STEENROD OPERATIONS IN CHOW THEORY

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Abstract. An action of the Steenrod Algebra is constructed on the mod p Chow theory of varieties over a field of characteristic different from p. This action should agree with the action of the Steenrod algebra on motivic cohomology used by Voevodsky to prove the Milnor conjecture [9]. The construction, however, uses only basic functorial properties of Edidin-Graham equivariant intersection theory [1] and the Fulton-MacPherson refined Gysin homomorphism [2]. In particular, it does not use resolution of singularities.

1. Introduction

Let X be a complete complex algebraic variety, let $H_*^{BM}(X)$ denote its Borel-Moore homology with coefficients in the field $\mathbf{F}_p$, and let $S_\bullet : H_*^{BM}(X) \to H_*^{BM}(X)$ be the total Steenrod p-th power operation in Borel-Moore homology. In [2] (Example 19.1.8), Fulton notes that $S_\bullet$ preserves algebraic classes. (This fact had previously been discovered by Kawai [4].)

Fulton's argument is as follows: Consider a subvariety V of X. By Hironaka, there is a resolution of singularities $\pi : M \to V$. If μ_M is the orientation class of M, then the cycle class cl(V) is equal to $\pi_* \mu_M$. Let $\psi : H^*X \to H^*(TM, TM - 0)$ be the Thom isomorphism, and let $S^\bullet$ be the total Steenrod operation in singular cohomology. Let $w = \psi^{-1} S^\bullet \psi(1)$. Fulton gives a formula

$$S_\bullet(\mathrm{cl(V)}) = \pi_* \mathrm{S}_\bullet(\mu_\mathrm{M}) = \pi_*(\mathrm{w}(\mathrm{T_M})^{-1} \cap \mu_\mathrm{M}). \tag{1}$$

As w can be expressed in terms of the Chern classes of a TM [6], it is tempting to use (1) as a definition of $S_\bullet$ in the mod p Chow groups $A_* X \otimes \mathbf{F}_p$. The problem, as Fulton notes, is whether $\pi_*(w(T_M)^{-1} \cap \mu_M)$ is independent of the resolution M.

In this paper, $S_\bullet$ is defined for quasi-projective varieties over a field k of characteristic not equal to p using the equivariant extension of Fulton-MacPherson Intersection Theory developed by Edidin and Graham [1]. (The definition actually works for any algebraic space over k with an injective morphism into a smooth algebraic space, and it can be extended to all all varieties over k by using a Chow envelope argument.) The construction loosely follows the construction of cohomology operations given by Steenrod and Epstein [7]. The definition is then shown to agree with (1) proving that $\pi_*(w(T_M)^{-1} \cap \mu_M)$ is indeed independent of M. The paper ends with a demonstration of the Adem relations in the "algebraic Steenrod algebra" generated by the graded components of $S_\bullet$. One aspect of our method that may be interesting is that, while Fulton's question definitely involves resolutions of singularities, our construction of $S_\bullet$ does not.

It should be noted that Voevodsky has defined similar operations in the context of motivic cohomology. Although the details of his construction have not yet fully appeared, these operations should agree with ours and, over a field where motivic

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cohomology is known to contain Chow groups (i.e., one that admits resolutions of singularities), Voevodsky's operations are more general. On the other hand, this paper fits into the Fulton-MacPherson framework for intersection theory, and the resulting simplicity may have some appeal.

1.1. **Conventions.** If Λ is a commutative ring, the Chow group (resp. the Chow ring) with coefficients in Λ is simply $A_*X \otimes \Lambda$ (resp. $A^*X \otimes \Lambda$). Until section 7, all work will be over an arbitrary ring Λ. Mention of Λ will however be suppressed from the notation with A_*X written instead of $A_*X \otimes \Lambda$. In section 7, a prime p will be chosen and Λ henceforth be set to $\mathbf{F}_p$.

Unfortunately, the numerical index $*$ sometimes grows too large in this paper to be conveniently written as a subscript. In these cases, the notation $A(*, X)$ will substitute for A_*X.

All schemes (or algebraic spaces) will be assumed to be over a field k whose spectrum will be written pt. And Sp_k will denote the category of algebraic spaces over k.

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2. Equivariant Intersection Theory I

In this section, the relevant portion of the Edidin-Graham Equivariant Intersection Theory is reviewed.

For any two algebraic spaces X and Y with the action of an algebraic group G, $X \times^G Y$ will denote the quotient of $X \times Y$ under the diagonal G-action — under the assumption that the quotient exists. The advantage of working with algebraic spaces is that, if the action of G on $X \times Y$ is set theoretically free, a quotient will always exist as an algebraic space. In [1], this fact is proved and it is shown that the intersection theory of Fulton and MacPherson [2] extends to the category of algebraic spaces unchanged. The basic reference for algebraic spaces is [5].

Let G be a g-dimensional algebraic group and let X be an n dimensional algebraic space. Edidin and Graham define equivariant Chow groups using an algebraic analogue of the Borel construction. Let V be an l dimensional linear representation of G, and let S be a Zariski closed subset of V outside of which G acts freely. Let $U = V - S$. The theory is based on the following:

Definition-Proposition 2.1 ([1]). *Assume that* $\operatorname{codim}_S V > \dim X - i$. *Then the group* $A(i + l - g, U \times^G X)$ *is independent of both* U *and* V. *This group, written* $A_i^G X$, *is called the* i*-th equivariant Chow group of* X.

U will be called an EG-approximation, and its quotient U/G will be called a BG-approximation. Note that the possibility exists for non-trivial Chow groups in negative dimensions. But clearly $A_i^G X = 0$ for $i > \dim X$.

$A_i^G X$ can be viewed as Chow homology. Edidin and Graham define Chow cohomology groups $A_G^i X$ as operational Chow groups (see [1] 2.6). Under their definition, there is an obvious product $A_G^i X \otimes A_G^j X \to A_G^{i+j} X$ giving $A_G^* X$ a ring structure. And, if X is smooth of pure dimension d, $A_G^i X = A_{d-i}^G X$. $A_*^G X$ is naturally an A_G^*X module. Moreover, any G-equivariant map $Y \to X$ induces a ring homomorphism $A_G^*X \to A_G^*Y$ making A_*^GY into an A_G^*X-module.

The Edidin-Graham theory builds on Totaro's definition of the Chow group of the classifying space BG of an algebraic group [8]. Totaro sets $CH^iBG = A^i\,(U/G)$. Since pt is smooth of dimension 0, we have $CH^iBG = A_{-i}^G\mathrm{pt} = \mathrm{A}^{\mathrm{i}}_{\mathrm{G}}\mathrm{pt}$. We will also write A^iBG for this group.

Following Edidin and Graham, we write X_G for $U \times^G X$ with U left unspecified. These spaces are sometimes called mixed quotients. When A_iX_G is written, it can be assumed that $\operatorname{codim}_S V > \dim X - i$. Similarly, if $f : X \to Y$ is a G-equivariant morphism, $f_G : X_G \to Y_G$ will denote the evident morphism of mixed quotients. The key to many of the functorial properties of equivariant intersection theory is the following

Proposition 2.2 ([1] Proposition 2)**.** *Let* **P** *be one of the following properties of morphisms: proper, flat, smooth, regular embedding or l.c.i. Then, if $X \to Y$ is a G-morphism having* **P**, *so is $X_G \to Y_G$.*

The proposition is used to show that equivariant Chow groups have the same functorial properties for **P** morphisms as do Fulton's classical Chow groups [2]. In particular, flat pull-back, l.c.i pull-back, proper push-forward and the refined Gysin homomorphism are all defined in the equivariant context.

Similarly, Edidin and Graham show that if a morphism $E \to X$ is a vector bundle, so is $E_G \to X_G$. This allows one to define equivariant Chern classes $c_i^G(E) \in A_G^iX$. It is particularly important that representations of G over k are equivariant vector bundles over pt, and thus have Chern classes in A^*BG.

We will need certain results about change of groups that are standard in the literature of equivariant cohomology but, to my knowledge, not fully covered in the literature of equivariant Chow theory.

2.3. Let $\rho : H \to G$ be a group homomorphism and suppose that G acts on X. There is a restriction morphism $\rho_H^G : A_*^GX \to A_*^HX$. If G and H are finite and ρ is an injection, then there is also transfer morphism $\mathrm{tr}_H^G : A_*^HX \to A_*^GX$. These are both functorial (e.g. $\rho_K^H \circ \rho_H^G = \rho_K^G$). And they commute with the pull-backs and push-forwards of equivariant intersection theory: For example, if $f : X \to Y$ is a morphism of G-spaces and if

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

is a pull-back diagram, then the following diagram

$$\begin{array}{ccc} A_*^HY' & \xrightarrow{f^!} & A_*^HX' \\ \downarrow{\scriptstyle \mathrm{tr}_H^G} & & \downarrow{\scriptstyle \mathrm{tr}_H^G} \\ A_*^GY' & \xrightarrow{f^!} & A_*^GX' \end{array}$$

commutes.

2.4. If $\rho : G \to G$ is an inner automorphism, then $\rho^* = \mathrm{id}$.

2.5. If V is a G-equivariant vector bundle, let $\rho^G_H V$ be the vector bundle obtained by restriction of the G-action to H. Let $\alpha \in A^G_* X$. Then $\rho^G_H(\alpha \cap c^G_i(V)) = (\rho^G_H \alpha) \cap c^H_i(\rho^G_H V)$.

The (somewhat tedious) verification of the two proceeding paragraphs will be sketched in the appendix using a definition of equivariant Chow groups which is equivalent to Definition 2.1 but more convenient for questions of functoriality.

3. Equivariant Cycle Class

In this section, a few general results on Chow groups necessary for the construction of $S_\bullet$ are collected. There are two main goals: (i) to explain why equivariant cycles give classes in equivariant Chow groups (this fact is mentioned in the proof of [1] Proposition 1), and (ii) to show that two equivariant cycles are equivalent if they belong to the same equivariant rational family. As we will only require the theory of cycles equivariant under a finite group action for the construction of Steenrod operations, we will restrict to this case.

Definition 3.1. The group $Z_k(X|\mathbf{P}^1;\Lambda)$ is the subgroup of (k+1)-dimensional cycles $\sum_i \lambda_i[V_i]$ in $X \times \mathbf{P}^1$ such that all V_i map dominantly to $\mathbf{P}^1$.

Following the Conventions of (1.1), the ring Λ will be suppressed in the notation, and the group of Λ- cycles $Z_k(X;\Lambda) = Z_k X \otimes \Lambda$ will be written as $Z_k X$.

Following the notation in [2], for $P \in \mathbf{P}^1$ and V a variety mapping dominantly to $\mathbf{P}^1$, let $V(P)$ be the fiber above P. The definition is extended to $Z_*(X|\mathbf{P}^1)$ by linearity.

Both $Z_k(\_)$ and $Z_k(\_|\mathbf{P}^1)$ can be viewed as presheaves in the étale topology. That $Z_k(\_)$ is actually a sheaf in the étale topology is proved in [3] for $\Lambda = \mathbf{Z}$. The argument is easily extended to the case of arbitrary ring Λ. It is also easily extended to show that $Z_k(\_|\mathbf{P}^1)$ is sheaf.

Proposition 3.2. *Two cycles α_0 and α_∞ in $Z_k X$ are rationally equivalent $\Leftrightarrow$ there is a cycle $\beta \in Z_k(X|\mathbf{P}^1)$ with $\beta(0) = \alpha_0$ and $\beta(\infty) = \alpha_\infty$.*

Proof. First consider the case $\Lambda = \mathbf{Z}$. By [2] Example 1.6.2, α_0 and α_∞ are equivalent if and only if there is a positive cycle $Z \in Z_k(X|\mathbf{P}^1)$ and a positive cycle γ with

$$Z(0) = \alpha_0 + \gamma \qquad Z(\infty) = \alpha_\infty + \gamma.$$

To prove ($\Rightarrow$), set $\beta = Z - \gamma \times [\mathbf{P}^1]$. To prove ($\Leftarrow$), write β as $\beta^+ - \beta^-$ where each term either positive or 0. Then, setting $Z = \beta^+$, shows that $\alpha_0 + \beta^-(0) = \alpha_\infty + \beta^-(\infty)$. This shows that α_0 and α_∞ are equivalent.

For arbitrary Λ, the proposition is equivalent to the statement that

$$Z_k(X|\mathbf{P}^1) \to Z_k X \to A_k X \to 0$$

is exact — with the first map given by $\beta \mapsto \beta(\infty) - \beta(0)$. Thus, since it holds for $\Lambda = \mathbf{Z}$, it holds for arbitrary Λ by the right exactness of $\otimes$. □

Suppose a finite group G acts on X through the action $a : G \times X \to X$. Let $p_2 : G \times X \to X$ be the projection. Note that both p_2 and the action are automatically flat morphisms.

3.3. If F is a presheaf of abelian groups in the étale topology, the fixed sections of F are defined as

$$F^G = \ker(F(X) \stackrel{a^* - p_2^*}{\to} F(G \times X)).$$

In particular, the groups $Z_*(X)^G$ and $Z_*(X|\mathbf{P}^1)^G$ are defined. When G is an finite algebraic group coming from a finite group, acting properly on X, these notions agree with the usual ones (e.g. [2] Example 1.7.6). This is in fact the only case relevant in what follows.

Now assume that G acts freely on X and has a quotient Y. If G is finite and étale, the fact that $Z(\_)^G$ and $Z_*(\_|\mathbf{P}^1)$ are étale sheaves implies that the pull-back $\pi : X \to Y$ induces isomorphisms $Z_*(Y) \xrightarrow{\sim} Z_*(X)^G$ and $Z_*(Y|\mathbf{P}^1) \xrightarrow{\sim} Z_*(X|\mathbf{P}^1)^G$. By abuse notation, elements of these two groups will be identified.

Proposition 3.4. *Let G act freely on X. Two cycles α_0 and α_∞ give the same class in $A_*(X/G)$ if there is a cycle $\beta \in \mathbf{Z}_*(X|\mathbf{P}^1)^G$ such that $\beta(0) = \alpha_0$ and $\beta(\infty) = \alpha_\infty$*

Proof. This is simply a matter of noticing that the diagram

$$\begin{array}{ccc} Z_k(Y|\mathbf{P}^1) & \xrightarrow{\pi^*} & Z_k(X|\mathbf{P}^1) \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} \\ Z_kY & \xrightarrow{\pi^*} & Z_kX \end{array}$$

commutes where σ is either $\beta \mapsto \beta(0)$ or $\beta \mapsto \beta(\infty)$. □

3.5. Let G now act properly, but not necessarily freely, on X. Let α be a cycle in $(Z_kX)^G$ (resp. $Z_k(X|\mathbf{P}^1)^G$). For any EG-model U of dimension d, $\alpha \mapsto [U] \times \alpha$ gives a map $(Z_kX)^G \to Z_{k+d}(U \times X)^G$ (resp. $Z_k(X|\mathbf{P}^1)^G \to Z_{k+d}(U \times X|\mathbf{P}^1)^G$). But, since the action of G on $U \times X$ is free, $Z_*(U \times X)^G = Z_*(U \times^G X)$ (resp. $Z_*(U \times X|\mathbf{P}^1)^G = Z_*(U \times^G X|\mathbf{P}^1)$). Thus, to every $\alpha \in Z_k(X)^G$, there is a G-equivariant cycle class $\mathrm{cl^G}(\alpha) \in \mathrm{A_k^GX}$.

3.6. It follows from Proposition 3.4, that if $\beta \in Z_k(X|\mathbf{P}^1)^G$, then $[U] \times \beta(0)$ and $[U]\times\beta(\infty)$ represent the same class in $A_{k+d}(X\times^G U)$ and thus in $A_k^G X$. This implies that two equivariant cycles α_0 and α_∞ in $(Z_kX)^G$ have the same G-equivariant cycle class if there is a cycle $\beta \in Z_k(X|\mathbf{P}^1)^G$ with $\beta(0) = \alpha_0$ and $\beta(\infty) = \alpha_\infty$. In other words, equivariant rational equivalences induce equivalences in $A_*^G X$.

The proof of the following will be postponed until the appendix where transfer will be discussed. We use the symbol 1 to denote the trivial group.

Proposition 3.7. *Let $Z \in Z_kX$.* $\mathrm{tr}_1^G Z = \mathrm{cl^G}(\sum_{\mathrm{g\in G}}[\mathrm{gZ}])$

4. The Fundamental Operation

Let $S(n)$ denote the symmetric group on n letters and $C(n)$ denote the cyclic group with n elements viewed as a subgroup in the obvious way. Let $\tilde{R}$ denote the regular $S(n)$-representations on $\mathbf{C}^n$ and let R denote the reduced regular representation, i.e., the cokernel of the map $\mathbf{1} \to \tilde{R}$ from the trivial representation. One can view these representations as bundles over $BS(n)$.

For a cycle $\alpha \in Z_kX$ let $\alpha^{\times n} \in Z_{nk}X^n$ denote its n-fold exterior product with itself. ([2] 1.10). As $\alpha^{\times n}$ is $S(n)$ invariant, it gives a class $[\alpha^{\times n}] \in Z_{nk}^G X^n$ for any $G \leq S(n)$.

4.1. If $[V] \in Z_i(X|\mathbf{P}^1)$ and $[W] \in Z_j(Y|\mathbf{P}^1)$ are two classes corresponding to subvarieties V and W, then V and W are both flat over $\mathbf{P}^1$. This implies that

$V \times_{\mathbf{P}^1} W$ is flat over $\mathbf{P}^1$. Thus every component of $V \times_{\mathbf{P}^1} W$ maps dominantly to $\mathbf{P}^1$. Consequently $[V \times_{\mathbf{P}^1} W] \in Z_{i+j}(X \times Y|\mathbf{P}^1)$. And linearity gives a product

$$Z_i(X|\mathbf{P}^1) \otimes Z_j(Y|\mathbf{P}^1) \to Z_{i+j}(X \times Y|\mathbf{P}^1)$$

written $\beta \otimes \gamma \mapsto \beta \times_{\mathbf{P}^1} \gamma$. In particular, any $\beta \in Z_k(X|\mathbf{P}^1)$ can be crossed with itself to obtain a class $\beta_{\mathbf{P}^1}^{\times n} \in Z_{nk}(X^n|\mathbf{P}^1)$.

Proposition 4.2. *The map $\alpha \mapsto \alpha^{\times n}$ factors through rational equivalence to give a map $P_G^n : A_k X \to A_{nk}^G X^n$ for any $G \leq S(n)$.*

Proof. Two cycles α_0 and α_∞ are equivalent iff there a cycle $\beta \in \mathbf{Z}_k(X, \mathbf{P}^1)$ with $\beta(0) = \alpha_0$ and $\beta_\infty = \alpha_\infty$. In this case, $\beta_{\mathbf{P}^1}^{\times n}$ is a cycle in $Z_{nk}(X^n, \mathbf{P}^1)$, with $\beta_{\mathbf{P}^1}^{\times n}(0) = \alpha_0^{\times n}$ and $\beta_{\mathbf{P}^1}^{\times n}(\infty) = \alpha_\infty^{\times n}$. □

Proposition 4.3. *Consider a morphism $f : X \to Y$ and a cycle $\alpha \in A_k Y$. Let $f_G^{\times n}$ denote the obvious G-equivariant morphism from X^n to Y^n.*

(i) *For f proper, $f_G^{\times n}$ is also proper, and $(f_G^{\times n})_* P_G^n(\alpha) = P_G^n(f_* \alpha)$.*
(ii) *For f is flat, $f_G^{\times n}$ is also flat, and $(f_G^{\times n})^* P_G^n(\alpha) = P_G^n(f^* \alpha)$.*
(iii) *For f a regular embedding (resp. l.c.i. morphism), $f_G^{\times n}$ is also a regular embedding (resp. lci. morphism), and $(f_G^{\times n})^! P_G^n(\alpha) = P_G^n(f^! \alpha)$. (Here $f^!$ is Fulton's refined Gysin homomorphism* [2]*).*

Proof. (i) is easy as it is true on the level of cycles, that is, in $Z_{nk}(X^n)$. (ii): If f is flat then it is easy to see that $f^{\times n}$ is also. Thus $f_G^{\times n}$ is flat by Proposition 2.2. It is easy to see that the required commutativity actually holds on the level of cycles.

Once (iii) is proved for regular embedding the statement for l.c.i. morphisms will be a consequence of (ii). So assume f is a regular embedding of codimension d. Let N be the normal bundle, and let

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

be a pull-back diagram. Let $N' = N_{|X'}$.

The refined Gysin of [2], $f^! : A_k Y' \to A_{k-d} X$, is constructed as the composition of the specialization homomorphism $\sigma : A_k Y' \to A_k N'$ with the isomorphism $(\pi^*)^{-1} : A_k N' \to A_{k-d} X'$.

σ is defined on the level of cycles by the rule $\sigma[V] = [C_{V \cap X} V]$. (Here $[C_{V \cap X} V]$ is the normal cone of $V \cap X$ in V — a subscheme of N'). One then sees that σ commutes with P_G^n on the level of cycles, i.e., the diagram

$$\begin{array}{ccc} Z_k Y' & \longrightarrow & Z_{nk} Y'^n \\ \downarrow & & \downarrow \\ Z_k N' & \longrightarrow & Z_{nk} N'^n \end{array}$$

commutes.

On the other hand, the isomorphism $\pi^* : A_{k-d}X' \to A_k N'$ is simply a flat pull-back. Thus the diagram

$$\begin{array}{ccc} A_{k-d}X' & \longrightarrow & A_{n(k-d)}X'^n \\ \downarrow & & \downarrow \\ A_k N' & \longrightarrow & A_{nk}N'^n \end{array}$$

also commutes by (ii). The result then follows by splicing the two diagrams together. □

If $V \to X$ is a vector bundle, $P^n_G(V)$ is the product of n-copies of V over n copies of X as a G-equivariant bundle.

Corollary 4.4. *Let $\alpha \in A_*X$ and let $e(V)$ represent the Euler class of V. Then $P^n_G(\alpha \cap e(V)) = P^n_G(\alpha) \cap e(P^n_G V)$.*

Proof. Let $i : X \to V$ be the inclusion of the 0-section. $\alpha \cap e(V) = i^* i_* \alpha$. The corollary then follows from the successive application of (ii) and (iii) of the previous proposition. □

5. Construction

Consider a pair (W, X) where $X \to W$ is an embedding of X into a smooth connected algebraic space W of dimension d. Set $m = (n-1)d$, and form the $S(n)$-equivariant pull-back diagram

$$\begin{array}{ccc} X & \longrightarrow & X^n \\ \downarrow & & \downarrow \\ W & \xrightarrow{\Delta} & W^n \end{array}$$

Definition 5.1. For $G \leq S(n)$, $d^W_G : A^G_k X^n \to A^G_{k-m} X$ is the map given by $d^W_G(\alpha) = \Delta^!_G \alpha$. We will write D^W_G for $d^W_G \circ P^n_G$.

Remark 5.2. One way to keep track of the degrees in the definition is to reindex the groups. For a pair (W, X), define $A^k_G[W, X] = A^G_{d-k} X$. Then the re-indexed map $d^W_G : A^k_G[W^n, X^n] \to A^k_G[W, X]$ preserves the degree, and D^W_G maps $A^k[W, X]$ to $A^{pk}_G[W, X]$. This is useful for proving the Adem relations but for most of this paper we prefer to keep the usual grading.

Proposition 5.3. *Let $X \to W_1 \to W_2$ be a sequence of embeddings where each W_i is smooth. Then*

$$d^{W_1}_G \cap e(R \otimes TW_2|_X) = d^{W_2}_G \cap e(R \otimes TW_1|_X)$$

Proof. Consider the commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X \\ \downarrow{\scriptstyle j} & & \downarrow \\ W_1 & \xrightarrow{\Delta_1} & W_1^n \\ \downarrow{\scriptstyle g} & & \downarrow \\ W_2 & \xrightarrow{\Delta_2} & W_2^n. \end{array}$$

The equivariant normal bundle N_i for W_i in W_i^p is $R \otimes TW_i$. Set $E = g^*N_2/N_1$ — the excess normal bundle. By the Excess Intersection Formula ([2] Theorem 6.3),

$$\begin{aligned} \Delta_2^!(\alpha) &= e(E|_X) \cap \Delta_1^!(\alpha) \\ &= e((g^*N_2/N_1)|_X) \cap \Delta_1^!(\alpha) \end{aligned}$$

Multiplying both sides by $e(N_1|_X)$ the result follows from the multiplicative property of the Euler class. □

Proposition 5.4. *Let*

$$\begin{array}{ccc} U & \xrightarrow{j} & X \\ \downarrow & & \downarrow \\ V & \longrightarrow & W \end{array}$$

be a commutative diagram of injective maps with the horizontal maps open embeddings and with W (and thus V) smooth.

(i) *For $\alpha \in \mathbf{A}_*^G X^n$, $j^*(d_G^W \alpha) = d_G^V((j^{\times n}) * \alpha)$.*
(ii) *For $\alpha \in A_* X$, $j^*(D_G^W \alpha) = D_G^V(j^* \alpha)$.*

Proof. (i) implies (ii) directly. In the case $V = W$, (i) follows from Theorem 6.2 (a) of [2] — the commutativity of flat pull-back with the refined Gysin. The general case then follows from Theorem 6.3 of [2] (the Excess Intersection Formula). □

6. Chow Theory of Cyclic Groups

Let n be an integer not divisible by char (k), and let μ_n be the group of n-th roots of unity. The canonical one-dimensional representation of μ_n gives rise to an equivariant line bundle L over $\mathrm{pt_k}$. Set $l = c_1(L) \in A^1 B\mu_n$. The following result is given in [8] for $k = \mathbf{C}$.

Theorem 6.1. *(i) $A^* B\mu_n = \Lambda[l]/(nl)$. (ii) If X is an algebraic space with trivial μ_n action, then $A_*^{\mu_n} X = A_* X \otimes A^* B\mu_n$.*

Proof. Let $E = \mathbf{A}^{r+1} - \{0\}$ with the diagonal action of μ_n. Write $B = E/\mu_n$, and note that $\mathbf{P}^r = E/\mathbf{G}_m$. Now let $\mathbf{G}_m$ act on $\mathbf{A}^1 \times E$ on the right by the formula: $(a, x)\lambda = (\lambda^n a, \lambda x)$. The resulting quotient $(\mathbf{A}^1 \times E)/\mathbf{G}_m$ is then the total space of the line bundle $\mathcal{O}(n)$ on $\mathbf{P}^r$. B embeds as an open subset of $\mathcal{O}(n)$ via the map $j : e \mapsto (1, e)$. The complement of $j(B)$ is the 0-section which we will simply write as $\mathbf{P}^r$. For any X we then have an exact sequence of groups

$$A_i(\mathbf{P}^r \times X) \to A_i(\mathcal{O}(n) \times X) \to A_i(B \times X) \to 0.$$

For $X = \mathrm{pt}$ the theorem follows from the elementary computation of the map on the left hand side: Both of the groups are Λ for $i \in [0, r]$ and the maps are multiplication by n. The generating class $h \in A_r\mathcal{O}(n)$ is the first Chern class of the pull-back of the canonical line bundle from $\mathbf{P}^r$. Since the representation of $\mathbf{G}_m$ restricts to the canonical representation of μ_n, h restricted to B is l. This proves (i).

For X arbitrary, there are Kunneth formulae $A_*(\mathbf{P}^r \times X) = A_*\mathbf{P}^r \otimes A_* X$ and $A_*(\mathcal{O}(n) \times X) = A_*\mathcal{O}(n) \otimes A_* X$. (ii) follows from (i) and the application of these formulae. □

6.2. We now make two assumptions that will be in force for the rest of the paper. Pick a prime p not equal to the characteristic of k, and set $\Lambda = \mathbf{F}_p$. Chow groups written without explicit coefficients will thus be taken modulo p. Now, if $k^\times$ contains the group $\mu_p(\overline{k})$ then $C(p) \cong \mu_p$. But of course the two groups are not naturally isomorphic. The isomorphisms themselves are in correspondence with the primitive p-th roots of unity. Therefore let us assume that such a primitive root ζ as been chosen in $\overline{k}$. We can then write $A^*BC(p)_{\overline{k}} = \mathbf{F}_p[l]$.

Remark 6.3. Note that even without choosing ζ there is a natural correspondence between one-dimensional representations of $C(p)$ and μ_p. The first Chern class then gives a correspondence between μ_p and $A^1BC(p)$. In other words, over a field containing the p-th roots of unity, $A^1BC(p) = \mathbf{Z}/p(1)$. This implies that $A^kBC(p) = \mathbf{Z}/p(k)$.

Proposition 6.4. *Let $r = [k(\mu_p) : k]$ be the index of the extension obtained by adjoining the p-th roots of unity to k. Then $A^*BC(p) \cong \mathbf{F}_p[\epsilon]$ where ϵ is the Euler class of an r-dimensional $C(p)$ representation.*

Proof. Let $M = k(\mu_p)$ and set $G = \mathrm{Gal}(M/k)$. The action of G on μ_p gives an identification of G with a subgroup of $(\mathbf{Z}/p)^\times$.

The representation R splits over M into a direct sum $\oplus_{i=1}^{p-1} L^{\otimes i}$ — here L is the one-dimensional representation associated to ζ. Under the natural identification of $A^1BC(p)_L$ with μ_p, $c_1(L^{\otimes i}) = \zeta^i$. Let S be a coset of $(\mathbf{Z}/p)^\times$ modulo G. The r-dimensional representation $V = \otimes_{i\in S} L^{\otimes i}$ is defined over k as it is fixed by G. All of the Chern classes of V_S vanish except for the Euler class which is nontrivial and thus a generator of $A^1BC(p)$. We claim that $\epsilon = e(V)$ generates $A^*BC(p)$.

Let $\rho : A^*BC(p)_k \to A^*BC(p)_L$ be the pull-back, and let $N : A^*BC(p)_L \to A^*BC(p)_k$ be the norm map. $N \circ \rho$ is simply multiplication by r. As r is relatively prime to p, ρ is a split injection. We will henceforth abuse this injection and consider $A^*BC(p)$ as a subgroup of $A^*BC(p)_L$. Clearly ϵ maps to some non-zero multiple of l^r. Thus to prove the proposition it is enough to show that $A^*BC(p) \subset F_p[l^r]$.

Now, under the identification of G with $(\mathbf{Z}/p)^\times$, the action of an element α on $l^i \in A^iBC(p)_L$ is simply multiplication by α^i. It follows that $(A^*BC(p)_L)^G = \mathbf{F}_p[l^r]$. Thus $A^*BC(p) \subset F_p[l^r]$. $\square$

Remark 6.5. If X is a space on which $C(p)$ acts trivially, $A_*^{C(p)}X$ is a direct summand of $A_*^{C(p)}X_L$. This follows from the same norm argument used in the proof of the proposition.

Let V be a vector bundle on X and assume that both V and X have trivial $C(p)$ actions. Let $\lambda_1, \cdots, \lambda_d$ be the Chern roots of V — let $c_*(V) = \prod(1 + \lambda_i)$. (We will employ the standard abuses of the splitting principle here and in the sequel.) Set $w(V) = \displaystyle\prod_{i=1}^{d}(1 + \lambda_i^{p-1})$. This is corresponds to the class w of the introduction. In order to keep track of the degrees, we also consider the polynomial $w(V,t) = \prod_{i=1}^{d}(1 + t\lambda_i^{p-1})$.

It is convenient to localize the Chow ring $A^*BC(p)$ by inverting the element ϵ. Note that, as ϵ is a non-zero divisor, $A^*BC(p)$ injects into $(A^*B\mu_p)_\epsilon$. Under the standing assumption that X has trivial $C(p)$ action, ϵ is also a non-zero divisor on $A_*^{C(p)}X$. Let η be the Euler class $e(R)$. By Wilson's Theorem, it follows that

$\eta = -l^{p-1}$. Note that localizing by η always has the same effect as localizing by ϵ. As it is generally harmless, we will sometimes write a formula in the localized groups without mentioning explicitly which group we are working in.

Proposition 6.6. *With V and X be as above, $e(R \otimes V) = \eta^d w(V, 1/\eta)$.*

Proof. The Chern roots of R are simply $l, 2l, \dots, (p-1)l$. Therefore,

$$\begin{aligned} e(R \otimes V) &= \prod_{i=1}^{d} \prod_{a=1}^{p-1} (\lambda_i + al) \\ &= \prod_{i=1}^{d} (\lambda_i^{p-1} - l^{p-1}) \\ &= \prod_{i=1}^{d} (\lambda_i^{p-1} + e(R)) \\ &= e(R)^d w(V, e(R)^{-1}). \end{aligned}$$

□

In the sequel, let $e(R \otimes V)(t)$ be the (unique) polynomial such that $e(R \otimes V)(\eta) = e(R \otimes V)$.

7. Definition and Basic Properties

In what follows, we will write C for $C(p)$ considered as a subgroup of $S = S(p)$ in the obvious way. We will also write D^W for D_C^W. Note that, if $C \leq G \leq S(p)$, the restriction map $A_*^G X \to \mathbf{A}_*^C X$ is a split injection. (The splitting is given up to a factor of $[G : C]$ by the transfer). Thus to compute D_G for all such G it suffices to compute D^W.

We first show that D^W is a group homomorphism, beginning with the Chow group analogue of a lemma of Steenrod. Of course, we keep the standing assumption that X is a space with trivial C action.

Lemma 7.1. *Let $T : A_*^C X^p \to A_* X^p$ be the transfer. Then $\Delta_C^! T = 0$.*

Proof. There is a commutative diagram

$$\begin{array}{ccccc} & & A_* X^p & \xrightarrow{T} & A_*^C X^p \\ & & \downarrow{\scriptstyle \Delta^!} & & \downarrow{\scriptstyle \Delta_C^!} \\ A_*^C X & \xrightarrow{\rho^*} & A_* X & \xrightarrow{T} & A_*^C X \end{array}$$

It follows from the computation of $A_*^C X$ that ρ is surjective. However $T\rho^* = 0$ (because it is multiplication by p). Therefore the T on the bottom row is 0. The result then follows from the commutativity of the diagram. □

Theorem 7.2. *D^W is a group homomorphism.*

Proof. Let Z_0 and Z_1 be two cycles.

$$(Z_0 + Z_1)^{\times p} = Z_0^{\times p} + Z_1^{\times p} + \Gamma$$

where Γ lies in the image of T (by Proposition 3.7). Therefore

$$\Delta^!_C((Z_0 + Z_1)^{\times p}) = \Delta^!_C(Z_0^{\times p}) + \Delta^!_C(Z_1^{\times p})$$

by the lemma. The theorem then follows from the definition of D^W. □

Now we can view $D^W(\alpha)$ as a polynomial $\sum b_i l^i$ in the variable l with coefficients in A_*X. We can do this even if k does not contain all of the p-th roots of unity because $A^C_* X \subset A^C_* X_{\overline{k}}$. Following Steenrod, we have the following

Theorem 7.3. *All terms of $b_i l^i$ of $D^W(\alpha)$ with i not divisible by $p-1$ are* 0.

Proof. Let G be the normalizer of C in S. $G \cong C(p-1) \ltimes C$ with $C(p-1)$ acting through the identification $C(p-1) \cong (\mathbf{Z}/p)^*$. Thus $C(p-1)$ acts on A^*BC. Under the above identification, with $k \in (\mathbf{Z}/p)^*$, $k_*w = kw$. Thus $k_*w^i = k^i w^i$. This gives the action of $C(p-1)$ on A^iBG. Note that the action is only trivial if i is a multiple of $p-1$.

Now recall that $d^W_C = \Delta^!_C \alpha^{\times p}$ is the restriction of $\Delta^!_G \alpha^{\times p}$. Therefore $C(p-1)$ must act trivially on $D^W_C(\alpha)$ (by the statement of paragraph 2.4 which is proved in Proposition 11.2). Hence, all terms $b_i w^i$ with i not a multiple of $p-1$ must be 0. □

7.4. Thus $D^W(\alpha)$ can be viewed as a polynomial in the Euler class η. To better keep track of degrees, we let $D^W(\alpha, t)$ be the polynomial in $A_*X \otimes \mathbf{F}_p[t]$ such that $D^W(\alpha, \eta) = D^W(\alpha)$.

Definition 7.5. The total Steenrod operation of the pair (W, X) is the Laurent series (with finitely many terms)

$$S^W_\bullet \alpha(t) = t^{d-k} D^W \alpha(1/t).$$

with $d = \dim W$ and $\alpha \in A_k X$. We simply write $S^W_\bullet \alpha$ for $S^W_\bullet \alpha(1)$. This is the total Steenrod operation. We define individual operations S^W_i by setting $S^W_\bullet \alpha(t) = \sum S^W_i(\alpha) t^i$. Note that S^W_i lowers the degree of α by $(p-1)i$.

Remark 7.6. To agree with the topological notation, we define $P^W_i \alpha = S^W_i \alpha$ for $p \neq 2$. For $p = 2$, we define $\mathrm{Sq}^W_{2i}\, \alpha = S^W_i$. The odd order operations obviously do not exist in the context of Chow groups (although they definitely do exist in the context of motivic cohomology) because the cycle class map sends Chow groups to even dimensional cohomology.

Remark 7.7. Since $D^W \alpha$ is a polynomial in t, $S^W_i \alpha = 0$ for $i > d - k$.

Remark 7.8. We can also formulate the definition for $\alpha \in A^k[W, X]$ by following through the reindexing. We write $S^\bullet_W \alpha(t) = t^k D^W \alpha(1/t)$ and set $S^\bullet_W \alpha(t) = \sum S^i_W t^i$. Then, of course, $S^i_W \alpha \in A^{k+(p-1)i}[W, X]$.

From Proposition 5.4, we have

Proposition 7.9. *If $j : U \to X$ is an inclusion of a Zariski open set, then* $j^*(S^W_\bullet \alpha) = S_\bullet(j^*\alpha)$.

Proposition 7.10. *If W_1 and W_2 are two smooth spaces containing X, then*

$$S^{W_1}_\bullet \alpha \cap w(TW_2) = S^{W_2}_\bullet \alpha \cap w(TW_1)$$

Proof. First assume that $W_1 \subset W_2$. Then the proposition follows by applying successively Proposition 5.3, Proposition 6.6, and the definitions. In general, we can reduce to the case of $W_1 \subset W_2$ by embedding both W_1 and W_2 in $W_1 \times W_2$ and using the fact that the $w(TW_i)$ are invertible in $A^*(X)$. □

Definition 7.11. If X is smooth, define $S^\bullet\alpha = S^X_\bullet\alpha$. It is natural to consider this as an operation on A^*X so that S^i raises degrees by $(p-1)i$. Of course, there is no difference between $S^\bullet$ and $S^\bullet_X$.

Remark 7.12. Suppose $f : X \to Y$ is a morphism of smooth varieties. Then from Proposition 4.3 and the functoriality of the Gysin it follows that $f^! D^Y(\alpha) = D^X(f^!\alpha)$.

Definition 7.13. For any X embedded in any smooth W, define

$$S_\bullet\alpha(t) = S^W_\bullet\alpha(t) \cap w(TW, t)^{-1}.$$

It follows from Proposition 7.10 that $S_\bullet\alpha$ is independent of W.

8. Functorialities

Let $i : X \to Y$ be a closed embedding with Y embedded in a smooth variety W. Then $D^W_G(i_*(\alpha)) = i_*(D^W_G(\alpha))$. This follows from [2], Theorem 6.2. Thus $S^W_\bullet(i_*\alpha) = i_*(S^W_\bullet\alpha)$. From the projection formula, it follows easily that $S_\bullet(i_*\alpha) = i_*(S_\bullet\alpha)$.

Lemma 8.1. *If $X \to U$ and $Y \to V$ are embeddings of X and Y into smooth varieties, then*

$$D^{U\times V}(\alpha \times \beta) = D^U\alpha \times D^V\beta.$$

Proof. This follows directly from Example 6.5.2 of [2]. □

Theorem 8.2 (Cartan Formula). *If $\alpha \in A_*X$ and $\beta \in A_*Y$ then*

$$\begin{aligned} S^{U\times V}_\bullet(\alpha \times \beta) &= S^U_\bullet\alpha \times S^V_\bullet\beta, \\ S_\bullet(\alpha \times \beta) &= S_\bullet\alpha \times S_\bullet\beta. \end{aligned}$$

Proof. The first equation follows directly from the lemma. The second is a consequence of the multiplicative property of w — $w(T(U \times V)) = w(TU) \cup w(TV)$. □

Applying Remark 7.12 to the embedding $X \to X \times X$, we have the following theorem as a corollary.

Theorem 8.3 (Cartan Formula). *For X smooth, $D^X(\alpha \cup \beta) = D^X(\alpha) \cup D^X(\beta)$, and $S^\bullet(\alpha \cup \beta) = S^\bullet\alpha \cup S^\bullet\beta$.*

Proposition 8.4. *Let X be a smooth space contained in a smooth space W with normal bundle $N = (TW_{|X})/TX$. Let $\alpha \in A^kX$.*

(i) $D^X_G[X] = [X]$ *for any group* $G \leq S(n)$.
(ii) $S^\bullet[X] = [X]$.
(iii) $S^W_\bullet[X] = [X] \cap w(N)$ *and* $S_\bullet[X] = [X] \cap w(TX)^{-1}$.
(iv) $S^i\alpha = \begin{cases} \alpha^p & : \ i = k \\ 0 & : \ i > k. \end{cases}$

Proof. (i) is a direct consequence of the following fact: If $j : A \to B$ is a regular embedding of smooth varieties, then $j^![B] = A$. (i)⇒(ii) by the definition of $S^\bullet$ and (ii)⇒ (iii) by Proposition 7.10.

The second line of (iv) follows from Remark 7.7. For the first line, note that $S^k\alpha$ is simply the constant coefficient in $D^X(\alpha, t)$, that is, $S^k\alpha = D^X_C(\alpha, 0)$ where C is the cyclic group with p elements. From the functoriality of D^X_G as a functor of the group G under the restriction map, it follows that $D^X_C(\alpha, 0) = D^X_{\{1\}}(\alpha)$. But $D^X_{\{1\}}(\alpha) = \alpha^p$. □

Remark 8.5. A consequence of (iv) of the above is that, if $\alpha \in A^1X$ and X is smooth, $S^\bullet(\alpha) = \alpha + \alpha^p$. This implies that $D^X(\alpha) = \alpha(\eta + \alpha^{p-1})$.

Corollary 8.6. *For any X embedded in a smooth space W of dimension d and $\alpha \in A_kX$, $S^W_i\alpha = 0$ for $i \notin [0, d-k]$ and S^W_0 is the identity.*

Proof. That $S^W_i\alpha$ vanishes for $i > d - k$ is Remark 7.7. To prove the rest of the corollary, first note that by linearity it suffices to consider the case $\alpha = [V]$ for V an irreducible subspace. Let V_{sing} be the singular locus of V and let $W_{\text{sm}} = W - V_{\text{sing}}$ (resp. $V_{\text{sm}} = V - V_{\text{sing}}$, $X_{\text{sm}} = X - X_{\text{sing}}$).

The smoothness of V_{sm} and Proposition 8.4 together imply that the corollary holds for $S^{W_{\text{sm}}}_i[V_{\text{sm}}]$ considered as an element of A_*V_{sm}. Then the covariant funtoriality of $S^{W_{\text{sm}}}_\bullet$ shows that the corollary holds for $S^{W_{\text{sm}}}_i[V_{\text{sm}}]$ considered to be in A_*X_{sm}.

Let $j : X_{\text{sm}} \to X$ be the inclusion. Then by Proposition 5.4, $S^{W_{\text{sm}}}_i[V_{\text{sm}}] = j^*(S^W_i[V])$. The result then follows from the fact that, for $r \geq k$, $A_rX = A_rX_{\text{sm}}$. □

Lemma 8.7. *Let V be a vector bundle on X of rank r and let $\alpha \in A_*X$. $S^W_\bullet(\alpha \cap e(V)) = S^W_\bullet(\alpha) \cap e(V)w(V)$.*

Proof. By Corollary 4.4, $P(\alpha\cap e(V)) = P(\alpha)\cap(P(V))$. It follows from the compatibility of the refined Gysin with pull-back of vector bundles ([2] Proposition 6.3) that $D^W(\alpha\cap e(V)) = D^W(\alpha)\cap e(V\otimes\tilde{R})$. But, now, $e(V\otimes\tilde{R}) = e(V)e(V\otimes R)$, and this is just $e(V)\eta^r w(V, 1/\eta)$. By the definition of $S^W_\bullet$, we thus have $S^W_\bullet(\alpha\cap e(V))(t) = S^W_\bullet(\alpha) \cap e(V)w(V, t)$ as desired. □

Remark 8.8. For a line bundle L with $c = c_1(L)$, this reduces to the statement that $S^W_\bullet(\alpha \cap c) = S^W_\bullet(\alpha) \cap (c + c^p)$.

Lemma 8.9. $\binom{-1-(p-1)k}{k} = 0$ *unless* $k = 0$.

Proof. $\binom{-1-(p-1)k}{k} = (-1)^k\binom{pk}{k}$. □

For a cycle $\alpha \in A_*X$, we write $[\alpha]_k$ for the component in degree k.

Lemma 8.10. $(S_\bullet[\mathbf{P}^n])_0 = 0$ *for* $n \neq 0$.

Proof. $S_\bullet[\mathbf{P}^n](t) = [\mathbf{P}^n] \cap w(T\mathbf{P}^n, t)^{-1}$. And $w(T\mathbf{P}^n, t) = (1 + th^{p-1})^{n+1}$. Thus, clearly, $(S_\bullet[\mathbf{P}^n])_0 = 0$ unless $n = (p-1)k$ for some integer k. And, in this case, $(S_\bullet[\mathbf{P}^n])_0 = \binom{-n-1}{k}$. The result then follows from Lemma 8.9. □

Proposition 8.11. *Suppose $f : X \to Y$ factors as a closed embedding $g : X \to \mathbf{P}^n \times Y$ followed by the projection $p_2 : \mathbf{P}^n \times Y \to Y$. Then $S_\bullet(f_*\alpha) = f_*(S_\bullet\alpha)$.*

Proof. Since we know that $S_\bullet$ is covariantly functorial for closed embeddings, we need only show that $S_\bullet$ commutes with p_{2*}. Let $\alpha \in A_k(Y \times \mathbf{P}^n)$. We can write $\alpha = \sum_{i+j=k} \beta_i \otimes [\mathbf{P}^j]$. Then $p_{2*}(\alpha) = \beta_k$. And using the Lemma and the Cartan formula, it is easy to see that $p_{2*}(S_\bullet \alpha) = S_\bullet \beta_k$. □

Corollary 8.12. *The definition of $S_\bullet$ agrees with the topological definition given in the introduction.*

Proof. Let $\pi : M \to X$ be a resolution of singularities, in particular, a projective map with M smooth. Then the functoriality of $S_\bullet$ for projective morphisms and the computation of Proposition 8.4 shows that $S_\bullet[X] = \pi_*([M] \cap w(TM)^{-1})$. □

Of course, the proof also shows that $\pi_*([M] \cap w(TM)^{-1})$ is independent of M.

Remark 8.13. We have defined $S_\bullet$ for a space X embedded in a smooth space W. In particular, $S_\bullet$ is defined for any quasi-projective X. Using Chow envelopes, one can define $S_\bullet$ on any scheme X and show that it is covariant for proper morphisms.

9. Adem Relations

In this section, we verify that the Adem relations hold for the S_i^W and the S_i. In order to unify the presentation, we use the reindexing of Remarks 5.2 and 7.8. Therefore, we will be considering throughout the groups $A^k[W, X]$ and the operations $S_W^\bullet = t^k D^W \alpha(1/t)$. We will also use the symbol ∞ to stand for an "infinite" W. In practice, this means that we will write $A^k[\infty, X] = A_{-k}X$. And we will write $S_\infty^\bullet(\alpha)$ for the reindexed $S_\bullet(\alpha)$. We will speak of W as being "finite", if it is not ∞. When X is smooth, D written without a superscript will mean D^X.

For W finite and $\alpha \in A^k[W, X]$, we have $D^W(\alpha) = \eta^k S_W^\bullet(1/\eta)$. We generalize this formula by defining $D^\infty = \eta^k S_\infty^\bullet(1/\eta)$. Note that, while $D^W(\alpha) \in A_C^*[W, X]$ for W finite, $D^\infty(\alpha)$ may only live in the localized module $A_C^*[W, X]_\eta$.

Now let C_1 and C_2 be two copies of the group $C(p)$ and suppose that E_i are EC_i models for $i = 1, 2$ over an algebraically closed field k. (In proving the Adem relations, we can work over an algebraically closed field without loosing generality). Let $B_i = E_i/C_i$ for $i = 1, 2$. We suppose that a large integer N has been chosen and that $A^k B_i = A^k BC_i$ for $k \leq N$. (N can be as large as we want). Let us write $A^* BC_i = \mathbf{F}[l_i]$. We will also write $\eta_i = -l_i^{p-1}$. Let R_i' be the subalgebra of $A^* BC_i$ generated by η_i. In general, we write R' for the subalgebra of $A^* BC(p)$ generated by η.

We want to compute the map $D : A^k B_1 \to A^* B_1 \otimes A^* BC_2$. By Remark 8.5, $D(l_1) = l_1(\eta_2 + l_1^{p-1}) = l_1(\eta_2 - \eta_1)$. As D is an algebra homomorphism, this computes it completely. However, we record the fact that $D(\eta_1) = \eta_1(\eta_2 - \eta_1)^{p-1}$. These two formulae hold for the finite varieties B_i. There is then an induced algebra map $D : A^* BC \to A^* BC \otimes A^* BC$ given by $D(l) = l \otimes 1(1 \otimes l - l \otimes 1)^{p-1}$.

Now for any space X embedded in a smooth (finite) space W there is a sequence

$$A^k[W, X] \to A^{pk}[W \times B_1, X \times B_1] \to A^{p^2 k}[W \times B_1 \times B_2, X \times B_1 \times B_2]$$

gotten by applying D^W and then $D^{W \times B_1}$ in succession. By taking $N > p^2 k$, this defines a map $D^2 : A^*[W, X] \to A^*[W, X] \otimes R' \otimes R'$. Let s be the automorphism of $R' \otimes R'$ which switches the factors, i.e., $s(l_1) = l_2$, $s(l_2) = l_1$.

Theorem 9.1. *For W finite, $D^2 = s(D^2)$.*

Sketch. Let C^2 be the product of C_1 and C_2 and let sw be the automorphism of C^2 which switches the factors. D^2 can be thought of as a map $D^2 : A^*[W,X] \to A^*_{C^2}[W,X]$. Then $\mathrm{sw}^*(D^2) = D^2$. This is shown in the topological context on pages 116 and 117 of [7]. The proof in the context of equivariant intersection theory uses the same reasoning and the same commutative diagrams. (And all of the necessary funtorial properties of equivariant intersection theory are at our disposal.) I will omit it.

What is left is to show that $\mathrm{sw}^* = s$. But this is clear from the way sw acts on one-dimensional representations. □

Let $m : R' \otimes R' \to R'$ be the multiplication map. Let $R = R'_\eta$. R inherits a grading from $A^*BC(p)$ so that η has degree $p-1$. Extend D, m, and s to R. Write $M = A^*[W,X]$ and write $D^M = D^W$. The Cartan formula shows us that D^2 can be commuted as the composition

$$M \xrightarrow{D^M} M \otimes R \xrightarrow{D^M \otimes D} M \otimes R \otimes R \otimes R \xrightarrow{1 \otimes s \otimes 1} M \otimes R \otimes R \otimes R \xrightarrow{1 \otimes 1 \otimes m} M \otimes R \otimes R. \tag{2}$$

9.2. Let ι be the involution on R taking η to $1/\eta$. Let M be any graded Abelian group with a map $D^M : M \to M \otimes R$ multiplying degrees by p. We write $D^M(\alpha)(1/\eta)$ for $(\mathrm{id} \otimes \iota) \circ D^M(\alpha)$. Define $S_M(\alpha) = \eta^k D^M(\alpha)(1/\eta)$ and write $S_M(\alpha) = \sum_i S^i_M(\alpha)\eta^i$. Define $D^2 : M \to M \otimes R \otimes R$ as in Equation 2.

The Adem relations will follow from the following

Theorem 9.3. *With the M as above, suppose that*

(i) *for any α, $S^i_M(\alpha) = 0$ for $i < 0$,*
(ii) *$s(D^2) = D^2$.*

Then the S^i_M satisfy the Adem relations. That is, for $0 < b < pc$,

$$S^b_M S^c_M = \sum_{i=0}^{[b/p]} (-1)^{b+i} \binom{(p-1)(c-i)-1}{b-pi} S^{b+c-i}_M S^i_M.$$

Proof. We will write u for the generator of the first factor of R and v for the other so that $R \otimes R = \mathbf{F}[u, v, u^{-1}, v^{-1}]$. In other words, we write u for η_1 and v for η_2. We will simply write S for S_M. Note that S^i raises degrees by $(p-1)i$.

We then compute

$$\begin{aligned}
D^2(\alpha) &= D(u^k S(\alpha)(1/u)) \\
&= D(u^k) \sum_{i \geq 0} S^i(\alpha) u^{-i} \\
&= \sum_{i \geq 0} D(S^i(\alpha)) D(u^{k-i}) \\
&= \sum_{i \geq 0} v^{k+(p-1)i} \sum_{j \geq 0} S^j S^i(\alpha) v^{-j} D(u^{k-i}) \\
&= \sum_{i,j \geq 0} S^j S^i(\alpha) v^{k+(p-1)i-j} u^{k-i} (v-u)^{(p-1)(k-i)}.
\end{aligned}$$

Thus under the assumption, we have that

$$\sum_{i,j \geq 0} S^j S^i(\alpha)(v-u)^{(p-1)(k-i)} [v^{k+(p-1)i-j} u^{k-i} - u^{k+(p-1)i} v^{k-i}] = 0.$$

We reduce this by dividing through by $(v-u)^{(p-1)k}v^k u^k$ to obtain the equation

$$\sum_{i,j\geq 0} S^j S^i(\alpha)(v-u)^{-(p-1)i}[v^{(p-1)i-j}u^{-i} - u^{(p-1)i-j}v^{-i}] = 0.$$

We then change variables, writing the equation in terms of $a = u/v$ and v, to obtain

$$\sum_{i,j\geq 0} S^j S^i(\alpha)v^{-i-j}(1-a)^{-(p-1)i}[a^{-i} - a^{pi-i-j}] = 0.$$

Thus, with $m = b + c$,

$$\sum_{i+j=m} S^j S^i(\alpha)(1-a)^{-(p-1)i}[a^{-i} - a^{pi-m}] = 0.$$

Multiplying through by a^m, we have

$$\sum_{i+j=m} S^j S^i(\alpha)(1-a)^{-(p-1)i}[a^j - a^{pi}] = 0.$$

In other words,

$$\sum_{i+j=m} S^j S^i(\alpha)(1-a)^{-(p-1)i}a^j = \sum_{i+j=m} S^j S^i(\alpha)(1-a)^{-(p-1)i}a^{pi}.$$

Now, multiply both sides through by $(1-a)^{(p-1)c-1}a^{-b}$ to get

$$\sum_{i+j=m} S^j S^i(\alpha)(1-a)^{(p-1)(c-i)-1}a^{j-b} = \sum_{i+j=m} S^j S^i(\alpha)(1-a)^{(p-1)(c-i)-1}a^{pi-b}.$$

The Adem relations follow from considering constant coefficient of each side of the equation expanded out as a formal power series in a.

On the left side, the coefficient is simply $S^a S^b$. This follows from Lemma 8.9. Note that on the right the coefficient will be 0 unless $pi \leq b$. Since $0 < b < pc$, this implies that $i < c$. At any rate, the coefficient is

$$\sum_{i+j=m} S^j S^i(-1)^{a-pi}\binom{(p-1)(c-i)-1}{b-pi}.$$

□

The Theorem directly implies the Adem relations for finite W. For $W = \infty$, one needs to know the following

Theorem 9.4. *If D^W satisfies the conditions of Theorem 9.3 for any finite W, then D^∞ does also.*

The proof of this is a combinatorial game with the Chern roots of TW. It will be omitted. As a corollary, we obtain

Theorem 9.5. *Let $\mathcal{A}(p)$ be the Steenrod Algebra at the prime p, and let β be the Bockstein element. Then $\mathcal{A}(p)$ modulo the two-sided ideal generated by β acts on $A^*[W, X]$ for W either a smooth algebraic space in which X is embedded or $W = \infty$.*

10. Appendix 1: Equivariant Intersection Theory II

In this section, we give a definition of equivariant Chow groups that is more functorial than the one given in Section 2 and thus more convenient for the verification of certain functorial properties. The definition follows previous functorial definitions worked out in [8] and [1]. We work with an algebraic group G of dimension g acting on a space X over a field k. For a equidimensional space, we write $|E|$ as a shorthand for $\dim E$.

Definition 10.1. $\mathcal{E}G$ is the category whose objects are smooth, equidimensional, free G-torsors $E \to B$. The morphisms are G-equivariant maps $E' \to E$. Let $t^X : \mathcal{E}G \to \mathrm{Sp}_k$ be the functor given by $E \mapsto E \times^G X$.

Remark 10.2. If E and F are two spaces in $\mathcal{E}G$, $E \times F$ will the the space with the diagonal G action. In fact, if F is any smooth space with a G-action, and E is in $\mathcal{E}G$, $E \times F$ with the diagonal action is in $\mathcal{E}G$.

Proposition 10.3. *Let $f : E \to F$ be a morphism in $\mathcal{E}G$. Let P be any one of the following properties of f: proper, flat, smooth, regular embedding, l.c.i., affine bundle, or vector bundle. Then if f has P, so does $t^X(f)$.*

Proof. For the first 5 properties, the proof is the same as the proof of Proposition 2.2 given in [1]. For the last two, the proof is the same as the proof of Lemma 1 in [1]. □

As all objects of $\mathcal{E}G$ are smooth and quasi-projective, all morphisms are l.c.i. Thus $t^X(f)$ is l.c.i. for any morphism f. In particular, one can define a functor $a_n^X : \mathcal{E}G^{\mathrm{op}} \to \mathrm{Ab}$ given by

$$\begin{array}{rcl} E & \mapsto & A(n+|E|-g, E\times^G X) \\ f & \mapsto & f^! \times^G \mathrm{id}. \end{array}$$

Theorem 10.4. $A_n^G X = \varprojlim_{\mathcal{E}G^{\mathrm{op}}} A(n+|E|-g, E\times^G X)$.

Note that, as any EG-model is in $\mathrm{ob}\,\mathcal{E}G$, there is a natural map $c : \varprojlim_{\mathcal{E}G^{\mathrm{op}}} a_n^X \to A_n^G X$.

The proof of the theorem will be accomplished with two lemmas. A form of the first was used by Totaro in [8]. In the lemmas, and in the proof of Theorem 10.4, U will be an EG model contained in a G-linear representation V of dimension l with $S = V - U$ and with $\mathrm{codim}_S V > \dim X - n$.

Lemma 10.5. *For any $E \in \mathrm{ob}\,\mathcal{E}G$, consider the diagram*

$$U \overset{p_1}{\leftarrow} U \times E \overset{p_2}{\rightarrow} E.$$

$a_n^X(p_2)$ is an isomorphism.

Proof. Let p_2' be the morphism $V \times E \to E$. Since this morphism is a vector bundle, $a_n^X(p_2')$ is an isomorphism. But let $i : U \times E \to V \times E$ be the inclusion. The condition on the codimension of S insures that $a_n^X(i)$ is an isomorphism. Thus it follows that $a_n^X(p_2)$ is also an isomorphism. □

Lemma 10.6. *Let p_1, $p_2 : U \times U \to U$ be the two projections. $a_n^X(p_1) = a_n^X(p_2)$.*

Proof. Let $\Delta : U \to U \times U$ be the diagonal. Since $p_1\Delta = \mathrm{id}$, the lemma will follow once we know that $a_n^X(\Delta)$ is an isomorphism. But this is clear as, for example, $a_n^X(p_1)$ is an isomorphism (by the proof of Definition-Proposition 2.1 [1]) and $p_1\Delta = \mathrm{id}$. □

Proof of Theorem 10.4. Recall that $A_n^G X \cong A(n+l-g, U \times^G X)$. For $\alpha \in A_n^G X$ and and an object E in $\mathcal{E}G$, set $\alpha(E) = a_n^X(p_2)^{-1} a_n^X(p_1)$. This association gives a map $\sigma : A_n^G X \to \varprojlim_{\mathcal{E}G^{\mathrm{op}}} a_n^X$ provided that it is functorial in E, and it is rather obvious that it is. What needs to be checked now is that $c\sigma$ and σc are both the identity. That $\sigma c = \mathrm{id}$ is easily checked. That $c\sigma$ is an isomorphism follows from Lemma 10.6. □

11. Appendix 2: Change of Groups

Here, transfer and restriction in the context of Equivariant Intersection Theory are investigated.

As the group under consideration will vary in this section the notation should be modified to vary as well. For any algebraic group G, write t_G^X for the functor $t^X : \mathcal{E}G \to \mathrm{Sp}_k$ and $a_{n,G}^X$ for the functor $a_n^X : \mathcal{E}G \to \mathrm{Ab}$. We will also restrict to the case of finite groups as they are all that it is needed in the construction of Steenrod operations.

11.1. Let $\rho : H \to K$ be a morphism of algebraic groups. There a functor $\rho_? : \mathcal{E}H \to \mathcal{E}K$ defined by $E \mapsto E \times^H K$. For any space X, $t_H^X = t_K^X \circ \rho_?$. (In other words, $(E \times^H K) \times^K X = E \times^H X$.) Thus there is a natural map $\rho^* : \varprojlim_{\mathcal{E}K} a_{n,K}^X \to \varprojlim_{\mathcal{E}H} a_{n,H}^X$. From the identifications of the last section, this map defines a restriction $\rho_H^K = \rho^* : A_n^K X \to A_n^H X$.

Proposition 11.2. *Let $\rho : H \to K$ and $\sigma : K \to L$ be maps of algebraic groups. Let X be an L-space.*

(i) $(\sigma \circ \rho)^* = \rho^* \circ \sigma^*$.
(ii) *ρ^* commutes with the proper push forward, flat pull-back, and the Gysin homomorphism.*
(iii) *If $H = K$ and ρ is an inner automorphism, then $\rho^* = id$.*

Proof. For (i) it suffices to note that $\sigma_? \rho_? = (\sigma\rho)_?$. (ii) is easy to check as the operations on the X variable are independent of the operation on the G variable. (iii) follows from the fact that when ρ is an inner automorphism there is a (fairly obvious) natural isomorphism between $\rho_?$ and $\mathrm{id}_{\mathcal{E}H}$. isomorhism □

Now, suppose ρ is an injection. Then restriction induces a map $\rho^? : \mathcal{E}G \to \mathcal{E}H$. The natural map, $E \times^H X \to E \times^G X$ induces a natural transformation $t_H^X \circ \rho^? \to t_G^X$. Push-forward induces a map $a_{n,H}^X \circ \rho^? \to a_{n,G}^X$. And this map in turn induces the transfer $\mathrm{tr}_H^G = \rho_* : A_n^H X \to A_n^G X$.

The proof of the following is not difficult.

Proposition 11.3. *Assume $\rho : H \to G$ and $\sigma : G \to L$ are two inclusions, then* $\mathrm{tr}_H^G \circ \mathrm{tr}_G^L = \mathrm{tr}_H^L$.

Proposition 11.4. *With ρ as in the previous lemma and with f and l.c.i. morphism,*

(i) *If*

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

is a pull-back then the following diagram

$$\begin{array}{ccc} A_*^H Y' & \xrightarrow{f^!} & A_*^H X' \\ \downarrow{\scriptstyle \mathrm{tr}_H^G} & & \downarrow{\scriptstyle \mathrm{tr}_H^G} \\ A_*^G Y' & \xrightarrow{f^!} & A_*^G X' \end{array}$$

commutes.

(ii) *If* $[G:H]=d$, $\rho^* \circ \rho_*(\alpha) = d\alpha$.

(iii) *For* $Z \in Z_k X$. $\mathrm{tr}_1^G[Z] = \mathrm{cl}^{\mathrm{G}}(\sum_{\mathrm{g}\in\mathrm{G}}[\mathrm{gZ}])$

Proof. (ii) Given $E \in \mathcal{E}H$, $E \times^H G$ is also in $\mathcal{E}H$. $\rho^* \circ \rho_*$ is then induced by the map

$$(E \times^H G) \times^H X \to E \times^H X.$$

The equality then follows from the fact that the degree of this map is d.

(i) Let $E \in \mathcal{E}G$. Then the transfer is induced by the proper push-forward trough the map $E \times^H X \to E \times^G X$. To check what is claimed amounts to checking that proper push-forward commutes with the Gysin homomorphism. This is a can be seen by writing out the ensuing diagram and applying parts (a) and (c) of Theorem 6.2 of [2].

(iii) Let U be an EG-model. The transfer described is then induced by the map $p : U \times X \to U \times^G X$. Set $R = \sum_{g\in G}[gZ]$. Then $p^*p_*(Z) = p^*(R)$ on the level of cycles. This shows that the $p_*(Z) = R$ proving the claim. □

With change of groups properly defined, the proof of Paragraph 2.3 is now a simple diagram chase. After the definitions are expanded, both sides represent the same Chern class acting on the same incarnation of α.

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