

# $S$ -modules in the category of schemes

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## Introduction

The goal of this paper is to give a theory of  $E_\infty$ -ring spectra and  $S$ -modules [6] in Morel and Voevodsky's category of algebraic spaces [16, 20]. This is an alternative to the work of Jardine, who worked out an approach to the same problem using symmetric spectra [9]. Briefly, Morel and Voevodsky defined a notion of spaces over an arbitrary field  $k$ , and used it to do homotopy theory in the category of smooth schemes over  $k$ . Passing to stable homotopy theory, they defined the notion of  $\mathbb{T}$ -spectra over  $k$ , with a smash product that is commutative, associative and unital up to homotopy. One would also like a notion of spectra that are strict rings and modules under the smash product. In classical homotopy theory, one approach to strict ring- and module-spectra was given by Elmendorf, Kriz, Mandell and May [6]. This is the notion of  $S$ -modules, which is a category of spectra that has a strictly associative, commutative, and unital smash product. Furthermore, there is a model structure on the category of  $S$ -modules whose homotopy category is equivalent to the usual homotopy category of spectra. The purpose of this paper is to work out this idea of  $S$ -modules in the context of smooth schemes over a field  $k$ . As an example, I give a construction of algebraic Morava  $K(n)$ -theories, thus proving the foundational part of a conjecture of Voevodsky [19], [8]. An alternative approach using symmetric prespectra is also possible.  $S$ -modules are interesting since they have a rich structure (see [6]). It also appears useful to have algebraic analogues of the constructions from topology in place for future investigations of homological operations, equivariant generalized cohomology theories, etc.

In the paper, I will introduce several categories of prespectra and spectra over  $k$ . The most important will be  $Spectra(\mathcal{U})$ , a closed model category of coordinate-free spectra in a sense similar to Lewis, May and Steinberger [12], where  $\mathcal{U} = \mathbb{A}^\infty$  is the universe in the algebraic context. This is needed for the construction of  $S$ -modules. To compare this category to Morel and Voevodsky's definition of  $\mathbb{T}$ -spectra [16], I also consider a decomposition  $\mathcal{B}$  of the universe into one-dimensional subspaces. I introduce the intermediate categories of  $\mathcal{B}$ -spectra and  $\mathcal{B}_{full}$ -spectra, which are

indexed on a sequence of subspaces of  $\mathcal{U}$  specified by  $\mathcal{B}$ . The difference between them is that a  $\mathcal{B}$ -spectrum has a sequence of chosen structure maps, whereas a  $\mathcal{B}_{full}$ -spectrum has structure maps corresponding to all complements of one indexing space inside another. I will show that the category of  $\mathcal{B}_{full}$ -spectra is equivalent to the category of  $\mathcal{B}$ -spectra, which is in turn equivalent to the category of  $\mathbb{T}$ -spectra. Also, there is a model structure on these equivalent categories of spectra. I will show that the homotopy category of  $\mathcal{U}$ -spectra is equivalent to the homotopy category of  $\mathcal{B}_{full}$ -spectra. Hence, the definitions are compatible with that of [16].

In Section 1, I recall some of the basic constructions used to do homotopy theory in the category of smooth schemes, due to Morel and Voevodsky [16]. They defined a model structure on  $Spc(k)$ , the category of spaces over  $k$  in two steps as follows. First, there is the simplicial model structure due to Jardine [9]. The  $\mathbb{A}^1$ -local structure on  $Spc(k)$ , which is the structure with which one actually works, is derived from the simplicial structure by  $\mathbb{A}^1$ -localization in the sense of Bousfield [1]. In the second section, I define the categories of coordinate-free prespectra and spectra over  $k$ , denoted  $Prespectra(\mathcal{U})$  and  $Spectra(\mathcal{U})$ . This is parallel to the approach to spectra described in Lewis, May and Steinberger [12]. For a general field  $k$ , the universe  $\mathcal{U}$  is not equipped with an inner product, making the notion of orthogonal complement ambiguous. To solve this problem, one first gives a definition of spectra which does not depend on orthogonal complements. Namely, I define coordinate-free spectra to be indexed on the cofinite subspaces of the universe, rather than on the finite subspaces as in the classical case. Unfortunately, I cannot show that the categories of spectra indexed on the cofinite subspaces of the universe is equivalent on the nose to a category of spectra indexed on a sequence. This is because no decreasing sequence of cofinite subspaces of  $\mathcal{U}$  is cofinal in the poset of cofinite subspaces of  $\mathcal{U}$ . This difficulty was pointed out to me by C. Weibel. To solve this problem, I introduce in Section 3 the auxiliary categories of  $\mathcal{B}$ -prespectra,  $\mathcal{B}$ -spectra,  $\mathcal{B}_{full}$ -prespectra and  $\mathcal{B}_{full}$ -spectra, for  $\mathcal{B}$  a decomposition of the universe into finite-dimensional subspaces.

They are intermediates between coordinate-free spectra and the classical definition of  $\mathbb{T}$ -spectra, and will play a role in the comparison between the two.

In Sections 4 and 5, I construct a model structure on the category of coordinate-free spectra over a field  $k$ . Analogously as in the case of spaces over  $k$ , one first defines the simplicial model structure on the category of spectra, by starting with the simplicial model structure on  $k$ -spaces and stabilizing. Then, one does Bousfield localization at all maps of the form  $\mathbb{A}_+^1 \wedge X \rightarrow X$  to define the  $\mathbb{A}^1$ -local model structure on  $Spectra(\mathcal{U})$ . This is the model structure on the category of spectra over  $k$  with which we will work. It allows one to compare the category of coordinate-free spectra with Morel and Voevodsky's definition in [16]. Namely, in Section 6, I show that the homotopy category associated with the  $\mathbb{A}^1$ -local model structure on  $\mathcal{U}$ -spectra is equivalent to the  $\mathbb{A}^1$ -local homotopy category of  $\mathbb{T}$ -spectra [16], via the categories of  $\mathcal{B}$ -spectra and  $\mathcal{B}_{full}$ -spectra. It turns out that the categories of  $\mathcal{B}$ -spectra are Quillen equivalent when we vary  $\mathcal{B}$ , and by taking a colimit of the argument, they are also Quillen equivalent to  $Spectra(\mathcal{U})$  (Theorem 6.3).

In Section 7, I prove directly that the homotopy category of  $Spectra(\mathcal{U})$  is equivalent to the stable homotopy category of  $\mathbb{A}^1$ -symmetric spectra constructed by Jardine [9] (Proposition 7.2). Section 8 gives the definitions of change of universe functors, parallel to those given in [12]. These are parametrized by what I call extended injections, which are injections from one universe into another which come with a compatible system of "preimages of a splitting". (This is a compromise between several simpler notions which do not work.) The extended injections serve as substitutes of linear isometries for the algebraic category. In particular, given a universe  $\mathcal{U}$ , the space  $\mathcal{L}(2)$  of extended injections from  $\mathcal{U} \oplus \mathcal{U}$  to  $\mathcal{U}$  parametrizes internal smash products of  $\mathcal{U}$ -spectra. In Section 9, I show that the algebraic space of these injections is  $\mathbb{A}^1$ -contractible. Thus, the internal smash products are well-defined up to  $\mathbb{A}^1$ -homotopies with all higher coherences. In Section 10, I construct twisted half-smash products and twisted function spectra, using a method analogous to that of Cole in [6]. Given a  $k$ -space  $A$  mapping into a space of extended injections, and a

coordinate-free spectrum  $E$ , the twisted half-smash product  $A \times E$  is the analogue to the corresponding construction in classical stable homotopy theory. Roughly speaking, it is the “gluing together” of the change of universe functors parametrized by  $A$ . This will be the technical basis for the construction of  $\mathbb{L}$ -spectra and  $S$ -modules.

The next three sections of the paper deal with  $\mathbb{L}$ -spectra and  $S$ -modules. In Section 11, I construct the category of  $\mathbb{L}$ -spectra, and a smash product on  $\mathbb{L}$ -spectra that is associative and commutative on the point-set level. I also construct the  $\mathbb{A}^1$ -local stable homotopy category of  $\mathbb{L}$ -spectra, and show that it is equivalent to the  $\mathbb{A}^1$ -local stable homotopy category of spectra. The smash product of  $\mathbb{L}$ -spectra, however, is not unital on the point-set level, but unital only up to homotopy. This is proven in Section 12. This situation is completely analogous to [6], but the technical details are more subtle (see Lemma 12.4, and its proof in Section 16). In Section 13, following [6], I define the category of  $S$ -modules, whose smash products are unital on the point-set level, in addition to being associative and commutative. Again, one shows that the  $\mathbb{A}^1$ -local stable homotopy category of  $S$ -modules is equivalent to the  $\mathbb{A}^1$ -local stable homotopy category of spectra. The proofs of the Quillen equivalences between  $S$ -modules,  $\mathbb{L}$ -spectra, and  $Spectra(\mathcal{U})$  are different from [6], because the closed model structures are different.

This gives a symmetric monoidal category of spectra over  $k$ . In particular, in Section 14 I consider the category of  $S$ -algebras, i. e. monoids in the symmetric monoidal category of  $S$ -modules. One can also look at modules over a given  $S$ -algebra. A special example of an  $S$ -algebra is that of the algebraic cobordism spectrum  $MGL$ . Using the theory of strict modules, I show how one can construct algebraic versions of the usual spectra derived from the cobordism spectrum  $MU$ , including algebraic Morava  $K$ -theories  $K(n)^{alg}$ .

The remaining sections of the paper contain technical details left over from the earlier sections. In Section 15, I prove that the model structures constructed on the categories of spectra,  $\mathbb{L}$ -spectra, and  $S$ -modules are indeed model structures, using the small objects argument. Section 16 contains the proofs of Lemmas 12.4

and 13.3, about the extended injections operad. Finally, in the Appendix, I record some facts about Morel and Voevodsky’s category  $Spc(k)$ : most importantly, that it is “compactly generated”, i. e. every object is a directed homotopy colimit of small objects. I also show that the  $\mathbb{A}^1$ -local homotopy category of  $k$ -spaces [16] is generated by small objects.

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## 1. Preliminaries

We first recall some of the basic notions about the category of algebraic spectra defined by Morel and Voevodsky [16, 20]. Let  $k$  be a field, and  $Sm/k$  be the category of smooth schemes of finite type over  $k$ . We cannot do homotopy theory directly in  $Sm/k$  since it does not have all small colimits. To solve this problem, one first endows  $Sm/k$  with the Nisnevich topology. This is a Grothendieck topology on  $Sm/k$ , and is the subtopology of the étale topology on  $Sm/k$ , where the coverings are generated by finite families of étale maps  $\{U_i\} \rightarrow X$ , such that for every point  $x$  of the scheme  $X$ , the induced map of schemes

$$\prod_i (U_i \times_X Spec(\mathcal{O}_{X,x}^h) \rightarrow Spec(\mathcal{O}_{X,x}^h))$$

has a section (Nisnevich [17]). Here,  $\mathcal{O}_{X,x}^h$  denotes the henselization of the local ring of  $X$  at  $x$ . Equivalently, it is generated by covering diagrams of the form

$$(1.1) \quad \begin{array}{ccc} p^{-1}(U) & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & X \end{array}$$

where  $i$  is an open embedding,  $p$  is an étale map, and the map  $p^{-1}(X \setminus U) \rightarrow X \setminus U$  is an isomorphism (see [16], Proposition 3.1.4).

The Nisnevich topology on  $Sm/k$  gives it the structure of a Grothendieck site. Define the category of  $k$ -spaces by

$$Spc(k) = \Delta^{op}Sh(Sm/k)_{Nis},$$

the category of simplicial sheaves over  $Sm/k$  in the Nisnevich topology. Then  $Spc(k)$  is a complete and cocomplete category, and is the algebraic analogue of the category of topological spaces. Let  $\mathbb{A}^1$  denote the affine line over  $k$ . Two maps  $f, g : X \rightarrow Y$  in  $Spc(k)$  are  $\mathbb{A}^1$ -homotopic if there is a map  $h : \mathbb{A}^1 \times X \rightarrow Y$  in  $Spc(k)$ , such that  $\{0\} \times X \rightarrow \mathbb{A}^1 \times X \xrightarrow{h} Y$  is  $f$ , and  $\{1\} \times X \rightarrow \mathbb{A}^1 \times X \xrightarrow{h} Y$  is  $g$ .

To do homotopy theory in any category  $\mathcal{C}$ , we need a model structure on  $\mathcal{C}$ . Recall that a category  $\mathcal{C}$  is a model category if it is complete and cocomplete, and comes with three classes of morphisms called the cofibrations, weak equivalences, and fibrations, with the following axioms.

CM1. The classes of cofibrations, weak equivalences, and fibrations contain the identity on each object, and are closed under compositions.

CM2. If  $f$  and  $g$  are composable maps, and two of the three maps  $f$ ,  $g$ , and  $gf$  are weak equivalences, then so is the third.

CM3. The classes of cofibrations, weak equivalences, and fibrations are closed under retracts.

CM4. Given the following commutative square

$$\begin{array}{ccc} A & \longrightarrow & E \\ i \downarrow & & \downarrow p \\ B & \longrightarrow & G \end{array}$$

where  $i$  is a cofibration and  $p$  a fibration, a lifting  $B \rightarrow E$  exists whenever either  $i$  or  $p$  is a weak equivalence. (In that case, we say that  $i$  has the left lifting property with respect to  $p$ , and  $p$  has the right lift property with respect to  $i$ .)

CM5. Let  $f : A \rightarrow B$  be any map in  $\mathcal{C}$ .

a.  $f$  can be factored into a composition  $p \cdot i$ , where  $p$  is a fibration and  $i$  an acyclic cofibration (a cofibration which is also a weak equivalence).

b.  $f$  can be factored into a composition  $q \cdot j$ , where  $j$  is a cofibration and  $q$  is an acyclic fibration (a fibration which is also a weak equivalence).

A cofibration (resp. fibration) is said to be acyclic if it is also a weak equivalence.

The category  $Spc(k)$  has a model structure defined by Morel and Voevodsky in two steps. First, one constructs the simplicial model category structure, which exists for the category of simplicial sheaves over any site (Jardine [9]). Then, one constructs an  $\mathbb{A}^1$ -local model structure, which is the structure that one really works with. It is derived from the simplicial structure by adding a class of  $\mathbb{A}^1$ -weak equivalences to the simplicial weak equivalences in the manner of Bousfield [1].

The simplicial model structure on  $Spc(k)$  is defined as follows. Recall that for a site  $\mathcal{S}$ , a point  $x^*$  of  $\mathcal{S}$  is just a functor from the category of sheaves on  $\mathcal{S}$  to the category of sets, which commutes with finite limits and all colimits.

DEFINITION 1.2. *Let  $f : X \rightarrow Y$  be a map in  $Spc(k)$ .*

1.  *$f$  is a simplicial cofibration if it is a monomorphism.*
2.  *$f$  is a simplicial weak equivalence if for every point  $x^*$  of the site  $Sm/k$ , the induced map  $f^* : x^*(X) \rightarrow x^*(Y)$  is a weak equivalence of simplicial sets.*
3.  *$f$  is a simplicial fibration if it has the right lifting property with respect to all simplicial acyclic cofibrations.*

Together, these three classes of maps given the simplicial model category structure on  $Spc(k)$  [9, 16]. Denote by  $\mathcal{H}_s(Spc(k))$  the homotopy category associated with this structure.

Consider the affine line  $\mathbb{A}^1$  over  $k$ . We also have the  $\mathbb{A}^1$ -local model structure on  $Spc(k)$ , the structure with which we in fact want to work. It is given by the following definitions (see also Bousfield [1]).

DEFINITION 1.3. A  $k$ -space  $X \in \mathit{Spc}(k)$  is  $\mathbb{A}^1$ -local if for any  $k$ -space  $Y$ , the map

$$\mathit{Hom}_{\mathcal{H}_s(\mathit{Spc}(k))}(Y, X) \rightarrow \mathit{Hom}_{\mathcal{H}_s(\mathit{Spc}(k))}(Y \times \mathbb{A}^1, X)$$

induced by the projection map  $Y \times \mathbb{A}^1 \rightarrow Y$  is a bijection.

DEFINITION 1.4. Let  $f : X \rightarrow Y$  be a map in  $\mathit{Spc}(k)$ .

1.  $f$  is an  $\mathbb{A}^1$ -cofibration if it is a monomorphism.
2.  $f$  is an  $\mathbb{A}^1$ -weak equivalence if for any  $\mathbb{A}^1$ -local  $k$ -space  $Z$ , the map of simplicial sets

$$f^* : \mathit{Hom}_{\mathcal{H}_s(\mathit{Spc}(k))}(Y, Z) \rightarrow \mathit{Hom}_{\mathcal{H}_s(\mathit{Spc}(k))}(X, Z)$$

is a bijection.

3.  $f$  is an  $\mathbb{A}^1$ -fibration if it has the right lifting property with respect to all acyclic  $\mathbb{A}^1$ -cofibrations.

There is another characterization of  $\mathbb{A}^1$ -weak equivalences by Proposition 2.2.9 of [16]:  $f : X \rightarrow Y$  is an  $\mathbb{A}^1$ -weak equivalence if and only if for any  $\mathbb{A}^1$ -local and simplicially fibrant  $Z$ , the map of internal  $\mathit{Hom}$ -objects in  $\mathit{Spc}(k)$

$$f^* : \underline{\mathit{Hom}}_{\mathit{Spc}(k)}(Y, Z) \rightarrow \underline{\mathit{Hom}}_{\mathit{Spc}(k)}(X, Z)$$

is a simplicial weak equivalence.

The  $\mathbb{A}^1$ -local model structure allows one to do homotopy theory in the category  $\mathit{Spc}(k)$ . Here,  $\mathbb{A}^1$  plays the role of the unit interval. For  $k$  a subfield of  $\mathbb{C}$ , there is a complex topological realization functor  $t_{\mathbb{C}}$  from  $\mathit{Sm}/k$  to the category of topological spaces, which extends to a functor on  $\mathit{Spc}(k)$ . Namely,  $t_{\mathbb{C}}$  takes a smooth scheme  $X$  to its  $\mathbb{C}$ -points  $X(\mathbb{C})$ . If  $k$  is contained in  $\mathbb{R}$ ,  $t_{\mathbb{C}}$  gives a  $\mathbb{Z}/2$ -equivariant topological space, with a  $\mathbb{Z}/2$ -action by complex conjugation.

To define spectra over  $k$ , we really need to consider  $k$ -spaces with basepoints. Let  $\mathit{Spc}(k)_{\bullet} = \mathit{Spec}(k) \downarrow \mathit{Spc}(k)$  be the category of  $k$ -spaces with basepoints, an object of which is a  $k$ -space  $X$  with a given map  $\mathit{Spec}(k) \rightarrow X$ . There is a smash product

$$\wedge : \mathit{Spc}(k)_{\bullet} \times \mathit{Spc}(k)_{\bullet} \rightarrow \mathit{Spc}(k)_{\bullet}.$$

There are also internal  $Hom$ -spaces, i. e. for every  $X \in Spc(k)_\bullet$ , the functor  $- \wedge X : Spc(k)_\bullet \rightarrow Spc(k)_\bullet$  has a right adjoint  $\underline{Hom}(X, -)$ .

We say that a map in  $Spc(k)_\bullet$  is a simplicial cofibration, simplicial weak equivalence, or simplicial fibration if it is a simplicial cofibration, simplicial weak equivalence, or simplicial fibration in  $Spc(k)$ . This defines the simplicial model structure on  $Spc(k)_\bullet$ . Likewise, the  $\mathbb{A}^1$ -local model structure on  $Spc(k)$  gives rise to the  $\mathbb{A}^1$ -local model structure on  $Spc(k)_\bullet$  (see [16]).

The zero-dimensional sphere in  $Spc(k)_\bullet$  is

$$S^0 = Spec(k) \amalg Spec(k).$$

There are two notions of the circle in the category  $Spc(k)_\bullet$ . The simplicial circle is

$$S_s^1 = \mathbb{A}^1 / \{0, 1\}$$

the affine line with the two points 0 and 1 identified. If  $k \subseteq \mathbb{R}$ , this corresponds by  $t_{\mathbb{C}}$  to the fixed circle  $S^1$ . There is also the “twisted” circle

$$S_t^1 = \mathbb{A}^1 \setminus \{0\}$$

with the basepoint at 1. For  $k \subset \mathbb{R}$ , this corresponds to  $S^\alpha$ , the circle with  $\mathbb{Z}/2$ -action by complex conjugation.

Consider the  $k$ -space

$$\mathbb{T} = \mathbb{A}^1 / (\mathbb{A}^1 \setminus \{0\}).$$

There is a canonical  $\mathbb{A}^1$ -weak equivalence

$$S_s^1 \wedge S_t^1 \simeq \mathbb{P}^1 \simeq \mathbb{T}.$$

Also, observe that  $\mathbb{T}^{\wedge n} \cong \mathbb{A}^n / (\mathbb{A}^n \setminus \{0\})$ . This allows the notion of  $\mathbb{T}$ -prespectra over  $k$ , given by Voevodsky in [20]. (However, our  $\mathbb{T}$ -prespectra are in fact called  $\mathbb{T}$ -spectra in [20]).

**DEFINITION 1.5.** *A  $\mathbb{T}$ -prespectrum over  $k$  is a collection of  $k$ -spaces  $E = \{E_n\}$ ,  $n \in \mathbb{Z}$ , with structure maps*

$$E_n \rightarrow \Omega^{\mathbb{T}} E_{n+1} = \underline{Hom}_{Spc(k)_\bullet}(\mathbb{T}, E_{n+1}).$$

A map  $f : E \rightarrow G$  of  $\mathbb{T}$ -prespectra is a sequence of maps

$$f_n : E_n \rightarrow G_n$$

that are compatible with the structure maps.

Analogous to the topological case (see [12]), we define a  $\mathbb{T}$ -spectrum as follows.

**DEFINITION 1.6.** *A  $\mathbb{T}$ -spectrum is a prespectrum whose structure maps are isomorphisms of based  $k$ -spaces. A map of  $\mathbb{T}$ -spectra is a map of  $\mathbb{T}$ -prespectra.*

Let  $Prespectra(k)$  and  $Spectra(k)$  denote the categories of  $\mathbb{T}$ -prespectra and  $\mathbb{T}$ -spectra over  $k$ , respectively. As in the case of topological spectra, there is an obvious forgetful functor  $R$  from the category  $Spectra(k)$  to the category  $Prespectra(k)$ . This has a left adjoint  $L$ , the spectrification functor. However,  $L$  is much simpler to describe than in the topological case. Since  $\mathbb{T}$  is a quotient of a smooth scheme by another, it is a small object in  $Spc(k)_\bullet$ , i. e. there is a natural bijection

$$Hom_{Spc(k)_\bullet}(\mathbb{T}, \text{colim}_i X_i) \cong \text{colim}_i Hom_{Spc(k)_\bullet}(\mathbb{T}, X_i)$$

for any directed system of based  $k$ -spaces  $X_i$  (see Lemma 17.2 in the Appendix). This means that if  $D = \{D_n\}$  is a  $\mathbb{T}$ -prespectrum, then

$$(1.7) \quad (LD)_n = \text{colim}_k \Omega^{\mathbb{T}^{\wedge k}} D_{n+k}.$$

This is similar to the topological spectrification functor when restricted to inclusion prespectra [12].

## 2. Coordinate-free Spectra

The definition of  $\mathbb{T}$ -spectra from the previous section is analogous to the notion of a topological spectrum indexed on the integers, which assumes a given basis for the universe. However, we really want a notion of coordinate-free spectra over  $k$ , analogous to the notion of coordinate-free topological spectra in Lewis, May, and Steinberger [12].

To define a coordinate-free  $k$ -spectrum, we first need the notion of a universe over  $k$ , i. e. an infinite-dimensional  $k$ -vector space, over whose subspaces the spectrum would be indexed. The natural definition is to set the universe  $\mathcal{U}$  to be the infinite-dimensional affine space over  $k$ :

$$\mathcal{U} = \mathbb{A}^\infty = \operatorname{colim}_n \mathbb{A}^n.$$

We can think of the variety  $\mathbb{A}^n$  to be the vector space  $k^n$  over  $k$ . So  $\mathbb{A}^\infty$  is  $\bigoplus_{\mathbb{N}} k$  as a vector space over  $k$ . (However, note that it is not in fact in  $Sm/k$ , an example of the lack of colimits in  $Sm/k$ .)

In the topological case, given a universe  $\mathcal{U} = \mathbb{R}^\infty$ , one indexes a  $\mathcal{U}$ -spectrum on the finite-dimensional subspaces of  $\mathcal{U}$ . In particular,  $\mathcal{U}$  comes equipped with an inner product, so we can subtract one finite-dimensional subspace from a larger one by taking orthogonal complements. This allows for well-defined structure maps

$$E_U \rightarrow \Omega^{V-U} E_V$$

for all  $U \subset V$ . We do not have this inner product when  $k$  is an arbitrary field. Rather, for general  $k$ , we will index an  $\mathcal{U}$ -spectrum on the subspaces of  $\mathcal{U}$  of finite codimension instead of subspaces of finite dimension.

Let  $U$  be a  $k$ -vector space, and  $V, Z$  linear subspaces of  $U$  such that  $V \cap Z = 0$ . Then we denote the internal direct sum  $V + Z \subseteq U$  by  $V \dot{+} Z$ .

**DEFINITION 2.1.** *For a fixed universe  $\mathcal{U}$ , we say that a linear subspace  $V \subset \mathcal{U}$  is cofinite if  $\mathcal{U}/V$  is finite-dimensional as a vector space over  $k$ . In other words,  $V \cong \mathbb{A}^\infty$ , and there is a linear injection  $V \rightarrow \mathcal{U}$ , with  $\mathcal{U}/V \cong \mathbb{A}^n$  for some  $n$ .*

For a universe  $\mathcal{U}$ , we also define the following category.

**DEFINITION 2.2.** *For a universe  $\mathcal{U}$ , the category  $\mathcal{C}(\mathcal{U})$  of cofinite subspaces of  $\mathcal{U}$  is as follows. The objects are the cofinite subspaces  $V \subseteq \mathcal{U}$ . For  $U, V$  objects of  $\mathcal{C}(\mathcal{U})$ , a morphism  $Z : U \rightarrow V$  is a finite-dimensional space  $Z \subset \mathcal{U}$ , such that  $V \dot{+} Z = U$ . If  $Z : U \rightarrow V$ , and  $T : V \rightarrow W$ , then  $W \subseteq V \subseteq U$ , and  $(Z \dot{+} T) \dot{+} W = U$ . We*

define the composition

$$T \cdot Z = T \dot{+} Z : U \rightarrow W.$$

We index coordinate-free  $k$ -prespectra on the objects of  $\mathcal{C}(\mathcal{U})$ , and require a structure map for every morphism in  $\mathcal{C}(\mathcal{U})$ .

DEFINITION 2.3. *Let  $\mathcal{U}$  be a  $k$ -universe. A  $\mathcal{U}$ -prespectrum  $D$  is a collection of based  $k$ -spaces  $D_U$ , for all cofinite  $U \subseteq \mathcal{U}$ , together with maps*

$$\rho_{V,Z}^U : D_U \rightarrow \Omega^Z D_V = \underline{\text{Hom}}_{\text{Spc}(k)_\bullet}(Z/(Z \setminus \{0\}), D_V)$$

for all  $Z : U \rightarrow V$  in  $\mathcal{C}(\mathcal{U})$ . We require that  $\rho_{U,0}^U = \text{Id}$ , and for  $Z : U \rightarrow V$ ,  $T : V \rightarrow W$  in  $\mathcal{C}(\mathcal{U})$ ,

$$\rho_{W,Z \dot{+} T}^U = (\Omega^Z \rho_{W,T}^V) \cdot \rho_{V,Z}^U : D_U \rightarrow \Omega^{Z \dot{+} T} D_W.$$

A map  $f : D \rightarrow D'$  of  $\mathcal{U}$ -prespectrum is a collection of maps  $f_U : D_U \rightarrow D'_U$  that are compatible with all structure maps.

We also have the category of coordinate-free spectra over  $\mathcal{U}$ .

DEFINITION 2.4. *A  $\mathcal{U}$ -spectrum is a  $\mathcal{U}$ -prespectrum  $E$ , all of whose structure maps  $\rho_{V,Z}^U$  are isomorphisms of based  $k$ -spaces. A map of  $\mathcal{U}$ -spectra is a map of  $\mathcal{U}$ -prespectra.*

For a universe  $\mathcal{U}$ , denote the categories of coordinate-free prespectra and spectra indexed on  $\mathcal{U}$  by  $\text{Prespectra}(\mathcal{U})$  and  $\text{Spectra}(\mathcal{U})$ , respectively. As in the categories of topological prespectra and spectra, all colimits and limits in  $\text{Prespectra}(\mathcal{U})$  and all limits in  $\text{Spectra}(\mathcal{U})$  can be formed spacewise. For a based  $k$ -space  $X$ , and a  $\mathcal{U}$ -spectrum  $E$ , the function spectrum  $F(X, E)$  is given by

$$F(X, E)_U = \underline{\text{Hom}}_{\text{Spc}(k)_\bullet}(X, E_U)$$

for each  $U \in \mathcal{C}(\mathcal{U})$ .

There is an obvious forgetful functor

$$R : \text{Spectra}(\mathcal{U}) \rightarrow \text{Prespectra}(\mathcal{U}).$$

It has a left adjoint, which we denote by  $L$ . Since each  $Z/(Z \setminus \{0\})$  is a small object in the category of based  $k$ -spaces for each finite-dimensional  $Z$ , this functor is similar to the case for  $\mathbb{T}$ -spectra. Let  $D$  be a  $\mathcal{U}$ -prespectrum. For each  $U \in \mathcal{C}(\mathcal{U})$ ,

$$(LD)_U = \operatorname{colim}_{(V,Z) \in U \downarrow \mathcal{C}(\mathcal{U})} \Omega^Z D_V.$$

Here, the colimit is taken over the category  $U \downarrow \mathcal{C}(\mathcal{U})$ , an object is a cofinite subspace  $V$  along with a map  $Z : U \rightarrow V$ , i. e. a finite subspace  $Z$  such that  $Z \dot{+} V = U$ . A morphism  $T : (V, Z) \rightarrow (V', Z')$  is a finite subspace  $T$  such that  $V' \dot{+} T = V$ ,  $Z \dot{+} T = Z'$ .

The spectrification functor  $L$  allows us to define colimits in  $Spectra(\mathcal{U})$  by first taking colimits in  $Prespectra(\mathcal{U})$  and then applying  $L$ . Also, the smash product  $X \wedge E$  of a based  $k$ -space  $X$  with a spectrum  $E$  is given by first taking smash products spacewise, then applying  $L$ .

For cofinite  $W \subseteq U$  in  $\mathcal{U}$  and  $Z : U \rightarrow W$  in  $\mathcal{C}(\mathcal{U})$ , let  $GL_W(Z)$  be the group of all linear maps  $g : U \rightarrow U$  such that  $g$  is the identity on  $W$  and sends  $Z$  to  $Z$ . This is an object of  $Spc(k)_\bullet$  since it is in fact a smooth scheme. If  $W' \subseteq W \subseteq U$  are cofinite,  $T : W \rightarrow W'$  in  $\mathcal{C}(\mathcal{U})$ , then there is an inclusion of schemes  $GL_W(Z) \subseteq GL_{W'}(Z \dot{+} T)$ . Define the general linear group on  $U$  by

$$GL(U) = \operatorname{colim}_{(W,Z) \in U \downarrow \mathcal{C}(\mathcal{U})} GL_W(Z).$$

For every  $g \in GL(U)$ , one can find subspaces  $W, Z \subset U$ , such that  $Z$  is finite-dimensional, and  $W \dot{+} Z = U$ , such that  $g$  is the identity on  $W$  and sends  $Z$  to  $Z$ . For each  $U$ ,  $GL(U)$  is an object of  $Spc(k)_\bullet$ .

For any  $\mathcal{U}$ -spectrum  $E$ , there is a canonical action of  $GL(U)$  on  $E_U$ . Suppose that  $g \in GL(U)$ . Choose  $W, Z$  as above. Define the action of  $g$  on  $E_U$  by the following diagram.

$$\begin{array}{ccc} E_U & \xrightarrow[\cong]{\rho_{W,Z}^U} & \Omega^Z E_W \\ \downarrow g & & \downarrow \Omega^{g^{-1}|_Z} Id \\ E_U & \xrightarrow[\cong]{\rho_{W,Z}^U} & \Omega^Z E_W \end{array}$$

Here  $\Omega^{g^{-1}|_Z} Id$  is the map on  $\Omega^Z E_W$  induced by the action of  $g^{-1}$  on  $Z/(Z \setminus \{0\})$  and the identity on  $E_W$ . It is easy to check that this action is independent of the choice of  $W$  and  $Z$ .

We make the following easy observation.

LEMMA 2.5. *For every morphism  $Z : U \rightarrow W$  and  $g \in GL(U)$  such that  $g$  sends  $W$  to  $W$ , the following diagram commutes:*

$$\begin{array}{ccc} E_U & \xrightarrow{\rho_{W,Z}^U} & \Omega^Z E_W \\ \downarrow g & & \downarrow \Omega^{g^{-1}|_Z} g|_W \\ E_U & \xrightarrow{\rho_{W,g(Z)}^U} & \Omega^{g(Z)} E_W. \end{array}$$

Here,  $\Omega^{g^{-1}|_Z} g|_W$  denotes  $\underline{Hom}(g^{-1}|_Z, g|_W)$ .

PROOF. We find  $V$  cofinite in  $W$ , and finite  $T$  such that  $T \dot{+} V = W$  and  $T \dot{+} Z = T \dot{+} g(Z)$ . This is possible since we can choose  $T$  to be  $(Z + g(Z)) \cap W$ , and  $V$  a complement of  $T$  in  $W$ . Additionally, we can choose  $V, T$  such that  $g$  is the identity on  $V$  and sends  $T$  to  $T$ . Then in the following diagram

$$\begin{array}{ccccc} E_U & \longrightarrow & \Omega^Z E_W & \longrightarrow & \Omega^{Z \dot{+} T} E_V \\ \downarrow g & & \downarrow \Omega^{g^{-1}|_Z} g|_W & & \downarrow \Omega^{g^{-1}|_{Z+T}} Id \\ E_U & \longrightarrow & \Omega^{g(Z)} E_W & \longrightarrow & \Omega^{Z \dot{+} T} E_V \end{array}$$

both the right hand square and the outside square commute. Therefore, the left hand square commutes also.  $\square$

Thus, for a  $\mathcal{U}$ -spectrum  $E$ , most of the structure maps of  $E$  are in some sense “redundant”.

### Examples:

1. *The suspension spectrum.*

There is a forgetful functor  $\Omega_V^{\mathcal{U}}$  from the category of  $\mathcal{U}$ -spectra to the category of based  $k$ -spaces, which assigns to a spectrum  $E$  the space  $E_V$ . This has a left adjoint

$$\Sigma_V^{\mathcal{U}} : Spc(k)_\bullet \rightarrow Spectra(\mathcal{U}),$$

the  $V$ -th shift desuspension of the suspension spectrum.

Let  $X \in \mathit{Spc}(k)_\bullet$ . For each  $Z \cong \mathbb{A}^n$ , denote  $Z/(Z \setminus \{0\}) \wedge X$  by  $\Sigma^Z X$ . Then for  $U \subset \mathcal{U}$  cofinite, define the based  $k$ -space

$$D_V(X)_U = \bigvee_{Z+U=V} \Sigma^Z X,$$

and  $D_V(X)_U = * = \mathit{Spec}(k)$  otherwise. If  $W \subseteq U \subseteq V$ ,  $W \dot{+} T = U$ , then for any  $Z$  with  $Z \dot{+} U = V$ ,  $(T \dot{+} Z) \dot{+} W = V$ . So we have a canonical map

$$\Sigma^T D_V(X)_V = \bigvee_{Z+U=V} \Sigma^{T+Z} X \rightarrow \bigvee_{Z'+W=V} \Sigma^{Z'} X = D_V(X)_W.$$

These maps are compatible, so we have a  $\mathcal{U}$ -prespectrum

$$D_V(X) = \{D_V(X)_U\}.$$

We define the spectrum  $\Sigma_V^{\mathcal{U}} X$  to be  $LD_V(X)$ .

Now for any  $\mathcal{U}$ -prespectrum  $E$  and based  $k$ -space  $X$ , suppose we have a map  $f : X \rightarrow E_V$ . Then for any cofinite  $U \subset V$ , and  $Z \dot{+} U = V$ , we have  $\Sigma^Z f : \Sigma^Z X \rightarrow \Sigma^Z E_V \rightarrow E_U$ . Therefore,  $f$  gives a map  $D_V(X)_U \rightarrow E_U$ . Conversely, given a map of  $\mathcal{U}$ -prespectra  $g : D_V(X) \rightarrow E$ , taking the  $V$ -th space gives a map of based  $k$ -spaces  $X \rightarrow E_V$ . So  $X \mapsto D_V(X)$  is the left adjoint to the  $V$ -th space functor from  $\mathit{Prespectra}(\mathcal{U})$ -prespectra to  $\mathit{Spc}(k)_\bullet$ , which means that  $\Sigma_V^{\mathcal{U}}$  is the left adjoint to the functor  $\Omega_V^{\mathcal{U}} : \mathit{Spectra}(\mathcal{U}) \rightarrow \mathit{Spc}(k)_\bullet$ .

## 2. Algebraic cobordism.

Consider the algebraic cobordism spectrum  $MGL$  constructed as a  $\mathbb{T}$ -spectrum in [20] as follows. Let  $X$  be a smooth scheme over  $k$ , and  $\gamma : E \rightarrow X$  a bundle over  $X$ , and let  $s(X)$  denote the 0-section of  $\gamma$  in  $E$ . Then the Thom space  $X^\gamma$  of  $\gamma$  is the based  $k$ -space given by the pushout square

$$\begin{array}{ccc} E \setminus s(X) & \longrightarrow & E \\ \downarrow & & \downarrow \\ * & \longrightarrow & X^\gamma \end{array}$$

Let  $1$  be the one-dimensional trivial bundle over  $X$ . Then as in the topological case, there is a natural isomorphism  $X^{\gamma \oplus 1} \cong \mathbb{T} \wedge X^\gamma$ . There are also Grassmannians

$G(m, n)$  with canonical  $n$ -dimensional bundles  $\gamma_{m,n}$ . Then we define

$$BGL(n) = \operatorname{colim}_m G(m, n)$$

with canonical bundle  $\gamma_n = \operatorname{colim}_m \gamma_{m,n}$ . We can define a  $\mathbb{T}$ -prespectrum whose  $n$ -th space is  $BGL(n)^{\gamma_n}$ , with structure map

$$BGL(n)^{\gamma_n} \rightarrow \Omega^{\mathbb{T}} BGL(n+1)^{\gamma_{n+1}}$$

adjoint to the map classifying  $\gamma_n \oplus 1$  on  $BGL(n)$ . Spectrification gives  $MGL$ .

In the coordinate-free context, we need to construct  $MGL_V$  for all cofinite subspaces  $V \subset \mathcal{U}$ . Let  $Z, U \subset \mathcal{U}$  be finite-dimensional. Let the Grassmannian  $G(U \oplus Z, |Z|)$  be the  $k$ -space of all  $|Z|$ -dimensional subspaces of the external direct sum  $U \oplus Z$ . There is a canonical  $|Z|$ -dimensional bundle  $\gamma_{U \oplus Z, |Z|}$  over  $G(U \oplus Z, |Z|)$ . If  $U, U' \subset \mathcal{U}$  are finite-dimensional,  $U \subseteq U'$ , then there is a natural map

$$G(U \oplus Z, |Z|)^{\gamma_{U \oplus Z, |Z|}} \rightarrow G(U' \oplus Z, |Z|)^{\gamma_{U' \oplus Z, |Z|}}.$$

Set

$$BGL_{\mathcal{U} \oplus Z}(|Z|)^{\gamma_{|Z|}} = \operatorname{colim}_{U \subset \mathcal{U} \text{ finite}} G(U \oplus Z, |Z|)^{\gamma_{U \oplus Z, |Z|}}.$$

Given  $Z \dot{+} T = Z'$ , there is a canonical map of based  $k$ -spaces

$$(2.6) \quad \Sigma^T BGL_{\mathcal{U} \oplus Z}(|Z|)^{\gamma_{|Z|}} \rightarrow BGL_{\mathcal{U} \oplus Z'}(|Z'|)^{\gamma_{|Z'|}}$$

which classifies the bundle  $\gamma_{|Z|} \oplus T$  over  $BGL_{\mathcal{U} \oplus Z}(|Z|)$ . We define the algebraic Thom prespectrum to be the  $\mathcal{U}$ -prespectrum whose  $V$ -th space is

$$(2.7) \quad \bigvee_{Z \dot{+} V = \mathcal{U}} BGL_{\mathcal{U} \oplus Z}(|Z|)^{\gamma_{|Z|}}.$$

Analogously as in the case of suspension prespectra, for  $W \dot{+} T = V$ , there is a canonical structure map from  $\Sigma^T$  of the  $V$ -th space to the  $W$ -th space. On each summand  $\Sigma^T BGL_{\mathcal{U} \oplus Z}(|Z|)^{\gamma_{|Z|}}$  for  $Z \dot{+} V = \mathcal{U}$ , we have

$$\Sigma^T BGL_{\mathcal{U} \oplus Z}(|Z|)^{\gamma_{|Z|}} \rightarrow BGL_{\mathcal{U} \oplus (T \dot{+} Z)}(|T \dot{+} Z|)^{\gamma_{|T \dot{+} Z|}}$$

which classifies the bundle  $\gamma_{|Z|} \oplus T$  over  $BGL_{\mathcal{U} \oplus Z}(|Z|)$ . Taking adjoints and applying the spectrification functor gives

$$MGL_V = \operatorname{colim}_{(W,T:V \rightarrow W) \in V \downarrow \mathcal{C}(\mathcal{U})} \Omega^T \left( \bigvee_{Z \dot{+} W = \mathcal{U}} BGL_{\mathcal{U} \oplus Z}(|Z|)^{\gamma_{|Z|}} \right).$$

This gives the coordinate-free version of the algebraic cobordism spectrum  $MGL$ . It will be shown in Section 6 that the homotopy type of this construction agrees with the one in [20].

### 3. Coordinatized Prespectra

In some sense, coordinate-free prespectra are really not what one would expect. In this section, we introduce a notion of prespectra which are not coordinate-free, but are more closely parallel to the classical case [12].

DEFINITION 3.1. *Let  $\mathcal{B} = \{U_i\}_{i \geq 1}$  be a fixed decomposition*

$$\mathcal{U} = \bigoplus_{i=1}^{\infty} U_i$$

*of the universe  $\mathcal{U} = \mathbb{A}^{\infty}$  into finite-dimensional subspaces. For  $n \in \mathbb{N}$ , let  $V_n = \bigoplus_{i > n} U_i$ . A  $\mathcal{B}$ -prespectrum  $D$  is a collection of based  $k$ -spaces  $D_n = D_{V_n}$ , with given structure map*

$$\rho_n : D_n \rightarrow \Omega^{U_{n+1}} D_{n+1} = \operatorname{Hom}_{\operatorname{Spc}(k)_\bullet}(U_{n+1}/(U_{n+1} \setminus \{0\}), D_{n+1}).$$

*A map  $f : D \rightarrow D'$  of  $\mathcal{B}$ -prespectra is a collection of maps  $f_n : D_n \rightarrow D'_n$  that commute with the structure maps.*

*A  $\mathcal{B}$ -spectrum  $E$  is a  $\mathcal{B}$ -prespectrum  $\{E_n\}$ , such that each structure map  $\rho_n$  is an isomorphism.*

We will let  $\operatorname{Prespectra}(\mathcal{B})$  and  $\operatorname{Spectra}(\mathcal{B})$  denote the categories of  $\mathcal{B}$ -prespectra and  $\mathcal{B}$ -spectra, respectively. For each  $\mathcal{B}$ , there is an obvious forgetful functor  $R_{\mathcal{B}}$  from  $\operatorname{Spectra}(\mathcal{B})$  to  $\operatorname{Prespectra}(\mathcal{B})$ . Let  $L_{\mathcal{B}}$  be the left adjoint to this forgetful functor.

For any  $\mathcal{B}$ -prespectrum  $D = \{D_n\}$ ,

$$(L_{\mathcal{B}}D)_n = \operatorname{colim}_l \Omega^{\oplus_{i=n+1}^{n+l} U_i} D_{n+l}.$$

This is similar to the spectrification functor from  $Prespectra(k)$  to  $Spectra(k)$ .

We will use the following elementary categorical lemma to compare the categories of  $\mathcal{B}$ -spectra for different decompositions  $\mathcal{B}$ , as well as to  $Spectra(\mathcal{U})$ .

LEMMA 3.2. *Let  $\mathcal{C}$  be a small directed category, and  $\mathcal{D} = \{D_i\} \subset \mathcal{C}$ ,  $f_i : D_i \rightarrow D_{i+1}$  be a sequence of morphisms in  $\mathcal{C}$  satisfying the following two conditions.*

1. *For every  $X \in \mathcal{C}$ , there is some  $D_i \in \mathcal{D}$ , and a map  $f : X \rightarrow D_i$  in  $\mathcal{C}$ .*
2. *For every pairs of maps  $g_0, g_1 : X \rightarrow Y$  in  $\mathcal{C}$ , there is some  $D_n \in \mathcal{D}$  and  $f : Y \rightarrow D_n$ , such that  $f \cdot g_0 = f \cdot g_1$ .*

*Then given a functor  $F$  from  $\mathcal{C}$  to any cocomplete category  $\mathcal{C}'$ , the canonical map  $\operatorname{colim}_{\mathcal{C}} F \rightarrow \operatorname{colim}_{\mathcal{D}} F$  is an isomorphism in  $\mathcal{C}'$ .*

Given  $\mathcal{B} = \{U_i\}$ , we can choose an refinement of  $\mathcal{B}$  into a decomposition  $\mathcal{B}' = \{U'_j\}$  of the universe  $\mathcal{U}$  into 1-dimensional subspaces. That is to say, there is an increasing sequence  $\{j_1, \dots\} \subset \mathbb{N}$ ,  $j_1 \geq 1$ , such that

$$U_1 = \bigoplus_{j=1}^{j_1} U'_j$$

$$U_i = \bigoplus_{j=j_{i-1}+1}^{j_i} U'_j \text{ for } i > 1.$$

This can be done by decomposing each  $U_i$  into 1-dimensional subspaces. There is a forgetful functor  $R(\mathcal{B}, \mathcal{B}')$  from the category of  $\mathcal{B}'$ -spectra to  $\mathcal{B}$ -spectra. Given a  $\mathcal{B}'$ -spectrum  $E$ , we define  $(R(\mathcal{B}, \mathcal{B}')E)_n = E_{j_n}$ . The structure map

$$(R(\mathcal{B}, \mathcal{B}')E)_n = E_{j_n} \rightarrow \Omega^{U_{n+1}}(R(\mathcal{B}, \mathcal{B}')E)_{n+1} = \Omega^{\oplus_{j_{n+1}}^{j_{n+1}+1} U'_j} E_{j_{n+1}}$$

is given by the composition of the structure maps of  $E$ . Conversely, a  $\mathcal{B}$ -spectrum  $G$  gives rise naturally to a  $\mathcal{B}'$ -spectrum  $L(\mathcal{B}, \mathcal{B}')G$ . Namely, for each  $n$ , let  $m$  be the smallest number such that  $m = j_i$  for some  $i$ , and  $m \geq n$ . Then define

$$(L(\mathcal{B}, \mathcal{B}')G)_n = \Omega^{\oplus_{j=n+1}^m U'_j} G_m.$$

The structure map

$$(L(\mathcal{B}, \mathcal{B}')G)_n \rightarrow \Omega^{U'_{n+1}}(L(\mathcal{B}, \mathcal{B}')G)_{n+1}$$

is the identity for  $m \geq n+1$ , and the structure map of  $G$  if  $m = n$ . It is easy to check that  $R(\mathcal{B}, \mathcal{B}')$  and  $L(\mathcal{B}, \mathcal{B}')$  are inverse equivalences, so the category of  $\mathcal{B}$ -spectra is equivalent to the category of  $\mathcal{B}'$ -spectra. Hence, we can always assume that in the decomposition  $\mathcal{B} = \{U_i\}$ , every  $U_i$  has dimension one.

We can also consider the forgetful functor

$$PreR(\mathcal{B}, \mathcal{B}') : Prespectra(\mathcal{B}') \rightarrow Prespectra(\mathcal{B})$$

analogous to the forgetful functor on the level of spectra. Let  $PreL(\mathcal{B}, \mathcal{B}')$  be the right adjoint to it. Given a  $\mathcal{B}$ -prespectrum  $D$ ,  $PreL(\mathcal{B}, \mathcal{B}')D$  is given as follows. For each  $n$ , consider the largest  $i$  such that  $j_i \leq n$ . Set

$$(PreL(\mathcal{B}, \mathcal{B}')D)_n = \Sigma^{\oplus_{j=j_{i+1}}^n U'_j} D_i.$$

If  $n+1 < j_{i+1}$ , the structure map  $(PreL(\mathcal{B}, \mathcal{B}')D)_n \rightarrow \Omega(PreL(\mathcal{B}, \mathcal{B}')D)_{n+1}$  is just adjoint to the identity map on  $\Sigma^{\oplus_{j=i+1}^{n+1} U'_j} D_i$ . If  $n+1 = j_{i+1}$ , then define the structure map to be adjoint to the structure map in  $D$ . The composition of the structure maps of  $PreL(\mathcal{B}, \mathcal{B}')D$  is the same as the structure maps in  $D$ . Since the sequence  $\{j_i\}$  is cofinal in  $\mathbb{N}$  in the sense of Lemma 3.2, it is easy to check that the following diagram commutes

$$\begin{array}{ccc} Prespectra(\mathcal{B}) & \xrightarrow{PreL(\mathcal{B}, \mathcal{B}')} & Prespectra(\mathcal{B}') \\ L_{\mathcal{B}} \downarrow & & L_{\mathcal{B}'} \downarrow \\ Spectra(\mathcal{B}) & \xrightarrow{L(\mathcal{B}, \mathcal{B}')} & Spectra(\mathcal{B}'). \end{array}$$

The definition of  $\mathcal{B}$ -spectra is also compatible with the definition of  $\mathbb{T}$ -spectra (see Definition 1.6).

**PROPOSITION 3.3.** *For any decomposition  $\mathcal{B}$ , there is an equivalence between the categories  $Spectra(\mathcal{B})$  and  $Spectra(k)$ .*

PROOF. Given  $\mathcal{B}$ , choose a refinement  $\mathcal{B}' = \{U'_j\}$  of  $\mathcal{B}$ , such that each  $U'_j$  has dimension one. Then the categories  $Spectra(\mathcal{B})$  and  $Spectra(\mathcal{B}')$  are equivalent. Also,  $Spectra(\mathcal{B}')$  is equivalent to  $Spectra(k)$  by definition.  $\square$

In fact, for a decomposition  $\mathcal{B}'$  of  $\mathcal{U}$  into one-dimensional subspaces, there is by definition an equivalence of categories on the prespectra level

$$Prespectra(\mathcal{B}') \xrightarrow{\cong} Prespectra(k).$$

We can define the suspension spectrum of a  $k$ -space and the algebraic cobordism spectrum in this coordinatized context. Let  $\mathcal{B} = \{U_i\}$  be a given decomposition of  $\mathcal{U}$  into one-dimensional spaces. Denote  $V_n = \bigoplus_{i>n} U_i$  and  $W_n = \bigoplus_{i\leq n} U_i$ . There is a forgetful functor  $\Omega_{V_n}^{\mathcal{B}}$  from  $Spectra(\mathcal{B})$  to  $Spc(k)_\bullet$ , which is just taking the  $n$ -th space  $E_n = E_{V_n}$ . Then  $\Omega_{V_n}^{\mathcal{B}}$  has the left adjoint  $\Sigma_{V_n}^{\mathcal{B}}$ , the  $V_n$ -th shift desuspension of the suspension spectrum. For a based  $k$ -space  $X$ , we first take the  $\mathcal{B}$ -prespectrum whose  $n$ -th space is  $\Sigma^{W_n} X$ . The structure maps are adjoint to the identities on  $\Sigma^{W_n} X$ . Then  $\Sigma_{V_n}^{\mathcal{B}} X$  is the spectrification of this  $\mathcal{B}$ -prespectrum.

For the case of  $MGL$ , we consider the Thom prespectrum indexed on  $\mathcal{B}$ , whose  $n$ -th space is  $BGL_{\mathcal{U}\oplus W_n}(|W_n|)^{\gamma|W_n|}$ . The structure maps are the canonical maps

$$\Sigma^{U_{n+1}} BGL_{\mathcal{U}\oplus W_n}(|W_n|)^{\gamma|W_n|} \rightarrow BGL_{\mathcal{U}\oplus W_{n+1}}(|W_{n+1}|)^{\gamma|W_{n+1}|}$$

classifying the bundle  $\gamma_{|W_n|} \oplus U_{n+1}$  over  $BGL_{\mathcal{U}\oplus W_n}(|W_n|)$  (see (2.6)). Spectrification gives  $MGL_{\mathcal{B}}$ , the  $Spectra(\mathcal{B})$  version of the algebraic cobordism spectrum. In Section 6, we will see that the homotopy categories of  $Spectra(\mathcal{B})$  and  $Spectra(\mathcal{U})$  are equivalent, and  $MGL_{\mathcal{B}}$  corresponds to the coordinate-free version of  $MGL$  in this equivalence.

For purposes of comparison with  $Spectra(\mathcal{U})$ , we also define the following intermediate categories of prespectra and spectra. Let  $\mathcal{B} = \{U_i\}_{i\geq 1}$  be a given decomposition of  $\mathcal{U}$  into one-dimensional subspaces, and let  $V_n = \bigoplus_{i>n} U_i$  for  $n \in \mathbb{N}$ . Let  $\mathcal{B}_{full}$  be the full subcategory of  $\mathcal{C}(\mathcal{U})$  on the objects  $V_n$ .

DEFINITION 3.4. A  $\mathcal{B}_{full}$ -prespectrum  $E$  is a prespectrum indexed on  $\mathcal{B}_{full}$ , i. e. a sequence of based  $k$ -spaces  $E_{V_n}$ , for all  $n \geq 0$ , together with structure maps

$$\rho_{n,Z}^m : E_{V_m} \rightarrow \Omega^Z E_{V_n}$$

for all maps  $Z : V_m \rightarrow V_n$  in  $\mathcal{C}(\mathcal{U})$ , with the usual compatibility conditions. A  $\mathcal{B}_{full}$ -spectrum is a  $\mathcal{B}_{full}$ -prespectrum, all of whose structure maps are isomorphisms.

We denote the categories of  $\mathcal{B}_{full}$ -prespectra and  $\mathcal{B}_{full}$ -spectra by  $Prespectra(\mathcal{B}_{full})$  and  $Spectra(\mathcal{B}_{full})$ , respectively.

PROPOSITION 3.5. For any decomposition  $\mathcal{B}$  of  $\mathcal{U}$  into one-dimensional subspaces, there is a canonical equivalence of categories

$$Spectra(\mathcal{B}) \simeq Spectra(\mathcal{B}_{full}).$$

PROOF. There is an obvious forgetful functor from  $Spectra(\mathcal{B}_{full})$  to  $Spectra(\mathcal{B})$ , by restricting structure maps to ones of the form

$$E_{V_n} \rightarrow \Omega^{U_{n+1}} E_{V_{n+1}}$$

for all  $n \in \mathbb{N}$ .

To give the inverse functor, we need to extend a  $\mathcal{B}$ -spectrum  $D$  to a  $\mathcal{B}_{full}$ -spectrum. For a given  $V_n$ ,  $n \in \mathbb{N}$ , define the subcategory  $\mathcal{D}_{V_n}$  of  $V_n \downarrow \mathcal{B}_{full}$ , where the objects are pairs  $(V_j, Z)$ ,  $Z \dot{+} V_j = V_n$ , and the morphisms are only of the form

$$U_{j+1} \dot{+} \cdots \dot{+} U_l : (V_j, Z) \rightarrow (V_l, Z \dot{+} U_{j+1} \dot{+} \cdots \dot{+} U_l)$$

for  $j \leq l$ . Suppose  $V_n \dot{+} T = V_m$  for some  $n \geq m$ . Consider the sequence

$$S = \{V_j, U_{m+1} \dot{+} \cdots \dot{+} U_j : V_m \rightarrow V_j\}_{j \geq m}$$

in  $\mathcal{D}_{V_m}$ , along with maps  $U_{j+1} : V_j \rightarrow V_{j+1}$ . The sequence  $S$  satisfies the conditions of Lemma 3.2 in  $\mathcal{D}_{V_m}$ . (In fact, it satisfies the conditions of Lemma 3.2 in  $V_m \downarrow \mathcal{B}_{full}$ .) This is because for  $V_r, V_s \in V_m \downarrow \mathcal{B}_{full}$  and any two finite-dimensional spaces  $Z_0, Z_1 :$

$V_r \rightarrow V_s$ , we can choose  $V_j$  such that  $j \geq s$ , and  $(Z_0 + Z_1) \cap V_s \subseteq U_{s+1} \dot{+} \cdots \dot{+} U_j$ . Then

$$Z_0 \dot{+} U_{s+1} \dot{+} \cdots \dot{+} U_j = Z_1 \dot{+} U_{s+1} \dot{+} \cdots \dot{+} U_j$$

so  $U_{s+1} \dot{+} \cdots \dot{+} U_j$  is a morphism satisfying Condition 2 of Lemma 3.2. Thus, for a  $\mathcal{B}$ -spectrum  $D$ , we have

$$D_{V_m} = \operatorname{colim}_{(V_j, Z) \in \mathcal{D}_{V_m}} \Omega^Z D_{V_j}.$$

Likewise, the sequence

$$S' = \{V_j, U_{m+1} \dot{+} \cdots \dot{+} U_j : V_m \rightarrow V_j\}_{j \geq m}$$

and

$$S' = \{V_j, T \dot{+} V_{n+1} \dot{+} \cdots \dot{+} V_j : V_m \rightarrow V_j\}_{j \geq n}$$

satisfy the conditions of Lemma 3.2 in  $\mathcal{D}_{V_m}$ . Thus, we get a canonical structure isomorphism

$$\begin{aligned} D_{V_m} &\cong \operatorname{colim}_{(V_j, Z) \in \mathcal{D}_{V_m}} \Omega^Z D_{V_j} \\ &\cong \operatorname{colim}_j \Omega^{T \dot{+} U_{n+1} \dot{+} \cdots \dot{+} U_j} D_{V_j} \\ &\cong \Omega^T \operatorname{colim}_j \Omega^{U_{n+1} \dot{+} \cdots \dot{+} U_j} D_{V_j} \\ &\cong \Omega^T D_{V_n}. \end{aligned}$$

The next to last isomorphism is due to the fact that  $T/(T \setminus \{0\})$  is a small object in  $\operatorname{Spc}(k)_\bullet$  (see Lemma 17.4 in the Appendix). Therefore, we have made  $D$  into a  $\mathcal{B}_{full}$ -spectrum.

Finally, by arguments similar as above, the sequence  $\{V_n, U_{n+1} : V_n \rightarrow V_{n+1}\}$  satisfies the conditions of Lemma 3.2 in the category  $\mathcal{B}_{full}$ . Using this, it is easy to check that the functor given above and the forgetful functor are inverses to each other.  $\square$

Given a decomposition  $\mathcal{B} = \{U_i\}$  of the universe into one-dimensional subspaces, let  $V_i = \bigoplus_{j>i} U_j$ . There is a forgetful functor

$$R_{\mathcal{B}}^{\mathcal{U}} : \operatorname{Prespectra}(\mathcal{U}) \rightarrow \operatorname{Prespectra}(\mathcal{B}_{full}).$$

Namely, given an  $\mathcal{U}$ -prespectrum  $D$ , we restrict to  $D_{V_i}$  and structure maps with respect to  $U_{i+1} : V_i \rightarrow V_{i+1}$ . The functor  $R_{\mathcal{B}}^{\mathcal{U}}$  restricts to a functor

$$R_{\mathcal{B}}^{\mathcal{U}} : \text{Spectra}(\mathcal{U}) \rightarrow \text{Spectra}(\mathcal{B}_{full}).$$

The functor  $\mathcal{R}_{\mathcal{B}}^{\mathcal{U}}$  has a left adjoint  $L_{\mathcal{B}}^{\mathcal{U}}$ . For a  $\mathcal{B}_{full}$ -prespectrum  $D$ , we define the  $\mathcal{U}$ -prespectrum  $L_{\mathcal{B}}^{\mathcal{U}}D$  as follows. For  $W \in \mathcal{C}(\mathcal{U})$ , set

$$(3.6) \quad L_{\mathcal{B}}^{\mathcal{U}}D_W = \text{colim}_{Z \dot{+} W = V_i} \Sigma^Z D_{V_i}.$$

If  $W \subset V_j \subseteq V_i$  for some  $j \geq i$ , suppose  $Z \dot{+} W = V_i$ . Recall the Noether isomorphism, which gives a canonical bijective correspondance between the subspaces of  $V_i/W$  and the subspaces of  $V_i$  containing  $W$ . Hence, there is a unique  $Z' \subseteq Z$  corresponding to  $V_{i+1}/W$ , such that  $W \dot{+} Z' = V_{i+1}$ . If  $U$  is finite-dimensional, and  $Z' \dot{+} U = Z$ , then it is elementary to check that  $V_{i+1} \dot{+} U = V_i$ . Hence, there is a map

$$\Sigma^Z D_{V_i} \rightarrow \bigvee_{Z' \dot{+} U = Z} \Sigma^{Z' \dot{+} U} D_{V_i} \rightarrow \Sigma^{Z'} D_{V_j}$$

where the second map is a wedge of structure maps of  $D$ . This gives the colimit used in (3.6). The structure maps of  $L_{\mathcal{B}}^{\mathcal{U}}D$  are defined similarly. In particular,  $L_{\mathcal{B}}^{\mathcal{U}}D_{V_i} = D_{V_i}$  for a  $\mathcal{B}_{full}$ -prespectrum  $D$ . Thus, it is straightforward to check that  $L_{\mathcal{B}}^{\mathcal{Y}} : \text{Prespectra}(\mathcal{B}_{full}) \rightarrow \text{Prespectra}(\mathcal{U})$  is indeed the left adjoint to  $R_{\mathcal{B}}^{\mathcal{U}}$ . Applying the spectrification functor gives  $L_{\mathcal{B}}^{\mathcal{U}}E$  from  $\text{Spectra}(\mathcal{B})$  to  $\text{Spectra}(\mathcal{U})$ .

On the level of spectra, we have the canonical equivalence of categories

$$\text{Spectra}(\mathcal{B}) \simeq \text{Spectra}(\mathcal{B}_{full})$$

by Proposition 3.5. Thus,  $(L_{\mathcal{B}}^{\mathcal{U}}, R_{\mathcal{B}}^{\mathcal{U}})$  becomes a pair of adjoint functors between the categories  $\text{Spectra}(\mathcal{B})$  and  $\text{Spectra}(\mathcal{U})$ .

#### 4. The Stable Simplicial Model Structure

In this section and the next, we describe the construction of the stable homotopy category of smooth schemes, along with some of its properties. This was given by

Morel and Voevodsky [16] and Jardine [9], but we give a slightly different construction here. We will state the theorems in this section, and defer their proofs to Section 15.

We construct the structure of a model category on  $Spectra(\mathcal{U})$ . As in the case of  $k$ -spaces, this is done in two steps. First, one uses the simplicial model structure on  $Spc(k)_\bullet$  [16] to construct the simplicial model structure on the category of spectra. The  $\mathbb{A}^1$ -local structure on  $Spectra(\mathcal{U})$  follows from the simplicial structure by adding a class of  $\mathbb{A}^1$ -weak equivalences to the simplicial weak equivalences, similar to the case of  $Spc(k)_\bullet$ .

We begin by recalling the simplicial model structures on  $Prespectra(k)$  and  $Spectra(k)$ . The category of based  $k$ -spaces has the simplicial model category structure, given in Definition 1.2. Recall the definition of the simplicial model structure on  $Prespectra(k)$ , due to Bousfield and Friedlander [2]. First, there is the levelwise simplicial model structure on  $Prespectra(k)$ .

**DEFINITION 4.1 ([2]).** *A map of  $\mathbb{T}$ -prespectra  $f : D \rightarrow E$  is a levelwise simplicial weak equivalence or a levelwise simplicial fibration if each map of  $k$ -spaces  $f_n : D_n \rightarrow E_n$  is a simplicial weak equivalence or simplicial fibration, respectively. It is a simplicial cofibration if it satisfies the left lifting property with respect to all levelwise simplicial acyclic fibrations.*

The classes of simplicial cofibrations, levelwise simplicial weak equivalences and levelwise simplicial fibrations form a model structure on  $Prespectra(k)$ . The stable simplicial structure on  $Prespectra(k)$  is defined as follows.

**DEFINITION 4.2 ([2]).** *Let  $f : D \rightarrow E$  be a map in  $Prespectra(k)$ .*

1.  *$f$  is a stable simplicial cofibration if it is a simplicial cofibration in the levelwise simplicial model structure.*

2.  *$f$  is a stable simplicial weak equivalence if it induces a bijection*

$$\operatorname{colim}_m \mathcal{H}_s(Spc(k)_\bullet)(\mathbb{T}^m, D_{n+m}) \xrightarrow{\cong} \operatorname{colim}_m \mathcal{H}_s(Spc(k)_{\text{bullet}})(\mathbb{T}^m, E_{n+m})$$

for all  $n$ .

3.  $f$  is a stable simplicial fibration if it has the right lifting property with respect to all stable simplicial acyclic cofibrations of prespectra.

**THEOREM 4.3 ([2]).** *The classes of stable simplicial cofibrations, stable simplicial weak equivalences, and stable simplicial fibrations give a model structure on  $\text{Prespectra}(k)$ .*

When there is no possibility of confusion, we will say the simplicial model structure to mean the stable simplicial model structure. We give an alternative formulation of stable simplicial weak equivalences.

**DEFINITION 4.4.** *A map  $f : D \rightarrow D'$  of  $\mathbb{T}$ -prespectra over  $k$  is an eventual spacewise equivalence if there is some  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,*

$$f_n : D_n \rightarrow D'_n$$

*is a simplicial weak equivalence in  $\text{Spc}(k)_\bullet$ .*

We also define a class  $\mathbb{W}$  of maps generated by eventual spacewise equivalences, which turns out to coincide with the class of stable simplicial weak equivalences of  $\mathbb{T}$ -prespectra.

**DEFINITION 4.5.** *We define the class  $\mathbb{W}$  of maps in  $\text{Prespectra}(k)$  to be the smallest class containing all eventual spacewise equivalences, and satisfying the following conditions:*

1. *For composable maps  $f, g$ , if two out of the three maps  $f$ ,  $g$ , and  $gf$  are in  $\mathbb{W}$ , then so is the third.*
2. *For a directed system of prespectra  $D^\alpha$  and maps  $f^\alpha : D^\alpha \rightarrow D^{\alpha'}$  for any  $f : \alpha \rightarrow \alpha'$ , such that each map is in  $\mathbb{W}$ , the induced map  $D^\beta \rightarrow \text{colim}_\alpha D^\alpha$  is in  $\mathbb{W}$  for each  $\beta$ .*

The first result of this section is the following, which we will prove in Section 15.

**PROPOSITION 4.6.** *The class of stable simplicial weak equivalences in the category  $\text{Prespectra}(k)$  coincides with  $\mathbb{W}$ .*

The simplicial model structure on  $Prespectra(k)$  gives rise to the simplicial model structure on  $Spectra(k)$ .

**DEFINITION 4.7.** *Let  $f : E \rightarrow G$  be a map of spectra.  $f$  is a simplicial cofibration, simplicial weak equivalence, or simplicial fibration of spectra if it is a simplicial cofibration, simplicial weak equivalence, or simplicial fibration in the category of pre-spectra.*

The proof of the next theorem is by the usual small objects argument, which we defer also to Section 15.

**THEOREM 4.8.** *The classes of simplicial cofibrations, weak equivalences, and fibrations of spectra define the structure of a model category on  $Spectra(k)$ .*

In particular, let  $L : Prespectra(k) \rightarrow Spectra(k)$  be the spectrification functor. We will see in Lemma 15.1 that for any  $\mathbb{T}$ -prespectrum  $D$ , the canonical map  $D \rightarrow LD$  is a simplicial weak equivalence of  $\mathbb{T}$ -prespectra, so the simplicial homotopy categories of  $\mathbb{T}$ -prespectra and  $\mathbb{T}$ -spectra are equivalent.

We can also formulate the levelwise and stable simplicial model structures for  $Prespectra(\mathcal{U})$  and  $Spectra(\mathcal{U})$  in an analogous fashion.

**DEFINITION 4.9.** *A map  $f : D \rightarrow E$  of  $\mathcal{U}$ -prespectra is a levelwise simplicial weak equivalence or fibration if for every  $V \in \mathcal{C}(\mathcal{U})$ , the map  $f_V : D_V \rightarrow E_V$  is a simplicial weak equivalence or simplicial fibration of based  $k$ -spaces, respectively. It is a simplicial cofibration in the levelwise model structure if it satisfies the left lifting property with respect to all levelwise simplicial acyclic fibrations.*

The proof of the following proposition is by the small objects argument (see Section 15).

**PROPOSITION 4.10.** *The classes of simplicial cofibrations, levelwise simplicial weak equivalences, and levelwise simplicial fibrations give a model category structure on  $Prespectra(\mathcal{U})$ .*

There is also the stable simplicial model structure on  $\text{Prespectra}(\mathcal{U})$ .

DEFINITION 4.11. *Let  $f : D \rightarrow E$  be a map in  $\text{Prespectra}(\mathcal{U})$ .*

1.  *$f$  is a stable simplicial cofibration if it is a simplicial cofibration in the levelwise model structure;*

2.  *$f$  is a stable simplicial weak equivalence if it induces a bijection*

$$\begin{aligned} \text{colim}_{W \dot{+} Z=V} \mathcal{H}_s(\text{Spc}(k)_\bullet)(Z/(Z \setminus \{0\}), D_W) \\ \cong \text{colim}_{W \dot{+} Z=V} \mathcal{H}_s(\text{Spc}(k)_\bullet)(Z/(Z \setminus \{0\}), E_W) \end{aligned}$$

for every  $V \in \mathcal{C}(\mathcal{U})$ .

2.  *$f$  is a stable simplicial fibration if it satisfies the right lifting property with respect to all stable simplicial acyclic cofibrations.*

The proof of the following theorem is similar to Bousfield and Friedlander [2].

THEOREM 4.12. *The classes of simplicial cofibrations, stable simplicial weak equivalences and stable simplicial fibrations form a model structure on  $\text{Prespectra}(\mathcal{U})$ .*

The model structure on  $\mathcal{U}$ -spectra is defined from the model structure on  $\mathcal{U}$ -prespectra.

DEFINITION 4.13. *A map  $f : D \rightarrow E$  of  $\mathcal{U}$ -spectra is a stable simplicial cofibration, stable simplicial weak equivalence, or stable simplicial fibration if it is a stable simplicial cofibration, stable simplicial weak equivalence, or stable simplicial fibration of  $\mathcal{U}$ -prespectra.*

We also have the following model structure theorem. Its proof is analogous to that of Theorem 4.8 (see Section 15).

THEOREM 4.14. *The classes of stable simplicial cofibrations, stable simplicial weak equivalences, and stable simplicial fibrations give a model structure on the category  $\text{Spectra}(\mathcal{U})$ .*

We denote by  $\mathcal{H}_s(\mathcal{U})$  the homotopy category associated with the stable simplicial model structure on  $\text{Spectra}(\mathcal{U})$ .

We will now start working on a comparison between  $Spectra(\mathcal{U})$ ,  $Spectra(\mathcal{B})$ , and  $Spectra(\mathcal{B}_{full})$ . Let  $\mathcal{B} = \{U_i\}$  be a fixed decomposition of  $\mathcal{U}$  into one-dimensional subspaces. The canonical identification of  $Prespectra(\mathcal{B})$  and  $Spectra(\mathcal{B})$  via Proposition 3.3 respectively with  $Prespectra(k)$  and  $Spectra(k)$  give the stable simplicial model structures on  $Prespectra(\mathcal{B})$  and  $Spectra(\mathcal{B})$ . Let  $\mathcal{H}_s(\mathcal{B})$  denote the homotopy category of the stable simplicial model structure on  $Spectra(\mathcal{B})$ .

There are also the categories  $Prespectra(\mathcal{B}_{full})$  and  $Spectra(\mathcal{B}_{full})$ . We refine the levelwise and stable simplicial model structures on  $Prespectra(\mathcal{B}_{full})$  similarly as for  $Prespectra(\mathcal{U})$  (see Definitions 4.9 and 4.11). By arguments similar to that of [2], these are indeed model category structures. We define a map of  $\mathcal{B}_{full}$ -spectra to be a stable simplicial cofibration, stable simplicial weak equivalence, or stable simplicial fibration if it is in the corresponding class of maps in  $Prespectra(\mathcal{B}_{full})$ . This is a model structure by arguments similar to that for Theorem 4.8. Consider the canonical equivalence

$$Spectra(\mathcal{B}) \simeq Spectra(\mathcal{B}_{full})$$

by Proposition 3.5. This equivalence is the identity on the level of the  $E_{V_i}$ 's, as well as on maps between spectra. Hence, the equivalence preserves the levelwise simplicial model category structures, as well as eventual spacewise weak equivalences. Using the fact that the sequence  $\{V_i, U_{i+1} : V_i \rightarrow V_{i+1}\}$  is cofinal in  $\mathcal{B}_{full}$  in the sense of Lemma 3.2, and the homotopy sets definition of stable simplicial weak equivalences in Definition 4.2, one sees that the equivalence also preserves stable simplicial weak equivalences, hence the stable simplicial model structures.

We recall some results in the theory of model categories. The following result is Theorem 9.7 in Dwyer and Spanlinski [5].

LEMMA 4.15 (Dwyer and Spalinski [5]). *Let  $\mathcal{C}$  and  $\mathcal{D}$  be two model categories, with two functors  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{C}$  which form an adjoint pair (with  $F$  being the left adjoint). If  $F$  preserves cofibrations and  $G$  preserves fibrations, then  $F$  and  $G$  pass to a pair of functors between the homotopy categories of  $\mathcal{C}$  and  $\mathcal{D}$ .*

The functors  $F, G$  are said to be a Quillen adjoint pair if they satisfy the condition of the lemma. By adjunction, the condition that  $F, G$  are a Quillen adjoint pair is also equivalent to each of the following two conditions: 1. the left adjoint  $F$  preserves cofibrations and acyclic cofibrations; 2. the right adjoint  $G$  preserves fibrations and acyclic fibrations.

A Quillen adjoint pair  $F : \mathcal{C} \rightarrow \mathcal{D}$ ,  $G : \mathcal{D} \rightarrow \mathcal{C}$  is a Quillen equivalence if the functors induced on the homotopy categories are equivalences of categories. Let  $G : \mathcal{D} \rightarrow \mathcal{C}$  be a functor between two model categories. Recall that if a map  $f$  in  $\mathcal{D}$  is a weak equivalence if and only if  $Gf$  is a weak equivalence in  $\mathcal{C}$ , then  $G$  is said to create weak equivalences in  $\mathcal{D}$ . There is the following result ([18], [14]).

LEMMA 4.16 ([14], Lemma 2.2). *Given a Quillen adjoint pair*

$$F : \mathcal{C} \rightarrow \mathcal{D}, G : \mathcal{D} \rightarrow \mathcal{C}$$

*if  $G$  creates the weak equivalences in  $\mathcal{D}$  and the unit of the adjunction  $X \rightarrow GFX$  is a weak equivalence for all cofibrant  $X \in \mathcal{C}$ , then  $(F, G)$  is a Quillen equivalence.*

PROPOSITION 4.17. *The functors*

$$R_{\mathcal{B}}^{\mathcal{U}} : \text{Spectra}(\mathcal{U}) \rightarrow \text{Spectra}(\mathcal{B})$$

$$L_{\mathcal{B}}^{\mathcal{U}} : \text{Spectra}(\mathcal{B}) \rightarrow \text{Spectra}(\mathcal{U})$$

*for a Quillen adjoint pair with respect to the stable simplicial model structures.*

PROOF. In the levelwise simplicial model structure,  $R_{\mathcal{B}}^{\mathcal{U}}$  preserves weak equivalences and fibrations, since these are just defined spacewise. Hence,  $L_{\mathcal{B}}^{\mathcal{U}}$  preserves cofibrations in the levelwise simplicial structure, which are the same as cofibrations in the stable simplicial model structure.

We need that  $L_{\mathcal{B}}^{\mathcal{U}}$  also preserves stable simplicial acyclic cofibrations. In fact,  $L_{\mathcal{B}}^{\mathcal{U}}$  preserves stable simplicial weak equivalences. The stable simplicial model structures on  $\text{Spectra}(\mathcal{B})$  and  $\text{Spectra}(\mathcal{B}_{full})$  are preserved by the equivalence of categories between them. Also, by Proposition 4.6, the stable simplicial weak equivalences of

$\mathcal{B}$ -spectra are generated by eventual spacewise equivalences. Thus, it suffices to consider an eventual spacewise equivalence  $f : E \rightarrow E'$  of  $\mathcal{B}_{full}$ -prespectra. So there exists some  $n \in \mathbb{N}$ , such that for all  $i \geq n$ ,  $f_i : E_i \rightarrow E'_i$  is a simplicial weak equivalence of based  $k$ -spaces. Then for all  $W \subset V_i$  for some  $i \geq n$ , applying  $L_{\mathcal{B}}^{\mathcal{U}}$  gives that  $(L_{\mathcal{B}}^{\mathcal{U}})_W$  is a directed colimit of simplicial weak equivalences of based  $k$ -spaces, which is therefore a simplicial weak equivalence of based  $k$ -spaces by [16]. Note that  $V_i \downarrow \mathcal{C}(\mathcal{U}) = \{W \mid V_i \subset W\}$  is cofinal in  $\mathcal{C}(\mathcal{U})$  in the sense of Lemma 3.2. Thus, it is easy to check by the homotopy sets characterization in Definition 4.11 that  $L_{\mathcal{B}}^{\mathcal{U}}f$  is a stable simplicial weak equivalence of  $\mathcal{U}$ -spectra.  $\square$

Thus,  $(L_{\mathcal{B}}^{\mathcal{U}}, R_{\mathcal{B}}^{\mathcal{U}})$  pass to a pair of adjoint functors

$$\begin{aligned} L_{\mathcal{B}}^{\mathcal{U}} : \mathcal{H}_s(\mathcal{B}) &\rightarrow \mathcal{H}_s(\mathcal{U}) \\ R_{\mathcal{B}}^{\mathcal{U}} : \mathcal{H}_s(\mathcal{U}) &\rightarrow \mathcal{H}_s(\mathcal{B}). \end{aligned}$$

## 5. The $\mathbb{A}^1$ -local Model Structure

The stable simplicial homotopy category  $\mathcal{H}_s(\mathcal{U})$  is not yet the stable homotopy category of  $\mathcal{U}$ -spectra with which we would like to work.

**DEFINITION 5.1.** *Two maps  $f, g : E \rightarrow G$  of  $\mathcal{U}$ -spectra are  $\mathbb{A}^1$ -homotopic if there is a map  $h : E \wedge \mathbb{A}_+^1 \rightarrow G$ , such that the composition of  $h$  with the two maps  $E = E \wedge S^0 \rightarrow E \wedge \mathbb{A}_+^1$ , given by sending the non-base point of  $S^0$  to 0 and 1, are  $f$  and  $g$  respectively.*

We would like  $\mathbb{A}^1$ -homotopy equivalences of  $\mathcal{U}$ -spectra to be weak equivalences. This motivates the following definitions, analogous to those given for  $k$ -spaces by Morel and Voevodsky [16].

**DEFINITION 5.2.** *A  $\mathcal{U}$ -spectrum  $G$  is said to be  $\mathbb{A}^1$ -local if for every  $\mathcal{U}$ -spectrum  $E$ , the map  $E \wedge \mathbb{A}_+^1 \rightarrow E$  induced by the projection  $\mathbb{A}_+^1 \rightarrow S^0$  induces a bijection of morphism sets*

$$\mathcal{H}_s(E, G) \rightarrow \mathcal{H}_s(E \wedge \mathbb{A}_+^1, G).$$

DEFINITION 5.3. *Let  $f : E \rightarrow E'$  be a map of  $\mathcal{U}$ -spectra.*

1.  *$f$  is an  $\mathbb{A}^1$ -cofibration if it is a (stable) simplicial cofibration;*
2.  *$f$  is an  $\mathbb{A}^1$ -weak equivalence if for every  $\mathbb{A}^1$ -local  $G$ , the map of morphism sets*

$$\mathcal{H}_s(E', G) \rightarrow \mathcal{H}_s(E, G)$$

*is a bijection;*

3.  *$f$  is an  $\mathbb{A}^1$ -fibration if it satisfies the right lifting property with respect to every acyclic  $\mathbb{A}^1$ -cofibration.*

There is the following alternative characterization of  $\mathbb{A}^1$ -weak equivalences of spectra, the proof of which is analogous to Proposition 2.2.9 of Morel and Voevodsky [16].

PROPOSITION 5.4. *A map  $f : E \rightarrow E'$  of  $\mathcal{U}$ -spectra is an  $\mathbb{A}^1$ -weak equivalence if and only if for any  $\mathbb{A}^1$ -local and simplicially fibrant spectrum  $G$ , the map of function spectra*

$$f^* : F(E', G) \rightarrow F(E, G)$$

*is a simplicial weak equivalence.*

We have the following theorem.

THEOREM 5.5. *The classes of  $\mathbb{A}^1$ -cofibrations,  $\mathbb{A}^1$ -weak equivalences and  $\mathbb{A}^1$ -fibrations define a model structure on  $\text{Spectra}(\mathcal{U})$ .*

PROOF. Analogous to that of Morel and Voevodsky for  $k$ -spaces ([16], Theorem 2.2.21). The first three model structure axioms are clear. The factorization of any map into a cofibration and an acyclic fibration is immediate since a simplicial acyclic fibration is an  $\mathbb{A}^1$ -local acyclic fibration (Lemma 2.2.9 of [16]). The other factorization uses the formal fact that  $\mathbb{A}^1$ -local acyclic cofibrations are preserved under pushouts and the bounding argument in Corollary 2.2.18 of [16], done spacewise. These allow the usual small objects argument to give the factorization of any map into an acyclic cofibration and a fibration. Finally, the part of CM4 which is not by definition follows formally by an argument due to Joyal [11].  $\square$

This gives the  $\mathbb{A}^1$ -local model structure on  $Spectra(\mathcal{U})$ , which is the structure with which one really wants to work. In particular, all  $\mathbb{A}^1$ -homotopy equivalences of spectra are weak equivalences in this structure. Let  $\mathcal{H}_{\mathbb{A}^1}(\mathcal{U})$  denote the homotopy category associated with this model structure. In the next section, we will show that this homotopy category is equivalent to the  $\mathbb{A}^1$ -local stable homotopy category of  $\mathbb{T}$ -spectra defined by Morel and Voevodsky [16]. We first give some properties of homotopy colimits of  $\mathcal{U}$ -spectra. Given a small category  $\mathcal{C}$ , and a functor  $E : \mathcal{C} \rightarrow Spectra(\mathcal{U})$ , we can define the homotopy colimit  $\text{hocolim}_{c \in \mathcal{C}} E_i$  similarly as for  $k$ -spaces ([3], [16]). In the simplicial homotopy category  $\mathcal{H}_s$ , morphisms into a homotopy colimit are determined by morphisms into objects of the diagram in the following sense.

**DEFINITION 5.6.** *Suppose  $\mathcal{C}$  is a small category, and  $\mathcal{D}$  is a category in which we can form homotopy colimits. Let  $S : \mathcal{C} \rightarrow \mathcal{D}$  be a functor. Then we say that for an object  $Y \in \mathcal{D}$ , the morphism set  $\mathcal{D}(\text{hocolim}_{\mathcal{C}} S_c, Y)$  is determined by  $\mathcal{D}(S_c, Y)$  for all  $c \in \mathcal{C}$ , if for any functor  $T : \mathcal{C} \rightarrow \mathcal{D}$  and natural transformation  $\eta : S \rightarrow T$  that induces a bijection  $\mathcal{D}(T_c, Y) \cong \mathcal{D}(S_c, Y)$  for all  $c \in \mathcal{C}$ ,  $\eta$  induces a bijection*

$$\mathcal{D}(\text{hocolim}_{\mathcal{C}} T_c, Y) \xrightarrow{\cong} \mathcal{D}(\text{hocolim}_{\mathcal{C}} S_c, Y).$$

We record the following standard argument.

**PROPOSITION 5.7.** *Suppose  $\mathcal{C}$  is a small category, and*

$$S : \mathcal{C} \rightarrow Spectra(\mathcal{U})$$

*is a functor. Then for any spectrum  $G$ , the morphism set in the simplicial homotopy category*

$$\mathcal{H}_s(\text{hocolim}_{\mathcal{C}} S_c, G)$$

*is determined by  $\mathcal{H}_s(S_c, G)$  for all  $c \in \mathcal{C}$ .*

PROOF. Denote

$$E = \operatorname{hocolim}_{\mathcal{C}} S_{\mathcal{C}}.$$

Define the simplicial object  $E^\bullet$  as follows: the  $n$ -th simplicial stage is

$$E^n = \coprod_{(f_1, \dots, f_n)} \operatorname{Target}(E(f_n))$$

where  $(f_1, \dots, f_n)$  runs through all  $n$ -tuples of composable morphisms in  $\mathcal{C}$ . Then the homotopy colimit  $E$  is the simplicial realization of  $E^\bullet$ .

This gives the following spectral sequence of mapping groups:

$$(5.8) \quad E_1 = \mathcal{H}_s(\coprod_{(f_1, \dots, f_n)} \operatorname{Target}(S(f_n)), G) \Rightarrow \mathcal{H}_s(E, G)$$

which is conditionally convergent. Let  $T$  be another functor, and let  $\eta : S \rightarrow T$  be a natural transformation that induces a bijection

$$\mathcal{H}_s(S(X), G) \cong \mathcal{H}_s(T(X), G)$$

for a spectrum  $G$  and  $X \in \mathcal{C}$ . Then we have a conditionally convergent spectral sequence similar to (5.8)

$$E_1 = \mathcal{H}_{\mathbb{A}^1}(\coprod_{(f_1, \dots, f_n)} \operatorname{Target}(T(f_n)), G) \Rightarrow \mathcal{H}_{\mathbb{A}^1}(\operatorname{hocolim}_{c \in \mathcal{C}} T(c), G)$$

The natural transformation  $\eta$  induces an isomorphism of spectral sequences.  $\square$

**Comment:** This spectral sequence argument is generic. Namely, the proposition also holds if we replace the morphism sets in  $\mathcal{H}_s$  by morphism sets in the  $\mathbb{A}^1$ -local stable homotopy category  $\mathcal{H}_{\mathbb{A}^1}$ . Thus, for any  $G$ , we also get a spectral sequence

$$E_1 = \mathcal{H}_{\mathbb{A}^1}(\coprod_{(f_1, \dots, f_n)} \operatorname{Target}(E(f_n)), G) \Rightarrow \mathcal{H}_{\mathbb{A}^1}(E, G).$$

So the morphism set  $\mathcal{H}_{\mathbb{A}^1}(\operatorname{hocolim}_{i \in \mathcal{I}} E_i, G)$  is also determined by the morphism sets  $\mathcal{H}_{\mathbb{A}^1}(E_i, G)$ .

Recall the following lemma of Morel and Voevodsky [16].

LEMMA 5.9 (Morel and Voevodsky [16], Lemma 2.1.14). *Suppose  $\mathcal{I}$  is a small directed category, and  $X : \mathcal{I} \rightarrow \operatorname{Spc}(k)_\bullet$  be a functor. Then the canonical map*

$$\operatorname{hocolim}_{i \in \mathcal{I}} X_i \rightarrow \operatorname{colim}_{i \in \mathcal{I}} X_i$$

is a simplicial weak equivalence of based  $k$ -spaces.

The same is true in the category of spectra.

LEMMA 5.10. *Let  $\mathcal{I}$  be a small directed category, and*

$$E : \mathcal{I} \rightarrow \text{Spectra}(\mathcal{U})$$

be a functor. Then the canonical map

$$\text{hocolim}_{i \in \mathcal{I}} E_i \rightarrow \text{colim}_{i \in \mathcal{I}} E_i$$

is a simplicial weak equivalence of  $\mathcal{U}$ -spectra.

PROOF. The homotopy colimit and colimit of spectra are given by first taking the homotopy colimit and colimit in the category of prespectra, then applying the spectrification functor  $L$ . In  $\text{Prespectra}(\mathcal{U})$ , the homotopy colimit and colimit are just taken spacewise. Denote the homotopy colimit and the colimit in the category of prespectra by  $\text{hocolim}^{\text{Prespectra}(\mathcal{U})} E_i$  and  $\text{colim}^{\text{Prespectra}(\mathcal{U})} E_i$ , respectively. Then by Lemma 5.9, in  $\text{Prespectra}(\mathcal{U})$ , the map

$$\text{hocolim}_{i \in \mathcal{I}}^{\text{Prespectra}(\mathcal{U})} E_i \rightarrow \text{colim}_{i \in \mathcal{I}}^{\text{Prespectra}(\mathcal{U})} E_i$$

is a spacewise simplicial weak equivalence. Hence, it is a simplicial weak equivalence of  $\mathcal{U}$ -prespectra. However, by Lemma 15.1, for any prespectrum  $D$ , the unit map  $D \rightarrow LD$  is a simplicial weak equivalence of prespectra. Therefore, the map

$$\text{hocolim}_{i \in \mathcal{I}} E_i \rightarrow \text{colim}_{i \in \mathcal{I}} E_i$$

is a simplicial weak equivalence of  $\mathcal{U}$ -spectra.  $\square$

The next proposition follows from the above lemma.

PROPOSITION 5.11. *Let  $\mathcal{I}$  be a small directed category.*

1. *Suppose  $E, E' : \mathcal{I} \rightarrow \text{Spectra}(\mathcal{U})$  are two functors, with a natural transformation  $f : E \rightarrow E'$ . Suppose that for every  $i \in \mathcal{I}$ ,  $f_i : E_i \rightarrow E'_i$  is an  $\mathbb{A}^1$ -weak equivalence of spectra. Then the induced map*

$$\text{colim}_{i \in \mathcal{I}} E_i \rightarrow \text{colim}_{i \in \mathcal{I}} E'_i$$

is an  $\mathbb{A}^1$ -weak equivalence of spectra.

2. Suppose  $E : \mathcal{I} \rightarrow \text{Spectra}(\mathcal{U})$  is a functor such that, for all morphisms  $i \rightarrow j$  in  $\mathcal{I}$ ,  $E_i \rightarrow E_j$  is an  $\mathbb{A}^1$ -weak equivalence of spectra. Then for every  $i \in \mathcal{I}$ , the map

$$E_i \rightarrow \text{colim}_{i \in \mathcal{I}} E_i$$

is an  $\mathbb{A}^1$ -weak equivalence of spectra.

PROOF. Let  $G$  be any  $\mathbb{A}^1$ -local spectrum. By the comment after Proposition 5.7, for all  $\mathbb{A}^1$ -local  $G$ ,  $f$  induces a bijection in the  $E_\infty$ -term

$$\mathcal{H}_s(\text{hocolim}_{i \in \mathcal{I}} E_i, G) \xrightarrow{\cong} \mathcal{H}_s(\text{hocolim}_{i \in \mathcal{I}} E'_i, G).$$

So the map induced on the homotopy colimits is an  $\mathbb{A}^1$ -weak equivalence.

By Lemma 5.11, the homotopy colimit over  $\mathcal{I}$  is naturally simplicially weakly equivalent to the colimit. But a simplicial weak equivalence is an  $\mathbb{A}^1$ -weak equivalence. Hence, the map induced on the colimits

$$\text{colim}_{i \in \mathcal{I}} E_i \rightarrow \text{colim}_{i \in \mathcal{I}} E'_i$$

is an  $\mathbb{A}^1$ -weak equivalence.

For the second part of the proposition, fix  $i \in \mathcal{I}$ . Consider the small directed category  $\mathcal{I}' = i \downarrow \mathcal{I}$  of all objects  $j$  together with a morphism  $i \rightarrow j$ . The claim then follows by applying the first part of the proposition to the constant functor  $E_i$  and the functor  $E$  from  $\mathcal{I}'$  to  $\text{Spectra}(\mathcal{U})$ .  $\square$

There is also the following equivalent characterization of  $\mathbb{A}^1$ -weak equivalences of  $\mathcal{U}$ -spectra. We say that a map of spectra is an elementary  $\mathbb{A}^1$ -weak equivalence if it is of the form

$$\Sigma_V^{\mathcal{U}} X \wedge \mathbb{A}_+^1 \rightarrow \Sigma_V^{\mathcal{U}} X$$

induced by the projection  $\mathbb{A}_+^1 \rightarrow S^0$ , where  $X \in \text{Spc}(k)_\bullet$  is a finite colimit of based smooth schemes, and  $V \subset \mathcal{U}$  is cofinite. Here,  $\Sigma_V^{\mathcal{U}} X$  denotes the  $V$ -th shift desuspension of the suspension  $\mathcal{U}$ -spectrum of  $X$ . We will show that every  $\mathbb{A}^1$ -weak equivalence of spectra can be built up from the elementary  $\mathbb{A}^1$ -weak equivalences.

PROPOSITION 5.12. *The class of  $\mathbb{A}^1$ -weak equivalences of  $\mathcal{U}$ -spectra is the smallest class  $\mathbb{W}_{\mathbb{A}^1}$  of maps of spectra containing all simplicial weak equivalences and satisfying the following conditions:*

1.  $\mathbb{W}_{\mathbb{A}^1}$  contains all elementary  $\mathbb{A}^1$ -weak equivalences of the form  $\Sigma_V^{\mathcal{U}}X \wedge \mathbb{A}_+^1 \rightarrow \Sigma_V^{\mathcal{U}}X$ .
2. If two of three maps  $f$ ,  $g$  and  $g \cdot f$  are in  $\mathbb{W}_{\mathbb{A}^1}$ , then so is the third.
3. Suppose we have a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ f \downarrow & & \downarrow f' \\ B & \xrightarrow{i'} & Y \end{array}$$

where  $f$  is in  $\mathbb{W}_{\mathbb{A}^1}$ . If  $i$  is a cofibration, then  $f'$  is also in  $\mathbb{W}_{\mathbb{A}^1}$ . If  $f$  is a fibration, then  $f'$  is in  $\mathbb{W}_{\mathbb{A}^1}$  and a cofibration.

4. Suppose  $\mathcal{I}$  is a small directed category and  $E : \mathcal{I} \rightarrow \text{Spectra}(\mathcal{U})$  is a functor such that, for all  $i \rightarrow j$  in  $\mathcal{I}$ , the map  $E_i \rightarrow E_j$  is in  $\mathbb{W}_{\mathbb{A}^1}$ . Then the induced map  $E_i \rightarrow \text{colim}_{i \in \mathcal{I}} E_i$  is in  $\mathbb{W}_{\mathbb{A}^1}$  for each  $i \in \mathcal{I}$ .

To prove the proposition, we first show that to say that a spectrum is local with respect to all maps of the form  $E \wedge \mathbb{A}_+^1 \rightarrow E$  is equivalent to the statement that it is local with respect to all elementary  $\mathbb{A}^1$ -weak equivalences.

LEMMA 5.13. *A spectrum  $G$  is  $\mathbb{A}^1$ -local if and only if for every  $X \in \text{Spc}(k)_\bullet$  which is a finite colimit of based smooth schemes and every cofinite  $V \subset \mathcal{U}$ , the map*

$$\mathcal{H}_s(\Sigma_V^{\mathcal{U}}X, G) \rightarrow \mathcal{H}_s(\Sigma_V^{\mathcal{U}}X \wedge \mathbb{A}_+^1, G)$$

induced by  $\mathbb{A}_+^1 \rightarrow S^0$  is a bijection.

PROOF. For any spectrum  $E$ , we have

$$E = \text{colim}_{V \in \mathcal{C}(\mathcal{U})} \Sigma_V^{\mathcal{U}}E_V.$$

By Lemma 5.10, the canonical map

$$\text{hocolim}_{V \in \mathcal{C}(\mathcal{U})} \Sigma_V^{\mathcal{U}}E_V \rightarrow \text{colim}_{V \in \mathcal{C}(\mathcal{U})} \Sigma_V^{\mathcal{U}}E_V$$

is a simplicial weak equivalence. Smashing with  $\mathbb{A}_+^1$  commutes with this homotopy colimit, so by Proposition 5.7, the morphism set  $\mathcal{H}_s(E, G)$  is determined by the morphism sets  $\mathcal{H}_s(\Sigma_V^{\mathcal{U}} E_V, G)$  in the sense of Definition 5.6. Hence,  $G$  is  $\mathbb{A}^1$ -local if and only if for every  $X \in Spc(k)_\bullet$  and every cofinite  $V \subset \mathcal{U}$ , the map

$$\mathcal{H}_s(\Sigma_V^{\mathcal{U}} X, G) \rightarrow \mathcal{H}_s(\Sigma_V^{\mathcal{U}} X \wedge \mathbb{A}_+^1, G)$$

is a bijection.

It remains to show we can make  $X$  a finite colimit of smooth schemes. By Lemma 17.1 in the Appendix, Every based  $k$ -space is a directed colimit of finite colimits of based smooth schemes, and the functor  $\Sigma_V^{\mathcal{U}}$  commutes with directed colimits. Hence, by Lemma 5.10, we have that every suspension spectrum  $\Sigma_V^{\mathcal{U}} X$  is  $\mathbb{A}^1$ -weak equivalent to some

$$\text{hocolim}_{i \in \mathcal{I}} \Sigma_V^{\mathcal{U}} X_i$$

for some small directed category  $\mathcal{I}$ , and the  $X_i$ 's are all finite colimits of based smooth schemes. Applying Proposition 5.7 again finishes the proof. □

**PROOF OF PROPOSITION 5.12.** Since it is the class of weak equivalences in a proper model category structure, the class of  $\mathbb{A}^1$ -weak equivalences of spectra clearly satisfies conditions 1, 2, and 3 of the proposition. By Proposition 5.11, condition 4 is also satisfied. Conversely, any elementary  $\mathbb{A}^1$ -weak equivalence is by definition in  $\mathbb{W}_{\mathbb{A}^1}$ . We will show that if  $E \rightarrow E'$  is an  $\mathbb{A}^1$ -weak equivalence of spectra, then it can be constructed from the elementary  $\mathbb{A}^1$ -weak equivalences via the conditions defining the class  $\mathbb{W}_{\mathbb{A}^1}$  in the proposition.

Note that the collection of elementary  $\mathbb{A}^1$ -weak equivalences form a set, since the site  $Sm/k$  is small. Thus, we can use a transfinite small object argument, done with respect to the set of elementary  $\mathbb{A}^1$ -weak equivalences, to construct the  $\mathbb{A}^1$ -localization functor  $L_{\mathbb{A}^1}$  on  $Spectra(\mathcal{U})$ . (This is similar to the standard construction of Bousfield [1], see also Dror-Farjoun [4].) For any  $E \in Spectra(\mathcal{U})$ ,  $L_{\mathbb{A}^1} E$  is the

universal  $\mathbb{A}^1$ -local spectrum to which  $E$  maps. The map  $E \rightarrow L_{\mathbb{A}^1}E$  is constructed as a directed colimit of pushouts of elementary  $\mathbb{A}^1$ -weak equivalences, so it is in  $\mathbb{W}_{\mathbb{A}^1}$ .

Now suppose that  $f : E \rightarrow E'$  is a  $\mathbb{A}^1$ -weak equivalence. Then there is the following diagram:

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ \downarrow & & \downarrow \\ L_{\mathbb{A}^1}E & \xrightarrow{L_{\mathbb{A}^1}f} & L_{\mathbb{A}^1}E' \end{array}$$

The bottom map

$$L_{\mathbb{A}^1}f : L_{\mathbb{A}^1}E \rightarrow L_{\mathbb{A}^1}E'$$

is an  $\mathbb{A}^1$ -weak equivalence since the other three maps are. However, a map between  $\mathbb{A}^1$ -local spectra is an  $\mathbb{A}^1$ -weak equivalence if and only if it is an isomorphism in  $\mathcal{H}_s$ . Hence,  $L_{\mathbb{A}^1}f$  is in  $\mathbb{W}_{\mathbb{A}^1}$ . so  $f$  is in  $\mathbb{W}_{\mathbb{A}^1}$  by condition 2 of the proposition.  $\square$

We also have the following ‘‘homotopy sets’’ characterization of  $\mathbb{A}^1$ -weak equivalences of spectra.

**PROPOSITION 5.14.** *A map of  $\mathcal{U}$ -spectra  $f : G \rightarrow G'$  is an  $\mathbb{A}^1$ -weak equivalence if and only if for every  $X \in \text{Spc}(k)_\bullet$  a finite colimit of based smooth schemes, and  $V \subset \mathcal{U}$  cofinite, the map*

$$\mathcal{H}_{\mathbb{A}^1}(\Sigma_V^{\mathcal{U}}X, G) \rightarrow \mathcal{H}_{\mathbb{A}^1}(\Sigma_V^{\mathcal{U}}X, G')$$

*induced by  $f$  is a bijection.*

**PROOF.** Given a spectrum  $E$ , we have  $E = \text{colim}_{U \in \mathcal{C}(\mathcal{U})} \Sigma_U^{\mathcal{U}}E_U$ . By Lemma 5.10, there is a natural  $\mathbb{A}^1$ -weak equivalence

$$\text{colim}_{U \in \mathcal{C}(\mathcal{U})} \Sigma_U^{\mathcal{U}}E_U \rightarrow \text{hocolim}_{U \in \mathcal{C}(\mathcal{U})} \Sigma_U^{\mathcal{U}}E_U.$$

By the comment after Proposition 5.7, the  $\mathcal{H}_{\mathbb{A}^1}(E, G)$ , is determined by the mapping groups  $\mathcal{H}_{\mathbb{A}^1}(\Sigma_U^{\mathcal{U}}E_U, G)$  in the sense of Definition 5.6 by a spectral sequence. In particular, if the induced map

$$\mathcal{H}_{\mathbb{A}^1}(E, G) \rightarrow \mathcal{H}_{\mathbb{A}^1}(E, G')$$

is a bijection for every suspension spectrum, then  $f : G \rightarrow G'$  induces an isomorphism of spectral sequences for every spectrum  $E$ . Hence, it is an  $\mathbb{A}^1$ -weak equivalence.

Now let  $E = \Sigma_V^{\mathcal{U}} X$  for some  $X \in \text{Spc}(k)_\bullet$ . By Lemma 17.3 in the Appendix, every based  $k$ -space is simplicially weak equivalent, and hence  $\mathbb{A}^1$ -weak equivalent, to a homotopy colimit of finite colimits of based smooth schemes. Since  $\Sigma_V^{\mathcal{U}}$  commutes with colimits and homotopy colimits, we have that  $\Sigma_V^{\mathcal{U}} X$  is  $\mathbb{A}^1$ -weak equivalent to a homotopy colimit of spectra of the form  $\Sigma_V^{\mathcal{U}} X_i$ , where each  $X_i \in \text{Spc}(k)_\bullet$  is a finite colimit of based smooth schemes. Again, by the comment after Proposition 5.7, if

$$\mathcal{H}_{\mathbb{A}^1}(\Sigma_V^{\mathcal{U}} X_i, G) \rightarrow \mathcal{H}_{\mathbb{A}^1}(\Sigma_V^{\mathcal{U}} X_i, G')$$

is a bijection for every  $X_i \in \text{Spc}(k)_\bullet$  that is a finite colimit of based smooth schemes, then  $f$  induces an isomorphism of spectral sequences.

The statement of the proposition follows.  $\square$

## 6. Comparison with Coordinatized Spectra

Now we can compare  $\text{Spectra}(\mathcal{U})$  with  $\text{Spectra}(k)$ . Let  $\mathcal{B} = \{U_i\}$  be any decomposition of  $\mathcal{U}$  into one-dimensional subspaces. Let  $V_n = \bigoplus_{i>n} U_i$  for  $n \in \mathbb{N}$ . The  $\mathbb{A}^1$ -local model structure on  $\text{Spectra}(\mathcal{B})$  is defined from the stable simplicial model structure on  $\text{Spectra}(\mathcal{B})$  in a manner similar to Definition 5.3.

LEMMA 6.1. *The pair of adjoint functors*

$$L_{\mathcal{B}}^{\mathcal{U}} : \text{Prespectra}(\mathcal{B}) \rightarrow \text{Prespectra}(\mathcal{U})$$

$$R_{\mathcal{B}}^{\mathcal{U}} : \text{Prespectra}(\mathcal{U}) \rightarrow \text{Prespectra}(\mathcal{B})$$

*form a Quillen adjoint pair with respect to the  $\mathbb{A}^1$ -local model structures on  $\mathcal{U}$ -prespectra and  $\mathcal{B}$ -prespectra.*

PROOF. Cofibrations in the  $\mathbb{A}^1$ -local model structure are the same as cofibrations in the stable simplicial model structure, which are preserved by  $L_{\mathcal{B}}^{\mathcal{U}}$ . We also

claim that  $R_{\mathcal{B}}^{\mathcal{U}}$  takes an  $\mathbb{A}^1$ -local  $\mathcal{U}$ -spectrum to an  $\mathbb{A}^1$ -local  $\mathcal{B}$ -spectrum. By Proposition 4.17,  $(L_{\mathcal{B}}^{\mathcal{U}}, R_{\mathcal{B}}^{\mathcal{U}})$  is a pair of adjoining functors between the stable simplicial homotopy categories  $\mathcal{H}_s(\mathcal{B})$  and  $\mathcal{H}_s(\mathcal{U})$ . If  $G$  is an  $\mathbb{A}^1$ -local  $\mathcal{U}$ -spectrum, then for any  $\mathcal{B}$ -spectrum  $E$ , the map of morphism sets

$$\mathcal{H}_s(\mathcal{B})(E, R_{\mathcal{B}}^{\mathcal{U}}G) \rightarrow \mathcal{H}_s(\mathcal{B})(E \wedge \mathbb{A}_+^1, R_{\mathcal{B}}^{\mathcal{U}}G)$$

is the same as

$$\mathcal{H}_s(\mathcal{U})(L_{\mathcal{B}}^{\mathcal{U}}E, G) \rightarrow \mathcal{H}_s(\mathcal{U})(L\mathcal{B}^{\mathcal{U}}(E \wedge \mathbb{A}_+^1), G).$$

This is a bijection since  $L_{\mathcal{B}}^{\mathcal{U}}$  commutes with  $- \wedge \mathbb{A}_+^1$ . Thus,  $R_{\mathcal{B}}^{\mathcal{U}}G$  is an  $\mathbb{A}^1$ -local  $\mathcal{B}$ -spectrum.

Therefore, if  $f : E \rightarrow E'$  is an  $\mathbb{A}^1$ -local weak equivalence of  $\mathcal{B}$ -spectrum, then for every  $\mathbb{A}^1$ -local  $\mathcal{U}$ -spectrum  $G$ , the map of morphism sets

$$\mathcal{H}_s(\mathcal{U})(L_{\mathcal{B}}^{\mathcal{U}}E', G) \rightarrow \mathcal{H}_s(\mathcal{U})(L_{\mathcal{B}}^{\mathcal{U}}E, G)$$

is the same as

$$\mathcal{H}_s(\mathcal{U})(E', R_{\mathcal{B}}^{\mathcal{U}}G) \rightarrow \mathcal{H}_s(\mathcal{B})(E, R_{\mathcal{B}}^{\mathcal{U}}G)$$

which is a bijection. So  $L_{\mathcal{B}}^{\mathcal{U}}f$  is an  $\mathbb{A}^1$ -weak equivalence of  $\mathcal{U}$ -spectra.  $\square$

Thus,  $(L_{\mathcal{B}}^{\mathcal{U}}, R_{\mathcal{B}}^{\mathcal{U}})$  pass to a pair of adjoint functors between the  $\mathbb{A}^1$ -local homotopy categories  $\mathcal{H}_{\mathbb{A}^1}(\mathcal{B})$  and  $\mathcal{H}_{\mathbb{A}^1}(\mathcal{U})$ .

In fact, the right adjoint  $R_{\mathcal{B}}^{\mathcal{U}}$  also preserves  $\mathbb{A}^1$ -weak equivalences.

**LEMMA 6.2.** *For any decomposition  $\mathcal{B} = \{U_i\}$  of  $\mathcal{U}$  into one-dimensional subspaces, the functor  $R_{\mathcal{B}}^{\mathcal{U}} : \text{Spectra}(\mathcal{U}) \rightarrow \text{Spectra}(\mathcal{B})$  preserves  $\mathbb{A}^1$ -weak equivalences.*

**PROOF.** We use the homotopy sets characterization of  $\mathbb{A}^1$ -weak equivalences of Proposition 5.14, which also holds for  $\mathcal{B}$ -spectra by a similar argument. Suppose  $f : E \rightarrow E'$  is an  $\mathbb{A}^1$ -weak equivalence of  $\mathcal{U}$ -spectra. Then by Lemma 6.1, for any  $V_n = \bigoplus_{i>n} U_i$ , and  $X \in \text{Spc}(k)_{\bullet}$ , the map of morphism sets

$$\mathcal{H}_{\mathbb{A}^1}(\mathcal{B})(\Sigma_{V_n}^{\mathcal{B}} X, R_{\mathcal{B}}^{\mathcal{U}}E) \rightarrow \mathcal{H}_{\mathbb{A}^1}(\mathcal{B})(\Sigma_{V_n}^{\mathcal{B}} X, E')$$

is the same as

$$\mathcal{H}_{\mathbb{A}^1}(\mathcal{U})(L_{\mathcal{B}}^{\mathcal{U}}\Sigma_{V_n}^{\mathcal{B}}X, E) \rightarrow \mathcal{H}_{\mathbb{A}^1}(\mathcal{U})(L_{\mathcal{B}}^{\mathcal{U}}\Sigma_{V_n}^{\mathcal{B}}X, E').$$

Now consider the diagram of forgetful functors

$$\begin{array}{ccc} \text{Spectra}(\mathcal{U}) & \xrightarrow{R_{\mathcal{B}}^{\mathcal{U}}} & \text{Spectra}(\mathcal{B}) \\ & \searrow \Omega^{\mathcal{U}} & \swarrow \Omega^{\mathcal{B}} \\ & \text{Spc}(k)_{\bullet} & \end{array}$$

This diagram commutes, so the diagram of left adjoints also commute. Hence, we have that

$$\Sigma_{V_n}^{\mathcal{U}}X \cong L_{\mathcal{B}}^{\mathcal{U}}\Sigma_{V_n}^{\mathcal{B}}X.$$

So the map of morphism groups is

$$\mathcal{H}_{\mathbb{A}^1}(\mathcal{U})(\Sigma_{V_n}^{\mathcal{U}}X, E) \rightarrow \mathcal{H}_{\mathbb{A}^1}(\mathcal{U})(\Sigma_{V_n}^{\mathcal{U}}X, E').$$

This is a bijection, so  $R_{\mathcal{B}}^{\mathcal{U}}f$  is an  $\mathbb{A}^1$ -weak equivalence of  $\mathcal{B}$ -spectra.  $\square$

The main result of this section is the following theorem, which states that in the  $\mathbb{A}^1$ -local homotopy categories, coordinate-free spectra are the same as coordinatized spectra.

**THEOREM 6.3.** *For any decomposition  $\mathcal{B}$  of  $\mathcal{U}$  into one-dimensional subspaces, the adjoint functors  $(L_{\mathcal{B}}^{\mathcal{U}}, R_{\mathcal{B}}^{\mathcal{U}})$  between  $\text{Spectra}(\mathcal{B})$  and  $\text{Spectra}(\mathcal{U})$  form a pair of Quillen equivalences with respect to the  $\mathbb{A}^1$ -local model structures. Therefore, there is an canonical equivalence of homotopy categories*

$$\mathcal{H}_{\mathbb{A}^1}(\mathcal{B}) \simeq \mathcal{H}_{\mathbb{A}^1}(\mathcal{U}).$$

In the following, let  $\mathcal{B} = \{U_i\}$  be a fixed decomposition of  $\mathcal{U}$  into one-dimensional subspaces. Let  $V_n = \bigoplus_{i>n} U_i$  for  $n \in \mathbb{N}$ . We have the pair of adjoint functors

$$\begin{aligned} L_{\mathcal{B}}^{\mathcal{U}} &: \text{Prespectra}(\mathcal{B}) \rightarrow \text{Prespectra}(\mathcal{U}) \\ R_{\mathcal{B}}^{\mathcal{U}} &= \text{Prespectra}(\mathcal{U}) \rightarrow \text{Prespectra}(\mathcal{B}) \end{aligned}$$

which is a Quillen adjoint pair with respect to the  $\mathbb{A}^1$ -local model structures on  $\mathcal{U}$ -prespectra and  $\mathcal{B}$ -prespectra.

We make the following definition.

**DEFINITION 6.4.** *Let  $\mathcal{B} = \{U_i\}$ ,  $\mathcal{B}' = \{U'_i\}$  be two decompositions of  $\mathcal{U}$  into one-dimensional subspaces, with  $V_n = \bigoplus_{i>n} U_i$ ,  $V'_n = \bigoplus_{i>n} U'_i$ . We say that  $\mathcal{B}$  is dominated by  $\mathcal{B}'$  if for every  $n \in \mathbb{N}$ , there is some  $j \geq n$  such that  $V'_j \subseteq V_n$ .*

Note that for any two decompositions  $\mathcal{B}$  and  $\mathcal{B}'$ , by taking  $\{V_i \cap V'_i\}$  and refining it, there is always a decomposition  $\mathcal{B}''$  which dominates both  $\mathcal{B}$  and  $\mathcal{B}'$ .

If  $\mathcal{B}$  is dominated by  $\mathcal{B}'$ , we can define the categories of  $(\mathcal{B} \cup \mathcal{B}')_{full}$ -prespectra and  $(\mathcal{B} \cup \mathcal{B}')_{full}$ -spectra to be indexed on the full subcategory of  $\mathcal{C}(\mathcal{U})$  generated by the objects  $V_i, V'_i$ . Note that  $(V'_i, U'_i : V'_{i-1} \rightarrow V'_i)$  satisfies the conditions of Lemma 3.2 in  $(\mathcal{B} \cup \mathcal{B}')_{full}$ , so there is a canonical equivalence of categories

$$(6.5) \quad Spectra(\mathcal{B}') \simeq Spectra((\mathcal{B} \cup \mathcal{B}')_{full}).$$

The proof of this is analogous to that of Proposition 3.5. In the following, we will identify these two categories.

We also have a pair of adjoint functors

$$\begin{aligned} L_{\mathcal{B}}^{\mathcal{B}'} &: Prespectra(\mathcal{B}) \rightarrow Prespectra(\mathcal{B} \cup \mathcal{B}'_{full}) \\ R_{\mathcal{B}}^{\mathcal{B}'} &: Prespectra(\mathcal{B} \cup \mathcal{B}'_{full}) \rightarrow Prespectra(\mathcal{B}). \end{aligned}$$

On the level of spectra, by the equivalence (6.5),  $(L_{\mathcal{B}}^{\mathcal{B}'}, R_{\mathcal{B}}^{\mathcal{B}'})$  is a pair of adjoint functors between  $Spectra(\mathcal{B})$  and  $Spectra(\mathcal{B}')$ . By arguments similar to that of Proposition 4.17, these form a pair of Quillen adjoints, both in the stable simplicial and  $\mathbb{A}^1$ -local model structures.

**PROPOSITION 6.6.** *For any decompositions  $\mathcal{B}, \mathcal{B}'$  of  $\mathcal{U}$  into one-dimensional subspaces, with  $\mathcal{B}$  dominated by  $\mathcal{B}'$ , the Quillen adjoints  $(L_{\mathcal{B}}^{\mathcal{B}'}, R_{\mathcal{B}}^{\mathcal{B}'})$  between  $Spectra(\mathcal{B})$  and  $Spectra(\mathcal{B}')$  is a pair of adjoint Quillen equivalences in the  $\mathbb{A}^1$ -local model structure.*

Proposition 6.6 is the main step in the proof of Theorem 6.3.

PROOF OF THEOREM 6.3. We need to show that for any cofibrant  $\mathcal{B}_{full}$ -spectrum  $E$ , the unit of adjunction

$$(6.7) \quad u : E \rightarrow R_{\mathcal{B}}^{\mathcal{U}} L_{\mathcal{B}}^{\mathcal{U}} E$$

is an  $\mathbb{A}^1$ -weak equivalence. There is a directed system of categories of  $\mathcal{B}'$ -spectra, ordered by domination. Namely, if  $\mathcal{B}_1$  is dominated by  $\mathcal{B}_2$ , where both are decompositions of  $\mathcal{U}$  into one-dimensional subspaces, then

$$L_{\mathcal{B}_1}^{\mathcal{B}_2} : Spectra(\mathcal{B}_1) \rightarrow Spectra(\mathcal{B}_2).$$

If  $\mathcal{B}_2$  is dominated by  $\mathcal{B}_3$ , then

$$L_{\mathcal{B}_1}^{\mathcal{B}_3} = L_{\mathcal{B}_2}^{\mathcal{B}_3} L_{\mathcal{B}_1}^{\mathcal{B}_2}.$$

If  $E$  is a  $\mathcal{B}_1$  spectrum, then the unit of adjunction

$$E \rightarrow R_{\mathcal{B}_1}^{\mathcal{B}_3} L_{\mathcal{B}_1}^{\mathcal{B}_3} E$$

is the composition

$$\begin{aligned} E &\rightarrow R_{\mathcal{B}_1}^{\mathcal{B}_2} L_{\mathcal{B}_1}^{\mathcal{B}_2} E \\ &\rightarrow R_{\mathcal{B}_1}^{\mathcal{B}_2} (R_{\mathcal{B}_2}^{\mathcal{B}_3} L_{\mathcal{B}_2}^{\mathcal{B}_3} (L_{\mathcal{B}_1}^{\mathcal{B}_2} E)) \\ &= R_{\mathcal{B}_1}^{\mathcal{B}_3} L_{\mathcal{B}_1}^{\mathcal{B}_3} E. \end{aligned}$$

By Proposition 6.6 and the fact that  $R_{\mathcal{B}_1}^{\mathcal{B}_2}$  preserves  $\mathbb{A}^1$ -weak equivalences (Lemma 6.2), each map in the composition is an  $\mathbb{A}^1$ -weak equivalence. Now consider the directed system of decompositions that dominate  $\mathcal{B}$ . Then for a  $\mathcal{B}$ -spectrum  $E$ , the unit of adjunction (6.7) is the directed colimit of this system of  $\mathbb{A}^1$ -local weak equivalences, so it is itself an  $\mathbb{A}^1$ -equivalence.

We also need to show that for any  $\mathcal{U}$ -spectrum  $E$ , the counit of adjunction

$$(6.8) \quad c : L_{\mathcal{B}}^{\mathcal{U}} R_{\mathcal{B}}^{\mathcal{U}} E \rightarrow E$$

is an  $\mathbb{A}^1$ -weak equivalence of  $\mathcal{U}$ -spectra. For this, note that

$$E = \operatorname{colim}_{V \in \mathcal{C}(\mathcal{U})} \Sigma_V^{\mathcal{U}} E_V$$

where  $\Sigma_V^{\mathcal{U}}$  denotes the  $V$ -th shift desuspension of the  $\mathcal{U}$ -suspension spectrum. Since directed colimits preserve  $\mathbb{A}^1$ -weak equivalences of  $\mathcal{U}$ -spectra, it suffices to show (6.8)

for  $E = \Sigma_V^{\mathcal{U}} X$ , where  $V \in \mathcal{C}(\mathcal{U})$ , and  $X$  is a based  $k$ -space. This is because  $L_{\mathcal{B}}^{\mathcal{U}} R_{\mathcal{B}}^{\mathcal{U}}$  also commutes with directed colimits.

For  $X \in \text{Spc}(k)_{\bullet}$ , we can assume without loss of generality that  $V = \mathcal{U}$ . Let  $\Sigma^{\mathcal{B}} X$  denotes the suspension  $\mathcal{B}$ -spectrum of  $X$ . By the proof of Lemma 6.2,

$$\Sigma^{\mathcal{U}} X \cong L_{\mathcal{B}}^{\mathcal{U}} \Sigma^{\mathcal{B}} X.$$

Also, consider the unit of adjunction

$$\Sigma^{\mathcal{B}} X \xrightarrow{u} R_{\mathcal{B}}^{\mathcal{U}} L_{\mathcal{B}}^{\mathcal{U}} \Sigma^{\mathcal{B}} X \cong R_{\mathcal{B}}^{\mathcal{U}} \Sigma^{\mathcal{U}} X.$$

By the above, this is an  $\mathbb{A}^1$ -weak equivalence. Also,  $L_{\mathcal{B}}^{\mathcal{U}}$  preserves  $\mathbb{A}^1$ -weak equivalences. So we have the diagram

$$\begin{array}{ccc} L_{\mathcal{B}}^{\mathcal{U}} \Sigma^{\mathcal{B}} X & \xrightarrow{\cong} & \Sigma^{\mathcal{U}} X \\ & \searrow \cong & \nearrow c \\ & L_{\mathcal{B}}^{\mathcal{U}} u & L_{\mathcal{B}}^{\mathcal{U}} R_{\mathcal{B}}^{\mathcal{U}} \Sigma^{\mathcal{U}} X \end{array} .$$

By looking at the adjoint diagram, we see that this diagram commutes. Thus,  $c$  is an  $\mathbb{A}^1$ -weak equivalence.  $\square$

For each  $n \in \mathbb{N}$ , denote  $Z_n = U_1 \dot{+} \cdots \dot{+} U_n$ , and  $T_n = U'_1 \dot{+} \cdots \dot{+} U'_n$ .

To prove Proposition 6.6, we will use the following lemma.

LEMMA 6.9. *There is a sequence of  $\mathbb{A}^1$ -paths  $U_n(t)$  in  $T_{r(n)}$  for some  $r(n) \geq n$ , such that*

1.  $U_n(t) = U_n$ ,  $U_n(1) = U'_n$  for all  $n$ ;
2. for each  $n$  and all  $t \in \mathbb{A}^1$ , the set  $\{U_1(t), \dots, u_n(t)\}$  is linearly independent;
3. the sequence  $\mathcal{B}_t = \{U_n(t)\}$  is dominated by  $\mathcal{B}'$  for all  $t \in \mathbb{A}^1$ .

We defer the proof of Lemma 6.9 for now.

PROOF OF PROPOSITION 6.6. To show that the unit of adjunction

$$(6.10) \quad c : E \rightarrow R_{\mathcal{B}}^{\mathcal{B}'} L_{\mathcal{B}}^{\mathcal{B}'} E$$

is an  $\mathbb{A}^1$ -weak equivalence of  $\mathcal{B}$ -spectra for any  $\mathcal{B}$ -spectrum  $E$ , it suffices to show it for  $E = \Sigma_{V_n}^{\mathcal{B}} X$  for some based  $k$ -space  $X$ , where  $\Sigma_{V_n}^{\mathcal{B}}$  denotes the  $V_n$ -th shift desuspension

of the suspension  $\mathcal{B}$ -spectrum of  $X$ . This is because  $E = \operatorname{colim}_{\mathcal{B}} \Sigma_{V_n}^{\mathcal{B}} E_{V_n}$ , and the unit map commutes with directed colimits.

Assume without loss of generality that  $V_n = \mathcal{U}$ , so  $E = \Sigma^{\mathcal{B}} X$ . Thinking of  $E$  has a  $\mathcal{B}_{full}$ -spectrum, we have to consider the maps of based  $k$ -spaces for each  $i$

$$(6.11) \quad \operatorname{colim}_{V_j \dot{+} Z = V_i} \Omega^Z \left( \bigvee_{T \dot{+} V_j = \mathcal{U}} \Sigma^T X \right) \rightarrow \operatorname{colim}_{V'_j \dot{+} Z' = V'_i} \Omega^{Z'} \left( \bigvee_{T' \dot{+} V'_j = \mathcal{U}} \Sigma^{T'} X \right).$$

Assume first that  $V_i = V_0 = \mathcal{U}$ . Then by the equivalences of categories

$$\operatorname{Spectra}(\mathcal{B}) \simeq \operatorname{Spectra}(\mathcal{B}_{full})$$

$$\operatorname{Spectra}(\mathcal{B}') \simeq \operatorname{Spectra}(\mathcal{B}'_{full})$$

the map (6.11) corresponds canonically to

$$(6.12) \quad \operatorname{colim}_n \Omega^{Z_n} \Sigma^{Z_n} X \rightarrow \operatorname{colim}_n \Omega^{T_n} \Sigma^{T_n} X$$

where  $Z_n = \dot{+}_{i=1}^n U_i$ , and  $T_n = \dot{+}_{i=1}^n U'_i$ . This comes from the fact that for each  $n$ , there is some  $r(n) \geq n$ , such that  $Z_n \subseteq T_{r(n)}$ .

Given Lemma 6.6, consider the map

$$\varphi : \mathbb{A}^1 \times \mathcal{U} \rightarrow \mathcal{U}$$

such that  $\varphi_t$  maps  $U_i$  onto  $U_i(t)$  isomorphically. So  $\varphi_t$  is an isomorphism for each  $t$ , and induces an isomorphism

$$\bar{\varphi}_t : \operatorname{colim}_n \Omega^{Z_n} \Sigma^{Z_n} X \xrightarrow{\cong} \operatorname{colim}_n \Omega^{Z_n(t)} \Sigma^{Z_n(t)} X$$

where  $Z_n(t) = U_1(t) \dot{+} \cdots \dot{+} U_n(t)$ . Also, for each  $t$  and  $n$ ,

$$Z_n(t) \subseteq T_j$$

for some  $j \geq n$  (depending on  $n$  and  $t$ ), so we have a canonical map

$$\psi_t : \operatorname{colim}_n \Omega^{Z_n(t)} \Sigma^{Z_n(t)} X \rightarrow \operatorname{colim}_n \Omega^{T_n} \Sigma^{T_n} X.$$

Together, we have a map

$$\psi \cdot \bar{\varphi} : \mathbb{A}^1 \times \operatorname{colim}_n \Omega^{Z_n} \Sigma^{Z_n} X \rightarrow \operatorname{colim}_n \Omega^{T_n} \Sigma^{T_n} X.$$

For  $t = 0$ ,  $\varphi_0$  is the identity, and  $\psi_0$  is the map (6.12). For  $t = 1$ ,  $\varphi_1$  is an isomorphism, and  $\psi_1$  is the identity. Thus, (6.12) is  $\mathbb{A}^1$ -homotopic via  $\psi \cdot \varphi$  to an isomorphism, so

(6.12) is an  $\mathbb{A}^1$ -weak equivalence of based  $k$ -spaces. This shows that for  $V_i = V_0 = \mathcal{U}$ , the map (6.11) is an  $\mathbb{A}^1$ -weak equivalence of based  $k$ -spaces.

For (6.11) on  $V_1$ , note that the map (6.11) depends on  $\mathcal{B}$ ,  $\mathcal{B}'$  cofinally, in the sense that if we change finitely many of the  $V_j$ 's and  $V_j'$ 's, the map remains unchanged. We have that  $V_1 \subset V_k'$  for some  $k$ , so we can assume that  $V_1' = V_1$ ,  $V_j'$  are unchanged for  $j \geq k$ , and choose  $V_1' \subset V_2' \subset \cdots \subset V_k'$  so that each  $V_j'$  has codimension one higher than  $V_{j-1}'$ , and choose the new  $U_j'$  to be a complement of  $V_j'$  in  $V_{j-1}'$  for each  $1 \leq j \leq k$ . Then repeating the argument as before, we get that the map (6.11) for  $V_1$  is canonically isomorphic to

$$(6.13) \quad \operatorname{colim}_n \Omega^{U_2 + \cdots + U_n} \Sigma^{Z_n} X \rightarrow \operatorname{colim}_n \Omega^{U_2' + \cdots + U_n'} \Sigma^{T_n} X.$$

By the same argument as above, we get that (6.13) is an  $\mathbb{A}^1$ -weak equivalence of based  $k$ -spaces. Repeating for all  $V_i$ , we get that (6.11) is a spacewise  $\mathbb{A}^1$ -weak equivalence, so an  $\mathbb{A}^1$ -weak equivalence of  $\mathcal{B}$ -spectra.

We also need to show that the counit of adjunction

$$(6.14) \quad c : L_{\mathcal{B}}^{\mathcal{B}'} R_{\mathcal{B}}^{\mathcal{B}'} E \rightarrow E$$

is an  $\mathbb{A}^1$ -weak equivalence of  $\mathcal{B}'$ -spectra for all fibrant  $\mathcal{B}'$ -spectra  $E$ . Note that  $E \cong \operatorname{colim}_{\mathcal{B}'} \Sigma_{V_i}^{\mathcal{B}'} E_i$ , so again it suffices to show it for  $E = \Sigma_{V_i}^{\mathcal{B}'} X$  for some  $X \in \operatorname{Spc}(k)$ . Then

$$E \cong L_{\mathcal{B}}^{\mathcal{B}'} \Sigma_{V_i}^{\mathcal{B}} X.$$

We have the diagram

$$\begin{array}{ccc} L_{\mathcal{B}}^{\mathcal{B}'} \Sigma_{V_i}^{\mathcal{B}} X & \xrightarrow{\cong} & E \\ & \searrow & \nearrow c \\ & L_{\mathcal{B}}^{\mathcal{B}'} R_{\mathcal{B}}^{\mathcal{B}'} E & \end{array} .$$

This diagram commutes. The map  $u$  is an  $\mathbb{A}^1$ -weak equivalence of  $\mathcal{B}$ -spectra, and  $L_{\mathcal{B}}^{\mathcal{B}'}$  preserves  $\mathbb{A}^1$ -weak equivalences. Hence,  $c$  is an  $\mathbb{A}^1$ -weak equivalence of  $\mathcal{B}'$ -spectra.  $\square$

It remains to prove Lemma 6.9.

PROOF OF LEMMA 6.9. We first find bases  $\{d_i\}$  and  $\{e_i\}$  for  $\mathcal{U}$  corresponding to  $\mathcal{B}$  and  $\mathcal{B}'$ , in the sense that  $U_i = \langle d_i \rangle$ , and  $U'_i = \langle e_i \rangle$  for  $i \geq 1$ . Then  $V_n = \langle d_{n+1}, d_{n+2}, \dots \rangle$ , and  $V'_n = \langle e_{n+1}, e_{n+2}, \dots \rangle$ . We need to find a sequence of paths  $U_n(t)$  in  $T_{r(n)}$  for some  $r(n) \geq n$ , satisfying the given conditions. To get  $U_1(t)$ , we have that  $V'_{r(1)} \subset V_1$  for some  $r(1) \in \mathbb{N}$ . Let

$$R_1 = V_1 \cap \langle e_1, e_2, \dots, e_{r(1)} \rangle.$$

Then by the Noether isomorphism, which gives a canonical bijective correspondence between subspaces of  $\mathcal{U}$  containing  $V'_{r(1)}$  and subspaces of  $\mathcal{U}/V'_{r(1)}$ , we have that

$$V_1 = R_1 \dot{+} V'_{r(1)}.$$

We will find a homotopy  $(d_1(t), R_1(t))$ , which is  $(d_1, R_1)$  for  $t = 0$ , and

$$(e_1, \langle e_2, \dots, e_{r(1)} \rangle)$$

for  $t = 1$ . Note that we can assume that  $d_1 \in \langle e_1, \dots, e_{r(1)} \rangle$  by making  $r(1)$  large enough. We also want the homotopy  $(d_1(t), R_1(t))$  to be contained in the space  $\langle e_1, \dots, e_{r(1)} \rangle$ . To do this, we find an  $\mathbb{A}^1$ -path  $f_1(t)$  in  $SL_{r(1)}$ , such that  $f_1(1)$  is the identity, and  $f_1(0)$  takes  $e_1$  to  $d_1$ , and  $\langle e_2, \dots, e_{r(1)} \rangle$  to  $R_1$ . This is possible because  $SL_{r(1)}$  is path-connected. Given  $f_1(t)$ , we take the projection

$$f_1(t) \mapsto (f_1(t)(e_1), f_1(t)(\langle e_2, \dots, e_{r(1)} \rangle)).$$

This gives the homotopy  $(d_1(t), R_1(t))$ . We set  $V_1(t) = R_1(t) \dot{+} V'_{r(1)}$ . Note that  $V'_{r(1)} \subset V_1(t)$ , and  $V_1(t) \dot{+} \langle d_1(t) \rangle = \mathcal{U}$  for all  $t \in \mathbb{A}^1$ .

To continue, we work by induction. Suppose we have constructed a path  $f_n(t)$  in  $SL_{r(n)}$  such that  $f_n(1)$  is the identity, and  $f_n(0)$  takes  $e_i$  to  $d_i$  for  $i \leq n$  and  $\langle e_{n+1}, \dots, e_{r(n)} \rangle$  to  $R_n = V_n \cap \langle e_1, \dots, e_{r(n)} \rangle$ . By projecting to  $d_n(t) = f_n(t)(e_n)$ , and  $R_n(t) = f_n(t) \langle e_{n+1}, \dots, e_{r(n)} \rangle$ , we get homotopies

$$(d_1(t), \dots, d_n(t), R_1(t), \dots, R_n(t), V_1(t), \dots, V_{n+1}(t))$$

with  $R_i(t) = V_i(t) \cap \langle e_1, \dots, e_{r(i)} \rangle$  and  $R_i(t) \dot{+} V'_{r(i)} = V_i(t)$  by the Noether isomorphism for each  $i \leq n$  and  $t \in \mathbb{A}^1$ . Suppose that

1. We have

$$\begin{aligned} (d_1(0), \dots, d_n(0), V_1(0), \dots, V_n(0)) &= (d_1, \dots, d_n, V_1, \dots, V_n) \\ (d_1(1), \dots, d_n(1), V_1(1), \dots, V_n(1)) &= (e_1, \dots, e_n, V'_1, \dots, V'_n); \end{aligned}$$

2. the  $d_i(t)$ 's are linearly independent for all  $t \in \mathbb{A}^1$ ;

3. For all  $t \in \mathbb{A}^1$ ,

$$\langle d_1(t), \dots, d_n(t) \rangle \dot{+} V_n(t) = \mathcal{U}.$$

Suppose further that the homotopies satisfy the condition that

$$(6.15) \quad V'_{r(i)} \subset \langle d_{i+1}(t), \dots, d_n(t) \rangle \dot{+} V'_{r(n)}$$

for all  $i \leq n$ .

We need to find the next homotopy  $(d_{n+1}(t), R_n(t))$ , with

$$\begin{aligned} (d_{n+1}(0), R_{n+1}(0)) &= (d_{n+1}, R_{n+1} = V_{n+1} \cap \langle e_1, \dots, e_{r(n+1)} \rangle) \\ (d_{n+1}(1), R_{n+1}(1)) &= (e_{n+1}, \langle e_{r(n)+1}, \dots, e_{r(n+1)} \rangle) \end{aligned}$$

for some large  $r(n+1)$ . We choose  $r(n+1)$  such that

$$d_1, \dots, d_{n+1} \in \langle e_1, \dots, e_{r(n+1)} \rangle$$

and  $V'_{r(n+1)} \subset V_n$ . By construction of

$$(d_1(t), \dots, d_n(t), R_1(t), \dots, R_n(t), V_1(t), \dots, V_n(t))$$

we have a homotopy of isomorphisms

$$j_n(t) : \langle e_1, \dots, e_{r(n)} \rangle \rightarrow \langle e_1, \dots, e_{r(n)} \rangle$$

with  $j_n(t) = f(n)(t)^{-1}$  taking  $d_i(t)$  to  $e_i$  for  $i \leq n$ , and  $R_n(t)$  to  $\langle e_{n+1}, \dots, e_{r(n)} \rangle$ .

We define

$$i_{n+1}(t) : \langle e_1, \dots, e_{r(n+1)} \rangle \rightarrow \langle e_1, \dots, e_{r(n+1)} \rangle$$

a path of isomorphisms, which is  $j_n(t) = f_n(t)^{-1}$  on  $\langle e_1, \dots, e_{r(n)} \rangle$  and the identity on  $\langle e_{r(n)+1}, \dots, e_{r(n+1)} \rangle$ . We need to find the next path  $f_{n+1}(t)$  in  $SL_{r(n+1)}$ , with  $f_{n+1}(1)$  the identity, and  $f_{n+1}(0)$  taking  $e_i$  to  $d_i$  for  $i \leq n+1$  and  $\langle e_{n+2}, \dots, e_{r(n+1)} \rangle$  to  $R_{n+1}$ . To do this, it suffices to find a path  $f'_{n+1}(t)$  in  $SL_{r(n+1)-n}$ , the special linear

group of the space  $\langle e_{n+1}, \dots, e_{r(n+1)} \rangle$ . We want that  $f'_{n+1}(1)$  the identity, and  $f'_{n+1}(0)$  taking  $e_{n+1}$  to  $i_{n+1}(0)(d_{n+1})$ , and  $\langle e_{n+2}, \dots, e_{r(n+1)} \rangle$  to  $i_{n+1}(0)(R_{n+1})$ . Taking the inclusion of  $SL_{r(n+1)-n}$  into  $SL_{r(n+1)}$  by adding on the identity map on  $\langle e_1, \dots, e_n \rangle$ ,  $f'_{n+1}(t)$  is a path in  $SL_{r(n+1)}$ . We set the path  $f_{n+1}(t)$  in  $SL_{r(n+1)}$  to be

$$f_{n+1}(t) = i^{n+1}(t)^{-1} \cdot f'_{n+1}(t).$$

Then  $f_{n+1}(t)$  has the desired properties, and we set  $d_{n+1}(t) = f_{n+1}(t)(e_{n+1})$ , and  $R_{n+1}(t) = f_{n+1}(t)(\langle e_{n+2}, \dots, e_{r(n+1)} \rangle)$ . This gives the next part of the homotopy, with  $V_{n+1}(t) = R_{n+1}(t) \dot{+} V'_{r(n+1)}$ .

We need to check that

$$(6.16) \quad d_{n+1}(t) \dot{+} R_{n+1}(t) = R_n(t) \dot{+} \langle e_{r(n)}, \dots, e_{r(n+1)} \rangle$$

to show that (6.15) is satisfied. If we add  $V'_{r(n+1)}$  to both sides of (6.16), we get  $V_n(t)$  on both sides. But both are also contained in  $\langle e_1, \dots, e_{r(n+1)} \rangle$ , so by the Noether isomorphism, (6.16) holds, so (6.15) is satisfied. This gives the construction of the paths, with  $U_n(t) = \langle d_n(t) \rangle$ .  $\square$

Recall the examples of suspension spectra and the algebraic cobordism spectrum. We can now show that their coordinate-free definitions are equivalent to the usual ones, with respect to the Quillen equivalence of categories

$$(6.17) \quad L_{\mathcal{B}}^{\mathcal{U}} : \text{Spectra}(\mathcal{B}) \rightarrow \text{Spectra}(\mathcal{U})$$

$$(6.18) \quad R_{\mathcal{B}}^{\mathcal{U}} : \text{Spectra}(\mathcal{U}) \rightarrow \text{Spectra}(\mathcal{B}).$$

**PROPOSITION 6.19.** *Let  $\mathcal{B}$  be any decomposition of  $\mathcal{U}$  into one-dimensional subspaces.*

1. *For  $X$  a based  $k$ -space, let  $\Sigma_{V_n}^{\mathcal{B}} X$  and  $\Sigma_{V_n}^{\mathcal{U}} X$  be the  $V_n$ -th shift desuspensions of the  $\mathcal{B}$ -suspension spectrum and of the  $\mathcal{U}$ -suspension spectrum of  $X$ , respectively. Then*

$$L_{\mathcal{B}}^{\mathcal{U}} \Sigma_{V_n}^{\mathcal{B}} X \simeq \Sigma_{V_n}^{\mathcal{U}} X$$

and

$$R_{\mathcal{B}}^{\mathcal{U}} \Sigma_{V_n}^{\mathcal{U}} X \simeq \Sigma_{V_n}^{\mathcal{B}} X$$

in the  $\mathbb{A}^1$ -local model structures.

2. Let  $MGL_{\mathcal{B}}$  and  $MGL_{\mathcal{U}}$  be the algebraic cobordism spectra in  $\text{Spectra}(\mathcal{B})$  and  $\text{Spectra}(\mathcal{U})$ , respectively. Then

$$L_{\mathcal{B}}^{\mathcal{U}} MGL_{\mathcal{B}} \simeq MGL_{\mathcal{U}}$$

and

$$R_{\mathcal{B}}^{\mathcal{U}} MGL_{\mathcal{U}} \simeq MGL_{\mathcal{B}}.$$

PROOF. 1. The statement

$$\Sigma_{V_n}^{\mathcal{U}} X \simeq L_{\mathcal{B}}^{\mathcal{U}} \Sigma_{V_n}^{\mathcal{B}} X$$

is shown in the proof of Lemma 6.2. So

$$R_{\mathcal{B}}^{\mathcal{U}} \Sigma_{V_n}^{\mathcal{U}} X \simeq R_{\mathcal{B}}^{\mathcal{U}} L_{\mathcal{B}}^{\mathcal{U}} \Sigma_{V_n}^{\mathcal{B}} X \xrightarrow{\simeq} \Sigma_{V_n}^{\mathcal{B}} X$$

via the counit of adjunction.

2. By definition, the  $V$ -th space of  $MGL_{\mathcal{U}}$  is

$$\text{colim}_{W \dagger T = V} \Omega^T \left( \bigvee_{Z \dagger W = \mathcal{U}} BGL_{\mathcal{U} \oplus Z}(|Z|)^{\gamma(|Z|)} \right).$$

For  $\mathcal{B} = \{U_i\}$ ,  $V_n = \oplus_{i > n} U_i$ , suppose  $\mathcal{B}$  is dominated by  $\mathcal{B}' = \{U'_i\}$ , with  $V'_n = \oplus_{j > n} U'_j$ .

We have

$$(MGL_{\mathcal{B}})_{V_n} = \text{colim}_{j \geq n} \Omega^{U_{n+1} + \dots + U_j} (BGL_{\mathcal{U} \oplus (U_{n+1} + \dots + U_j)}(j - n))^{\gamma_{j-n}}.$$

There is a similar formula for  $MGL_{\mathcal{B}'}$ . Via the equivalence of categories between  $\mathcal{B}'$ -spectra and  $(\mathcal{B} \cup \mathcal{B}')_{full}$ -spectra, this is canonically isomorphic to  $MGL_{(\mathcal{B} \cup \mathcal{B}')_{full}}$ .

For each  $n$ ,  $V_n \subset W'_j$  for some  $m$ . This gives a natural map

$$f : MGL_{\mathcal{B}} \rightarrow R_{\mathcal{B}}^{\mathcal{B}'} MGL_{\mathcal{B}'}$$

Also, consider the isomorphism  $\mathcal{U} \rightarrow \mathcal{U}$ , which takes  $U_i$  isomorphically onto  $U'_i$ .

This induces a spacewise isomorphism

$$g : MGL_{\mathcal{B}} \xrightarrow{\cong} R_{\mathcal{B}}^{\mathcal{B}'} MGL_{\mathcal{B}'}$$

Using Lemma 6.9, and arguments similar to that of Proposition 6.6, we get for each  $n \in \mathbb{N}$  an homotopy

$$\mathbb{A}_+^1 \wedge (MGL_{\mathcal{B}})_{V_n} \rightarrow (R_{\mathcal{B}}^{\mathcal{B}'} MGL_{\mathcal{B}'})_{V_n}$$

with is  $f_{V_n}$  at  $t = 0$  and the isomorphism  $g_{V_n}$  at  $t = 1$ . Hence,  $f$  is an  $\mathbb{A}^1$ -weak equivalence of  $\mathcal{B}$ -spectra.

Now take the directed system of decompositions of  $\mathcal{U}$  into one-dimensional subspaces which dominate  $\mathcal{B}$ , partially ordered by domination. The for every such  $\mathcal{B}'$ , we have a map

$$MGL_{\mathcal{B}} \rightarrow R_{\mathcal{B}}^{\mathcal{B}'} MGL_{\mathcal{B}'}$$

This gives a diagram  $\mathcal{D}$  with objects  $R_{\mathcal{B}}^{\mathcal{B}'} MGL_{\mathcal{B}'}$ , and for  $\mathcal{B}_1$  dominated by  $\mathcal{B}_2$ , the map

$$R_{\mathcal{B}}^{\mathcal{B}_1} MGL_{\mathcal{B}_1} \rightarrow R_{\mathcal{B}}^{\mathcal{B}_1} R_{\mathcal{B}_1}^{\mathcal{B}_2} MGL_{\mathcal{B}_2} = R_{\mathcal{B}}^{\mathcal{B}_2} MGL_{\mathcal{B}_2}.$$

Each map is an  $\mathbb{A}^1$ -weak equivalence of  $\mathcal{B}$ -spectra, since by Lemma 6.2,  $R_{\mathcal{B}_1}^{\mathcal{B}_2}$  preserves  $\mathbb{A}^1$ -weak equivalences. Thus,

$$MGL_{\mathcal{B}} \rightarrow \operatorname{colim}_{\mathcal{D}} R_{\mathcal{B}}^{\mathcal{B}'} MGL_{\mathcal{B}'} \cong R_{\mathcal{B}}^{\mathcal{U}} MGL_{\mathcal{U}}$$

is an  $\mathbb{A}^1$ -weak equivalence.

The other part is formal. Since  $L_{\mathcal{B}}^{\mathcal{U}}$  preserves  $\mathbb{A}^1$ -weak equivalences, we have

$$L_{\mathcal{B}}^{\mathcal{U}} MGL_{\mathcal{B}} \simeq L_{\mathcal{B}}^{\mathcal{U}} R_{\mathcal{B}}^{\mathcal{U}} MGL_{\mathcal{U}} \xrightarrow{\cong} MGL_{\mathcal{U}}$$

by the counit of adjunction. □

## 7. Characterization of $\mathbb{A}^1$ -weak Equivalences

Next, we show that our definition of  $\mathbb{A}^1$ -weak equivalences of spectra agrees with Jardine's definition of stable  $\mathbb{A}^1$ -weak equivalences [9]. First, recall Jardine's definition of the stable homotopy category. Consider the spacewise  $\mathbb{A}^1$ -local model structure on the category of  $\mathcal{U}$ -prespectra, where fibrations and weak equivalences are

spacewise  $\mathbb{A}^1$ -fibrations and  $\mathbb{A}^1$ -weak equivalences. This is a model category structure, so it has a fibrant replacement functor  $\Gamma : \text{Prespectra}(\mathcal{U}) \rightarrow \text{Prespectra}(\mathcal{U})$ . Namely,  $\Gamma$  takes a prespectrum  $D$  to a prespectrum  $\Gamma D$  which is spacewise  $\mathbb{A}^1$ -fibrant. There is a natural map  $D \rightarrow \Gamma D$  which is a spacewise  $\mathbb{A}^1$ -weak equivalence and a cofibration. The other functor Jardine uses is the spectrification functor  $L : \text{Prespectra}(\mathcal{U}) \rightarrow \text{Spectra}(\mathcal{U})$ . For a prespectrum  $D$ ,

$$(LD)_V = \text{colim}_{(W,Z) \in V \downarrow \mathcal{C}(\mathcal{U})} \Omega^Z D_W$$

where  $V \downarrow \mathcal{C}(\mathcal{U})$  denotes the comma category whose objects are objects  $W$  in  $\mathcal{U}$  together with a map  $Z : V \rightarrow W$ .

DEFINITION 7.1 (Jardine [9]). *A map of prespectra  $f : E \rightarrow E'$  is a stable  $\mathbb{A}^1$ -equivalence if*

$$L\Gamma f : L\Gamma E \rightarrow L\Gamma E'$$

*is a spacewise  $\mathbb{A}^1$ -weak equivalence.*

PROPOSITION 7.2. *A map of spectra  $f : E \rightarrow E'$  is an  $\mathbb{A}^1$ -weak equivalence in the sense of Definition 5.3 if and only if it is a stable  $\mathbb{A}^1$ -equivalence in the sense of Jardine's definition 7.1.*

The proposition follows from the next lemma.

LEMMA 7.3. *A spacewise  $\mathbb{A}^1$ -weak equivalence of  $\mathcal{U}$ -spectra is an  $\mathbb{A}^1$ -weak equivalence of spectra in our sense (Definition 5.3).*

PROOF. Let  $\mathcal{H}_s(\text{Spc}(k)_\bullet)$  denote the homotopy category of based  $k$ -spaces associated with the simplicial model structure. Let  $f : E \rightarrow E'$  be a spacewise  $\mathbb{A}^1$ -weak equivalence of spectra. Then  $f$  is the map

$$f : E = \text{colim}_V \Sigma_V^{\mathcal{U}} E_V \rightarrow E' = \text{colim}_V \Sigma_V^{\mathcal{U}} E'_V$$

induced by the maps  $\Sigma_V^{\mathcal{U}} f_V$ . By Proposition 5.12, directed colimits preserve  $\mathbb{A}^1$ -weak equivalences. Thus, it suffices to show that  $\Sigma_V^{\mathcal{U}} f_V$  induces a bijection

$$\mathcal{H}_s(\Sigma_V^{\mathcal{U}} E'_V, G) \cong \mathcal{H}_s(\Sigma_V^{\mathcal{U}} E_V, G)$$

for every  $\mathbb{A}^1$ -local spectrum  $G$  and  $V \subset \mathcal{U}$  cofinite. By adjunction, this is equivalent to showing that the map  $f_V$  induces a bijection in the simplicial homotopy category of based  $k$ -spaces

$$\mathcal{H}_s(\mathrm{Spc}(k)_\bullet)(E'_V, G_V) \cong \mathcal{H}_s(\mathrm{Spc}(k)_\bullet)(E_V, G_V).$$

But if  $G$  is an  $\mathbb{A}^1$ -local spectrum, then for every based  $k$ -space  $X$ , we have a bijection

$$\mathcal{H}_s(\Sigma_V^{\mathcal{U}} X, G) \cong \mathcal{H}_s(\Sigma_V^{\mathcal{U}} X \wedge \mathbb{A}_+^1, G).$$

By adjunction, this corresponds to a bijection

$$\mathcal{H}_s(\mathrm{Spc}(k)_\bullet)(X, G_V) \cong \mathcal{H}_s(\mathrm{Spc}(k)_\bullet)(X \wedge \mathbb{A}_+^1, G_V).$$

Hence, each  $G_V$  is an  $\mathbb{A}^1$ -local space, and the lemma follows.  $\square$

**PROOF OF PROPOSITION 7.2:** For this proof, we work with both  $\mathcal{U}$ -spectra and  $\mathbb{T}$ -spectra, indexed on the integers. This is possible because by Theorem 6.3, for any choice of decomposition  $\mathcal{B}$  of  $\mathcal{U}$  into one-dimensional subspaces, the adjoint functors  $(L_{\mathcal{B}}^{\mathcal{U}}, R_{\mathcal{B}}^{\mathcal{U}})$  are Quillen equivalences, and both preserve  $\mathbb{A}^1$ -weak equivalences. We say that a map  $f : E \rightarrow E'$  of  $\mathcal{U}$ -prespectra is an eventual spacewise  $\mathbb{A}^1$ -weak equivalence, if there is some  $V \subset \mathcal{U}$  cofinite, such that for all  $W \subset V$ ,

$$f_W : E_W \rightarrow E'_W$$

is an  $\mathbb{A}^1$ -weak equivalence of based  $k$ -spaces. Jardine [9] has shown that if a map is a spacewise  $\mathbb{A}^1$ -weak equivalence, then it is a stable  $\mathbb{A}^1$ -equivalence in his definition. Since for a prespectrum  $D$ ,  $LD$  only depends on  $D_W$  for  $W \in V \downarrow \mathcal{C}(\mathcal{U})$ , an eventual spacewise  $\mathbb{A}^1$ -weak equivalence of prespectra is an  $\mathbb{A}^1$ -equivalence in his sense. Consider an elementary  $\mathbb{A}^1$ -weak equivalence of spectra

$$E \wedge \mathbb{A}_+^1 \rightarrow E$$

where  $E$  is any spectrum. The spectrum  $E \wedge \mathbb{A}_+^1$  is obtained by applying the spectrification functor  $L$  to the prespectrum  $\{E_V \wedge \mathbb{A}_+^1\}$ . The space  $E_V \wedge \mathbb{A}_+^1$  is  $\mathbb{A}^1$ -weak equivalent to  $E_V$  for every  $V$ , so the prespectrum  $\{E_V \wedge \mathbb{A}_+^1\}$  is stably  $\mathbb{A}^1$ -equivalent to  $E$  in Jardine's sense.

Now for  $n \in \mathbb{N}$ , we recall the  $n$ -th partial spectrification functor  $L_n$  on the category of  $\mathbb{T}$ -spectra, defined in the proof of Lemma 15.1. For any prespectrum  $D$ , set

$$\begin{aligned} (L_n D)_m &= D_m \text{ if } m \geq n \\ &= \Omega^{n-m} D_n \text{ if } m < n. \end{aligned}$$

There is a natural transformation  $Id \rightarrow L_n$  and a natural transformation  $L_n \rightarrow L_{n+1}$  for each  $n$ . For any prespectrum  $D$ , the maps  $D \rightarrow L_n D$  and  $L_n D \rightarrow L_{n+1} D$  are clearly eventual spacewise  $\mathbb{A}^1$ -equivalences. But  $LD = \text{colim}_n L_n D$ , and the unit map  $D \rightarrow LD$  is the directed colimit of the maps  $L_n D \rightarrow L_{n+1} D$ . Since Jardine's stable  $\mathbb{A}^1$ -equivalences are closed under directed colimits, the unit map is a stable  $\mathbb{A}^1$ -equivalence in his sense for any prespectrum. Hence, an elementary  $\mathbb{A}^1$ -local equivalence of spectra is an  $\mathbb{A}^1$ -equivalence by Jardine's definition. But since his stable  $\mathbb{A}^1$ -equivalences satisfy the axioms of Proposition 5.12, all of our  $\mathbb{A}^1$ -weak equivalences of spectra are stable  $\mathbb{A}^1$ -equivalences in the sense of Jardine's definition.

Conversely, to show that any stable  $\mathbb{A}^1$ -equivalence in Jardine's definition is an  $\mathbb{A}^1$ -weak equivalence in our definition, consider the following commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & E' \\ \downarrow & & \downarrow \\ L\Gamma E & \longrightarrow & L\Gamma E' \end{array}$$

where  $E \rightarrow E'$  is a map of spectra which is a stable  $\mathbb{A}^1$ -equivalence in the sense of Jardine's definition. The bottom map is a spacewise  $\mathbb{A}^1$ -weak equivalence, so it is an  $\mathbb{A}^1$ -weak equivalence by Lemma 7.3. Likewise, the maps  $E \rightarrow \Gamma E$  and  $E' \rightarrow \Gamma E'$  are spacewise  $\mathbb{A}^1$ -equivalences. Also, as shown in Lemma 15.1, the map  $\Gamma D \rightarrow L\Gamma D$  is an  $\mathbb{A}^1$ -weak equivalence in our definition since it is a simplicial weak equivalence. So the vertical maps are  $\mathbb{A}^1$ -weak equivalences in our sense. Therefore, so is the top map  $E \rightarrow E'$ .  $\square$

In particular, the proof of the proposition shows that although Jardine considers all prespectra and we only consider spectra, the resulting homotopy categories are

equivalent, since every prespectrum  $D$  is  $\mathbb{A}^1$ -weak equivalent to a spectrum (in both ours and Jardine's sense), namely  $LD$ .

## 8. Change of Universe

To define smash products of spectra, we need to define change of universe functors. In the classical case, if  $i : \mathcal{U} \rightarrow \mathcal{V}$  is a linear isometry between two universes, then we have an adjoint pair of change of universe functors  $i^*$  from the category of  $\mathcal{V}$ -spectra to the category of  $\mathcal{U}$ -spectra, and  $i_*$  from the category of  $\mathcal{U}$ -spectra to the category of  $\mathcal{V}$ -spectra. In the algebraic case, where the universes do not come with inner products, the analogue of a linear isometry is the following.

DEFINITION 8.1. *For two  $k$ -universes  $\mathcal{U}, \mathcal{V}$ , an extended injection*

$$(i, F) : \mathcal{U} \rightarrow \mathcal{V}$$

*is a linear injection  $i : \mathcal{U} \rightarrow \mathcal{V}$ , together with a given map  $F$  from the objects of  $\mathcal{C}(\mathcal{U})$  to the objects of  $\mathcal{C}(\mathcal{V})$ , satisfying the following conditions:*

1. *For every cofinite  $U \subset \mathcal{U}$ ,*

$$F(U) \cap i(\mathcal{U}) = i(U).$$

2. *The dimension of  $\mathcal{V}/F(U)$  is the same as the dimension of  $\mathcal{U}/U$ .*
3. *If  $U \subset U'$  are cofinite in  $\mathcal{U}$ , then  $F(U) \subset F(U')$ .*

In particular, if  $i : \mathcal{U} \rightarrow \mathcal{V}$  is split by a surjection  $j : \mathcal{V} \rightarrow \mathcal{U}$ , then  $i$ , along with  $F(U) = j^{-1}(U)$ , is an extended injection.

Note that given an extended injection  $(i, F) : \mathcal{U} \rightarrow \mathcal{V}$ , cofinite  $U \subset U'$  in  $\mathcal{C}(\mathcal{U})$ , and  $Z \dot{+} U' = U$  for a finite-dimensional  $Z$  in  $\mathcal{U}$ , we have  $i(Z) \dot{+} F(U') = F(U)$ . Hence,  $F$  determines a faithful functor  $\mathcal{C}(\mathcal{U}) \rightarrow \mathcal{C}(\mathcal{V})$  by sending a morphism  $Z$  to  $i(Z)$ .

Let  $(i, F) : \mathcal{U} \rightarrow \mathcal{V}$  be an extended injection. We can define the change of universe functor  $i^* : \text{Spectra}(\mathcal{V}) \rightarrow \text{Spectra}(\mathcal{U})$  directly. Namely, given a  $\mathcal{V}$ -spectrum  $E$ , we

set

$$(8.2) \quad (i^* E)_U = E_{F(U)}$$

for every cofinite  $U \subset \mathcal{U}$ . For  $Z \dot{+} U = U'$  in  $\mathcal{U}$ , we just define the structure map  $\rho_{U,Z}^{U'}$  in  $i^* E$  to be

$$(8.3) \quad (i^* E)_{U'} = E_{F(U')} \xrightarrow{\rho_{F(U),i(Z)}^{F(U')}} \Omega^{i(Z)} E_{F(U)} \xrightarrow{\Omega^{i|_Z}} \Omega^Z (i^* E)_U.$$

Here,  $\Omega^{i|_Z}$  is the functor  $\underline{Hom}(i(Z)/(i(Z) \setminus \{0\}), -) \rightarrow \underline{Hom}(Z/(Z \setminus \{0\}), -)$  induced by  $i|_Z$ . (Note that the functor  $F$  is suppressed in this notation.)

The functor  $i^*$  has a left adjoint  $i_*$ . To give a formulation for  $i_*$ , we first define the subcategory  $\mathcal{C}(\mathcal{V}, F)$  of  $\mathcal{C}(\mathcal{V})$ . Here, the objects are cofinite subspaces  $V$  such that  $V \subset F(i^{-1}(V)) = F(i^{-1}(V \cap i(\mathcal{U})))$ . For two such  $V, V'$ , a morphism  $Z : V \rightarrow V'$  is a finite-dimensional subspace  $Z$  in  $\mathcal{V}$ , such that  $Z \dot{+} V' = V$ , and  $i^{-1}(Z) \dot{+} i^{-1}(V') = i^{-1}(V)$ .

For every  $V \in \mathcal{C}(\mathcal{V})$ ,  $W = V \cap F(i^{-1}(V))$  is cofinite in  $\mathcal{V}$ , and  $W \cap i(\mathcal{U}) = V \cap i(\mathcal{U})$ , so  $W \subset F(i^{-1}(W))$ . Therefore,  $\mathcal{C}(\mathcal{V}, F)$  is cofinal in  $\mathcal{C}(\mathcal{V})$  in the sense of Lemma 3.2. In particular, this means that if we have a spectrum indexed on  $\mathcal{C}(\mathcal{V}, F)$ , then by the methods of Section 3, we in fact have a spectrum indexed on  $\mathcal{C}(\mathcal{V})$ .

Let  $E$  be a  $\mathcal{U}$ -spectrum. For  $V \in \mathcal{C}(\mathcal{V}, F)$ , we define

$$(8.4) \quad D_V = \bigvee_{Z \dot{+} V = F(i^{-1}(V))} \Sigma^Z E_{i^{-1}(V)}$$

For  $T : V \rightarrow V'$  in  $\mathcal{C}(\mathcal{V}, F)$ , note that

$$(T \cap i(\mathcal{U})) \dot{+} (V' \cap i(\mathcal{U})) = V \cap i(\mathcal{U})$$

by applying  $i$  to the equation  $i^{-1}(T) \dot{+} i^{-1}(V') = i^{-1}(V)$ , and

$$(T \cap i(\mathcal{U})) \dot{+} (T \cap F(i^{-1}(V'))) = T.$$

Also, for any  $Z$  such that  $Z \dot{+} V = F(i^{-1}(V))$ ,  $Z' = Z \dot{+} (T \cap F(i^{-1}(V)))$  satisfies  $Z' \dot{+} V' = F(i^{-1}(V'))$ . We define the structure map  $\rho_{V',Z}^V$  to be

$$\begin{aligned} \Sigma^T \bigvee_{Z \dot{+} V = F(i^{-1}(V))} \Sigma^Z E_{i^{-1}(V)} &\cong \bigvee_{Z \dot{+} V = F(i^{-1}(V))} \Sigma^{T+Z} E_{i^{-1}(V)} \\ &\cong \bigvee_{Z \dot{+} V = F(i^{-1}(V))} \Sigma^{(T \cap i(\mathcal{U}) \dot{+} (T \cap F(i^{-1}(V)))) \dot{+} Z} E_{i^{-1}(V)} \\ &\rightarrow \bigvee_{Z \dot{+} V = F(i^{-1}(V))} \Sigma^{(T \cap F(i^{-1}(V))) \dot{+} Z} E_{i^{-1}(V')} \\ &\rightarrow \bigvee_{Z' \dot{+} V' = F(i^{-1}(V'))} \Sigma^{Z'} E_{i^{-1}(V')}. \end{aligned}$$

Here, the third map is  $\Sigma^{i|T}$  applied to the structure map  $\rho_{i^{-1}(V'), i^{-1}(T)}^{i^{-1}(V)}$ .

This defines a prespectrum  $\{D_V\}$  over the category  $\mathcal{C}(\mathcal{V}, F)$ . By the above observation, after spectrification, it gives a spectrum indexed on  $\mathcal{C}(\mathcal{V})$ , which is our  $i_* E$ .

**PROPOSITION 8.5.** *There is a natural bijection*

$$(8.6) \quad \text{Hom}_{\mathcal{U}}(E, i^* G) \cong \text{Hom}_{\mathcal{V}}(i_* E, G).$$

**PROOF.** Similar as in the case for topological spectra: it suffices to show that there is a natural bijection between  $\text{Hom}_{\mathcal{U}}(E, i^* G)$  and  $\text{Hom}_{\text{Prespectra}(\mathcal{C}(\mathcal{V}, F))}(D, G)$ , where  $D$  is the prespectrum indexed over  $\mathcal{C}(\mathcal{U}, F)$  defined above. Given a map  $f : E \rightarrow i^* G$  of  $\mathcal{U}$ -spectra, and  $V \in \mathcal{C}(\mathcal{U}, F)$ ,  $Z \dot{+} V = F(i^{-1}(V))$ , the map

$$f_{i^{-1}(V)} : E_{i^{-1}(V)} \rightarrow G_{F(i^{-1}(V))} \cong \Omega^Z G_V$$

gives a map  $\Sigma^Z E_{i^{-1}(V)} \rightarrow G_V$ , hence by (8.4), we have a map  $D_V \rightarrow G_V$ . Conversely, given a map of  $\mathcal{C}(\mathcal{V}, F)$ -prespectra  $g : D \rightarrow G$ , suppose  $U \subset \mathcal{U}$  is cofinite. Then  $F(U)$  is cofinite in  $\mathcal{V}$ ,  $F(U) \cap i(\mathcal{U}) = i(U)$ . Therefore we have the map

$$g_{F(U)} : E_U = \bigvee_{Z \dot{+} F(U) = F(U)} \Sigma^Z E_U \rightarrow G_U.$$

This gives a map of  $\mathcal{U}$ -spectra  $E \rightarrow i^* G$ . □

### 9. The Space of Extended Injections

Given two  $\mathcal{U}$ -spectra  $E, E'$ , there is a  $\mathcal{U} \oplus \mathcal{U}$ -prespectrum whose  $(U \oplus U')$ -th space is just  $E_U \wedge E'_{U'}$ . Spectrification gives the external smash product  $E \overline{\wedge} E'$ , which is a  $\mathcal{U} \oplus \mathcal{U}$ -spectrum. Given a fixed extended injection  $(j, F) : \mathcal{U} \oplus \mathcal{U} \rightarrow \mathcal{U}$ , we can define an internal smash product on  $Spectra(\mathcal{U})$ . Namely,

$$E \wedge_{(j,F)} E' = j_*(E \overline{\wedge} E').$$

To show that the internal smash product is well-defined up to  $\mathbb{A}^1$ -homotopy, we need to consider  $\mathcal{I}(\mathcal{U}, \mathcal{V})$ , the  $k$ -space of extended injections from  $\mathcal{U}$  to  $\mathcal{V}$ . For finite-dimensional  $U \cong \mathbb{A}^n$  and  $V \cong \mathbb{A}^m$ , an extended injection  $(i, F) : U \rightarrow V$  is equivalent to a split injection  $(i, j)$ , where  $j$  is a surjection  $V \rightarrow U$ , since  $V = i(U) \dot{+} F(0)$ , so we can just define  $j$  to be  $i^{-1}$  on  $i(U)$  and 0 on  $F(0)$ . The relation that  $j \cdot i = Id$  is algebraic for linear maps  $i, j$ . So for  $U, V$  finite-dimensional, the space of extended injections  $\mathcal{I}(U, V)$  in  $Spc(k)$  is a subspace of  $\underline{Hom}(U, V) \times \underline{Hom}(V, U)$ . Choose sequences of finite subspaces  $U_k \subset \mathcal{U}$  and  $V_l \subset \mathcal{V}$ , such that  $U_k \dot{+} Z_k = U_{k+1}$  and  $V_l \dot{+} T_l = V_{l+1}$  for finite-dimensional  $Z_i, T_i$ , and such that  $\mathcal{U} = \text{colim}_k U_k$  and  $\mathcal{V} = \text{colim}_l V_l$ . Then as an object of  $Spc(k)$ ,

$$(9.1) \quad \mathcal{I}(\mathcal{U}, \mathcal{V}) = \lim_k \text{colim}_l \mathcal{I}(U_k, V_l).$$

Note that if  $(g, F)$  and  $(g', F')$  are extended injections  $\mathcal{U} \rightarrow \mathcal{V}$ , then  $(g \oplus g', F \oplus F')$  is an extended injection  $\mathcal{U} \oplus \mathcal{U} \rightarrow \mathcal{V} \oplus \mathcal{V}$ . Here, for  $U \oplus U'$  cofinite in  $\mathcal{U} \oplus \mathcal{U}$ ,  $(F \oplus F')(U \oplus U') = F(U) \oplus F'(U')$ . The conditions of Definition 8.1 are easy to check.

The following proposition leads to a proof that the internal smash product of  $\mathcal{U}$ -spectra to be well-defined up to  $\mathbb{A}^1$ -homotopy.

**PROPOSITION 9.2.** *The  $k$ -space  $\mathcal{I}(\mathcal{U}, \mathcal{V})$  of extended injections from  $\mathcal{U}$  to  $\mathcal{V}$  is  $\mathbb{A}^1$ -contractible.*

**LEMMA 9.3.** *For any  $k$ -universe  $\mathcal{U}$ , there is a path*

$$H : \mathbb{A}^1 \rightarrow \mathcal{I}(\mathcal{U}, \mathcal{U})$$

such that  $H(0) = Id$  and  $H(1) = (j, F)$  where  $\mathcal{U}/i(\mathcal{U})$  is infinite-dimensional, and  $\bigcap_{U \in \mathcal{C}(\mathcal{U})} F(U)$  is infinite-dimensional.

PROOF. We think of  $\mathbb{A}^1$  as  $k$ . Choose a basis  $e_1, e_2, \dots$  of  $\mathcal{U}$ , and denote the subspace  $\langle e_{n+1}, e_{n+2}, \dots \rangle$  by  $V_n$ . First consider the path  $h_0 : \mathbb{A}^1 \rightarrow \mathcal{I}(\mathcal{U}, \mathcal{U})$ , where the image  $(h_0(t), F_{h_0}(t))$  at time  $t \in \mathbb{A}^1$  is given by

$$\begin{aligned} h_0(t)(e_i) &= 2te_i + (1 - 2t)e_{i+1} \\ F_{h_0}(t)(V_i) &= V_i. \end{aligned}$$

let  $(j, G) = (h_0(1), F_{h_0}(1))$ .  $j$  is an injection of codimension 1, and it is split by a map  $f : \mathcal{U} \rightarrow \mathcal{U}$  which sends  $e_1$  to 0 and fixes  $e_i + e_{i+1}$ . Let  $F : \mathcal{C}(\mathcal{U}) \rightarrow \mathcal{C}(\mathcal{U})$  be given by  $F(V_n) = f^{-1}(V_n) = j(V_n) \dot{+} \langle e_1 \rangle$ .

We will show that there is a path in  $\mathcal{I}(\mathcal{U}, \mathcal{U})$  connecting  $Id$  and  $(j, F)$  by defining a path  $h_1 : \mathbb{A}^1 \rightarrow \mathcal{I}(\mathcal{U}, \mathcal{U})$  connecting  $(j, F)$  and  $(j, G)$  as follows. At time  $t$ , define the injection  $h_1(t) = j : \mathcal{U} \rightarrow \mathcal{U}$ . To define  $F_{h_1}(t)$ , it suffices to define  $F_{h_1}(t)(V_i)$  for all  $i \geq 0$ . For  $i = 0$ ,  $V_0 = \mathcal{U}$ , so we must have  $F_{h_1}(t)(V_0) = \mathcal{U}$  for all  $t$ . For  $i = 1$ , define the path  $g_1 : \mathbb{A}^1 \rightarrow \mathcal{U}$ ,  $g_1(t) = te_1 - (1-t)e_2$ . Then set  $F_{h_1}(t)(V_1) = \langle g_1(t) \rangle \dot{+} j(V_1)$ , so  $F_{h_1}(0)(V_1) = V_1 = G(V_1)$ , and  $F_{h_1}(1)(V_1) = \langle e_1 \rangle \dot{+} j(V_1) = F(V_1)$ . In general, suppose we have path  $g_i$  in  $\mathcal{U}$ , such that  $F_{h_1}(t)(V_i) = \langle g_i(t) \rangle \dot{+} j(V_i)$ . Consider the path  $-e_i + (1-t)(e_i + e_{i+1})$  in  $\langle e_i, e_{i+1} \rangle$  connecting  $-e_i$  and  $e_{i+1}$ . Define

$$g_{i+1}(t)(e_{i+1}) = g_i(t) + (1-t)(e_i + e_{i+1})$$

and  $F_{h_1}(t)(V_{i+1}) = \langle g_{i+1}(t) \rangle \dot{+} j(V_{i+1})$ . Then  $F_{h_1}(t)(V_{i+1})$  has the same codimension in  $\mathcal{U}$  as  $V_{i+1}$ , and  $F_{h_1}(t)(V_{i+1}) \subset F_{h_1}(t)(V_i)$ . Therefore,  $h_1$  gives a path in  $\mathcal{I}(\mathcal{U}, \mathcal{U})$  connecting the identity and  $(j, F)$ .

Now choose a basis  $\{e_{i,k}\}_{i,k \geq 1}$  for  $\mathcal{U}$ . By the above, the identity on each  $\mathcal{U}_k = \langle e_{i,k} \rangle$  is homotopic, via extended injections, to an extended injection  $(j_k, F_k)$  which is a split injection of codimension 1. Denote the homotopy by  $H_k$ , and  $V_{i,k} = \langle e_{i+1,k}, e_{i+2,k} \rangle$ . The sum  $j = \bigoplus_k j_k$  is a split injection of infinite codimension, with  $F = \bigoplus_k F_k$  which takes  $U \subset \mathcal{U}$  cofinite to  $j(U) \dot{+} \langle e_{1,k} \rangle$ . On the injections,  $H(t) = \bigoplus_k H_k(t)$  gives an  $\mathbb{A}^1$ -homotopy between the identity on  $\mathcal{U}$  and  $j$ . Note also

that the subcategory of cofinite subspaces with objects

$$V_{i_1, k_1} \dot{+} \cdots \dot{+} V_{i_n, k_n} \dot{+} (\dot{+}_{k \neq k_n} \mathcal{U}_k)$$

is cofinal in the category  $\mathcal{C}(\mathcal{U})$ . So for  $t$ , define

$$\begin{aligned} F_H(t)(V_{i_1, k_1} \dot{+} \cdots \dot{+} V_{i_n, k_n} \dot{+} (\dot{+}_{k \neq k_n} \mathcal{U}_k)) = \\ F_{k_1}(V_{i_1}) \dot{+} \cdots \dot{+} F_{k_n}(V_{i_n}) \dot{+} (\dot{+}_{k \neq k_n} \mathcal{U}_k). \end{aligned}$$

This satisfies the axioms for an extended injection, so we have a homotopy between the identity on  $\mathcal{U}$  and  $(j, F)$ , an extended injection of infinite codimension.  $\square$

**PROOF OF PROPOSITION 9.2.** Choose a basis  $\{a_i\}$  of the universe  $\mathcal{U}$ , and consider the two split injections  $j_1, j_2 : \mathcal{U} \rightarrow \mathcal{U} \oplus \mathcal{U}$ , which map  $\mathcal{U}$  onto the first and second summands, respectively. We first show that  $j_1$  and  $j_2$  are homotopic in  $\mathcal{I}(\mathcal{U}, \mathcal{U} \oplus \mathcal{U})$ . Denote  $b_i = j_1(a_i)$ , and  $c_i = j_2(a_i)$ , then  $\{b_1, c_1, b_2, c_2, \dots\}$  form a basis for  $\mathcal{U} \oplus \mathcal{U}$ . Consider the identity matrix and the matrix which switches the two basis vectors in  $GL_2(k)$ . Since they have the same determinant, there is an algebraic map  $h : \mathbb{A}^1 \rightarrow GL_2(k)$ , which connects the two matrices. Suppose that  $h(t) = (h_{s,t}(k))$  where  $s, t = 1, 2$ . Define the homotopy

$$H : \mathbb{A}^1 \times \mathcal{U} \rightarrow \mathcal{U} \oplus \mathcal{U}$$

to be

$$H(t, a_i) = h_{1,1}tb_i + h_{2,1}(t)c_i.$$

This gives a homotopy between the injections  $j_1$  and  $j_2$ . It remains to define the functor  $F_H(t) : \mathcal{C}(\mathcal{U}) \rightarrow \mathcal{C}(\mathcal{U} \oplus \mathcal{U})$  at time  $t$ . For  $V_n = \langle a_{n+1}, a_{n+2}, \dots \rangle$  in  $\mathcal{U}$ , take

$$\begin{aligned} F_H(t)(V_n) = \\ j_1(V_n) \dot{+} j_2(V_n) \dot{+} \langle h_{1,2}tb_1 + h_{2,2}tc_1, \dots, h_{1,2}tb_n + h_{2,2}tc_n \rangle. \end{aligned}$$

Since  $h$  is a path in  $GL_2(k)$ ,  $F_H(t)(V_n) \cap H(t)(\mathcal{U}) = H(t)(V_n)$ , and  $(H, F_H)$  is an extended injection.

Now let  $\{d_i\}$  be a basis for the universe  $\mathcal{U}$ . We define the homotopy of extended injections  $(h_1, F_{h_1}) : \mathbb{A}^1 \rightarrow \mathcal{I}(\mathcal{V}, \mathcal{V})$  to be the homotopy of Lemma 9.3 composed with an automorphism of  $\mathcal{V}$ , such that  $h_1(0)$  is the identity on  $\mathcal{V}$ , and  $h_1(1)$  is a split injection that sends  $d_i$  to  $d_{2i}$ . Also, let  $(h_2, F_{h_2}) : \mathbb{A}^1 \rightarrow \mathcal{I}(\mathcal{U}, \mathcal{U} \oplus \mathcal{U})$  be the homotopy

between  $j_1$  and  $j_2$  given above. Let  $(g_0, G_0)$  be a fixed extended injection from  $\mathcal{U}$  to  $\mathcal{V}$ . First, define the homotopy  $H_1 : \mathbb{A}^1 \times \mathcal{I}(\mathcal{U}, \mathcal{V}) \rightarrow \mathcal{I}(\mathcal{U}, \mathcal{V})$  by

$$H_1(t, (g, G)) = (h_1, F_{h_1}) \cdot (g, G).$$

$H_1$  gives a homotopy equivalence between  $\mathcal{I}(\mathcal{U}, \mathcal{V})$  and  $(h_1(1), F_{h_1(1)}) \cdot \mathcal{I}(\mathcal{U}, \mathcal{V})$ .

Consider the following map

$$f : \mathcal{V} \oplus \mathcal{V} \rightarrow \mathcal{V}, \quad f(j_1(d_i)) = d_{2i}, \quad f(j_2(d_i)) = d_{2i-1}.$$

$f$  is an isomorphism, so it is an extended injection, given by the splitting  $f^{-1}$ . We define  $H_2 : \mathbb{A}^1 \times \mathcal{I}(\mathcal{U}, \mathcal{V}) \rightarrow \mathcal{I}(\mathcal{U}, \mathcal{V})$  to be

$$H_2(t, (g, G)) = f \cdot ((g, G) \oplus (g_0, G_0)) \cdot (h_2(t), F_{h_2(t)}).$$

Then  $H_2$  gives a homotopy equivalence between  $(h_1(1), F_{h_1(1)}) \cdot \mathcal{I}(\mathcal{U}, \mathcal{V})$  and the point  $f \cdot j_2 \cdot (g_0, G_0)$ .  $\square$

For a fixed universe  $\mathcal{U}$ , consider the operad  $\mathcal{L}$ , where  $\mathcal{L}(n)$  is the  $k$ -space of extended injections  $\mathcal{I}(\mathcal{U}^{\oplus n}, \mathcal{U})$ , with the obvious composition maps and  $\Sigma_n$ -action. This is the algebraic analogue of the linear isometries operad in the topological case [12]. By the above proposition, each  $k$ -space  $\mathcal{L}(n)$  is  $\mathbb{A}^1$ -contractible. Therefore, the internal smash product of  $\mathcal{U}$ -spectra are well-defined up to  $\mathbb{A}^1$ -homotopy.

## 10. Twisted Half-Smash Products and Twisted Function Spectra

Our next goal is to construct twisted half-smash products and twisted function spectra. Namely, for a  $k$ -space  $A \in \text{Spc}(k)$  which is equipped with a map  $\varphi : A \rightarrow \mathcal{I}(\mathcal{U}, \mathcal{V})$ , we would like a pair of adjoint functors

$$\begin{aligned} A \times - & : \text{Spectra}(\mathcal{U}) \rightarrow \text{Spectra}(\mathcal{V}) \\ F[A, -] & : \text{Spectra}(\mathcal{V}) \rightarrow \text{Spectra}(\mathcal{U}) \end{aligned}$$

which are just the change of universe functors when  $A$  is  $\text{Spec}(k)$ . Intuitively, one thinks of  $A \times E$  as constructed by “gluing together”  $\varphi(a)_* E$  over all  $a \in A$ , with

some suitable structure as an algebraic space. Likewise,  $F[A, G]$  is constructed from the collection of  $\varphi(a)^*G$ , over all  $a \in A$ .

Our construction is parallel to that of Cole in [6]. For two  $k$ -universes  $\mathcal{U}$  and  $\mathcal{V}$ , we consider a category  $\mathcal{S}(\mathcal{V}, \mathcal{U})$ . An object  $\mathcal{E}$  of this category is a system  $\mathcal{E}(U)$  of  $\mathcal{V}$ -spectra, one for each cofinite  $U \subset \mathcal{U}$ . If  $U' \subset U$  are cofinite in  $\mathcal{U}$ , and  $T \dot{+} U' = U$ , then there is a structure isomorphism of  $\mathcal{V}$ -spectra

$$\Sigma^T \mathcal{E}(U') \xrightarrow{\cong} \mathcal{E}(U).$$

Morphisms in  $\mathcal{S}(\mathcal{V}, \mathcal{U})$  are systems of morphisms of  $\mathcal{V}$ -spectra that commute with all such structure maps.

There is a function object functor

$$F(-, -) : \mathcal{S}(\mathcal{V}, \mathcal{U}) \times \text{Spectra}(\mathcal{V}) \rightarrow \text{Spectra}(\mathcal{U}).$$

If  $\mathcal{E}$  is in  $\mathcal{S}(\mathcal{V}, \mathcal{U})$ , and  $G$  is a  $\mathcal{V}$ -spectrum, then for  $U \subset \mathcal{U}$  cofinite,

$$F(\mathcal{E}, G)_U = F_{\text{Spectra}(\mathcal{V})}(\mathcal{E}(U), G)$$

as a based  $k$ -space. It is easy to check that this gives a  $\mathcal{U}$ -spectrum, and that this functor preserves inverse limits. It has a left adjoint

$$\wedge : \mathcal{S}(\mathcal{V}, \mathcal{U}) \times \text{Spectra}(\mathcal{U}) \rightarrow \text{Spectra}(\mathcal{V}).$$

If  $E$  is a  $\mathcal{U}$ -spectrum, then

$$\mathcal{E} \wedge E = \text{colim}_{U \in \mathcal{C}(\mathcal{U})} \mathcal{E}(U) \wedge E_U.$$

Here, the colimit is taken over the maps

$$\mathcal{E}(U) \wedge E_U \cong \Sigma^Z \mathcal{E}(U') \wedge E_U \cong \mathcal{E}(U') \wedge \Sigma^Z E_U \rightarrow \mathcal{E}(U') \wedge E_{U'}$$

for  $Z : U \rightarrow U'$  in  $\mathcal{C}(\mathcal{U})$ .

Given  $\varphi : A \rightarrow \mathcal{I}(\mathcal{U}, \mathcal{V})$ , we will construct an element  $\mathcal{M}(A)$  of  $\mathcal{S}(\mathcal{V}, \mathcal{U})$  and define

$$(10.1) \quad A \times E = \mathcal{M}(A) \wedge E$$

$$(10.2) \quad F[A, G] = F(\mathcal{M}(A), G).$$

To define  $\mathcal{M}(A)$ , we first introduce the notion of a continuous prespectrum. For a  $\mathcal{U}$ -prespectrum  $D$ , if  $U' \subset U$  are cofinite in the universe  $\mathcal{U}$ , then there is a structure map

$$\bigvee_{Z \dot{+} U' = U} \Sigma^Z D_{U'} \rightarrow D_U.$$

However, observe that the collection of all  $Z$ 's with  $Z \dot{+} U' = U$  has a natural structure as a object of  $\text{Spc}(k)$ . Let  $n$  be the difference of the codimensions of  $U'$  and  $U$ . Consider the infinite Grassmannian  $BGL(n, U)$ , which one may think of as the  $k$ -space of all  $n$ -dimensional subspaces of  $U$ . This has a canonical  $n$ -dimensional bundle  $\gamma_n$  over it. Denote the Grassmannian

$$G(n, U', U) = \{Z \mid Z \dot{+} U' = U\}$$

as a subspace of  $BGL(n, U)$ . The idea of a continuous prespectrum is to use the  $k$ -space structure on  $G(n, U', U)$ , instead of taking a wedge over it (thought of as a discrete set of points).

**DEFINITION 10.3.** *A continuous  $\mathcal{U}$ -prespectrum  $D$  is a collection of based  $k$ -spaces  $D_U$  for all cofinite  $U \subset \mathcal{U}$ , such that for all  $U' \subset U$ , there is a structure map*

$$(10.4) \quad D_U \wedge G(n, U', U)^{\gamma_n} \rightarrow D_{U'}$$

*which satisfies the obvious associativity conditions. Here  $G(n, U', U)^{\gamma_n}$  denotes the Thom space of the canonical bundle  $\gamma^n$  over  $G(n, U', U)$ . Maps of continuous  $\mathcal{U}$ -spectra are systems of maps of based  $k$ -spaces which commute with all structure maps.*

In particular, there is a forgetful functor from the category of continuous prespectra to the category of prespectra. Namely, for a fixed  $Z \dot{+} U' = U$ , the structure map  $\Sigma^Z D_U \rightarrow D_{U'}$  is the structure map of the continuous prespectrum, restricted to the fiber of  $\gamma_n$  at  $Z \in G(n, U', U)$ .

On the spectrum level, the notions of continuous and discrete spectra are the same. Namely, we have the following.

**LEMMA 10.5.** *A spectrum is a continuous prespectrum.*

PROOF. Let  $E$  be a  $\mathcal{U}$ -spectrum. Then for each cofinite  $U \subset \mathcal{U}$ , there is a natural algebraic action of  $GL(U)$  on  $E_U$ . If  $U' \subset U$ , let  $GL(U', U)$  denote the subspace of  $GL(U)$  consisting of all  $g$  which are the identity on  $U'$ . Choose a fixed  $Z_0$  such that  $Z_0 \dot{+} U' = U$ . Then for any  $Z \dot{+} U' = U$ , and  $g \in GL(U)$  such that  $g$  is the identity on  $U'$  and sends  $Z_0$  to  $Z$ , the following diagram commutes:

$$\begin{array}{ccc} E_U & \xrightarrow{\cong} & \Omega^{Z_0} E_{U'} \\ \downarrow g & & \downarrow \\ E_U & \xrightarrow{\cong} & \Omega^Z E_{U'} \end{array}$$

In particular, the structure map  $\rho_{U', Z}^U : E_U \xrightarrow{\cong} \Omega^Z E_{U'}$  is invariant if we compose  $g$  by an element of  $GL(Z_0)$ . So  $\rho_{U', Z}^U$  is determined by the action of  $GL(U', U)$  on  $E_U$  and the particular structure map  $E_U \xrightarrow{\cong} \Omega^{Z_0} E_{U'}$ . Hence, the structure maps  $\{\rho_{U', Z}^U \mid Z \dot{+} U' = U\}$  are parametrized by  $GL(U', U)/GL(Z_0)$ , which is canonically isomorphic to  $G(n, U', U)$ . Therefore, we can put the structure maps of  $E$  together to give a single structure map

$$E_U \rightarrow \underline{Hom}_{Spc(k)_\bullet}(G(n, U', U)^{\gamma_n}, E_{U'}).$$

□

Therefore, there is a forgetful functor from  $Spectra(\mathcal{U})$  to  $ContPrespectra(\mathcal{U})$ . It has a left adjoint  $L$ . Since the category of continuous prespectra is a full subcategory of the category of prespectra,  $L$  is the same as the spectrification functor from the  $Prespectra(\mathcal{U})$  to  $Spectra(\mathcal{U})$ .

**Comment:** For  $X \in Spc(k)_\bullet$ , and  $U \subset \mathcal{U}$  cofinite, we can construct the  $U$ -th shift desuspension of the continuous suspension prespectrum of  $X$ . Namely, we define the  $U'$ -th space to be  $Spec(k) = *$  if  $U'$  is not contained in  $U$ . If  $U' \subset U$ , then we define  $U'$ -th space to be

$$G(n, U', U)^{\gamma_n} \wedge X.$$

It is straightforward to check that this gives a continuous prespectrum. We have the following diagram of categories, where all the arrows are the left adjoints to the

forgetful functors.

$$\begin{array}{ccc}
 Spc(k)_\bullet & \longrightarrow & Prespectra(\mathcal{U}) \\
 \downarrow & & \downarrow \\
 ContPrespectra(\mathcal{U}) & \longrightarrow & Spectra(\mathcal{U})
 \end{array}$$

Because the adjoint diagram of forgetful functors commute, this diagram commutes, i. e. the suspension spectrum of a  $k$ -space is the same whether one goes through the discrete or continuous suspension prespectrum.

Now we construct  $\mathcal{M}A$  as a continuous prespectrum. Fix  $U \subset \mathcal{U}$  cofinite. For  $V \subset \mathcal{V}$ , define the  $k$ -spaces

$$(10.6) \quad A_{U,V} = \{a \in A \mid V \subset F_a(U)\}.$$

More precisely, we take the Grassmannian  $BGL(n, \mathcal{V})$ , which one may think of as the space of all  $n$ -dimensional subspaces of  $\mathcal{V}$ . There is an algebraic map from  $A$  to  $BGL(n, \mathcal{V})$ , given by  $a \mapsto F_a(U)$ . Also, let  $BGL(n, V, \mathcal{V})$  be the Grassmannian of all  $n$ -dimensional subspaces of  $\mathcal{V}$  which contain  $V$ . Then  $A_{U,V}$  is defined by the pullback square

$$\begin{array}{ccc}
 A_{U,V} & \longrightarrow & A \\
 \downarrow & & \downarrow \\
 BGL(n, V, \mathcal{V}) & \longrightarrow & BGL(n, \mathcal{V}).
 \end{array}$$

Likewise, we define

$$(10.7) \quad \tilde{A}_{U,V} = \{(a, Z_a) \mid a \in A_{U,V}, Z_a \dagger V = F_a(U)\}.$$

$\tilde{A}_{U,V}$  has a natural algebraic structure as a pullback in  $Spc(k)$ .

There is a bundle  $\eta_{U,V}$  over  $\tilde{A}_{U,V}$ , with total “space”

$$\{(a, Z_a), v \mid (a, Z_a) \in \tilde{A}_{U,V}, v \in Z_a\}.$$

Define

$$(10.8) \quad T_{U,V} = \tilde{A}_{U,V}^{\eta_{U,V}}$$

to be the Thom space of this bundle. Then for fixed  $U$ , and  $V' \subset V$  in  $\mathcal{V}$ , the algebraic map

$$\tilde{A}_{U,V} \times G(n, V', V) \rightarrow \tilde{A}_{U',V}$$

which sends  $((a, Z_a), T)$  to  $(a, Z_a \dot{+} T)$  is covered by a map of bundles

$$\eta_{U,V} \times \gamma_n \rightarrow \eta_{U',V}.$$

Hence, we have

$$T_{U,V} \wedge G(n, V', V)^{\gamma_n} \rightarrow T_{U',V}.$$

So  $\{T_{U,V}\}$  forms a continuous  $\mathcal{V}$ -prespectrum. Let  $\mathcal{M}(A)(U)$  be the spectrification  $L\{T_{U,V}\}$ .

LEMMA 10.9. *If  $T \dot{+} U' = U$  for cofinite subspaces  $U', U \subset \mathcal{U}$ , then there is a canonical isomorphism*

$$\Sigma^T \mathcal{M}(A)(U') \xrightarrow{\cong} \mathcal{M}(A)(U).$$

PROOF. Fix  $U', U$  and  $T$  in  $\mathcal{U}$  as above. For each  $V \subset \mathcal{V}$ , there is a natural map

$$\tilde{A}_{U',V} \rightarrow \tilde{A}_{U,V}$$

which takes  $(a, Z_a)$  to  $(a, Z_a \dot{+} a(T))$ , and induces a map

$$\Sigma^T T_{U',V} \rightarrow T_{U,V}$$

which gives a map of prespectra. We construct the intermediate  $k$ -space between  $\tilde{A}_{U',V}$  and  $\tilde{A}_{U,V}$

$$\tilde{A}_{U,V}^{res} = \{(a, Z_a) \mid a \in A_{U,V}, Z_a \dot{+} V = F_a(U), a(T) \subset Z_a\} \subset \tilde{A}_{U,V}.$$

Then the above map factors through  $(\tilde{A}_{U,V}^{res})^{\gamma_n}$ . Denote by  $\mathcal{E}(U)^{res}$  the spectrification of the continuous prespectrum  $\{(\tilde{A}_{U,V}^{res})^{\eta_{U,V}}\}$ . Then we have

$$\Sigma^T \mathcal{E}(U') \xrightarrow{f} \mathcal{E}(U)^{res} \xrightarrow{g} \mathcal{E}(U).$$

We claim that each map is an isomorphism after spectrification.

By Lemma 17.1 in the Appendix, every  $k$ -space is a colimit of small objects in  $Spc(k)$ , and the maps are natural in  $A$ . Thus, we can assume that  $A$  is small in

$Spk(k)$ . Then there is some  $V_0 \subset \mathcal{V}$  such that  $A_{U',V_0} = A_{U,V_0} = A$ . At the  $V$ -th place, we have the map

$$\Sigma^T \tilde{A}_{U',V}^{\eta_{U',V}} \rightarrow (\tilde{A}_{U,V}^{res})^{\eta_{U,V}}$$

which sends  $(a, Z_a)$  to  $(a, Z_a \dot{+} a(T))$  on the base  $k$ -space, and adds  $a(T)$  to each fiber. If  $W \subset V$ ,  $W \dot{+} Z_0 = V$ , then the following diagram commutes

$$\begin{array}{ccc} \Sigma^{Z_0} \Sigma^T \tilde{A}_{U',V}^{\eta_{U',V}} & \longrightarrow & \Sigma^{Z_0} (\tilde{A}_{U,V}^{res})^{\eta_{U,V}} \\ \downarrow & & \downarrow \\ \Sigma^T \tilde{A}_{U',W}^{\eta_{U',W}} & \longrightarrow & (\tilde{A}_{U,W}^{res})^{\eta_{U,W}}. \end{array}$$

So for all  $V \subset V_0$ ,  $f_V$ , the map induced in the colimit over  $\mathcal{C}(\mathcal{V})$ , is an isomorphism. Hence,  $f$  is an isomorphism.

To show that  $g$  is an isomorphism, we first consider a small subspace  $K_{U,V}$  of  $\tilde{A}_{U,V}$ . Again, we can assume that  $A_{U,V} = A_{U',V} = A$ . Then there is some finite dimensional subspace  $Q \subset V$ , such that for all  $(a, Z_a) \in \tilde{A}_{U,V}$ ,  $Q \dot{+} Z_a$  contains  $a(T)$ . Choose a complement  $W \subset V$ ,  $W \dot{+} Q = V$ . Then there is a map

$$\Sigma^Q K_{U,V} \rightarrow (A_{U,W}^{res})^{\eta_{U,W}}$$

which sends  $(a, Z_a)$  to  $(a, Z_a \dot{+} Q)$  on the base  $k$ -spaces. This induces a map  $K_{U,V} \rightarrow \mathcal{E}(U)^{res}$ . This is independent of the choice of  $Q$  and  $W$ , since it is clear that the maps induced as above via  $Q_1, W_1$  and  $Q_2, W_2$  are the same if  $Q_1 \subset Q_2$  and  $W_2 \subset W_1$ . However, for any small  $k$ -space of choices of  $Q$  and  $W$ , there is a single finite-dimensional space  $Q_0$  with  $W_0 \dot{+} Q_0 = V$  such that  $Q \subset Q_0$  and  $W_0 \subset W$  for all  $Q, W$ . Therefore, we have a canonical map  $(\tilde{A}_{U,V}^{res})^{\eta_{U,V}} \rightarrow \mathcal{E}(U)^{res}$  by taking colimits over the small subspaces of  $\tilde{A}_{U,V}^{res}$ . Passing to the colimit in the source gives the inverse map to  $g$ .  $\square$

Thus,  $\mathcal{M}(A)$  is an object of  $\mathcal{S}(\mathcal{V}, \mathcal{U})$ , and we have constructed the twisted half-smash product and the twisted half-function spectrum by  $A$ . In particular, if  $A = \{(a, F_a)\}$  is a single point, i. e. isomorphic to  $Spec(k)$ , then for each  $U$ ,  $\{T_{U,V}\}$  is the continuous suspension prespectrum of  $S^0$ , with a shift desuspension by  $F_a(U)$ . So by

the previous comment,

$$\mathcal{M}(A)(U) = \Sigma_{F_a(U)}^{\mathcal{V}} S^0.$$

Therefore, if  $G$  is a  $\mathcal{V}$ -spectrum, for any cofinite  $U \in \mathcal{U}$ ,

$$F[A, G]_U = F(\Sigma_{F_a(U)}^{\mathcal{V}} S^0, G) = \underline{Hom}_{\mathit{Spc}(k)_\bullet}(S^0, G_{F_a(U)}) = G_{F_a(U)}$$

i. e.  $F[A, G] = a^*G$ . Hence, by the uniqueness of adjoints,  $A \times E = a_*E$  for any  $\mathcal{U}$ -spectrum  $E$ .

Note that for  $U = \mathcal{U}$ ,  $A_{\mathcal{U}, \mathcal{V}} = A$ , we have

$$\tilde{A}_{\mathcal{U}, \mathcal{V}}^{\eta_{\mathcal{U}, \mathcal{V}}} = A_+ \wedge G(n, \mathcal{V}, \mathcal{V})^{\gamma_n}.$$

This is exactly the continuous suspension spectrum of  $A_+$ , so

$$\mathcal{M}(A)(\mathcal{U}) \cong \Sigma^{\mathcal{V}} A_+.$$

In general, we have the following untwisting lemma.

LEMMA 10.10. *If  $U_0 \subset \mathcal{U}$ ,  $V_0 \subset \mathcal{V}$  have the same codimension, then*

$$\mathcal{M}(A)(U_0) \cong \Sigma_{V_0}^{\mathcal{V}} A_+.$$

*This isomorphism is natural in  $A$ .*

PROOF. Suppose we have fixed finite-dimensional  $T_0 \subset \mathcal{U}$ ,  $T_0 \dot{+} U_0 = \mathcal{U}$ , and  $Z_0 \subset \mathcal{V}$ ,  $Z_0 \dot{+} V_0 = \mathcal{V}$ . As before, we can assume that  $A$  is small, so there is some  $V \subset \mathcal{U}$ , such that for all  $W \subset V$ ,  $A_{U_0, W} = A$ . Choose  $W \subseteq V_0$ . There is a map

$$f : \tilde{A}_{U_0, W}^{\eta_{U_0, W}} \rightarrow A_+ \wedge G(n, W, V_0)^{\gamma_n}$$

which takes the fiber over  $(a, Z_a)$  to the fiber over  $(a, (Z_a \dot{+} a(T_0)) \cap V_0)$ . By the Noether isomorphism,  $(Z_a \dot{+} a(T_0)) \cap V_0$  is a well-defined complement of  $W$  in  $V_0$ , so  $f$  is a well-defined algebraic map. This map gives a map of prespectra. Similarly, there is a map going the other way

$$g : A_+ \wedge G(n, W, V_0)^{\gamma_n} \rightarrow \tilde{A}_{U_0, W}^{\eta_{U_0, W}}$$

which takes the fiber over  $(a, Z)$  to the fiber over  $(Z \dot{+} Z_0) \cap F_a(U_0)$ .

On the prespectrum level,  $f$  and  $g$  are not inverses. For  $(a, Z_a) \in \tilde{A}_{U_0, W}$  and  $Z \in G(n, W, V_0)$  with  $f(a, Z_a) = (a, Z)$ ,  $gf(a, Z_a) = (a, Z_a)$  if and only if  $Z_a \dot{+} a(T_0) = Z \dot{+} Z_0$ . But for any small subspace  $K_{U_0, W}$  of  $\tilde{A}_{U_0, W}^{\eta_{U_0, W}}$  and any small subspace  $G'(n, W, V_0)$  of  $G(n, W, V_0)$ , there is some finite-dimensional  $Q \subset W$  and cofinite  $W'$ , with  $Q \dot{+} W' = W$ , such that  $Q \dot{+} Z_a \dot{+} a(T_0) = A \dot{+} Z \dot{+} Z_0$  for all  $(a, Z_a) \in K_{U_0, W}$  and  $Z \in G'(n, W, V_0)$ . Thus, passing to colimits over  $W$ , we have induced maps

$$\operatorname{colim}_{\mathcal{C}(\mathcal{V})} \tilde{A}_{U_0, W}^{\eta_{U_0, W}} \xrightleftharpoons{\quad} \operatorname{colim}_{\mathcal{C}(\mathcal{V})} A_+ \wedge G(n, W, V_0)^{\gamma_n}.$$

So on the spectrum level  $g \cdot f = Id$ . By an analogous argument,  $f \cdot g = Id$  as well.  $\square$

The next result follows from Lemma 10.10.

**PROPOSITION 10.11 (Untwisting).** *Given a based  $k$ -space  $X$ , (an unbased)  $k$ -space  $A$ , and a  $\mathcal{V}$ -spectrum  $G$ , with  $\varphi : A \rightarrow \mathcal{I}(\mathcal{U}, \mathcal{V})$ ,  $V \subset \mathcal{V}$  and  $U \subset \mathcal{U}$  of the same codimension, there are isomorphisms*

$$\begin{aligned} A \times \Sigma_U^{\mathcal{U}} X &\cong A_+ \wedge \Sigma_V^{\mathcal{V}} X \\ \Omega_U^{\mathcal{U}} F[A, G] &\cong \underline{\operatorname{Hom}}_{\operatorname{Spc}(k)_\bullet}(A_+, \Omega_V^{\mathcal{V}} G). \end{aligned}$$

**PROOF.** Given a  $\mathcal{V}$ -spectrum  $G$ ,

$$\begin{aligned} \Omega_U^{\mathcal{U}} F[A, G] &= F_{\operatorname{Spectra}(\mathcal{V})}(\mathcal{M}(A)(U), G) \\ &\cong F_{\operatorname{Spectra}(\mathcal{V})}(\Sigma_V^{\mathcal{V}} A_+, G) \\ &\cong \underline{\operatorname{Hom}}_{\operatorname{Spc}(k)_\bullet}(A_+, \Omega_V^{\mathcal{V}} G). \end{aligned}$$

The first statement follows from the uniqueness of adjoints.  $\square$

We also have the following associativity properties of the twisted half-smash product.

**PROPOSITION 10.12.** *1. If  $A \rightarrow \mathcal{I}(\mathcal{U}, \mathcal{V})$ ,  $E$  is a  $\mathcal{U}$ -spectrum, and  $X \in \operatorname{Spc}(k)_\bullet$ , then*

$$(A \times E) \wedge X \cong A \times (E \wedge X)$$

naturally.

2. If  $A \rightarrow \mathcal{I}(\mathcal{U}, \mathcal{V})$ ,  $B \rightarrow \mathcal{I}(\mathcal{V}, \mathcal{W})$ , and  $E$  is a  $\mathcal{U}$ -spectrum, then

$$(B \times A) \times E \cong B \times (A \times E)$$

naturally.

PROOF. It is easy to check that if  $\mathcal{E}$  is an object of  $\mathcal{S}(\mathcal{V}, \mathcal{U})$ , then for any  $\mathcal{U}$ -spectrum  $E$  and based  $k$ -space  $X$ ,

$$(\mathcal{E} \wedge E) \wedge X \cong \mathcal{E} \wedge (E \wedge X) \cong (\mathcal{E} \wedge X) \wedge E$$

naturally. The first part of the proposition follows immediately.

Given  $A, B$  as in part 2, we have

$$B \times A \rightarrow \mathcal{I}(\mathcal{U}, \mathcal{V}) \times \mathcal{I}(\mathcal{V}, \mathcal{W}) \rightarrow \mathcal{I}(\mathcal{U}, \mathcal{W})$$

where the second map is the composition. We can define  $\mathcal{M}B \wedge \mathcal{M}A \in \mathcal{S}(\mathcal{W}, \mathcal{U})$  by

$$(\mathcal{M}B \wedge \mathcal{M}A)(U) = \mathcal{M}B \wedge \mathcal{M}A(U).$$

We claim that

$$\mathcal{M}(B \times A) \cong \mathcal{M}B \wedge \mathcal{M}A$$

canonically in  $\mathcal{S}(\mathcal{W}, \mathcal{U})$ . The second part of the proposition follows from this claim.

For fixed  $U \in \mathcal{C}(\mathcal{U})$  and  $V \in \mathcal{C}(\mathcal{V})$ , define the map

$$\tilde{B}_{V,W} \times \tilde{A}_{U,V} \rightarrow (\widetilde{B \times A})_{U,W}.$$

Namely,

$$(a, Z_a) \times (b, Z_b) \mapsto (b \cdot a, Z_b \dot{+} b(Z_a)).$$

This is covered by a map of bundles, so we have

$$\tilde{B}_{V,W}^{\eta_{V,W}} \wedge \tilde{A}_{U,V}^{\eta_{U,V}} \rightarrow (\widetilde{B \times A})_{U,W}^{\eta_{U,W}}.$$

Varying over  $W \in \mathcal{C}(\mathcal{W})$  and passing to the colimit, this gives a map of  $\mathcal{W}$ -spectra

$$\mathcal{M}B(V) \wedge \mathcal{M}A(U)_V \rightarrow \mathcal{M}(B \times A)(U).$$

For  $V, V' \in \mathcal{C}(\mathcal{V})$ , the  $\mathcal{W}$ -spectra maps commute with any  $T : V \rightarrow V'$  in  $\mathcal{C}(\mathcal{V})$ , so varying over  $V \in \mathcal{C}(\mathcal{V})$  gives a map

$$(\mathcal{M}B \wedge \mathcal{M}A)(U) \rightarrow \mathcal{M}(B \times A)(U).$$

By the usual small subspace and colimit argument, this is an isomorphism of  $\mathcal{W}$ -spectra. These isomorphisms also commute with the structure maps in  $\mathcal{S}(\mathcal{W}, \mathcal{U})$  for any  $Z : U \rightarrow U'$  in  $\mathcal{C}(\mathcal{U})$ .  $\square$

We will also need the following lemma, which says that the twisted half-smash product takes  $\mathbb{A}^1$ -weak equivalences of  $k$ -spaces in the first variable to  $\mathbb{A}^1$ -weak equivalences of spectra.

LEMMA 10.13. *If  $f : A \rightarrow B$  is an  $\mathbb{A}^1$ -weak equivalence of  $k$ -spaces, and  $B \rightarrow \mathcal{I}(\mathcal{U}, \mathcal{V})$ , then for any  $\mathcal{U}$ -spectrum  $E$ ,*

$$f \times E : A \times E \rightarrow B \times E$$

*is an  $\mathbb{A}^1$ -local weak equivalence of  $\mathcal{V}$ -spectra.*

PROOF. First, consider the case of a suspension spectrum  $E = \Sigma_U^{\mathcal{U}} X$ , where  $X \in \text{Spc}(k)_\bullet$ ,  $U \subset \mathcal{U}$  is cofinite. Let  $V \subset \mathcal{V}$  be cofinite, with codimension in  $\mathcal{V}$  the same as the codimension of  $U$  in  $\mathcal{U}$ . Then by the untwisting lemma,  $f \times E$  is just

$$\Sigma_V^{\mathcal{U}}(f \wedge X) : \Sigma_V^{\mathcal{U}}(A \wedge X) \rightarrow \Sigma_V^{\mathcal{U}}(B \wedge X).$$

But  $f \wedge X$  is an  $\mathbb{A}^1$ -local weak equivalence of spaces. Also, as shown in the proof of Proposition 7.2,  $\Sigma_V^{\mathcal{U}}$  takes  $\mathbb{A}^1$ -weak equivalences of  $k$ -spaces to  $\mathbb{A}^1$ -weak equivalences of spectra.

For a general spectrum  $E$ , we have  $E = \text{colim}_{U \in \mathcal{C}(\mathcal{U})} \Sigma_U^{\mathcal{U}} E_U$ . The twisted half-smash product commutes with directed colimits, so  $A \times E \cong \text{colim} A \times \Sigma_U^{\mathcal{U}} E_U$ . By Proposition 5.12,  $\mathbb{A}^1$ -weak equivalences are preserved by directed colimits. So  $f \times E$  is an  $\mathbb{A}^1$ -weak equivalence of spectra.  $\square$

### 11. The Category of $\mathbb{L}$ -spectra

Our next goal is to construct a category of spectra in which the smash product is commutative and associative on the nose. This is the category of  $\mathbb{L}$ -spectra. Let  $\mathcal{U}$  be a fixed  $k$ -universe. Let  $\mathcal{L}$  denote the operad of extended injections, i. e.

$$\mathcal{L}(n) = \mathcal{I}(\mathcal{U}^{\oplus n}, \mathcal{U})$$

for each  $n$ . The structure maps are the obvious compositions. As shown in Section 5,  $\mathcal{L}$  is a spacewise contractible operad. For algebraic  $\mathcal{U}$ -spectra  $E_1, \dots, E_n$ , we think of  $\mathcal{L}(n) \times (E_1 \bar{\wedge} \dots \bar{\wedge} E_n)$  as the  $n$ -fold internal smash product in  $Spectra(\mathcal{U})$ . Here,  $\bar{\wedge}$  denotes the external smash product.

The original construction of  $\mathbb{L}$ -spectra in [6] carries through formally in the algebraic category. Let  $\mathbb{L}$  denote the monad in the category of  $\mathcal{U}$ -spectra given by

$$(11.1) \quad \mathbb{L}E = \mathcal{L}(1) \times E.$$

**DEFINITION 11.2.** *An  $\mathbb{L}$ -spectrum is an algebra over the monad  $\mathbb{L}$ . In other words, it is an  $\mathcal{U}$ -spectrum  $E$ , together with an action  $\xi : \mathcal{L}(1) \times E \rightarrow E$ , which satisfies the usual associativity and unit diagrams.*

If  $X \in Spc(k)_\bullet$ , then the suspension spectrum  $\Sigma^{\mathcal{U}}X$  is naturally an  $\mathbb{L}$ -spectrum. The structure map is

$$\begin{aligned} \mathcal{L}(1) \times \Sigma^{\mathcal{U}}X &\cong \Sigma^{\mathcal{U}}(\mathcal{L}(1)_+ \wedge X) \\ &\rightarrow \Sigma^{\mathcal{U}}X \end{aligned}$$

where the isomorphism is by Proposition 10.11, and the second map just collapses  $\mathcal{L}(1)$  to a point. In particular, the sphere spectrum is an  $\mathbb{L}$ -spectrum.

The category of  $\mathbb{L}$ -spectra is complete and cocomplete, with colimits and limits constructed in  $Spectra(\mathcal{U})$ . We define the smash product of  $\mathbb{L}$ -spectra  $E \wedge_{\mathcal{L}} E'$  to be the coequalizer of the two maps

$$(\mathcal{L}(2) \times \mathcal{L}(1) \times \mathcal{L}(1)) \times (E \bar{\wedge} E') \rightrightarrows \mathcal{L}(2) \times (E \bar{\wedge} E')$$

where the two maps are induced by the  $\mathbb{L}$ -spectrum structures on  $E, E'$  and the operad structure map of  $\mathcal{L}$ .

PROPOSITION 11.3. *For  $\mathcal{U}$ -spectra  $E, E'$ , there is a natural isomorphism*

$$(11.4) \quad E \wedge_{\mathcal{L}} E' \cong E' \wedge_{\mathcal{L}} E.$$

PROOF. The proof in [6] carries through: the symmetric group  $\Sigma_2$  acts on  $\mathcal{L}(2)$ , so if  $\tau : \mathcal{U}^2 \rightarrow \mathcal{U}^2$  is the transposition, then there is an isomorphism

$$\mathcal{L}(2) \xrightarrow{\cong} \mathcal{L}(2) \times \{\tau\}.$$

This induces an isomorphism

$$\mathcal{L}(2) \times (E \overline{\wedge} E') \cong \mathcal{L}(2) \times \tau_*(E \overline{\wedge} E') \cong \mathcal{L}(2) \times (E' \overline{\wedge} E).$$

Likewise, there is an isomorphism

$$(\mathcal{L}(2) \times \mathcal{L}(1) \times \mathcal{L}(1)) \times (E \overline{\wedge} E') \cong (\mathcal{L}(2) \times \mathcal{L}(1) \times \mathcal{L}(1)) \times (E' \overline{\wedge} E)$$

which in addition switches the two copies of  $\mathcal{L}(1)$ . The two maps used to define  $E \wedge_{\mathcal{L}} E'$  commute with these isomorphisms, so there is an induced isomorphism on the coequalizer.  $\square$

We have the following important lemma, the algebraic version of Lemma I.5.4 in [6], which is due to Hopkins.

LEMMA 11.5. *For  $i, j \geq 1$ ,*

$$\mathcal{L}(i + j) \cong \mathcal{L}(2) \times_{\mathcal{L}(1) \times \mathcal{L}(1)} (\mathcal{L}(i) \times \mathcal{L}(j)).$$

PROOF. The original proof of Hopkins in [6] carries through: the diagram

$$\begin{array}{ccc} \mathcal{L}(2) \times \mathcal{L}(1) \times \mathcal{L}(1) \times \mathcal{L}(i) \times \mathcal{L}(j) & \begin{array}{c} \xrightarrow{\gamma \times Id} \\ \xrightarrow{Id \times \gamma} \end{array} & \mathcal{L}(2) \times \mathcal{L}(i) \times \mathcal{L}(j) \\ & & \downarrow \\ & & \mathcal{L}(i + j) \end{array}$$

is a split coequalizer in  $Spc(k)$ . Namely, choose isomorphisms  $f : \mathcal{U}^i \rightarrow \mathcal{U}$  and  $g : \mathcal{U}^j \rightarrow \mathcal{U}$ . Then there is a splitting

$$\alpha : \mathcal{L}(i + j) \rightarrow \mathcal{L}(2) \times \mathcal{L}(i) \times \mathcal{L}(j)$$

given by  $a \mapsto (a \cdot (f^{-1} \oplus g^{-1}), f, g)$ . The other splitting

$$\beta : \mathcal{L}(2) \times \mathcal{L}(i) \times \mathcal{L}(j) \rightarrow \mathcal{L}(2) \times \mathcal{L}(1) \times \mathcal{L}(1) \times \mathcal{L}(i) \times \mathcal{L}(j)$$

is given by  $(b, c, d) \mapsto (b, c \cdot f^{-1}, d \cdot g^{-1}, f, g)$ .

Hence, the diagram is a split coequalizer.  $\square$

This gives the associativity of the smash product of  $\mathbb{L}$ -spectra.

PROPOSITION 11.6. *For  $\mathbb{L}$ -spectra  $E, E'$  and  $E''$ , there is a natural isomorphism*

$$(11.7) \quad (E \wedge_{\mathcal{L}} E') \wedge_{\mathcal{L}} E'' \cong E \wedge_{\mathcal{L}} (E' \wedge_{\mathcal{L}} E'').$$

PROOF. Again, the proof in [6] carries through formally: for any  $\mathbb{L}$ -spectrum  $E$ ,

$$E \cong \mathcal{L}(1) \times_{\mathcal{L}(1)} E.$$

So

$$\begin{aligned} (E \wedge_{\mathcal{L}} E') \wedge_{\mathcal{L}} E'' & \cong \mathcal{L}(2) \times_{\mathcal{L}(1) \times \mathcal{L}(1)} (\mathcal{L}(2) \times_{\mathcal{L}(1) \times \mathcal{L}(1)} E \overline{\wedge} E') \overline{\wedge} (\mathcal{L}(1) \times_{\mathcal{L}(1)} E'') \\ & \cong (\mathcal{L}(2) \times_{\mathcal{L}(1) \times \mathcal{L}(1)} \mathcal{L}(2) \times \mathcal{L}(1)) \times_{\mathcal{L}(1) \times \mathcal{L}(1) \times \mathcal{L}(1)} (E \overline{\wedge} E' \overline{\wedge} E'') \\ & \cong \mathcal{L}(3) \times_{\mathcal{L}(1) \times \mathcal{L}(1) \times \mathcal{L}(1)} (E \overline{\wedge} E' \overline{\wedge} E'') \end{aligned}$$

by Lemma 11.5. The same is true for  $E \wedge_{\mathcal{L}} (E' \wedge_{\mathcal{L}} E'')$ .  $\square$

For a fixed  $\mathbb{L}$ -spectrum  $E$ , the functor  $E \wedge_{\mathcal{L}} -$  has a right adjoint, the function spectrum  $F_{\mathcal{L}}(E, -)$  in the category of  $\mathbb{L}$ -spectra, whose construction is similar to the topological case in [6].

The next goal is to construct a model structure on the category of  $\mathbb{L}$ -spectra. We begin by constructing the simplicial model structure on  $\mathbb{L}$ -spectra from the simplicial model structure on the category of spectra, analogous to the approach in [6].

DEFINITION 11.8. *Let  $f : E \rightarrow G$  be a map of  $\mathbb{L}$ -spectra.*

1.  *$f$  is a simplicial fibration of  $\mathbb{L}$ -spectra if it is a simplicial fibration of spectra;*

2.  $f$  is a simplicial weak equivalence of  $\mathbb{L}$ -spectra if it is a simplicial weak equivalence of spectra;
3.  $f$  is a simplicial cofibration of  $\mathbb{L}$ -spectra if it satisfies the left lifting property with respect to all simplicial acyclic fibrations of  $\mathbb{L}$ -spectra.

We have the following theorem.

**THEOREM 11.9.** *The classes of simplicial cofibrations, weak equivalences, and fibrations form a model structure on the category of  $\mathbb{L}$ -spectra.*

The proof of the theorem is by the small objects argument, which we defer to Section 15.

This gives the simplicial model structure on  $\mathbb{L}$ -spectra. Denote the homotopy category associated with this model structure by  $\mathcal{H}_s(\mathbb{L})$ . Just as in the cases of  $k$ -spaces and spectra, it is not the right homotopy category. Rather, we want to have an  $\mathbb{A}^1$ -local model structure. By the associativity of the twisted half-smash product, if  $E$  is an  $\mathbb{L}$ -spectrum, then so is  $E \wedge \mathbb{A}_+^1$ . Also, the map  $E \wedge \mathbb{A}_+^1 \rightarrow E$ , induced by the projection  $\mathbb{A}_+^1 \rightarrow S^0$ , is a map of  $\mathbb{L}$ -spectra for an  $\mathbb{L}$ -spectrum  $E$ . Therefore, we can take the  $\mathbb{A}^1$ -localization of the simplicial model structure on  $\mathbb{L}$ -spectra, similarly as for spectra. Specifically, we have the following definitions.

**DEFINITION 11.10.** *An  $\mathbb{L}$ -spectrum  $G$  is said to be  $\mathbb{A}^1$ -local if for any spectrum  $E$ , the map of morphism sets in  $\mathcal{H}_s(\mathbb{L})$  induced by  $E \wedge \mathbb{A}_+^1 \rightarrow E$*

$$\mathcal{H}_s(\mathbb{L})(\mathbb{L}E, G) \rightarrow \mathcal{H}_s(\mathbb{L})(\mathbb{L}E \wedge \mathbb{A}_+^1, G)$$

*is a bijection.*

**DEFINITION 11.11.** *A map  $f : E \rightarrow E'$  of  $\mathbb{L}$ -spectra is an  $\mathbb{A}^1$ -equivalence if for every  $\mathbb{A}^1$ -local  $\mathbb{L}$ -spectrum  $G$ , the induced map  $\mathcal{H}_s(\mathbb{L})(f, G)$  in the simplicial homotopy category is a bijection. An  $\mathbb{A}^1$ -cofibration of  $\mathbb{L}$ -spectra is a simplicial cofibration of  $\mathbb{L}$ -spectra. Finally, a map of  $\mathbb{L}$ -spectra is an  $\mathbb{A}^1$ -fibration if it has the right lifting*

property with respect to all cofibrations of  $\mathbb{L}$ -spectra which are also  $\mathbb{A}^1$ -weak equivalences.

We have the following theorem.

**THEOREM 11.12.** *The classes of  $\mathbb{A}^1$ -fibrations,  $\mathbb{A}^1$ -weak equivalences, and cofibrations of  $\mathbb{L}$ -spectra define a model structure on the category of  $\mathbb{L}$ -spectra.*

**PROOF.** Similar to the proof of Theorem 5.5 for the case of  $\mathcal{U}$ -spectra.  $\square$

Analogous to Proposition 5.11, small directed colimits of  $\mathbb{L}$ -spectra preserve  $\mathbb{A}^1$ -weak equivalences.

**PROPOSITION 11.13.** *Let  $\mathcal{I}$  be a small directed category.*

1. *Suppose  $E, E' : \mathcal{I} \rightarrow \mathbb{L}$ -spectra are functors, and  $f : E \rightarrow E'$  is a natural transformation such that  $f_i : E_i \rightarrow E'_i$  is an  $\mathbb{A}^1$ -weak equivalence of  $\mathbb{L}$ -spectra for every  $i \in \mathcal{I}$ . Then the map induced on the colimits*

$$\operatorname{colim}_{i \in \mathcal{I}} E_i \rightarrow \operatorname{colim}_{i \in \mathcal{I}} E'_i$$

*is an  $\mathbb{A}^1$ -weak equivalence of  $\mathbb{L}$ -spectra.*

2. *Suppose  $E : \mathcal{I} \rightarrow \mathbb{L}$ -spectra is a functor such that for every morphism  $i \rightarrow j$  in  $\mathcal{I}$ ,  $E_i \rightarrow E_j$  is an  $\mathbb{A}^1$ -weak equivalence of  $\mathbb{L}$ -spectra. Then for every  $i \in \mathcal{I}$ , the map*

$$E_i \rightarrow \operatorname{colim}_{i \in \mathcal{I}} E_i$$

*is an  $\mathbb{A}^1$ -weak equivalence of  $\mathbb{L}$ -spectra.*

**PROOF.** The proof is analogous to that of Proposition 5.11. For the first part, note that colimits and homotopy colimits of  $\mathbb{L}$ -spectra are created in the category of spectra. Therefore, for any functor  $E : \mathcal{I} \rightarrow \mathbb{L}$ -spectra, the canonical map

$$\operatorname{hocolim}_{i \in \mathcal{I}} E_i \rightarrow \operatorname{colim}_{i \in \mathcal{I}} E_i$$

is a simplicial weak equivalence of spectra, and hence an  $\mathbb{A}^1$ -weak equivalence of  $\mathbb{L}$ -spectra (see [16], Lemma 2.1.14). Thus, it suffices to show that  $f$  induces an  $\mathbb{A}^1$ -weak equivalence on the homotopy colimits. Let  $G$  be any  $\mathbb{A}^1$ -local  $\mathbb{L}$ -spectrum.

Then by a spectral sequence argument like that of Proposition 5.7, the morphism set  $\mathcal{H}_s(\mathbb{L})(\text{hocolim}_{i \in \mathcal{I}} E_i, G)$  is determined (in the sense of Definition 5.6) by the morphism sets  $\mathcal{H}_s(E_i, G)$ , where  $i$  ranges over the objects of  $\mathcal{I}$ . But for every  $i \in \mathcal{I}$ ,  $f$  induces an isomorphism

$$\mathcal{H}_s(\mathbb{L})(E_i, G) \cong \mathcal{H}_s(\mathbb{L})(E'_i, G).$$

Therefore,  $f$  induces an isomorphism of spectral sequences. So it induces an  $\mathbb{A}^1$ -weak equivalence

$$\text{hocolim}_{i \in \mathcal{I}} E_i \rightarrow \text{hocolim}_{i \in \mathcal{I}} E'_i.$$

For the second part, fix an object  $i$  of  $\mathcal{I}$ . Then the claim follows from the first part of the proposition by considering the functor  $E$  and the constant functor  $E_i$ .  $\square$

LEMMA 11.14. *An  $\mathbb{L}$ -spectrum  $G$  is  $\mathbb{A}^1$ -local if and only if for all  $\mathbb{L}$ -spectra of the form  $\mathbb{L}\Sigma_V^{\mathcal{U}}X$ , where  $X$  is a finite colimit of based smooth schemes in  $\text{Spc}(k)_\bullet$ , the map of morphism sets in the simplicial homotopy category*

$$\mathcal{H}_s(\mathbb{L})(\mathbb{L}\Sigma_V^{\mathcal{U}}X, G) \rightarrow \mathcal{H}_s(\mathbb{L})(\mathbb{L}\Sigma_V^{\mathcal{U}}X \wedge \mathbb{A}_+^1, G)$$

*is a bijection.*

PROOF. Similar to the proof of Lemma 5.13. For any spectrum  $E$ , we have  $E = \text{colim}_{V \in \mathcal{C}(\mathcal{U})} \Sigma_V^{\mathcal{U}} E_V$ . The functor  $\mathbb{L} : \mathcal{H}_s \rightarrow \mathcal{H}_s(\mathbb{L})$  is a left adjoint, so it commutes with colimits. By the previous proposition,  $\mathbb{A}^1$ -weak equivalences of  $\mathbb{L}$ -spectra are preserved by directed colimits, so  $G$  is  $\mathbb{A}^1$ -local if and only if for all  $X \in \text{Spc}(k)_\bullet$  and cofinite  $V \subset \mathcal{U}$ , the map of morphism sets

$$\mathcal{H}_s(\mathbb{L})(\mathbb{L}\Sigma_V^{\mathcal{U}}X, G) \rightarrow \mathcal{H}_s(\mathbb{L})(\mathbb{L}\Sigma_V^{\mathcal{U}}X \wedge \mathbb{A}_+^1, G)$$

is a bijection.

To show that, it suffices to check for all  $X \in \text{Spc}(k)_\bullet$  which are finite colimits of based smooth schemes. We use Lemma 17.3 in the Appendix, which states that every  $X \in \text{Spc}(k)_\bullet$  is simplicially weakly equivalent to a homotopy colimit of finite colimits of smooth schemes. The suspension spectrum functor and  $\mathbb{L}$  commute with homotopy colimits, so for every  $X \in \text{Spc}(k)_\bullet$ , there is some small category  $\mathcal{C}$  and a

functor  $S : \mathcal{C} \rightarrow \mathit{Spc}(k)_\bullet$  such that each  $S_c$  is a finite colimit of based smooth schemes, and  $\mathbb{L}\Sigma_V^{\mathcal{U}}X$  is simplicially weakly equivalent to

$$\mathrm{hocolim}_{c \in \mathcal{C}} \mathbb{L}\Sigma_V^{\mathcal{U}}S_c.$$

But by a spectral sequence argument like that of Proposition 5.7, the morphism set  $\mathcal{H}_s(\mathbb{L})(\mathrm{hocolim}_{i \in \mathcal{I}} E_i, G)$  is determined by  $\mathcal{H}_s(\mathbb{L})(E_i, G)$  for  $i \in \mathcal{I}$  in the sense of Definition 5.6.  $\square$

We say that a map of  $\mathbb{L}$ -spectra is an elementary  $\mathbb{A}^1$ -weak equivalence, if it is of the form

$$\mathbb{L}\Sigma_V^{\mathcal{U}}X \wedge \mathbb{A}_+^1 \rightarrow \mathbb{L}\Sigma_V^{\mathcal{U}}X$$

where  $X$  is a finite homotopy colimit of based smooth schemes in  $\mathit{Spc}(k)_\bullet$ , and  $V \subset \mathcal{U}$  is cofinite. Similar to the case of spectra, every  $\mathbb{A}^1$ -weak equivalence of  $\mathbb{L}$ -spectra can be constructed from elementary  $\mathbb{A}^1$ -weak equivalences of  $\mathbb{L}$ -spectra.

**PROPOSITION 11.15.** *The class of  $\mathbb{A}^1$ -local equivalences of  $\mathbb{L}$ -spectra is the smallest class  $\mathbb{W}_{\mathbb{A}^1}(\mathbb{L})$  of maps of  $\mathbb{L}$ -spectra containing all simplicial weak equivalences and satisfying the following conditions:*

1.  $\mathbb{W}_{\mathbb{A}^1}(\mathbb{L})$  contains all elementary  $\mathbb{A}^1$ -weak equivalences.
2. If two out of the three maps  $f$ ,  $g$ , and  $g \cdot f$  are in  $\mathbb{W}_{\mathbb{A}^1}(\mathbb{L})$ , then so is the third.
3.  $\mathbb{W}_{\mathbb{A}^1}(\mathbb{L})$  is closed under pushouts by cofibrations. Its intersection with the class of all cofibrations is closed under all pushouts.
4. Suppose  $\mathcal{I}$  is a small directed category,  $E$  is a functor from  $\mathcal{I}$  to the category of  $\mathbb{L}$ -spectra, and for all  $i \rightarrow j$  in  $\mathcal{I}$ , the map  $E_i \rightarrow E_j$  is in  $\mathbb{W}_{\mathbb{A}^1}(\mathbb{L})$ . Then for each  $i \in \mathcal{I}$ , the induced map  $E_i \rightarrow \mathrm{colim}_{j \in \mathcal{I}} E_j$  is in  $\mathbb{W}_{\mathbb{A}^1}(\mathbb{L})$ .

**PROOF.** Similar to that of Proposition 5.12 for spectra. For each  $\mathbb{L}$ -spectrum  $E$ , we construct the  $\mathbb{A}^1$ -localization of  $E$  by the small objects argument, in the manner of Bousfield [1].  $\square$

This gives the  $\mathbb{A}^1$ -local model structure on the category of  $\mathbb{L}$ -spectra. This is the model structure with which we want to work. Let  $\mathcal{H}_{\mathbb{A}^1}(\mathbb{L})$  denote the homotopy category of  $\mathbb{L}$ -spectra associated with the  $\mathbb{A}^1$ -local model structure.

We also have the following result, which is the analogue of Proposition 5.14 for  $\mathbb{L}$ -spectra.

**PROPOSITION 11.16.** *A map of  $\mathbb{L}$ -spectra  $f : E \rightarrow E'$  is an  $\mathbb{A}^1$ -weak equivalence if and only if for all  $X \in \text{Spc}(k)_\bullet$  such that  $X$  is a finite colimit of based smooth schemes and all cofinite  $V \subset \mathcal{U}$ , the induced map*

$$\mathcal{H}_{\mathbb{A}^1}(\mathbb{L})(\mathbb{L}\Sigma_V^{\mathcal{U}}X, E) \rightarrow \mathcal{H}_{\mathbb{A}^1}(\mathbb{L})(\mathbb{L}\Sigma_V^{\mathcal{U}}X, E')$$

*is a bijection.*

**PROOF.** For every  $\mathbb{L}$ -spectrum  $E$ , consider its simplicial cofibrant model  $C(E)$ , which is isomorphic to  $E$  in the simplicial homotopy category, and thus in the  $\mathbb{A}^1$ -local homotopy category as well. By the proof of Theorem 11.9,  $C(E)$  is constructed as a transfinite directed colimit of spectra  $E(\gamma)$ , where  $\gamma$  are ordinal numbers. In particular,  $E(0) = * = \mathbb{L}*$ . Given  $E(\gamma)$ , the next stage  $E(\gamma + 1)$  is constructed by a pushout diagram of the form

$$\begin{array}{ccc} \coprod A_i & \longrightarrow & E(\gamma) \\ \downarrow & & \downarrow \\ \coprod B_i & \longrightarrow & E(\gamma + 1) \end{array}$$

where  $A_i \rightarrow B_i$  are simplicial cofibrations in the category of spectra. If  $\gamma$  is a limit ordinal, the  $E(\gamma)$  is defined to be

$$E(\gamma) = \text{colim}_{\eta < \gamma} E(\eta).$$

The functor  $\mathbb{L}$  commutes with colimits, so by transfinite induction, the cofibrant model  $C(E)$  is a free  $\mathbb{L}$ -spectrum. Hence,  $f$  is an  $\mathbb{A}^1$ -weak equivalence if and only if for all spectra  $G$ , the induced map

$$\mathcal{H}_{\mathbb{A}^1}(\mathbb{L})(\mathbb{L}G, E) \rightarrow \mathcal{H}_{\mathbb{A}^1}(\mathbb{L})(\mathbb{L}G, E')$$

is a bijection.

The proof that we can reduce to the case where  $G$  is the suspension spectrum of a finite colimit of based schemes is like that of Proposition 5.14: a spectral sequence argument similar to Proposition 5.7.  $\square$

Let  $U$  denote the forgetful functor from the category of  $\mathbb{L}$ -spectra to the category of spectra. We will show that the adjoint functors  $\mathbb{L}$  and  $U$  pass to a pair of adjoint equivalences between the  $\mathbb{A}^1$ -local homotopy categories of spectra and  $\mathbb{L}$ -spectra.

We have the following lemma.

LEMMA 11.17. *The functors  $U$  and  $\mathbb{L}$  pass to a pair of adjoint functors between the  $\mathbb{A}^1$ -local stable homotopy categories*

$$\begin{aligned}\mathbb{L} : \mathcal{H}_{\mathbb{A}^1} &\rightarrow \mathcal{H}_{\mathbb{A}^1}(\mathbb{L}) \\ U : \mathcal{H}_{\mathbb{A}^1}(\mathbb{L}) &\rightarrow \mathcal{H}_{\mathbb{A}^1}\end{aligned}$$

PROOF. The forgetful functor preserves simplicial fibrations and weak equivalences by definition. Thus, it preserves simplicial acyclic fibrations. By the adjunction, the functor  $\mathbb{L}$ -preserves simplicial cofibrations. Therefore, by Lemma 4.15, the forgetful functor and  $\mathbb{L}$  pass to a pair of adjoint functors between the simplicial stable homotopy categories  $\mathcal{H}_s$  and  $\mathcal{H}_s(\mathbb{L})$ .

Suppose  $G$  is an  $\mathbb{A}^1$ -local  $\mathbb{L}$ -spectrum. Then for any spectrum  $E$ , the adjunction in the simplicial homotopy categories gives

$$\begin{aligned}\mathcal{H}_s(E, G) &= \mathcal{H}_s(\mathbb{L})(\mathbb{L}E, G) \\ &= \mathcal{H}_s(\mathbb{L})(\mathbb{L}E \wedge \mathbb{A}_+^1, G) \\ &= \mathcal{H}_s(E \wedge \mathbb{A}_+^1, G).\end{aligned}$$

So after forgetting,  $G$  is also  $\mathbb{A}^1$ -local in the category of spectra. Hence, if  $E \rightarrow E'$  is an  $\mathbb{A}^1$ -weak equivalence of spectra, then for any  $\mathbb{A}^1$ -local  $\mathbb{L}$ -spectrum  $G$ ,

$$\begin{aligned}\mathcal{H}_s(\mathbb{L})(\mathbb{L}E', G) &= \mathcal{H}_s(E', G) \\ &= \mathcal{H}_s(E, G) \\ &= \mathcal{H}_s(\mathbb{L})(\mathbb{L}E, G)\end{aligned}$$

by adjunction. Therefore,  $\mathbb{L}E \rightarrow \mathbb{L}E'$  is an  $\mathbb{A}^1$ -weak equivalence of  $\mathbb{L}$ -spectra. Since  $\mathbb{A}^1$ -cofibrations of both spectra and  $\mathbb{L}$ -spectra are just cofibrations in the respective simplicial model structures,  $\mathbb{L}$  also preserves cofibrations in the  $\mathbb{A}^1$ -local model structures. This shows that  $\mathbb{L}$  and  $U$  form a Quillen adjoint pair with respect to the  $\mathbb{A}^1$ -local model structures, so the statement follow from Lemma 4.15.  $\square$

**THEOREM 11.18.** *The functors  $\mathbb{L}$  and  $U$  pass to a pair of adjoining Quillen equivalences between the  $\mathbb{A}^1$ -local stable categories  $\mathcal{H}_{\mathbb{A}^1}(\mathbb{L})$  and  $\mathcal{H}_{\mathbb{A}^1}$ .*

**PROOF.** We first show that  $U$  creates  $\mathbb{A}^1$ -weak equivalences of  $\mathbb{L}$ -spectra. Every  $\mathbb{A}^1$ -weak equivalence of  $\mathbb{L}$ -spectra can be constructed recursively from elementary  $\mathbb{A}^1$ -weak equivalences of  $\mathbb{L}$ -spectra, via conditions 2,3, and 4 of Proposition 11.15. An elementary  $\mathbb{A}^1$ -weak equivalence of  $\mathbb{L}$ -spectra is an  $\mathbb{A}^1$ -weak equivalence of spectra. Also, the forgetful functor from the category of  $\mathbb{L}$ -spectra to the category of spectra preserves colimits. Thus, every  $\mathbb{A}^1$ -weak equivalence of  $\mathbb{L}$ -spectra is constructed in the category of spectra from elementary  $\mathbb{A}^1$ -weak equivalences, i. e. it is an  $\mathbb{A}^1$ -weak equivalence of spectra.

Conversely, suppose  $f : E \rightarrow E'$  is a map of  $\mathbb{L}$ -spectra, and  $Uf$  is an  $\mathbb{A}^1$ -weak equivalence of spectra. Then for every spectrum  $G$ ,  $f$  induces a bijection

$$\mathcal{H}_{\mathbb{A}^1}(G, E) \cong \mathcal{H}_{\mathbb{A}^1}(G, E').$$

By adjunction, this gives

$$\mathcal{H}_{\mathbb{A}^1}(\mathbb{L})(\mathbb{L})(G, E) \cong \mathcal{H}_{\mathbb{A}^1}(\mathbb{L})(G, E').$$

So by Proposition 11.16,  $f$  is an  $\mathbb{A}^1$ -weak equivalence of  $\mathbb{L}$ -spectra.

For any spectrum  $E$ , the unit of the adjunction  $E \rightarrow \mathbb{L}E$  is an  $\mathbb{A}^1$ -weak equivalence of spectra since  $\mathcal{L}(1)$  is  $\mathbb{A}^1$ -contractible. The theorem then follows from Lemma 4.16.  $\square$

## 12. Unital Properties of $\mathbb{L}$ -spectra

For any  $\mathbb{L}$ -spectrum  $E$ , there is a natural map

$$(12.1) \quad \lambda : S \wedge_{\mathcal{L}} E \rightarrow E$$

just as for topological spectra [6]. To define  $\lambda$ , note that  $S = \mathcal{L}(0) \rtimes S^0$ . So if  $E = \mathbb{L}E'$  for some spectrum  $E'$ , we define  $\lambda$  by

$$\begin{aligned} S \wedge_{\mathcal{L}} E &= \mathcal{L}(2) \times_{\mathcal{L}(1)^2} (\mathcal{L}(0) \rtimes S^0) \wedge (\mathcal{L}(1) \rtimes E') \\ &\cong (\mathcal{L}(2) \times_{\mathcal{L}(1)^2} \mathcal{L}(0) \times \mathcal{L}(1)) \rtimes (S^0 \wedge E') \\ &\rightarrow \mathcal{L}(1) \rtimes E'. \end{aligned}$$

For a general  $\mathbb{L}$ -spectrum  $E$ , we use the fact that the diagram

$$\mathbb{L}\mathbb{L}E \rightrightarrows \mathbb{L}E \longrightarrow E$$

is a coequalizer. So  $\lambda : S \wedge_{\mathcal{L}} E \rightarrow E$  can be defined to be the map induced by the  $\lambda$ 's on  $\mathbb{L}\mathbb{L}E$  and on  $\mathbb{L}E$ .

We have the following formal property of  $\lambda$  from [6].

**PROPOSITION 12.2** ([6], Proposition I.8.3.). *Let  $E$  and  $G$  be  $\mathbb{L}$ -spectra. Under the associativity isomorphism, the map*

$$\lambda : S \wedge_{\mathcal{L}} (E \wedge_{\mathcal{L}} G) \rightarrow E \wedge_{\mathcal{L}} G$$

*agrees with*

$$\lambda \wedge_{\mathcal{L}} Id : (S \wedge_{\mathcal{L}} E) \wedge_{\mathcal{L}} G \rightarrow E \wedge_{\mathcal{L}} G.$$

In general,  $\lambda$  is not an isomorphism. However, we do have that it is an isomorphism in the  $\mathbb{A}^1$ -local homotopy category.

**PROPOSITION 12.3.** *For an  $\mathbb{L}$ -spectrum  $E$ , the map*

$$\lambda : S \wedge_{\mathcal{L}} E \rightarrow E$$

*is an  $\mathbb{A}^1$ -weak equivalence of  $\mathbb{L}$ -spectra. Therefore, the functor  $S \wedge_{\mathcal{L}} -$  preserves  $\mathbb{A}^1$ -weak equivalences.*

The proposition follows from the next lemma, whose proof we defer to Section 16.

LEMMA 12.4. *The  $k$ -space  $\mathcal{L}(2) \times_{\mathcal{L}(1) \times \mathcal{L}(1)} \mathcal{L}(0) \times \mathcal{L}(1)$  is  $\mathbb{A}^1$ -contractible.*

PROOF OF PROPOSITION 12.3: First, assume that  $E = \mathbb{L}E'$  for some spectrum  $E'$ . The map

$$(12.5) \quad \lambda : S \wedge_{\mathcal{L}} E \rightarrow E$$

is just

$$(\mathcal{L}(2) \times_{\mathcal{L}(1)^2} \mathcal{L}(0) \times \mathcal{L}(1)) \rtimes E' \rightarrow \mathcal{L}(1) \rtimes E'.$$

So by Lemmas 12.4 and 10.13, it is an  $\mathbb{A}^1$ -weak equivalence of spectra. By the proof of Theorem 11.18, the forgetful functor creates  $\mathbb{A}^1$ -weak equivalences of  $\mathbb{L}$ -spectra, so it has to be an  $\mathbb{A}^1$ -weak equivalence of  $\mathbb{L}$ -spectra as well.

In the general case, let  $X \in \mathit{Spc}(k)_\bullet$  be a finite colimit of based smooth schemes. Consider the following diagram

$$\begin{array}{ccc} \mathcal{H}_{\mathbb{A}^1}(\mathbb{L})(\mathbb{L}\Sigma_V^{\mathcal{U}}X, S \wedge_{\mathcal{L}} E) & \xrightarrow{\lambda^*} & \mathcal{H}_{\mathbb{A}^1}(S \wedge_{\mathcal{L}} \mathbb{L}\Sigma_V^{\mathcal{U}}X, S \wedge_{\mathcal{L}} E) \\ \lambda_* \downarrow & & \downarrow \lambda_* \\ \mathcal{H}_{\mathbb{A}^1}(\mathbb{L})(\mathbb{L}\Sigma_V^{\mathcal{U}}X, E) & \xrightarrow{\lambda^*} & \mathcal{H}_{\mathbb{A}^1}(S \wedge_{\mathcal{L}} \mathbb{L}\Sigma_V^{\mathcal{U}}X, E). \end{array}$$

By the previous paragraph, the two horizontal maps are bijections. As we will see in the next section,  $\lambda : S \wedge_{\mathcal{L}} S \rightarrow S$  is an isomorphism. Hence, the right hand side is a bijection, whose inverse sends a map  $f : S \wedge_{\mathcal{L}} \mathbb{L}\Sigma_V^{\mathcal{U}}X \rightarrow E$  to

$$S \wedge_{\mathcal{L}} \mathbb{L}\Sigma_V^{\mathcal{U}}X \xrightarrow{\lambda^{-1} \wedge Id} S \wedge_{\mathcal{L}} S \wedge_{\mathcal{L}} \mathbb{L}\Sigma_V^{\mathcal{U}}X \xrightarrow{Id \wedge f} S \wedge_{\mathcal{L}} E.$$

Hence, the left hand side map is a bijection. So by Proposition 11.16,  $\lambda : S \wedge_{\mathcal{L}} E \rightarrow E$  is an  $\mathbb{A}^1$ -weak equivalence of  $\mathbb{L}$ -spectra for arbitrary  $\mathbb{L}$ -spectra  $E$ .  $\square$

Thus,  $\mathbb{L}$ -spectra are unital in  $\mathcal{H}_{\mathbb{A}^1}(\mathbb{L})$ . This leads to the following result.

LEMMA 12.6. *For any  $\mathbb{L}$ -spectrum  $E$ , the map  $E \rightarrow F_{\mathcal{L}}(S, E)$  adjoint to  $\lambda$  is an  $\mathbb{A}^1$ -weak equivalence of  $\mathbb{L}$ -spectra.*

PROOF. By Proposition 12.3, the functor  $S \wedge_{\mathcal{L}} -$  preserves  $\mathbb{A}^1$ -weak equivalences of  $\mathbb{L}$ -spectra. It also preserves cofibrations of  $\mathbb{L}$ -spectra, which are just simplicial cofibrations. Hence, the functors  $S \wedge_{\mathcal{L}} -$  and  $F_{\mathcal{L}}(S, -)$  pass to a pair of adjoints in the  $\mathbb{A}^1$ -local stable homotopy category  $\mathcal{H}_{\mathbb{A}^1}(\mathbb{L})$ . For any  $X \in \text{Spc}(k)_{\bullet}$  which is a finite colimit of smooth schemes, and  $V \subset \mathcal{U}$  cofinite, we have

$$\begin{aligned} \mathcal{H}_{\mathbb{A}^1}(\mathbb{L})(\mathbb{L}\Sigma_V^{\mathcal{U}}X, E) &\xrightarrow{\cong} \mathcal{H}_{\mathbb{A}^1}(\mathbb{L})(S \wedge_{\mathcal{L}} \mathbb{L}\Sigma_V^{\mathcal{U}}X, E) \\ &\cong \mathcal{H}_{\mathbb{A}^1}(\mathbb{L})(\mathbb{L}\Sigma_V^{\mathcal{U}}X, F_{\mathcal{L}}(S, E)). \end{aligned}$$

The conclusion follows by Proposition 11.16.  $\square$

### 13. The Category of $S$ -modules

Our category of  $\mathbb{L}$ -spectra has a smash product that is strictly associative and commutative. However, it is not strictly unital in the sense that there is not an isomorphism

$$E \cong S \wedge_{\mathcal{L}} E$$

for any  $\mathbb{L}$ -spectrum  $E$ , where  $S$  denotes the sphere spectrum. Rather, there is a natural map of  $\mathbb{L}$ -spectra going the other way:

$$\lambda : S \wedge_{\mathcal{L}} E \rightarrow E$$

for any  $\mathbb{L}$ -spectrum  $E$ .

This suggests the following definition of  $S$ -modules, analogous to that of [6], on which the smash product is strictly unital as well.

**DEFINITION 13.1.** *An  $S$ -module  $E$  is an  $\mathbb{L}$ -spectrum such that the map  $\lambda : S \wedge_{\mathcal{L}} E \rightarrow E$  is an isomorphism. The category of  $S$ -modules is the full subcategory of  $\mathbb{L}$ -spectra whose objects are the  $S$ -modules.*

First, the sphere spectrum  $S$  itself is an  $S$ -module.

**PROPOSITION 13.2.** *For any  $X \in \text{Spc}(k)_{\bullet}$ , the map  $\lambda : S \wedge_{\mathcal{L}} \Sigma^{\mathcal{U}}X \rightarrow \Sigma^{\mathcal{U}}X$  is an isomorphism. In particular,  $\lambda : S \wedge_{\mathcal{L}} S \rightarrow S$  is an isomorphism.*

We have the next technical lemma.

LEMMA 13.3. *The coequalizer  $\mathcal{L}(2) \times_{\mathcal{L}(1) \times \mathcal{L}(1)} \mathcal{L}(0) \times \mathcal{L}(0)$  is  $\text{Spec}(k)$ .*

We will defer the proof of Lemma 13.3 to Section 16. Given the lemma, Proposition 13.2 follows.

PROOF OF PROPOSITION 13.2. For a based  $k$ -space  $X \in \text{Spc}(k)_\bullet$ , the suspension spectrum is  $\Sigma^{\mathcal{U}}X = \mathcal{L}(0) \times X$ . For the based  $k$ -space  $S^0$ , let  $\eta : S^0 \wedge X \rightarrow X$  be the obvious unit isomorphism. Then  $\lambda$  on  $\Sigma^{\mathcal{U}}X$  is just

$$\gamma \times \eta : (\mathcal{L}(2) \times_{\mathcal{L}(1)^2} \mathcal{L}(0) \times \mathcal{L}(0)) \times (S^0 \wedge X) \rightarrow \mathcal{L}(0) \times X.$$

So by Lemma 13.3,  $\lambda$  is an isomorphism. In particular, this holds for the sphere spectrum  $S = \Sigma^{\mathcal{U}}S^0$ .

□

By the above proposition and Proposition 12.2, the functor  $S \wedge_{\mathcal{L}} -$  takes an  $\mathbb{L}$ -spectrum to an  $S$ -module. This functor has both a right and a left adjoint. The right adjoint is the function spectrum functor  $F_{\mathcal{L}}(S, -)$ . The left adjoint, as in [6], is just the forgetful functor from the category of  $S$ -modules to the category of  $\mathbb{L}$ -spectra. In particular, the category of  $S$ -modules is complete and cocomplete. The colimits are just colimits of  $\mathbb{L}$ -spectra, and the limits are given by first taking limits in the category of  $\mathbb{L}$ -spectra, then applying the functor  $S \wedge_{\mathcal{L}} -$ . For  $S$ -modules  $E$  and  $G$ , we define smash products and function spectra in the category of  $S$ -modules as follows, as in [6]:

$$(13.4) \quad E \wedge_S G = E \wedge_{\mathcal{L}} G$$

$$(13.5) \quad F_S(E, G) = S \wedge_{\mathcal{L}} F_{\mathcal{L}}(E, G).$$

We would like to construct a compatible model structure on the category of  $S$ -modules. Again, we start by constructing the simplicial model structure on the category of  $S$ -modules. We want the simplicial fibrations and weak equivalences to be inherited from the category of  $\mathbb{L}$ -spectra. The right adjoint of  $S \wedge_{\mathcal{L}} -$  is  $F_{\mathcal{L}}(S, -)$

instead of the forgetful functor. Therefore, instead of using the forgetful functor to define simplicial fibrations and weak equivalences from simplicial fibrations and weak equivalences of  $\mathbb{L}$ -spectra, we should use  $F_{\mathcal{L}}(S, -)$ . We make the following definition.

DEFINITION 13.6. *Let  $f : E \rightarrow G$  be a map of  $S$ -modules.*

1.  *$f$  is a simplicial fibration of  $S$ -modules if  $F_{\mathcal{L}}(S, f)$  is a simplicial fibration of  $\mathbb{L}$ -spectra.*

2.  *$f$  is a simplicial weak equivalence of  $S$ -modules if  $F_{\mathcal{L}}(S, f)$  is a simplicial weak equivalence of  $\mathbb{L}$ -spectra.*

3.  *$f$  is a simplicial cofibration of  $S$ -modules if it satisfies the left lifting property with respect to all simplicial acyclic fibrations of  $S$ -modules.*

THEOREM 13.7. *The classes of simplicial fibrations, weak equivalences and cofibrations define a model structure on the category of  $S$ -modules.*

The proof of the theorem is similar to that for  $\mathbb{L}$ -spectra. Again, we defer it to Section 15.

Let  $\mathcal{H}_s(S)$  denote the homotopy category associated with the simplicial model structure on the category of  $S$ -modules. Again, this is not the correct homotopy category, and we need to apply  $\mathbb{A}^1$ -localization, as in Definition 5.3.

DEFINITION 13.8. *An  $S$ -module  $G$  is  $\mathbb{A}^1$ -local if*

$$\mathcal{H}_s(S \wedge_{\mathcal{L}} \mathbb{L}E, G) \rightarrow \mathcal{H}_s(S)((S \wedge_{\mathcal{L}} \mathbb{L}E) \wedge \mathbb{A}_+^1, G)$$

*is a bijection for all spectra  $E$ .*

This allows us to define the  $\mathbb{A}^1$ -local model structure on  $S$ -modules.

DEFINITION 13.9. *A map  $f : E \rightarrow E'$  of  $S$ -modules is an  $\mathbb{A}^1$ -weak equivalence if for every  $\mathbb{A}^1$ -local  $S$ -module  $G$ , the induced map*

$$\mathcal{H}_s(S)(E', G) \rightarrow \mathcal{H}_s(S)(E, G)$$

is a bijection. It is an  $\mathbb{A}^1$ -cofibration of  $S$ -modules if it is a simplicial cofibration of  $S$ -modules. It is an  $\mathbb{A}^1$ -fibration of  $S$ -modules if it has the right lifting property with respect to all acyclic  $\mathbb{A}^1$ -cofibrations of  $S$ -modules.

We have the following model structure theorem for  $S$ -modules.

**THEOREM 13.10.** *The classes of  $\mathbb{A}^1$ -weak equivalences,  $\mathbb{A}^1$ -fibrations and cofibrations of  $S$ -modules define a model structure on the category of  $S$ -modules.*

**PROOF.** Similar to Theorems 5.5 and 11.12 in the cases of spectra and  $\mathbb{L}$ -spectra.  $\square$

This gives the  $\mathbb{A}^1$ -local model structure on  $S$ -modules, which is the right structure with which to work. Let  $\mathcal{H}_{\mathbb{A}^1}(S)$  denote the homotopy category associated with the  $\mathbb{A}^1$ -local model structure on  $S$ -modules.

Consider the adjoint functors  $S \wedge_{\mathcal{L}} -$  and  $F_{\mathcal{L}}(S, -)$ . We will show that they pass to equivalences of categories between  $\mathcal{H}_{\mathbb{A}^1}(\mathbb{L})$  and  $\mathcal{H}_{\mathbb{A}^1}(S)$ . First, note the following.

**LEMMA 13.11.** *The functors  $S \wedge_{\mathcal{L}} -$  and  $F_{\mathcal{L}}(S, -)$  are a Quillen adjoint pair.*

**PROOF.** By definition,  $F_{\mathcal{L}}(S, -)$  takes simplicial weak equivalences and fibrations of  $S$ -modules to simplicial weak equivalences and fibrations of  $\mathbb{L}$ -spectra. The adjunction and the fact that  $F_{\mathcal{L}}(S, -)$  preserves simplicial acyclic fibrations imply that  $S \wedge_{\mathcal{L}} -$  preserves simplicial cofibrations. Therefore, by Lemma 4.15,  $S \wedge_{\mathcal{L}} -$  and  $F_{\mathcal{L}}(S, -)$  pass to a pair of adjoint functors between the simplicial stable homotopy categories  $\mathcal{H}_s$  and  $\mathcal{H}_s(\mathbb{L})$ .

If  $G$  is an  $\mathbb{A}^1$ -local  $S$ -module, then for any spectrum  $E$ , the adjunction in the simplicial homotopy categories gives

$$\begin{aligned} \mathcal{H}_s(\mathbb{L})(\mathbb{L}E, F_{\mathcal{L}}(S, G)) &= \mathcal{H}_s(S)(S \wedge_{\mathcal{L}} \mathbb{L}E, G) \\ &= \mathcal{H}_s(S)(S \wedge_{\mathcal{L}} \mathbb{L}E \wedge \mathbb{A}_+^1, G) \\ &= \mathcal{H}_s(\mathbb{L})(\mathbb{L}E \wedge \mathbb{A}_+^1, F_{\mathcal{L}}(S, G)). \end{aligned}$$

So  $F_{\mathcal{L}}(S, -)$  takes  $G$  to an  $\mathbb{A}^1$ -local  $\mathbb{L}$ -spectrum. Hence, given an  $\mathbb{A}^1$ -weak equivalence of  $\mathbb{L}$ -spectra  $f : E \rightarrow E'$ , and an  $\mathbb{A}^1$ -local  $S$ -module  $G$ , we have

$$\begin{aligned} \mathcal{H}_s(S)(S \wedge_{\mathcal{L}} E', G) &= \mathcal{H}_s(\mathbb{L})(E', F_{\mathcal{L}}(S, G)) \\ &= \mathcal{H}_s(\mathbb{L})(E, F_{\mathcal{L}}(S, G)) \\ &= \mathcal{H}_s(S)(S \wedge_{\mathcal{L}} E, G). \end{aligned}$$

Thus,  $S \wedge_{\mathcal{L}} -$  takes an  $\mathbb{A}^1$ -weak equivalence of  $\mathbb{L}$ -spectra to an  $\mathbb{A}^1$ -weak equivalence of  $S$ -modules. Also, since  $\mathbb{A}^1$ -cofibrations of  $S$ -modules and  $\mathbb{L}$ -spectra are just simplicial cofibrations,  $S \wedge_{\mathcal{L}} -$  preserves cofibrations as well.  $\square$

Similar to the cases for spectra and  $\mathbb{L}$ -spectra in Propositions 5.11 and 11.13, we have the following result for  $S$ -modules.

**PROPOSITION 13.12.** *Let  $\mathcal{I}$  be a small directed category.*

1. *Suppose  $E, E' : \mathcal{I} \rightarrow S$ -modules are functors, and  $f : E \rightarrow E'$  is a natural transformation such that for every  $i \in \mathcal{I}$ ,  $f_i : E_i \rightarrow E'_i$  is an  $\mathbb{A}^1$ -weak equivalence of  $S$ -modules. Then the map induced by  $f$  on the colimits*

$$\operatorname{colim}_{i \in \mathcal{I}} E_i \rightarrow \operatorname{colim}_{i \in \mathcal{I}} E'_i$$

*is an  $\mathbb{A}^1$ -weak equivalence of  $S$ -modules.*

2. *Suppose  $E : \mathcal{I} \rightarrow S$ -modules is a functor, such that for every morphism  $i \rightarrow j$ ,  $E_i \rightarrow E_j$  is an  $\mathbb{A}^1$ -weak equivalence. Then for every  $i \in \mathcal{I}$ , the map*

$$E_i \rightarrow \operatorname{colim}_{i \in \mathcal{I}} E_i$$

*is an  $\mathbb{A}^1$ -weak equivalence of  $S$ -modules.*

**PROOF.** Note that by a spectral sequence argument similar to that of Proposition 5.7, small homotopy colimits of  $S$ -modules preserve  $\mathbb{A}^1$ -weak equivalences. Thus, for the first part, it suffices to show that for any  $E : \mathcal{I} \rightarrow S$ -modules, the canonical map

$$(13.13) \quad \operatorname{hocolim}_{i \in \mathcal{I}} E_i \rightarrow \operatorname{colim}_{i \in \mathcal{I}} E_i$$

is an  $\mathbb{A}^1$ -weak equivalence of  $S$ -modules. Homotopy colimits and colimits of  $S$ -modules are created in the category of  $\mathbb{L}$ -spectra, so by Proposition 11.13, it is an

$\mathbb{A}^1$ -weak equivalence of  $\mathbb{L}$ -spectra. By Lemma 13.11,  $S \wedge_{\mathcal{L}}$  takes this to an  $\mathbb{A}^1$ -weak equivalence of  $S$ -modules. But  $\text{hocolim}_{i \in \mathcal{I}} E_i$  and  $\text{colim}_{i \in \mathcal{I}} E_i$  are  $S$ -modules, so the map 13.13 itself is an  $\mathbb{A}^1$ -weak equivalence of  $S$ -modules.

For part two, fix  $i \in \mathcal{I}$ . Then the claim follows by applying part one of the proposition to the constant functor  $E_i$  and  $E$ .  $\square$

Say that a map of  $S$ -modules is an elementary  $\mathbb{A}^1$ -weak equivalence if it is of the form

$$S \wedge_{\mathcal{L}} \mathbb{L}\Sigma_V^{\mathcal{U}} X \wedge \mathbb{A}_+^1 \rightarrow S \wedge_{\mathcal{L}} \mathbb{L}\Sigma_V^{\mathcal{U}} X.$$

Note that the collection of isomorphism classes of elementary  $\mathbb{A}^1$ -weak equivalences of  $S$ -modules form a set. We have the following characterization of  $\mathbb{A}^1$ -weak equivalences of  $S$ -modules as maps built up from elementary  $\mathbb{A}^1$ -weak equivalences of  $S$ -modules, analogous to Propositions 5.12 and 11.15.

**PROPOSITION 13.14.** *The class of  $\mathbb{A}^1$ -weak equivalences of  $S$ -modules is the smallest class  $\mathbb{W}_{\mathbb{A}^1}(S)$  of maps of  $S$ -modules containing all simplicial weak equivalences and satisfying the following conditions:*

1.  $\mathbb{W}_{\mathbb{A}^1}(S)$  contains all the elementary  $\mathbb{A}^1$ -weak equivalences of  $S$ -modules.
2. If  $f$  and  $g$  are composable maps, and two out of the three maps  $f, g$  and  $g \cdot f$  are in  $\mathbb{W}_{\mathbb{A}^1}(S)$ , then so is the third.
3.  $\mathbb{W}_{\mathbb{A}^1}(S)$  is closed under pushouts by cofibrations. Its intersection with the class of cofibrations is closed under all pushouts.
4.  $\mathbb{W}_{\mathbb{A}^1}(S)$  is closed under small directed colimits.

**PROOF.** This is similar to the proofs in the cases of spectra and  $\mathbb{L}$ -spectra (see Proposition 5.12).  $\square$

We now return to the Quillen adjoints  $S \wedge_{\mathcal{L}} -$  and  $F_{\mathcal{L}} -$  between the categories of  $\mathbb{L}$ -spectra and  $S$ -modules.

**THEOREM 13.15.** *The functors  $S \wedge_{\mathcal{L}} -$  and  $F_{\mathcal{L}}(S, -)$  pass to a pair of adjoint Quillen equivalences between  $\mathcal{H}_{\mathbb{A}^1}(S)$  and  $\mathcal{H}_{\mathbb{A}^1}(\mathbb{L})$ .*

PROOF. First, we show that  $F_{\mathcal{L}}(S, -)$  creates  $\mathbb{A}^1$ -weak equivalences in the category of  $S$ -modules. Just as in the cases of spectra and  $\mathbb{L}$ -spectra, an  $S$ -module  $G$  is  $\mathbb{A}^1$ -local if and only if

$$\mathcal{H}_s(S)(S \wedge_{\mathcal{L}} \mathbb{L}\Sigma_V^{\mathcal{U}} X, G) \rightarrow \mathcal{H}_s(S)(S \wedge_{\mathcal{L}} \mathbb{L}\Sigma_V^{\mathcal{U}} X \wedge \mathbb{A}_+^1, G)$$

is a bijection for every  $X \in \text{Spc}(k)_{\bullet}$  which is a finite colimit of based smooth schemes, and every cofinite  $V \subset \mathcal{U}$ . By Proposition 13.14,  $\mathbb{A}^1$ -weak equivalences of  $S$ -modules are constructed recursively from elementary  $\mathbb{A}^1$ -weak equivalences. The map of  $\mathbb{L}$ -spectra  $\mathbb{L}\Sigma_V^{\mathcal{U}} X \wedge \mathbb{A}_+^1 \rightarrow \mathbb{L}\Sigma_V^{\mathcal{U}} X$  is an  $\mathbb{A}^1$ -weak equivalence of  $\mathbb{L}$ -spectra. By Proposition 12.3, the functor  $S \wedge_{\mathcal{L}} -$  preserves  $\mathbb{A}^1$ -weak equivalences of  $\mathbb{L}$ -spectra. Thus, an elementary  $\mathbb{A}^1$ -weak equivalence of  $S$ -modules is an  $\mathbb{A}^1$ -weak equivalence of  $\mathbb{L}$ -spectra. Since the forgetful functor from  $S$ -modules to  $\mathbb{L}$ -spectra is a left adjoint, it preserves colimits, and each  $\mathbb{A}^1$ -weak equivalence of  $S$ -modules is constructed in the category  $\mathbb{L}$ -spectra from elementary  $\mathbb{A}^1$ -weak equivalences. Thus, the forgetful functor takes an  $\mathbb{A}^1$ -weak equivalence of  $S$ -modules to an  $\mathbb{A}^1$ -weak equivalence of  $\mathbb{L}$ -spectra. By Lemma 12.6,  $E$  is  $\mathbb{A}^1$ -weak equivalent to  $F_{\mathcal{L}}(S, E)$  for any  $\mathbb{L}$ -spectrum  $E$ . So  $F_{\mathcal{L}}(S, -)$  also takes an  $\mathbb{A}^1$ -weak equivalence of  $S$ -modules to an  $\mathbb{A}^1$ -weak equivalence of  $\mathbb{L}$ -spectra.

Conversely, suppose  $f : E \rightarrow E'$  is a map of  $S$ -modules, and  $F_{\mathcal{L}}(S, f)$  is an  $\mathbb{A}^1$ -weak equivalence of  $\mathbb{L}$ -spectra. Then  $f$  itself is an  $\mathbb{A}^1$ -weak equivalence of  $\mathbb{L}$ -spectra. Applying  $S \wedge_{\mathcal{L}} -$  shows that  $f$  is an  $\mathbb{A}^1$ -weak equivalence of  $S$ -modules.

Now consider the unit of the adjunction  $E \rightarrow F_{\mathcal{L}}(S, S \wedge_{\mathcal{L}} E)$ , where  $E$  is any  $\mathbb{L}$ -spectrum. For each  $\mathbb{L}$ -spectrum  $E$ , we have shown that its simplicial cofibrant model  $C(E)$  is a free  $\mathbb{L}$ -spectrum. Since a simplicial weak equivalence is an  $\mathbb{A}^1$ -weak equivalence, we can assume that  $E = \mathbb{L}E'$  for some spectrum  $E'$ . In this case, we have

$$\begin{array}{ccc} E & \xrightarrow{\quad} & F_{\mathcal{L}}(S, S \wedge_{\mathcal{L}} E) \\ & \swarrow \lambda & \nearrow \\ & S \wedge_{\mathcal{L}} E & \end{array}$$

where  $\lambda$  is an  $\mathbb{A}^1$ -weak equivalence of  $\mathbb{L}$ -spectra. The map at the right side is an  $\mathbb{A}^1$ -weak equivalence of  $\mathbb{L}$ -spectra for all  $\mathbb{L}$ -spectra, so the unit of the adjunction is an  $\mathbb{A}^1$ -weak equivalence in the category of  $\mathbb{L}$ -spectra. The theorem follows from Lemma 4.16.

□

Thus, for the purposes of doing homotopy theory, we can use the stable homotopy categories of spectra,  $\mathbb{L}$ -spectra, or  $S$ -modules interchangeably. The category of  $S$ -modules is a symmetric monoidal category (not just up to homotopy). The constructions in the category of topological  $S$ -modules of [6] carry through.

#### 14. $S$ -algebras and their Modules

From the symmetric monoidal structure on the category of  $S$ -modules, we can make sense of monoids in this category. This section follows closely the ideas of [6].

**DEFINITION 14.1.** *An  $S$ -algebra  $E$  is a monoid in the category of  $S$ -modules, i. e. there is a multiplication map*

$$\mu : E \wedge_S E \rightarrow E$$

*which satisfies the usual associativity, and unit diagrams.*

In particular, the sphere spectrum  $S$  is an  $S$ -algebra, and the category of modules over the  $S$ -algebra  $S$  is the old category of  $S$ -modules.

Let  $E$  be a given  $S$ -algebra. Then we also have the notion of modules over  $E$ . A left  $E$  module  $M$  is an  $S$ -module with a multiplication map

$$E \wedge_S M \rightarrow E$$

satisfying the usual associativity and unit diagrams. The notion of a right  $E$ -module is defined similarly. Given a left  $E$ -module  $M$  and a right  $E$ -module  $N$ , the smash product over  $E$  is given by the coequalizer diagram in the category of  $S$ -modules:

$$M \wedge_S E \wedge_S N \rightrightarrows M \wedge_S N \longrightarrow M \wedge_E N$$

where the two maps are the module structure maps of  $M$  and  $N$ , respectively.  $M \wedge_E N$  is a priori an  $S$ -module. There is a canonical isomorphism

$$E \wedge_E N \xrightarrow{\cong} N$$

for any left  $E$ -module  $N$ , coming from the definition. A similar statement holds for right  $E$ -modules.

Given a spectrum  $X$ , a natural construction is that of the free (left)  $E$ -module  $\mathbb{F}_E X$  generated by  $X$ . If  $E = S$ , then

$$\mathbb{F}_S X = S \wedge_{\mathcal{L}} \mathbb{L}X.$$

For a general  $E$ , we have

$$\mathbb{F}_E X = E \wedge_S (S \wedge_{\mathcal{L}} \mathbb{L}X).$$

This is the left adjoint to the functor from (left)  $E$ -modules to spectra which takes an  $E$ -module  $M$  to the spectrum  $F_{\mathcal{L}}(S, M)$ . In particular, the sphere spectrum in the category of  $E$ -modules is

$$S_E = \mathbb{F}_E S.$$

There is an  $\mathbb{A}^1$ -local closed model structure on the category of  $E$ -modules, induced from the close model structure on  $S$ -modules, as in [6]. Let  $\mathcal{H}_{\mathbb{A}^1}(E)$  denote the homotopy category associated with this structure.

In the category of  $E$ -modules, we have the analogues of the algebraic constructions of taking quotients and localizations by elements of the coefficient ring of  $E$ , similarly as in the topological case [6]. We first consider the case of quotients. Let  $M$  be a (left)  $E$ -module. Let  $E_*$  denote the coefficient ring  $\pi_* E$  of  $E$ , and let  $x \in E_*$ . Then  $x$  can be realized as a map in the category of  $E$ -modules

$$S_E^{[x]} \xrightarrow{x} E.$$

We have a map of  $E$ -modules

$$\Sigma^{[x]} M \xrightarrow{x} E \wedge_E M \cong M$$

which represents multiplication by  $x$  on the  $E_*$ -module  $M_*$ . We define the  $E$ -module  $M/x$  to be the cofiber of this map, in the category of  $E$ -modules. Thus, given a countable sequence of elements  $(x_1, x_2, \dots)$ , we can define the  $E$ -module  $M/(x_1, x_2, \dots)$  inductively. For any finite  $n$ , we define  $M/(x_1, \dots, x_n)$  to be the quotient of  $M/(x_1, \dots, x_{n-1})$  by  $x_n$ . For the infinite case, we define

$$M/(x_1, \dots) = \text{hocolim}_n M/(x_1, \dots, x_n).$$

Note that this construction does not require the sequence  $(x_1, x_2, \dots)$  to be regular.

Likewise, we can define the localization of  $M$  at  $x \in E_*$ . Consider the sequence of elements  $x^n \in E_*$ . Each  $x^n$  can be realized as a map of  $E$ -modules

$$S_E^{|x|^n} \xrightarrow{x^n} E \rightarrow S_E^0$$

where the second map is an  $\mathbb{A}^1$ -weak equivalence of  $E$ -modules. Smashing with  $M$  and  $X_E^{-|x|^n}$  over  $E$  gives

$$S_E^0 \wedge_E M \xrightarrow{x^n} S_E^{-|x|^n} \wedge_E M$$

representing multiplication by  $x^n$  on  $M_*$ . Let  $j_0 = 0$  and  $j_n = j_{n-1} + |x|^n$ . Then smashing over  $E$  with  $S_E^{-j_{n-1}}$  gives maps of  $E$ -modules

$$S_E^{-j_{n-1}} \wedge_E M \rightarrow S_E^{-j_n} \wedge_E M.$$

We define the  $E$ -module  $M[x^{-1}]$  to be the homotopy colimit of this sequence of maps. Similarly, for a countable sequence of elements  $X = \{x_1, x_2, \dots\} \subset E_*$ , we can define the  $E$ -module  $M[X^{-1}]$ .

**Example:** Algebraic Morava  $K$ -theories.

Consider the algebraic cobordism spectrum  $MGL$ . We have the following theorem using the methods of [12, 15]. Namely, one defines an explicit  $\mathcal{L}(1)$ -action on the Thom prespectrum 2.7 which defines  $MGL$ . The  $V$ -th space of this Thom prespectrum is a wedge of spaces of the form  $BGL_{\mathcal{U} \oplus Z}(|Z|)^{\gamma|Z|}$ . For an extended injection  $(i, F) : \mathcal{U} \rightarrow \mathcal{U}$ ,  $(i, F)$  acts on  $\mathcal{U} \oplus \mathcal{U}$  by acting on each copy of  $\mathcal{U}$ , so it gives a map  $\mathcal{U} \oplus Z \rightarrow \mathcal{U} \oplus i(Z)$ . This induces an action on the Thom prespectrum, since we have that  $i(Z) \dot{+} F(V) = \mathcal{U}$  for all  $Z, V$  such that  $Z \dot{+} V = \mathcal{U}$ .

**THEOREM 14.2.** *MGL has a natural structure as an  $E_\infty$  ring spectrum, i. e. a monoid in the category of  $\mathbb{L}$ -spectra. Thus, applying  $S \wedge_{\mathcal{L}} -$  gives a model for MGL which has a natural structure as an S-algebra.*

There is a theory of algebraic orientations, similar to the case of complex and Real orientations in the topological case [7, 8]. In particular, recall that the coefficient ring  $MU_*$  of the complex cobordism spectrum  $MU$  is the Lazard ring, i. e. the universal ring for formal group laws. Also, recall that in the algebraic category, there are two different versions of the circle: the simplicial circle  $S_s^1$  and the twisted circle  $S_t^1$ . Hence,  $MGL$ , as a generalize algebraic cohomology theory, is really bigraded. We denote the grading of the coefficient ring of  $MGL$  by

$$MGL_{k,l} = \mathcal{H}_{\mathbb{A}^1}(S_s^{k-l} \wedge S_t^l, MGL)$$

in the category of the  $\mathbb{A}^1$ -local homotopy category of spectra [20]. There is a map of rings

$$MU_* \rightarrow MGL_{*,*}$$

coming from the formal group law on  $MGL_{*,*}$  that corresponds to the orientation on  $MGL$ . In particular, this map takes  $MU_{2k}$  to  $MGL_{2k,k}$ . ( $MGL_{*,*}$  is a graded-commutative ring with respect to the grading by the first index  $k$ ).

Let  $MU_{(p)}$  and  $MGL_{(p)}$  denote the localization of  $MU$  and  $MGL$  at a prime  $p$ . In the case of  $MGL$ , localization is defined in the manner analogous to the topological case. Namely, for each integer  $n$  relatively prime to  $p$ , we have a map of  $MGL$ -modules

$$MGL \xrightarrow{n} MGL$$

and  $MGL_{(p)}$  is defined as the homotopy colimit of this sequence of maps. By [6], the Morava  $K$ -theory spectrum  $K(n)$  in the topological case can be constructed as a  $MU$ -module by localizing  $MU$  at  $p$  in the category of  $MU$ -modules, then taking the quotient with respect to all polynomial generators of  $(MU_{(p)})_*$ , except one generator in dimension  $2(p^n - 1)$ , then localizing with respect to the remaining generator  $v_n$ .

In the algebraic category, we take quotients with respect to the images of the sequence of generators of  $(MU_{(p)})_*$  in  $(MGL_{(p)})_{*,*}$  (although these images do not form a regular sequence in  $MGL_{*,*}$ ). After localizing with respect to the image of  $v_n$ , we obtain the algebraic Morava  $K$ -theory spectra  $K(n)^{alg}$  as  $MGL$ -modules. Other spectra derived from  $MGL$  such as the algebraic versions of  $E(n)$ , and connected algebraic Morava  $K$ -theories  $k(n)^{alg}$  are defined in similar ways.

## 15. Proofs of the Model Structure Theorems

In this section, we prove the model structure theorems for spectra,  $\mathbb{L}$ -spectra, and  $S$ -modules. Our first goal is Proposition 4.6, which gives the alternative definition of the stable simplicial closed model structure on the category of algebraic prespectra. Recall Definition 4.5 of the class  $\mathbb{W}$  of maps of prespectra, generated by all eventual spacewise equivalences of prespectra. We would like to show that the class  $\mathbb{W}$  is the same as the class of weak equivalences in the stable simplicial model structure on  $Prespectra(k)$ .

Let  $L : Prespectra(k) \rightarrow Spectra(k)$  denote the spectrification functor. We have the following lemmas.

LEMMA 15.1. *If  $D$  is a prespectrum, then the unit of the adjunction  $\mu : D \rightarrow LD$  is in  $\mathbb{W}$ .*

PROOF. On each space  $D_n$  of  $D$ ,  $\mu$  is

$$\mu_n : D_n \rightarrow \operatorname{colim}_k \Omega^k D_{n+k}.$$

For each  $N \geq 0$ , define the prespectrum  $L_N D$ , with  $(L_N D)_n = D_n$  for  $n \geq N$ , and  $(L_N D)_n = \Omega^{N-n} D_N$  for  $n < N$ . There is an obvious map  $\mu^0 : D \rightarrow L_0 D$ , and  $\mu^N : L_{N-1} D \rightarrow L_N D$  for every  $N > 0$ .  $L_N D$  is a “partial spectrification” of  $D$ , in the sense that  $LD = \operatorname{colim}_N L_N D$ , and  $\mu : D \rightarrow LD$  is the colimit of all  $\mu^N$ ,  $N \geq 0$ . Clearly, each  $\mu^N$  is an eventual simplicial spacewise equivalence, so  $\mu$  is contained in  $\mathbb{W}$ . □

Recall there is a levelwise simplicial model structure on prespectra, where the fibrations and weak equivalences are just given spacewise. Each prespectrum is levelwise simplicially weak equivalent to a levelwise simplicially fibrant prespectrum, which can be made functorial. We recall a description of this fibrant replacement functor. By the simplicial model structure on  $Spc(k)_\bullet$ , for every based  $k$ -space  $X$ , the canonical map  $X \rightarrow *$  can be factored to

$$X \xrightarrow{i} FX \xrightarrow{p} *$$

where  $i$  is a simplicial acyclic cofibration,  $p$  is a simplicial fibration.  $F$  is given in [9] by the standard small object argument, which is a natural construction. Therefore,  $F : Spc(k)_\bullet \rightarrow Spc(k)_\bullet$  is a functor, and comes with a natural transformation  $Id \rightarrow F$ , such that for every object  $X \in Spc(k)_\bullet$ ,  $FX$  is simplicially fibrant, and the map  $X \rightarrow FX$  is a simplicial acyclic cofibration. Observe that by the construction of  $F$ , if a based  $k$ -space  $X$  is already simplicially fibrant, then there is a natural retraction

$$FX \rightarrow X$$

which is a simplicial weak equivalence.

For any  $X \in Spc(k)_\bullet$ , consider the simplicial acyclic cofibration  $i_X : X \rightarrow FX$ . We have

$$F\Omega i_X : F\Omega X \rightarrow F\Omega FX.$$

However, the loop of a simplicially fibrant space is again simplicially fibrant, so  $F\Omega FX \rightarrow \Omega FX$  naturally, and we have a natural map

$$\theta : F\Omega X \rightarrow \Omega FX$$

for every  $X$ .

For a prespectrum  $D = \{D_n\}$ , define  $FD = \{FD_n\}$ .  $FD$  is then a prespectrum, whose structure maps are

$$FD_i \xrightarrow{Fr_i} F\Omega D_{i+1} \xrightarrow{\theta} \Omega FD_{i+1}.$$

This gives the fibrant replacement functor

$$F : Prespectra(k) \rightarrow Prespectra(k)$$

in the levelwise simplicial model structure on  $Prespectra(k)$ . Applying the spectrification functor  $L$  gives the spectrum  $LFD$ , whose  $i$ -th space is

$$(LFD)_i = \operatorname{colim}_k \Omega^k FD_{i+k}.$$

LEMMA 15.2. *A map  $f : D \rightarrow E$  of prespectra is in the class  $\mathbb{W}$  if and only if the induced map*

$$LFf : LFD \rightarrow LFE$$

*is a spacewise simplicial weak equivalence.*

PROOF. First, suppose that  $LFf$  is a spacewise simplicial weak equivalence. Then it is certainly contained in  $\mathbb{W}$ . We show that  $D$  maps to  $LFD$  by a map in  $\mathbb{W}$ . The canonical map  $D \rightarrow FD$  is a spacewise simplicial weak equivalence by the definition of  $F$ , and the map  $FD \rightarrow LFD$  is in  $\mathbb{W}$  by Lemma 15.1. Therefore, we have a map of prespectra  $D \rightarrow LFD$  which is in  $\mathbb{W}$ . Likewise, there is a map  $E \rightarrow LFE$  contained in  $\mathbb{W}$ . Thus, by the commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{f} & E \\ \downarrow & & \downarrow \\ LFD & \xrightarrow{LFf} & LFE \end{array}$$

$f : D \rightarrow E$  is contained in  $\mathbb{W}$ .

Conversely, suppose that  $f$  is a map in  $\mathbb{W}$ . Since  $L$  and  $F$  preserve all diagrams, it suffices to assume that  $f$  is in fact an eventual spacewise simplicial equivalence of prespectra. So there is some  $N$  such that  $f_n : D_n \rightarrow E_n$  is a simplicial weak equivalence of  $k$ -spaces for all  $n \geq N$ . Thus,  $Ff_n$  is also a simplicial weak equivalence of  $k$ -spaces. Now for every  $m$ , we have

$$LFf_m : \operatorname{colim}_k \Omega^k FD_{m+k} \rightarrow \operatorname{colim}_k \Omega^k FE_{m+k}.$$

For  $k \geq N - m$ ,  $FD_{m+k} \rightarrow FE_{m+k}$  is a simplicial weak equivalence of  $k$ -spaces, and looping preserves weak equivalences between simplicially fibrant  $k$ -spaces by Lemma

2.1.13 of [16]. Therefore,  $LFf_m$  is a simplicial weak equivalence of  $k$ -spaces for all  $m$ .  $\square$

It is shown in Bousfield and Friedlander [2] that a map  $f$  of prespectra is a stable simplicial weak equivalence if and only if  $LFf$  is a spacewise simplicial weak equivalence. Proposition 4.6 follows. This gives the simplicial closed model structure on  $Prespectra(k)$ .

Let  $\alpha$  be a cardinal number greater than the cardinality of the power set of the morphism set of the site  $Sm/k$ . Recall the following definition of Jardine [10].

**DEFINITION 15.3** (Jardine). *Let  $\mathcal{T}$  be a small Grothendieck site, and let  $X$  be a simplicial sheaf over  $\mathcal{T}$ .  $X$  is said to be  $\alpha$ -bounded if for every  $U \in Sm/k$ , the section  $X(U)$  has cardinality less than  $\alpha$  at every simplicial stage.*

We define a prespectrum  $E$  to be  $\alpha$ -bounded if each  $E_n$  is an  $\alpha$ -bounded  $k$ -space. We record the following lemma, which was used in the proof of the stable simplicial model structure on  $Prespectra(k)$ , and is analogous to [10].

**LEMMA 15.4.** *A map  $p : E \rightarrow G$  of prespectra has the right lifting property with respect to all simplicial acyclic cofibrations of prespectra if it has the right lifting property with respect to all  $i : A \rightarrow B$  which are elementary simplicial acyclic cofibrations, such that  $B$  (and hence  $A$ ) is  $\alpha$ -bounded.*

Now we consider the model structure on  $Spectra(k)$ , namely Theorem 4.8.

**PROOF OF THEOREM 4.8:** The axioms CM1, CM2, and CM3 are clear. By Lemma 15.1, the functor  $L$  preserves simplicial acyclic cofibrations of  $\mathbb{T}$ -prespectra. Hence, the class of simplicial fibrations of spectra is exactly the class of spectra maps which have the right lifting property with respect to all simplicial acyclic cofibrations of spectra, and CM4 follows from CM5 formally, by an argument due to Joyal (see [10]).

To show the factorization of any  $f : E \rightarrow G$  as a simplicial acyclic cofibration  $i$  followed by a simplicial fibration  $p$ , we use the small object argument. Let  $\beta$  be a

cardinal number greater than  $2^\alpha$ . We construct a sequence of spectra  $E(\gamma)$  and maps  $E(\gamma) \rightarrow G$  for  $\gamma < \beta$ . Let  $E(0) = E$ . Suppose we have  $E(\gamma)$ . Then construct the prespectrum  $E'(\gamma + 1)$  as follows. Consider all diagrams of the form

$$\begin{array}{ccc} A_I & \longrightarrow & E(\gamma) \\ i_I \downarrow & & \downarrow f(\gamma) \\ B_I & \longrightarrow & G \end{array}$$

where  $i_I$  is chosen from a list of representatives of isomorphism classes of elementary simplicial acyclic cofibrations of prespectra, with  $B_I$   $\alpha$ -bounded. Define the prespectrum  $E(\gamma + 1)$  to be the pushout

$$\begin{array}{ccc} \coprod A_I & \longrightarrow & E(\gamma) \\ \coprod i_I \downarrow & & \downarrow \\ \coprod B_I & \longrightarrow & E'(\gamma + 1). \end{array}$$

Let  $f(\gamma + 1) : E(\gamma + 1) \rightarrow G$  be the naturally induced map. The map  $E(\gamma) \rightarrow E(\gamma + 1)$  is a simplicial acyclic cofibration. Define the spectrum  $E(\gamma + 1)$  to be  $LE'(\gamma + 1)$ . For limit ordinal  $\xi$ , set

$$E(\xi) = \operatorname{colim}_{\gamma < \xi} E(\gamma).$$

Each  $E(\gamma) \rightarrow E'(\gamma)$  is a simplicial acyclic cofibration of prespectra. But  $L$  preserves simplicial acyclic cofibrations, so

$$E(\gamma) \cong LE(\gamma) \rightarrow LE'(\gamma + 1) = E(\gamma + 1)$$

is a simplicial acyclic cofibration. Now let  $E(\beta) = \operatorname{colim}_{\gamma < \beta} E(\gamma)$ , with maps  $i : E \rightarrow E(\beta)$ ,  $p : E(\beta) \rightarrow G$ . Then  $i$  is a simplicial acyclic cofibration, and  $p$  has the right lifting property with respect to all simplicial acyclic cofibration of prespectra. So it is a simplicial fibration.

For the other factorization, we use a similar argument to factor  $f$  as  $q \cdot j$ , where  $j$  is a simplicial cofibration of spectra, and  $q$  has the right lifting property with respect to all simplicial cofibrations of spectra. Then by the fact that  $L$  preserves simplicial cofibrations, the usual adjunction argument gives that  $q$  is a simplicial acyclic fibration of prespectra, and hence of spectra.  $\square$

The analogous result for  $\mathcal{U}$ -spectra is Theorem 4.14. To prove it, we need the following lemma. Consider the spectrification functor  $L : Prespectra(\mathcal{U}) \rightarrow Spectra(\mathcal{U})$ .

LEMMA 15.5. *For  $E$  any  $\mathcal{U}$ -prespectrum, the unit of the spectrification functor  $E \rightarrow LE$  is a stable simplicial weak equivalence of  $\mathcal{U}$ -spectra.*

PROOF. On each cofinite  $V \subseteq \mathcal{U}$ , the unit map is

$$E_V \rightarrow (LE)_V = \operatorname{colim}_{(W,Z) \in V \downarrow \mathcal{C}(\mathcal{U})} \Omega^Z E_W.$$

For fixed cofinite  $U \subset \mathcal{U}$ , consider the map of homotopy groups

$$(15.6) \quad \begin{aligned} & \operatorname{colim}_{(V,T) \in U \downarrow \mathcal{C}(\mathcal{U})} \mathcal{H}_s(Spc(k)_\bullet)(T/(T \setminus \{0\}), E_U) \\ & \rightarrow \operatorname{colim}_{(V,T) \in U \downarrow \mathcal{C}(\mathcal{U})} \mathcal{H}_s(Spc(k)_\bullet)(T/(T \setminus \{0\}), (LE)_U). \end{aligned}$$

Since for each  $T$ ,  $T/(T \setminus \{0\})$  is small in  $Spc(k)_\bullet$ , the right hand side is

$$\begin{aligned} & \operatorname{colim}_{(V,T) \in U \downarrow \mathcal{C}(\mathcal{U})} \mathcal{H}_s(Spc(k)_\bullet)(T/(T \setminus \{0\}), \operatorname{colim}_{(W,Z) \in V \downarrow \mathcal{C}(\mathcal{U})} \Omega^Z E_W) \\ & \cong \operatorname{colim}_{(V,T) \in U \downarrow \mathcal{C}(\mathcal{U}), (W,Z) \in V \downarrow \mathcal{C}(\mathcal{U})} \mathcal{H}_s(Spc(k)_\bullet)(T/(T \setminus \{0\}), \Omega^Z E_W) \\ & \cong \operatorname{colim}_{(V,T) \in U \downarrow \mathcal{C}(\mathcal{U}), (W,Z) \in V \downarrow \mathcal{C}(\mathcal{U})} \mathcal{H}_s(Spc(k)_\bullet)((T \dot{+} Z)/((T \dot{+} Z) \setminus \{0\})) \end{aligned}$$

which is the same as the left hand side of (15.6).  $\square$

PROOF OF THEOREM 4.14. Analogous to that of Theorem 4.8.  $\square$

Using Lemma 15.5 and arguments similar to that for coordinatized spectra, we see that the stable simplicial homotopy categories of  $Prespectra(\mathcal{U})$  and  $Spectra(\mathcal{U})$  are equivalent.

We now turn to the case of  $\mathbb{L}$ -spectra and prove Theorem 11.9, which gives the simplicial model structure on  $\mathbb{L}$ -spectra, with weak equivalences and fibrations created in the category of spectra. To prove the theorem, we need to show that simplicial acyclic cofibrations satisfy the left lifting property with respect to all simplicial fibrations, and any map  $f$  of  $\mathbb{L}$ -spectra can be factored as  $f = p \cdot i$ , where  $p$  is a simplicial fibration,  $i$  is a simplicial cofibration, and either  $p$  or  $i$  can be made to be a simplicial weak equivalence.

First, note that the class of simplicial cofibrations is closed under pushouts and directed colimits. We will use the following lemma.

LEMMA 15.7. *If  $j : A \rightarrow B$  is a simplicial cofibration of spectra, then any pushout of  $\mathbb{L}j : \mathbb{L}A \rightarrow \mathbb{L}B$  is a simplicial cofibration of  $\mathbb{L}$ -spectra. If  $j : A \rightarrow B$  is a simplicial acyclic cofibration of spectra, then any pushout of  $\mathbb{L}j$  is a simplicial acyclic cofibration of  $\mathbb{L}$ -spectra.*

PROOF. For the first part, suppose we have a diagram

$$\begin{array}{ccc} \mathbb{L}A & \longrightarrow & X \\ \downarrow \mathbb{L}j & & \downarrow \\ \mathbb{L}B & \longrightarrow & Y \end{array}$$

where  $X \rightarrow Y$  is a simplicial acyclic fibration of  $\mathbb{L}$ -spectra. Then by adjunction, this diagram is equivalent to

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow j & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

in the category of spectra. Here, a lifting  $B \rightarrow X$  exists, so by the adjunction, a lifting  $\mathbb{L}B \rightarrow X$  exists in the original diagram.

For the simplicial acyclic cofibration statement, note that since colimits of  $\mathcal{L}$ -spectra are just colimits of the underlying spectra, a pushout diagram in the category of  $\mathbb{L}$ -spectra is a pushout diagram of spectra. Thus, it suffices to show that  $\mathbb{L}j$  is a simplicial acyclic cofibration of spectra. We have

$$\mathbb{L}A = \operatorname{colim}_{\mathcal{C}(\mathcal{U})}(\mathcal{M}\mathcal{L}(1)(U) \wedge A_U).$$

If  $j : A \rightarrow B$  is a spacewise inclusion, then  $\mathbb{L}j$  is a spacewise inclusion, since spectrification preserves spacewise inclusions. To show that it is a simplicial weak equivalence, by Lemma 15.4 it suffices to show that for an eventual spacewise equivalence  $j' : A' \rightarrow B'$  of prespectra, the map

$$\operatorname{colim}_{\mathcal{C}(\mathcal{U})}(\mathcal{M}\mathcal{L}(1)(U) \wedge A'_U) \rightarrow \operatorname{colim}_{\mathcal{C}(\mathcal{U})}(\mathcal{M}\mathcal{L}(1)(U) \wedge B'_U)$$

is a simplicial weak equivalence of spectra. But there is some  $U_0 \in \mathcal{C}(\mathcal{U})$  such that for all  $U \in \mathcal{C}(\mathcal{U})$ ,  $U \subseteq U_0$ ,  $A'_U \rightarrow B'_U$  is a simplicial acyclic cofibration of  $k$ -spaces. Hence, we have a simplicial weak equivalence of spectra.  $\square$

We show the factorizations by the usual small object argument. First, let  $\alpha$  be a cardinal number greater than the cardinality of the power set of the morphism set of  $Sm/k$ . Recall that a prespectrum  $E$  is  $\alpha$ -bounded if each  $k$ -space  $E_U$  is  $\alpha$ -bounded (see Definition 15.3).

**PROPOSITION 15.8.** *Any map  $f : E \rightarrow G$  of  $\mathbb{L}$ -spectra can be factored as  $f = p \cdot i$  where  $p$  is a simplicial acyclic fibration and  $i$  is a simplicial cofibration.*

**PROOF.** Let  $\beta$  be a cardinal number greater than  $2^\alpha$ . Let  $\mathcal{F}$  be the class of maps of the form  $\mathbb{L}A \rightarrow \mathbb{L}B$ , where  $B$  is  $\alpha$ -bounded, and  $A \rightarrow B$  is a simplicial cofibration of spectra. As usual, we construct the following diagram:

$$\begin{array}{ccccccc} E & \xrightarrow{i_0} & E(1) & \xrightarrow{i_1} & E(2) & \xrightarrow{f_2} & \cdots \\ f=f_0 \downarrow & & f_1 \downarrow & & \downarrow i_2 & & \\ G & \xrightarrow{=} & G & \xrightarrow{=} & G & \xrightarrow{=} & \cdots \end{array}$$

For an ordinal number  $\gamma < \beta$ , assume we have  $E(\gamma) \rightarrow G$ . Then  $E(\gamma + 1)$  is given by the following pushout diagram

$$\begin{array}{ccc} \coprod \mathbb{L}A_I & \longrightarrow & E(\gamma) \\ \downarrow & & \downarrow i_\gamma \\ \coprod \mathbb{L}B_I & \longrightarrow & E(\gamma + 1) \end{array}$$

where the indexing set  $I$  ranges over all isomorphism classes of commutative diagrams of the form

$$\begin{array}{ccc} \mathbb{L}A_I & \longrightarrow & E(\gamma) \\ \downarrow & & \downarrow f_\gamma \\ \mathbb{L}B_I & \longrightarrow & G \end{array}$$

with  $\mathbb{L}A_I \rightarrow \mathbb{L}B_I$  in  $\mathcal{F}$ . Also, if  $\xi < \beta$  is a limit ordinal, we define

$$E(\xi) = \operatorname{colim}_{\gamma < \xi} E(\gamma).$$

Define  $E' = \operatorname{colim}_{\gamma < \beta} E(\gamma)$ . Then we have  $i : E \rightarrow E'$  and  $p : E' \rightarrow G$  with  $f = p \cdot i$ . For each  $\gamma < \beta$ , the map  $E(\gamma) \rightarrow E(\gamma + 1)$  is the pushout of a simplicial cofibration, and  $i$  is the directed colimit of these maps. Hence,  $i$  is a simplicial cofibration of  $\mathbb{L}$ -spectra. Also, given any diagram

$$\begin{array}{ccc} A & \longrightarrow & E' \\ j \downarrow & & \downarrow \\ B & \longrightarrow & G \end{array}$$

of spectra, where  $j$  is a simplicial cofibration of spectra and  $B$  is  $\alpha$ -bounded, there is a lifting  $B \rightarrow E'$  adjoint to the lifting in  $\mathbb{L}$ -spectra that comes by construction. Thus, by the Lemma 15.4,  $E' \rightarrow G$  is a simplicial acyclic fibration of spectra, and so of  $\mathbb{L}$ -spectra as well.  $\square$

There is also the other factorization.

**PROPOSITION 15.9.** *Any map  $f : E \rightarrow G$  of  $\mathbb{L}$ -spectra can be factored as  $f = p \cdot i$ , where  $i$  is a simplicial acyclic cofibration and  $p$  is a simplicial fibration.*

**PROOF.** We use lemma 15.7. Let  $\mathcal{F}$  be the class of all acyclic cofibrations of the form  $\mathbb{L}A \rightarrow \mathbb{L}B$ , where  $A \rightarrow B$  is an acyclic cofibration of spectra. As before, we can construct  $E(\gamma)$  for all ordinals  $\gamma < \beta$ . By Lemma 15.7, each map  $E(\gamma) \rightarrow E(\gamma + 1)$  is a simplicial acyclic cofibration of  $\mathbb{L}$ -spectra. Define  $E' = \operatorname{colim}_{\gamma < \beta} E(\gamma)$ . Then  $i : E \rightarrow E'$  is a simplicial acyclic cofibration since simplicial weak equivalences of spectra are preserved by directed colimits. Also, in the category of spectra,  $p : E' \rightarrow G$  satisfies the right lifting property with respect to all simplicial acyclic cofibrations  $A \rightarrow B$  where  $B$  is  $\alpha$ -bounded. However, by Lemma 15.4 a map of spectra has the right lifting property with respect to all simplicial acyclic cofibrations if and only if it has the right lifting property with respect to all simplicial acyclic cofibrations whose target is  $\alpha$ -bounded. Hence, it is a simplicial fibration of spectra and therefore of  $\mathbb{L}$ -spectra as well.  $\square$

Finally, the left lifting property of acyclic cofibrations with respect to all fibrations follows formally, by the formal argument of Joyal (see [10]).

Finally, we have Theorem 13.7, which gives the simplicial model structure on the category of  $S$ -modules, with the weak equivalences and fibrations created in the category of  $\mathbb{L}$ -spectra via the functor  $F_{\mathcal{L}}(S, -)$ . To prove Theorem 13.7, we need that every map  $f : E \rightarrow G$  of  $S$ -modules can be factored into a simplicial cofibration and a simplicial acyclic fibration. We let  $\mathcal{F}$  be the class of maps of the form

$$S \wedge_{\mathcal{L}} (\mathbb{L}A) \rightarrow S \wedge_{\mathcal{L}} (\mathbb{L}B)$$

where  $A \rightarrow B$  is a simplicial cofibration of spectra, and  $B$  is  $\alpha$ -bounded. Each map in  $\mathcal{F}$  is in particular a simplicial cofibration of  $S$ -modules. By definition, simplicial cofibrations of  $S$ -modules are preserved under pushouts and directed colimits. By the usual small object argument, we can factor  $f$  as  $p \cdot i$ , where  $i : E \rightarrow E'$  is a directed colimit of pushouts of maps in  $\mathcal{F}$ , so  $i$  is a simplicial cofibration of  $S$ -modules. Also,  $p$  satisfies the right lifting property with respect to every map in  $\mathcal{F}$ . Given any simplicial cofibration of spectra  $j : A \rightarrow B$ , where  $B$  is  $\alpha$ -bounded, the commutative square of spectra

$$\begin{array}{ccc} A & \longrightarrow & F_{\mathcal{L}}(S, E') \\ \downarrow j & & \downarrow F_{\mathcal{L}}(S, p) \\ B & \longrightarrow & F_{\mathcal{L}}(S, G) \end{array}$$

is equivalent to the diagram of  $S$ -modules

$$\begin{array}{ccc} S \wedge_{\mathcal{L}} \mathbb{L}A & \longrightarrow & E' \\ S \wedge_{\mathcal{L}} \mathbb{L}j \downarrow & & \downarrow p \\ S \wedge_{\mathcal{L}} \mathbb{L}B & \longrightarrow & G. \end{array}$$

Hence, it has a lifting, so  $F_{\mathcal{L}}(S, p)$  is a simplicial acyclic cofibration of spectra, and therefore of  $S$ -modules as well.

For the other factorization, we let  $\mathcal{F}$  be the class of maps of the form

$$S \wedge_{\mathcal{L}} \mathbb{L}A \rightarrow S \wedge_{\mathcal{L}} \mathbb{L}B$$

where  $A \rightarrow B$  is a simplicial acyclic cofibration of spectra, and  $A$  is  $\alpha$ -bounded. Again, we use the small object argument to factor  $f$  as  $p \cdot i$ . To show that  $i$  is a simplicial acyclic cofibration of  $S$ -modules, we need the following.

LEMMA 15.10. *If  $j : A \rightarrow B$  is a simplicial acyclic cofibration of spectra, then any pushout of  $S \wedge_{\mathcal{L}} \mathbb{L}j$  is a simplicial acyclic cofibration of  $S$ -modules.*

PROOF. Suppose we have a pushout diagram

$$\begin{array}{ccc} S \wedge_{\mathcal{L}} \mathbb{L}A & \longrightarrow & E \\ \downarrow S \wedge_{\mathcal{L}} \mathbb{L}j & & \downarrow f \\ S \wedge_{\mathcal{L}} \mathbb{L}B & \longrightarrow & G \end{array}$$

in the category of  $S$ -modules. By adjunction,  $f$  is a simplicial cofibration of  $S$ -modules. To show that it is a weak equivalence, note that by Lemma 15.7,  $\mathbb{L}A \rightarrow \mathbb{L}B$  is a simplicial acyclic cofibration of spectra. Also, recall that

$$S \wedge_{\mathcal{L}} \mathbb{L}A = (\mathcal{L}(2) \times_{\mathcal{L}(1)^2} \mathcal{L}(0) \times \mathcal{L}(1)) \times (S^0 \wedge A).$$

Simplicial acyclic cofibrations of spectra are preserved by twisted half-smash products, so  $S \wedge_{\mathcal{L}} \mathbb{L}A \rightarrow S \wedge_{\mathcal{L}} \mathbb{L}B$  is a simplicial acyclic cofibration of spectra. Now pushouts in the category of  $S$ -modules are created in the category of spectra. But the pushout in the category of spectra is a simplicial acyclic cofibration of spectra. Thus, the pushout of  $S$ -modules  $f$  itself is a simplicial weak equivalence of spectra, and hence of  $\mathbb{L}$ -spectra. But  $F_{\mathcal{L}}(S, -)$  preserves simplicial weak equivalences of  $\mathbb{L}$ -spectra as well, so  $F_{\mathcal{L}}(S, f)$  is a simplicial weak equivalence of  $\mathbb{L}$ -spectra.  $\square$

Now  $f = p \cdot i$ , and  $i$  is a directed colimit of pushouts of maps in  $\mathcal{F}$ , so it is a simplicial acyclic cofibration of  $S$ -modules. Also, by the adjunction,  $F_{\mathcal{L}}(S, p)$  has the left lifting property with respect to any simplicial acyclic cofibration of spectra  $A \rightarrow B$ , where  $B$  is  $\alpha$ -bounded. Hence,  $p$  is a simplicial fibration of  $S$ -modules. Finally, the right lifting property of simplicial fibrations with respect to simplicial acyclic cofibrations follows formally.

## 16. Technical Results on the Extended Injections Operad

In this section, we prove several results on the extended injections operad  $\mathcal{L}$  used in the paper. Recall that for a  $k$ -universe  $\mathcal{U}$ ,  $\mathcal{L}(n)$  is the space of extended injections  $\mathcal{I}(\mathcal{U}^{\oplus(n)}, \mathcal{U})$ . By Proposition 9.2,  $\mathcal{L}(n)$  is contractible for all  $n$ .

We first prove Lemma 12.4, which states the  $k$ -space  $\mathcal{L}(2) \times_{\mathcal{L}(1) \times \mathcal{L}(1)} \mathcal{L}(0) \times \mathcal{L}(1)$  is  $\mathbb{A}^1$ -contractible.

PROOF. We do a modified version of the proof in [6]. Fix infinite-dimensional subspaces  $\mathcal{U}_1, \mathcal{U}_2 \subset \mathcal{U}$ , such that  $\mathcal{U} = \mathcal{U}_1 \dot{+} \mathcal{U}_2$ . Let  $i_1, i_2 : \mathcal{U} \rightarrow \mathcal{U}^2$  be inclusions into the first and second summands, with the obvious extended injection structures. For  $g \in \mathcal{L}(2)$ , denote  $g \cdot i_1$  and  $g \cdot i_2$  by  $g_1$  and  $g_2$ , respectively.

Given an extended injection  $(h, H) : \mathcal{U} \rightarrow \mathcal{V}$ , we define the complement of  $(h, H)$  to be

$$(16.1) \quad C(h, H) = \bigcap_{U \in \mathcal{C}(\mathcal{U})} H(U).$$

Of course,  $C(h, H)$  may be  $\{0\}$ . When there is no chance of confusion, we abbreviate  $C(h, H)$  by  $C(h)$ . We make the following definitions:

$$\begin{aligned} \mathcal{K}(2) &= \{g \in \mathcal{L}(2) \mid g_2(\mathcal{U}) \subset \mathcal{U}_2, \mathcal{U}_1 \subset C(g_2)\} \\ \mathcal{K}^0(1) &= \{f \in \mathcal{L}(1) \mid f(\mathcal{U}) \subset \mathcal{U}_2, \mathcal{U}_1 \subset C(f)\} \\ \hat{\mathcal{K}}(1) &= \mathcal{K}(2) \times_{\mathcal{L}(1) \times \mathcal{L}(1)} \mathcal{L}(0) \times \mathcal{L}(1) \\ \hat{\mathcal{L}}(1) &= \mathcal{L}(2) \times_{\mathcal{L}(1) \times \mathcal{L}(1)} \mathcal{L}(0) \times \mathcal{L}(1). \end{aligned}$$

First, it is easy to see that  $\mathcal{K}^0(1)$  is isomorphic to the  $k$ -space of extended injections  $\mathcal{I}(\mathcal{U}, \mathcal{U}_2)$ , so it is  $\mathbb{A}^1$ -contractible. Next, consider the structure map

$$\gamma : \mathcal{L}(2) \times \mathcal{L}(0) \times \mathcal{L}(1) \rightarrow \mathcal{L}(1)$$

given by  $\gamma(g, *, f) \mapsto g_2 \cdot f$ . It induces a map

$$\hat{\gamma} : \hat{\mathcal{K}}(1) \rightarrow \mathcal{K}^0(1).$$

We will show that  $\hat{\gamma}$  is an isomorphism of  $k$ -spaces.  $(g, *, f)$  and  $(g', *, f')$  in  $\mathcal{K}(2) \times \mathcal{L}(0) \times \mathcal{L}(1)$  represent the same element in  $\hat{\mathcal{K}}(1)$  if there are  $j_1, j_2 \in \mathcal{L}(1)$ , such that  $g' = g \cdot (j_1 \oplus j_2)$ , and  $j_2 \cdot f' = f$ . Choose linear isomorphisms  $k_1 : \mathcal{U} \rightarrow \mathcal{U}_1$  and  $k_2 : \mathcal{U} \rightarrow \mathcal{U}_2$ . Define the map

$$\varphi : \mathcal{K}^0(1) \rightarrow \hat{\mathcal{K}}(1)$$

by  $f \mapsto (k_1 \oplus k_2, *, k_2^{-1} \cdot f)$ . Then  $\varphi$  is the inverse to  $\hat{\gamma}$ . Thus,  $\hat{\mathcal{K}}(1)$  is  $\mathbb{A}^1$ -contractible.

Finally, the proof in [6] that the inclusion  $\hat{\mathcal{K}}(1) \rightarrow \hat{\mathcal{L}}(1)$  is a homotopy equivalence carries through, with the homotopies replaced by the  $\mathbb{A}^1$ -homotopies constructed in the proof of Proposition 9.2. Therefore,  $\hat{\mathcal{L}}(1)$  is  $\mathbb{A}^1$ -contractible.  $\square$

We also prove Lemma 13.3. Namely, the coequalizer  $\mathcal{L}(2) \times_{\mathcal{L}(1) \times \mathcal{L}(1)} \mathcal{L}(0) \times \mathcal{L}(0)$  is  $\text{Spec}(k)$ .

To prove the lemma, we first make the following observation.

LEMMA 16.2. *Suppose  $(h, H) : X \times \mathcal{U} \rightarrow \mathcal{U}$  is a system of extended injections parametrized over a small  $k$ -space  $X$ . Also, suppose  $U \subset \mathcal{U}$  is cofinite, and  $Z \subset \mathcal{U}$  is finite-dimensional, such that  $Z \subset \bigcap_{x \in X} H(x)(U)$ . Then there is a  $U' \subset U$  of codimension 1, such that  $Z \subset \bigcap_{x \in X} H(x)(U')$ .*

PROOF. Note that for any cofinite  $V \subset \mathcal{U}$ ,  $f(x) : \mathcal{U}/V \rightarrow \mathcal{U}/(H(x)(V))$  is an isomorphism. This gives a map

$$p_V : X \times \mathcal{U} \rightarrow \mathcal{U}/V.$$

For fixed  $x \in X$ ,  $p_V(x)$  is

$$\mathcal{U} \rightarrow \mathcal{U}/(H(x)(V)) \xrightarrow{f(x)^{-1}} \mathcal{U}/V.$$

In particular,  $p_V(Z) \subset U/V$  for every  $V \subset U$ . For each  $x \in X$ , the dimension of  $p_V(x)(Z)$  is at most the dimension of  $Z$ , and since  $X$  is small, the dimension of  $p_V(Z)$  is bounded as  $V$  varies over  $\mathcal{C}(\mathcal{U})$ . Choose  $V \subset U$  such that the codimension of  $V$  in  $U$  is strictly greater than the dimension of  $p_V(Z)$ . Then we can find a  $U' \subset U$  of codimension 1, such that  $V \subset U'$ , and  $p_V(Z) \subset U'/V$ . Then for every  $x \in X$ ,

$$p_V(x)(Z) \cong Z/(Z \cap H(x)(V)) \subset U'/V \cong H(x)(U')/H(x)(V).$$

Therefore,  $Z \subset H(x)(U')$  for every  $x \in X$ .  $\square$

PROOF OF LEMMA 13.3: Let  $X$  be a small  $k$ -space, and  $(f, F), (g, G) : X \times \mathcal{U}^2 \rightarrow \mathcal{U}$  two systems of extended injections from  $\mathcal{U}^2$  to  $\mathcal{U}$ , parametrized by  $X$ . We will show that there is an element of  $\mathcal{L}(1) \times \mathcal{L}(1)$  which takes  $(f(x), F(x))$  to  $(g(x), G(x))$  for

each  $x \in X$ . First,  $f$  is the (internal) direct sum of  $(f_1, F_1)$  and  $(f_2, F_2) : X \times \mathcal{U} \rightarrow \mathcal{U}$ , which are the restrictions of  $f$  to the first and second summands of  $\mathcal{U}^2$ , respectively. Likewise,  $g$  is the internal direct sum of  $(g_1, G_1)$  and  $(g_2, G_2) : X \times \mathcal{U} \rightarrow \mathcal{U}$ .

Given two systems of extended injections  $(h, H), (h', H') : X \times \mathcal{U} \rightarrow \mathcal{U}$ , we say that  $(h, H)$  and  $(h', H')$  are disjoint if

$$(16.3) \quad \text{Im}(h) \subset \cap_{x \in X} C(h'(x), H'(x))$$

$$(16.4) \quad \text{Im}(h') \subset \cap_{x \in X} C(h(x), H(x)).$$

Here,  $C(h(x), H(x))$  is the complement of the extended injection

$$(h(x), H(x))$$

as defined in 16.1.

We claim that there are extended injections  $(i, I)$  and  $(j, J) : \mathcal{U} \rightarrow \mathcal{U}$ , such that  $f_1 \cdot i$  and  $g_1 \cdot J$  are disjoint. We show this by inductively constructing two sequences of linearly independent elements  $d_1, d_2, \dots$  and  $e_1, e_2, \dots$  in  $\mathcal{U}$ , as well as a decreasing sequence of cofinite sets  $U_0, U_1, \dots$ , such that each  $U_{n+1}$  has codimension 1 in  $U_n$ . Let  $U_0$  be  $\mathcal{U}$ . Suppose we have  $d_1, \dots, d_{n-1}, e_1, \dots, e_{n-1}$  linearly independent, and  $U_0, \dots, U_{n-1}$ , such that

$$U_k \dot{+} \langle d_k \rangle = U_{k-1}$$

for each  $1 \leq k \leq n-1$ . Also,

$$g(X \times \langle e_1, \dots, e_{n-1} \rangle) \subset \bigcap_{x \in X} F(x)(U_{n-1}).$$

Denote  $V_n = \cap_{x \in X} F(x)(U_{n-1})$ . Since  $X$  is small, this is cofinite in  $\mathcal{U}$ . Also,

$$\bigcap_{x \in X} g(x)^{-1}(V_n)$$

is again cofinite by the fact that  $X$  is a small  $k$ -space. So there is some  $e_n \in \mathcal{U}$  outside of the span of  $e_1, \dots, e_{n-1}$ , such that  $g(x)(e_n) \in V_n$  for every  $x \in X$ . Since  $X$  is small,

$$g(X \times \langle e_1, \dots, e_n \rangle) \subset \bigcap_{x \in X} F(x)(U_{n-1})$$

is finite-dimensional. Hence by Lemma 16.2, there is some  $U_n \subset U_{n-1}$  of codimensional 1, such that

$$Z_n \subset \bigcap_{x \in X} F(x)(U_n).$$

Finally, choose  $d_n \in U_{n-1}$  such that

$$U_n \dot{+} \langle d_n \rangle = U_{n-1}.$$

Having chosen  $d_1, \dots, e_1, \dots$  and  $U_0, \dots$  as above, we choose a complement  $\bar{U} \subset \mathcal{U}$  such that  $\bar{U} \dot{+} \langle e_1, e_2, \dots \rangle = \mathcal{U}$ . Define extended injections  $(i, I)$  and  $(j, J) : \mathcal{U} \rightarrow \mathcal{U}$  as follows. Let  $\langle a_1, a_2, \dots \rangle$  be a fixed basis for  $\mathcal{U}$ . Then set

$$\begin{aligned} i(a_k) &= d_k \\ j(a_k) &= e_k \\ I(\langle a_k, a_{k+1}, \dots \rangle) &= U_k \\ J(\langle a_k, a_{k+1}, \dots \rangle) &= \langle e_k, e_{k+1}, \dots \rangle \dot{+} \bar{U}. \end{aligned}$$

Then

$$\begin{aligned} (16.5) \quad \text{Im}(g \cdot (Id \times j)) &= g(X \times \langle e_1, e_2, \dots \rangle) \\ &\subset \bigcap_{x \in X, k \geq 0} F(x)(U_k) = \bigcap_{x \in X} C(f(x) \cdot i, F(x) \cdot I). \end{aligned}$$

Now note that  $\text{Im}(g_1 \cdot (Id \times j)) \subset \text{Im}(g_1)$ , and

$$\bigcap_{x \in X} C(f_1(x), F_1(x)) \subset \bigcap_{x \in X} C(f_1(x) \cdot i, F_1(x) \cdot I).$$

Hence, by repeating this procedure, we can find extended injections  $i_1, i_2, j_1, j_2 : \mathcal{U} \rightarrow \mathcal{U}$ , such that  $f_1 \cdot i_1, f_2 \cdot i_2, g_1 \cdot j_1$  and  $g_2 \cdot j_2$  are pairwise disjoint. Hence, we may replace  $f$  by  $f \cdot (Id \times (i_1 \oplus i_2))$  and  $g$  by  $g \cdot (Id \times (j_1 \oplus j_2))$ .

Next, consider  $f_1, g_1 : \mathcal{U} \rightarrow \mathcal{U}$ . By the above construction,  $\text{Im}(f_1) \cap \text{Im}(g_1) = 0$  in particular. Define another system of extended injection  $(h_1, H_1) : X \times \mathcal{U} \rightarrow \mathcal{U}$  as follows. Let  $a_1, a_2, \dots$  be a fixed basis of  $\mathcal{U}$ . Denote also  $\mathcal{U}_1 = \langle a_1, a_3, \dots \rangle$ , and

$\mathcal{U}_2 = \langle a_2, a_4, \dots \rangle$ . Let  $(k_1, K_1), (m_1, M_1) : \mathcal{U} \rightarrow \mathcal{U}$  be given by

$$k_1(a_n) = a_{2n-1}$$

$$K_1(V) = k_1(V) \dot{+} \mathcal{U}_2$$

$$m_1(a_n) = a_{2n}$$

$$M_1(V) = k_2(V) \dot{+} \mathcal{U}_1.$$

For each  $x \in X$ , define

$$h_1(x)(a_{2n-1}) = f_1(x)(a_n)$$

$$h_1(x)(a_{2n}) = g_1(x)(a_n).$$

For a cofinite  $V \subset \mathcal{U}$ , if  $\mathcal{U}_2 \subset V$ , set

$$H_1(x)(V) = F_1(x)(k_1^{-1}(V \cap \text{Im}(f))).$$

If  $\mathcal{U}_1 \subset V$ , set

$$H_1(x)(V) = G_1(x)(m_1^{-1}(V \cap \text{Im}(g))).$$

And for a general cofinite  $V = \langle a_n, a_{n+1}, \dots \rangle \subset \mathcal{U}$ , there are cofinite  $V_1, V_2$  such that  $V = V_1 \cap V_2$  and  $\mathcal{U}_1 \subset V_1, \mathcal{U}_2 \subset V_2$ . Define then

$$H_1(x)(V) = H_1(x)(V_1) \cap H_1(x)(V_2).$$

Since  $f_1$  and  $g_1$  are disjoint, this gives an extended injection, and for every  $x \in X$ ,

$$f_1(x) = h_1(x) \cdot k_1$$

$$g_1(x) = h_1(x) \cdot m_1.$$

By a similar argument for  $f_2$  and  $g_2$ , there is a system of extended injections  $(h_2, H_2) : X \times \mathcal{U} \rightarrow \mathcal{U}$ , such that

$$f_2(x) = h_2(x) \cdot k_2$$

for every  $x \in X$ . Also,  $h_1$  and  $h_2$  are disjoint, so  $h_1 \oplus h_2 \in \mathcal{L}(2)$ . Hence,  $f(x) = (h_1(x) \dot{+}_2(x)) \cdot (k_1 \oplus k_2)$  for each  $x$ . But by the same argument,  $g(x)$  is also in the same orbit as  $h_1(x) \dot{+} h_2(x)$ , via  $m_1 \oplus m_2$ .  $\square$

## 17. Appendix: Small Objects in the Category of Simplicial Sheaves

In this appendix, we record some facts about the category of spaces over a field  $k$  used in the paper. In particular, in Lemma 17.3 below, we will show that in the categories of unbased and based  $k$ -spaces, every object is simplicially weakly equivalent to a homotopy colimit of finite colimit of smooth schemes, which are small objects. This fact is used to prove Propositions 5.14 and 11.16, that in the simplicial and  $\mathbb{A}^1$ -local stable homotopy categories, the morphism sets from all possible spectra are determined by morphism sets from spectra of the form  $\Sigma_V^{\mathcal{U}} X$ , where  $X$  is a finite colimit of smooth schemes. Also, since the isomorphism classes of finite colimits of smooth schemes form a set, Lemma 17.3 also allows us to use the bounded object argument to construct the  $\mathbb{A}^1$ -localization of a spectrum, as in Proposition 5.12.

Recall that the category of  $k$ -spaces is

$$Spc(k) = \Delta^{op} Sh(Sm/k)_{Nis}$$

the category of simplicial sheaves over the category  $Sm/k$  of smooth schemes over  $k$ , where  $Sm/k$  is a Grothendieck site by the Nisnevich topology.

We first consider the simpler category  $Sh(Sm/k)_{Nis}$  of sheaves over  $Sm/k$ . The following fact is standard, and we record it here for the sake of convenience.

LEMMA 17.1. *Every sheaf is a colimit of smooth schemes.*

PROOF. Recall that a presheaf on  $Sm/k$  is just a contravariant functor  $Sm/k \rightarrow Sets$ , and a smooth scheme  $Y$  is a presheaf by the representable functor  $Hom(-, Y)$ . There is the Yoneda Lemma (see [13]): for any presheaf  $F$  on  $Sm/k$  and smooth scheme  $Y$ , there is a natural bijection of sets

$$Nat(Hom(-, Y), F) \cong F(Y).$$

Now for any presheaf  $F$ , define the category  $Sm/k(F)$  to have objects pairs  $(X, t)$  where  $X \in Sm/k$ , and  $t \in F(X)$ . A morphism  $f : (X, t) \rightarrow (Y, z)$  is a morphism  $f : X \rightarrow Y$  in  $Sm/k$ , such that  $F(f) : F(Y) \rightarrow F(X)$  takes  $z$  to  $t$ . For a map  $f : (X, t) \rightarrow (Y, z)$  in  $Sm/k(F)$ , there is a map of presheaves  $Hom(-, X) \rightarrow Hom(-, Y)$ :

just the obvious natural transformation induced by  $f$ . For any  $(X, t)$ , by the Yoneda Lemma, there is a natural map of presheaves  $Hom(-, X) \rightarrow F$ , namely the natural transformation corresponding to  $t \in F(X)$ . It is easy to check that these maps commute with the natural transformations induced by morphisms in  $Sm/k(F)$ , so there is a map of prespectra

$$\operatorname{colim}_{(X,t) \in Sm/k(F)} Hom(-, X) \rightarrow F.$$

For fixed  $U \in Sm/k$ , consider the map of sets

$$\operatorname{colim}_{(X,t) \in Sm/k(F)} Hom(U, X) \rightarrow F(U).$$

A typical element of  $Hom(U, X)$  is a triple  $(f, X, t)$ , where  $f : U \rightarrow X$ , and  $(X, t) \in Sm/k$ . The natural map on the morphism sets induced by  $f$  takes the triple  $(Id, U, F(f)(t))$  to  $(f, X, t)$ . Hence, in the colimit, each  $(f, X, t)$  is identified with a unique triple  $(Id, U, F(f)(t))$ , where  $t \in F(X)$  for some  $X \in Sm/k$ . Therefore, the map is in fact an bijection, and  $F$  is a colimit of representable functors. Since  $Sm/k$  is a small site, this is a small colimit. Then in the category of presheaves on  $Sm/k$ ,

$$F \cong \operatorname{colim}_{(X,t) \in Sm/k(F)} Hom(-, X).$$

Passing to the category of sheaves, we note that the sheafification functor from presheaves to sheaves is the left adjoint to the forgetful functor, and thus commutes with all colimits. So applying sheafification to the above bijection gives that every sheaf  $F$  on  $Sm/k$  is a small colimit of representable sheaves, i. e. smooth schemes.  $\square$

Next, recall the definition of a small object in a category: an object  $X$  is small if for any small directed system  $\{S_i\}_{i \in \mathcal{I}}$ ,

$$Hom(X, \operatorname{colim}_{i \in \mathcal{I}} S_i) \cong \operatorname{colim}_{i \in \mathcal{I}} Hom(X, S_i).$$

LEMMA 17.2. *A smooth scheme is a small object in the category of sheaves over  $Sm/k$  with the Nisnevich topology.*

PROOF. Let  $\mathcal{I}$  be a small directed category, and  $S : \mathcal{I} \rightarrow Sh(Sm/k)_{Nis}$  a functor. Let  $X \in Sm/k$  be a smooth scheme, thought of as a representable sheaf. By the Yoneda Lemma again, on the level of presheaves

$$Hom(X, \operatorname{colim}_{i \in \mathcal{I}} S_i) = (\operatorname{colim}_{i \in \mathcal{I}} S_i)(X)$$

and

$$\operatorname{colim}_{i \in \mathcal{I}} Hom(X, S_i) = \operatorname{colim}_{i \in \mathcal{I}} S_i(X).$$

In the category of presheaves, we certainly have

$$(\operatorname{colim}_{i \in \mathcal{I}} S_i)(X) = \operatorname{colim}_{i \in \mathcal{I}} S_i(X).$$

Therefore, it suffices to show that a small directed colimit of sheaves in the category of presheaves on  $Sm/k$  is in fact a sheaf in the Nisnevich topology. Recall that the Nisnevich topology is generated by diagrams of the form (1.1). A presheaf  $F$  is a sheaf in the Nisnevich topology if and only if  $F$  takes a square of the form (1.1) to a pullback square of sets. But pullback squares are finite inverse limits of sets, and are stable under small directed colimits. Hence, if each  $S_i$  takes all such diagrams to pullback squares of sets, then so does  $\operatorname{colim}_{i \in \mathcal{I}} S_i$ . Therefore,  $\operatorname{colim}_{i \in \mathcal{I}} S_i$  is a sheaf in the Nisnevich topology.  $\square$

Therefore, finite colimits of smooth schemes are also small in the category of sheaves  $Sh(Sm/k)_{Nis}$ . Every sheaf is a colimit of finite colimits of smooth schemes.

The next lemma expresses  $k$ -spaces, i. e. simplicial sheaves, as homotopy colimits of finite colimits of smooth schemes.

LEMMA 17.3. *In  $\Delta^{op}Sh(Sm/k)_{Nis}$ , every object is simplicially weakly equivalent to a homotopy colimit of finite colimits of smooth schemes.*

PROOF. The category of simplicial sheaves  $\Delta^{op}Sh(Sm/k)_{Nis}$  is just the category of diagrams of sheaves over  $Sm/k$  indexed on  $\Delta^{op}$ . By Proposition 2.2.14 of [16], for a simplicial sheaf  $X = \{X_n\}$ ,

$$\operatorname{hocolim}_{\Delta^{op}} X_n \rightarrow X$$

is a simplicial weak equivalence. Putting the two statements together gives the lemma.  $\square$

We also record some facts about the  $\mathbb{A}^1$ -local homotopy category of  $k$ -spaces. We have the following lemma, which is not explicitly used in this paper.

LEMMA 17.4. *Finite colimits of smooth schemes are small in the  $\mathbb{A}^1$ -local homotopy category of  $\Delta^{op}Sh(Sm/k)_{Nis}$ .*

PROOF. It suffices to show that a smooth scheme is small in the  $\mathbb{A}^1$ -local homotopy category  $\mathcal{H}_{\mathbb{A}^1}(\Delta^{op}Sh(Sm/k)_{Nis})$ . Let  $X$  be a smooth scheme,  $\mathcal{I}$  a small directed category, and  $S : \mathcal{I} \rightarrow \Delta^{op}Sh(Sm/k)_{Nis}$  a functor. Every object in  $\Delta^{op}Sh(Sm/k)_{Nis}$  is cofibrant. Thus, the morphism set in the  $\mathbb{A}^1$ -local homotopy category

$$\mathcal{H}_{\mathbb{A}^1}(X, \operatorname{colim}_{i \in \mathcal{I}} S_i)$$

is the set of homotopy classes of maps from  $X$  to  $\Gamma(\operatorname{colim}_{i \in \mathcal{I}} S_i)$ , where  $\Gamma$  denotes the simplicial fibrant approximation functor. Recall that homotopy classes of maps from a cofibrant object  $A$  to a fibrant object  $X$  can be detected using cylinder objects, which are objects  $Cyl(A)$  that come with maps

$$A \amalg A \xrightarrow{i} Cyl(A) \xrightarrow{f} A$$

factoring the folding map, with  $i$  a cofibration and  $f$  a weak equivalence (see [5]). By Remark 4.23 of [5], if  $f, g : A \rightarrow X$  are homotopic, then there is a homotopy between them using any chosen cylinder object.

The functor  $\Gamma$  is constructed as a transfinite colimit, so it commutes with  $\operatorname{colim}_{i \in \mathcal{I}}$ , i. e.

$$\Gamma(\operatorname{colim}_{i \in \mathcal{I}} S_i) = \operatorname{colim}_{i \in \mathcal{I}} \Gamma S_i.$$

Now suppose that  $f, g : X \rightarrow \Gamma(\operatorname{colim}_{i \in \mathcal{I}} S_i)$  are homotopic maps, and  $X$  is a smooth scheme. Since  $X$  is small in  $\Delta^{op}Sh(Sm/k)_{Nis}$ ,

$$f \amalg g : X \amalg X \rightarrow \Gamma(\operatorname{colim}_{i \in \mathcal{I}} S_i)$$

lifts to some  $\Gamma S_i$ . We need to show that we can also lift the homotopy between  $f$  and  $g$ . We can choose our cylinder object to be  $X \times \mathbb{A}^1$ . Consider the diagram

$$\begin{array}{ccc} X \amalg X & \xrightarrow{f \amalg g} & \Gamma S_i \\ \downarrow & & \downarrow \\ X \times \mathbb{A}^1 & \longrightarrow & \Gamma(\operatorname{colim}_{i \in \mathcal{I}} S_i). \end{array}$$

Since  $X \times \mathbb{A}^1$  is also a small object in  $\Delta^{op}Sh(Sm/k)_{Nis}$ , the bottom map lifts to some  $\Gamma S_j$ ,  $j \in \mathcal{I}$ . Since  $\mathcal{I}$  is directed, there is some  $k \in \mathcal{I}$  such there is a commutative diagram

$$\begin{array}{ccc} X \amalg X & \longrightarrow & \Gamma S_k \\ \downarrow & \nearrow & \downarrow \\ X \times \mathbb{A}^1 & \longrightarrow & \Gamma(\operatorname{colim}_{i \in \mathcal{I}} S_i). \end{array}$$

Thus, we have

$$\begin{aligned} \mathcal{H}_{\mathbb{A}^1}(X, \operatorname{colim}_{i \in \mathcal{I}} S_i) &\cong \operatorname{colim}_{i \in \mathcal{I}} \mathcal{H}_s(X, \Gamma S_i) \\ &\cong \operatorname{colim}_{i \in \mathcal{I}} \mathcal{H}_s(X, S_i). \end{aligned}$$

□

Finally, we really need the above statements to hold in the category of based simplicial sheaves over  $Sm/k$ . We will show that a directed colimit in the unbased category of simplicial sheaves gives a directed colimit in the based category. Given a based simplicial sheaf  $X$ , we can think of it as an unbased simplicial sheaf. Then for some small directed category  $\mathcal{I}$  and functor  $S : \mathcal{I} \rightarrow \Delta^{op}Sh(Sm/k)_{Nis}$ , we have  $X = \operatorname{colim}_{i \in \mathcal{I}} S_i$  in the category of unbased simplicial sheaves. The basepoint of  $X$  is a map  $\operatorname{Spec}(k) \rightarrow X$ . Since  $\operatorname{Spec}(k)$  is a small object, it lifts to  $S_i$  for some  $i \in \mathcal{I}$ . Let the small directed category  $\mathcal{I}'$  be the comma category of objects in  $\mathcal{I}$  with a map from  $i$ . Then in the category of based simplicial sheaves,

$$X = \operatorname{colim}_{j \in \mathcal{I}'} S_j.$$

The remaining constructions carry through. In this sense, both the categories of unbased and based  $k$ -spaces are “compactly generated”, i. e. every object is a directed colimit of finite colimits of smooth schemes. In particular, note that the collection

of isomorphism classes of finite colimits of smooth schemes is a set, as opposed to a class.

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