Tagungsbericht 39/1999

Algebrische K-Theorie

26.9.-2.10.1999

The conference consisting of 22 talks was chaired by Daniel R. Grayson (Urbana), Uwe Jannsen (Köln) and Bruno Kahn (Paris).

Vortragsauszüge

Equivariant Chow groups and Cohomology of finite groups

Emmanuuel Peyre (Strasbourg, France)

The talk was devoted to the study of geometrically negligible classes in the cohomology of a finite group. Let G be a finite group and M be a G-module, an element γ of $H^i(G, M)$ is said to be geometrically negligible if and only if it satisfies the following condition: For any field extension K of $\mathbf{C}$, for any morphism $\rho : \mathrm{Gal}(\overline{K}/K) \to G$, where $\overline{K}$ is an algebraic closure of K, the lifting $\rho^*(\gamma)$ is trivial in the Galois cohomology group $H^i(K, M) := H^i(\mathrm{Gal}(\overline{K}/K), M)$. The group of negligible classes is denoted by $H^i(G, M)_n$.

From now on, we restrict ourselves to the case $M = \mathbf{Q}/\mathbf{Z}$. In that case, it is known that $H^i(G, \mathbf{Q}/\mathbf{Z})_n$ is trivial if $i \leq 2$. In degree 3, it is related to equivariant Chow groups, which are defined as follows: Let W be a faithful representation of G over $\mathbf{C}$ and U be the open subset of W on which G acts freely. By taking sufficiently many copies of a given faithful representation, one may assume that $\mathrm{codim}_W(W - U) > i$. Then for any smooth variety Y over $\mathbf{C}$ equipped with an action of G, one defines

$$\mathrm{CH}^i_G(Y) := \mathrm{CH}^i((Y \times U)/G)$$

and we put $\mathrm{CH}^i_G(\mathbf{C}) = \mathrm{CH}^i_G(\mathrm{Spec}\,\mathbf{C})$.

Theorem. *There is a canonical isomorphism from $CH^2_G(\mathbf{C})$ to $H^3(G, \mathbf{Q}/\mathbf{Z})$.*

This enables one to compare the group of geometrically negligible classes with the one of permutation negligible classes, introduced by Saltman and which is defined by

$$H^3(G, \mathbf{Q}/\mathbf{Z})_p = \ker(H^3(G, \mathbf{Q}/\mathbf{Z}) \to H^3(G, \mathbf{C}(W)^*).$$

Corollary. *If $2 \not| \, |G|$, then $H^3(G, \mathbf{Q}/\mathbf{Z})_p = H^3(G, \mathbf{Q}/\mathbf{Z})_n$.*

Saltman had proven that these groups may be different if G is a 2-group (for example the quaternion group).

In higher degrees, one needs equivariant $\mathcal{K}_i^M$-cohomology. If X is a smooth variety over a field k, the group $H^j(X, \mathcal{K}_i^M)$ is defined as the j-th cohomology group of the complex

$$0 \to K_i^M k(X) \to \bigoplus_{x \in X^{(1)}} K_{i-1}^M k(x) \to \cdots \to \bigoplus_{x \in X^{(i)}} \mathbf{Z} \to 0$$

where $X^{(i)}$ denotes the set of points of codimension i in X and $K_*^M k$ the Milnor K-theory ring of k. Equivariant $\mathcal{K}_i^M$-cohomology is defined in terms of the usual $\mathcal{K}_i^M$-cohomology as the equivariant Chow groups. Then Voevodsky's work implies the following result:

Theorem. *There are canonical isomorphisms*

$$H_G^2(\mathbf{C}, \mathcal{K}_3^M)_{(2^\infty)} \xrightarrow{\sim} H^4(G, \mathbf{Q}_2/\mathbf{Z}_2)_n \quad \textit{and} \quad H_G^2(\mathbf{C}, \mathcal{K}_4^M)_{(2^\infty)} \xrightarrow{\sim} H^5(G, \mathbf{Q}_2/\mathbf{Z}_2)_n.$$

Computing the Homology of Koszul Complexes

Bernhard Köck (Karlsruhe, Germany)

We presented a new method for computing the homology of generalized Koszul complexes. This method is based on simplicial techniques and on the theory of cross effect functors. The main application of these computations is a new and very natural proof of Riemann-Roch without denominators which does not use the somewhat complicated deformation to the normal cone any longer. Motivated by the plethysm problem in universal representation theory we constructed explicit and functorial short exact sequences which immediately imply the Adams-Riemann-Roch formula for regular immersions of codimension 1 (and codimension 2) and the Riemann-Roch formula for tensor power operations for regular immersions of arbitrary codimension.

Algebraic K-theory of non-linear projective spaces

Thomas Hüttemann (Bielefeld, Germany)

QUILLEN has shown that there is an isomorphism of K-groups $K_j(P_R^n) \cong \bigoplus_0^n K_j(R)$, where $K_j(R)$ is defined as the K-theory of the category of finitely generated projective R-modules, and $K_j(P_R^n)$ is the K-theory of the category of vector bundles of finite rank on the scheme $P_R^n = \text{Proj } R[X_0, \ldots, X_n]$. In the case of the projective line, it is well-known that the category of vector bundles is equivalent to a certain category of diagrams of modules. A "non-linear" version of this diagram category (replacing rings by monoids and modules by equivariant spaces) has been introduced by KLEIN, VOGELL, WALDHAUSEN and WILLIAMS, who have also shown that its K-theory (in the sense of WALDHAUSEN) is homotopy equivalent to $A(*) \times A(*)$, providing an analogue of QUILLEN's splitting result. In the talk I demonstrated how to extend the non-linear setting to higher dimensions. It is surprising that many constructions from algebraic geometry can be mimicked with "non-linear sheaves". The resulting mathmatical machinery is one ingredient for actual K-theoretical calculations. Another important tool is the existence of two QUILLEN closed model structures on a certain category of (non-linear) presheaves. We thus have the full strength of abstract homotopy theory available, which allows for a conceptual approach to the delicate finiteness conditions for sheaves.

A postscript version of the paper is available for download via internet at
`http://archiv.ub.uni-bielefeld.de/disshabi/1999/0006.ps`.

A field with u-invariant 9

Oleg Izhboldin (Bielefeld, Germany)

Let F be a field of characteristic $\neq 2$. Let $\varphi = a_1x_1^2 + ... + a_nx_n^2$ be a quadratic form over F (we assume that $\det\varphi \neq 0$). The form is called isotropic if there exist $\alpha_1, ..., \alpha_n \in F$ such that $a_1\alpha_1^2 + ... + a_n\alpha_n^2 = 0$ and at least one of the elements $\alpha_1, ..., \alpha_n$ is non zero. The u-invariant of the field F is defined as the maximal dimension of unisotropic quadratic forms over F:

$$u(F) = \sup\{\dim\varphi \mid \varphi \text{ is an unisotorpic form over } F\}.$$

For instance 1) $u(\mathbb{C}) = 1$; 2) $u(\mathbb{R}) = \infty$; 3) $u(\text{local field}) = 4$; 4) $u(\text{finite field}) = 2$; 5) $u(\mathbb{C}(t_1, ...t_n)) = 2^n$; 6) $u(\mathbb{Q}(i)) = 4$. Note that in all examples listed above the u-invariant is always a power of 2 or infinite. This observation was (probably) a reason for the well known Kaplansky conjecture (1953) stating that the only possible finite values of the u-invariant are powers of 2. In 1991 A. Merkurjev disproved this conjecture by constructing a field F with $u(F) = 2n$ for any $n \geq 1$. It is known that the u-invariant never equals 3, 5 or 7. We prove the following

Theorem. *There exists a field F with $u(F) = 9$.*

One of the steps of the proof of this theorem is based on the computation of the group $\mathrm{CH}^3(X_\varphi)$ where φ is a form of dimension ≥ 9 and X_φ is the projective quadratic given by the equation $\varphi = 0$. We compute the group $\mathrm{CH}^3(X_\varphi)$ completely, except for the case $\dim\varphi = 8$. Besides, we prove that for all forms of dimension ≥ 9 which are not 4-fold Pfister neighbors the sequence

$$0 \to H^4(F) \to H^4_{nr}(F(\varphi)/F) \to \mathrm{Tors}\mathrm{CH}^3(X_\varphi) \to 0$$

is exact. This completes the computation of the groups $H^4_{nr}(F(\varphi)/F)$ for forms of dimension ≥ 9.

Bloch's conductor formula in higher dimensions
(joint work with K.Kato)

Takeshi Saito (University of Tokyo, Japan/Max-Planck Bonn, Germany)

Let K be a local field and let X be a proper flat regular scheme over the integer ring O_K with smooth generic fiber. The conductor $\mathrm{Art}(X/O_K)$ is an invariant measuring the ramification of the Galois representation on the ℓ-adic etale cohomology. S. Bloch defined the self-intersection 0-cycle $(\Delta, \Delta) \in CH_0(X_F)$ supported on the closed fiber X_F and formulated

Conjecture.

$$\deg(\Delta, \Delta) = -\mathrm{Art}(X/O_K).$$

He proved it in the case $\dim X_K = 1$. The case $\dim X_K = 0$ is a classical equality between discriminant and conductor. The main result is

Theorem. *Conjecture is true if the characteristic of K is 0 and the reduced closed fiber $(X_F)_{\rm red}$ has simple normal crossings.*

Key ingredients in the proof are the following

1. Functorial propertity of a logarithmic localized Gysin map $G(X_K \times_K X_K) \to G(X_F)$ of the Grothendieck groups of coherent sheaves.

2. Lefschetz trace formula for log etale cohomology.

Combining these 2 new results with alteration by de Jong, we prove the theorem.

Semi-topological K-theory

Mark Walker (University of Nebraska Lincoln, USA)

In the first portion of this talk, I describe joint work with Dan Grayson in which we prove the "algebraic mapping space" $Hom(X \times \Delta^\bullet, Grass)^{h+}$ gives the algebraic K-theory of a smooth variety X over a field k. Here, $Grass$ is the ind-variety parameterizing finite dimensional subspaces of k^∞, $\Delta^\bullet$ is the standard cosimplicial scheme, and the superscript "$h+$" refers to forming the homotopy-theoretic group completion with respect to a certain Γ-space structure.

In the remainder of the talk, I describe joint work with Eric Friedlander concerning the so-called "semi-topological K-theory" of a complex variety X. This is defined in analogy to the construction of algebraic K-theory given above, by endowing the set $Hom(X, Grass)$ with the structure of a topological space. When X is a projective variety, this space may be defined by observing $Hom(X, Grass)$ has the structure of an inductive limit of quasi-projective varieties. Writing $\underline{Hom}(X, Grass)$ for this topological space, we define $K^{semi}(X)$ to be the space $\underline{Hom}(X, Grass)^{h+}$, where here again the superscript refers to forming a certain homotopy-theoretic group completion.

In general $K_0^{semi}(X) = \pi_0 K^{semi}(X)$ is the quotient of $K_0(X)$ by algebraic equivalence of vector bundles, and rationally we have a natural isomorphism $K_0^{semi}(X)_{\mathbb{Q}} \cong A^*(X)_{\mathbb{Q}}$ via the Chern character.

We prove several results for K^{semi}, including:

(1) the projective bundle formula $K_*^{semi}(PE) \cong K_*(X)^n$, where E is a rank n bundle over a projective variety X,

(2) a version of Poincare duality for smooth projective varieties, and

(3) a natural equivalence $K^{semi}(C) \sim K_{top}(C^{an})$, where C is Riemann surface.

Additionally, we establish the existence of a natural "double-square" commutative diagram

$$\begin{array}{ccccc} K_j^{alg}(X) & \longrightarrow & K_j^{semi}(X) & \longrightarrow & K_{top}^{-j}(X^{an}) \\ \downarrow & & \downarrow & & \downarrow \\ \bigoplus_q H_{\mathcal{M}}^{2q-j}(X,\mathbb{Z}(q)) & \longrightarrow & \bigoplus_q L^q H^{2q-j}(X) & \longrightarrow & \bigoplus_q H_{sing}^{2q-j}(X). \end{array}$$

Here, the vertical maps are total Segre class maps and the groups appearing along the bottom row are the motivic, morphic, and singular cohomology groups of X. We make four conjectures about this diagram for a smooth variety X, which are listed in increasing level of difficulty:

(1) The vertical maps are rational isomorphisms.

(2) The map $K_j^{alg} \to K_j^{sem}$ is an isomorphism for finite coefficients.

(3) The map $K_*^{semi} \to K_{top}^{-*}$ becomes an isomorphism after inverting the action of the Bott element.

(4) The map $K_j^{semi} \to K_{top}^{-j}$ is an isomorphism for $j \geq dim(X) - 1$.

Weil topology on varieties over finite fields

Steven Lichtenbaum (Brown University, USA)

We introduce a new Grothendieck topology on varieties over finite fields. This construction bears the same relation to the étale topology as the Weil group does to the Galois group, so we call it the "Weil topology".

Let X be a variety over a finite field k, and let $\bar{X} = X \times_k \bar{k}$. We define a topology T_X by letting the objects of T_X be schemes étale and of finite type over $\bar{X}$, and the morphisms be those which commute with the projection map to X and commute with the projection map to $\bar{k}$ up to an integral power of Frobenius on each connected component. The coverings in this topology are étale coverings.

We may identify sheaves for the Weil topology on $\bar{k}$ with Γ-modules, where Γ (isomorphic to $\mathbf{Z}$) is the subgroup of the absolute Galois group of k generated by Frobenius. Let $\pi : X \to k$, and F a Weil sheaf on X. We define $H_W^0(X, F)$ to be $H^0(\bar{X}, F)^\Gamma$ and $H_W^i(X, F)$ to be the derived functors of H_W^0.

We conjecture that if $\mathbf{Z}(n)$ is an étale motivic complex on X (say, Bloch's higher Chow groups to be specific) then the $H_W^i(X, \mathbf{Z}(n)$ are finitely generated abelian groups for all i, X and n, and that if X is connected, projective and smooth over k, these groups satisfy a version of Poincaré duality. We can check this for dimension $X = 1$.

Motives and splitting patterns of quadrics

Alexander Vishik, (MPI-Bonn, Germany)

Let k be some field, $char(k) \neq 2$, and $W(k)$ be the *Witt ring* of quadratic forms over k. Let $I \subset W(k)$ be the ideal of *even-dimensional* forms. We get a filtration: $W \supset I \supset I^2 \supset \cdots \supset I^n \supset \ldots$. The interesting problem is to describe the possible dimensions of anisotropic forms in I^n. The following statement is well-known.

Hauptsatz (Arason-Pfister).

Let $char(k) \neq 2$, $q \in I^n(k)$ is anisotropic form. Then $\dim(q)$ *is either* 0, *or* $\geq 2^n$.

Do we still have some restrictions on $\dim(q)$, if $\dim(q) \geq 2^n$? The following conjecture is well-known (was formulated, in particular, by B.Kahn):

Conjecture 1.

Let $char(k) \neq 2$, $q \in I^n(k)$ is anisotropic form. Then $\dim(q)$ is either 0, or 2^n, or $\geq 2^n + 2^{n-1}$.

This conjecture was settled for $n = 3$ by A.Pfister, and for $n = 4$ by D.Hoffmann. Our main result is:

Main Theorem. *Let $char(k) = 0$, and $\sqrt{-1} \in k^*$.*

Then Conjecture 1 is true for all n.

The main tool in the proof is the following lemma, describing possible dimensions of "binary" direct summand in the motive of quadric.

Main Lemma.

Let $char(k) = 0$, $\sqrt{-1} \in k^*$, and q be anisotropic form over k. Suppose that L is a direct summand in $M(Q)$, and $L|_{\overline{k}} = \mathbb{Z} \oplus \mathbb{Z}(i)[2i]$. Then $i = 2^r - 1$ for some r.

The Lemma above also permits to classify *splitting patterns* of forms of *height* $= 2$:

Theorem.

Let $char(k) = 0$, $\sqrt{-1} \in k^*$, and q be anisotropic form over k of height $= 2$. Then the splitting pattern of q is:

1) for $q \in I^n \backslash I^{n+1}(k)$ for $n \geq 1$: either $(2^m - 2^n, 2^{n-1})$, for $m > n \geq 1$, or $(2^{n-1}, 2^{n-1})$, or $(2^{n-2}, 2^{n-1})$ (for $n \geq 2$),

2) for $q \notin I(k)$: $(2^m - 2^n + 1, 2^{n-1} - 1)$ for some $m \geq n > 1$.

In the end, we formulate the conjecture, describing **all** possible values of $\dim(q)$ for anisotropic $q \in I^n$.

Conjecture 2.

Let $char(k) \neq 2$, $q \in I^n(k)$ be anisotropic. Then $\dim(q)$ is either $2^{n+1} - 2^{i+1}$, where $0 \leq i \leq n$, or is even number $\geq 2^{n+1}$.

This conjecture is a consequence of some natural conjecture about the structure of the indecomposable direct summand in $M(Q)$. Also, all the dimensions prescribed (by Conjecture) are really dimensions of some anisotropic forms from $I^n(k)$ (for suitable k)(examples are constructed).

K-theory of complete discrete valuation fields

Lars Hesselholt (MIT, USA)

This is joint work with Ib Madsen. Let A be a complete discrete valuation ring with field of fractions K of characteristic zero and perfect residue field k of characteristic $p > 2$. We prove that the canonical map

$$K_*(K, \mathbb{Z}/p\mathbb{Z}) \to K_*^{\mathrm{et}}(K, \mathbb{Z}/p\mathbb{Z})$$

is an isomorphism in degrees ≥ 1.

It is known that for the fields in question, the cyclotomic trace

$$K_*(K, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{TC}_*(A|K; p, \mathbb{Z}/p\mathbb{Z})$$

is an isomorphism in degrees ≥ 1, and it is the right hand side - topological cyclic homology - we evaluate. We define $T(A|K)$ to be the mapping cone of the transfer map $T(k) \to T(A)$ in topological Hochschild homology. There is a circle action on $T(A|K)$ and we define $\mathrm{TR}^n(A|K; p)$ to be the fixed set by the cyclic group of order p^{n-1}. The homotopy groups $\mathrm{TR}^{\cdot}_*(A|K; p)$ has a rich algebraic structure of which the de Rham-Witt pro-complex $W.\omega^*_{(A,M)}$ of A with the canonical log structure is the universal example. The main theorem is that when $\mu_p \subset K$, the canonical map

$$W.\omega^*_{(A,M)} \otimes S_{\mathbb{F}_p}(\mu_p) \xrightarrow{\sim} \mathrm{TR}^{\cdot}_*(A|K; p, \mathbb{Z}/p\mathbb{Z})$$

is a pro-isomorphism. The topological cyclic homology groups are given a long exact sequence

$$\cdots \longrightarrow \mathrm{TC}_s(A|K; p, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \mathrm{TR}_s(A|K; p, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{1-F} \mathrm{TR}_s(A|K; p, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \cdots$$

and main theorem implies that for $s \geq 0$,

$$\begin{aligned} \mathrm{TC}_{2s}(A|K; p, \mathbb{Z}/p\mathbb{Z}) &= H^0(K, \mu_p^{\otimes s}) \oplus H^2(K, \mu_p^{\otimes(s+1)}), \\ \mathrm{TC}_{2s+1}(A|K; p, \mathbb{Z}/p\mathbb{Z}) &= H^0(K, \mu_p^{\otimes(s+1)}). \end{aligned}$$

Hence the groups $K_*(K, \mathbb{Z}/p\mathbb{Z})$ and $K_*^{\mathrm{et}}(K, \mathbb{Z}/p\mathbb{Z})$ are abstractly isomorphic in degrees ≥ 1, and it is not hard to see that the canonical map is an isomorphism.

A counterexample to the conjectured description of Chow rings of finite groups

Burt Totaro (Chicago, USA)

I explain two new results on Chow rings of classifying spaces of algebraic groups G, extending the results of my earlier papers:

Torsion algebraic cycles and complex cobordism, J. AMS 10 (1997), 467-493. The Chow ring of a classifying space, Proc. Symp. Pure Math. (K-Theory, Seattle 1997), to appear.

First, we construct an analogue of Becker-Gottlieb transfer for Chow groups. The basic idea of the construction had been used for different purposes by Takeshi Saito. In particular, as Becker and Gottlieb showed for all topological cohomology theories, we find that for any reductive group G over a field of characteristic zero with maximal torus T, the restriction map $CH^*(BG) \to CH^*(BN(T))$ is split injective.

The second result of the lecture is the disproof of a conjecture from my paper in the Seattle conference volume. Namely, we find a finite group G for which the Chow ring $CH^*(BG)$ is not generated by transfers of Chern classes of representations of subgroups of G. The group, the finite Heisenberg group $1 \to \mathbb{Z}/2 \to G \to (\mathbb{Z}/2)^6 \to 1$, was suggested by the work by Schuster and Yagita, who found that the cobordism ring $MU^*(BG)$ is not generated by transfers of Chern classes for this group.

Schuster and Yagita find their interesting element of $MU^6(BG)$ using the Atiyah-Hirzebruch spectral sequence for complex cobordism. No such computational tool is available for Chow groups, and so, to find a corresponding interesting element of $CH^3(BG)$, we are forced to exhibit a suitable algebraic cycle more or less explicitly.

Mixed Tate motoves with finite coefficients and conjectures about the Galois groups of fields

Leonid Positselski (IAS, Princeton, USA)

In the common list of conjectures about motives with rational coefficients over a field, there is one saying that the embedding of the (hypothetical) abelian category of mixed Tate motives into the abelian category of all mixed motives should induce isomorphisms on the Ext spaces. The goal of the talk was to formulate analogue conjectures for mixed Tate motives with finite and integral coefficients and connect the former with conjectures about Galois groups of fields. Hoping that the conjectures proposed here might prove more accessible than other conjectures about motives, I was taking care to formulate them in a way not dependent on any existence of abelian categories of motives, i.e. essentially on the vanishing conjectures.

For a triangulated category $\mathcal{D}$ and two subsets $\mathcal{A}, \mathcal{B} \subset Ob\mathcal{D}$ we put $\mathcal{A} * \mathcal{B} = \{C \in Ob\mathcal{D} : \exists A \to B \to C \to A[1]$ with $A \in \mathcal{A}, B \in \mathcal{B}\}$, and $\mathcal{A}^{*\infty} = \{0\} \cup \bigcup_{n=1}^{\infty} \mathcal{A} * \mathcal{A} * \ldots * \mathcal{A}$ (n $\mathcal{A}$'s). It is easy to see that $*$ is associative. Further, a sequence of objects $\{E_i \in Ob\mathcal{D} : i \in \mathbb{Z}\}$ is called an *exceptional sequence* if $Hom^k_{\mathcal{D}}(E_i, E_j) = 0$ for all $i > j$ and $i = j$, $k \neq 0$.

Theorem 1. *Let $\mathcal{D}$ be a triangulated category generated by an exceptional sequence E_i and $\mathcal{M} = \{E_i\}^{*\infty}$. Then $\mathcal{M}$ is an abelian category and $\mathcal{D}^b(\mathcal{M}) = \mathcal{D}$ iff*
(i) $\mathcal{D}$ is of algebraic origin (see Beilinson "On the derived category of perverse sheaves", or A. Neeman "New axioms for triangulated categories"); and
(ii) $Hom^k_{\mathcal{D}}(E_i, E_j) = 0$ for all $k < 0$ and all $i \neq j$, $k = 0$, and all $End_{\mathcal{D}}(E_i) = Hom^0_{\mathcal{D}}(E_i, E_i)$ are division rings, and
(iii) either of the next two equivalent conditions hold
(iii') the composition $Hom^1_{\mathcal{D}}(\mathcal{M}, \mathcal{M}) \times \ldots \times Hom^1_{\mathcal{D}}(\mathcal{M}, \mathcal{M}) \longrightarrow Hom^k_{\mathcal{D}}(\mathcal{M}, \mathcal{M})$ is surjective for all $k \geq 2$; or
(iii") $\mathcal{D} = \bigcup_{a<b} \mathcal{M}[a] * \mathcal{M}[a+1] * \ldots * \mathcal{M}[b]$.

Conjecture. *For any field F, and any coefficient ring $C = \mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{Z}/m$, let $\mathcal{DM}(F, C)$ be the triangulated category of motives. Define the triangulated category of mixed Tate motives as the full subcategory $\mathcal{M} = \{C(i) : i \in \mathbb{Z}\}^{*\infty}$. Then the condition (iii) is satisfied for $\mathcal{M} \subset \mathcal{D}$.* (This is what Bloch and Kriz call the "$K(\pi, 1)$ conjecture". We will call it the "silly filtration conjecture" in view of (iii).)

Theorem 2. *Let E_i be an exceptional sequence in $\mathcal{D}$ such that $\mathcal{M} = \{E_i\}^{*\infty}$ satisfies the condition (iii). Further assume (for simplicity of notation) that there is an auto-equivalence $(1) : \mathcal{D} \to \mathcal{D}$ such that $E_{i+1} = E_i(1)$. Then $Hom^k_{\mathcal{D}}(E_i, E_j) = 0$ for all $k > j - i$ and the graded ring $\bigoplus_{k=0}^{\infty} Hom^k_{\mathcal{D}}(E_0, E_k)$ is quadratic, i.e. generated by $Hom^1_{\mathcal{D}}(E_0, E_k)$ with relations in degree 2.*

Remark. By the motivic conjectures one should have $Hom^i_{\mathcal{D}}(C, C(i)) = K^{\mathrm{Miln}}_i(F) \otimes C$, and Milnor's K-theory is indeed a quadratic ring. That is the only justification of the silly filtration conjecture that I am aware of.

One cannot express the condition (iii) in terms of the algebra $\bigoplus Hom^k(E_i, E_j)$ alone in general, though one can in some special cases.

Theorem 3. *Let E_i be an exceptional sequence in $\mathcal{D}$; assume that there is a shift functor as in theorem 2. Further assume that $Hom^k_{\mathcal{D}}(E_i, E_j) = 0$ for all $k \neq j - i$, and that the algebras $End_{\mathcal{D}}(E_i)$ are semi-simple. Then the condition (iii) holds for $\mathcal{M} = \{E_i\}^{*\infty}$ iff the graded algebra $\bigoplus_{k=0}^{\infty} Hom^k_{\mathcal{D}}(E_0, E_k)$ is Koszul (see Beilinson-Ginzburg-Soergel "Koszul duality patterns..." for the definition).*

In view of the Beilinson-Parshin conjecture, for coefficients $C = \mathbb{Q}$ and a field of $char(F) \neq 0$ it follows that the silly filtration conjecture is equivalent to the algebra $K_*^{\mathrm{Miln}}(F) \otimes \mathbb{Q}$ being Koszul (this was communicated to me by A. Beilinson).

Koszulity Conjecture (A. Vishnik & me, 1995). *For any prime l and a field F containing the l-th roots of unity, the algebra $K_*^{\mathrm{Miln}}(F) \otimes \mathbb{Q}$ is Koszul.*

The next theorem holds in the assumption of the Suslin-Voevodsky-Geisser-Levine result on the Milnor-Bloch-Kato conjecture implying the Beilinson-Lichtenbaum formulas for the motivic cohomology with finite coefficients. I also assume that the map $K_n^{\mathrm{Miln}}(F) \otimes \mathbb{Z}/l \to H^n(G_F, \mu_n^{\otimes l})$ is an isomorphism for $n = 2$ (as proven by Merkurjev-Suslin) and a monomorphism for $n = 3$ (which may be still unknown).

Theorem. *For any prime l, the union of the Milnor-Bloch-Kato conjecture and the "silly filtration" conjecture for $\mathbb{Z}/l$ coefficients is equivalent to the Koszulity conjecture.*

The result on Koszulity implying the MBK conjecture is due to A. Vishik and me (Math. Res Letters, 1995). The rest is in the spirit of theorem 3 above.

Triangular Witt groups

Paul Balmer (London, Canada)

1. Theorem. *Let $\mathcal{E}$ be an exact category with duality. Then the Witt group of $\mathcal{E}$ is an invariant of the derived bounded category:*

$$\mathrm{W}(\mathcal{E}) \cong \mathrm{W}\big(\mathrm{D^b}(\mathcal{E})\big).$$

Giving the precise meaning of this statement requires the definition of the Witt group of a triangulated category with duality. Actually, using the translation functor, a triangulated category with duality $(K, \#)$ immediately inherits a collection of dualities – namely the $T^n \circ \#$ for $n \in \mathbb{Z}$. Each of these dualities defines a Witt group. These groups will be denoted by $\mathrm{W}^n(K, \#)$ and they are periodic in the following way:

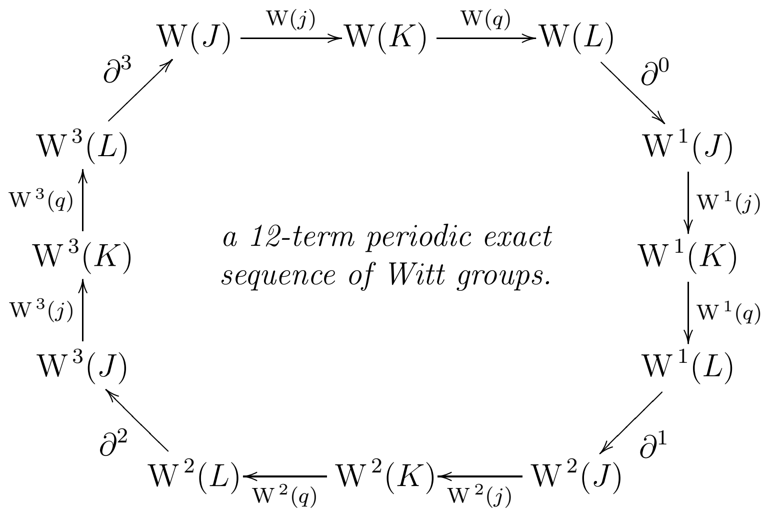
$$\mathrm{W}^{n+4}(K, \#) \cong \mathrm{W}^n(K, \#).$$

Those *translated* (or *shifted*) Witt groups are also *higher and lower* Witt groups, in the sense of the following result.

2. Theoreom. *Consider an exact sequence of triangulated categories with duality:*

$$0 \to J \xrightarrow{j} K \xrightarrow{q} L \to 0$$

such that $\frac{1}{2} \in K$ and such that $A \oplus B \simeq B \Rightarrow A \simeq 0$ in L. Then there exists



3. Localization of schemes. Let X be a scheme. Consider $\mathcal{E}(X)$ the category of vector bundles over X. Call lazily $\mathrm{D^b}\big(\mathcal{E}(X)\big)$ *the* derived category of X. Let $U \hookrightarrow X$ be an open subscheme. When is the derived category of U a *localization* of the derived category of X? This is true, for instance if X is separated and regular. When it happens to be true, one gets from theorems 1 and 2 a localization exact sequence, linking the Witt groups of X, of U, and of the category of those complexes on X which are acyclic on U.

In the case of affine schemes, this result was already established by A. Ranicki.

4. Spectral sequences. (Joined work with Charles Walter)

4.1. Theorem. *Let $K = D^0 \supset D^1 \supset \cdots \supset D^n \supset D^{n+1} = 0$ be a filtration of triangulated categories with dualities such that $\frac{1}{2} \in K$. Then there is a converging spectral sequence:*

$$E_1^{p,q} = \mathrm{W}^{p+q}(D^p/D^{p+1}) \quad \Longrightarrow \mathrm{W}^{p+q}(K)$$

where $E_1^{p,q} = 0$ when p is not between 0 and n.

4.2. Theorem. *Let X be a regular separated scheme of Krull dimension n, in which 2 is invertible. Then, there exists a spectral sequence*

$$E_1^{p,q} = \begin{cases} \bigoplus_{x \in X^{(p)}} \mathrm{W}\big(k(x)\big) & \textit{if } q \equiv 0 \textit{ mod } 4 \\ 0 & \textit{if } q \equiv 1,2,3 \textit{ mod } 4 \end{cases} \qquad \Longrightarrow \mathrm{W}^{p+q}(X).$$

The "first page" of this spectral sequence is then quite particular. In line $q = 0$, we have a Gersten complex between $p = 0$ and $p = n$ and 0 outside those columns. Then this line is repeated every 4 lines (up and down), the lines in between being all trivial. In low dimension, we obtain the following result.

4.3. Corollary. *Let X be a separated regular scheme of dimension ≤ 4 in which 2 is invertible. Then there exists an exact sequence:*

$$\bigoplus_{x \in X^{(3)}} \mathrm{W}\big(k(x)\big) \longrightarrow \bigoplus_{x \in X^{(4)}} \mathrm{W}\big(k(x)\big) \longrightarrow \mathrm{W}(X) \longrightarrow \bigoplus_{x \in X^{(0)}} \mathrm{W}\big(k(x)\big) \longrightarrow \bigoplus_{x \in X^{(1)}} \mathrm{W}\big(k(x)\big).$$

In particular, purity for Witt groups holds for such schemes when dimension is lower or equal to 3, that is:

$$0 \longrightarrow \mathrm{W}(X) \longrightarrow \bigoplus_{x \in X^{(0)}} \mathrm{W}\big(k(x)\big) \longrightarrow \bigoplus_{x \in X^{(1)}} \mathrm{W}\big(k(x)\big)$$

is an exact sequence.

Purity for global schemes in dimension 3 is new. So is the weak purity in dimension 4 (exactness at the points of codimension 0). Purity has recently been proved in all dimensions for regular local rings containing a field by M. Ojanguren and I. Panin.

On Poly(anna)logs[1]
(Joint work with Philippe Elbaz-Vincent.)

Herbert Gangl (MPI Bonn, Germany)

In an unpublished note, Kontsevich defined the "$1\frac{1}{2}$-logarithm", associated to a prime p, as the truncated power series of $-\log(1-x)$ (for which we propose the "truncated" letter $\pounds$) as a function from $\mathbb{Z}/p$ to $\mathbb{Z}/p$:

$$\pounds_1(x) = \pounds_1^{(p)}(x) = \sum_{k=1}^{p-1} \frac{x^k}{k} \qquad (\mathrm{mod}\ p).$$

We think to it as the "finite 1-logarithm". Kontsevich observed that it satisfies a functional equation which is known in the literature as the fundamental equation of information theory, and he asked for similar functional equations for the next case, i.e. for the function $\pounds_2(x) = \sum_{k=1}^{p-1} x^k/k^2$.

Cathelineau on the other hand, motivated by his studies on Hilbert's third problem (on scissors congruences), was led earlier to the fundamental equation of information by considering an "infinitesimal" analog of the dilogarithm function. This suggests looking at functional equations for infinitesimal polylogarithms (of level n) and proving these equations for the "finite polylogarithm" $\pounds_{n-1}(x) = \sum_{k=1}^{p-1} x^k/k^{n-1}$ (mod p) directly. For $n = 3$, Cathelineau has given the general functional equation for the infinitesimal trilogarithm. This enables us to answer Kontsevich's question: we can show that $\pounds_2$ satisfies the same equation. Even better, via a tangential procedure (given by Cathelineau for $n = 2$)—which allows to derive a functional equation for the infinitesimal n-logarithm from a functional equation for the n-logarithm—one obtains a recipe how to produce equations for $\pounds_{n-1}$ from the latter ones.

The finite polylogarithms have appeared in the literature prominently in the guise of "Mirimanoff polynomials" (cf. Ribenboim's 13 Lectures). One can prove that the product $\pounds_1(a)\pounds_1(b)$ can be expressed in terms of $\pounds_2$ only, and the special case $a = b$ amounts to an identity found by Mirimanoff which is crucial for proving his criteria for Fermat's last theorem. Others of Mirimanoff's identities can be reinterpreted in terms of functional equations of finite polylogarithms which might nurture the hope that further knowledge concerning the latter could provide more obstacles for a solution of FLT to exist... (but this may well be too Pollyanna![1])

[1]**Pollyanna.** The name of the heroine of stories written by Eleanor Hodgman Porter (1868-1920), American

Motivic cohomology over discrete valuation rings

Thomas Geisser (Tokyo, Japan)

Let X be a smooth variety over a Henselian discrete valuation ring R, with residue field k of characteristic p, and quotient field F of characteristic 0. Let $j : U \to X$ and $i : Y \to X$ be the inclusions of the generic and closed fibers. Let $\mathbb{Z}(n)$ the motivic complex defined by higher Chow groups for either the Zariski, Nisnevich or étale topology. Then there is an exact triangle in the derived category of sheaves on Y:

$$Ri^!\mathbb{Z}(n) \to i^*\mathbb{Z}(n) \to i^*Rj_*\mathbb{Z}(n) \to \ldots.$$

By recent results of Levine, the motivic complex on X has the localization property, or equivalently satisfies purity for the Zariski topology $Ri^!\mathbb{Z}(n)_{\text{Zar}} \cong \mathbb{Z}(n-1)_{\text{Zar}}[-2]$. To study the étale motivic complex we can restrict ourselves to finite coefficients (since rationally Zariski and étale motivic cohomology agree). In this case we have $\mathbb{Z}/m(n)_{\text{ét}} \cong \mu_m^{\otimes n}$ on U, because U is smooth over a field of characteristic 0. Conjecturally, a truncated version of purity should hold for the étale topology.

Conjecture 1.

$$\tau_{\leq m} Ri^!\mathbb{Z}/m(n)_{\text{ét}} \cong \mathbb{Z}/m(n-1)[-2]_{\text{ét}}.$$

In fact, truncation should not be necessary if $p \not| m$. The right hand sides agree with $\mu_m^{\otimes n-1}[-2]$ and $\nu^{n-1}[-n-1]$ for $p \not| m$ and $m = p$, respectively, because Y is smooth over a field of characteristic p. The conjectures is a consequence of the Beilinson-Lichtenbaum conjecture (stating that Zariski and étale motivic cohomology of field agree in degrees less than or equal to the weight). More precisely, the above conjecture is equivalent to the Beilinson-Lichtenbaum conjecture for all quotient fields of strictly Henselian local rings of X. Using purity of $\mu_m^{\otimes n}$ one sees that the prime to p part of the conjecture is equivalent to $\mathbb{Z}/m(n)_{\text{ét}} \cong \mu_m^{\otimes n}$. It also has the following strengthening:

Conjecture 2. *If $p \not| m$, then $i^*\mathbb{Z}/m(n)_{\text{Nis}} \cong \mathbb{Z}/m(n)_{\text{Nis}}$.*

We have to use the Nisnevich topology here in order to get Henselian local rings; the statement is wrong for the Zariski topology or if $p|m$. On the other hand, the first conjecture follows by étale sheafification. To study the above problems, we use a Gersten resolution:

Theorem 3. *Let $\epsilon : X_{\text{ét}} \to X_{\text{Zar}}$ be the change of sites. Then there is a long exact sequence of sheaves on Y_{Zar}*

$$0 \to R^s i^! \epsilon_* \mathbb{Z}(n) \to \bigoplus_{x \in Y^{(0)}} H^s_{x,\text{ét}}(X, \mathbb{Z}(n)) \to \bigoplus_{x \in Y^{(1)}} H^{s+1}_{x,\text{ét}}(X, \mathbb{Z}(n)) \to \ldots.$$

This is a Gersten resolution of on a smooth scheme over a field, hence Gabber's method for proving Gersten resolutions can be applied. Comparing this Gersten resolution to the resolution of $R^{s-2}\epsilon_*\mu_m^{\otimes n-1}$, and of ν^{n-1}, this theorem allows the following partial answers to the first conjecture:

children's author, used with allusion to her skill at the 'glad game' of finding cause for happiness in the most disastrous situations; **one who is unduly optimistic or achieves happiness through self-delusion**. [Oxford English Dictionary 2]

Theorem 4. *a) If $p \not| m$, then $\mu_m^{\otimes n}$ is a split direct summand of $\mathbb{Z}/m(n)_{\text{ét}}$. They agree, if purity holds for the strict Henselization of the generic point of the closed fiber. For example, they are equal if m has not divisor smaller than $n-1$.*

b) We have $R^{n+1}\mathbb{Z}/p(n)_{\text{ét}} \cong \nu^{n-1}$, and purity if and only if purity holds for the strict Henzelisation of the generic point of the closed fiber.

Proof. a) For the first statement note that the composition

$$\mu_m^{\otimes n} \cong \mathbb{Z}/m(1)^{\otimes n} \overset{\cup}{\to} \mathbb{Z}/m(n) \overset{c}{\to} \mu_m^{\otimes n}$$

is the identity. The second statement follows comparing Gersten resolutions. For the final statement, one uses the calculation of K-theory of a Henselian discrete valuation ring, and the degeneration of the spectral from motivic cohomology to K-theory up to small primes.

b) For the first statement, one uses the calculation of Milnor-K-theory of a Henselian discrete valuation ring of Bloch-Kato. The second statement follows using Gersten resolutions. Q.E.D.

Note that the theorem reduces the first conjecture to the Beilinson-Lichtenbaum conjecture for the quotient field of the strict Henselization of the closed fiber of the generic point.

Unipotent elements in semisimple simply connected groups in characteristic $p > 0$

Philippe Gille

Let k be a field with positive characteristic $p > 0$ and k_s a separable closure of k. Let G/k be a semisimple simply connected algebraic group. A subgroup U of $G(k)$ is called unipotent if every element of U is unipotent. We prove the following result, conjectured by J. Tits.

THEOREM. *Assume $[k : k^p] \leq p$. Then every unipotent subgroup U of $G(k)$ is contained in the unipotent radical of a k–parabolic subgroup of G.*

The case $p = 2$ is due to Tits and the theorem was also known if p is not a torsion prime of G and if G has type A_n or C_n. One says that a k–closed unipotent subgroup of G/k is k–embeddable in a Borel subgroup if there exists a k–Borel subgroup of G/k, which contains U. According to Borel–Tits, an unipotent subgroup U of $G(k)$ is contained in the unipotent radical of a k–parabolic subgroup of G if and only if each element is k_s–embeddable in a Borel subgroup. There is an other reduction to the case of a single element of order p. Thus we are reduced to the following case: $k = k_s$, U is generated by an element u of order p, G/k is split and almost simple, G has not type C_n. Let us set $K = k((t))$ and denote by K_{mod} a maximal tamely ramified closure of K. The key point is to associate to u a cohomology class $\gamma_\chi(u) \in H^1(K, G)$ in the following way. Let $\chi \in H^1(K, \mathbb{Z}/p\mathbb{Z})$ be a character such that the corresponding extension L/K is totally ramified (e.g. the Artin–Schreier extension $X^p - X = \frac{1}{t}$). Viewing u as a morphism $u : \mathbb{Z}/p\mathbb{Z} \to G$, one defines $\gamma_\chi(u) = u_*(\chi) \in H^1(L/K, G) \subset H^1(K, G)$. Using the Bruhat–Tits building of G on a suitable extension of L, we prove that the class $\gamma_\chi(u)_{K_{mod}} \in H^1(K_{mod}, G)$ detects if u lies in a Borel subgroup. Kato's p–cohomological dimensions of k and K satisfy $\dim_p(k) \leq 1$ and $\dim_p(K) \leq 2$. The absolute Galois group $\mathrm{Gal}(K_{mod})$ is a p–group, hence $H^1(K_{mod}, G) = 1$ by the main theorem of a previous work, and the proof is finished.

Natural indecomposable cycles for higher Chow groups on low dimensional Jacobians

Alberto Collino (Torino, Italy)

There is a basic indecomposable higher cycle $K \in CH^g(J(C),1)$ on the Jacobian $J(C)$ of a general hyperelliptic curve C of genus g, see [C1]. Consider K_t the translation of K associated with a point $t \in C$, we prove that in general $K - K_t$ is indecomposable if $g \geq 3$. Our tool is Lewis' condition for indecomposability [L]. We produce next on the jacobian $J(C)$ of any curve C of genus 3 a geometrically natural family of higher cycles, which we call the 4-configurations, and which have trivial image under the regulator map.

We show that when C becomes hyperelliptic the family in the limit contains a component of indecomposable cycles of type $K - K_t$. This is not sufficient to yield indecomposability of the general 4-configuration. Luckily N. Fakhruddin [F] has just proved that this property holds, by looking at the boundary of the family of 4-configurations when the curve C degenerates to the stable curve $C_1 + C_2$, here C_i is of genus i. Fakhruddin's boundary is an element in the usual Chow group $CH^2(J(C_2) \times C_1)$ and it is homologous to 0. Fakhruddin's computes that the abel-jacobi image of the cycle cannot be inside the specialization of the abel jacobi image of a cycle on the general jacobian of genus 3. This is the basic ingredient which yields indecomposability.

We view the facts above as instances of a *memory* , to the effect that $CH^m(A,n)$ should remember properties of $CH^m(B,n-1)$, where A and B are abelian varieties with $dimB = dimA + 1$. Our motivation for this expectation is Bloch's definition of higher Chow groups, for instance the first higher Chow group $CH^m(X,1)$ is a relative group of cycles of codimension m in the product $X \times \mathbb{P}^1$, *relative* to $X \times \{0,\infty\}$. Now $\mathbb{P}^1$ *relative* to $\{0,\infty\}$ is $\mathbb{C}^*$, and $A \times \mathbb{C}^*$ is degeneration of $A \times E$, E an elliptic curve.

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On l-adic polylogarithms and l-adic iterated integrals

Zdzisław Wojtkowiak (Nice, France)

Let $X = P^1_{\mathbb{Q}} \setminus \{0,1,\infty\}$. Let p be a path on $V(\mathbb{C})$ from $\vec{01}$ to $z \in V(\mathbb{C})$ and let $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Then $p^{-1}\sigma(p) \in \pi_1(X_{\bar{\mathbb{Q}}}, \vec{01})$. We embed $\pi_1(X_{\bar{\mathbb{Q}}}, \vec{01})$ into non-commutative formal power series $\mathbb{Q}_l\{\{X,Y\}\}$ in the following way: x(loop around 0) $\mapsto e^X$, y(loop around 1) $\mapsto e^Y$. Let $\wedge_p(\sigma)$ be the image of $p^{-1}\sigma(p)$ in $\mathbb{Q}_l\{\{X,Y\}\}$. $\wedge_p(\sigma)$ is a Lie element. We set

$$\log \wedge_p(\sigma) = l(z)X + l_1(z)Y + l_2[Y,X] + ... + l_n[[Y,X],X^{n-2}] + ...$$

$l_n(z)$ are functions from $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ to $Z_l(n)$. As classical polylogarithms they depend on the choice of a path p. The question when $\sum n_i l_n(z_i)$ is a cocycle leads to the same conditions as in Zagier's conjecture for polylogarithms. The l-adic polylogarithms satisfy the same functional equations as classical polylogarithms.

Noncommutative Chern characters of compact Lie group C^*-algebras and compact quantum groups

(Joint work with D.N. Diep and N.Q. Tho)

Aderemi O. Kuku (ICTP, Trieste, Italy)

In the talk, we at first, construct and study Chern characters from K-theory (Algebraic and Topological) of an involutive Banach algebra A to the periodic cyclic homology $HP_*(A)$ and entire cyclic homology $HE_*(A)$.

When A is the group C^*-algebra $C^*(G)$ of a compact Lie group G, we show that the Chern characters $K_*(C^*(G)) \to HE_*(G)$ and $K_*^{alg}(C^*(G)) \to HP_*(C^*(G))$ are isomorphisms that can be identified respectively with the classical Chern characters $K_*^W(C(T)) \to HE_*^{\bar{W}}(C(T))$ and $K_*(C(T)) \to HP_*(C(T))$ that are also isomorphisms when T is a maximal torus of G with Weyl group W.

When G is a complex algebraic group with compact real form and $C_\varepsilon^*(G)$ is the C^*-algebra compact quantum group, we prove that the Chern characters $K_*(C_\varepsilon^*(G)) \to HE_*(C_\varepsilon^*(G))$ and $K_*^{alg}(C_\varepsilon^*(G)) \to HP_*(C_\varepsilon^*(G))$ induce, respectively, isomorphisms $K_*(C_\varepsilon^*(G)) \otimes \mathbb{C} \simeq HE_*(C_\varepsilon^*(G))$, $K_*^{alg}(C_\varepsilon^*(G)) \otimes \mathbb{C} \to HP_*(C_\varepsilon^*(G))$ which can respectively be with the isomorphism
$K^*(N_T) \times \mathbb{C} \to H_{DR}^*(N_T)$ where N_T is the normaliser of T in G.

A remark on the rank conjecture

Rob de Jeu (Durham, UK)

For an Abelian group A, write $A_{\mathbb{Q}}$ for $A \otimes_{\mathbb{Z}} \mathbb{Q}$. Let F be an infinite field, and let $n \geq 1$ be an integer. Suslin proved that the natural map $H_n(GL_n(F), \mathbb{Z}) \to H_n(GL(F), \mathbb{Z})$ is an isomorphism. Viewing $K_n(F)_{\mathbb{Q}}$ as a subspace of $H_n(GL(F), \mathbb{Q})$, we get a filtration on $K_n(F)_{\mathbb{Q}}$ by setting

$$F_r^{\text{rank}} K_n(F)_{\mathbb{Q}} = \mathrm{Image}(H_n(GL_r(F), \mathbb{Q})) \bigcap K_n(F)_{\mathbb{Q}}$$

for $r \geq 1$, and $F_r^{\text{rank}} = 0$ for $r \leq 0$. Here the image is the image of the natural map $H_n(GL_r(F), \mathbb{Q}) \to H_n(GL(F), \mathbb{Q})$. Suslin conjectured that for an infinite field F, for all r, we have a direct sum decomposition

$$F_r^{\text{rank}} K_n(F)_{\mathbb{Q}} \oplus F_\gamma^{r+1} K_n(F)_{\mathbb{Q}} = K_n(F)_{\mathbb{Q}}.$$

Here $F_\gamma^m K_n(F)_{\mathbb{Q}} = \oplus_{j=m}^n K_n^{(j)}(F)$ is the m–th part of the gamma filtration on $K_n(F)_{\mathbb{Q}}$, $K_n^{(j)}(F)$ being the j–th eigenspace for the Adams operations ψ^k. We prove a statement about the action of certain operators on $H_n(GL(F), \mathbb{Z})$, which implies that for an infinite field F, the rank conjecture is equivalent to the equality

$$F_r^{\text{rank}} K_n(F) = \bigoplus_{j=1}^r K_n^{(j)}(F).$$

Let $CH^p(F,n)$ denote Bloch's higher Chow groups. One can define linear higher Chow groups by using only linear cycles in the definition, obtaining groups denoted $LCH^p(F,n)$ together with a natural map $LCH^p(F,n) \to CH^p(F,n)$. Suslin also conjectured that the natural maps $H_n(GL_p(F),\mathbb{Q}) \to H_n(GL_{p+1}(F),\mathbb{Q})$ are injective. Assuming this conjecture holds for F, or under slightly weaker assumptions, we can use our results to define maps

$$\psi_n^{(p)} : \frac{F_p^{\text{rank}} K_n(F)_{\mathbb{Q}}}{F_{p-1}^{\text{rank}} K_n(F)_{\mathbb{Q}}} \to \frac{\text{Image}(H_n(GL_p(F),\mathbb{Q}))}{\text{Image}(H_n(GL_{p-1}(F),\mathbb{Q}))} \to LCH^p(F,n)_{\mathbb{Q}} \to CH^p(F,n)_{\mathbb{Q}}.$$

The last group is isomorphic to $K_n^{(p)}(F)$, and if the rank conjecture holds for F, so is the first. We conjecture that, under the assumptions needed to define $\psi_n^{(p)}$, if the rank conjecture holds for the infinite field F, then
(i) the map $\psi_n^{(p)} : F_p^{\text{rank}} K_n(F)_{\mathbb{Q}}/F_{p-1}^{\text{rank}} K_n(F)_{\mathbb{Q}} \to CH^p(F,n)_{\mathbb{Q}}$ is an isomorphism;
(ii) $\psi_n^{(p)}$ is a non–zero rational multiple of the composition of the natural isomorphism $K_n^{(p)}(F) \cong F_p^{\text{rank}} K_n(F)_{\mathbb{Q}}/F_{p-1}^{\text{rank}} K_n(F)_{\mathbb{Q}}$ implied by the rank conjecture, and Bloch's comparison isomorphism $K_n^{(p)}(F) \cong CH^p(F,n)_{\mathbb{Q}}$.

This conjecture is an attempt to unify several other conjectures in this area. For example, it implies that the map $LCH^p(F,n)_{\mathbb{Q}} \to CH^p(F,n)_{\mathbb{Q}}$ is surjective, a conjecture of Gerdes, which is known if $p = n$ or $n-1$. We verify part (i) of this new conjecture in the case $p = n$, using results of Suslin and Nesterenko.

Class field theory for varieties over p-adic fields

Tamás Szamuely (Budapest, Hungary)

Let k be a p-adic field and let X be a smooth, projective, geometrically integral k-variety. Unramified class field theory of X, as introduced by S. Bloch and S. Saito, studies a reciprocity map

$$\phi_X : SK_1(X) \to \pi_1^{ab}(X)$$

where $SK_1(X)$ is defined as the cokernel of the natural map

$$\partial : \bigoplus_{x\in X_1} K_2 k(x) \to \bigoplus_{x\in X_0} k(x)^{\times}$$

induced by the tame symbol (where X_i=points of dimension i in X).

The map ϕ_X coincides with the reciprocity map of local class field theory for k in the case X is just a point; on the other hand, when X has good reduction with (smooth) special fibre Y, it specialises via the boundary maps in K-theory to the reciprocity map

$$\rho_Y : CH_0(Y) \to \pi_1^{ab}(Y)$$

defined by S. Lang in 1956.

The easiest way of defining ϕ_X is by imitating (following S. Saito) the construction of Lang: one picks a closed point x and sends the multiplicative group of the residue field $k(x)$ into $Gal(k(x))^{ab}$ by local class field theory, then pushes the image forward into $\pi_1^{ab}(X)$ by functoriality of the fundamental group. One then has to check that this map is trivial on the image of

δ which amounts to proving a certain reciprocity law on curves. As I explained in my lecture, both this reciprocity law and the classical one used in Lang's construction are special cases of a purely geometric statement which is an easy consequence of the covariant functoriality of the Bloch-Ogus spectral sequence for proper morphisms.

Thanks to the work of S. Saito and various collaborators, much is known about the image of ϕ_X; in particular, it is dense in the good reduction case and in the semi-stable case there is a description of the cokernel in terms of the dual graph of the special fibre. However, the kernel is much more difficult to determine. It cannot be expected to be trivial since Colliot-Thélène showed that even in the case of curves of genus at least one it contains a uniquely divisible subgroup of infinite rank; conversely, any uniquely divisible subgroup of $SK_1(X)$ is contained in the kernel. So the best that can happen is that the kernel itself is uniquely divisible. Unique l-divisibility for $l \neq p$ of the kernel is known for curves with good reduction and for curves of genus one. In the last part of my lecture I showed that assuming the Bloch-Kato conjecture in degree 3 (known for $l = 2$ and perhaps $l = 3$) we get unique l-divisibility ($l \neq p$) of the kernel for surfaces satisfying $H^2(X, \mathbf{Q_l}) = 0$ (so in particular those with potentially good reduction) and that in general for varieties with good reduction the problem is intimately related to the acyclicity in degree 3 of a certain Bloch-Ogus complex which holds trivially in dimension ≤ 2 and which is a special case of a conjecture of Kato's in general. The proofs of these results exploit Voevodsky's theory of motivic complexes.

Negative K-theory of exact categories

Marco Schlichting (Paris, France)

For any exact category $\mathcal{E}$ we construct an exact category $S\mathcal{E}$, called suspension of $\mathcal{E}$, such that the K-theory space $K(\tilde{\mathcal{E}})$ of the idempotent completion $\tilde{\mathcal{E}}$ of $\mathcal{E}$ has the same homotopy type as $\Omega K(S\mathcal{E})$. We therefore obtain a (in general non-connective) Ω-spectrum $\mathbb{K}(\mathcal{E})$ by setting $\mathbb{K}(\mathcal{E})_n = K(\widetilde{S^n\mathcal{E}})$ whose positive homotopy groups are the Quillen K-groups of $\mathcal{E}$. Our construction gives a definition of negative algebraic K-groups of an arbitrary exact category $\mathcal{E}$ as the negative homotopy groups of $\mathbb{K}(\mathcal{E})$, or equivalently as $\mathbb{K}_i(\mathcal{E}) = \pi_i\mathbb{K}(\mathcal{E}) = K_0(\widetilde{S^i\mathcal{E}})$ for $i \leq 0$. It coincides with Bass' negative K-theory of a ring A if $\mathcal{E}$ is $\mathcal{P}(A)$, the (split exact) category of finitely generated projective A modules. More generally, for any split exact category $\mathcal{E}$, our negative K-groups agree with those defined by Karoubi and Pedersen-Weibel. We conjecture that our negative K-groups coincide with those defined by Thomason-Trobaugh for the exact category of vector bundles on a quasi-compact and quasi-separated scheme supporting an ample family of line bundles.

We prove a localization theorem, extend Quillen's resolution and additivity theorems to negative K-theory and show that the negative K-theory of noetherian abelian categories is trivial. For an exact functor f, we construct an exact category $C(f)$ whose $\mathbb{K}$-theory is the homotopy cofiber of $\mathbb{K}(f)$. The long exact sequence in positive and negative K-theory associated to the localization of a ring with respect to a left denominator set of right non-zero divisors is an example of our setting. We also remark that the Pedersen-Weibel K-homology theory on finite simplicial complexes has an extension to exact categories within our framework.

Gersten's conjecture in the equicharacteristic case

Ivan Panin (Steklov Mathematical Institute, St.Petersburg, Russia)

The Gersten conjecture in K-theory ([Q], Conjecture 5.10) inspired many deep results [BO], [Ga], [Gr], [C-THK], [PS], [R], [V]. This conjecture was proved by Gersten for the case of discreat valuation rings with finite residue fields and for certain equi-characteristic discreat valuation rings. Quillen proved this conjecture for regular local rings of the geometric type. And later Sherman proved it for all equi-characteristic discreat valuation rings. We prove the following result

Theorem. *Let R be an equi-characteristic regular local ring. Then the Gersten complex*

$$0 \to K_*(R) \to K_*(K) \to \bigoplus_{ht(p)=1} K_{*-1}(k(p)) \to \ldots$$

is exact, where K is the quotient field of R.

The proof of this theorem is based on a theorem of D.Popescu, on a limit theorem of Grothendieck, and on the result of Quillen in the geometric case.

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Berichterstatter: Marco Schlichting (Paris)

E-mail Adressen der Teilnehmer

Pedro Luis del Angel <mat9e0@sp2.power.uni-essen.de,plar@xanum.uam.mx>

Paul Balmer <paul.balmer@ima.unil.ch,balmer@math.unice.fr>

Grzegorz Banaszak <banaszak@math.amu.edu.pl>

Alberto Collino <collino@dm.unito.it>

Bjrn Ian Dundas <bjoerndu@math.uio.no>

Ofer Gabber <gabber@ihes.fr>

Herbert Gangl <herbert@exp-math.uni-essen.de,herbert@mpim-bonn.mpg.de>

Thomas Geisser <geisser@ms.u-tokyo.ac.jp>

P. Gille <philippe.gille@math.u-psud.fr>

Daniel R. Grayson <dan@math.uiuc.edu>

Masana Harada <harada@kusm.kyoto-u.ac.jp>

Lars Hesselholt <larsh@math.mit.edu>

Thomas Huettemann <huette@Mathematik.Uni-Bielefeld.DE>

Oleg Izhboldin <oleg@mathematik.uni-bielefeld.de>

Uwe Jannsen <jannsen@kurims.kyoto-u.ac.jp,jannsen@mi.uni-koeln.de>

J. F. (Rick) Jardine <jardine@uwo.ca>

Rob de Jeu <jeu@cco.caltech.edu>

Bruno Kahn <kahn@math.jussieu.fr>

Bernhard Koeck <Bernhard.Koeck@math.uni-karlsruhe.de>

Aderemi O. Kuku <kuku@ictp.trieste.it>

Steven Lichtenbaum <slicht@brownvm.brown.edu>

Stefan J. Mueller-Stach <mueller-stach@uni-essen.de>

Ivan A. Panin <panin@pdmi.ras.ru,panin@lomi.spb.su>

Claudio Pedrini <pedrini@dima.unige.it>

Emmmanuel Peyre <peyre@math.u-strasbg.fr>

Leonid Positselski <posic@ias.edu>

Ulf Rehmann <rehmann@mathematik.uni-bielefeld.de>

Andreas Rosenschon <rosensch@dima.unige.it>

Takeshi Saito <t-saito@ms.u-tokyo.ac.jp>

Marco Schlichting <schlicht@math.jussieu.fr>

Victor P. Snaith <snaith@mcmaster.ca>

Michael Spiess <spiess@mathi.uni-heidelberg.de>

Ross Staffeldt <ross@nmsu.edu>

Tamas Szamuely <szamuely@math-inst.hu>

Burt Totaro <totaro@math.uchicago.edu>

Alexander Vishik <vishik@icarus.math.mcmaster.ca>

Mark E. Walker <mwalker@math.unl.edu>

Charles Weibel <weibel@math.rutgers.edu>

Zdzislaw Wojtkowiak <wojtkow@math.unice.fr>