Introduction.

Eric M. Friedlander, A. Suslin and V. Voevodsky.

Our original goal which finally led to this volume was the construction of "motivic cohomology theory" whose existence was conjectured by A. Beilinson and S. Lichtenbaum ([2], [3], [17], [18]). Even though this would seem to be achieved at the end of the third paper, our motivation evolved into a quest for a deeper understanding of various properties of algebraic cycles. Thus, several of the papers presented here do not deal directly with motivic cohomology but rather with basic questions about algebraic cycles.

In this introduction, we shall begin with a short reminder of A. Beilinson's formulation of motivic cohomology theory. We then proceed to briefly summarize the topic and contents of individual papers in the volume.

Let k be a field and Sm/k denote the category of smooth schemes over k. A. Beilinson conjectured that there should exist certain complexes $\mathbf{Z}(n)$ ($n \geq 0$) of sheaves in the Zariski topology on Sm/k which have the following properties:

1. $\mathbf{Z}(0)$ is the constant sheaf $\mathbf{Z}$.

2. $\mathbf{Z}(1)$ is the sheaf $\mathcal{O}^*$ placed in cohomological degree 1.

3. For a field F over k, one has

   $$H^n(Spec(F), \mathbf{Z}(n)) = K_n^M(F)$$

   where $K_n^M(F)$ is the n-th Milnor K-group of F.

4. For a smooth scheme X over k, one has

   $$\mathbf{H}^{2n}_{Zar}(X, \mathbf{Z}(n)) = A^n(X)$$

   where $A^n(X)$ is the Chow group of cycles of codimension n on X modulo rational equivalence.

5. For any smooth scheme X over k, there is a natural spectral sequence with the E_2-term of the form

   $$E_2^{p,q} = \mathbf{H}^p_{Zar}(X, \mathbf{Z}(q))$$

and differentials $d_r : E_r^{p,q} \to E_r^{p+r-1,q+2r-1}$ which converges to Quillen's K-groups $K_{2q-p}(X)$.

After tensoring with $\mathbf{Q}$ this spectral sequence degenerates and one has

$$\mathbf{H}^i_{Zar}(X, \mathbf{Z}(n)) \otimes \mathbf{Q} = gr^n_\gamma K_{2n-i}(X) \otimes \mathbf{Q}$$

where the groups on the right hand side are quotients of the γ-filtration in Quillen's K-theory of X.

Observe that the complexes $\mathbf{Z}(n)$ are determined rationally by the last of above properties.

The hypercohomology groups $\mathbf{H}^i_{Zar}(X, \mathbf{Z}(n))$ are usually denoted by $H^i_{\mathcal{M}}(X, \mathbf{Z}(n))$ and called *motivic cohomology* groups of X.

This definition of motivic cohomology is not "topology free". In particular one may consider the corresponding hypercohomology groups in the etale topology instead of the Zariski topology. S. Lichtenbaum ([17], [18]) has in fact suggested axioms for the etale analog of Beilinson's motivic cohomology. We emphasize that everywhere in this volume "motivic (co-)homology" mean motivic (co-)homology in the Zariski topology unless the etale topology is explicitly specified.

In addition to the axioms given above, Beilinson's original list contained two further axioms. These we state below in the form of conjectures.

Beilinson-Lichtenbaum Conjecture. For a field F over k and a prime l not equal to $char(k)$, one has

$$\mathbf{H}^i_{Zar}(Spec(F), \mathbf{Z}(n) \otimes \mathbf{Z}/l) = \begin{cases} H^i_{et}(F, \mu_l^{\otimes n}) & \text{for} \quad i \leq n \\ 0 & \text{for} \quad i > n \end{cases}$$

Beilinson-Soule Vanishing Conjecture. For a smooth scheme X over k, one has

$$\mathbf{H}^i_{Zar}(X, \mathbf{Z}(n)) = 0$$

for $i < 0$.

In conjunction with the spectral sequence relating motivic cohomology to algebraic K-theory, these two "axioms" imply the validity of highly nontrivial conjectures in algebraic K-theory.

All the approaches to motivic (co-)homology suggested in the last several years can be roughly divided into two types depending on which of property 3 and 4 was considered to be "more fundamental":

1. To construct motivic cohomology (usually of a field) as cohomology groups of certain complexes with terms being given by explicit generators and relations generalizing Milnor's definition of K^n_M ([3],[4], [13]).

2. To construct motivic cohomology of a scheme as cohomology of a complex defined in terms of algebraic cycles thus generalizing the classical definition of Chow groups ([5],[6],[8] [14][1]).

The basic problem with the first approach is that it is very difficult to prove functorial properties of theories constructed in such a way; for example, the proof that Milnor K-theory has transfer maps is quite non-trivial. On the other hand, an important advantage of this first approach is that from this point of view it is possible to construct natural looking complexes (see [13]) satisfying the Beilinson-Soule vanishing property (saying that $H^i_{\mathcal{M}}(X, \mathbf{Z}(n)) = 0$ for $i < 0$).

The second approach was pioneered by S. Bloch in [5] who introduced "higher" Chow groups with many good properties. In the papers of this volume, we develop an alternative (and independent) theory of motivic homology and cohomology based upon algebraic cycles. (The reader should be forewarned that many of our results apply only to varieties over a field for which resolution of singularities is valid.)

A major difficulty which must be confronted in this approach is that standard moving techniques used in the classical theory of algebraic cycles are not sufficient to prove basic properties of such a theory. S. Bloch recently solved this problem for his higher Chow groups by introducing an ingenious but very complicated moving technique based on blow-ups and M. Spivakovsky 's solution of Hironaka's polyhedron game ([7]). A similar problem was encountered by A. Suslin when attempting to prove some properties of "algebraic singular homology theory".

Our theory culminates in the fourth paper of this volume "Triangulated categories of motives over a field": this applies the results of the other papers to construct a consistent triangulated theory of mixed motives over a field[2]. We construct there a certain rigid tensor triangulated category $DM_{gm}(k)$ to-

[1]The construction of motivic cohomology given by D. Grayson does not refer explicitely to algebraic cycles, but uses instead some version of algebraic K-theory with supports.

[2]Two other constructions of triangulated categories of mixed motives have been proposed by M. Hanamura [15] and M. Levine [16].

gether with an invertible object $\mathbf{Z}(1)$ called the Tate object and two functors

$$M : Sch/k \to DM_{gm}(k)$$

$$M_c : (Sch/k)_{prop} \to DM_{gm}(k)$$

(where $(Sch/k)_{prop}$ is the category of schemes of finite type over k and proper morphisms) which satisfy triangulated analogs of functorial properties of homology and Borel-Moore homology respectively. Aside from further "motivic" applications, the value of this theory is that it provides a natural categorical framework for different kinds of "algebraic cycle (co-)homology" type theories.

In particular Bloch's higher Chow groups $CH^i(X, j)$ and Suslin's algebraic singular homology $H_i^{alg}(X, \mathbf{Z})$ admit the following descriptions in terms of $DM_{gm}(k)$:

$$CH^i(X, j) = Hom_{DM}(\mathbf{Z}(d-i)[2d-2i+j], M_c(X))$$

(where $d = dim(X)$)

$$H_i^{alg}(X, \mathbf{Z}) = Hom_{DM}(\mathbf{Z}[i], M(X)).$$

Similarly, one defines motivic cohomology by

$$H^i_{\mathcal{M}}(X, \mathbf{Z}(j)) \equiv Hom_{DM}(M(X), \mathbf{Z}(j)[i]).$$

In the case of a smooth scheme X, the motivic Poincare duality theorem asserts that cohomology is canonically isomorphic to Borel-Moore homology and thus isomorphic to higher Chow groups.

We prove that our motivic cohomology theory satisfies properties (1)-(4) listed above. The comparison with higher Chow groups together with the recent construction of a "motivic spectral sequence" by S. Bloch and S. Lichtenbaum [9] and work in progress by Eric M. Friedlander, M. Levine, A. Suslin and M. Walker gives us the fifth property[3].

As was mentioned above, attempting to develop such a theory one encounters certain difficulties related to the fact that we do not yet know how to work efficiently with algebraic cycles up to "higher homotopies".

There are four main technical tools which we use to overcome these difficulties:

[3]We would like to mention that the existence of a "motivic spectral sequence" is not strictly speaking a part of the theory discussed here, but rather of a yet to be constructed stable homotopy theory of schemes.

1. a theory of sheaves of relative cycles.
2. a theory of sheaves and presheaves with transfers.
3. the Nisnevich and cdh-topologies on the category of schemes.
4. the Friedlander-Lawson moving lemma for families of algebraic cycles.

The theory of sheaves of relative cycles is developed in the first paper of this volume “Relative cycles and Chow sheaves”. The basic idea of this theory comes from two independent sources - one is Lawson homology theory and the classical theory of Chow varieties and another is the sheaf theoretic approach to finite relative cycles used in [22] and [21].

An important aspect of the theory of relative cycles is that it is “elementary” in the sense that it only uses very basic properties of schemes. We develope this theory in the very general context of schemes of finite type over an arbitrary Noetherian scheme, although applications considered in this volume concern varieties over a field.

Another of the main tools appearing in this paper is the cdh-topology. This plays an important role in the theory, because different kinds of “localization” sequences for sheaves of relative cycles become exact in the cdh-topology.

The theory of presheaves with transfers and more specifically homotopy invariant presheaves with transfers is the main theme of the second paper “Cohomological theory of preshaves with transfers”. The idea that transfers should play an important role in motivic theory can be traced to two sources. One is the use of transfers in the proof of a rigidity theorem by A. Suslin in [20]; another is the theory of qfh-sheaves considered in [22]. These were first combined in [21], leading to a proof of A. Suslin's conjecture relating the algebraic singular homology with finite coefficients and the etale cohomology. The results on presheaves with transfers obtained in the second paper together with the Nisnevich (or “completely decomposable”) topology [19] enable us to prove the Mayer-Vietoris exact sequence for algebraic singular homology; this result is an example of a solution to a “moving” probem inaccessible to either classical methods or S. Bloch's “moving by blow-ups” techniques.

The Friedlander-Lawson moving lemma for families of algebraic cycles ([11]) appears in the third paper in which we construct a bivariant theory called bivariant cycle cohomology[4]. In this paper, duality between cohomology and homology is studied, following duality studied by Eric M. Friedlander and H.B. Lawson in the context of Lawson homology [10]. While the definition of our bivariant theory (at least in the case of a smooth first argument) is very elementary, to prove basic properties such as Mayer-Vietoris and localization requires all the machinery of the preceding papers together with the moving lemma for families of algebraic cycles. As before, we have had two main sources of inspiration. One is the theory of morphic cohomology which is a bivariant analog of Lawson homology developed by Eric M. Friedlander and H.B. Lawson. Another is the approach to localization problems based on the theory of presheaves with transfers and the cdh-topology which was developed by V. Voevodsky.

The fifth and last paper in the volume gives a proof of the fact that the groups of the Borel-Moore homology component of the bivariant cycle cohomology are canonically isomorphic (in appropriate cases) to Bloch's higher Chow groups, thereby providing a link between our theory and Bloch's original approach to motivic (co-)homology.

The reader will find separate introductions at the beginning of each paper in this volume. To obtain a broader view of motivic cohomology and its relationship to Grothendieck's original goal of a good category of motives, the reader may wish to consult [1].

V. Voevodsky is pleased to acknowledge encouragement and useful conversations with A. Beilinson, A. Goncharov, and D. Kazhdan.

References

[1] *Motives*, volume 54-55 of *Proc. of Symp. in Pure Math.* AMS, 1994.

[2] A. Beilinson. Height pairing between algebraic cycles. In *K-theory, Arithmetic and Geometry.*, volume 1289 of *Lecture Notes in Math.*, pages 1–26. Springer-Verlag, 1987.

[3] A. Beilinson, R. MacPherson, and V. Schechtman. Notes on motivic cohomology. *Duke Math. J.*, pages 679–710, 1987.

[4] We should mention that the word "bivariant" is used by us in its "naive" sense and not in the sense of [12].

[4] A. A. Beilinson, A. B. Goncharov, V. V. Schechtman, and A. N. Varchenko. Aomoto dilogarithms, mixed Hodge structures and motivic cohomology of pairs of triangles on the plane. In *The Grothendieck festchrift*, volume 1, pages 135–172. Birkhauser, Boston, 1990.

[5] S. Bloch. Algebraic cycles and higher K-theory. *Adv. in Math.*, 61:267–304, 1986.

[6] S. Bloch. Algebraic cycles and the Lie algebra of mixed Tate motives. *JAMS*, 4:771–791, 1991.

[7] S. Bloch. The moving lemma for higher Chow groups. *J. Algebr. Geom.*, 3(3):537–568, Feb. 1994.

[8] S. Bloch and I. Kriz. Mixed Tate motives. *Ann. of Math.*, 140:557–605, 1994.

[9] S. Bloch and S. Lichtenbaum. A spectral sequence for motivic cohomology. *Preprint*, 1994.

[10] Eric M. Friedlander and H. Blaine Lawson. Duality relating spaces of algebraic cocycles and cycles. *Topology*, 36(2):533–565, 1997.

[11] Eric M. Friedlander and H. Blaine Lawson. Moving algebraic cycles of bounded degree. *Invent. Math.*, 132(1):91–119, 1998.

[12] W. Fulton and R. MacPherson. *Catergorical framework for the study of singular spaces*. Number 243 in Mem. AMS. AMS, 1981.

[13] A. B. Goncharov. Geometry of configurations, polylogarithms, and motivic cohomology. *Adv. Math.*, 114(2):197–318, 1995.

[14] Daniel R. Grayson. Weight filtrations via commuting automorphisms. *K-Theory*, 9(2):139–172, 1995.

[15] Masaki Hanamura. Mixed motives and algebraic cycles. I. *Math. Res. Lett.*, 2(6):811–821, 1995.

[16] Marc Levine. *Mixed motives*. American Mathematical Society, Providence, RI, 1998.

[17] S. Lichtenbaum. Values of zeta-functions at non-negative integers. In *Number theory*, volume 1068 of *Lecture Notes in Math.*, pages 127–138. Springer-Verlag, 1983.

[18] S. Lichtenbaum. New results on weight-two motivic cohomology. In *The Grothendieck festchrift*, volume 3, pages 35–55. Birkhauser, Boston, 1990.

[19] Y. Nisnevich. The completely decomposed topology on schemes and associated descent spectral sequences in algebraic K-theory. In *Algebraic K-theory: connections with geometry and topology*, pages 241–342. Kluwer Acad. Publ., Dordrecht, 1989.

[20] A. Suslin. On the K-theory of algebraically closed fields. *Invent. Math*, 73:241–245, 1983.

[21] Andrei Suslin and Vladimir Voevodsky. Singular homology of abstract algebraic varieties. *Invent. Math.*, 123(1):61–94, 1996.

[22] V. Voevodsky. Homology of schemes. *Selecta Mathematica, New Series*, 2(1):111–153, 1996.