n$. Since the complexes $\text{Ind}_{GM_n}^{GL_n} D_*^n$ and C_*^n are GL_n -resolutions of the modules $\text{Ind}_{GM_n}^{GL_n} \mathbb{A}$ and $\mathbb{A}$, respectively, the complex K_*^n is a resolution of the module $\tilde{K}^n$ as well. Therefore, $H_*(GL_n, K_*^n) = H_*(GL_n, \tilde{K}^n)$. Standard homological algebra shows that $H_p(GL_n, \tilde{K}^n) = H_{p+1}(GL_n, GM_n)$. □

Now we can find out the E^1 -term of the hyperhomology spectral sequence of GL_n with coefficients in K_*^n .

Notation 2.4. From now on we often denote the pair $(GL_k \times X, GM_k \times X)$ by $(GL_k, GM_k) \times X$.

Proposition 2.5. *if $p > 0$, then*

$$E_{p,q}^1 = H_p(GL_n, K_q^n) = H_{p+1}((GL_{n-q-1}, GM_{n-q-1}) \times (F^*)^{q+1}).$$

Proof. A short exact sequence of coefficients

$$(2.6) \quad 0 \longrightarrow K_q^n \longrightarrow \text{Ind}_{GM_n}^{GL_n} D_q^n \longrightarrow C_q^n \longrightarrow 0$$

and calculations of the homology of GL_n with coefficients in $\text{Ind}_{GM_n}^{GL_n} D_q^n$ and C_q^n makes the proof similar to one of Proposition 2.3. $\square$

Let us look at short exact sequence (2.6). It gives us a long exact sequence of GL_n -homology groups. Consider its final terms. Since the map $H_1(GM_{n-q} \times (F^*)^q) \rightarrow H_1(GL_{n-q} \times (F^*)^q)$ is an epimorphism for any q , we have the following short exact sequence.

$$(2.7) \quad 0 \longrightarrow H_0(GL_n, K_q^n) \longrightarrow (D_q^n)_{GM_n} \longrightarrow (C_q^n)_{GL_n} \longrightarrow 0.$$

Assembling together these sequences for all q we obtain a short exact sequence of complexes.

$$(2.8) \quad 0 \longrightarrow H_0(GL_n, K_*^n) \longrightarrow (D_*^n)_{GM_n} \longrightarrow (C_*^n)_{GL_n} \longrightarrow 0.$$

The homology long exact sequence corresponding to (2.8) together with (2.5) and Proposition 1.4 gives us the following.

Lemma 2.6.

$$E_{0,q}^2 = H_0(GL_n, K_q^n) = \begin{cases} 0 & q < n \\ \text{Coker}(H_{n+1}(D_*^n)_{GM_n} \rightarrow H_{n+1}(C_*^n)_{GL_n}) & q = n. \end{cases}$$

$\square$

Definition 2.7. We define an $\mathbb{A}$ -module $\wp^n(F)$ (or $\wp^n$ for shortness) as

$$\text{Coker}((H_{n+1}(D_*^n)_{GM_n} \longrightarrow H_{n+1}(C_*^n)_{GL_n}).$$

Let us mention that in even dimensions the considered module is generated by symbols $(a_1, \dots, a_n)$, where $a_i \in F^*$ and $a_i \neq a_j$ for $i \neq j$. In odd dimensions it is generated by linear combinations of these symbols, lying in the kernel of augmentation map.

We are now ready to prove the Theorem mentioned in the introduction.

Theorem 2.8. *Assume that the base field F is infinite and $n!$ is invertible in the coefficient ring $\mathbb{A}$. Then there exists a naturally defined first quadrant spectral sequence converging to $E_p^\infty = H_{p+1}(GL_n, GM_n)$ and with the E^1 -term, which has the following form for $p > 0$:*

$$E_{p,q}^1 = \begin{cases} H_{p+1}(GL_{n-q-1} \times (F^*)^{q+1}, GM_{n-q-1} \times (F^*)^{q+1}) & \text{if } q < n-2 \\ 0 & \text{otherwise.} \end{cases}$$

Moreover,

$$E_{0,q}^2 = \begin{cases} 0 & q < n \\ \wp^n(F) & q = n \end{cases}$$

Proof. One can easily see that almost everything was already proven above. We just need to check vanishing of the E^1 -term of the considered spectral sequence in columns $n-1$ and n . But in these columns the E^1 -term should have the homology of pairs $(GL_1, GM_1) \times (F^*)^{n-1}$ and $(GL_0, GM_0) \times (F^*)^n$, which are obviously trivial. $\square$

Remark 2.9. From now on we will call the constructed above spectral sequence “The Main Spectral Sequence”.

3. STRUCTURE OF $\wp^n(F)$

The main purpose of this section is to measure the deviation between $\mathbb{A}$ -modules $\wp^n(F)$ defined above and $\mathbb{A} \otimes \wp^n(F)_{\text{cl}}$, where $\wp^n(F)_{\text{cl}}$ denotes the “classical” pre-Bloch group. We will abuse the notation and define these modules by $\wp^n(F)_{\text{cl}}$ as well.

Really, it seems that only a group $\wp^2(F)_{\text{cl}}$ appeared in publications (See [1, 9, 13]) so, we need to clarify what we do mean under “classical” for $n > 2$.

Let us fix some integer $n \geq 2$ and an infinite field F . Consider a complex

$$(3.1) \quad P_*^n(F): P_0^n(F) \xleftarrow{d} P_1^n(F) \xleftarrow{d} \dots$$

where $P_k^n(F)$ is the free abelian group spanned by $(k+1)$ -tuples of rational points of the projective space $\mathbb{P}^{n-1}(F)$, which are in general position (i.e. the matrix built from these points satisfy GP-condition). Differential operator d is defined in the standard way.

The complex $P_*^n(F)$ has a canonical structure of GL_n -module and clearly acyclic since the field F is infinite. One can easily check that the lowest non-trivial homology group (in positive dimension) of the complex $(P_*^n(F))_{GL_n}$ appears in dimension $n+1$.

Definition 3.1. We set $\wp^n(F)_{\text{cl}} = H_{n+1}(P_*^n(F)_{GL_n})$.

Remark 3.2. The definition above is the most natural generalization of the definition which can be found, for example in papers of Suslin [13] and Sah [9] for dimension 2. We hope it gives us enough motivation to call the groups of this series “classical”.

We shall also introduce one auxiliary object.

Definition 3.3. Let us define $\widehat{\wp}^n(F)$ as $H_{n+1}((C_*^n(F))_{GL_n})$, where $C_*^n(F)$ is the complex defined in the previous section.

A priori we have the following canonical maps between these modules:

$$(3.2) \quad \wp^n(F) \xleftarrow{\varphi_1} \widehat{\wp}^n(F) \xrightarrow{\varphi_2} \wp^n(F)_{\text{cl}}$$

where φ_1 is the “projectivization” mapping induced by the natural map $\mathbb{A}^n(F) \setminus \{0\} \rightarrow \mathbb{P}^{n-1}(F)$ and φ_2 was introduced in the definition of $\wp^n(F)$. Let $\mathbf{a} = (a_1, \dots, a_n)^T$ be an n -column such that $a_1, \dots, a_n \in F^*$ and $a_i \neq a_j$ for $i \neq j$. (From now on and up to the end of this section we consider only n -tuples satisfying this condition.) We reserve a notation $[\mathbf{a}]$ for the element of $(C_{n+1}^n(F))_{GL_n}$ given by the matrix $(e_1, e_2, \dots, E, \mathbf{a})$. The module $\widehat{\wp}^n(F)$ is generated by elements of the form $[\mathbf{a}]$ in even dimensions and by $[\mathbf{a}] - [\mathbf{b}]$ in odd ones. (Abusing the notation we use the symbol $[\mathbf{a}]$ to denote the corresponding element of $\widehat{\wp}^n(F)$ as well.)

Lemma 3.4. *For any $n > 1$, $k \in F^*$, and an n -column $\mathbf{a} = (a_1, \dots, a_n)^T$ the following relation holds in $\widehat{\wp}^n(F)$*

$$[\mathbf{a}] - [k\mathbf{a}] = M(k),$$

where $M(k) \in \widehat{\wp}^n(F)$ is a constant depending only on k .

Proof. Let

$$(3.3) \quad X = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & ka_1 & kb_1 \\ 0 & 1 & & 0 & 1 & ka_2 & kb_2 \\ \vdots & & \ddots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & ka_n & kb_n \end{pmatrix} - \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & a_1 & b_1 \\ 0 & 1 & & 0 & 1 & a_2 & b_2 \\ \vdots & & \ddots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & a_n & b_n \end{pmatrix}$$

Consider $d(X) = \sum_{i=0}^{n+2} (-1)^i d_i(X)$, where d_i is the face operator throwing away the i th column.

It can be easily checked that for any $0 \leq i < n+1$

$$(3.4) \quad d_i((e_1, e_2, \dots, e_n, E, \mathbf{a}, \mathbf{b})) = [\mathbf{f}_i(\mathbf{a}, \mathbf{b})],$$

where $\mathbf{f}_i: F^{*n} \times F^{*n} \rightarrow F^{*n}$ are homogeneous rational functions of $\mathbf{a}, \mathbf{b}$ such that $\deg_{\mathbf{a}} \mathbf{f}_i = -1$ and $\deg_{\mathbf{b}} \mathbf{f}_i = 1$. This implies that all these terms annihilate in $d(X)$. So that, we have $d(X) = (-1)^{n+1}(d_{n+1}(X) - d_{n+2}(X))$. This shows that the element

$$(3.5) \quad ([k\mathbf{a}] - [\mathbf{a}]) - ([k\mathbf{b}] - [\mathbf{b}]) \in C_{n+1}^n(F)_{GL_n}$$

belongs to the image of the differential d and therefore, vanishes in the homology group $\widehat{\wp}^n(F)$. $\square$

In case of odd n we can prove even a stronger result.

Lemma 3.5. *If n is odd, then $M(k) = 0$ for any k .*

Proof. Let $k, l \in F^*$. Consider

$$(3.6) \quad d((e_1, e_2, \dots, e_n, E, l\mathbf{a}, k\mathbf{b}) - (e_1, e_2, \dots, e_n, E, \mathbf{a}, \mathbf{b})).$$

The proof of the previous lemma tells us that this expression equals to

$$(3.7) \quad \underbrace{M(k/l) - M(k/l) + \cdots - M(k/l)}_{n+1 \text{ times}} + M(k) - M(l).$$

Therefore, in the module $\widehat{\wp^n}(F)$ we have $M(k) = M(l)$. Since $M(k)$ is obviously “exponential” ($M(kl) = M(k) + M(l)$), this completes the proof. $\square$

Corollary 3.6. *If $n > 1$ is odd, then the map $\widehat{\wp^n}(F) \xrightarrow{\varphi_1} \wp^n(F)_{\text{cl}}$ is an isomorphism.*

This corollary allows us to define the canonical map $\wp^n(F)_{\text{cl}} \rightarrow \wp^n(F)$ for odd n . Now we are going to construct the map $\wp^n(F)_{\text{cl}} \rightarrow \wp^n(F)$, provided that n is even.

Proposition 3.7. *Let $n > 0$ be an even integer. Then for any $k \in F^*$ the element $2^{\frac{n}{2}}\varphi_1(M(k))$ vanishes in $\wp^n(F)$.*

Proof. Let’s consider the group ring $\mathbb{Z}[\Sigma_n]$. The following construction supplies C_{n+1}^n with a structure of $\mathbb{Z}[\Sigma_n]$ -module. Let $[\mathbf{x}] = (e_1, \dots, e_n, E, \mathbf{x}) \in C_{n+1}^n$ and $\Xi = \sum_i c_i \xi_i \in \mathbb{Z}[\Sigma_n]$. ($\mathbf{x}$ here is an n -column and ξ_i are n -permutations.

We set

$$(3.8) \quad \Xi[\mathbf{x}] = \sum_i c_i (e_1, e_2, \dots, e_n, E, \xi_i \mathbf{x}) \in C_{n+1}^n.$$

Now, we construct a product map $\mathbb{Z}[\Sigma_k] \otimes \mathbb{Z}[\Sigma_l] \xrightarrow{\times} \mathbb{Z}[\Sigma_{k+l}]$ in the following way. Let $\sigma \in \Sigma_k$ and $\tau \in \Sigma_l$ be permutations acting on sets $\{1, \dots, k\}$ and $\{1, \dots, l\}$ respectively. Then $\sigma \times \tau \in \Sigma_{k+l}$ acts on the set $\{1, \dots, k+l\}$ by the formula

$$(3.9) \quad \sigma \times \tau(i) = \begin{cases} \sigma(i) & \text{for } i \leq k \\ k + \tau(i - k) & \text{for } i > k \end{cases}$$

We extend this definition to group rings by linearity. Let now γ be only nontrivial element in Σ_2 . Set $\Gamma = 1 + \gamma \in \mathbb{Z}[\Sigma_2]$ and consider $\Gamma^m = \Gamma \times \Gamma \times \cdots \times \Gamma \in \mathbb{Z}[\Sigma_{2m}]$. The following lemma can be easily checked by direct calculation.

Lemma 3.8. *Let n be a positive even. Then for any n -tuple $\mathbf{x}$ an element $\Gamma^{\frac{n}{2}}[\mathbf{x}]$ belongs to the image of the map $D_{n+1}^n \rightarrow C_{n+1}^n$ and its $\mathbb{Z}$ -augmentation is $2^{\frac{n}{2}}$.*

Consider now the element $\Gamma^{\frac{n}{2}}[k\mathbf{x}] - \Gamma^{\frac{n}{2}}[\mathbf{x}] \in \widehat{\wp^n}(F)$. Since it comes from D_{n+1}^n , its image in $\wp^n(F)$ is 0. On the other hand it should be equal to $2^{\frac{n}{2}}M(k)$. $\square$

Finally, we shall show that in dimension 2 the “classical” case coincides with our module $\wp^2(F)$.

Proposition 3.9. *Assume that 2 is invertible in the coefficient ring. Then $\wp^2(F) = \wp^2(F)_{\text{cl}}$*

Proof. Let us define a map $\wp^n(F) \xrightarrow{\beta} \wp^n(F)_{\text{cl}}$ setting $\beta([a/b]) = [b/a]_{\text{cl}}$. It's enough to check that the relation $[a/b] + [b/a] = 0$ existing in $\wp^2(F)$ holds in $\wp^2(F)_{\text{cl}}$ as well. Suslin showed (Lemma 1.2 [13]) that this relation holds after multiplication by 2. $\square$

4. APPLICATION TO LOW DIMENSIONAL CASES

Now we have introduced all the players and are ready to get some applications of the developed machinery. The main objectives of this section is to give a homological interpretation of groups $\wp^n(F)$ and to show some of thier possible applications to the investigation of groups $H_*(GL_n)$. The most important properties of groups $\wp^n(F)$ are summarized in Conjectures 0.1, 0.2.

Remark 4.1. In case of algebraically closed field F the assertion of Conjecture 0.2 easily follows from the divisibility of groups $\wp^n(F)_{\text{cl}}$ which look more geometrically motivated.

Let's, first, make one step down. It was shown in [8, 12] that there exists a sequence of groups $S_n(F)$ explicitly given by generators and relations such that

$$(4.1) \quad S_n(F) = \bigoplus_{i \geq 0} H_{n-2i}(GL_{n-2i}, GL_{n-2i-1}).$$

These groups may be decomposed as direct sum of certain Milnor K -groups that obviously implies that groups S_* of an algebraically closed field are divisible. These facts about groups $S_n(F)$ gave us basic motivation for Conjectures 0.1, 0.2.

In this section we shall prove Conjecture 0.1 for $n < 5$. (For $n = 4$ we had to assume in addition, that the field F is algebraically closed.) Conjecture 0.2 was proven for $n = 2$ in [9, 13]. Unfortunately, we do not know how to prove 0.2 for $n > 2$. Rather, we are going to show the importance of this conjecture for the investigation of the homology of linear groups.

Before proceeding, let us make some remarks concerning a definition of a product on homology and the “indecomposable part of the homology” which appear in Conjecture 0.1.

For integer $i, j > 0$ we can define a cup-product

$$(4.2) \quad H_i(GL_n, GM_n) \otimes H_j(F^*) \xrightarrow{\smile} H_{i+j}(GL_{n+1}, GM_{n+1})$$

as a composition of the external homological product and the natural map induced by the inclusion

$$(4.3) \quad F^* \times GL_n \mapsto \begin{pmatrix} F^* & 0 \\ 0 & GL_n \end{pmatrix} \subset GL_{n+1}.$$

One can easily check that this product is well-defined. The same construction defines a product on birelative homology groups as well.

We call the factor-group

$$H_k(GL_n, \{GM_n, GL_{n-1}\}) / \left(\bigoplus_{i+j=k} H_i(GL_{n-1}, \{GM_{n-1}, GL_n\}) \smile H_j(F^*) \right)$$

the indecomposable part of homology $H_k(GL_n, \{GM_n, GL_{n-1}\})_{\text{ind}}$.

4.1. The case $n = 2$. Theorem 2.8 implies the following fact as an immediate corollary.

Theorem 4.2.

$$H_i(GL_2, GM_2) = \begin{cases} 0 & i < 3 \\ \wp^2(F) & i = 3. \end{cases}$$

Proof. Really, if $n = 2$, then the E^1 -term of The Main Spectral Sequence has no nonzero entries but in the 0th row. $\square$

4.2. The case $n = 3$. In this case the E^1 -term of The Main Spectral Sequence is transgressive. Theorem 4.2 and the relative version of the Künneth formula [14] allow us to write down its E^2 -term in terms of relative homology groups.

$$(4.4) \quad \begin{array}{ccccccc} H_4(GL_2, GM_2) \oplus \wp^2 \otimes F^* & & & & & & \\ \wp^2 & & & & \text{Z E R O} & & \\ 0 & & 0 & & & & \\ 0 & & 0 & 0 & \wp^3 & \dots & \end{array}$$

This implies the following long exact sequence:

$$(4.5) \quad \dots \longrightarrow E_{0,q}^2 \xrightarrow{d^{(q)}} H_q((GL_2, GM_2) \times (F^*)) \longrightarrow H_q(GL_3, GM_3) \longrightarrow \dots$$

The statement of Conjecture 0.1 for $n = 3$ read as follows.

Theorem 4.3. $\wp^3(F) = H_4(GL_3, \{GM_3, GL_2\})_{\text{ind}}$.

Proof. We want to construct the following commutative diagram with exact columns.

$$\begin{array}{ccc}
 H_4(GL_2, GM_2) & \longrightarrow & H_4(GL_2, GM_2) \oplus (\wp^2 \otimes F^*) \\
 \downarrow & & \downarrow \\
 H_4(GL_3, GM_3) & \xlongequal{\quad} & H_4(GL_3, GM_3) \\
 \downarrow & & \downarrow \\
 H_4(GL_3, \{GM_3, GL_2\}) & \longrightarrow & \wp^3(F) \\
 \downarrow & & \downarrow \\
 H_3(GL_2, GM_2) & \xrightarrow{\sim} & \wp^2(F) \\
 \downarrow & & \downarrow \\
 H_3(GL_3, GM_3) & \xlongequal{\quad} & H_3(GL_3, GM_3) \\
 \downarrow & & \downarrow \\
 0 & & 0
 \end{array}
 \tag{4.6}$$

Let us remind that we denoted by $\tilde{K}^n$ the kernel of the map $\text{Ind}_{GM_n}^{GL_n} \mathbb{A} \rightarrow \mathbb{A}$ and by K_k^n the kernel of the map $\text{Ind}_{GM_n}^{GL_n} D_k^n \rightarrow C_k^n$. Let us also denote by $i: GL_n \hookrightarrow GL_{n+1}$ the standard embedding: $A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$. Consider the diagram:

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 \text{Ind}_{GL_2}^{GL_3} \tilde{K}^2 & & K_0^3 \\
 \downarrow & & \downarrow \\
 \text{Ind}_{GL_2}^{GL_3} \text{Ind}_{GM_2}^{GL_2} \mathbb{A} & \xrightarrow{\varphi} & \text{Ind}_{GM_3}^{GL_3} D_0^3 \\
 \downarrow & & \downarrow \\
 \text{Ind}_{GL_2}^{GL_3} \mathbb{A} & \xrightarrow{\psi} & C_0^3
 \end{array}
 \tag{4.7}$$

where $\varphi(g_1 \otimes g_2 \otimes 1) = g_1 i(g_2) \otimes e_1$ and $\psi(g \otimes 1) = g \otimes e_1$. Since the group GL_2 doesn't act on the vector e_1 , one can easily check that these maps are well defined, the diagram above commutes and has exact columns. The maps φ and ψ induce a map of the kernels $\text{Ind}_{GL_2}^{GL_3} \tilde{K}^2 \xrightarrow{\xi} K_0^3$.

This map ξ (together with the identity map in the lowest degree) induces a morphism of two-term complexes:

$$(4.8) \quad \begin{array}{ccc} (\tilde{K}^3 & \longleftarrow & \text{Ind}_{GL_2}^{GL_3} \tilde{K}_2) \\ \text{id} \downarrow & & \xi_* \downarrow \\ (\tilde{K}^3 & \longleftarrow & K_0^3). \end{array}$$

Consider the hyperhomology $H_*(GL_3, (\tilde{K}^3 \leftarrow \text{Ind}_{GL_2}^{GL_3} \tilde{K}_2))$. Using the standard machinery of homological algebra one can check that the corresponding spectral sequence converges to $H_*(GL_3, \{GM_3, GL_2\})$.

The E^1 -term of one of the associated spectral sequences has only two nontrivial rows and has the groups $H_*(GL_3, GM_3)$ in the 0th row and the groups $H_*(GL_2, GM_2)$ in the first one.

Let us now look at the hyperhomology spectral sequence of GL_3 with coefficients in the complex $(\tilde{K}^3 \leftarrow K_0^3)$. The map $K_0^3 \rightarrow \tilde{K}^3$ is an epimorphism. Consider the following GL_3 -resolution of $\text{Ker}(K_0^3 \rightarrow \tilde{K}^3)$: $K_1^3 \leftarrow K_2^3 \leftarrow \dots$. Since $H_i(GL_3, K_q^3) = 0$ for $i > 0$, $q > 0$, we have:

$$(4.9) \quad H_q(GL_3, (\tilde{K}^3 \leftarrow K_0^3)) \simeq H_0(GL_3, K_{q-1}^3).$$

The E^1 -term of the other spectral sequence associated to the latter two-term complex has two nonzero rows

$$(4.10) \quad \begin{array}{ccccc} \dots & H_*((GL_2, GM_2) \times F^*) & \dots \\ \dots & H_*((GL_3, GM_3)) & \dots \end{array}$$

Evidently, the map of the corresponding spectral sequences induced by ξ is a canonical isomorphism in the 0-th rows and induced by the inclusion $(GL_2, GM_2) \hookrightarrow (GL_2, GM_2) \times F^*$ in the first rows. We have constructed all the maps in diagram (4.6). The desired properties now follow from the functoriality.

The standard diagram chase in (4.6) gives us the exact sequence

$$(4.11) \quad \wp^2 \otimes F^* \xrightarrow{\gamma} H_4(GL_3, \{GM_3, GL_2\}) \longrightarrow \wp^3(F) \longrightarrow 0.$$

Now we want to check that γ is the product map. The homomorphism γ is induced by the augmentation map $K_0^3 \rightarrow \tilde{K}^3$ sending any column to 1. One can easily verify that this map defines a morphism of stabilizers.

$$(4.12) \quad \text{Stab}_{GL_3} e_1 = F^* \times GL_2 \longrightarrow \text{Stab}_{GL_3} 1 = GL_3,$$

given as follows

$$(4.13) \quad F^* \times GL_2 \mapsto \begin{pmatrix} F^* & 0 \\ 0 & GL_2 \end{pmatrix}$$

The latter morphism induces the desired product on homology. This fact, together with diagram (4.6), finishes the proof of Theorem 4.3. $\square$

Before proceeding with the case $n = 4$ we shall prove the following theorem.

Theorem 4.4. *The natural embedding $(GL_2, GM_2) \hookrightarrow (GL_3, GM_3)$ induces an isomorphism $H_3(GL_2, GM_2) \simeq H_3(GL_3, GM_3)$, provided that $3!$ is invertible in the coefficient ring.*

Proof. Let us look at the end terms of long exact sequence (4.5) associated to the case $n = 3$.

$$(4.14) \quad \wp^3 \xrightarrow{d} H_3(GL_2, GM_2) \longrightarrow H_3(GL_3, GM_3) \longrightarrow 0.$$

This proves that the map under consideration is an epimorphism. We shall now prove that the map d is zero, which requires more work. First of all, we introduce a new complex $C_*^{n|1}$ defined as follows. We set $C_q^{n|1} = C_q^n$ and

$$(4.15) \quad d^{n|1}(a_0|a_1, \dots, a_q) = \sum_{i=1}^q (-1)^i (a_0|a_1, \dots, \hat{a}_i, \dots, a_q).$$

(Since the differential doesn't affect the 0th column we will often separate it by “|”.) We also define a complex $\bar{D}_*^{n|1}$ as the following modification of D_*^n . We set $\bar{D}_k^{n|1}$ to be a free $\mathbb{A}$ -module generated by all classes of D -equivalence of matrices having the form:

$$(*|\underbrace{m, \dots, m}_{p \text{ times}}, \underbrace{a, \dots, a}_{q \text{ times}})$$

Here p and q are nonnegative integers, $p + q = k$; the symbol “ m ” means a monomial column, the symbol “ a ” means an affine column, and the star denotes a column of any of these two types. In addition, we assume that the maximal affine submatrix $(a, \dots, a)$ satisfies SGP-condition.

We introduce also a subcomplex of $\bar{D}_*^{n|1}$ denoted by $D_*^{n|1}$. For any k , we set $D_k^{n|1} = D_k^n / F_0 D_k^n$. Less formally, generators of $D_*^{n|1}$ are the generators of $\bar{D}_*^{n|1}$, having the form

$$(m|m, \dots, m, a, \dots, a).$$

Differential operators in all of the introduced complexes are given by formula (4.15).

Using the results of sections 1 and 2 one can easily check the following list of properties of complexes $C_*^{n|1}$, $\bar{D}_*^{n|1}$ and $D_*^{n|1}$.

a) Complexes $C_*^{n|1}$, $\bar{D}_*^{n|1}$, and $D_*^{n|1}$ are acyclic.

b) They have canonical structure of GL_n - (resp. GM_n -) modules.

c) Associated hyperhomology spectral sequence converges to 0 and has E^1 -term coinciding with the E^1 -term of the spectral sequence associated to the complex C_*^n (resp. D_*^n).

d) There is a natural epimorphism $\varphi: \text{Ind}_{GM_n}^{GL_n} D_*^{n|1} \twoheadrightarrow C_*^{n|1}$.

e) A map ψ , given by the formula

$$(4.16) \quad \psi(a_0, a_1, \dots, a_q) = \sum_{i=0}^q (-1)^i (a_i | a_0, a_1, \dots, \hat{a}_i, \dots, a_q)$$

induces morphisms of complexes $C_*^n \rightarrow C_*^{n|1}$ and $D_*^n \rightarrow \bar{D}_*^{n|1}$.

f) The diagram

$$(4.17) \quad \begin{array}{ccccc} \text{Ind}_{GM_n}^{GL_n} D_*^n & \xrightarrow{\varphi} & C_*^n & \longrightarrow & 0 \\ 1 \otimes \psi \downarrow & & \psi \downarrow & & \\ \text{Ind}_{GM_n}^{GL_n} \bar{D}_*^{n|1} & \xrightarrow{\varphi} & C_*^{n|1} & \longrightarrow & 0 \end{array}$$

commutes. This condition shows that ψ defines a morphism of cones

$$(4.18) \quad \text{Cone}(D_*^n \rightarrow C_*^n) \xrightarrow{\psi} \text{Cone}(\bar{D}_*^{n|1} \rightarrow C_*^{n|1}).$$

(Let us remind that we denote by $\text{Cone}(D_*^n \rightarrow C_*^n)$ the *Cone* of the morphism $\text{Ind}_{GM_n}^{GL_n} D_*^n \rightarrow C_*^n$.) We can also see that from the “homological point of view” there is no difference between complexes $(D_*^{n|1})_{GM_n}$ and $(\bar{D}_*^{n|1})_{GM_n}$.

Lemma 4.5. *If $n!$ is invertible in the coefficient ring, then the natural embedding $(D_*^{n|1})_{GM_n} \hookrightarrow (\bar{D}_*^{n|1})_{GM_n}$ induces an isomorphism*

$$(\bar{D}_*^{n|1})_{GM_n} \simeq (D_*^{n|1})_{GM_n} \oplus Q_*,$$

where the complex Q_* is contractible.

Proof. Decomposition is obvious. Contracting homotopy for Q_* can be cooked up by standard methods. $\square$

Corollary 4.6. *Under the same conditions as before, we have:*

$$H_*((\bar{D}_*^{n|1})_{GM_n}) \simeq H_*((D_*^{n|1})_{GM_n}).$$

Consider the spectral sequence associated to the $\text{Cone}(\bar{D}^{3|1} \rightarrow C^{3|1})$. The E^2 -term of this spectral sequence looks as follows.

$$(4.19) \quad \begin{array}{ccccccc} H_3(GL_2, GM_2) & 0 & \cdots & & & & \\ 0 & 0 & 0 & \cdots & & & \\ 0 & 0 & 0 & \wp^{3|1} & \cdots & & \end{array}$$

(We denote by $\wp^{n|1}$ an analog of the group $\wp^n$ which appears in the case of $n|1$ -complexes.)

Since the spectral sequence above converges to 0, we obtain an isomorphism $d_3^{3|1}: \wp^{3|1} \xrightarrow{\sim} H_3(GL_2, GM_2)$. Morphism ψ gives us a morphism of the E^2 -terms of corresponding hyperhomology spectral sequences which is the identity map on the first columns. So, we have the following commutative diagram:

$$(4.20) \quad \begin{array}{ccccc} \wp^3 & \xrightarrow{d} & H_3(GL_2, GM_2) & \longrightarrow & H_3(GL_3, GM_3) \longrightarrow 0 \\ \psi_* \downarrow & & \nearrow d_3^{3|1} & & \\ \wp^{3|1} & \xrightarrow{\simeq} & & & \end{array}$$

We are going to extend this diagram up to the following one:

$$(4.21) \quad \begin{array}{ccccc} \wp^3 & \xrightarrow{d} & H_3(GL_2, GM_2) & \longrightarrow & H_3(GL_3, GM_3) \longrightarrow 0 \\ \psi_* \downarrow & & \nearrow d_3^{3|1} & & \\ \wp^{3|1} & \xrightarrow{\simeq} & & & \\ \pi_* \downarrow & & \nearrow \simeq & & \\ \wp^2 & & & & \end{array}$$

and show that the vertical path $\wp^3 \rightarrow \wp^{3|1} \rightarrow \wp^2$ is the zero map. Let us first define the map π_* . This map is induced by the projection map $\mathbb{P}^2(F) \xrightarrow{\pi} \mathbb{P}^1(F)$ with the center at the point e_3 . More precisely, in any class of GL_3 -equivalence we can always find a matrix A of the form

$$(4.22) \quad A = \left(\begin{array}{c|ccc} 0 & w_{11} & \cdots & w_{1k} \\ 0 & w_{21} & \cdots & w_{2k} \\ 1 & w_{31} & \cdots & w_{3k} \end{array} \right)$$

We set

$$(4.23) \quad \pi(A) = \begin{pmatrix} w_{11} & \cdots & w_{1k} \\ w_{21} & \cdots & w_{2k} \end{pmatrix}$$

The same construction works for the terms of $(D_*^{3|1})_{GM_3}$ as well. One can easily check that constructed map is well-defined. The map just constructed gives us a morphism of bicomplexes:

$$(4.24) \quad \begin{array}{ccccccc} D_1^{3|1} & \xleftarrow{d^{3|1}} & D_2^{3|1} & \xleftarrow{d^{3|1}} & D_3^{3|1} & \xleftarrow{\quad} & \dots \\ \downarrow \pi_* & \searrow & \downarrow \pi_* & \searrow & \downarrow \pi_* & \searrow & \\ & D_0^2 & \xleftarrow{d} & D_1^2 & \xleftarrow{d} & D_2^2 & \xleftarrow{\quad} \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ C_1^{3|1} & \xleftarrow{d^{3|1}} & C_2^{3|1} & \xleftarrow{d^{3|1}} & C_3^{3|1} & \xleftarrow{\quad} & \dots \\ \downarrow \pi_* & \searrow & \downarrow \pi_* & \searrow & \downarrow \pi_* & \searrow & \\ & C_0^2 & \xleftarrow{d} & C_1^2 & \xleftarrow{d} & C_2^2 & \xleftarrow{\quad} \dots \end{array}$$

Comparing the E^2 -terms of the corresponding hyperhomology spectral sequences one can easily verify that the bottom triangle of Diagram (4.21) commutes.

Lemma 4.7. *The map $\wp^3 \xrightarrow{\psi} \wp^{3|1} \xrightarrow{\pi_*} \wp^2$ is zero.*

Proof. We can write down this map explicitly. Let $\mathbf{c}$ denote a 2-column. Denoting the generator $(e_1, e_2, E, \mathbf{c})$ of $\wp^2(F)$ by $(\mathbf{c})^*$, we have

$$(4.25) \quad \begin{aligned} \pi_* \psi \begin{pmatrix} 1 & 0 & 0 & 1 & a_1 \\ 0 & 1 & 0 & 1 & a_2 \\ 0 & 0 & 1 & 1 & a_3 \end{pmatrix} &= \begin{pmatrix} a_2 \\ a_3 \end{pmatrix}^* - \begin{pmatrix} a_1 \\ a_3 \end{pmatrix}^* + \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^* \\ &\quad - \begin{pmatrix} a_3 - a_1 \\ a_3 - a_2 \end{pmatrix}^* + \begin{pmatrix} a_3^{-1} - a_1^{-1} \\ a_3^{-1} - a_2^{-1} \end{pmatrix}^* \end{aligned}$$

On the other hand, we have

$$(4.26) \quad \begin{aligned} \partial \begin{pmatrix} 1 & 0 & 1 & a_3^{-1} - a_1^{-1} & a_3 - a_1 \\ 0 & 1 & 1 & a_3^{-1} - a_2^{-1} & a_3 - a_2 \end{pmatrix} &= \begin{pmatrix} -a_1 a_2 \\ -a_1 a_3 \end{pmatrix}^* - \begin{pmatrix} -a_1 a_2 \\ -a_2 a_3 \end{pmatrix}^* + \\ &\quad \begin{pmatrix} -a_1 a_3 \\ -a_2 a_3 \end{pmatrix}^* - \begin{pmatrix} a_3 - a_1 \\ a_3 - a_2 \end{pmatrix}^* + \begin{pmatrix} a_3^{-1} - a_1^{-1} \\ a_3^{-1} - a_2^{-1} \end{pmatrix}^* \end{aligned}$$

From the previous section we already know that in $\wp^2(F)$ there exists a relation

$$(4.27) \quad \begin{pmatrix} x \\ y \end{pmatrix}^* = \begin{pmatrix} ax \\ ay \end{pmatrix}^*.$$

Therefore, the image of $\pi_*\psi$ is a boundary. This observation finishes the proof of the lemma and the proof of Theorem 4.4 as well. $\square$

$\square$

We obtain as an easy corollary of Theorem 4.4 the following fact, proven in [13] with $\mathbb{Z}$ -coefficients in a different way.

Corollary 4.8. *The kernel of the homomorphism $H_3(GL_2) \rightarrow H_3(GL_3)$ lies in $H_3(GM_2)$.*

Proof. It is clear from the following commutative diagram with exact rows

$$(4.28) \quad \begin{array}{ccccc} H_3(GM_2) & \longrightarrow & H_3(GL_2) & \longrightarrow & H_3(GL_2, GM_2) \\ & \downarrow & \downarrow & & \simeq \downarrow \\ H_3(GM_3) & \longrightarrow & H_3(GL_3) & \longrightarrow & H_3(GL_3, GM_3) \end{array}$$

$\square$

The last case when we could get through with direct calculations is the next one.

4.3. The case $n = 4$. Throughout this section we shall assume in addition, that the base field F is algebraically closed. Using theorem 2.8 and the relative Künneth formula [14] we can write down the terms of interest in the E^1 -term of Spectral Sequence (2.8), in the case $n = 4$.

(For the shortness we will often denote $H_m(GL_n, GM_n)$ by $H_m(n)$.)

$$(4.29) \quad \begin{array}{ccc} \bigoplus_{i+j=5} (H_i(3) \otimes H_j(F^*)) & \bigoplus_{i+j+k=5} (H_i(3) \otimes H_j(F^*) \otimes H_k(F^*)) & \\ H_4(3) \oplus F^* \otimes H_3(3) & H_4(2) \oplus (F^* \times F^*) \otimes H_3(2) & \\ H_3(3) & H_3(2) & \text{Z E R O} \\ 0 & 0 & \\ * & * & * \end{array}$$

(We have $E_{p,q}^1 = 0$ for $p > 0$ and $q > 1$.)

Remark 4.9. All the Tor-summands in the considered dimensions happen to be zero. This is implied by vanishing the groups $H_m(GL_n, GM_n)$ for $m < 3$ and unique divisibility of the group

$$H_3(GL_3, GM_3) \simeq H_3(GL_2, GM_2) = \wp^2(F)$$

(See [13] and also [4]+[5].)

One can easily check that the differential operator $d_1: E_{21}^1 \rightarrow E_{20}^1$ ($H_3(2) \rightarrow H_3(3)$) is zero. In order to investigate the other homomorphisms we use the following construction. Consider a complex $C_*^{4|2}$ having the same generators as C_*^4 and the differential, given by the formula:

$$(4.30) \quad d(a_0, a_1 | a_2, \dots, a_q) = \sum_{i=2}^q (-1)^i (a_0, a_1 | a_2, \dots, \hat{a}_i, \dots, a_q).$$

We can also define a complex $D_*^{4|2}$, setting $D_q^{4|2} = D_q^4 / F_1 D_q^4$ and with a differential, given by Formula (4.30) as well. Less formally, the module $D_q^{4|2}$ is spanned by $n \times (q+1)$ -matrices with two separated monomial columns. In the same way as before, we construct $\text{Cone}(D_*^{4|2} \rightarrow C_*^{4|2})$. We also construct homomorphisms $C_*^{4|2} \xrightarrow{\varphi} C_*^4$ and $D_*^{4|2} \xrightarrow{\varphi} D_*^4$, given by the formula

$$(4.31) \quad \varphi: (a_0, a_1 | a_2, \dots, a_q) = (a_0, a_1, a_2, \dots, a_q) + (a_1, a_0, a_2, \dots, a_q),$$

which induce a morphism of cones

$$(4.32) \quad \text{Cone}(D_*^{4|2} \rightarrow C_*^{4|2}) \xrightarrow{\varphi} \text{Cone}(D_*^4 \rightarrow C_*^4).$$

The E^1 -term of the hyperhomology spectral sequence of GL_4 with coefficients in the cone $\text{Cone}(D_*^{4|2} \rightarrow C_*^{4|2})$ looks as follows:

$$(4.33) \quad E_{p,q}^1 = \begin{cases} \tilde{E}_{p,q+1}^1 & q > 0 \\ 0 & q = 0 \end{cases}$$

Here $\tilde{E}^1$ denotes the E^1 -term of the spectral sequence associated to the case $n = 4$. (See (4.29))

One can easily check that the latter spectral sequence converges to zero and has no nontrivial differentials, but coming out of the 0th row. So that, all the differentials in (4.29) vanish on the image of φ_* . We can also conclude that any element lying in the image of φ_* belongs to the image of the corresponding differential (from the 0th row) and, therefore, goes to zero in $E_{*,*}^\infty$. The homomorphism φ_* acts on the groups

$$(4.34) \quad \bigoplus_{i+j+k=N} (H_i(GL_3, GM_3) \otimes H_j(F^*) \otimes H_k(F^*))$$

as an endomorphism $\text{id} + s$, where s swaps the two copies of F^* .

Now we apply this discussion to the calculation of differentials in the spectral sequence (4.29).

Lemma 4.10. *The group $\wp^2 = H_3(GL_2, GM_2)$ is a direct summand of $\wp^4$, provided that 6 is invertible in the coefficient ring.*

Proof. Since $\wp^2 = H_3(GL_2, GM_2) = E_{2,1}^2 \subset \text{Im } \varphi_*$, there is no nontrivial differentials in spectral sequence (4.33) going out of the term of dimension $(2, 1)$ and the map

$$(4.35) \quad \wp^4 = E_{0,4}^3 \xrightarrow{d_3} E_{2,1}^3 = E_{2,1}^1 = H_3(GL_2, GM_2) = \wp^2$$

is an epimorphism. One can easily check that the map

$$(4.36) \quad H_3(GL_2, GM_2) = \wp^2 \xrightarrow{\varphi_*} \wp^2 = H_3(GL_2, GM_2)$$

is multiplication by 2. (The group on the left-hand side is an entry of the spectral sequence, corresponding to the case $\{4|2\}$. The right-hand side one is the corresponding entry of (4.29)).

So that, we have the following commutative diagram

$$(4.37) \quad \begin{array}{ccc} \wp^{4|2} & \xrightarrow{\simeq} & H_3(GL_2, GM_2) \\ \varphi_* \downarrow & & \downarrow \times 2 \\ \wp^4 & \xrightarrow{d_3} & H_3(GL_2, GM_2), \end{array}$$

which gives us a section of the differential d_3 . □

Lemma 4.11. *The map*

$$\begin{aligned} & H_4(GL_2, GM_2) \oplus (F^* \times F^*) \otimes H_3(GL_2, GM_2) \\ & \xrightarrow{d_1} H_4(GL_3, GM_3) \oplus F^* \otimes H_3(GL_3, GM_3) \end{aligned}$$

is given by a matrix $\begin{pmatrix} 0 & 0 \\ 0 & \psi \end{pmatrix}$, such that the following sequence

$$0 \longrightarrow \text{Im } \varphi_* \longrightarrow (F^* \times F^*) \otimes H_3(GL_2, GM_2) \xrightarrow{\psi} F^* \otimes H_3(GL_3, GM_3)$$

is exact.

Proof. One can easily see that the map

$$(4.38) \quad d_1: \bigoplus_{i+j+k=N} H_i(F^*) \otimes H_j(F^*) \otimes H_k(2) \longrightarrow \bigoplus_{l+m=N} H_l(F^*) \otimes H_m(3)$$

looks as follows

$$(4.39) \quad d_1(f \otimes g \otimes h) = f \otimes (g \smile h) - g \otimes (f \smile h).$$

This shows immediately that $d_1(H_4(GL_2, GM_2)) = 0$.

Since $i_*: H_3(GL_2, GM_2) \rightarrow H_3(GL_3, GM_3)$ is an isomorphism (See Theorem 4.4), each of the terms $F^* \otimes H_3(GL_2, GM_2)$ maps isomorphically onto $F^* \otimes H_3(3)$. (These maps are different by their signs.) Therefore, cokernel of d_1 is $H_4(GL_3, GM_3)$. The map

$$(4.40) \quad (F^* \times F^*) \otimes H_3(GL_2, GM_2) / \text{Im } \varphi \longrightarrow F^* \otimes H_3(GL_3, GM_3)$$

is an isomorphism. This implies the coincidence of $\text{Ker } d_1$ and $\text{Im } \varphi_*$. □

Corollary 4.12. *Since $\text{Ker}(E_{1,3}^1 \xrightarrow{d_1} E_{0,3}^1) \subset \text{Im } \varphi_*$, the natural projection $E_{1,3}^2 \rightarrow E_{1,3}^\infty$ is zero.*

Now we are able to draw the E^4 -term of the spectral sequence associated to the case $n = 4$.

$$\begin{array}{rcccl}
 \bigoplus_{i+j=5} H_i(3) \otimes H_j(F^*) / \text{Im } d_1 & * & & & \\
 (4.41) & H_4(3) & 0 & \text{Z E R O} & \\
 & H_3(3) & 0 & \vdots & \\
 & 0 & 0 & 0 & \dots \\
 & 0 & 0 & 0 & \wp^4 / \wp^2 \dots
 \end{array}$$

This immediately implies two corollaries.

Corollary 4.13. *We obtained the following stabilization result:*

$$H_3(GL_2, GM_2) \simeq H_3(GL_3, GM_3) \simeq \dots \simeq H_3(GL, GM).$$

Corollary 4.14. *The following sequence*

$$\bigoplus_{i+j=5} H_i(3) \otimes H_j(F^*) \longrightarrow H_5(4) \longrightarrow \wp^4 / \wp^2 \longrightarrow H_4(3) \longrightarrow H_4(4) \longrightarrow 0$$

is exact.

We can, finally, prove the main theorem of this subsection

Theorem 4.15. *Assume that $3!$ is invertible in the coefficient ring. Then the group $\wp^4(F)$ is a direct sum of the group $H_3(GL_2, GM_2)$ and the indecomposable component $H_5(GL_4, \{GL_3, GM_4\})_{\text{ind}}$.*

Proof. We have already established that the group $H_3(GL_2, GM_2) \simeq \wp^2(F)$ is a direct summand of $\wp^4(F)$. Let us now consider the following diagram.

$$\begin{array}{ccc}
 H_5(GL_3, GM_3) & \longrightarrow & \bigoplus_{i+j=5} H_i(GL_3, GM_3) \otimes H_j(F^*) \\
 \downarrow & & \downarrow \mu \\
 H_5(GL_4, GM_4) & \equiv & H_5(GL_4, GM_4) \\
 \downarrow & & \downarrow \\
 H_5(GL_4, \{GL_3, GM_4\}) & \longrightarrow & \wp^4(F)/\wp^2(F) \\
 \downarrow & & \downarrow \\
 H_4(GL_3, GM_3) & \equiv & H_4(GL_3, GM_3) \\
 \downarrow & & \downarrow \\
 H_4(GL_4, GM_4) & \equiv & H_4(GL_4, GM_4) \\
 \downarrow & & \downarrow \\
 0 & & 0
 \end{array}
 \tag{4.42}$$

Using the technique similar to the developed in the proof of the case $n = 3$, we can show that this diagram is commutative, has exact columns, and the map μ coincides with the $\smile$ -product on one-related homology groups. The standard diagram chase gives us the following exact sequence

$$\begin{array}{l}
 (4.43) \\
 \bigoplus_{\substack{i+j=5 \\ j>0}} H_i(GL_3, GM_3) \otimes H_j(F^*) \xrightarrow{\eta} H_5(GL_4, \{GL_3, GM_4\}) \longrightarrow \wp^4/\wp^2 \longrightarrow 0.
 \end{array}$$

(The map η here is obtained as a composition of μ and the projection map from the homology long exact sequence. We should now check that the image of the map η coincides with the image of the $\smile$ -product on bi-relative homology groups. Consider the following commutative diagram,

whose left vertical path is exact:

$$\begin{array}{ccc}
 & H_4(GL_2, GM_2) \otimes H_1(F^*) & \\
 & \downarrow & \\
 & H_4(GL_3, GM_3) \otimes H_1(F^*) & \xrightarrow{\mu} H_5(GL_4, GM_4) \\
 (4.44) \quad & \downarrow & \downarrow \\
 & H_4(GL_3, \{GM_3, GL_2\}) \otimes H_1(F^*) & \xrightarrow{\sim} H_5(GL_4, \{GM_4, GL_3\}) \\
 & \downarrow & \\
 & 0 &
 \end{array}$$

Since the left vertical arrow of the commutative square above is an epimorphism and all elements coming from $H_4(GL_2, GM_2) \otimes H_1(F^*)$ go to zero in the group $H_5(GL_4, \{GM_4, GL_3\})$, we obtain the desired result. $\square$

Proposition 4.16. *Assume that the claim of Conjecture 0.2 holds for $n = 2, 4$. (It is known only for $n = 2$. See [4, 13]). Let F be an algebraically closed field and $p > n$ be a prime. Then the natural mapping $H_4(GL_3(F), \mathbb{Z}/p) \rightarrow H_4(GL_4(F), \mathbb{Z}/p)$ is a monomorphism.*

Proof. One can easily show that Conjecture 0.2 implies vanishing of the group $H_5(GL_4, \{GM_4, GL_3\}, \mathbb{Z}/p)$. Let's now look at the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
 (4.45) \quad H_5(GM_4, GM_3, \mathbb{Z}/p) & \longrightarrow & H_4(GM_3, \mathbb{Z}/p) & \xrightarrow{i_*} & H_4(GM_4, \mathbb{Z}/p) \\
 \downarrow & & \downarrow & & \downarrow \\
 H_5(GL_4, GL_3, \mathbb{Z}/p) & \longrightarrow & H_4(GL_3, \mathbb{Z}/p) & \xrightarrow{i_*} & H_4(GL_4, \mathbb{Z}/p)
 \end{array}$$

Any class x in $H_4(GL_3, \mathbb{Z}/p)$ belonging to the kernel of i_* lifts up to some class x' in $H_5(GM_4, GM_3, \mathbb{Z}/p)$. It can be easily checked that the map i_* in the upper row is a monomorphism. Therefore, the image of x' in $H_4(GM_3, \mathbb{Z}/p)$ and, moreover $H_4(GL_3, \mathbb{Z}/p)$ is zero. $\square$

The latter proposition shows the significant importance of Conjecture 0.2 for investigation of the homology of linear groups. The questions connected to the estimation of kernels of maps induced by natural embedding seem to be the most difficult in this field.

5. CONSTRUCTION OF THE HOMOMORPHISM

$$\xi: H_n(GL_n(F)) \rightarrow \wp^{n-1}(F)$$

In this section we generally follow the strategy proposed by Suslin in [13]. Some changes of the construction make our approach more convinient.

Remark 5.1. In order to make our computations fully compatible with ones in [13] we've chosen here a different embedding of $GL_{n-1} \hookrightarrow GL_n$, given by the formula $GL_{n-1} \mapsto \begin{pmatrix} GL_{n-1} & 0 \\ 0 & 1 \end{pmatrix} \in GL_n$. This, of course, doesn't change the maps induced on homology.

Consider the complex $P_*^n(F)$ defined in section 3. Let us define another differential d' such that

$$(5.1) \quad d'(x_0, \dots, x_p) = \sum_{i=0}^{p-1} (-1)^i (x_0, \dots, \hat{x}_i, \dots, x_p).$$

One can easily prove

Lemma 5.2. *The following complexes are acyclic:*

$$\begin{array}{ccccccc} 0 & \longleftarrow & \mathbb{Z} & \xleftarrow{\epsilon} & P_0^n & \xleftarrow{d} & P_1^n & \xleftarrow{d} & \dots \\ & & & & 0 & \xleftarrow{d'} & P_1^n & \xleftarrow{d'} & \dots \end{array}$$

We define the homomorphism $\nu: P_m^n \rightarrow P_m^n$ by the formula:

$$(5.2) \quad \nu(x_0, \dots, x_p) = \sum_{i=0}^p (-1)^i (x_0, \dots, \hat{x}_i, \dots, x_p, x_i).$$

Direct computation shows that $d'\nu = \nu d$. Consider now the following morphism of complexes

$$(5.3) \quad \begin{array}{ccccccc} P_1^n & \xleftarrow{d} & P_2^n & \xleftarrow{d} & P_3^n & \xleftarrow{d} & \dots \\ \nu \downarrow & & \nu \downarrow & & \nu \downarrow & & \\ P_1^n & \xleftarrow{d'} & P_2^n & \xleftarrow{d'} & P_3^n & \xleftarrow{d'} & \dots \end{array}$$

and denote its cone by $\mathcal{P}_*$. Let also define the “natural” augmentation τ as a composition $\tau: \mathcal{P}_0 = P_1^n \xrightarrow{d'} P_0^n \xrightarrow{\epsilon} \mathbb{Z}$. One can easily check that the complex $0 \leftarrow \mathbb{Z} \xleftarrow{\tau} \mathcal{P}_*$ is acyclic. This complex has a natural GL_n -module structure that determines a canonical homomorphism $H_n(GL_n(F)) \rightarrow H_n((\mathcal{P}_*)_{GL_n(F)})$. The groups $(P_m^n)_{GL_n(F)} = \bar{P}_m^n$ can be easily computed. In particular, $\bar{P}_m^n = \mathbb{Z}$ for $m \leq n$, $\bar{P}_{n+1}^n = \coprod \mathbb{Z}[a_1, \dots, a_{n-1}]$, where $[a_1, \dots, a_{n-1}]$ is the orbit of

$$(5.4) \quad \begin{pmatrix} & & & & & 1 & \\ e_1 & e_2 & \cdots & e_{n-1} & E & a_1 & e_n \\ & & & & & a_2 & \\ & & & & & \vdots & \\ & & & & & a_{n-1} & \end{pmatrix}$$

and summation is over all vectors $[a_1, \dots, a_{n-1}]$ for which the matrix above satisfy GP-condition. In the same way, the group $\bar{P}_{n+1}^n$ can be written as $\coprod \mathbb{Z} \begin{bmatrix} a_1 & \dots & a_{n-1} \\ b_1 & \dots & b_{n-1} \end{bmatrix}$ under the same conditions. Let us define the following homomorphisms:

(5.5)

$$\omega_1: (\mathcal{P}_n^n)_{GL_n} = \mathbb{Z} \oplus \coprod \mathbb{Z} [a_1, \dots, a_{n-1}] \rightarrow (\bar{P}_n^{n-1}) = \coprod \mathbb{Z} [x_1, \dots, x_{n-2}]$$

and

$$(5.6) \quad \omega_2: (\mathcal{P}_{n+1}^n)_{GL_n} = \coprod \mathbb{Z} [a_1, \dots, a_{n-1}] \oplus \coprod \mathbb{Z} \begin{bmatrix} a_1 & \dots & a_{n-1} \\ b_1 & \dots & b_{n-1} \end{bmatrix} \rightarrow (\bar{P}_{n+1}^{n-1}) = \coprod \mathbb{Z} \begin{bmatrix} x_1 & \dots & a_{n-2} \\ y_1 & \dots & y_{n-2} \end{bmatrix}$$

setting

$$(5.7) \quad \omega_1(1) = 0$$

$$(5.8) \quad \omega_1([a_1, \dots, a_{n-1}]) = \langle a_1, \dots, a_{n-2} \rangle$$

$$(5.9) \quad \omega_2 \left(\begin{bmatrix} a_1 & \dots & a_{n-1} \\ b_1 & \dots & b_{n-1} \end{bmatrix} \right) = \left\langle \begin{bmatrix} a_1 & \dots & a_{n-2} \\ b_1 & \dots & b_{n-2} \end{bmatrix} \right\rangle$$

$$(5.10) \quad \omega_2([a_1 \cdots a_{n-1}]) = \left\langle \begin{bmatrix} a_1 & \dots & a_{n-2} \\ \frac{a_1(1-a_{n-1})}{a_1-a_{n-1}} & \dots & \frac{a_{n-2}(1-a_{n-1})}{a_{n-2}-a_{n-1}} \end{bmatrix} \right\rangle$$

The above notation $\left\langle \begin{bmatrix} a_1 & \dots & a_{n-2} \\ b_1 & \dots & b_{n-2} \end{bmatrix} \right\rangle$ means the class of matrix

$$(5.11) \quad \begin{pmatrix} & & & & & 1 & 1 \\ e_1 & e_2 & \dots & e_n & E & a_1 & b_1 \\ & & & & & \vdots & \vdots \\ & & & & & a_{n-2} & b_{n-2} \end{pmatrix}$$

in $(P_{n+1}^{n-1})_{GL_{n-1}}$.

Direct computation shows that the following diagram commutes.

$$(5.12) \quad \begin{array}{ccc} \bar{P}_{n+1}^n & & \\ \downarrow \nu & \searrow \omega_2 & \\ \bar{P}_{n+1}^n & \xleftarrow{d'} & \bar{P}_{n+2}^n \\ \downarrow \omega_1 & & \downarrow \omega_2 \\ \bar{P}_n^{n-1} & \xleftarrow{d} & \bar{P}_{n+1}^{n-1} \end{array}$$

This gives us a well-defined homomorphism $\omega: H_n((\mathcal{P}_*^n)_{GL_n}) \rightarrow \wp_{\text{cl}}^{n-1}$.

Combining two homomorphisms obtained in this section with homomorphism constructed in section 3 we get the following map:

$$(5.13) \quad \xi: H_n(GL_n) \rightarrow H_n((\mathcal{P}_*^n)_{GL_n}) \xrightarrow{\omega} \wp_{\text{cl}}^{n-1} \xrightarrow{\alpha} \wp^{n-1}.$$

Since the action of GL_{n-1} does not affect the vector e_n , we obtain a structure of GL_{n-1} -module on the low row of the bicomplex $\mathcal{P}_*^n$. This defines a canonical homomorphism $H_n(GL_{n-1}) \rightarrow \wp^{n-1}$. On the other hand, the standard GL_{n-1} -resolution of $\mathbb{Z}$ maps to the resolution $P_0^{n-1} \xleftarrow{d} P_1^{n-1} \xleftarrow{d} \dots$. This gives us an alternative way to define the mapping $H_n(GL_{n-1}) \rightarrow \wp^{n-1}$. Repeating almost word by word arguments from the proof of Lemma 3.4. [13], we can see that both constructions define the same map and the diagram

$$(5.14) \quad \begin{array}{ccc} H_n(GL_{n-1}) & \xrightarrow{i_*} & H_n(GL_n) \\ & \searrow & \swarrow \xi \\ & \wp_{\text{cl}}^{n-1} & \end{array}$$

commutes.

The homomorphism constructed in this section allows us to make some estimations of kernels of maps $H_n(GL_{n-1}) \xrightarrow{i_*} H_n(GL_n)$. It can be considered as a generalization of Proposition 3.1 [13] for higher dimensions. (See also Corollary 4.2.) The following proposition holds.

Proposition 5.3. *For $3 \leq n \leq 5$ the kernel of the natural homomorphism $H_n(GL_{n-1}) \xrightarrow{i_*} H_n(GL_n)$ maps into decomposable part of $H_n(GL_{n-1}, GM_{n-1})$. (Here i_* denotes, as usually, a homomorphism induced on homology by natural group inclusion.)*

Proof. Consider the following diagram:

$$(5.15) \quad \begin{array}{ccccc} & & H_n(GM_{n-1}) & & \\ & & \downarrow & & \\ & & H_n(GL_{n-1}) & \xrightarrow{i_*} & H_n(GL_n) \\ & & \downarrow & & \downarrow \\ \bigoplus_{i+j=n} H_i(n-2) \otimes H_j(F^*) & \longrightarrow & H_n(GL_{n-1}, GM_{n-1}) & \longrightarrow & \tilde{\wp}^{n-1}(F), \end{array}$$

where $\tilde{\wp}^i$ denotes $\wp^4/\wp^2$ for $i = 4$ and coincide to $\wp^i$ otherwise. Proposition 5.3 and results of section 4 imply commutativity of this diagram and

exactness of its rows and columns. Diagram chasing now shows that an element belonging to $\text{Ker}(H_n(GL_{n-1}) \rightarrow H_n(GL_n))$ should either come from $H_n(GM_{n-1})$ or map to the decomposable part of $H_n(GL_{n-1}, GM_{n-1})$. $\square$

Remark 5.4. One can easily see that for $n=3$ there is no nonzero decomposable elements in $H_3(CL_2, GM_2)$ and our result coincides with Corollary 4.2.

REFERENCES

- [1] Bloch, S. *Higher regulators, algebraic K-theory, and zeta functions of elliptic curves*, Lecture Notes, Univ. of California, Irvine, Calif., 1978.
- [2] Brown, K. *Cohomology of groups*. Springer-Verlag New York, Heidelberg, Berlin, 1982.
- [3] Cathelineau, J.-L. *Birapport et groupoides*. Enseign. Math. (2), 41 (1995) no. 3–4, pp.257–280
- [4] Dupont, J.L. and C.-H. Sah *Scissors congruences II*, Jour. Pure and Applied Algebra. 25 (1982) pp.159–195.
- [5] Dupont, J.L. and E.T. Poulsen *Generation of $\mathbb{C}(x)$ by a restricted set of operations*, Jour. Pure and Applied Algebra. 25 (1982) pp.155–157.
- [6] Elbaz-Vincent Ph. *K_3 indécomposable des anneaux et homologie de SL_2* . Thèse de doctorat, Nice, 1995.
- [7] Kallen, van der W. *Homology stability for genaral linear group*. Invent. Math. 1980 V.60 (3) pp.269–295.
- [8] Nesterenko, Yu.P. and A.A. Suslin *Homology of the full linear group over a local ring and Milnor’s K-theory*, Math. USSR Izvestija, vol. 34, N 1 (1990) pp.121–145.
- [9] Sah, C.-H. *Homology of classical Lie groups made discrete III*, Jour. Pure and Applied Algebra. 56 (1989) pp.269–312.
- [10] Sah, C.-H and Wagoner, J.B. *Second homology of Lie groups made discrete*, Comm. Algebra 5 (1977), no. 6, pp.611–642.
- [11] Serre, J.-P. *Arbres, Amalgames, SL_2* , Astér. no. 46. Soc. Math. France, 1977.
- [12] Suslin, A.A. *Homology of GL_n , characteristic classes and Milnor’s K-theory*, Proc. Steklov Inst. Math. N 3, 165 (1985) pp.207–226.
- [13] Suslin, A.A. *K_3 of field and the Bloch group*, Proc. Steklov Inst. Math. N 3 (1991) pp.217–239.
- [14] Switzer, R.M. *Algebraic topology – homotopy and homology*, Springer-Verlag Berlin, New York, Heidelberg, 1975.
- [15] Yagunov, S. *Homology of bi-Grassmannian complexes*. K-theory Vol. 12, No. 3 (1997) pp.277–292.
- [16] Yagunov, S. *Geometrically Originated Complexes and the Homology of the Pair (GL_n, GM_n)* . Ph.D. Thesis, Northwestern University, Evanston, 1997.
- [17] Yagunov, S.A. *On a spectral sequence converging to homology of GL_n relatively to GM_n* . Preprint POMI RAN #8, 1998. (in Russian)

MAX PLANK INSTITUTE FÜR MATHEMATIK, VIVATSGASSE 7, 53111 BONN, GERMANY.

Email address: yagunov@mpim-bonn.mpg.de