THE EQUI-CHARACTERISTIC CASE OF THE GERSTEN CONJECTURE

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Abstract. One of the well-known problem in the algebraic K-theory is the Gersten conjecture. The geometric case of this conjecture was proved by D. Quillen. The equi-characteristic case of the conjecture is proved in this preprint. This covers the result of D. Quillen. Actually we use the result of D. Quillen and certain results of D. Popescu and A. Grothendieck.

Introduction

The Gersten conjecture in K-theory ([Q, Conj. 5.10] and/or [Ge]) inspired many deep results [BO], [Ga], [Gr], [GS], [C-THK], [PS], [R], [V]. The geometric case of the Gersten conjecture in K-theory was proved by D. Quillen in his fundamental paper [Q, Th. 5.11]. The case of equicharacteristic 1-dimensional rings was done by C. C. Sherman in [Sh]. One of the goal of the present paper is to prove the equi-characteristic case of the Gersten conjecture for rings of any dimension. Namely the following result holds

Theorem A. *Let R be an equi-characteristic regular local ring. Then the Gersten complex*

$$0 \longrightarrow K_*(R) \longrightarrow K_*(K) \longrightarrow \underset{ht(\mathfrak{p})=1}{\oplus} K_{*-1}(k(\mathfrak{p})) \longrightarrow \underset{ht(\mathfrak{q})=1}{\oplus} K_{*-2}(k(\mathfrak{q})) \longrightarrow \dots$$

is exact, where K is the quotient field of R.

The proof of Theorem A is based on a theorem of D. Popescu, on a theorem of A. Grothendieck and on the result of D. Quillen in the geometric case.

Theorem (D. Popescu [P])**.** *Let R be an equi-characteristic regular local ring. Then there exists a perfect field k contained in R and for each such a field k the ring R coincides with a filtering inductive limit of finitely generated smooth k-algebras.*

Grothendieck limit theorem (A. Grothendieck [SGA], Exp. VII, Th. 5.7)**.** *Let k be a field and let* Sch$/k$ *be the category of Noetherian k-schemes. Let F be a presheaf of abelian groups on* Sch$/k$ *commuting with filtering projective limits of Noetherian affine schemes, i.e., the canonical map $\varinjlim F(S^\alpha) \to F(\varinjlim S^\alpha)$ is an isomorphism for each filtering inductive system of Noetherian k-algebras with a Noetherian limit $S = \varinjlim S^\alpha$. Let $F^\sim$ be the Zariski associated sheaf of F. Then for an inductive filtering system of Noetherian k-algebras R^β with a Noetherian limit $R = \varinjlim R^\beta$ and for each integer $p \geq 0$ the canonical map $\varinjlim H^p_{\mathrm{Zar}}(X^\beta, F^\sim) \to H^p_{\mathrm{Zar}}(X, F^\sim)$ is an isomorphism (here $X^\beta = \mathrm{Spec}\,(R^\beta)$, $X = \mathrm{Spec}\,(R)$).*

Theorem (D. Quillen [Q], Th. 5.11)**.** *Let k be a field and let R be an essentially smooth local k-algebra, i.e., $R = A_{\mathfrak{p}}$ for a finitely generated smooth k-algebra A and a prime ideal $\mathfrak{p} \subset A$. Then the Gersten complex*

$$0 \longrightarrow K_*(R) \longrightarrow K_*(K) \longrightarrow \bigoplus_{ht(\mathfrak{p})=1} K_{*-1}(k(\mathfrak{p})) \longrightarrow \bigoplus_{ht(\mathfrak{q})=1} K_{*-2}(k(\mathfrak{q})) \longrightarrow \dots$$

is exact, where K is the quotient field of K.

The paper is organized as follows. Section 1 is devoted to the proof of Theorem A. Proofs of all lemmas stated in §1 are given in §§2–4. Section 5 contains some other results which can be proved by the method of this paper. Section 6 contains a sketch of a proof of the Grothendieck limit theorem. Section 7 contains a general result concerning a weak form of the Gersten conjecture.

§1 Proof of Theorem A

We first state several lemmas which will be proved in §§2–4 below. First two lemmas are about the geometric case. To state them consider an essentially smooth local k-algebra S for a field k. In the other words, the ring S is of the form $A_{\mathfrak{p}}$, where A is a finitely generated smooth k-algebra and $\mathfrak{p}$ is a prime ideal in A. Let $\underline{\underline{K}}_*$ be the Zariski sheaf associated with the K-functor. Let $X = \mathrm{Spec}\,(S)$ and $X_f = \mathrm{Spec}\,(S_f)$.

1.1. Lemma. *Let $f \in S$ be a local parameter. Then the canonical map*

$$K_*(S_f) \to \underline{\underline{K}}_*(S_f)$$

is an isomorphism. In the other words, $K_*(S_f) = H^0_{\mathrm{Zar}}(X_f, \underline{\underline{K}}_*)$.

1.2. Lemma. *If $p \geq 1$, then $H^p_{\mathrm{Zar}}(X_f, \underline{\underline{K}}_*) = 0$.*

Other five lemmas are devoted to the equi-characteristic case. To state them let R be a regular local ring (equi-characteristic) and let $f \in R$ be a local parameter. Let $X = \mathrm{Spec}\,(R)$ and $X_f = \mathrm{Spec}\,(R_f)$ and let $Z \subseteq X$ be the vanishing locus of f.

1.3. Lemma. *The map $K_*(R_f) \to \underline{\underline{K}}_*(R_f)$ is an isomorphism.*

1.4. Lemma. *$H^p_{\mathrm{Zar}}(X_f, \underline{\underline{K}}_*) = 0$ for each $p \geq 1$.*

1.5. Lemma. *The sequence*

$$0 \longrightarrow K_*(X) \longrightarrow K_*(X_f) \xrightarrow{d_z} K_{*-1}(Z) \longrightarrow 0$$

is exact.

1.6. Lemma. *Assume the Gersten conjecture holds for the scheme Z, then the group $K_{*-1}(Z)$ is a subgroup of the group $K_{*-1}(k(Z))$ and the composite map $\underline{\underline{K}}_*(X_f) \longrightarrow K_*(k(X)) \xrightarrow{\partial_z} K_{*-1}(k(Z))$ takes value in the subgroup $K_{*-1}(Z)$ of the group $K_{*-1}(k(Z))$.*

1.7. Lemma. *Under the hypotheses of Lemma 1.6 the diagram*

$$\begin{tikzcd}
\underline{\underline{K}}_*(X_f) \arrow[r, "\partial_z"] & K_{*-1}(Z) \\
K_*(X_f) \arrow[u, "\mathrm{can}"'] \arrow[r, "d_z"] & K_{*-1}(Z) \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

commutes, where the map d_z is the boundary map in the localization sequence for the open subscheme X_f of the scheme X.

These seven lemmas will be proved below in §§2–4. Now we give

1.8. Proof of Theorem A. We proceed by the induction on the dimension of the ring R. If $\dim(R) = 1$ and $f \in R$ is a local parameter then Lemma 1.5 proves Theorem A in this case. In fact, the ring R_f is the quotient field K of R in this case and the ring R/fR is the only residue field. One may assume now that $\dim(R) \geq 2$ and theorem A holds for every regular local ring (equi-characteristic) of dimension strictly smaller than $d = \dim(R)$.

The vanishing locus Z of the local parameter $f \in R$ is a regular local scheme of dimension $d-1$ and X_f is a regular scheme of dimension $d-1$. Let $g_*(X)$ be the Gersten complex of the scheme X, i.e.,

$$g_*(X) = \big(0 \to K_*(k(X)) \longrightarrow \underset{ht(\mathfrak{p})=1}{\oplus} K_{*-1}(k(\mathfrak{p})) \longrightarrow \underset{ht(\mathfrak{q})=2}{\oplus} K_{*-2}(k(\mathfrak{q})) \longrightarrow \dots\big).$$

Let $g_*(X_f)$ be the Gersten complex of the scheme X_f, i.e.,

$$g_*(X_f) = \big(0 \longrightarrow K_*(k(X)) \longrightarrow \underset{\substack{ht\mathfrak{p}=1 \\ f\notin\mathfrak{p}}}{\oplus} K_{*-1}(k(\mathfrak{p})) \longrightarrow \underset{\substack{ht\mathfrak{q}=2 \\ f\notin\mathfrak{q}}}{\oplus} K_{*-2}(k(\mathfrak{q})) \longrightarrow \dots\big).$$

Let $g^z_*(X)$ be a subcomplex of the complex $g_*(X)$ "supporting" on the subscheme Z, i.e.,

$$g^z_*(X) = \big(0 \longrightarrow 0 \longrightarrow K_{*-1}(k(Z)) \longrightarrow \underset{\substack{ht\mathfrak{q}=2 \\ f\in\mathfrak{q}}}{\oplus} K_{*-2}(k(\mathfrak{q})) \longrightarrow \dots\big).$$

Let $\underline{\underline{g}}_*(X)$, $\underline{\underline{g}}_*(X_f)$, and $\underline{\underline{g}}^z_*(X)$ be the corresponding complexes of sheaves on schemes X, X_f, and X, respectively. Clearly the complex $g^z_*(X)$ coincides with the Gersten complex $g_{*-1}(Z)$ of Z sheafted by "-1", i.e., $g^z_*(X) = g_{*-1}(Z)[-1]$. Thus one has the short exact sequence of complexes

$$(\dagger) \qquad 0 \longrightarrow g_{*-1}(Z)[-1] \longrightarrow g_*(X) \longrightarrow g_*(X_f) \longrightarrow 0.$$

Since $\dim(Z) = d-1$ the inductive assumption shows that the complex $g_{*-1}(Z)$ is a resolution of the group $K_{*-1}(Z)$. In particular, $H^p(g_{*-1}(Z)) = 0$ for $p \geq 1$. Thus, $H^p(g_{*-1}(Z)[-1]) = 0$ for $p \geq 2$ and $H^1(g_{*-1}(Z)[-1]) = K_{*-1}(Z)$.

Observe as well that the complex $g_*(X_f)$ is the complex of global sections of the sheaf complex $\underline{\underline{g}}_*(X_f)$. The complex $\underline{\underline{g}}_*(X_f)$ consists of flasque sheaves and it is a resolution of the sheaf K of the scheme X because dim $(X$ $) < d$. Thus,

$H^p(g_*(X_f)) = H^p_{\text{Zar}}(X_f, \underline{\underline{K}}_*)$. Lemmas 1.3 and 1.4 shows now that $H^p(g_*(X_f)) = 0$ for $p \geq 1$ and $H^0(g_*(X_f)) = \underline{\underline{K}}_*(X_f)$.

Now, the long cohomology sequence associated with the short exact sequence (†) shows that $H^p(g_*(X)) = 0$ for $p \geq 2$ and groups $H^0(g_*(X))$ and $H^1(g_*(X))$ are determined by the exact sequence

$$0 \to H^0(g_*(X)) \to \underline{\underline{K}}_*(X_f) \xrightarrow{\partial_z} K_{*-1}(Z) \to H^1(g_*(X)) \to 0.$$

Lemma 1.7 shows that the diagram

$$\begin{tikzcd}
\underline{\underline{K}}_*(X_f) \arrow[r, "\partial_z"] & K_{*-1}(Z) \\
K_*(X_f) \arrow[u, "\text{can}"'] \arrow[r, "d_z"] & K_{*-1}(Z) \arrow[u, "\text{id}"']
\end{tikzcd}$$

commutes. Lemma 1.3 shows that the map "can" in this diagram is an isomorphism. Thus Lemma 1.5 proves that $H^1(g_*(X)) = 0$ and the map "can" induces an isomorphism $K_*(X) \xrightarrow{\sim} H^0(g_*(X))$. Thus the complex $g_*(X)$ is a resolution of the group $K_*(X)$. Theorem A is proved. □

§2. PROOFS OF LEMMAS 1.1 AND 1.2

Let $X = \text{Spec}\,(S)$ and $X_f = \text{Spec}\,(S_f)$, and $Z = \text{Spec}\,(S/fS)$. Recall that for an irreducible scheme Y the Gersten complex $g_*(Y)$ is the complex

$$0 \to K_*(k(Y)) \xrightarrow{\partial} \bigoplus_{\text{codim}(y)=1} K_{*-1}(k(y)) \xrightarrow{\partial} \bigoplus_{\text{codim}(z)=2} K_*(k(z)) \xrightarrow{\partial} \dots .$$

2.1. Proof of Lemma 1.1. The Gersten complexes $g_{*-1}(Z)$ and $g_*(X)$ are resolutions of groups $K_{*-1}(Z)$ and $K_*(X)$ respectively by the geometric case of the Gersten conjecture proved by Quillen (see the introduction). The sheaf Gersten complex $\underline{\underline{g}}_*(X_f)$ is a resolution of the sheaf $\underline{\underline{K}}_*$ on the scheme X_f again by the geometric case of the Gersten conjecture. Since $\Gamma(X_f, \underline{\underline{g}}_*(X_f)) = g_*(X_f)$ hence $H^0(g_*(X_f)) = \underline{\underline{K}}_*(X_f)$. Therefore the short exact sequence of complexes (†) from §1

$$0 \to g_{*-1}(Z)[-1] \to g_*(X) \to g_*(X_f) \to 0$$

gives rise (in part) to a short exact sequence of groups

$$(*) \qquad 0 \to K_*(X) \to \underline{\underline{K}}_*(X_f) \xrightarrow{\partial_z} K_{*-1}(Z) \to 0.$$

On the other hand, the long exact localization sequence of K-groups for the open embedding $X_f \hookrightarrow X$ gives rise to a short exact sequence (use again the geometric case of the Gersten conecture, more specifically use the injective of the map $K_*(X) \to K_*(X_f)$)

$$(**) \qquad 0 \to K_*(X) \to K_*(X_f) \xrightarrow{d_z} K_{*-1}(Z) \to 0.$$

The canonical morphism of complexes $(**) \to (*)$ is the identity on $K_*(X)$ and on $K_{*-1}(Z)$. Thus it is an isomorphism on the middle terms, i.e., the map $K_*(X_f) \to$ $\underline{\underline{K}}_*(X_f)$ is an isomorphism. The lemma is proved.

Proof of Lemma 1.2. Consider once again the short exact sequence

$$0 \longrightarrow g_{*-1}(Z)[-1] \longrightarrow g_*(X) \longrightarrow g_*(X_f) \longrightarrow 0.$$

By the geometric case of the Gersten conjecture the complexes $g_{*-1}(Z)$ and $g_*(X)$ are resolutions of the groups $K_{*-1}(Z)$ and $K_*(X)$, respectively. Thus $H^p(g_*(X_f)) = 0$ for $p > 0$.

On the other hand, $g_*(X_f)$ is a complex of global sections of the sheaf complex $\underline{\underline{g}}_*(X_f)$ and the last one is a flasque resolution of the sheaf $\underline{\underline{K}}_*$ on the scheme X_f (again use the geometric case of the Gersten conjecture). Thus $H^p(g_*(X_f)) = H^p_{\mathrm{Zar}}(X_f, \underline{\underline{K}}_*)$ for each $p \geq 0$. Hence $H^p_{\mathrm{Zar}}(X_f, \underline{\underline{K}}_*) = 0$ for $p > 0$. □

§3. Certain observations

This section contains certain observations concerning the theorem of D. Popescu mentioned in the introduction. We use these observations in §4 below to prove Lemmas 1.3–1.7.

Let R be a regular local ring (equi-characteristic). Then there exists a perfect subfield k in R. The theorem of D. Popescu mentioned in the introduction states that R is a filtering inductive limit of finitely generated smooth k-algebras: $R = \varinjlim R^{\alpha}$.

3.1. Let $\mathfrak{M} \subset R$ be the only maximal ideal in R. Let $\varphi_{\alpha} : R^{\alpha} \to R$ be the canonical homomorphism and let $\varphi_{\alpha\beta} : R^{\alpha} \to R^{\beta}$ be the transition homomorphism of the inductive system ($\beta \geq \alpha$). Set $\mathfrak{p}_{\alpha} = \varphi_{\alpha}^{-1}(\mathfrak{M})$. Then $\mathfrak{p}_{\alpha}$ is a prime ideal in R^{α}. Denote by $R^{\alpha}_{\mathfrak{p}_{\alpha}}$ the localization of R^{α} with respect to the prime ideal $\mathfrak{p}_{\alpha}$. Set $\mathfrak{M}(\alpha) = \mathfrak{p}_{\alpha} \cdot R^{\alpha}_{\mathfrak{p}_{\alpha}}$. It is the only maximal ideal in $R^{\alpha}_{\mathfrak{p}_{\alpha}}$. Let $\psi_{\alpha\beta} : R^{\alpha}_{\mathfrak{p}_{\alpha}} \to R^{\beta}_{\mathfrak{p}_{\beta}}$ ($\beta \geq \alpha$) denote the only ring homomorphism making commutative the diagram

$$\begin{tikzcd}
R^{\alpha} \arrow[r, "\varphi_{\alpha\beta}"] \arrow[d] & R^{\beta} \arrow[d] \\
R^{\alpha}_{\mathfrak{p}_{\alpha}} \arrow[r, "\psi_{\alpha\beta}"] & R^{\beta}_{\mathfrak{p}_{\beta}}.
\end{tikzcd}$$

Let $\psi_{\alpha} : R^{\alpha}_{\mathfrak{p}_{\alpha}} \to R$ denote the only ring homomorphism making commutative the diagram

$$\begin{tikzcd}
R^{\alpha} \arrow[r, "\varphi_{\alpha}"] \arrow[d] & R \arrow[d, "\mathrm{id}"] \\
R^{\alpha}_{\mathfrak{p}_{\alpha}} \arrow[r, "\psi_{\alpha\beta}"] & R.
\end{tikzcd}$$

Clearly the following lemma holds

3.2. Lemma. *The system of transition maps* $\{\psi_{\alpha\beta} : R^{\alpha}_{\mathfrak{p}_{\alpha}} \to R^{\beta}_{\mathfrak{p}_{\beta}}\}$ *is a filtering inductive system and the canonical homomorphism*

$$\varinjlim R^{\alpha}_{\mathfrak{p}_{\alpha}} \to R$$

is an isomorphism. For each index α *the ring* $R^{\alpha}_{\mathfrak{p}_{\alpha}}$ *is an essentially smooth local* k*-algebra over the perfect field* k*.*

Now let $f \in R$ be a local parameter in R. Let α be an index such that the element f can be lifted up to an element of R^{α}. Choose and fix a lift $f \in R^{\alpha}$

of the element f, i.e., $\psi_\alpha(f_\alpha) = f$. Increasing the index α one may assume that $f_\alpha \in \mathfrak{M}(\alpha)$. Since $f \notin \mathfrak{M}^2$ hence $f_\alpha \notin \mathfrak{M}(\alpha)^2$. Thus f_α is a local parameter in $R^\alpha_{\mathfrak{p}_\alpha}$. For each $\beta \geq \alpha$ set $f_\beta = \psi_{\alpha\beta}(f_\alpha)$. Clearly $f_\beta \in \mathfrak{M}(\beta)$ and $f_\beta \notin \mathfrak{M}(\beta)^2$ and the following lemma holds.

3.3. Lemma. *The canonical homomorphisms*

$$\varinjlim_{\beta\geq\alpha} R^\beta_{\mathfrak{p}_\beta} \longrightarrow R, \quad \varinjlim_{\beta\geq\alpha} R^\beta_{\mathfrak{p}_\beta, f_\beta} \longrightarrow R_f$$

are isomorphisms, and for each $\beta \geq \alpha$ the element f_β is a local parameter of the ring $R^\beta_{\mathfrak{p}_\beta}$.

3.4. Notation. *Set* $X^\beta_{\mathfrak{p}_\beta} = \operatorname{Spec}(R^\beta_{\mathfrak{p}_\beta})$, $X^\beta_{\mathfrak{p}_\beta, f_\beta} = \operatorname{Spec}(R^\beta_{\mathfrak{p}_\beta, f_\beta})$, $X = \operatorname{Spec}(R)$, $X_f = \operatorname{Spec}(X_f)$.

§4. Proof of Lemmas 1.3–1.7

We keep the notation of Section 3 in this Section.

4.1. Proof of Lemma 1.3. For a commutative ring S we write $\underline{\underline{K}}_*(S)$ for the group $\underline{\underline{K}}_*(\operatorname{Spec} S)$ (the evaluation of the sheaf $\underline{\underline{K}}_*$ on the scheme $\operatorname{Spec} S$). Now consider the commutative diagram

$$\begin{tikzcd}
\varinjlim_{\beta\geq\alpha} K_*(R^\beta_{\mathfrak{p}_\beta, f_\beta}) \arrow[r] \arrow[d] & \varinjlim_{\beta\geq\alpha} \underline{\underline{K}}_*(R^\beta_{\mathfrak{p}_\beta, f_\beta}) \arrow[d] \\
K_*(R_f) \arrow[r, "\psi_{\alpha\beta}"] & \underline{\underline{K}}_*(R_f),
\end{tikzcd}$$

where the vertical arrows are induced by homomorphisms $R^\beta_{\mathfrak{p}_\beta, f_\beta} \to R_f$ ($\beta \geq \alpha$) (see Lemma 3.3). Observe that both vertical arrows are isomorphisms. In fact the left one is an isomorphism because K-groups commute with filtering inductive system of rings and because of Lemma 3.3. The right-hand side arrow is an isomorphism by the Grothendieck limit theorem (see the introduction) applied to the functor $K_* : \mathrm{Sch}/k \to Ab$.

The top arrow in the diagram is an isomorphism as well. In fact each ring $R^\beta_{\mathfrak{p}_\beta}$ is an essentially smooth local k-algebra (see Lemma 3.2). Thus the top arrow is an isomorphism by Lemma 1.1. Therefore the bottom arrow is an isomorphism as well. Lemma 1.3 is proved.

4.2. Proof of Lemma 1.4. By Lemma 3.3 one has $R_f = \varinjlim_{\beta\geq\alpha} R^\beta_{\mathfrak{p}_\beta, f_\beta}$. The Grothendieck limit theorem (see the introduction) shows that

$$H^p(X_f, \underline{\underline{K}}_*) = \varinjlim_{\beta\geq\alpha} H^p(X^\beta_{\mathfrak{p}_\beta, f_\beta}, \underline{\underline{K}}_*).$$

The right-hand side of this equality vanishes for $p > 0$ by Lemma 1.2. Lemma 1.4 is now proved.

4.3. Proof of Lemma 1.5. Consider the long sequence of K-groups with respect to the open embedding $X_f \hookrightarrow X$ and the closed subscheme $Z = \mathrm{Spec}\,(R/fR)$ of the scheme X

$$\ldots \to K_*(Z) \to K_*(X) \to K_*(X_f) \xrightarrow{d_z} K_{*-1}(Z_1) \to K_{*-1}(X) \to K_{*-1}(X_f) \to \ldots .$$

To prove the lemma it suffices to check that the maps $K_*(X) \to K_*(X_f)$ are injective. For that consider the commutative diagram (see Lemma 3.3)

$$\begin{tikzcd}
\varinjlim_{\beta\geq\alpha} K_*(X^{\beta}_{\mathfrak{p}_\beta}) \arrow[r] \arrow[d, "\wr"] & \varinjlim_{\beta\geq\alpha} K_*(X^{\beta}_{\mathfrak{p}_\beta, f_\beta}) \arrow[d, "\wr"] \\
K_*(X) \arrow[r, "\psi_{\alpha\beta}"] & K_*(X_f).
\end{tikzcd}$$

The vertical arrows are isomorphisms because K-groups commute with filtering inductive systems of rings and because of Lemma 3.3. The top arrow is a monomorphism. In fact the Gersten conjecture holds in the geometric case due to the result of Quillen mentioned in the introduction and because the ring $R^{\beta}_{\mathfrak{p}(\beta)}$ is an essentially smooth local k-algebra by Lemma 3.2. Thus the map $K_*(X^{\beta}_{\mathfrak{p}_\beta}) \to K_*(X^{\beta}_{\mathfrak{p}_\beta, f_\beta})$ is injective for each index $\beta \geq \alpha$. So, the top arrow in the diagram is injective. Therefore, the map $K_*(X) \to K_*(X_f)$ is injective as well. Lemma 1.5 is proved. $\square$

4.4. Proof of Lemma 1.6. The Gersten complex $g_*(X)$ of the scheme X is a complex (see §2 for the notation). Thus $d \circ d = 0$. Therefore for each divisor Y in Z the map

$$\sum_{X \supset Z' \supset Y} \partial^{z'}_Y \circ \partial_{Z'} : K_*(k(X)) \to K_{*-2}(k(Y))$$

vanishes, where the sum runs over all irreducible divisors Z' in X containing Y. For each divisor Z' different from Z the composite map $\underline{\underline{K}}_*(X_f) \to K_*(k(X)) \overset{\partial_{Z'}}{\to} K_{*-1}(k(Z'))$ vanishes. Thus the map $\sum_{X \supset Z' \supset Y} \partial^{z'}_Y \circ \partial_{Z'}$ when restricted to $\underline{\underline{K}}_*(X_f)$ coincides with $\partial^{z}_Y \circ \partial_Z : \underline{\underline{K}}_*(X_f) \to K_{*-2}(k(Y))$. And therefore this last composite vanishes. So, one has the inclusion $\partial_Z(\underline{\underline{K}}_*(X_f)) \subset \bigcap_Y \mathrm{Ker}\,\big[\partial_Y : K_{*-1}(k(Z)) \to K_{*-2}(k(Y))\big]$. By the very assumption of Lemma 1.6 the group $K_{*-1}(Z)$ is a subgroup of the group $K_{*-1}(k(Z))$ and the right-hand side of the mentioned intersection of kernels coincides with this subgroup. Thus the inclusion $\partial_Z(\underline{\underline{K}}(X_f)) \subset K_{*-1}(Z)$ is proved. $\square$

4.5. Proof of Lemma 1.7. Consider the diagram

$$\begin{tikzcd}
K_*(k(X)) \arrow[rrr, "d"] & & & K_{*-1}(k(Z)) \\
& \underline{\underline{K}}_*(X_f) \arrow[r, "\partial_Z"] \arrow[ul] & K_{*-1}(Z) \arrow[ur] & \\
& K_*(X_f) \arrow[r, "d_Z"] \arrow[u, "\mathrm{can}"'] \arrow[dl] & K_{*-1}(Z) \arrow[u, "\mathrm{id}"'] \arrow[dr] & \\
K_*(k(X)) \arrow[uuu, "\mathrm{id}"] \arrow[rrr, "d"] & & & K_{*-1}(k(Z)) \arrow[uuu, "\mathrm{id}"']
\end{tikzcd}$$

where the map d and ∂_Z are the boundary maps in the localization sequences. All the squares in this diagram commute except may be the inner square. The map $K_{*-1}(Z) \to K_{*-1}(k(Z))$ is injective by the very hypotheses of Lemma 1.7. Therefore the inner square in the diagram commutes as well. Lemma 1.7 is proved. □

§5. Some other results

This section contains some other results which can be proved by the method of this paper.

5.1. Theorem B. *Let R be an equi-characteristic regular local ring and let A be an Azumaya algebra over R. Then the Gersten complex*

$$0 \to K_*(A) \to K_*(A \underset{R}{\otimes} K) \to \underset{ht\mathfrak{p}=1}{\oplus} K_{*-1}(A(\mathfrak{p})) \to \underset{ht\mathfrak{q}=2}{\oplus} K_{*-2}(A(\mathfrak{q})) \to \dots$$

is exact (here K is the quotient field of R and $A(\mathfrak{p}) = A \underset{R}{\otimes} k(\mathfrak{p})$).

5.2. Theorem C. *Let R be an equi-characteristic regular local ring. Let $p = \operatorname{char}(R)$ and let A be a torsion $\operatorname{Gal}(R)$-module (prime to p-torsion). Let $X = \operatorname{Spec}(R)$ and let $\eta = \operatorname{Spec}(K)$, where K is the quotient field of R. Then the Cousin complex*

$$0 \to H^*_{\mathrm{et}}(X, A) \to H^*_{\mathrm{et}}(\eta, A) \to \underset{x \in X^{(1)}}{\oplus} H^{*+1}_x(X, A) \to \underset{x \in X^{(2)}}{\oplus} H^{*+2}_y(X, A) \to \dots$$

is exact and moreover there is a canonical isomorphism $H^{*+r}_z(X, A) \overset{\sim}{=} H^{*-r}_{\mathrm{et}}(z, A(-r))$ for each codimension r point z of the scheme X.

5.3. Theorem D. *Let R be an equi-characteristic regular local ring and let k be a subfield in R. Let T be a smooth variety over k. If k is perfect, then the Gersten complex*

$$0 \to K_*(T_R) \to K_*(T_K) \to \underset{ht\mathfrak{p}=1}{\oplus} K_{*-1}(T_{k(\mathfrak{p})}) \to \underset{ht\mathfrak{q}=2}{\oplus} K_{*-2}(T_{k(\mathfrak{q})}) \to \dots$$

is exact (here for a k-algebra L we write T_L for $T \underset{k}{\otimes} L$ and K is the fraction field of R).

5.4. Theorem E. *Let R be an equi-characteristic regular local ring and let k be a subfield in R. Let $p = \operatorname{char}(k)$ and let A be a torsion $\operatorname{Gal}(k)$-module (prime to p-torsion). Let T be a smooth variety over k. If k is perfect, then the Cousin complex*

$$0 \to H^*_{\mathrm{et}}(T_R, A) \to H^*_{\mathrm{et}}(T_K, A) \to \underset{ht\mathfrak{p}=1}{\oplus} H^{*+1}_{T(\mathfrak{p})}(T_R, A) \to \underset{ht\mathfrak{q}=2}{\oplus} H^{*+2}_{T(\mathfrak{q})}(T_R, A) \to \dots$$

is exact. Moreover there is a canonical isomorphism $H^{*+r}_{T(\mathfrak{q})}(T_R, A) \overset{\sim}{=} H^{*-r}\,(T(\mathfrak{q}), A(-r))$ for each prime height r ideal $\mathfrak{q}$ in R (here $T(\mathfrak{p}) = T \underset{k}{\otimes} k(\mathfrak{p})$).

Remark. There is a general statement about the exactness of a Cousin complex, but it is not entirely clear whether the terms of the Cousin complex coincide with the expected ones in interesting particular cases (see 7.2).

§6. The Grothendieck limit theorem

We give a sketch of a proof of the Grothendieck limit theorem stated in the introduction. This proof was communicated to the author by A. Suslin. Let k be a field and let Sch/k be the category of Noetherian schemes over k. Let Ab be the category of abelian groups. Let $\{R^\alpha/\alpha \in A\}$ be a filtering inductive system of Noetherian k-algebras such that the limit $R_\infty = \varinjlim R^\alpha$ is Noetherian as well. Let $X^\alpha = \operatorname{Spec} R^\alpha$ and $X_\infty = \operatorname{Spec} R_\infty$. Let $f_{\alpha\beta} : R^\alpha \to R^\beta$ be the transition map (here $\beta \geq \alpha$) and let $f_\alpha : R^\alpha \to R_\infty$ be the canonical map. Let $\varphi_{\alpha\beta} : X^\beta \to X^\alpha$ and $\varphi_\alpha : X_\infty \to X^\alpha$ be the respective morphisms of schemes.

6.1. Definition. *Let $F : \mathrm{Sch}/k \to Ab$ be a presheaf. One says in this section that F is continuous if F commutes with filtering projective limits of Noetherian affine schemes, i.e., the canonical map $\varinjlim F(S^\beta) \to F(\varinjlim S^\beta)$ is an isomorphism for a filtering inductive system of Noetherian k-algebras S^β with a Noetherian limit $S = \varinjlim S^\beta$.*

6.2. Lemma. *Let $F : \mathrm{Sch}/k \to Ab$ be a continuous presheaf and let $F^\sim$ be the Zariski sheaf on Sch/k associated with F. Then $F^\sim$ is continuous as well.*

6.3. Construction. *Let $G : \mathrm{Sch}/k \to Ab$ be a presheaf. Define a presheaf G_∞ on the Zariski topology of the scheme X_∞ as follows. A section of G_∞ on an open $U \subset X_\infty$ is a couple (s^α, U^α) such that $\varphi_\alpha^{-1}(U^\alpha) = U$ and $s^\alpha \in \Gamma(U^\alpha, G)$. Two couples (s^α, U^α) and (s^β, U^β) are identified if there exists an index γ with $\gamma \geq \alpha$ and $\gamma \geq \beta$ such that the couples $(\varphi^*_{\alpha\gamma}(s^\alpha), \varphi^{-1}_{\alpha\gamma}(U^\alpha))$ and $(\varphi^*_{\beta\gamma}(s^\beta), \varphi^{-1}_{\beta\gamma}(U^\beta))$ coincide.*

6.4. Properties. *The correspondence $G \mapsto G_\infty$ has the following properties*

1. $\Gamma(X_\infty, G_\infty) = \varinjlim \Gamma(X^\alpha, G)$;
2. *if G is a sheaf then G_∞ is a sheaf on X_∞;*
3. *if G is a continuous sheaf then $G\big|_{X_\infty} = G_\infty$ on the Zariski topology on X_∞;*
4. *if G an injective sheaf then G_∞ is a flasque sheaf on X_∞;*
5. *the functor $G \mapsto G_\infty$ is exact (on the category of Zariski sheaves): if $0 \to G' \to G \to G'' \to 0$ is an exact sequence of Zariski sheaves on Sch/k then $0 \to G'_\infty \to G_\infty \to G''_\infty \to 0$ is an exact sequence of Zariski sheaves on X_∞.*

6.5. Proposition. *Let $F : \mathrm{Sch}/k \to Ab$ be a continuous sheaf. Then the canonical map*

$$\varinjlim H^p\left(X^\alpha, F\big|_{X^\alpha}\right) \to H^p\left(X_\infty, F\big|_{X_\infty}\right) \quad (p \geq 0)$$

is an isomorphism.

Proof. Let $F \to I^{\cdot}$ be an injective resolution of F on the big Zariski site on Sch/k. Then $F_\infty \to I^{\cdot}_\infty$ is a flasque resolution of the sheaf F_∞ on the scheme X_∞ (by (2), (4), and (5)). Thus, $H^p(X_\infty, F_\infty) = H^p(\Gamma(X_\infty, I^{\cdot}_\infty))$. By property (1) one has

$$H^p(\Gamma(X_\infty, I^{\cdot}_\infty)) = H^p(\varinjlim \Gamma(X^\alpha, I^{\cdot})) = \varinjlim H^p(\Gamma(X^\alpha, I^{\cdot})) = \varinjlim H^p\left(X^\alpha, F\big|_{X^\alpha}\right).$$

Since $F_\infty = F\big|_{X_\infty}$ (see (3)) hence one gets finally

$$H^p\left(X_\infty, F\big|_{X_\infty}\right) = \varinjlim H^p\left(X^\alpha, F\big|_{X^\alpha}\right).$$

Proposition 6.5 is proved. □

Joining this theorem with Lemma 6.2 one gets the following corollary.

6.6. Theorem (Grothendieck). *Let $F : \mathrm{Sch}/k \to Ab$ be a continuous presheaf and let $F^{\sim}$ be the Zariski sheaf associated with F. Then*

$$\varinjlim H^p\left(X^\alpha, F^{\sim}\big|_{X^\alpha}\right) = H^p\left(X_\infty, F^{\sim}\big|_{X_\infty}\right).$$

This is the Grothendieck limit theorem stated in the introduction. □

§7. A weak form of the Gersten conjecture

This section contains a general result concerning the exactness of a Cousin complex. Let k and Sch/k be the same as in §6. If X is a scheme and F is a sheaf on X (in the Zariski topology) then the Cousin complex of F on X is the complex (cf. [C-THK, §1])

$$0 \to \bigoplus_{x\in X^{(0)}} F(x) \to \bigoplus_{y\in X^{(1)}} H^1_y(X,F) \to \bigoplus_{z\in X^{(2)}} H^2_z(X,F) \to \dots,$$

where for a point $t \in X$ one denotes by $H^p_t(X,F)$ the group $\varinjlim H^p_{U\cap\{\overline{t}\}}(U,F)$ with the limit running over all open U containing the point t.

7.1. Notation. *The complex is written as $C(X,F)$. For a closed sbscheme $Y \subseteq X$ let $C_Y(X,F)$ denote a subcomplex of the complex $C(X,F)$ "supporting" on the subscheme Y.*

7.2. Definition. *One says that a presheaf $F : \mathrm{Sch}/k \to Ab$ satisfies a weak Gersten conjecture for k-schemes (respectively, a weak Gersten conjecture in the k-geometric case) if the complex $C(X,F^{\sim})$ is a resolution of the group $F(X)$ for each regular local scheme $X \in \mathrm{Sch}/k$ (respectively, for each essentially smooth local k-scheme) and for the Zariski sheaf $F^{\sim}$ associated with F.*

7.3. Theorem. *Let k be a perfect field. Let $F : \mathrm{Sch}/k \to Ab$ be a presheaf satisfying the weak Gersten conjecture in the k-geometric case. If F is continuous (see 6.1) then F satisfies the weak Gersten conjecture for k-schemes.*

Proof. Since F satisfies the weak Gersten conjecture in the k-geometric case hence for an essentially smooth local k-scheme $X \in \mathrm{Sch}/k$ and for an irreducible closed subset $Y \subset X$ of codimension r one has $H^i_Y(X,F^{\sim}) = H^i(C_Y(X,F^{\sim}))$. On the other hand $H^i(C_Y(X,F^{\sim})) = 0$ for $i < r$ because Y does not contain any points $x \in X^{(i)}$ with $i < r$. Hence $H^i_Y(X,F^{\sim}) = 0$ for all $i < r$.

Now let $X = \mathrm{Spec}\,(R)$ be a regular local scheme from Sch/k. Recall that there is a local-global spectral sequence

$$E_1^{p,q} = \coprod_{x\in X^{(q)}} H^{p+q}_x(X,F^{\sim}) \Rightarrow H^{p+q}(X,F^{\sim}).$$

The complex $C(X,F^{\sim})$ coincides with the complex $E_1^{p,*}$. Thus to verify that the complex $C(X,F^{\sim})$ is a resolution of the group $F(X)$ it suffices to check the vanishing of the groups $H^i_y(X,F^{\sim})$ for each point $y \in X$ with $i < \mathrm{codim}_X(y)$. Localizing X one may assume that y is the only closed point of X and $i < \dim\,(X) = d$.

Since the field k is perfect Lemma 3.2 states that R is a filtering inductive limit of essentially smooth local k-algebras: $R = \varinjlim R^\alpha$. Let $f_1, \dots, f_d$ be a system elements $f_1, \dots, f_d$ up to elements of R^α. Increasing the index α one may assume that $f_{1,\alpha}, \dots, f_{d,\alpha}$ belongs to the maximal ideal $\mathfrak{M}^\alpha$ of the ring R^α. Further it is easy to observe that $f_{1,\alpha}, \dots, f_{d,\alpha}$ is a partial system of local parameters in the ring R^α. Moreover for an index $\beta \geq \alpha$ the images $f_{1,\beta}, \dots, f_{d,\beta}$ of the elements $f_{1,\alpha}, \dots, f_{d,\alpha}$ in the ring R^β are a partial system of local parameters in the ring R^β. Let Z^β be the vanishing locus of elements $f_{1,\beta}, \dots, f_{d,\beta}$ and let $X^\beta = \operatorname{Spec}(R^\beta)$. Then $H^i_{Z^\beta}(X^\beta, F^\sim) = 0$ for each $i < d$. Using Theorem 6.6 one can show that $H^i_y(X, F^\sim) = \varinjlim_{\beta \geq \alpha} H^i_{Z^\beta}(X^\beta, F^\sim)$. Thus $H^i_y(X, F^\sim) = 0$ for each $i < d$. Theorem 7.3 is proved. □

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