Cohomological theory of presheaves with transfers.

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1 Introduction

Let k be a field and Sm/k be the category of smooth schemes over k . In this paper we study contravariant functors from the category Sm/k to additive

categories equipped with *transfer maps*. More precisely we consider contravariant functors $F : (Sm/k)^{op} \rightarrow A$ together with a family of morphisms $\phi_{X/S}(Z) : F(X) \rightarrow F(S)$ given for any smooth curve $X \rightarrow S$ over a smooth scheme S over k and a relative divisor Z on X over S which is finite over S . If these maps satisfy some natural properties (see definition 3.1) such a collection of data is called a *pretheory* over k . Some examples of pretheories are given in §3.4.

A pretheory F is called homotopy invariant if for any smooth scheme X one has $F(X \times \mathbf{A}^1) = F(X)$. For any pretheory F with values in an abelian category we define a family of homotopy invariant pretheories $\underline{h}_i(F)$. Applying this construction to particular pretheories F one can get algebraic cycle homology and bivariant morphic homology of Friedlander and Gabber (see [1]) or algebraic singular homology and algebraic Lawson homology defined in [7].

In the first section we prove some elementary facts about divisors with proper support on relative curves (see also [7]). In Section 3 we define pretheories and prove some of their basic properties.

In the third section we study local behavior of homotopy invariant pretheories. In most cases we assume that our pretheory takes values in the category of abelian groups and study the sheaf F_{Zar} in the Zariski topology on Sm/k associated with the presheaf F . The main theorem (Theorem 4.27) of this section states that if F is a homotopy invariant pretheory over a perfect field k with values in the category of abelian groups then the functors $U \rightarrow H_{Zar}^i(U, F_{Zar})$ have canonical structures of homotopy invariant pretheories.

As an application we show in §4.6 that for any such pretheory F the sheaf F_{Zar} has the Gersten resolution.

In the next section we consider the sheaves in the Nisnevich topology (see [5]) associated with pretheories with values in the category of abelian groups. It turns out that the Nisnevich topology has several serious advantages over the Zariski topology in the context of pretheories. In particular for any reasonable (see Definition 5.1), not necessarily homotopy invariant, pretheory F with values in the category of abelian groups the functors $U \rightarrow H_{Nis}^i(U, F_{Nis})$ have canonical structures of pretheories (Theorem 5.3). On the other hand, using the results of Section 4 we show that for any homotopy invariant pretheory F over a perfect field with values in the category of abelian groups one

has canonical isomorphisms

$$H_{Nis}^i(-, F_{Nis}) \cong H_{Zar}^i(-, F_{Zar})$$

for all $i \geq 0$ (Theorem 5.7). It allows us to reformulate all the results of the previous section for the Nisnevich topology. In the last paragraph we apply our technique to prove the existence of a long exact sequence of Mayer-Vietoris type for algebraic singular homology (Theorem 5.17). Though apparently we are only interested in open coverings the use of the Nisnevich topology in the proof is crucial.

In paragraph 5.4 we consider the behavior of homotopy invariant pretheories with respect to blow-ups. Proposition 5.21 has important applications to algebraic cycle cohomology (see [2]).

In the last paragraph we consider the etale topology. The first result is the rigidity Theorem 5.25 (which is just a version of the rigidity theorem proven in [7]). It says that the sheaf in the etale topology on Sm/k associated with a homotopy invariant pretheory with values in the category of torsion abelian groups of torsion prime to characteristic of the field k is locally constant. For rational coefficients we get in the case of a perfect field k isomorphisms (Theorem 5.28):

$$H_{et}^i(-, F_{et}) \otimes \mathbf{Q} \cong H_{Zar}^i(-, F_{Zar}) \otimes \mathbf{Q}.$$

Most results of this paper are formulated for smooth schemes over a perfect field. This restriction on the base scheme is important. In fact almost all the results of this paper are wrong even for smooth schemes over the spectrum of a discrete valuation ring of characteristic zero.

The idea to consider presheaves with transfer (instead of qfh-sheaves) which allows us to obtain the “correct” answers with integral coefficients is due completely to my conversations with A.Suslin. I am also very grateful to D. Kazhdan for his patient interest in my work and to Yu. Tschinkel who told me about the existence of the Nisnevich topology.

2 Divisors with compact support on relative curves.

Let S be a smooth scheme. A curve $p : X \rightarrow S$ over S is an equi-dimensional morphism of finite type of relative dimension one. The morphism $\emptyset \rightarrow S$ will also be considered as a curve over S .

For any curve $X \rightarrow S$ over a smooth scheme S denote by $c_{equi}(X/S, 0)$ the free abelian group generated by closed integral subschemes of X which are finite over S and surjective over an irreducible component of S .

Let Z be any closed subscheme of X which is finite over S . Consider the cycle $Cycl_X(Z)$ of X in Z . It follows immediately from our definition that it belongs to $c_{equi}(X/S, 0)$ if and only if any irreducible component of Z dominates an irreducible component of S . In particular $Cycl_X(Z)$ belongs to $c_{equi}(X/S, 0)$ for any closed subscheme Z in X which is finite and flat over S .

As was shown in [8] for any morphism of smooth schemes $f : S' \rightarrow S$ one can define a homomorphism

$$cycl(f) : c_{equi}(X/S, 0) \rightarrow c_{equi}(X \times_S S'/S', 0)$$

such that the following conditions hold:

1. For a composable pair of morphisms $S'' \xrightarrow{g} S' \xrightarrow{f} S$ one has

$$cycl(f \circ g) = cycl(g) \circ cycl(f)$$

2. For any dominant morphism $f : S' \rightarrow S$ of smooth schemes and an integral closed subscheme Z in $c_{equi}(X/S, 0)$ one has

$$cycl(f)(Z) = Cycl_{X \times_S S'}(Z \times_S S')$$

3. For any morphism of smooth schemes $f : S' \rightarrow S$ and a closed subscheme Z in X which is finite and flat over S one has:

$$cycl(f)(Cycl_X(Z)) = Cycl_{X \times_S S'}(Z \times_S S').$$

It can be shown easily (using Platication Theorem, see [6] or [8, Th. 2.2.2]) that homomorphisms $cycl(f)$ are uniquely determined by the conditions (1)-(3) given above.

Let $g : X_1 \rightarrow X_2$ be a morphism of curves over a smooth scheme S and Z be an integral closed subscheme in X_1 which belongs to $c_{equi}(X_1/S, 0)$. Since it is proper over S its image $g(Z)$ in X_2 is a closed subscheme which belongs to $c_{equi}(X_2/S, 0)$. The morphism $Z \rightarrow g(Z)$ is a finite surjective morphism

of integral schemes. Denote its degree (i.e. the degree of the corresponding function fields extension) by $n(Z, g)$.

We define the direct image homomorphism

$$g_* : c_{\text{equi}}(X_1/S, 0) \rightarrow c_{\text{equi}}(X_2/S, 0)$$

setting

$$g_*(Z) = n(Z, g)g(Z).$$

The following proposition is a particular case of [8, Prop. 3.6.2].

Proposition 2.1 *Let $X_1 \rightarrow S$, $X_2 \rightarrow S$ be a pair of curves over a smooth scheme S , $g : X_1 \rightarrow X_2$ be a morphism over S and $f : S' \rightarrow S$ be a morphism of smooth schemes. Then the following diagram commutes:*

$$\begin{array}{ccc} c_{\text{equi}}(X_1/S, 0) & \xrightarrow{\text{cycl}(f)} & c_{\text{equi}}(X_1 \times_S S'/S', 0) \\ g_* \downarrow & & \downarrow g_* \\ c_{\text{equi}}(X_2/S, 0) & \xrightarrow{\text{cycl}(f)} & c_{\text{equi}}(X_2 \times_S S'/S', 0). \end{array}$$

The following technique provides us with a method of construction of elements in the groups $c_{\text{equi}}(X/S, 0)$ which will be used extensively in this paper.

Let D be a Cartier divisor on X . Denote by $\mathcal{L}_D$ the corresponding line bundle on X . There is a canonical trivialization $t_D : \mathcal{O}_U \rightarrow (\mathcal{L}_D)_{|U}$ of $\mathcal{L}_D$ on $U = X - D$. If D is effective this trivialization extends to a canonical section $s_D : \mathcal{O}_X \rightarrow \mathcal{L}_D$ of $\mathcal{L}_D$ on X . Suppose that X is normal and connected. Then for any line bundle $\mathcal{L}$ on X , open subset U of X and a trivialization t of $\mathcal{L}$ on U there exists a unique Cartier divisor $D(\mathcal{L}, U, t)$ and an isomorphism $\mathcal{L} \cong \mathcal{L}_{D(\mathcal{L}, U, t)}$ such that $t = t_{\mathcal{L}_D}$ (see [4, 21.1.4]). Note that $D(\mathcal{L}, U, t)$ considered as a closed subset of X lies in $X - U$.

Let $\bar{p} : \bar{X} \rightarrow S$ be a proper normal curve over a smooth scheme S and $X \subset \bar{X}$ be an open subscheme in $\bar{X}$ which is quasi-affine and smooth over S . Denote by X_∞ the reduced scheme $\bar{X} - X$.

Proposition 2.2 *Let $\mathcal{L}$ be a line bundle on $\bar{X}$, U be an open neighborhood of X_∞ in $\bar{X}$ and $t : \mathcal{O}_U \rightarrow \mathcal{L}_{|U}$ be a trivialization of $\mathcal{L}$ on U . Consider the Cartier divisor $D(\mathcal{L}, U, t)$ defined by the triple $(\mathcal{L}, U, t)$. Then $\text{Cycl}_X(D(\mathcal{L}, U, t))$ belongs to $c_{\text{equi}}(X/S, 0)$.*

Proof: Let

$$\text{Cycl}_X(D(\mathcal{L}, U, t)) = \sum n_i Z_i$$

where Z_i are integral closed subschemes of X . Since $\bar{p} : \bar{X} \rightarrow S$ is proper and X is quasi-affine over S the open subset U is dense in all fibers of $\bar{p}$ and $\bar{X} - U$ is finite over S . Since $D(\mathcal{L}, U, t)$ as a closed subset in $\bar{X}$ belongs to $\bar{X} - U$ the closed subschemes Z_i are finite over S . On the other hand each of Z_i is of pure codimension one in X and being finite over S it must dominate an irreducible component of S .

Lemma 2.3 *Under the assumption of Proposition 2.2 consider a morphism of smooth schemes $f : S' \rightarrow S$. Then for any triple $(\mathcal{L}, U, t)$ as above we have:*

$$\text{cycl}(f)(\text{Cycl}_X(D(\mathcal{L}, U, t))) = \text{Cycl}_{X \times_S S'}(D(f_X^*(\mathcal{L}), f_X^{-1}(U), f_X^*(t)))$$

where f_X is the projection $X \times_S S' \rightarrow X$.

Proof: It follows from the fact that $D(\mathcal{L}, U, t)$ is flat over S by the third property of base change homomorphisms.

Let $\mathcal{Z}_0, \mathcal{Z}_1$ be two elements of $c_{\text{equi}}(X/S, 0)$. We say that they are equivalent if there is an element $\mathcal{Z}$ in $S \times \mathbf{A}^1$ such that

$$\text{cycl}(i_0)(\mathcal{Z}) = \mathcal{Z}_0$$

$$\text{cycl}(i_1)(\mathcal{Z}) = \mathcal{Z}_1$$

where $i_0, i_1 : S \rightarrow S \times \mathbf{A}^1$ are the closed embeddings which correspond to the points 0 and 1 of $\mathbf{A}^1$ respectively. We denote the group of equivalence classes of elements of $c_{\text{equi}}(X/S, 0)$ with respect to this equivalence relation by $h_0(X/S)$. It follows immediately from the definition that homomorphisms $\text{cycl}(f)$ give us homomorphisms $h_0(X/S) \rightarrow h_0(X \times_S S'/S')$ and by Proposition 2.1 the groups $h_0(X/S)$ are covariantly functorial with respect to morphisms $X_1 \rightarrow X_2$ of curves over S .

Definition 2.4 *Let $p : X \rightarrow S$ be a curve over a smooth scheme S . A good compactification of X is a pair $(\bar{p} : \bar{X} \rightarrow S, j : X \rightarrow \bar{X})$ where $\bar{X}$ is a proper curve over S and j is an open embedding over S such that the closed subset $X_\infty = \bar{X} - X$ in $\bar{X}$ has an open neighborhood which is affine over S .*

Note that if $(\bar{p} : \bar{X} \rightarrow S, j : X \rightarrow \bar{X})$ is a good compactification of a curve $p : X \rightarrow S$ and $f : S' \rightarrow S$ is a morphism of smooth schemes then $(\bar{p}' : \bar{X} \times_S S' \rightarrow S', j \times_S Id_{S'} : X \times_S S' \rightarrow \bar{X} \times_S S')$ is a good compactification of the curve $p' : X \times_S S' \rightarrow S'$.

The following lemma shows that if a normal curve over a smooth scheme has a good compactification then we may always choose it in such a way that the corresponding proper curve is also normal.

Lemma 2.5 *Let $p : X \rightarrow S$ be a normal curve over a smooth scheme S and $(\bar{p} : \bar{X} \rightarrow S, j : X \rightarrow \bar{X})$ be a good compactification of X . Let further $\bar{X}'$ be the normalization of $\bar{X}$. Then the open embedding $j : X \rightarrow \bar{X}$ defines an open embedding $j' : X \rightarrow \bar{X}'$ and the pair $(\bar{p}' : \bar{X}' \rightarrow S, j' : X \rightarrow \bar{X}')$ is a good compactification of X .*

Proof: Since the morphism $\bar{X}' \rightarrow \bar{X}$ is finite it is also affine which implies that the preimage of the affine open neighborhood of X_∞ in $\bar{X}'$ is an affine open neighborhood of $X'_\infty = \bar{X}' - X$ in $\bar{X}'$.

Proposition 2.6 *Let $p : X \rightarrow S$ be a smooth quasi-affine curve over a smooth scheme S and $(\bar{p} : \bar{X} \rightarrow S, j : X \rightarrow \bar{X})$ be a good compactification of X . Let X_∞ be the reduced scheme $\bar{X} - X$.*

Let further $\mathcal{L}$ be a line bundle on $\bar{X}$ and $s : \mathcal{O}_{X_\infty} \rightarrow \mathcal{L}_{|X_\infty}$ be a trivialization of $\mathcal{L}$ over X_∞ . Then the following statements hold.

1. *For any two extensions $\tilde{s}_1, \tilde{s}_2$ of the trivialization s to an open neighborhood U of X_∞ the cycles $Cycl_X(D(\mathcal{L}, U, \tilde{s}_1))$, $Cycl_X(D(\mathcal{L}, U, \tilde{s}_2))$ give the same element in $h_0(X/S)$.*
2. *If S is affine there exists an open neighborhood U of X_∞ in $\bar{X}$ and an extension $\tilde{s} : \mathcal{O}_U \rightarrow \mathcal{L}_{|U}$ of the trivialization s to a trivialization of $\mathcal{L}$ on U .*

Proof: The second statement follows trivially from the fact that X_∞ has an affine open neighborhood in $\bar{X}$.

To prove the first one consider the curve

$$\bar{X} \times \mathbf{A}^1 \rightarrow S \times \mathbf{A}^1$$

over $S \times \mathbf{A}^1$ and let $\mathcal{L}'$ be the pull-back of $\mathcal{L}$ with respect to the projection $\bar{X} \times \mathbf{A}^1 \rightarrow \bar{X}$. The trivializations $\tilde{s}_1, \tilde{s}_2$ define trivializations $\tilde{s}'_1, \tilde{s}'_2$ of $\mathcal{L}'$ on

$U \times \mathbf{A}^1$ which coincide on $X_\infty \times \mathbf{A}^1$. Consider the section $t\tilde{s}_1' + (1-t)\tilde{s}_2'$ of $\mathcal{L}'$ on $U \times \mathbf{A}^1$ where t is the regular function given by the projection $U \times \mathbf{A}^1 \rightarrow \mathbf{A}^1$. Its restriction to $X_\infty \times \mathbf{A}^1$ is equal to the trivialization induced by s and hence it defines a trivialization h of $\mathcal{L}'$ on an open neighborhood V of $X_\infty \times \mathbf{A}^1$ in $\bar{X} \times \mathbf{A}^1$. The divisor $D(\mathcal{L}', V, h)$ associated with h gives us by Proposition 2.2 an element $\mathcal{Z}$ in $c_{equi}(X \times \mathbf{A}^1/S \times \mathbf{A}^1, 0)$. Consider the closed embeddings $i_0, i_1 : S \rightarrow S \times \mathbf{A}^1$ which correspond to the points 0 and 1 on $\mathbf{A}^1$ respectively. Lemma 2.3 implies that

$$cycl(i_0)(\mathcal{Z}) = Cycl_X(D(\mathcal{L}, U, \tilde{s}_2))$$

and

$$cycl(i_1)(\mathcal{Z}) = Cycl_X(D(\mathcal{L}, U, \tilde{s}_1)).$$

Let $p : X \rightarrow S$ be a smooth quasi-affine curve over an affine smooth scheme S and $(\bar{p} : \bar{X} \rightarrow S, j : X \rightarrow \bar{X})$ be a good compactification of X . Denote by $Pic(\bar{X}, X_\infty)$ the set of isomorphism classes of pairs of the form $(\mathcal{L}, s)$ where $\mathcal{L}$ is a line bundle on $\bar{X}$ and s is a trivialization of $\mathcal{L}$ on X_∞ . Tensor multiplication of line bundles and their sections gives a structure of abelian group on $Pic(\bar{X}, X_\infty)$. Proposition 2.6 implies that there is a well defined homomorphism

$$Pic(\bar{X}, X_\infty) \rightarrow h_0(X/S).$$

Lemma 2.3 implies immediately that this homomorphism commutes in the obvious sense with base change homomorphisms $cycl(-)$.

Note that if we consider pairs of the form $(\mathcal{O}_{\bar{X}}, s)$ we get a homomorphism

$$\mathcal{O}^*(X_\infty) \rightarrow h_0(X/S).$$

Remark 2.7 It can be shown (see [7]) that for a smooth quasi-affine curve $X \rightarrow S$ over a smooth affine scheme S and a good compactification of X the homomorphism $Pic(\bar{X}, X_\infty) \rightarrow h_0(X/S)$ is in fact an isomorphism. We will not use this fact in the present paper.

3 Pretheories.

3.1 Categories of pretheories.

Let k be a field. We denote by Sm/k the category of smooth schemes of finite type over k .

Definition 3.1 *A pretheory (F, ϕ, A) over a field k is the following collection of data.*

1. *An additive category A .*
2. *A contravariant functor $F : Sm/k \rightarrow A$.*
3. *For any object S of Sm/k and any smooth curve $p : X \rightarrow S$ over S a homomorphism of abelian groups*

$$\phi_{X/S} : c_{equi}(X/S, 0) \rightarrow Hom(F(X), F(S)).$$

These data should satisfy the following conditions.

1. *For any object S of Sm/k , any smooth curve $p : X \rightarrow S$ over S and any S -point $i : S \rightarrow X$ of X one has $\phi_{X/S}(i(S)) = F(i)$.*
2. *Let $f : S_1 \rightarrow S_2$ be a morphism of smooth schemes of finite type over k and $p : X_2 \rightarrow S_2$ be a smooth curve over S_2 . Consider the Cartesian square*

$$\begin{array}{ccc} X_1 & \xrightarrow{g} & X_1 \\ \downarrow & & \downarrow \\ S_1 & \xrightarrow{f} & S_2. \end{array}$$

Then for any $\mathcal{Z}$ in $c_{equi}(X_2/S_2, 0)$ one has

$$F(f) \circ \phi_{X_2/S_2}(\mathcal{Z}) = \phi_{X_1/S_1}(f^*(\mathcal{Z})) \circ F(g).$$

3. *For any pair X, Y of objects of Sm/k the canonical morphism*

$$F(X \amalg Y) \rightarrow F(X) \oplus F(Y)$$

is an isomorphism.

Definition 3.2 Let $(F_1, \phi_1, A), (F_2, \phi_2, A)$ be a pair of pretheories with values in an additive category A . A morphism of pretheories

$$T : (F_1, \phi_1, A) \rightarrow (F_2, \phi_2, A)$$

is a morphism $T : F_1 \rightarrow F_2$ of presheaves on the category Sm/k such that for any smooth scheme S over k , any smooth curve $p : X \rightarrow S$ over S and any element $\mathcal{Z} \in c_{equi}(X/S, 0)$ the following diagram commutes:

$$\begin{array}{ccc} F_1(X) & \xrightarrow{\phi_1(\mathcal{Z})} & F_1(S) \\ T_X \downarrow & & \downarrow T_S \\ F_2(X) & \xrightarrow{\phi_2(\mathcal{Z})} & F_2(S). \end{array}$$

Denote by $Preth(k, A)$ the category of pretheories over k with values in an additive category A . It is obviously an additive category. Moreover if A is an abelian category one has the following obvious proposition.

Proposition 3.3 Let k be a field and A be an abelian category. Then the category $Preth(k, A)$ is abelian and the forgetfull functor

$$Preth(k, A) \rightarrow Preshv(Sm/k, A)$$

to the category of A -valued presheaves on Sm/k is exact.

Let (F, ϕ, A) be a pretheory over a field k and U be an object of Sm/k . We define a pretheory (F_U, ϕ_U, A) as follows:

1. The presheaf F_U takes a smooth scheme U of finite type over k to the object $F(U \times_k X)$.
2. Let $p : X \rightarrow S$ be a smooth curve over a smooth scheme S of finite type over k and $\mathcal{Z}$ be an element of $c_{equi}(X/S, 0)$. Then we define $(\phi_U)_{X/S}(\mathcal{Z})$ to be the homomorphism $\phi(p^*(\mathcal{Z}))$ where p is the projection $S \times U \rightarrow S$.

One can easily see that (F_U, ϕ_U, A) is indeed a pretheory and this construction gives us a functor

$$(Sm/k)^{op} \times Preth(k, A) \rightarrow Preth(k, A).$$

For any pretheory (F, ϕ, A) with values in an abelian category A we denote by (F_{-1}, ϕ_{-1}, A) the pretheory

$$\text{coker}((F_{\mathbf{A}^1}, \phi_{\mathbf{A}^1}, A) \rightarrow (F_{\mathbf{A}^1 - \{0\}}, \phi_{\mathbf{A}^1 - \{0\}}, A)).$$

Consider a separable field extension $k \subset L$ and let $f : \text{Spec}(L) \rightarrow \text{Spec}(k)$ be the corresponding morphism of schemes. Then for any pretheory (F, ϕ, Ab) over k where Ab is the category of abelian groups the inverse image $p^*(F)$ of the presheaf F has a canonical structure of a pretheory over L . It gives us an exact functor $p^* : \text{Preth}(k, Ab) \rightarrow \text{Preth}(L, Ab)$. If U is a smooth scheme over k then one has a canonical isomorphism:

$$p^*((F_U, \phi_U, Ab)) \cong p^*((F, \phi, Ab))_{U_L}.$$

In particular there is a canonical isomorphism

$$p^*((F, \phi, Ab)_{-1}) = p^*((F, \phi, Ab))_{-1}.$$

3.2 Homotopy invariant pretheories.

Definition 3.4 *Let k be a field and (F, ϕ, A) be a pretheory over k . It is called homotopy invariant if for any object X of Sm/k the projection $X \times \mathbf{A}^1 \rightarrow X$ induces an isomorphism $F(X) \rightarrow F(X \times \mathbf{A}^1)$.*

For any field k and an additive category A denote by $\text{HIPreth}(k, A)$ the full subcategory in the category $\text{Preth}(k, A)$ which consists of homotopy invariant pretheories. In the following proposition we summarize basic properties of homotopy invariant pretheories which follow directly from our definitions.

Proposition 3.5 *Let k be a field and A be an abelian category. Then the category $\text{HIPreth}(k, A)$ is abelian and the inclusion functor $\text{HIPreth}(k, A) \rightarrow \text{Preth}(k, A)$ is exact. Moreover any subpretheory of a homotopy invariant pretheory is a homotopy invariant pretheory and any extension of homotopy invariant pretheories is a homotopy invariant pretheory.*

Let F be a presheaf on the category Sm/k with values in an abelian category. Let $\Delta^\bullet = (\Delta^n, \partial_i^n, s_i^n)$ be the standard cosimplicial object in Sm/k (see [2, §4]). Denote by $\underline{\mathcal{C}}_*(F)$ the complex of presheaves such that $\underline{\mathcal{C}}_n(F)(U) = F(U \times \Delta^n)$ and the differential is given by the alternated sums

of morphisms induced by ∂_i^n . Denote by $\underline{h}_i(F)$ the cohomology presheaves $H^{-i}(\underline{C}_*(F))$ of this complex.

If (F, ϕ, A) is a pretheory with values in an abelian category A then $\underline{C}_*(F)$ is a complex of pretheories (or equivalently a pretheory with values in the category of complexes over A) and in particular the presheaves $\underline{h}_i(F)$ have canonical structures of pretheories.

Proposition 3.6 *For any pretheory (F, ϕ, A) over k where A is an abelian category the pretheories $\underline{h}_i(F, \phi, A)$ are homotopy invariant.*

Proof: Since a pretheory is homotopy invariant if and only if the underlying presheaf is it is sufficient to show that for any presheaf of abelian groups F on Sm/k the presheaves $\underline{h}_i(F)$ are homotopy invariant. Let U be a smooth scheme over k . We have to show that the morphisms $\underline{h}_i(F)(U) \rightarrow \underline{h}_i(F)(U \times \mathbf{A}^1)$ induced by the projection $U \times \mathbf{A}^1 \rightarrow U$ are isomorphisms.

Denote by $i_0, i_1 : U \rightarrow U \times \mathbf{A}^1$ the closed embeddings $Id_U \times \{0\}$ and $Id_U \times \{1\}$ respectively. Let us show first that the morphisms

$$i_0^* : \underline{h}_i(F)(U \times \mathbf{A}^1) \rightarrow \underline{h}_i(F)(U)$$

$$i_1^* : \underline{h}_i(F)(U \times \mathbf{A}^1) \rightarrow \underline{h}_i(F)(U)$$

coincide. It is sufficient to prove that the corresponding morphisms of complexes of abelian groups $\underline{C}_*(F)(U \times \mathbf{A}^1) \rightarrow \underline{C}_*(F)(U)$ are homotopic.

Define homomorphisms

$$s_n : F(U \times \mathbf{A}^1 \times \Delta^n) \rightarrow F(U \times \Delta^{n+1})$$

by the formula

$$s_n = \sum_{i=0}^n (-1)^i (Id_U \times \psi_i)^*$$

where $\psi_i : \Delta^{n+1} \rightarrow \Delta^n \times \mathbf{A}^1$ is the linear isomorphism taking v_j to $v_j \times 0$ if $j \leq i$ or to $v_{j-1} \times 1$ if $j > i$ (here $v_j = (0, \dots, 1, \dots, 0)$ is the j -th vertex of Δ^{n+1} (resp. Δ^n)). A straightforward computation shows that $sd + ds = i_1^* - i_0^*$.

Consider now the morphism

$$Id_U \times \mu : U \times \mathbf{A}^1 \times \mathbf{A}^1 \rightarrow U \times \mathbf{A}^1$$

where $\mu : \mathbf{A}^1 \times \mathbf{A}^1 \rightarrow \mathbf{A}^1$ is given by multiplication of functions. Applying the previous result to the embeddings

$$i_0, i_1 : U \times \mathbf{A}^1 \rightarrow (U \times \mathbf{A}^1) \times \mathbf{A}^1$$

we conclude that the homomorphism

$$\underline{h}_i(F)(U \times \mathbf{A}^1) \rightarrow \underline{h}_i(F)(U \times \mathbf{A}^1)$$

induced by the composition

$$U \times \mathbf{A}^1 \xrightarrow{pr_1} U \xrightarrow{i_0} U \times \mathbf{A}^1$$

is the identity homomorphism which implies immediately the assertion of the proposition.

Note that we have a canonical morphism:

$$(F, \phi, A) \rightarrow \underline{h}_0(F, \phi, A).$$

The following proposition follows easily from the definition of $\underline{h}_0(-)$.

Proposition 3.7 *For any pretheory (F, ϕ, A) and any homotopy invariant pretheory (G, ψ, A) the homomorphism*

$$Hom(\underline{h}_0(F, \phi, A), (G, \psi, A)) \rightarrow Hom((F, \phi, A), (G, \psi, A))$$

induced by the projection

$$(F, \phi, A) \rightarrow \underline{h}_0(F, \phi, A).$$

is an isomorphism.

Thus $\underline{h}_0(-)$ is a functor $Preth(k, A) \rightarrow HIPreth(k, A)$ which is left adjoint to the inclusion functor. In particular it is right exact. The functors $\underline{h}_i(-)$ are in some sense the left derived functors of $\underline{h}_0(-)$. More precisely one has:

Proposition 3.8 *Let*

$$0 \rightarrow (F_1, \phi_1, A) \rightarrow (F_2, \phi_2, A) \rightarrow (F_3, \phi_3, A) \rightarrow 0$$

be a short exact sequence of pretheories with values in an abelian category A . Then there is a canonical long exact sequence of homotopy invariant pretheories of the form:

$$\dots \rightarrow \underline{h}_i(F_1, \phi_1, A) \rightarrow \underline{h}_i(F_2, \phi_2, A) \rightarrow \underline{h}_i(F_3, \phi_3, A) \rightarrow \underline{h}_{i-1}(F_1, \phi_1, A) \rightarrow \dots$$

Proof: It follows immediately from the obvious fact that $F \rightarrow \underline{C}_*(F)$ is an exact functor from the category of presheaves to the category of complexes of presheaves.

Unfortunately one can not apply usual techniques from homological algebra to define $\underline{h}_i(-)$ as the left derived functors of $\underline{h}_0$ because even if A is the category of abelian groups there is not enough pretheories (F, ϕ, A) such that $\underline{h}_i(F, \phi, A) = 0$ for all $i > 0$.

We will often use below the following simple results.

Proposition 3.9 *Let (F, ϕ, A) be a pretheory such that A is an abelian category and U be a smooth scheme of finite type over k . Then there are canonical isomorphisms of pretheories*

$$(\underline{h}_i(F, \phi, A))_U \cong \underline{h}_i((F, \phi, A)_U)$$

in particular if (F, ϕ, A) is a homotopy invariant pretheory then $(F, \phi, A)_U$ is also homotopy invariant.

Proposition 3.10 *Let $k \subset L$ be a separable extension of fields and $f : \text{Spec}(L) \rightarrow \text{Spec}(k)$ be the corresponding morphism. Then the inverse image functor $f^* : \text{Preth}(k, \text{Ab}) \rightarrow \text{Preth}(L, \text{Ab})$ takes homotopy invariant pretheories to homotopy invariant pretheories and commutes with the corresponding functors $\underline{h}_i$.*

Proposition 3.11 *Let (F, ϕ, A) be a pretheory over k . It is homotopy invariant if and only if for any smooth scheme S of finite type over k and any smooth curve $X \rightarrow S$ over S the morphism $\phi_{X/S}$ can be factored through the canonical epimorphism $c_{\text{equi}}(X/S, 0) \rightarrow h_0(X/S)$ (see the definition of $h_0(-)$ after Lemma 2.3).*

Proof: The “only if” part is obvious. Suppose that (F, ϕ, A) is a pretheory such that the morphisms $\phi_{X/S}$ can be factored through $h_0(X/S)$. Let us show that it is homotopy invariant.

Let $i : X \rightarrow X \times \mathbf{A}^1$ be the embedding which corresponds to the point $\{0\}$ of $\mathbf{A}^1$. It is obviously sufficient to show that the following composition is equal to the identity morphism:

$$F(X \times \mathbf{A}^1) \xrightarrow{F(i)} F(X) \xrightarrow{F(pr_1)} F(X \times \mathbf{A}^1).$$

Consider the curve $(X \times \mathbf{A}^1) \times \mathbf{A}^1$ over $X \times \mathbf{A}^1$ and let

$$\mathcal{Z} \in h_0((X \times \mathbf{A}^1) \times \mathbf{A}^1 / X \times \mathbf{A}^1)$$

be the element which corresponds to the divisor $(X \times \mathbf{A}^1) \times \{0\}$. Then by the properties (1) and (2) of pretheories our composition is equal to the composition:

$$F(X \times \mathbf{A}^1) \xrightarrow{F(pr_1 \times Id_{\mathbf{A}^1})} F((X \times \mathbf{A}^1) \times \mathbf{A}^1) \xrightarrow{\phi(\mathcal{Z})} F(X \times \mathbf{A}^1).$$

The element $\mathcal{Z}$ coincides in $h_0((X \times \mathbf{A}^1) \times \mathbf{A}^1 / X \times \mathbf{A}^1)$ with the element represented by the graph of the morphism

$$Id_X \times \Delta : X \times \mathbf{A}^1 \rightarrow X \times \mathbf{A}^1 \times \mathbf{A}^1$$

and hence by the property (1) of pretheories we have $\phi(\mathcal{Z}) = F(Id_X \times \Delta)$. Therefore our composition is equal to

$$F((pr_1 \times Id_{\mathbf{A}^1}) \circ (Id_X \times \Delta)) = F(Id_{X \times \mathbf{A}^1}) = Id_{F(X \times \mathbf{A}^1)}.$$

Proposition is proven.

3.3 Elementary properties of transfer maps.

Proposition 3.12 *Let k be a field and (F, ϕ, A) be a homotopy invariant pretheory over k . Let S be an object of Sm/k , $j : U \rightarrow X$ be an open embedding of smooth curves over S and a be an element of $c_{equi}(U/S, 0)$. Then one has*

$$\phi_{X/S}(j_*(a)) = \phi_{U/S}(a) \circ F(j).$$

Proof: Let Z be the closed subset $X - U$ of X . Consider the smooth curve $W = X \times \mathbf{A}^1 - Z \times \{0\}$ over $S \times \mathbf{A}^1$. Let $p : S \times \mathbf{A}^1 \rightarrow S$ be the canonical projection. Then there exists a unique element $\tilde{a}$ in $c_{equi}(W/S \times \mathbf{A}^1, 0)$ such that its image in $c_{equi}(X \times \mathbf{A}^1/S \times \mathbf{A}^1, 0)$ equals $cycl(p)(j_*(a))$. Let $i_0, i_1 : S \rightarrow S \times \mathbf{A}^1$ be the morphisms which correspond to the points $\{0\}$ and $\{1\}$ of $\mathbf{A}^1$ respectively. Then the fiber of W over i_0 is canonically isomorphic to U and the fiber of W over i_1 is canonically isomorphic to X and with respect to these isomorphisms we have

$$\begin{aligned} cycl(i_0)(\tilde{a}) &= a \\ cycl(i_1)(\tilde{a}) &= j_*(a). \end{aligned}$$

Let $q : W \rightarrow X$ be the obvious projection. Consider the following diagram:

$$\begin{array}{ccccc} & & F(X) & \xrightarrow{\phi_{X/S}(j_*(a))} & F(S) \\ & Id \nearrow & \uparrow & & \uparrow F(i_1) \\ F(X) & \xrightarrow{F(q)} & F(W) & \xrightarrow{\phi_{W/S \times \mathbf{A}^1}(\tilde{a})} & F(S \times \mathbf{A}^1) \\ F(j) & \searrow & \downarrow & & \downarrow F(i_0) \\ & & F(U) & \xrightarrow{\phi_{U/S}(a)} & F(S) \end{array}$$

It is commutative by the property (2) of pretheories. Since F is homotopy invariant $F(i_0), F(i_1)$ are isomorphisms and we have

$$\phi_{U/S}(a) \circ F(j) = \phi_{X/S}(j_*(a)).$$

The proposition is proven.

Let $k \subset E_1 \subset E_2$ be finite separable extensions of k and $f : Spec(E_2) \rightarrow Spec(E_1)$ be the corresponding morphism of smooth schemes over k . For any pretheory (F, ϕ, A) over k we define a morphism

$$F_*(f) : F(Spec(E_2)) \rightarrow F(Spec(E_1))$$

as follows. Consider the smooth curve $\mathbf{A}_{E_2}^1$ over $Spec(E_1)$. The point $\{0\}$ of $\mathbf{A}^1$ gives as an element in $c_{equi}(\mathbf{A}_{E_2}^1/Spec(E_1), 0)$ and we set $F_*(f)$ to be the composition

$$F(Spec(E_2)) \xrightarrow{F(p)} F(\mathbf{A}_{E_2}^1) \xrightarrow{\phi_{\mathbf{A}_{E_2}^1/Spec(E_1)}(\{0\})} F(Spec(E_1))$$

where $p : \mathbf{A}_{E_2}^1 \rightarrow Spec(E_2)$ is the canonical projection.

Proposition 3.13 *Let C be a smooth curve over k and x be a closed point of C . Suppose that the residue field k_x of x is separable over k and consider the following diagram of smooth schemes over k*

$$\begin{array}{ccc} \text{Spec}(k_x) & \xrightarrow{i} & C \\ p_0 \searrow & & \downarrow p \\ & & \text{Spec}(k) \end{array}$$

Then for any homotopy invariant pretheory (F, ϕ, A) over k the following diagram commutes

$$\begin{array}{ccccc} & F(C) & \xrightarrow{F(i)} & F(\text{Spec}(k_x)) & \\ \phi_{C/\text{Spec}(k)}(x) \downarrow & \downarrow & \swarrow & F_*(p_0) & \\ & F(\text{Spec}(k)) & & & \end{array}$$

Proof: Since k_x is separable over k the embedding $\text{Spec}(k_x) \rightarrow C$ is a smooth pair over k and hence there exists an open neighborhood U of x in C and an étale morphism $f : U \rightarrow \mathbf{A}_k^1$ such that $f^{-1}(\{0\}) = x$. By Proposition 3.12 we may suppose that $U = C$.

Consider a Cartesian square of the form

$$\begin{array}{ccccc} & & X & & \\ \tilde{f} \swarrow & & & \searrow & pr \\ \mathbf{A}_k^2 & & & & C \\ g \searrow & & & \swarrow & f \\ & & \mathbf{A}_k^1 & & \end{array}$$

where g is the morphism given by multiplication of functions (i.e $g(u, v) = uv$).

The composition

$$X \xrightarrow{\tilde{f}} \mathbf{A}_k^2 \xrightarrow{pr_1} \mathbf{A}_k^1$$

defines a smooth curve over $\mathbf{A}_k^1$ (here pr_1 is the projection $(u, v) \mapsto u$). Consider the reduced closed subscheme $Z = \tilde{f}^{-1}(\{v = 0\})$ in X . One can easily see that it is integral and belongs to $c_{\text{equi}}(X/\mathbf{A}_k^1, 0)$. The fibers of $X/\mathbf{A}_k^1$ over the points $\{0\}$ and $\{1\}$ of $\mathbf{A}_k^1$ are isomorphic to $\mathbf{A}_{k_x}^1$ and C respectively and the intersections of these fibers with Z are the points $\{0\}$ (on $\mathbf{A}_{k_x}^1$) and x (on C).

Let a be an element of $F(C)$. Consider the element $\phi_{X/\mathbf{A}_k^1}(Z)(F(pr)(a))$. By the property (2) of pretheories its restriction to the fiber over $\{0\}$ is equal to $F_*(p_0)(F(i)(a))$ and its restriction to the fiber over $\{1\}$ is equal to $\phi_{S/Spec(k)}(x)(a)$. Therefore these two elements coincide since F is homotopy invariant.

Proposition 3.14 *Let $k \subset E_1 \subset E_2$ be finite separable extensions of a field k and $d = [E_2 : E_1]$. Consider the diagram*

$$\begin{array}{ccc} Spec(E_2) & \xrightarrow{f} & Spec(E_1) \\ p_2 & \searrow & \downarrow p_1 \\ & & Spec(k) \end{array}$$

Then for any homotopy invariant pretheory (F, ϕ, A) over k one has

$$F_*(p_2) \circ F(f) = d F_*(p_1).$$

Proof: Consider the smooth curve $\mathbf{A}_{E_1}^1$ over $Spec(k)$. Since E_2 is separable over k it has a primitive element and hence there exists a closed point x on $\mathbf{A}_{E_1}^1$ such that the field of functions of x is isomorphic to E_2 . Let us identify $Spec(E_1)$ with the point $\{0\}$ of $\mathbf{A}_{E_1}^1$. Then we have two morphisms $f_1, f_0 : Spec(E_2) \rightarrow \mathbf{A}_{E_1}^1$ where f_0 corresponds to the morphism f with respect to our identification and f_1 corresponds to the point x . Since the compositions of f_1 and f_0 with the canonical projection $\mathbf{A}_{E_1}^1 \rightarrow Spec(E_1)$ coincide these two morphisms are homotopic over $Spec(k)$ and hence $F(f_0) = F(f_1)$ since F is homotopy invariant.

On the other hand as an element in $h_0(\mathbf{A}_{E_1}^1/Spec(k))$ the point x is equal to $d\{0\}$. Since F is homotopy invariant we conclude by Proposition 3.13 that

$$\begin{aligned} dF_*(p_1) &= d\phi_{\mathbf{A}_{E_1}^1/Spec(k)}(\{0\}) = \phi_{\mathbf{A}_{E_1}^1/Spec(k)}(x) = \\ &= F_*(p_2) \circ F(f_1) = F_*(p_2) \circ F(f). \end{aligned}$$

The proposition is proven.

Corollary 3.15 *Let $k \subset E_1 \subset E_2$ be finite separable extensions of a field k , $d = [E_2 : E_1]$ and $f : Spec(E_2) \rightarrow Spec(E_1)$ be the corresponding morphism. Then for any homotopy invariant pretheory (F, ϕ, A) over k one has*

$$F_*(f)F(f) = d Id_{F(Spec(E_1))}.$$

Lemma 3.16 *Let k be a field and C be a smooth curve over k . Then the group $h_0(C/\text{Spec}(k))$ is generated by closed points x of C whose residue fields k_x are separable over k .*

Proof: Let z be a closed point of C such that the field k_z is not separable over k . Let further $\bar{C}$ be a smooth compactification of C . Denote by $x_1, \dots, x_k$ the closed points $\bar{C} - C$. To show that the element of $h_0(C/\text{Spec}(k))$ which corresponds to z is a sum of elements which correspond to closed points of C with separable residue fields it is sufficient to find a rational function f on $\bar{C}$ such that the following conditions hold:

1. $f = 1$ at the points $x_1, \dots, x_k$.
2. The divisor $D(f)$ of f is of the form $\sum q_i - z - \sum p_j$ where q_i, p_j are closed points of C such that the fields k_{q_i}, k_{p_j} are separable extensions of k .

Note first that since C is smooth over k there is infinitely many closed points on C with separable residue fields. Therefore there exists a finite set $p_1, \dots, p_n$ of such points and a meromorphic function g on $\bar{C}$ such that $g(x_i) = 0$ for all $i = 1, \dots, k$ and g has poles in the points $z, p_1, \dots, p_n$ with multiplicities $[k_z : k], [k_{p_1} : k], \dots, [k_{p_k} : k]$ respectively.

Consider the function g as a morphism $\bar{C} \rightarrow \mathbf{P}_k^1$. It is smooth in the points $p_1, \dots, p_k$ and hence it is smooth everywhere but in a finite set of closed points $s_1, \dots, s_t$ (note that since the extension k_z/k is not separable and $g(z) = \infty$ is a point with separable residue field on $\mathbf{P}^1$ we have $z \in \{s_1, \dots, s_t\}$). We may assume that k is infinite (since finite fields are perfect). Then there is a k -point y on $\mathbf{P}_k^1$ such that $y \neq 0, \infty$ and g is smooth over y . We set $f = (y - g)/y$. One can easily see that this function satisfies the conditions (1)-(2) from above.

Proposition 3.17 *Let $f : X_1 \rightarrow X_2$ be a morphism of smooth curves over a field k and (F, ϕ, A) be a homotopy invariant pretheory over k . Then the following diagram commutes*

$$\begin{array}{ccc} c_{\text{equi}}(X_1/\text{Spec}(k), 0) & \xrightarrow{\phi_{X_1/\text{Spec}(k)}} & \text{Hom}(F(X_1), F(\text{Spec}(k))) \\ f_* \downarrow & & \downarrow \\ c_{\text{equi}}(X_2/\text{Spec}(k), 0) & \xrightarrow{\phi_{X_2/\text{Spec}(k)}} & \text{Hom}(F(X_2), F(\text{Spec}(k))). \end{array}$$

Proof: It follows immediately from Propositions 3.13, 3.14 and Lemma 3.16.

Definition 3.18 *Let k be a field and (F, ϕ, A) be a pretheory over k . It is said to be of homological type if for any smooth scheme S over k , any morphism $f : X_1 \rightarrow X_2$ of smooth curves over S and any element a in $c_{equi}(X_1/S, 0)$ one has*

$$\phi_{X_2/S}(f_*(a)) = \phi_{X_1/S}(a) \circ F(f).$$

3.4 Examples of pretheories.

Etale cohomology. For any field k consider the functor $\mathcal{H}_{et}(-, F)$ from the category Sm/k to the derived category of $Gal(\bar{k}/k)$ -modules which takes a smooth scheme X to the object $\mathbf{R}p_*(p^*(F))$ where F is a sheaf on the small etale site $Spec(k)_{et}$ and $p : X_{et} \rightarrow Spec(k)_{et}$ is the morphism of sites defined by the structural morphism of X . This functor has a canonical structure of a pretheory. If F is a sheaf of $\mathbf{Z}[1/\text{char}(k)]$ -modules this pretheory is homotopy invariant. A fortiori, for any $i \geq 0$ the functor from Sm/k to the category of abelian groups which takes X to the group $H_{et}^i(X, F)$ has a canonical structure of pretheory which is homotopy invariant if F is a sheaf of $\mathbf{Z}[1/\text{char}(k)]$ -modules.

If S is the shift functor on the derived category of $Gal(\bar{k}/k)$ -modules then for any locally constant n -torsion sheaf F such that n is prime to characteristic of k we have (by Kunnet formula for the etale cohomology) a canonical isomorphism

$$(\mathcal{H}_{et}(-, F))_{-1} \cong S^{-1} \circ \mathcal{H}_{et}(-, F \otimes \mu_n^{-1}).$$

In particular

$$(H_{et}^i(-, F))_{-1} \cong H_{et}^{i-1}(-, F \otimes \mu_n^{-1})$$

where μ_n is the sheaf of n -th roots of unit.

Algebraic K-theory. For any field k consider the functor K_i from Sm/k to the catgeory of abelian groups which takes a smooth scheme X to the Quillen's K-group $K_i(X)$. Since for any smooth scheme S over k and any smooth curve $p : X \rightarrow S$ over S an element of $c_{equi}(X/S, 0)$ is a formal linear combination of closed subschemes in X which are flat and finite over S one

can define transfer homomorphisms

$$\phi_{X/S} : c_{equi}(X/S, 0) \rightarrow Hom(K_i(X), K_i(S)).$$

Unfortunately, as was shown by Mark Walker these homomorphisms do not satisfy in general the second axiom of the definition of pretheory. It can be proven though that if we consider the separated in the Zariski topology presheaves $\tilde{K}_i(-)$ which correspond to $K_i(-)$ we get a pretheory which is clearly homotopy invariant.

Note also that we have canonical isomorphisms $(\tilde{K}_i)_{-1} \cong \tilde{K}_{i-1}$.

Algebraic De Ram cohomology. Let k be a field of characteristic zero. One can define a structure of pretheory on the functor which takes a smooth scheme $p : X \rightarrow Spec(k)$ over k to the object $\mathbf{R}p_*(\Omega^\bullet)$ of the derived category of k -vector spaces. In particular we get structures of pretheories on the functors $X \mapsto H_{DR}^k(X)$.

qfh-sheaves. Let F be a qfh-sheaf of abelian groups on the category Sch/k of schemes of finite type over a field k (see [10], [7]). Then one can easily see that the restriction of F to the category Sm/k has a canonical structure of a pretheory and the same holds for all functors of the form $X \mapsto \mathbf{H}_{qfh}^i(X, K)$ where K is a complex of sheaves in the qfh-topology.

4 Zariski sheaves associated with pretheories.

4.1 Technical lemmas.

Definition 4.1 *Let S be a smooth scheme. A standard triple*

$$(\bar{p} : \bar{X} \rightarrow S, X_\infty, Z)$$

over S is the following collection of data:

1. *a proper normal curve $\bar{p} : \bar{X} \rightarrow S$ over S ,*
2. *a pair of reduced closed subschemes Z, X_∞ in X .*

These data should satisfy the following conditions:

1. the closed subset $Z \cup X_\infty$ has an open neighborhood in $\bar{X}$ which is affine over S ,
2. $Z \cap X_\infty = \emptyset$,
3. the scheme $X = \bar{X} - X_\infty$ is quasi-affine and smooth over S .

One can easily see that if $(\bar{p} : \bar{X} \rightarrow S, X_\infty, Z)$ is a standard triple then for any closed reduced subscheme Z' of Z the triple $(\bar{p} : \bar{X} \rightarrow S, X_\infty, Z')$ is also a standard triple.

Lemma 4.2 *Let S be a smooth scheme and $(\bar{p} : \bar{X} \rightarrow S, X_\infty, Z)$ be a standard triple over S . Denote the scheme $\bar{X} - X_\infty$ by X and let $p : X \rightarrow S$ be the corresponding morphism.*

Let $f : X' \rightarrow X$ be an etale morphism. Denote by W_f the closed subspace in X which consists of points x such that f is not finite over x and let Z_0 be a closed subspace in Z . Suppose that the following conditions hold:

1. $W_f \subset Z$
2. the morphism $f^{-1}(Z_0) \rightarrow Z_0$ is finite.

Denote by $\bar{f} : \bar{X}' \rightarrow \bar{X}$ the finite morphism with normal $\bar{X}'$ associated with f . Then the triple

$$(\bar{p} \circ \bar{f} : \bar{X}' \rightarrow S, \bar{X}' - X', f^{-1}(Z_0))$$

is a standard triple over S .

Proof: Note first that since the morphism $f^{-1}(Z_0) \rightarrow Z_0 \rightarrow S$ is finite the subspace $f^{-1}(Z_0)$ is closed in $\bar{X}'$. On the other hand $f^{-1}(Z_0)$ lies in X' and hence $f^{-1}(Z_0) \cap (\bar{X}' - X') = \emptyset$.

The scheme $X' \rightarrow X \rightarrow S$ is smooth and quasi-affine over S since etale morphisms are smooth and quasi-affine. It is sufficient to verify that $f^{-1}(Z_0) \cup (\bar{X}' - X')$ has an affine open neighborhood in $\bar{X}'$. Since $W_f \subset Z$ the image $\bar{f}(f^{-1}(Z_0) \cup (\bar{X}' - X'))$ of this subscheme lies in $X_\infty \cup Z$ and hence has an affine open neighborhood since $\bar{f}$ is affine morphism.

Lemma 4.3 *Let $S = \text{Spec}(\mathcal{O})$ be a smooth affine semi-local scheme over a field k , $\bar{p} : \bar{X} \rightarrow S$ be a projective curve over S and Z be a closed subscheme in $\bar{X}$. Then Z has an affine open neighborhood in $\bar{X}$ if and only if the fibers of Z over the closed points of S are finite.*

Proof: The “only if” part is obvious. Suppose that the fibers of Z over the closed points of S are finite and let $z_1, \dots, z_n$ be the corresponding points of X . Since X is projective over S and S is affine there exists a quasi-projective scheme $\tilde{X}$ over k and an affine morphism $X \rightarrow \tilde{X}$. Our lemma follows now from the fact that any finite set of points in a quasi-projective scheme has an affine open neighborhood.

Let S be a smooth affine scheme and $T = (\bar{p} : \bar{X} \rightarrow S, X_\infty, Z)$ be a standard triple over S . Denote by $p : X \rightarrow S$ the scheme $\bar{X} - X_\infty$ over S . Consider the smooth curve $X \times_S X \rightarrow X$ over X . The diagonal $\Delta : X \rightarrow X \times_S X$ gives a relative Cartier divisor on $X \times_S X$ over X . Denote by $\mathcal{L}_\Delta$ be the corresponding line bundle.

Definition 4.4 *A splitting of T over an open subset U of X is a trivialization of the restriction of $\mathcal{L}_\Delta$ to $U \times_S Z$.*

Lemma 4.5 *Let k be a field, S be a smooth scheme over k , $T = ((\bar{p} : \bar{X} \rightarrow S, X_\infty, Z)$ a standard triple over S and $j : U \rightarrow X$ be an affine open subset in X such that T splits over U . Then for any homotopy invariant pretheory (F, ϕ, A) over k there is a morphism $F(X - Z) \rightarrow F(U)$ such that the composition $F(X) \rightarrow F(X - Z) \rightarrow F(U)$ coincides with the morphism $F(j) : F(X) \rightarrow F(U)$.*

Proof: Consider the line bundle $\mathcal{M}$ on $U \times_S \bar{X}$ which corresponds to the obvious U -point of $U \times_S \bar{X}$ and let s be its section which defines our divisor. Its restriction to $U \times_S X_\infty$ gives a trivialization $\tilde{s}$ of $\mathcal{M}$ there. For a splitting t of T over U let $\tilde{t}$ be the trivialization of $\mathcal{M}$ on $U \times_S (X_\infty \amalg Z)$ equal to $\tilde{s}$ on $U \times_S X_\infty$ and to t on $U \times_S Z$.

Suppose now that U is affine. Then by Proposition 2.6 the pair $(\mathcal{M}, \tilde{t})$ defines an element A_t in $h_0(U \times_S (X - Z)/U)$ such that the direct image of A_t to $h_0(U \times_S X)$ coincides with the class of tautological U -point of $U \times_S X$. In particular if (F, ϕ, A) is a homotopy invariant pretheory over k then any splitting t of T over an affine open subset U gives us a morphism $F(X - Z) \rightarrow F(U)$ and the composition $F(X) \rightarrow F(X - Z) \rightarrow F(U)$ coincides by Proposition 3.12 with the morphism $F(j)$.

Lemma 4.6 *Let k be a field, S be a smooth scheme over k , $T = ((\bar{p} : \bar{X} \rightarrow S, X_\infty, Z)$ be a standard triple over S and U be an affine open subset in X such that*

T splits over U . Suppose further that $Z = Z_1 \amalg Z_2$ where Z_1, Z_2 are reduced closed subschemes of X such that $Z_1 \cap Z_2 = \emptyset$.

Then for any homotopy invariant pretheory (F, ϕ, A) over k the canonical morphism of complexes

$$\begin{array}{ccccccc} 0 \rightarrow & F(X) & \rightarrow & F(X-Z_1) \oplus F(X-Z_2) & \rightarrow & F(X-Z) & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow & F(U) & \rightarrow & F((X-Z_1) \cap U) \oplus F((X-Z_2) \cap U) & \rightarrow & F((X-Z) \cap U) & \rightarrow 0 \end{array}$$

is homotopic to zero.

Proof: We have to define morphisms

$$\begin{aligned} f_i &: F(X - Z) \rightarrow F((X - Z_i) \cap U) \\ g_i &: F(X - Z_i) \rightarrow F(U) \end{aligned}$$

which satisfy the usual axioms of a homotopy. As in the proof of Lemma 4.5 we will first construct some elements

$$\begin{aligned} A_i &\in h_0(((X - Z_i) \cap U) \times_S (X - Z) / ((X - Z_i) \cap U)) \\ B_i &\in h_0(U \times_S (X - Z_i) / U) \end{aligned}$$

and then will use the transfer maps with respect to these elements to define the morphisms we need.

Consider the line bundle $\mathcal{M}$ on $U \times_S \bar{X}/U$ which corresponds to the obvious U -point of $U \times_S \bar{X}$ and let s be its section which defines our divisor. Its restriction to $U \times_S X_\infty$ gives a trivialization s_∞ of $\mathcal{M}$ there. By our assumption the restriction of $\mathcal{M}$ to the closed subscheme $U \times_S Z$ is trivial. Since U is affine there is an affine open neighborhood V of the subscheme $U \times_S (X_\infty \amalg Z)$ in $U \times_S \bar{X}$ and a trivialization t of $\mathcal{M}$ on V whose restriction to $U \times_S X_\infty$ is equal to s_∞ .

Suppose for simplicity of the notations that $i = 1$. Consider the smooth curve $((X - Z_1) \cap U) \times_S (X - Z)$ over $(X - Z_1) \cap U$. Consider now the restriction of $\mathcal{M}$ to the subscheme $((X - Z_1) \cap U) \times_S (X_\infty \amalg Z)$. Since the corresponding divisor on $((X - Z_1) \cap U) \times_S \bar{X}$ does not intersect $((X - Z_1) \cap U) \times_S (X_\infty \amalg Z_1)$ we have the canonical trivialization s_1 of $\mathcal{M}$ there. Consider now the trivialization r of $\mathcal{M}$ on $((X - Z_1) \cap U) \times_S (X_\infty \amalg Z)$ which is equal to s_1 on $((X - Z_1) \cap U) \times_S (X_\infty \amalg Z_1)$ and to the restriction of t on $((X - Z_1) \cap U) \times_S Z_2$.

We claim that r can be extended to an open neighborhood V_0 of $((X - Z_1) \cap U) \times_S (X_\infty \amalg Z)$ in $((X - Z_1) \cap U) \times_S \tilde{X}$ (of course the only problem is that if $\text{codim}(Z_1) \geq 1$ the scheme $(X - Z_1) \cap U$ is not affine). Since V is affine there exists a regular function h on V whose restriction to $U \times_S (X_\infty \amalg Z_2)$ is equal to one and whose restriction to $U \times_S Z_1$ is equal to s/t . Let Y be the divisor of h . Then the open subset $V_0 = ((X - Z_1) \cap U) \times_S \tilde{X} \cap (V - Y)$ is an open neighborhood of $((X - Z_1) \cap U) \times_S (X_\infty \amalg Z)$. Consider now the trivialization th of $\mathcal{M}$ on V_0 . It is obviously an extension of the form we need.

We define now A_1 to be the element which corresponds to r by Proposition 2.6(a). The definition of A_2 is similar.

Let us now construct the elements B_i . Since U is affine it is sufficient to define a trivialization of $\mathcal{M}$ on $U \times_S (X_\infty \amalg Z_i)$. We set it to be equal to the restriction of t .

Denote by C the element in $h_0(U \times_S (X - Z)/U)$ which corresponds to the trivialization t of $\mathcal{M}$.

We got morphisms:

$$\tilde{C} : F(X - Z) \rightarrow F(U)$$

$$\tilde{A}_i : F(X - Z) \rightarrow F((X - Z_i) \cap U)$$

$$\tilde{B}_i : F(X - Z_i) \rightarrow F(U).$$

Let us define the morphism g_2 to be equal to the morphism $\tilde{B}_2$, the morphism g_1 to be equal to zero, the morphism f_1 to be equal to $\tilde{A}_1$ and the morphism f_2 to be equal to the difference $\delta - \tilde{A}_2$ where δ is the composition

$$F(X - Z) \xrightarrow{\tilde{C}} F(U) \rightarrow F((X - Z_2) \cap U).$$

One can verify easily now using Proposition 3.12 that we indeed constructed a homotopy of our morphism to the zero morphism.

Proposition 4.7 *Let S be a smooth scheme over a field k and $T_i = (\bar{p}_i : \bar{X}_i \rightarrow S, X_{\infty,i}, Z_i)$, $i = 1, 2$ be two standard triples over S . Denote the schemes $\bar{X}_i - X_{\infty,i}$ by X_i .*

Let $\bar{f} : \bar{X}_1 \rightarrow \bar{X}_2$ be a finite morphism such that the following conditions hold.

1. $\bar{f}^{-1}(X_{\infty,2}) \subset X_{\infty,1}$.

2. Let $f : X_1 \rightarrow X_2$ be the morphism defined by $\bar{f}$ then $f^{-1}(Z_2) = Z_1$ and the morphism $f^{-1}(Z_2) \rightarrow Z_2$ is an isomorphism.

Let (F, ϕ, A) be a homotopy invariant pretheory over k of homological type such that A is an abelian category and V be an open affine subset of X_2 such that T_2 splits over V . Then there is a morphism

$$\psi : F(X_1 - Z_1) \rightarrow F(V - V \cap Z_2)$$

satisfying the following conditions.

1. The composition

$$F(X_1) \rightarrow F(X_1 - Z_1) \rightarrow F(V - V \cap Z_2)$$

is zero.

2. The following diagram is commutative:

$$\begin{array}{ccc} F(X_2 - Z_2) & & \\ \downarrow & \searrow & \\ F(X_1 - Z_1) & \rightarrow & F(V - V \cap Z_2)/F(V) \\ & \searrow & \downarrow \\ & & F(f^{-1}(V) - f^{-1}(V) \cap Z_1)/F(f^{-1}(V)) \end{array}$$

Proof: Let $\mathcal{M}$ be the line bundle on $X_2 \times_S \bar{X}_1$ which is the preimage of the diagonal line bundle on $X_2 \times_S \bar{X}_2$ with respect to the morphism $Id \times_S \bar{f}$ and let s be a section of $\mathcal{M}$ which defines the corresponding divisor $\tilde{X}_2 \rightarrow X_2 \times_S \bar{X}_1$ where $\tilde{X}_2 = \bar{f}^{-1}(X_2)$. Since the restriction of $\bar{f}$ to Z_1 is an isomorphism a splitting of T_2 over V gives us a trivialization t of $\mathcal{M}$ on $V \times_S Z_1$.

Let U be an affine open neighborhood of $V \times_S (Z_1 \amalg X_{1,\infty})$ in $V \times_S \bar{X}$. There exists a regular function h on U which is equal to 1 on $V \times_S X_{1,\infty}$ and to the regular function s/t on $V \times_S Z_1$. Let Y be the divisor of h . Its intersection with $V \times_S X_{1,\infty}$ is empty and its intersection with $V \times_S Z_1$ is $(V \cap Z_2) \times_S Z_1$. Hence the open subset $V_0 = ((V - V \cap Z_2) \times_S \bar{X}_1) \cap (U - Y)$ is a neighborhood of $(V - V \cap Z_2) \times_S (Z_1 \amalg X_{1,\infty})$ in $(V - V \cap Z_2) \times_S \bar{X}_1$ and h gives an invertible function on U_0 .

Let C be the corresponding element of $H_0((V - V \cap Z_2) \times_S (X_1 - Z_1)/(V - V \cap Z_2))$. We define our morphism ψ to be the composition

$$F(X_1 - Z_1) \xrightarrow{F(pr_2)} F((V - V \cap Z_2) \times_S (X_1 - Z_1)) \xrightarrow{\phi(C)} F(V - V \cap Z_2).$$

One can easily verify now that it satisfies the conditions of our proposition (we need the fact that F is of homological type to prove the commutativity of the upper triangle in our diagram).

Corollary 4.8 *In the notations of Proposition 4.7 suppose that A is the category of abelian groups. Then the following statements hold.*

1. *Let $a \in F(X_2 - Z_2)$ be an element such that its restriction to $F(X_1 - Z_1)$ can be extended to $F(X_1)$. Then for any closed point z of Z_2 there exists an open neighborhood V of z in X_2 such that the restriction of a to $V - V \cap Z_2$ can be extended to V .*
2. *Let a be an element in $F(X_1 - Z_1)$. Then for any closed point z of Z_2 there exists an open neighborhood V of z in X_2 and an element $a_0 \in F(V - V \cap Z_2)$ such that the images of a and a_0 in $F(f^{-1}(V) - f^{-1}(V) \cap Z_1)/F(f^{-1}(V))$ coincide.*

Proposition 4.9 *Let W be a smooth quasi-projective variety over a field k . Let $Y \subset W$ be a closed reduced subscheme in W such that $Y \neq W$ and $\{y_1, \dots, y_n\}$ be a finite set of closed points of Y . Then there exists an affine open neighborhood U of $\{y_1, \dots, y_n\}$ in W and a standard triple $(\bar{p} : \bar{X} \rightarrow S, X_\infty, Z)$ over a smooth affine variety S such that the pair $(U, U \cap Y)$ is isomorphic to the pair $(X = \bar{X} - X_\infty, Z)$.*

Proof: See [11, Remark 4.13].

4.2 Pretheories in a neighborhood of a smooth subscheme.

Consider the inclusion of categories $i : Sm/k \rightarrow Sch/k$ where Sch/k is the category of Noetherian schemes over k . Then for any presheaf F on Sm/k with values in the category of abelian groups the restriction of the presheaf $i^*(F)$ on Sch/S to Sm/k is equal to F .

For any pretheory (F, ϕ, A) over k such that A is the category of abelian groups and any Noetherian scheme X over k we will use the notation $F(X)$ instead of $i^*(F)(X)$.

We denote by F_{Zar} the sheaf in the Zariski topology on Sch/k associated with the presheaf $i^*(F)$.

Let k be a field, $i : Z \rightarrow X$ be a smooth pair over k and F be a presheaf of abelian groups on Sm/k . Denote the open subscheme $X - Z$ by U and let $j : U \rightarrow X$ be the corresponding open embedding. Consider the exact sequence of presheaves on X_{Zar} :

$$F \rightarrow j_*j^*(F) \rightarrow coker \rightarrow 0.$$

Let $coker_{Zar}$ be the sheaf in the Zariski topology associated with the presheaf $coker$. One can easily see that the canonical morphism

$$coker_{Zar} \rightarrow i_*i^*(coker_{Zar})$$

is an isomorphism. We denote by $F_{(X,Z)}$ the sheaf $i^*(coker_{Zar})$ on Z_{Zar} .

Remark 4.10 Since the associated sheaf functor is exact we have an exact sequence of the form

$$F_{Zar} \rightarrow (j_*j^*F)_{Zar} \rightarrow (coker)_{Zar} \rightarrow 0$$

but in general the canonical morphism $(j_*j^*F)_{Zar} \rightarrow j_*j^*(F_{Zar})$ is not an isomorphism since the associated sheaf functor does not commute with the direct image functor. For example if F is a sheaf on X_{Zar} we have $(j_*j^*(H_{Zar}^n(-, F)))_{Zar} \cong R^n j_*(F)$ but $(H_{Zar}^n(-, F))_{Zar} = 0$.

Proposition 4.11 *Let Z be a smooth scheme over a field k and (F, ϕ, A) be a homotopy invariant pretheory over k such that A is the category of abelian groups. Then there is a canonical isomorphism of sheaves on Z_{Zar} of the form*

$$(F_{-1})_{Zar} \rightarrow F_{(Z \times \mathbf{A}^1, Z \times \{0\})}.$$

Proof: Let us first define a morphism

$$(F_{-1})_{Zar} \rightarrow F_{(Z \times \mathbf{A}^1, Z \times \{0\})}.$$

Let $i : Z \rightarrow Z \times \mathbf{A}^1$, $j : Z \times (\mathbf{A}^1 - \{0\}) \rightarrow Z \times \mathbf{A}^1$ be the obvious closed and open embeddings. Since the associated sheaf functor commutes with the inverse image functor it is sufficient to define a morphism of presheaves

$$F_{-1} \rightarrow i^*(coker(F \rightarrow j_*j^*(F)))$$

on Z_{zar} . Almost by definition a section of the right hand side presheaf over an open subset U of Z is an equivalence class of pairs of the form (V, a) where V is an open subset in $Z \times \mathbf{A}^1$ such that $U \subset Z \cap V$ and a is an element of $F(U - U \cap (Z \times \{0\}))$. Two pairs $(V_1, a_1), (V_2, a_2)$ correspond to the same section of our presheaf if there exists an open subset V_0 in $V_1 \cap V_2$ such that $U \subset V_0 \cap (Z \times \{0\})$ and the images of a_1 and a_2 in $F(V_0 - V_0 \cap (Z \times \{0\}))/F(V_0)$ coincide.

Since a section of F_{-1} over an open subset U of Z is by definition an element of $F(U \times (\mathbf{A}^1 - \{0\}))/F(U \times \mathbf{A}^1)$ the definition of our morphism is obvious.

To prove that the corresponding morphism of associated sheaves in the Zariski topology on Z is an isomorphism it is sufficient to show that if Z is a local smooth scheme over k and V is an open neighborhood of $Z \times \{0\}$ in $Z \times \mathbf{A}^1$ then the morphism

$$F(Z \times (\mathbf{A}^1 - \{0\}))/F(Z \times \mathbf{A}^1) \rightarrow F(U - U \cap (Z \times \{0\}))/F(U)$$

is an isomorphism.

Note that since U is a neighborhood of $Z \times \{0\}$ and we may assume Z to be local the triple $T_1 = (\mathbf{P}_Z^1 \rightarrow Z, \mathbf{P}_Z^1 - U, Z \times \{0\})$ is a standard triple over Z by Lemma 4.3 and we may apply Proposition 4.7 to the standard triples $T_1, T_2 = (\mathbf{P}_Z^1 \rightarrow Z, \mathbf{P}_Z^1 - \mathbf{A}_Z^1, Z \times \{0\})$ and the identity morphism $Id_{\mathbf{P}_Z^1}$. Since the standard triple T_2 splits over $\mathbf{A}_Z^1$ it implies the result we need.

Up to the end of this section (F, ϕ, A) denote a homotopy invariant pretheory of homological type over a field k such that A is the category of abelian groups.

Proposition 4.12 *Let $f : X_1 \rightarrow X_2$ be an etale morphism of smooth schemes over k and Z be a closed subscheme in X_2 such that the morphism $f^{-1}(Z) \rightarrow Z$ is an isomorphism. Then for any (F, ϕ, A) as above the following conditions hold:*

1. *Let $a \in F(X_2 - Z)$ be an element such that the preimage of a to $X_1 - f^{-1}(Z)$ can be extended to X_1 . Then for any closed point x of X_2 there exists an open neighborhood U in X_2 such that the restriction of a to $U - U \cap Z$ can be extended to U .*

2. For any element $a \in F(X_1 - f^{-1}(Z))$ and any closed point x of X_2 there exists an open neighborhood U of x on X_2 and an element $a' \in F(U - U \cap Z)$ such that the images of a and a' in $F(f^{-1}(U - U \cap Z))/F(f^{-1}(U))$ coincide.

Proof: Denote by $\bar{f} : X'_1 \rightarrow X_2$ the finite morphism with normal X'_1 associated with f and let Y be the closed subspace $\bar{f}(X'_1 - X_1)$ in X_2 . By Proposition 4.9 for any closed point x of X_2 there is an open neighborhood U and a standard triple $(\bar{p}_1 : \bar{W}_1 \rightarrow S, W_{1,\infty}, R_1)$ over a smooth scheme S such that the pair $(U, U \cap (Z \cup Y))$ is isomorphic to the pair $(W_1 = \bar{W}_1 - W_{1,\infty}, R_1)$.

Let g be the morphism $X_1 \times_{X_2} U \rightarrow X_2$. Lemma 4.2 implies that there is a standard triple $(\bar{p}_2 : \bar{W}_2 \rightarrow S, W_{2,\infty}, R_2)$ such that the pair $(W_2 = \bar{W}_2 - W_{2,\infty}, R_2)$ is isomorphic to the pair $(X_1 \times_{X_2} U, (X_1 \times_{X_2} U) \cap f^{-1}(Z))$. To finish the proof it is sufficient now to apply Corollary 4.8 to the standard triples $(\bar{p}_1 : \bar{W}_1 \rightarrow S, W_{1,\infty}, U \cap Z)$, $(\bar{p}_2 : \bar{W}_2 \rightarrow S, W_{2,\infty}, R_2)$.

Corollary 4.13 *Let $f : X_1 \rightarrow X_2$ be an etale morphism of smooth schemes over k and Z be a closed subscheme in X_2 such that the morphism $f^{-1}(Z) \rightarrow Z$ is an isomorphism. Then for any (F, ϕ, A) as above the canonical morphism of sheaves*

$$F_{(X_2, Z)} \rightarrow F_{(X_1, f^{-1}(Z))}$$

on $Z_{\text{zar}} = f^{-1}(Z)_{\text{zar}}$ is an isomorphism.

Theorem 4.14 *Let k and (F, ϕ, A) be as above, X be a smooth scheme over k and $Z \subset X$ be a smooth closed subscheme of X . Let further x be a closed point of X and d be the codimension of Z in x . Then there exists an open neighborhood U and a family of isomorphisms*

$$F_{(U \times Y, (U \cap Z) \times Y)} \cong F_{(\mathbf{A}^d \times (U \cap Z) \times Y, (U \cap Z) \times Y)}$$

of sheaves on $((U \cap Z) \times Y)_{\text{zar}}$ given for all smooth schemes Y over k which are natural with respect to Y (here $(U \cap Z) \times Y$ is considered as a closed subscheme of $\mathbf{A}^d \times (U \cap Z) \times Y$ by means of the embedding which corresponds to the point $(0, \dots, 0)$ of $\mathbf{A}^d$).

In particular for $d = 1$ we have isomorphisms

$$F_{(U \times Y, (U \cap Z) \times Y)} \cong (F_{-1})_{\text{zar}}.$$

Proof: The second statement follows from the first one by Proposition 4.11.

Since Z is a smooth subscheme on a smooth variety X there exists an open neighborhood U of x and an etale morphism $p : U \rightarrow \mathbf{A}^n$ such that $Z \cap U = p^{-1}(\mathbf{A}^{n-d} \times \{0\})$. We may suppose that $U = X$. Let $p_0 : Z \rightarrow \mathbf{A}^{n-d}$ be the restriction of p to Z . Consider the Cartesian square

$$\begin{array}{ccc} W_0 & \rightarrow & Z \times \mathbf{A}^d \\ q \downarrow & & \downarrow p_0 \times Id_{\mathbf{A}^d} \\ X & \xrightarrow{p} & \mathbf{A}^n = \mathbf{A}^{n-d} \times \mathbf{A}^d. \end{array}$$

Then $q^{-1}(Z) \cong Z \times_{\mathbf{A}^{n-d}} Z$ and since p_0 is etale the complement to the diagonal $q^{-1}(Z(-\Delta))$ is a closed subset in $q^{-1}(Z)$ and, hence in W_0 . Let W be the complement $W_0 - (q^{-1}(Z) - \Delta)$.

To finish the proof it is sufficient now to apply Corollary 4.13 to both arrows in the diagram

$$\begin{array}{ccc} W \times Y & \rightarrow & Z \times \mathbf{A}^d \times Y \\ \downarrow & & \\ U \times Y & & \end{array}$$

4.3 Pretheories on curves over a field.

Theorem 4.15 *Let k be a field and (F, ϕ, A) be a homotopy invariant pretheory over k such that A is the category of abelian groups. Then for any open subscheme U in $\mathbf{A}_k^1$ one has:*

$$H_{Zar}^i(U, F_{Zar}) = \begin{cases} F(U) & \text{for } i = 0 \\ 0 & \text{for } i \neq 0 \end{cases}$$

Proof: Consider an open covering of the form $U = U_1 \cup U_2$. Let Z_i be the reduced closed subscheme $U - U_i$. Then

$$T = (\mathbf{P}_k^1 \rightarrow Spec(k), \mathbf{P}_k^1 - U, Z_1 \coprod Z_2)$$

is a standard triple over $Spec(k)$ which obviously splits over U . Hence the sequence

$$0 \rightarrow F(U) \rightarrow F(U_1) \oplus F(U_2) \rightarrow F(U_1 \cap U_2) \rightarrow 0$$

is exact by Lemma 4.6. Therefore, F considered as a functor on the category of Zariski open subsets of U is a sheaf which implies the statement of the theorem for $i = 0$. To prove the statement for $i > 0$ note that using induction on the number of open subsets in a covering we can show that Čech cohomology of any open U' in U with coefficients in F are zero in positive dimensions. Then the standard spectral sequence which relates usual cohomology and Čech cohomology shows that $H_{Zar}^i(U, F) = 0$ for $i > 0$.

Corollary 4.16 *Let k be a field and (F, ϕ, A) be a homotopy invariant pretheory over k such that A is the category of abelian groups. Then the morphism $\mathbf{A}_k^1 \rightarrow \text{Spec}(k)$ induces an isomorphism*

$$F_{Zar}(\text{Spec}(k)) \rightarrow F_{Zar}(\mathbf{A}_k^1).$$

4.4 Pretheories on semi-local schemes.

Let W be an object of Sch/k . We say that W is a smooth semi-local scheme over k if there exists a smooth affine scheme X of finite type over k and a finite set $\{x_1, \dots, x_n\}$ of points of X such that W is the inverse limit of open neighborhoods of this set.

Proposition 4.17 *Let k be a field, (F, ϕ, A) be a homotopy invariant pretheory over k , W be a smooth quasi-projective variety over k and $\{x_1, \dots, x_n\}$ be a finite set of points of W . Then for any nonempty open subset U of W there exists an open neighborhood V of $\{x_1, \dots, x_n\}$ and a morphism $F(U) \rightarrow F(V)$ such that the following diagram commutes:*

$$\begin{array}{ccc} F(W) & \rightarrow & F(U) \\ \downarrow & \swarrow & \\ F(V) & & \end{array}$$

Proof: Let N be the reduced closed subscheme $W - U$ of W . By Lemma 4.9 there exists an open neighborhood V_0 of the set $\{x_1, \dots, x_n\}$ and a standard triple $T = (\bar{p} : \bar{X} \rightarrow S, X_\infty, Z)$ over a smooth affine variety S such that the pair $(V_0, V_0 \cap N)$ is isomorphic to the pair $(X = \bar{X} - X_\infty, Z)$. Since the Picard group of an affine semi-local scheme is zero there exists an open neighborhood V of the set $\{x_1, \dots, x_n\}$ in X such that T splits over V . The statement of our proposition follows now from Lemma 4.5.

Corollary 4.18 *Let k be a field, (F, ϕ, A) be a homotopy invariant pretheory over k such that A is the category of abelian groups and W be a smooth semi-local scheme over k . Then for any nonempty open subscheme U of W the morphism $F(W) \rightarrow F(U)$ is a splitting monomorphism.*

Corollary 4.19 *Let k be a field, (F, ϕ, A) be a homotopy invariant pretheory over k such that A is the category of abelian groups and X be a smooth variety over k . Then for any nonempty open subscheme U of X the morphism $F_{Zar}(X) \rightarrow F_{Zar}(U)$ is a monomorphism.*

Proposition 4.20 *Let k be a field, (F_i, ϕ_i, A) , $i = 1, 2$ be two homotopy invariant pretheories over k such that A is the category of abelian groups and $f : F_1 \rightarrow F_2$ be a morphism of pretheories such that for any field extension $k \subset E$ the morphism $f_E : F_1(\text{Spec}(E)) \rightarrow F_2(\text{Spec}(E))$ is an isomorphism. Then the morphism of associated sheaves $f_{Zar} : (F_1)_{Zar} \rightarrow (F_2)_{Zar}$ is an isomorphism.*

Proof: Consider the exact sequence of presheaves

$$0 \rightarrow \ker(f) \rightarrow F_1 \rightarrow F_2 \rightarrow \text{coker}(f) \rightarrow 0.$$

By Proposition 3.5 the presheaves $\ker(f)$ and $\text{coker}(f)$ have canonical structures of homotopy invariant pretheories. Since the associated sheaf functor is exact it is sufficient to show that $\ker(f)_{Zar} = \text{coker}(f)_{Zar} = 0$. It follows from our assumption and Corollary 4.18.

Proposition 4.21 *Let k be a field and (F, ϕ, A) be a homotopy invariant pretheory over k such that A is the category of abelian groups. Then for any smooth scheme X over k one has:*

$$F_{Zar}(X \times \mathbf{A}^1) = F_{Zar}(X).$$

Proof: It follows from Corollary 4.16 and Corollary 4.19.

For a presheaf of abelian groups F on Sch/k we denote by $s(F)$ the separated in the Zariski topology presheaf associated with F . By definition it means that for any noetherian scheme X over k we have

$$s(F)(X) = F(X) / \cup (\ker : F(X) \rightarrow \oplus F(U_i))$$

where the union is taken over the partially ordered set of the Zariski open coverings $\{U_i\}$ of X .

Proposition 4.22 *Let k be a field and (F, ϕ, A) be a homotopy invariant pretheory over k such that A is the category of abelian groups. Then the presheaf $s(F)$ has a unique structure of a homotopy invariant pretheory such that the canonical morphism of presheaves $F \rightarrow s(F)$ is a morphism of pretheories.*

Proof: Note first that $s(F)$ is obviously homotopy invariant. Let S be a smooth scheme over k , $p : X \rightarrow S$ be a smooth curve over S and Z be a closed reduced subscheme of X which belongs to $c_{equi}(X/S, 0)$. To prove our proposition it is sufficient to show that if S is local and $a \in F(X)$ is an element such that the image of a in $s(F)(X)$ is equal to zero then $\phi_{X/S}(Z)(a) = 0$.

By Corollary 4.18 our condition implies that there is an open neighborhood U of Z in X such that the restriction of a to U is equal to zero. Thus $\phi_{X/S}(Z)(a) = 0$ by Proposition 3.12.

Proposition 4.23 *Let W be a smooth semi-local scheme over a field k and $W = U \cup V$ be an open covering of W . Then for any pretheory (F, ϕ, A) over k such that A is the category of abelian groups the following sequence is exact:*

$$0 \rightarrow F(W) \rightarrow F(U) \oplus F(V) \rightarrow F(U \cap V) \rightarrow 0.$$

Proof: It follows from Lemma 4.6 in exactly the same way as in the proof of Proposition 4.17.

Proposition 4.24 *Let k be a field, (F, ϕ, A) be a homotopy invariant pretheory over k such that A is the category of abelian groups and W be a smooth semi-local scheme over k . Then one has $F_{Zar}(W) = F(W)$.*

Proof: By Proposition 4.22 and Corollary 4.19 we may suppose that F is a separated presheaf. Let $W = \cup U_i$ be an open covering of W . It is sufficient to show that for any collection of elements $a_i \in F(U_i)$ such that the restrictions of a_i and a_j to $U_i \cap U_j$ coincide there is an element a in $F(W)$ such that its restrictions to U_i are equal to a_i .

We use induction on the number k of open subsets. For $k = 2$ our statement follows from Proposition 4.23.

For an arbitrary finite covering $W = \cup U_i$ let J be the subset in the set of closed points of W which consists of points which belong to $U_1 \cup U_2$.

Proposition 4.23 implies that there is a closed subscheme Z in W such that Z does not contain points from J and an element $b \in F(U_1 \cup U_2 - (U_1 \cup U_2) \cap Z)$ such that the restriction of b to $U_l - U_l \cap Z_l$ coincides with the restrictions of a_l for $l = 1, 2$.

One can easily see that the collection of open subsets $((U_1 \cup U_2 - (U_1 \cup U_2) \cap Z), U_3, \dots, U_k)$ is a covering of W and we can use the inductive assumption to assign an element in $F(W)$ to the collection of sections $(b, a_3, \dots, a_k)$ on this open covering. Since F is separated Corollary 4.18 implies that this element satisfies the conditions we need. Proposition is proven.

Proposition 4.25 *Let k be a field and (F, ϕ, A) be a homotopy invariant pretheory over k such that A is the category of abelian groups. Then the restriction of the sheaf F_{Zar} to the category Sm/k has a unique structure of pretheory such that the morphism of presheaves $F \rightarrow F_{Zar}$ is a morphism of pretheories.*

Proof: By Proposition 4.22 we may assume that F is separated. Let S be a smooth scheme over k , $p : X \rightarrow S$ be a smooth curve over S and Z be a reduced irreducible subscheme of X which belongs to $c_{equi}(X/S, 0)$. Consider a section $a \in F_{Zar}(X)$. Since F is separated, Proposition 4.24 implies easily that there is an open covering $\{U_i \rightarrow S\}$ of S and open neighborhoods V_i of closed subschemes $Z \cap p^{-1}(U_i)$ in $p^{-1}(U_i)$ such that the restrictions a_i of a to V_i belong to the subgroups $F(V_i)$ of $F_{Zar}(V_i)$. Let b_i be the elements $\phi_{V_i/U_i}(Z \cap V_i)(a_i)$ in $F(U_i)$. Since F is separated Corollary 4.19 implies easily that the restrictions of b_i and b_j to $U_i \cap U_j$ coincide and hence there is a global element $b \in F(S)$. We set $\phi_{X/Z}(Z)(a) = b$. One can verify easily (using again the fact that F is separated and Corollary 4.19) that it gives a structure of pretheory on F_{Zar} .

Proposition 4.26 *Let k be a field and (F, ϕ, A) be a homotopy invariant pretheory over k such that A is the category of abelian groups. Then the pretheory F_{Zar} is a homotopy invariant pretheory of homological type.*

Proof: It follows easily from Proposition 3.17, Proposition 4.21 and Corollary 4.19.

4.5 Zariski cohomology with coefficients in pretheories.

Theorem 4.27 *Let k be a perfect field and (F, ϕ, A) be a homotopy invariant pretheory over k such that A is the category of abelian groups. Then for all $i \geq 0$ the presheaves $H_{Zar}^i(-, F_{Zar})$ on Sm/k have canonical structures of homotopy invariant pretheories of homological type.*

Proof: We use induction on i . For $i = 0$ our statement is proven in Proposition 4.26. Suppose that the theorem is proven for all $i < n$.

Lemma 4.28 *Let k be a perfect field and (F, ϕ, A) be a homotopy invariant pretheory over k such that A is the category of abelian groups. Then under the inductive assumption for any smooth semi-local scheme W over k one has*

$$H_{Zar}^n(W, F_{Zar}) = 0.$$

Proof: We use induction on the number j of closed points of W . For $j = 1$ our statement is trivial. Suppose that W is a smooth semi-local scheme, $\{x_1, \dots, x_j\}$ the set of closed points of W and a an element of $H_{Zar}^n(W, F_{Zar})$.

By the inductive assumption on j there is a closed subscheme Z_1 of W which does not contain x_1 and a closed subscheme Z_2 of W which does not contain any of the points $x_2, \dots, x_j$ such that a becomes zero on $U_1 = W - Z_1$ and $U_2 = W - Z_2$. One can easily see that $Z_1 \cap Z_2 = \emptyset$ and hence we have a long exact sequence

$$\begin{aligned} \dots \rightarrow H^{n-1}(U_1, F_{Zar}) \oplus H^{n-1}(U_2, F_{Zar}) &\rightarrow H^{n-1}(U_1 \cap U_2, F_{Zar}) \rightarrow \\ &\rightarrow H^n(W, F_{Zar}) \rightarrow H^n(U_1, F_{Zar}) \oplus H^n(U_2, F_{Zar}) \rightarrow \dots \end{aligned}$$

But since $H^{n-1}(-, F_{Zar})$ has a structure of homotopy invariant pretheory the homomorphism

$$H^{n-1}(U_1, F_{Zar}) \oplus H^{n-1}(U_2, F_{Zar}) \rightarrow H^{n-1}(U_1 \cap U_2, F_{Zar})$$

is surjective by Proposition 4.23 which implies that $a = 0$.

Lemma 4.29 *Let k be a perfect field and (F, ϕ, A) be a homotopy invariant pretheory over k such that A is the category of abelian groups. Then under the inductive assumption the presheaf $U \rightarrow H_{Zar}^n(U, F_{Zar})$ on Sm/k has a canonical structure of pretheory.*

Proof: Let S be a smooth variety over k and $p : X \rightarrow S$ be a smooth curve over S . Consider an element A of $c_{equi}(X/S, 0)$ and let $Z = Supp(A)$ be the support of A which is a closed subset in X finite and surjective over S .

Let $\mathcal{C}_A$ be the full subcategory in the category X_{Zar} of open subsets of X which consists of open subsets U such, that

$$p^{-1}(p((X - U) \cap Z)) \cap Z = (X - U) \cap Z.$$

One can easily see that $\mathcal{C}_A$ is closed under finite intersections. Consider the Grothendieck topology on $\mathcal{C}_A$ such that coverings are families of the form $\{U_i \rightarrow U\}$ satisfying $U = \cup U_i$ and denote by $\alpha : X_{Zar} \rightarrow \mathcal{C}_A$ the obvious morphism of sites.

For any object U of $\mathcal{C}_A$ let $\beta^{-1}(U)$ be the open subset $S - p((X - U) \cap Z)$ of S . One can easily see that $\beta^{-1}(U \cap V) = \beta^{-1}(U) \cap \beta^{-1}(V)$ and that if $U = \cup U_i$ then $\beta^{-1}(U) = \cup \beta^{-1}(U_i)$. Therefore we have a morphism of sites $\beta : S_{Zar} \rightarrow \mathcal{C}_A$.

Lemma 4.28 implies easily that $R^i \alpha_*(F_{Zar}) = 0$ for $i \leq n$. Therefore by the Leray spactral sequence for the morphism α we have

$$H_{Zar}^n(X, F_{Zar}) = H^n(\mathcal{C}_A, \alpha_*(F_{Zar})).$$

On the other hand we have a canonical morphism

$$H^n(\mathcal{C}_A, \alpha_*(F_{Zar})) \rightarrow H_{Zar}^n(S, \beta^* \alpha_*(F_{Zar})).$$

To define a homomorphism

$$\phi_{X/S}^n(A) : H_{Zar}^n(X, F_{Zar}) \rightarrow H_{Zar}^n(S, F_{Zar})$$

it is sufficient now to define a homomorphism $\beta^* \alpha_* F_{Zar} \rightarrow F_{Zar}$ of sheaves on S_{Zar} or, equivalently, a homomorphism $\alpha_* F_{Zar} \rightarrow \beta_* F_{Zar}$ of sheaves on $\mathcal{C}_A$. It means that for any open subset U of X which belongs to $\mathcal{C}_A$ we have to define a homomorphism

$$F_{Zar}(U) \rightarrow F_{Zar}(\beta^{-1}(U)).$$

We obviously have $U \cap Z = p^{-1}(\beta^{-1}(U)) \cap Z$ and hence $U \cap Z \cap p^{-1}(\beta^{-1}(U))$ is finite over $\beta^{-1}(U)$. Therefore A gives us an element A_U in $c_{equi}(p^{-1}(\beta^{-1}(U)) \cap U / \beta^{-1}(U), 0)$ and we can define our homomorphism to be the composition

$$F_{Zar}(U) \rightarrow F_{Zar}(p^{-1}(\beta^{-1}(U)) \cap U) \rightarrow F_{Zar}(\beta^{-1}(U))$$

where the last arrow is the morphism which corresponds to A_U with respect to the canonical structure of pretheory on F_{Zar} defined in Proposition 4.25.

One can easily verify now that our construction indeed provides a structure of a pretheory on the presheaf $H_{Zar}^n(-, F_{Zar})$ on Sm/k .

Lemma 4.30 *Under the assumptions of Lemma 4.29 the pretheory $H_{Zar}^n(-, F_{Zar})$ is of homological type.*

Proof: Let S be a smooth scheme over k , $f : X_1 \rightarrow X_2$ be a morphism of smooth curves over S and A be an element of $c_{equi}(X_1/S, 0)$. Then f defines a morphism of sites $\psi : \mathcal{C}_A \rightarrow \mathcal{C}_{f_*(A)}$ such that the following diagram is commutative

$$\begin{array}{ccc} (X_1)_{Zar} & \xrightarrow{f} & (X_2)_{Zar} \\ \alpha_1 \downarrow & & \downarrow \alpha_2 \\ \mathcal{C}_A & \xrightarrow{\psi} & \mathcal{C}_{f_*(A)} \\ \beta_1 \nearrow & & \uparrow \beta_2 \\ & & S_{Zar}. \end{array}$$

We only have to verify that the homomorphisms

$$\beta_1^* \psi^* F_{Zar} \rightarrow F_{Zar}$$

$$\beta_2^* F_{Zar} \rightarrow F_{Zar}$$

coincide with respect to the canonical isomorphism $\beta_1^* \psi^* \cong \beta_2^*$. One can easily see that it means exactly that F_{Zar} is a pretheory of homological type which was proven in Proposition 4.26.

We will need the following technical result.

Lemma 4.31 *Let X be a smooth scheme of finite type over a perfect field k , $Z \subset X$ be a closed subset of X such that $\text{codim}(Z) \geq 1$ and x be a point of X . Then there is an open neighborhood U of x in X and a sequence of reduced closed subschemes $\emptyset \subset Y_0 \subset Y_1 \subset \dots \subset Y_k$ in U satisfying the following conditions:*

1. *The schemes $Y_i - Y_{i-1}$ are smooth divisors on $U - Y_{i-1}$.*
2. *$U \cap Z$ is a subset of Y_k .*

Proof: We may suppose that x is a closed point of X and that X is connected.

We proceed by induction on $\dim(X)$. Since X is smooth over k there is an open neighborhood U of x and a smooth morphism $p : U \rightarrow V$ of relative dimension one such that V is a smooth scheme of dimension $\dim(X) - 1$ over k . Consider the closure Z' of $p(Z \cap U)$ in V .

If $\text{codim}(Z') \geq 1$ then by the inductive assumption there is a sequence of closed subschemes in V of the form $\emptyset \subset Y'_0 \subset \dots \subset Y'_n$ such that the conditions of our lemma hold for this sequence with respect to Z' . One can easily see that these conditions also hold for the sequence $\emptyset \subset p^{-1}(Y'_0) \subset \dots \subset p^{-1}(Y'_n)$ of closed subschemes of U with respect to Z .

Suppose now that $\text{codim}(Z') = 0$ i.e. $Z' = V$. Let Z_0 be the closed subset of $Z \cap U$ which is the union of irreducible components of Z of codimension > 1 and the closed subset of singular points of Z . Then the codimension of the closure of $p(Z_0)$ is greater than zero hence there is a sequence $\emptyset \subset Y_0 \subset \dots \subset Y_n$ of closed subschemes of U satisfying the conditions of the lemma with respect to Z_0 . One can easily see now that the sequence

$$\emptyset \subset Y_0 \subset \dots \subset Y_n \subset Y_n \cup (Z \cap U)$$

satisfies the conditions of the lemma with respect to Z . Lemma is proven.

To finish the proof of the theorem it is sufficient to show now that under the inductive assumption for any smooth scheme X over k the canonical projection $X \times \mathbf{A}^1 \rightarrow X$ induces an isomorphism

$$H_{Zar}^n(X, F_{Zar}) \rightarrow H_{Zar}^n(X \times \mathbf{A}^1, F_{Zar}).$$

Since the projection $p : X \times \mathbf{A}^1 \rightarrow X$ splits our morphism is a monomorphism. The Leray spectral sequence for p together with the inductive assumption shows that it is sufficient to verify that any element $a \in H_{Zar}^n(X \times \mathbf{A}^1, F_{Zar})$ becomes zero on an open covering $X = \cup U_i$ of X . By Theorem 4.15 there is a nonempty open subscheme U of X such that a equals zero on $U \times \mathbf{A}^1$. Together with Lemma 4.31 it implies that one has to show that for a local smooth X and a smooth divisor Z on X the morphism $H_{Zar}^n(X \times \mathbf{A}^1, F_{Zar}) \rightarrow H_{Zar}^n((X - Z) \times \mathbf{A}^1, F_{Zar})$ is a monomorphism.

Lemma 4.32 *Consider the open embedding $j : (X - Z) \times \mathbf{A}^1 \rightarrow X \times \mathbf{A}^1$ and suppose that X is a local scheme. Then under our assumptions we have*

$$R^m j_*(F_{Zar}) = 0$$

for $m < n$.

Proof: Let $i : Z \times \mathbf{A}^1 \rightarrow X \times \mathbf{A}^1$ be the complimentary closed embedding. By definition $R^m j_*(F_{Zar})$ is the sheaf associated with the presheaf $j_* j^*(H_{Zar}^m(-, F_{Zar}))$. Since the sheaf associated with the presheaf $H_{Zar}^m(-, F_{Zar})$ is zero we have $R^m j_*(F_{Zar}) = i_* i^*(R^m j_*(F_{Zar}))$ and

$$i^*(R^m j_*(F_{Zar})) = (H_{Zar}^m(-, F_{Zar}))_{(X \times \mathbf{A}^1, Z \times \mathbf{A}^1)}.$$

By our assumption and Lemma 4.30 the presheaf $H_{Zar}^m(-, F_{Zar})$ has a structure of a homotopy invariant pretheory of homological type. Therefore since X is local we have (by Theorem 4.14) an isomorphism

$$i^*(R^m j_*(F_{Zar})) \cong ((H_{Zar}^m(-, F_{Zar}))_{-1})_{Zar}.$$

Denote by $p_0 : Z \times (\mathbf{A}^1 - \{0\}) \rightarrow Z$ the canonical projection. The right hand side sheaf is isomorphic by definition to $R^m(p_0)_*(F_{Zar})$. Again by the inductive assumption and Corollary 4.19 this sheaf on Z_{Zar} is separated. On the other hand it is zero in the generic point by Theorem 4.15. Lemma is proven.

Consider the exact sequence of sheaves on $X \times \mathbf{A}^1$

$$0 \rightarrow F_{Zar} \rightarrow j_* j^*(F_{Zar}) \rightarrow coker \rightarrow 0$$

(the first morphism is a monomorphism by Corollary 4.19). Since the restriction of a to $(X - Z) \times \mathbf{A}^1$ is equal to zero Lemma 4.32 and the Leray spectral sequence for the open embedding j imply that the image of a in $H_{Zar}^n(X \times \mathbf{A}^1, j_* j^*(F_{Zar}))$ is equal to zero. Therefore a belongs to the image of the homomorphism

$$H_{Zar}^{n-1}(X \times \mathbf{A}^1, coker) \rightarrow H_{Zar}^n(X \times \mathbf{A}^1, F_{Zar}).$$

Since $coker \cong i_* i^* coker$ and the higher direct images for closed embeddings are zero the left hand side group is isomorphic to $H_{Zar}^{n-1}(Z \times \mathbf{A}^1, i^* coker)$. We

may suppose that X is local. Then by Proposition 4.16 and Theorem 4.14 the sheaf $i^*(\text{coker})$ is isomorphic to the sheaf $(F_{-1})_{Zar}$.

If $n > 1$ it implies immediately that $a = 0$ by the induction. For $n = 1$ we have a homomorphism

$$H^0(X \times \mathbf{A}^1, j_* j^*(F_{Zar})) \rightarrow H^0(X \times \mathbf{A}^1, \text{coker})$$

The left hand side group is equal to $F_{Zar}(X - Z)$ since F_{Zar} is homotopy invariant and the right hand side group is isomorphic to $F_{Zar}(Z \times (\mathbf{A}^1 - \{0\}))/F_{Zar}(F)$. Therefore our homomorphism is surjective by Theorem 4.14 since X and Z are local. Theorem is proven.

Corollary 4.33 *Let k be a perfect field and (F, ϕ, A) be a homotopy invariant pretheory over k such that A is the category of abelian groups. Let further X be a smooth scheme over k and $j : U \rightarrow X$ be an open subscheme of X such that $X - U$ is a smooth divisor on X . Then $R^n j_*(F_{Zar}) = 0$ for $n > 0$.*

Proof: See Lemma 4.32.

4.6 The Gersten resolution.

Proposition 4.34 *Let k be a perfect field and (F, ϕ, A) be a homotopy invariant pretheory over k such that A is the category of abelian groups. Then for any $i \geq 0$ one has a canonical isomorphism of pretheories*

$$(H_{Zar}^i(-, F_{Zar}))_{-1} \cong H_{Zar}^i(-, (F_{-1})_{Zar}).$$

Proof: Let us first consider the case $i = 0$. We have to show that the canonical morphism of presheaves $(F_{-1})_{Zar} \rightarrow (F_{Zar})_{-1}$ is an isomorphism. By Proposition 4.26 both presheaves have canonical structures of homotopy invariant pretheories and obviously this morphism is a morphism of pretheories. Our statement follows now from Theorem 4.15 and Proposition 4.20.

Suppose now that $i > 0$. One can easily see that it is sufficient to show that for any smooth scheme X over k one has $R^i p_*(F_{Zar}) = 0$ for $i > 0$ where p is the projection $X \times (\mathbf{A}^1 - \{0\}) \rightarrow X$. It follows from Theorem 4.27, Proposition 4.18 and Theorem 4.15.

Lemma 4.35 *Let k , (F, ϕ, A) be as in Proposition 4.34 and S be a smooth local scheme over k then one has:*

$$H_{Zar}^i(S \times (\mathbf{A}^n - \{0\}), F_{Zar}) / H_{Zar}^i(S, F_{Zar}) = \begin{cases} F_{-n}(S) & \text{for } i = n - 1 \\ 0 & \text{for } i \neq n - 1 \end{cases}$$

Proof: It follows easily by induction from Proposition 4.34, Theorem 4.27 and the Mayer-Vietoris long exact sequence for the Zariski cohomology.

For any point x of a smooth scheme X over k we denote by $H_x^i(X, F)$ the group $\lim_{\rightarrow} H_{\{\bar{x}\} \cap U}^i(U, F)$ where limit is taken over the partially ordered set of open neighborhoods of x in X .

Lemma 4.36 *Let k be a perfect field, X be a smooth scheme over k and x be a point of codimension p on X . Then for any F as above one has:*

$$H_x^i(X, F) = \begin{cases} F_{-p}(\text{Spec}(k_x)) & \text{for } i = p \\ 0 & \text{for } i \neq p \end{cases}$$

Proof: Let us recall that for any open embedding $j : U \rightarrow X$ and any presheaf F on X_{Zar} we defined in §4.2 a sheaf $F_{(X, X-U)}$ on $(X - U)_{Zar}$ as $i^*(\text{coker} : F \rightarrow j_* j^*(F))_{Zar}$ where $i : X - U \rightarrow X$ is the corresponding closed embedding.

Let X_x be the local scheme of x in X . For $p = 0$ or $i = 0$ our statement follows from Corollary 4.17. Assume that $i, p > 0$. Consider the sheaf $H^{i-1}(-, F)_{(X_x, x)}$ on $\text{Spec}(k_x)_{Zar}$. Clearly it is the constant sheaf which corresponds to the abelian group $H_x^i(X, F)$. Applying Theorem 4.14 to $H^{i-1}(-, F)$ we conclude that

$$H_x^i(X, F) = (H^{i-1}(-, F))_{(\mathbf{A}_{k_x}^p, (0, \dots, 0))}(\text{Spec}(k_x)) = H_{(0, \dots, 0)}^i(\mathbf{A}_{k_x}^p, F).$$

We have the long exact sequence for the cohomology with supports

$$\dots \rightarrow H_{(0)}^i(\mathbf{A}_{k_x}^p, F) \rightarrow H^i(\mathbf{A}_{k_x}^p, F) \rightarrow H^i(\mathbf{A}_{k_x}^p - \{0\}, F) \rightarrow H_{(0)}^{i+1}(\mathbf{A}_{k_x}^p, F) \rightarrow \dots$$

By Theorem 4.27 we have isomorphisms $H^i(\mathbf{A}_{k_x}^p, F) \rightarrow H^i(\text{Spec}(k_x), F)$ and therefore homomorphisms $H^i(\mathbf{A}_{k_x}^p, F) \rightarrow H^i(\mathbf{A}_{k_x}^p - \{0\}, F)$ are injective. Our statement follows now from Lemma 4.35.

Theorem 4.37 *Let k be a perfect field, (F, ϕ, A) be a homotopy invariant pretheory over k such that A is the category of abelian groups and F is a sheaf in the Zariski topology. Let further X be a smooth scheme of dimension n over k . Denote by $X^{(k)}$ the set of points of X of codimension k . Then there is a canonical exact sequence of sheaves on X_{Zar} of the form:*

$$0 \rightarrow F \rightarrow \bigoplus_{x \in X^{(0)}} (i_x)_*(F) \rightarrow \dots \rightarrow \bigoplus_{x \in X^{(n)}} (i_x)_*(F_{-n}) \rightarrow 0.$$

Proof: For any sheaf F we have a spectral sequence

$$(E_1^{pq} = \bigoplus_{x \in X^{(p)}} H_x^{p+q}(X, F)) \Rightarrow H^{p+q}(X, F).$$

Consider the corresponding spectral sequence of sheaves on X_{Zar} . It converges to the sheaves in the Zariski topology associated with the presheaves $H^{p+q}(-, F)$ which equal zero for $p+q \neq 0$ and F for $p+q = 0$. On the other hand Lemma 4.36 implies that it degenerates to a complex of the form

$$\bigoplus_{x \in X^{(0)}} (i_x)_*(F) \rightarrow \dots \rightarrow \bigoplus_{x \in X^{(n)}} (i_x)_*(F_{-n}) \rightarrow 0$$

which immediately implies the statement of our theorem.

5 Sheaves in Nisnevich and etale topologies associated with pretheories.

5.1 Pretheoretical structures on Nisnevich cohomology.

Definition 5.1 *Let k be a field and (F, ϕ, A) be a pretheory over k such that A is the category of abelian groups. It is called a reasonable pretheory if the following condition holds. Let S be a smooth local scheme over k , $f : X_1 \rightarrow X_2$ be an etale morphism of smooth curves over S and Z be a reduced irreducible subscheme of X_1 which belongs to $c_{equi}(X/S, 0)$ such that $Z \rightarrow f(Z)$ is an isomorphism. Then one has*

$$\phi_{X_2/S}(f(Z)) = \phi_{X_1/S}(Z) \circ F(f).$$

Proposition 5.2 *Let k be a field and (F, ϕ, A) be a homotopy invariant pretheory over k such that A is the category of abelian groups. Then F is reasonable.*

Proof: It follows from Corollary 4.19 and Proposition 3.17.

Theorem 5.3 *Let k be a field and (F, ϕ, A) be a reasonable pretheory over k . Then the presheaves $U \rightarrow H_{Nis}^i(U, F_{Nis})$ have canonical structures of (reasonable) pretheories.*

Proof: Let us consider first the case $i = 0$. Denote by $s_{Nis}(F)$ the separated in the Nisnevich topology presheaf associated with F . Let us show that it has a structure of a reasonable pretheory.

It is sufficient to show that for any smooth henselian local scheme S over k , smooth curve $p : X \rightarrow S$ over S and a reduced irreducible subscheme Z in $c_{equi}(X/S, 0)$ we have $\phi_{X/S}(Z)(a) = 0$ for any $a \in \ker(F(X) \rightarrow s_{Nis}(F)(X))$. It follows immediately from the fact any scheme finite over S is a local henselian scheme and the fact that F is reasonable.

We may assume now that F is a separated presheaf in the Nisnevich topology. Let $p : X \rightarrow S$ be a smooth curve over a smooth scheme S over k , Z be a closed subscheme in X which belongs to $c_{equi}(X/S, 0)$ and a be an element in $F_{Nis}(X)$. We want to construct an element $b = \phi_{X/S}(Z)(a)$ in $F_{Nis}(S)$.

Let $\{U_i \rightarrow X\}$ be a finite Nisnevich covering of X such that the restrictions of a to U_i belong to $F(U_i)$. Since a scheme finite over a local henselian scheme is a local henselian scheme there is a Nisnevich covering $\{V_j \rightarrow S\}_{1 \leq j \leq m}$ of S such that for any $j = 1, \dots, m$ the covering

$$\{U_i \times_S V_j \rightarrow X \times_S V_j\}$$

splits over $Z_j = Z \times_S V_j$.

Let $\tilde{Z}_j$ be the image of such a splitting, i.e. a closed subscheme in $\coprod_i U_i \times_S V_j$ which maps isomorphically on Z_j . Then $\coprod_i U_i \times_S V_j \rightarrow V_j$ is a smooth curve over V_j and $\tilde{Z}_j$ belongs to $c_{equi}((\coprod_i U_i \times_S V_j)/V_j, 0)$. Consider the elements $b_j = \phi(\tilde{Z}_j)(F(pr_1)(\coprod a_i))$.

One can easily see that b_j define a global element b in $F_{Nis}(S)$ and that this construction indeed provides us with a structure of a reasonable pretheory on F_{Nis} .

Consider now the case $i > 0$. Let $p : X \rightarrow S$ be a smooth curve over a smooth scheme S over k and Z be a reduced closed subscheme of X which belongs to $c_{equi}(X/S, 0)$. Consider the following diagram of morphisms of

sites:

$$\begin{array}{ccc} Z_{Nis} & \xrightarrow{i} & X_{Nis} \\ & p_0 \searrow & \downarrow p \\ & & S_{Nis} \end{array}$$

Since p_0 is finite the higher direct images in the Nisnevich topology with respect to p_0 are zero. It implies that one has only to construct a morphism of sheaves $(p_0)_* i^*(F_{Nis}) \rightarrow F_{Nis}$ on S_{Nis} . It can be easily done using our theorem for $i = 0$ and the fact that a scheme finite over a henselian local scheme is a disjoint union of henselian local schemes.

5.2 Cohomology of affine space and the comparison theorem.

Proposition 5.4 *Let k be a field, (F, ϕ, A) be a homotopy invariant pretheory over k such that A is the category of abelian groups and U be an open subscheme in $\mathbf{A}_k^1$. Then one has:*

$$H_{Nis}^i(U, F_{Nis}) = \begin{cases} F(U) & \text{for } i = 0 \\ 0 & \text{for } i \neq 0 \end{cases}$$

Proof: Let us consider first the case $i = 0$. First of all by Theorem 4.15 we have $F \cong F_{Zar}$ on U_{Zar} . Hence the morphism $F(U) \rightarrow F_{Nis}(U)$ is a monomorphism by Corollary 4.18 since any Nisnevich covering contains at least one non-empty open subset. It implies that we may replace F by $s_{Nis}(F)$ and assume that F is a separated presheaf in the Nisnevich topology.

One can easily see now that it is sufficient to show that for any element a in $F_{Nis}(U)$ and any closed point x of U there is an open neighborhood V such that the restriction of a to V belongs to $F(V) \subset F_{Nis}(V)$. It follows immediately from Proposition 4.12.

To prove our proposition for $i > 0$ it is obviously sufficient to consider the case $i = 1$. The corresponding result again follows easily from Proposition 4.12.

Proposition 5.5 *Let k be a field and (F, ϕ, A) be a homotopy invariant pretheory over k such that A is the category of abelian groups. Then the canonical morphism of presheaves $F_{Zar} \rightarrow F_{Nis}$ is an isomorphism.*

Proof: Note first that Theorem 5.3 together with Proposition 5.4 imply that F_{Nis} is a homotopy invariant pretheory and our morphism is obviously a morphism of pretheories. To finish the proof it is sufficient to apply Proposition 4.20 to this morphism.

Theorem 5.6 *Let k be a perfect field, (F, ϕ, A) be a homotopy invariant pretheory over k such that A is the category of abelian groups and X be a smooth scheme over k . Then the canonical projection $X \times \mathbf{A}^1 \rightarrow X$ induces isomorphisms*

$$H_{Nis}^i(X, F_{Nis}) \rightarrow H_{Nis}^i(X \times \mathbf{A}^1, F_{Nis}).$$

Proof: The proof is strictly parallel to the proof of Theorem 4.27 with considerable simplification due to Theorem 5.3.

Theorem 5.7 *Let k be a perfect field, (F, ϕ, A) be a homotopy invariant pretheory over k such that A is the category of abelian groups and X be a smooth scheme over k . Then the canonical morphisms $H_{Zar}^i(X, F_{Zar}) \rightarrow H_{Nis}^i(X, F_{Nis})$ are isomorphisms.*

Proof: The case $i = 0$ is proven in Proposition 5.5. Suppose that $i > 0$. By Theorems 5.6 and 5.3 the presheaves $H_{Nis}^i(-, F_{Nis})$ are homotopy invariant pretheories. It is obviously sufficient to show that the sheaves in the Zariski topology associated with these presheaves are zero. It follows immediately from Corollary 4.18 since the Nisnevich and the Zariski cohomology coincide for fields.

Remark 5.8 Unfortunately the analogs of the previous results do not hold for the flat version of the Nisnevich topology. Consider the following example. Let F be the cokernel of the morphism of sheaves $\mathbf{G}_m \xrightarrow{z^2} \mathbf{G}_m$ in the Zariski topology. Our results imply that F has a canonical structure of a homotopy invariant pretheory of homological type.

Consider the Nis-flat covering of $\mathbf{A}^1$ by the open subset $U_0 = \mathbf{A}^1 - \{0\}$ and the flat morphism $U_1 = \mathbf{A}^1 \xrightarrow{z^2} \mathbf{A}^1$. Let a be the canonical section of F on U_0 . Together with the zero section on U_1 they give us a nontrivial global section of F_{Nis-fl} on $\mathbf{A}^1$. It implies immediately that our theory does not work in this situation.

5.3 Applications to Suslin homology.

We will show in this section that our results on cohomology of sheaves associated with pretheories imply some important vanishing properties for the sheaves $\underline{h}_i(-)_{Zar}$ (see the end of §3.2). As an application we will prove the Mayer-Vietoris theorem for the Suslin homology.

Theorem 5.9 *Let k be a perfect field and (F, ϕ, A) be a pretheory over k with values in the category of abelian groups. Then the following conditions are equivalent.*

1. *The sheaves $(\underline{h}_i(F))_{Zar}$ are zero for all $i \leq n$.*
2. *For any homotopy invariant pretheory G which is a sheaf in the Nisnevich topology one has $Ext^i(F_{Nis}, G) = 0$ for all $i \leq n$.*

Proof: Note that by Proposition 5.5 one has $(\underline{h}_i(F))_{Zar} = (\underline{h}_i(F))_{Nis}$ and that (by 4.26) these presheaves are homotopy invariant pretheories.

(1 $\Rightarrow$ 2). Consider the reduced singular simplicial complex $\tilde{\underline{C}}_*(F)$ of F where

$$\tilde{\underline{C}}_0(F)(X) = F(X) = \underline{C}_0(F)(X)$$

$$\tilde{\underline{C}}_k(F)(X) = (\text{coker} : F(X) \rightarrow F(X \times \mathbf{A}^k)) = (\underline{C}_k(F)/F)(X) \text{ for } k > 0.$$

Clearly the obvious morphism $\underline{C}_*(F) \rightarrow \tilde{\underline{C}}_*(F)$ is a quasi-isomorphism. The reason why $\tilde{\underline{C}}_*$ is sometimes more convenient to work with is the following lemma.

Lemma 5.10 *Let G be a Nisnevich sheaf which is a homotopy invariant pretheory. Then for any $j > 0$ and any $i \geq 0$ one has*

$$Ext^i((\tilde{\underline{C}}_j(F))_{Nis}, G) = 0.$$

Proof: It follows easily from the fact that the presheaf $\tilde{\underline{C}}_j(F)$ is contractible (and hence so is the associated sheaf) and Theorem 5.6.

Lemma 5.10 implies that for any G as above and any i the obvious morphism

$$Hom_D((\underline{C}_*(F))_{Nis}, G[i]) = Hom_D((\tilde{\underline{C}}_*(F))_{Nis}, G[i]) \rightarrow Ext^i(F_{Nis}, G)$$

is an isomorphism (here Hom_D are morphisms in the derived category of Nisnevich sheaves on Sm/k). If $(\underline{h}_j(F))_{Nis} = 0$ for $j \leq n$ then the left hand side groups are obviously zero for all $i \leq n$.

(2- > 1) Let $(\underline{h}_k(F))_{Nis}$ be the first nontrivial cohomology sheaf of $\underline{C}_*(F)$. Then there is a nonzero morphism in the derived category

$$(\underline{C}_*(F))_{Nis} \rightarrow (\underline{h}_k(F))_{Nis}[k].$$

Since $(\underline{h}_k(F))_{Nis}$ is a homotopy invariant pretheory it gives us as above a nontrivial element in $Ext^k(F_{Nis}, (\underline{h}_k(F))_{Nis})$ and hence by our condition $k > n$.

Corollary 5.11 *Let k be a perfect field and F be a pretheory over k with values in the category of abelian groups such that $F_{Nis} = 0$. Then for any $i \geq 0$ one has $\underline{h}_i(F)_{Zar} = 0$.*

Corollary 5.12 *Let k be a perfect field and*

$$0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$$

be a sequence of reasonable pretheories such that the corresponding sequence of associated sheaves in the Nisnevich topology is exact. Then one has a canonical long exact sequence of sheaves of the form

$$\dots \rightarrow (\underline{h}_i(F))_{Zar} \rightarrow (\underline{h}_i(G))_{Zar} \rightarrow (\underline{h}_i(H))_{Zar} \rightarrow (\underline{h}_{i-1}(F))_{Zar} \rightarrow \dots$$

Proof: It follows immediately from Corollary 5.11 and Theorem 5.3.

Let k be a field, X be a scheme of finite type over k and U be a smooth scheme over k . Denote by $c_{equi}(X/Spec(k), 0)(U)$ the free abelian group generated by integral closed subschemes of $X \times U$ which are finite over U and surjective over an irreducible component of U . As was shown in [8] for any morphism of smooth schemes $f : U' \rightarrow U$ one can define a homomorphism

$$cycl(f) : c_{equi}(X/Spec(k), 0)(U) \rightarrow c_{equi}(X/Spec(k), 0)(U')$$

such that $c_{equi}(X/Spec(k), 0)(-)$ becomes a presheaf on the category Sm/k and the obvious analogs of properties (1)-(3) given at the beginning of §2 hold.

Proposition 5.13 *Let k be a field and X be a scheme of finite type over k . Then the presheaf $c_{\text{equi}}(X/\text{Spec}(k), 0)$ on Sm/k is a sheaf in the étale (and a fortiori in the Nisnevich) topology.*

Proof: It follows immediately from our definition of $c_{\text{equi}}(X/\text{Spec}(k), 0)$ and property (2) of the base change homomorphisms.

Proposition 5.14 *For any scheme of finite type X over a field k the presheaf of abelian groups $c_{\text{equi}}(X/\text{Spec}(k), 0)$ on Sm/k has a canonical structure of a pretheory of homological type.*

Proof: We have to define for any smooth curve C/S over a smooth scheme S over k a pairing of the form

$$c_{\text{equi}}(C/S, 0) \otimes c_{\text{equi}}(X/\text{Spec}(k), 0)(C) \rightarrow c_{\text{equi}}(X/\text{Spec}(k), 0)(S).$$

This is a special case of the correspondence pairings considered in [8, Section 3.7]. The fact that these pairings give us a structure of a pretheory follows from [8, Th. 3.7.3].

Proposition 5.15 *Let k be a field, X a separated scheme over k and $X = U \cup V$ an open covering of X . Then the sequence*

$$0 \rightarrow c_{\text{equi}}(U \cap V) \rightarrow c_{\text{equi}}(U) \oplus c_{\text{equi}}(V) \rightarrow c_{\text{equi}}(X) \rightarrow 0$$

is exact in the Nisnevich topology (we abbreviated $c_{\text{equi}}(-/\text{Spec}(k), 0)$ to $c_{\text{equi}}(-)$).

Proof: It is a special case of [8, Prop. 4.3.7].

Definition 5.16 *Let k be a field and X a scheme of finite type over k . The groups $\underline{h}_i(c_{\text{equi}}(X/\text{Spec}(k), 0))(\text{Spec}(k))$ are called the Suslin homology of X . We denote them by $H_i^S(X/k)$.*

Theorem 5.17 *Let k be a perfect field and X a scheme of finite type over k . Then for any open covering $X = U \cup V$ of X there is a canonical long exact sequence of the form*

$$\dots \rightarrow H_i^S(U \cap V) \rightarrow H_i^S(U) \oplus H_i^S(V) \rightarrow H_i^S(X) \rightarrow H_{i-1}^S(U \cap V) \rightarrow \dots$$

Proof: It follows from Corollary 5.12 and Proposition 5.15.

5.4 Cohomology of blowups.

For any smooth scheme X over a field k denote by $\mathbf{Z}_{\text{Nis}}(X)$ the sheaf of abelian groups on $(Sm/k)_{\text{Nis}}$ freely generated by the sheaf of sets representable by X . Let $Ab((Sm/k)_{\text{Nis}})$ be the category of sheaves of abelian groups on $(Sm/k)_{\text{Nis}}$. Then for any object F of this category we have canonical isomorphisms:

$$Ext_{Ab((Sm/k)_{\text{Nis}})}^i(\mathbf{Z}_{\text{Nis}}(X), F) \cong H_{\text{Nis}}^i(X, F).$$

Proposition 5.18 *Let k be a field, $f : X \rightarrow Y$ an etale morphism of smooth schemes over k and Z a closed subscheme in Y such that $f^{-1}(Z) \rightarrow Z$ is an isomorphism. Then the canonical morphism of sheaves*

$$\mathbf{Z}_{\text{Nis}}(X)/\mathbf{Z}_{\text{Nis}}(X - f^{-1}(Z)) \rightarrow \mathbf{Z}_{\text{Nis}}(Y)/\mathbf{Z}_{\text{Nis}}(Y - Z)$$

on $(Sm/k)_{\text{Nis}}$ is an isomorphism.

Proof: Note first that $(Y - Z) \amalg X \rightarrow Y$ is a Nisnevich covering of Y . Therefore our morphism is an epimorphism. To prove that it is a monomorphism it is sufficient to prove that it is a monomorphism on the corresponding freely generated presheaves (since the associated sheaf functor is exact). Consider the diagram:

$$\begin{array}{ccccccc} 0 \rightarrow \mathbf{Z}(X - f^{-1}(Z)) & \xrightarrow{i} & \mathbf{Z}(X) & \rightarrow & \mathbf{Z}(X)/\mathbf{Z}(X - f^{-1}(Z)) & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 \rightarrow \mathbf{Z}(Y - Z) & \rightarrow & \mathbf{Z}(Y) & \rightarrow & \mathbf{Z}(Y)/\mathbf{Z}(Y - Z) & \rightarrow & 0 \end{array}$$

where $\mathbf{Z}(-)$ is the presheaf which sends a smooth scheme U to the direct sum of free abelian groups generated by sets of morphisms from the connected components of U to $-$. We have to show that $\ker(\mathbf{Z}(f))$ lies in $\text{Im}(i)$. Let W be a connected scheme. Then $\ker(\mathbf{Z}(f))$ is the group of formal sums of the form $\sum_{i \in I} n_i g_i$, where g_i are morphisms from W to X such that there exists a decomposition $I = \amalg I_k$ satisfying $f \circ g_i = f \circ g_j$ for $i, j \in I_k$ and $\sum_{i \in I_k} n_i = 0$ for any k . Therefore, we have to prove only that if $f \circ g = f \circ h$ for some $g, h : W \rightarrow X$ then either $g = h$ or g and h can be factored through $X - f^{-1}(Z)$. Let g, h be such morphisms. Then there exists a morphism $g \times h : W \rightarrow X \times_Y X$ whose compositions with the projections are the morphisms g and h respectively. To finish the proof it is sufficient to notice that since f is an etale morphism the diagonal in $X \times_Y X$ is a connected

component i.e. there is a decomposition of the form $X \times_Y X = \Delta(X) \amalg X_0$. But since $f^{-1}(Z) \rightarrow Z$ is an isomorphism the projections $pr_1, pr_2 : X_0 \rightarrow X$ can be factored through $X - f^{-1}(Z)$. Proposition is proven.

Proposition 5.19 *Let X be a smooth scheme over a field k and $Z \rightarrow X$ be a smooth subscheme of X . Denote by $p_Z : X_Z \rightarrow X$ the blowup of X in Z and consider the following Cartesian square:*

$$\begin{array}{ccc} p_Z^{-1}(Z) & \xrightarrow{i} & X_Z \\ q \downarrow & & \downarrow p_Z \\ Z & \rightarrow & X \end{array}$$

Then the canonical morphism of sheaves $\ker(\mathbf{Z}_{\text{Nis}}(q)) \rightarrow \ker(\mathbf{Z}_{\text{Nis}}(p_Z))$ on Sm/k is an isomorphism (and a fortiori the same holds for any stronger topology).

Proof: Obviously our morphism is a monomorphism. It is sufficient to show that it is an epimorphism. For any $X \in ob(Sm/k)$ let $L(X)$ be the sheaf of sets representable by X . Note that there is an obvious long exact sequence of the form

$$\dots \rightarrow \mathbf{Z}_{\text{Nis}}(L(X_Z) \times_{L(X)} L(X_Z)) \rightarrow \mathbf{Z}_{\text{Nis}}(L(X_Z)) \rightarrow \mathbf{Z}_{\text{Nis}}(L(X))$$

It is sufficient to show that the morphism

$$L(i) \times L(i) \amalg L(\Delta) : L(p_Z^{-1}(Z) \times_Z p_Z^{-1}(Z)) \amalg L(X_Z) \rightarrow L(X_Z) \times_{L(X)} L(X_Z)$$

is an epimorphism. It is obvious since as a scheme $X_Z \times_X X_Z$ is a union of the diagonal and the image of $p_Z^{-1}(Z) \times_Z p_Z^{-1}(Z)$.

Remark 5.20 The statement of the proposition above is false for sheaves in the Nisnevich (or Zariski) topology on Sch/k which gives in particular an example of the fact that functor of the inverse image on the categories of sheaves associated with the canonical embedding of categories $Sm/k \rightarrow Sch/k$ is not left exact. Nevertheless the proposition is still valid for all schemes if we consider qfh- or cdh-topology (see [10], [8]).

Proposition 5.21 *In the notations of Proposition 5.19 suppose that k is a perfect field and let (F, ϕ, A) be a homotopy invariant pretheory over k such that A is the category of abelian groups. Then for any $i \geq 0$ one has:*

$$Ext^i(\text{coker}(\mathbf{Z}_{\text{Nis}}(\text{p}_Z)), F_{\text{Nis}}) = 0$$

Proof: One can easily see that our problem is local with respect to X . Denote by U the open subscheme $X - Z$ of X and consider the diagram of sheaves on $(Sm/k)_{\text{Nis}}$:

$$\begin{array}{ccccccc} 0 \rightarrow & \mathbf{Z}_{\text{Nis}}(U) & \rightarrow & \mathbf{Z}_{\text{Nis}}(X_Z) & \rightarrow & \text{coker}_1 & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow & \mathbf{Z}_{\text{Nis}}(U) & \rightarrow & \mathbf{Z}_{\text{Nis}}(X) & \rightarrow & \text{coker}_2 & \rightarrow 0 \end{array}$$

One can easily see that $\text{coker}(\mathbf{Z}(\text{p}_Z)) = \text{coker}(\text{coker}_1 \rightarrow \text{coker}_2)$. Together with Proposition 5.18 and the fact that Z is smooth it implies that locally with respect to X we have:

$$\text{coker}(\mathbf{Z}(\text{p}_Z)) \cong \text{coker}(\mathbf{Z}(Z \times \mathbf{A}_{(Z \times \{0\})}^d) \rightarrow \mathbf{Z}(Z \times \mathbf{A}^d))$$

where d is the codimension of Z . We have a projection $Z \times \mathbf{A}_{(Z \times \{0\})}^d \rightarrow Z \times \mathbf{P}^{d-1}$ which is a locally trivial in the Zariski topology fibration with the fiber $\mathbf{A}^1$. By Theorem 5.6 it implies that we have:

$$\begin{aligned} Ext^i(\mathbf{Z}(Z \times \mathbf{A}_{(Z \times \{0\})}^d), F) &= H^i(Z \times \mathbf{P}^{d-1}, F) \\ Ext^i(\mathbf{Z}(Z \times \mathbf{A}^d), F) &= H^i(Z, F). \end{aligned}$$

Our statement follows now trivially from Proposition 5.19.

Corollary 5.22 *Let k be a perfect field, X be a smooth scheme over k and Z be a smooth closed subscheme in X . Denote by $p : X_Z \rightarrow X$ the blow-up of Z in X . Then for any homotopy invariant pretheory F with values in the category of abelian groups there is a long exact sequence of the form:*

$$\begin{aligned} \dots \rightarrow H^i(X, F_{\text{Zar}}) &\rightarrow H^i(X_Z, F_{\text{Zar}}) \oplus H^i(Z, F_{\text{Zar}}) \rightarrow \\ &\rightarrow H^i(p^{-1}(Z), F_{\text{Zar}}) \rightarrow H^{i+1}(X, F_{\text{Zar}}). \end{aligned}$$

(Note that it is true for both the Zariski and the Nisnevich cohomology groups by Theorem 5.7)

Proof: Consider the following diagram of morphisms of sheaves in the Nisnevich topology:

$$\begin{array}{ccc} \mathbf{Z}_{\text{Nis}}(p_Z^{-1}(Z)) & \xrightarrow{i} & \mathbf{Z}_{\text{Nis}}(X_Z) \\ q \downarrow & & \downarrow \\ \mathbf{Z}_{\text{Nis}}(Z) & \rightarrow & \mathbf{Z}_{\text{Nis}}(X) \end{array} \quad \mathbf{Z}_{\text{Nis}}(p_Z)$$

The left vertical arrow is an epimorphism since $p^{-1}(Z) \rightarrow Z$ is a locally trivial in Zariski topology bundle over Z . This fact and Proposition 5.19 imply immediately that we have an exact sequence of the form:

$$0 \rightarrow \mathbf{Z}_{\text{Nis}}(p_Z^{-1}(Z)) \rightarrow \mathbf{Z}_{\text{Nis}}(X_Z) \oplus \mathbf{Z}_{\text{Nis}}(Z) \rightarrow \mathbf{Z}_{\text{Nis}}(X) \rightarrow \text{coker}(\mathbf{Z}_{\text{Nis}}(p_Z)) \rightarrow 0$$

and the corollary follows from Proposition 5.21.

Remark 5.23 It can be shown (see [9]) that the exact sequence from Corollary 5.22 splits canonically and one has:

$$H^i(X_Z, F_{Zar}) = H^i(X, F_{Zar}) \oplus \bigoplus_{j=1}^{c-1} H^{i-2j}(Z, (F_{-j})_{Zar}).$$

5.5 Sheaves in the etale topology associated with pretheories.

Proposition 5.24 *Let k be a field and (F, ϕ, A) be a homotopy invariant pretheory over k such that A is the category of $\mathbf{Q}$ -vector spaces. Then the canonical morphism of presheaves on Sm/k*

$$F_{\text{Nis}} \rightarrow F_{\text{et}}$$

is an isomorphism.

Proof: It follows trivially from Corollary 4.18 and the results of §3.3.

The corresponding result with integral coefficients is wrong. Instead we have the following “rigidity” theorem.

Theorem 5.25 *Let k be a field of exponential characteristic p and (F, ϕ, A) be a homotopy invariant pretheory over k such that A is the category of torsion abelian groups of torsion prime to p . Then the sheaf F_{et} on Sm/k is locally constant.*

Proof: The proof is strictly parallel to the proof of the rigidity theorem in [7].

Corollary 5.26 *Let k be a field of exponential characteristic p and (F, ϕ, A) be a homotopy invariant pretheory over k such that A is the category of $\mathbf{Z}[1/p]$ -modules. Consider the exact sequence of sheaves*

$$0 \rightarrow (F_{tors})_{et} \rightarrow F_{et} \rightarrow F_{et} \otimes \bar{\mathbf{Q}} \rightarrow (F_{cotor})_{et} \rightarrow 0.$$

Then the sheaves $(F_{tors})_{et}, (F_{cotor})_{et}$ are locally constant.

Proof: It follows from Theorem 5.25 and Proposition 3.5.

Proposition 5.27 *Let X be a scheme and F be a sheaf of $\mathbf{Q}$ -vector spaces on X_{et} then the canonical morphisms*

$$H_{Nis}^i(X, F) \rightarrow H_{et}^i(X, F)$$

are isomorphisms.

Proof: It follows immediately from the Leray spectral sequence and the fact that the etale cohomology of any henselian local scheme are torsion groups.

Proposition 5.28 *Let k be a perfect field and (F, ϕ, A) be a homotopy invariant pretheory over k such that A is the category of abelian groups. Then the canonical morphisms of pretheories*

$$H_{Zar}^i(-, F_{Zar} \otimes \mathbf{Q}) \rightarrow H_{et}^i(-, F_{et} \otimes \mathbf{Q})$$

are isomorphisms.

Proof: It follows from Theorem 5.7, Proposition 5.28 and Proposition 5.24.

Corollary 5.29 *Let k be a perfect field of exponential characteristic p , (F, ϕ, A) a homotopy invariant pretheory over k such that A is the category of $\mathbf{Z}[1/p]$ -modules and X a smooth scheme over k . Then the canonical morphisms*

$$H_{et}^i(X, F_{et}) \rightarrow H_{et}^i(X \times \mathbf{A}^1, F_{et})$$

are isomorphisms.

Proof: It follows from our comparison theorem for the rational coefficients, Theorem 5.25 and homotopy invariance of the étale cohomology with locally constant torsion coefficients.

Proposition 5.30 *Let k be a perfect field and $F : (Sm/k)^{op} \rightarrow A$ a contravariant functor from the category Sm/k to a $\mathbf{Q}$ -linear additive category A . Suppose that there are two pretheory structures ϕ^1, ϕ^2 of on F . Then ϕ^1 and ϕ^2 coincide semi-locally. In particular if F is a separated presheaf of $\mathbf{Q}$ -vector spaces in the Zariski topology, then ϕ^1 and ϕ^2 coincide.*

Proof: Let S be a smooth affine variety over k and $p : X \rightarrow S$ a smooth curve over S . We have to show that for any finite set $\{s_1, \dots, s_n\}$ of points of S and any element A in $c_{equi}(X/S, 0)$ there is an open neighborhood U of $\{s_1, \dots, s_n\}$ such that the following two compositions coincide

$$\begin{aligned} F(X) &\xrightarrow{\phi_{X/S}^1(A)} F(S) \rightarrow F(U) \\ F(X) &\xrightarrow{\phi_{X/S}^2(A)} F(S) \rightarrow F(U). \end{aligned}$$

By Proposition 4.17 it is sufficient to show that there exists a nonempty open subset U satisfying this property. It means that we may suppose that $S = Spec(K)$ where K is a field, X is a curve over K and A is a closed point of X . In this case our statement follows easily from the results of §3.3.

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