

Motivic cohomology are isomorphic to higher Chow groups.

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In this short paper we show that the motivic cohomology groups defined in [3],[4] are isomorphic to the motivic cohomology groups defined in [1,] for smooth schemes over any field. In view of [1, Corollary 11.2] it implies that motivic cohomology of [3],[4] are isomorphic to higher Chow groups. Because of the homotopy invariance property of higher Chow groups this implies that the Cancellation Theorem [4, Theorem 2.1] holds over any field. Both facts were previously known only under the resolution of singularities assumption. The only new element in the proof is Proposition 5.

Recall that the motivic complex $\mathbf{Z}(q)$ of weight q was defined in [4],[3] as $C_*(\mathbf{Z}_{tr}(\mathbf{G}_m^{\wedge q}))[-q]$. In [1, §11] Friedlander and Suslin defined complexes which we will denote $\mathbf{Z}_{tr}^{SF}(q)$ as $C_*(z_{equi}(\mathbf{A}^q, 0))[-2q]$. The hypercohomology with coefficients in $\mathbf{Z}_{tr}^{SF}(q)$ are shown in [1] to coincide with the higher Chow groups for smooth varieties over all fields. In this paper we prove the following result.

Theorem 1 *For any field k the complexes of sheaves with transfers $\mathbf{Z}_{tr}(q)$ and $\mathbf{Z}_{tr}^{SF}(q)$ on Sm/k are quasi-isomorphic in the Zariski topology.*

Corollary 2 *For any field k , any smooth scheme X over k and any $p, q \in \mathbf{Z}$ one has a natural isomorphism*

$$Hom_{DM^{eff,-}(k)}(M(X), \mathbf{Z}(q)[p]) = CH^q(X, 2q - p).$$

It is clearly sufficient to proof the theorem for a perfect field k . Since both complexes are complexes of presheaves with transfers with homotopy invariant ([5, Prop. 3.6]) cohomology presheaves it is sufficient to show that they are quasi-isomorphic in the Nisnevich topology i.e. that the corresponding objects of $DM^{eff,-}(k)$ are isomorphic. We construct this isomorphism as a composition of two. One is of the form

$$C_*(\mathbf{Z}_{tr}(\mathbf{P}^n)/\mathbf{Z}_{tr}(\mathbf{P}^{n-1})) \rightarrow C_*(\mathbf{Z}_{tr}(\mathbf{G}_m^{\wedge n}))[n] \quad (1)$$

and the second one of the form

$$C_*(\mathbf{Z}_{tr}(\mathbf{P}^n)/\mathbf{Z}_{tr}(\mathbf{P}^{n-1})) \rightarrow C_*(z_{equi}(\mathbf{A}^n, 0)) \quad (2)$$

The construction of the first one is well known. Given a covering $X = \cup_{i \in I} U_i$ of X by a finite collection of Zariski open subsets we can form in the usual way a complex of the form

$$0 \rightarrow \mathbf{Z}_{tr}(\cap_{i \in I} U_i) \rightarrow \dots \rightarrow \oplus_{i \in I} \mathbf{Z}_{tr}(U_i) \rightarrow \mathbf{Z}_{tr}(X) \rightarrow 0 \quad (3)$$

Lemma 3 *The complex (3) is exact in the Nisnevich topology.*

Consider the covering $\mathbf{P}^n = \cup_{i=0, \dots, n} \mathbf{A}^n$ and let C_n be the corresponding complex of the form (3). For $m < n$ the embedding $\mathbf{P}^m \rightarrow \mathbf{P}^n$ gives a morphism of complexes $C_m \rightarrow C_n$. In particular C_n/C_{n-1} is a complex which “starts” with $\mathbf{Z}_{tr}(\mathbf{P}^n)/\mathbf{Z}_{tr}(\mathbf{P}^{n-1})$ and ends by $\mathbf{Z}_{tr}((\mathbf{A}^1 - \{0\})^n)$. By Lemma 3 it defines an element in $Ext^n(\mathbf{Z}_{tr}(\mathbf{P}^n)/\mathbf{Z}_{tr}(\mathbf{P}^{n-1}), \mathbf{Z}_{tr}((\mathbf{A}^1 - \{0\})^n))$. Projecting this element with respect to the morphism $\mathbf{Z}_{tr}((\mathbf{A}^1 - \{0\})^n) \rightarrow \mathbf{Z}_{tr}(\mathbf{G}_m^{\wedge n})$ we get an element

$$c_n \in Ext^n(\mathbf{Z}_{tr}(\mathbf{P}^n)/\mathbf{Z}_{tr}(\mathbf{P}^{n-1}), \mathbf{Z}_{tr}(\mathbf{G}_m^{\wedge n}))$$

We define (1) as the morphism in $DM^{eff,-}$ corresponding to c_n . An inductive argument using only the Mayer-Vietoris and the homotopy invariance properties of motives implies the following.

Proposition 4 *The morphism (1) is an isomorphism.*

We define (2) as the morphism given by the obvious homomorphisms of presheaves with transfers

$$\mathbf{Z}_{tr}(\mathbf{P}^n)/\mathbf{Z}_{tr}(\mathbf{P}^{n-1}) \rightarrow z_{equi}(\mathbf{A}^n, 0) \quad (4)$$

Proposition 5 *The morphism of complexes of presheaves with transfers*

$$C_*(\mathbf{Z}_{tr}(\mathbf{P}^n)/\mathbf{Z}_{tr}(\mathbf{P}^{n-1})) \rightarrow C_*(z_{equi}(\mathbf{A}^n, 0))$$

induced by (4) is a quasi-isomorphism in the Nisnevich topology.

Proof: Define F_n as the subpresheaf in $z_{equi}(\mathbf{A}^n, 0)$ such that for a smooth connected X the group $F_n(X)$ is generated by closed irreducible subschemes Z of $X \times \mathbf{A}^n$ which are equidimensional over X of relative dimension 0 and

which do not intersect $X \times \{0\}$. We have the following diagram of morphisms of presheaves

$$\begin{array}{ccc}
\mathbf{Z}_{tr}(\mathbf{P}^{n-1}) & = & \mathbf{Z}_{tr}(\mathbf{P}^{n-1}) \\
\downarrow & & \downarrow \\
\mathbf{Z}_{tr}(\mathbf{P}^n - \{0\}) & \rightarrow & \mathbf{Z}_{tr}(\mathbf{P}^n) \\
\downarrow & & \downarrow \\
F_n & \rightarrow & z_{equi}(\mathbf{A}^n, 0)
\end{array} \tag{5}$$

The statement of the proposition follows from the diagram (5) and Lemmas 6,7,8.

Lemma 6 *The morphism of Nisnevich sheaves*

$$\mathbf{Z}_{tr}(\mathbf{P}^n)/\mathbf{Z}_{tr}(\mathbf{P}^n - \{0\}) \rightarrow z_{equi}(\mathbf{A}^n, 0)/F_n$$

is an isomorphism.

Proof: Let S be a henselian local scheme and Z be an irreducible closed subset of $Z \times \mathbf{A}^n$ which is equidimensional of relative dimension zero over Z . If Z does not belong to $F_n(S)$ then the intersection of Z with $S \times \{0\}$ is nonempty. Since it is closed it must contain the closed point of S and thus the image of Z in S contains the closed point. But then Z is finite over S by [2, Th.4.2(c)] and thus it is closed in $S \times \mathbf{P}^n$. This proves surjectivity. The proof of injectivity is similar.

Lemma 7 *The complex of presheaves $C_*(F_n)$ is exact.*

Proof: Let us show that F_n is contactible i.e. that there exist a collection of natural in X homomorphisms

$$\phi_X : F_n(X) \rightarrow F_n(X \times \mathbf{A}^1)$$

such that the composition of ϕ_X with the restriction to $X \times \{0\}$ is zero and to $X \times \{1\}$ is identity. Consider the morphism $\pi : X \times \mathbf{A}^n \times \mathbf{A}^1 \rightarrow X \times \mathbf{A}^n$ given by the formula $\pi(x, r, t) = (x, rt)$. An element of $F_n(X)$ is a cycle \mathcal{Z} on $X \times \mathbf{A}^n$ which does not intersect $X \times \{0\}$. Since π is flat over $X \times (\mathbf{A}^n - \{0\})$ the cycle $\pi^*(\mathcal{Z})$ is well defined and one checks immediately that it belongs to $F_n(X \times \mathbf{A}^1)$. One further verifies that the homomorphisms we constructed are compatible with the functoriality in X and satisfy the required conditions for the restrictions to $X \times \{0\}$ and $X \times \{1\}$.

The homomorphisms ϕ_X define for any X a homomorphism $C_*(F_n)(X) \rightarrow C_*(F_n)(X \times \mathbf{A}^1)$. By [5, Prop.3.6] the restriction maps $C_*(F_n)(X \times \mathbf{A}^1) \rightarrow C_*(F_n)(X)$ corresponding to $X \times \{0\}$ and $X \times \{1\}$ are a quasi-isomorphisms. This implies that our homomorphism is a quasi-isomorphism which equals zero on cohomology. Thus $C_*(F_n)(X)$ is exact.

Lemma 8 *The morphism*

$$C_*(\mathbf{Z}_{tr}(\mathbf{P}^{n-1})) \rightarrow C_*(\mathbf{Z}_{tr}(\mathbf{P}^n - \{0\}))$$

is a quasi-isomorphism in the Nisnevich topology.

Proof: It is sufficient to show that the morphism $M(\mathbf{P}^{n-1}) \rightarrow M(\mathbf{P}^n - \{0\})$ is an isomorphism in $DM^{eff,-}(k)$. This morphism is a section of the morphism $M(p)$ where $p : \mathbf{P}^n - \{0\} \rightarrow \mathbf{P}^{n-1}$ is a locally trivial in the Zariski topology bundle whose fibers are affine spaces. Any such p gives an isomorphism of motives because of the homotopy invariance and the Mayer-Vietoris properties.

References

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