

Algebraic Morava K-theories and the
higher degree formula

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CHAPTER 1

Introduction

In this thesis we will use homotopy-theoretical methods in algebraic geometry. At the end of 1996 V. Voevodsky published the paper [9] in which he proved the Milnor conjecture. To get such a result he employed standard homotopy theory techniques in an algebraic context. Referring to [9], Theorem 2.25 gives sufficient conditions for the Milnor conjecture to hold; in particular the second one states that $H_B^{n+1,n}(\check{C}(X_{\underline{a}}), \mathbb{Z}_{(2)}) = 0$ where H_B is the Beilinson motivic cohomology and $\check{C}(X_{\underline{a}})$ is a certain simplicial smooth scheme. By ordinary algebraic geometry techniques, which include two theorems of M. Rost ([6], [7]), it is possible to prove that $H_B^{2^n-1, 2^n-1}(\check{C}(X_{\underline{a}}), \mathbb{Z}) = 0$, implying that any class in $H_B^{2^n-1, 2^n-1}(\check{C}(X_{\underline{a}}), \mathbb{Z}/2)$ which is a restriction of an integral homology class is 0. On the other hand, by properties of $X_{\underline{a}}$ and a transfer argument, it is sufficient to prove that the classes in

$$H_B^{n+1,n}(\check{C}(X_{\underline{a}}), \mathbb{Z}/2)$$

which are reduction of integral classes, are zero. In algebraic topology, understanding $H_B^{*,*}(\quad, \mathbb{Z}/2)$ as ordinary cohomology with $\mathbb{Z}/2$ coefficients, this problem could be solved as follows: we have the cohomological operation $Q_{n-2}Q_{n-3} \cdots Q_1 : H^{n+1}(X, \mathbb{Z}/2) \rightarrow H^{2^n-1}(X, \mathbb{Z}/2)$ which we want to show to be injective proving that

$H^{n+1}(X, \mathbb{Z}_{(2)}) = 0$. The injectivity of such operation can be reduced to the inductive assumption $H^i(X, \mathbb{Z}_{(2)}) = 0$ for $i < n + 1$ and to the study of the so called Margolis homology groups of $H^{n+1}(X, \mathbb{Z}/2)$ for the X we are interested in. The problem was therefore reduced to creating an environment in which the Beilinson motivic cohomology could be representable by some object. In addition it might be possible to do other useful constructions. In fall 1998 all the details of such construction were published in [5]. The vanishing of the Margolis groups of the cohomology of a space X , can be rephrased as the vanishing of a certain map $\Phi_1^{*,*}(X_+) \rightarrow \Phi_1^{*-2d, *-d}(X_+)$ where Φ_1 is the fiber of $\mathbf{H}_{\mathbb{Z}/q} \xrightarrow{Q_t} \Sigma^{2q^t-1, q^t-1} \mathbf{H}_{\mathbb{Z}/q}$. This can be considered as a first approximation to algebraic Morava K-theory.

A few weeks after V. Voevodsky proposed that I investigate a general construction of the Morava K-theories, M. Rost, during a conference in Berkeley in May 1998, announced that he knew of a program that was likely to prove the Milnor conjecture at odd primes, such result goes under the name of Bloch-Kato conjecture. Rost conjectured the existence of an “higher degree formula” (Voevodsky’s “degree formula” being the $r = 1$ case): let k be a field of characteristic 0, q a prime number, X and Y be projective, smooth schemes of dimension $r(q^t - 1)$ and r and t positive integers. If $f : Y \rightarrow X$ is dominant, then there exists a certain characteristic number t_r such that $t_r(Y) \equiv \deg(f) \cdot t_r(X) \pmod{I_r(X)}$ where $I_r(X)$ is the reduction modulo q of $J_r(X)$ an ideal of \mathbb{Z} associated to X . In his program, M. Rost deals with varieties Y cobordant to q fold products of projective, smooth, irreducible varieties Y_i of dimension $q^n - 1$ and such that $s_{q^n-1}(Y_i) \not\equiv 0 \pmod{q^2}$ for

each i . If the Chern classes c_h can be expressed as the h -th elementary symmetric polynomials in the t_i , then s_k is the characteristic class corresponding to the k -th Newton polynomial in variables t_i . The characteristic number $t_r(Y)$ should be nonzero for such varieties and this is exactly the place where algebraic Morava K-theories appear. The point being again that the cohomological operations Q_n detect varieties Y_i with $s_{q^n-1}(Y_i) \neq 0$ in a precise sense.

At this stage, it is hard to formulate a possible definition for what algebraic Morava K-theories should be: in topology these spectra $k(t)$ are characterized by $H^*(k(t), \mathbb{Z}/q) = \frac{\mathcal{A}_{top}}{\mathcal{A}_{top}Q_t}$ or equivalently $\pi_*(k(t)) = \mathbb{Z}/q[v_t]$ with v_t being an homotopy class in dimension $2(q^t - 1)$. Currently, in the motivic stable homotopy category and for a prime q , two algebraic Morava K-theories are known: one has a canonical **MGI**-module structure in the sense of [1] and the “right” cohomology (Theorem 2.1), but the homotopy is not known; the other, considered here in the first chapter, has the same cohomology, admits a resolution of Eilenberg MacLane spectra. No ring structures are known in either cases and very few things can be deduced about their homotopy. In the second chapter the higher degree formula is established by showing that any map **MGI** $\rightarrow \Phi_r$ which is nonzero when composed with the resolution morphisms $\Phi_r \rightarrow \Phi_{r-1} \rightarrow \cdots \rightarrow \mathbf{H}_{\mathbb{Z}/q}$ can be used to define the characteristic number t_r . At this point it becomes essential to have a resolution by motivic Eilenberg-MacLane spectra of our model of algebraic Morava K-theory. In the fourth chapter we prove that varieties X satisfying a certain condition (Proposition 4.1) will have the property that $I_r(X) = 0$. This makes the higher

degree formula a nontrivial result. All the norm varieties are known to satisfy this condition. For any such variety, the higher degree formula provides a surprising evidence of how strongly topological properties (characteristic numbers) can affect algebro-geometrical invariants (degree of maps), showing in particular that for certain varieties the degree of a map between them can be cobordism invariant.

CHAPTER 2

The construction of an algebraic Morava K-theory spectrum

In this section we are going to define what will be called the algebraic Morava K-theories. Throughout this paper $Spec k$ will always denote the spectrum of a field of characteristic 0, the number q a fixed prime number and t will be a nonnegative integer. By construction, these spectra will come equipped with a $\mathbf{H}_{\mathbb{Z}/q}$ -pro nilpotent resolution. Po Hu in [2], uses a slightly different definition of spectra from the one proposed by Voevodsky in [11]. Her definition, makes the category of **MGI**-module spectra closed under colimits. From this she has built a spectrum $k'(t)$, which comes equipped with a **MGI**-module structure, by taking the homotopy colimit of

$$(2.2.1) \quad \mathbf{MGI} \xrightarrow{\cdot q} \mathbf{E}_0 \xrightarrow{\eta_1} \mathbf{E}_1 \xrightarrow{\eta_2} \cdots \xrightarrow{\eta_{q^t-2}} \mathbf{E}_{q^t-2} \xrightarrow{\eta_{q^t}} \mathbf{E}_{q^t} \xrightarrow{\eta_{q^t+1}} \cdots$$

with \mathbf{E}_i defined by the exact triangle

$$(2.2.2) \quad \Sigma^{2(i-1), i-1} \mathbf{E}_{i-1} \xrightarrow{\cdot a_{i-1}} \mathbf{E}_{i-1} \xrightarrow{\eta_i} \mathbf{E}_i$$

induced by the **MGI**-module multiplication of certain homotopy class $a_{i-1} \in \pi_{2i,i}(\mathbf{MGI})$.

Of course, in order for this to make sense we need to define what the homotopy classes a_i are. Now, $\mathbf{MGI}^{*,*}(\mathbb{P}^\infty)$ is the power series ring $\mathbf{MGI}^{*,*}(\mathbf{S}^0)[[x]]$ with $x \in \mathbf{MGI}^{2,1}(\mathbb{P}^\infty)$ canonically defined by the identity $\mathbb{P}^\infty \rightarrow \mathbf{MGI}_{2,1} = \mathbb{P}^\infty$. From

the Kunneth theorem, we have

$$(2.2.3) \quad \mathbf{MGI}^{*,*}(\mathbb{P}^\infty \times \mathbb{P}^\infty) = \mathbf{MGI}^{*,*}(\mathbb{P}^\infty) \hat{\otimes} \mathbf{MGI}^{*,*}(\mathbb{P}^\infty) = \mathbf{MGI}^{*,*}(\mathbf{S}^0)[[x_1, x_2]].$$

Thus the multiplication $\mathbb{P}^\infty \times \mathbb{P}^\infty \xrightarrow{m} \mathbb{P}^\infty$ induces a map

$$\mathbf{MGI}^{*,*}(\mathbf{S}^0)[[x]] \xrightarrow{m^*} \mathbf{MGI}^{*,*}(\mathbf{S}^0)[[x_1, x_2]].$$

The power series in two variables m^*x is defined to be the formal group law associated with \mathbf{MGI} . It gives a canonical map $\mathbf{MU}_*(\mathbf{S}^0) = L \rightarrow \mathbf{MGI}_{*,*}(\mathbf{S}^0)$ from the Lazard ring L to the motivic homotopy of \mathbf{MGI} . We denote by a_i the images under this map of the polynomial generators of \mathbf{MU}_* ; notice that $a_i \in \mathbf{MGI}_{2i,i}$. We need the following result:

Theorem 2.1. *The spectra $k'(t)$, as defined above, has the property that*

$$H^{*,*}(k'(t), \mathbb{Z}/q) \cong \frac{\mathcal{A}^{**}}{\mathcal{A}^{**}Q_t}$$

and there exist an exact triangle $\Sigma^{2(q^t-1), q^t-1}k'(t) \rightarrow k'(t) \rightarrow C_0$, with $H^{*,*}(C_0, \mathbb{Z}/q) \cong \mathcal{A}^{**}$.

Proof. The motivic cohomology of \mathbf{BGI} is known to be a polynomial algebra over the Chern classes with $H^{*,*}(\mathit{Spec} k, \mathbb{Z}/q)$ as coefficient ring ([4], Theorem 4.1.10). To prove the Theorem, we will need the next result which gives a partial description of the action on the motivic Steenrod algebra on the cohomology of \mathbf{MGI} :

Definition 2.1. Let \mathbf{E} and \mathbf{F} any two spectra; for a class $\alpha \in [\mathbf{E}; \Sigma^{p,q}\mathbf{F}]$ the invariant is defined as the integer $\|\alpha\| = p - 2q$.

Proposition 2.1. Let $I = (i_1, i_2, \dots, i_k)$. Then $H^{*,*}(\mathbf{MGL}, \mathbb{Z}/q)$ is a free $\frac{\mathcal{A}^{**}}{\mathcal{A}^{**}(Q_0, \dots, Q_t, \dots)}$ -module over a basis $\{m'_I\}$ with $m'_I \in H^{2(\Sigma^{i_j}), (\Sigma^{i_j})}(\mathbf{MGL}, \mathbb{Z}/q)$ and $i_j \neq q^n - 1$ for any positive integer n .

Proof. The Chern classes c_i have invariant 0 because they are in bidegree $(2i, i)$, hence $Q_j c_i$ is zero for any i and j because $\|Q_j\| = 1$, and there are no classes with invariant 1 in the cohomology of \mathbf{MGL} (see Voevodsky [9], Theorem 3.21). Therefore the action of the motivic Steenrod algebra on $H^{*,*}(\mathbf{MGL}, \mathbb{Z}/q)$ factors through $\frac{\mathcal{A}^{**}}{\mathcal{A}^{**}(Q_0, \dots, Q_t, \dots)}$. This action is described just by looking at Chern classes and then extend by linearity on $H^{*,*}$. A typical element in $\frac{\mathcal{A}^{**}}{\mathcal{A}^{**}(Q_0, \dots, Q_t, \dots)}$ can be written as $a(r_1, r_2, \dots)$ in the Milnor basis [3], with $a \in H^{*,*}$. The action of that operation on a Chern class c_i is $(a(r_1, \dots)) \cdot c_i = a((r_1, \dots)c_i)$, but $(r_1, \dots)c_i$ is a polynomial with \mathbb{Z}/q coefficients in the Chern classes, because it has 0 invariant, and these are the only such classes in $H^{*,*}(\mathbf{MGL}, \mathbb{Z}/q)$. It follows that, since $H^{*,*}(\mathbf{MGL}, \mathbb{Z}/q)$ is a free left $H^{*,*}$ -module, the only relations in the action of $\frac{\mathcal{A}^{**}}{\mathcal{A}^{**}(Q_0, \dots, Q_t, \dots)}$ could possibly arise only topologically. However, it is well known that it is not the case. \square

The next step is to study the cohomology of the \mathbf{E}_i ; to do this we have to understand the behaviour of a_i^* in cohomology. There are two different phenomena happening: the first one is described by

Lemma 2.1. *Assume $i \neq q^n - 1$. The cohomology long exact sequence associated to*

$$(2.2.4) \quad \Sigma^{2i,i} \mathbf{E}_i \xrightarrow{\cdot a_i} \mathbf{E}_i \xrightarrow{\eta_{i+1}} \mathbf{E}_{i+1}$$

splits in short exact sequences of the kind

$$(2.2.5) \quad 0 \longleftarrow H^{*+2i,*+i}(\mathbf{E}_i, \mathbb{Z}/q) \xleftarrow{(\cdot a_i)^*} H^{*,*}(\mathbf{E}_i, \mathbb{Z}/q) \longleftarrow H^{*,*}(\mathbf{E}_{i+1}, \mathbb{Z}/q) \longleftarrow 0$$

Proof. To see what $\cdot a_i$ induces in cohomology we can argue as follows: let $I = (i_1, i_2, \dots, i_k)$, then, in topology, $\{m_1^{i_1} m_2^{i_2} \dots m_k^{i_k}\}$ is a basis over \mathbb{Z} of $H_*(\mathbf{MU}, \mathbb{Z})$, and their duals are the cohomological classes denoted by the symbol m'_I . The Hurewicz homomorphism $\pi_*(\mathbf{MU}) \xrightarrow{h} H_*(\mathbf{MU}, \mathbb{Z})$ is known to send a_i to classes $m_i \in H_{2i}(\mathbf{MU}, \mathbb{Z})$ for $i \neq p^n - 1$ and a_{p^n-1} to pm_{p^n-1} for any n and any prime p . In our context, let's define m_i to be the images of a_i for $i \neq p^n - 1$ and m_{p^n-1} to be $h(a_{p^n-1})/p$; the last definition makes sense because of the following:

Lemma 2.2. *$H_{*,*}(\mathbf{MGl}, \mathbb{Z})$ is a free left $H_{*,*}$ -module over the m_I .*

Proof. Using similar methods as those of Voevodsky in [10], it is possible to prove that the motive of the Grassmannian $G_n(\mathbb{A}_k^{n+m})$ of n -planes in the $(n+m)$ -dimensional affine space splits as $\oplus \mathbb{Z}(\ast)[2\ast]$ with one direct summand for each topological homology class of the same Grassmannian. Since motivic homology

of a scheme X is represented in the category DM (see [10]) as $H_{i,j}(X, \mathbb{Z}) = \text{Hom}_{DM}(\mathbb{Z}(j)[i]; M(X))$ where $M(X)$ represents the motive of X , the Lemma follows at once. \square

The previous Lemma implies that the topological realization map

$$H_{2i,i}(\mathbf{MGI}, \mathbb{Z}/q) \rightarrow H_{2i}(\mathbf{MU}, \mathbb{Z}/q)$$

is an isomorphism for all i , therefore $h(a_{q^n-1})$ must be divisible by q . As in topology, denote with m'_I the dual class to $m_1^{i_1} \cdots m_k^{i_k} \in H^{*,*}(\mathbf{MGI}, \mathbb{Z}/q)$, being $I = (i_1, \dots, i_k)$; by brevity we let $m'_i = m'_{(0, \dots, \overset{i}{1})}$ and assume that $i \neq q^n - 1$. Denote by c the generator of $\tilde{H}_{*,*}((\mathbf{P}^1)^{\wedge i}, \mathbb{Z}/q)$ as $H_{*,*}$ -module and by $\cdot a_i$ the map $(\mathbf{P}^1)^{\wedge i} \rightarrow \mathbf{MGI}$; then by duality, $1 = \langle m'_i, m_i \rangle = \langle m'_i; (\cdot a_i)_* c \rangle = \langle (\cdot a_i)^* m'_i; c \rangle$, hence $(\cdot a_i)^* m'_i \in \tilde{H}^{2i,i}((\mathbf{P}^1)^{\wedge i}, \mathbb{Z}/q)$ is going to be the generator of the reduced cohomology of $(\mathbf{P}^1)^{\wedge i}$. Using the \mathbf{MGI} -module structure of \mathbf{E}_i we can make $\cdot a_i$ in a map of spectra $\Sigma^{2i,i} \mathbf{E}_i \xrightarrow{\cdot a_i} \mathbf{E}_i$. By what just mentioned, m'_i is going to hit the bottom cohomology class of $\Sigma^{2i,i} \mathbf{E}_i$, which we call $\Sigma^{2i,i} \tau$. From the \mathcal{A}^{**} linearity of $(\cdot a_i)^*$ in cohomology, we right away get that $\frac{\mathcal{A}^{**}}{\mathcal{A}^{**}(Q_0, \dots, Q_t, \dots)} \Sigma^{2i,i} \tau \xleftarrow{(\cdot a_i)^*} \frac{\mathcal{A}^{**}}{\mathcal{A}^{**}(Q_0, \dots, Q_t, \dots)} m'_i$ is an isomorphism. $(\cdot a_i)_*$ preserves the left $H_{*,*}(\mathbf{MGI}, \mathbb{Z}/q)$ -module structure, therefore in cohomology $m'_{(0, \dots, \overset{j}{1}, 0, \dots, \overset{i}{1})}$ is going to map to $\Sigma^{2i,i} m'_j$ because

$$\begin{aligned}
(2.2.6) \quad & \langle a_i^* (m'_{(0, \dots, \overset{j}{1}, 0, \dots, \overset{i}{1})}), \Sigma^{2i, i} m_j \rangle = \langle m'_{(0, \dots, \overset{j}{1}, 0, \dots, \overset{i}{1})}, (a_i)_* (\Sigma^{2i, i} m_j) \rangle \\
& = \langle m'_{(0, \dots, \overset{j}{1}, 0, \dots, \overset{i}{1})}, m_j (a_i)_* (\Sigma^{2i, i} 1) \rangle = \langle m'_{(0, \dots, \overset{j}{1}, 0, \dots, \overset{i}{1})}, m_j m_i \rangle = 1
\end{aligned}$$

Similarly, $m'_{J+(0, \dots, \overset{i}{1})}$ hits $\Sigma^{2i, i} m'_j$, and this implies that $(a_i)^*$ is surjective and the same is true with finite coefficients. This proves Lemma 2.1. \square

The previous Lemma also shows that

$$H^{*,*}(\mathbf{E}_i, \mathbb{Z}/q) \cong H^{*,*}(\mathbf{E}_{i-1}, \mathbb{Z}/q)/M_i$$

where M_i is the $H^{*,*}$ -submodule generated by the m'_l with the nonzero i -th entry.

On the other hand, when we take the cone \mathbf{E}_{q^n} of the multiplication by $a_{q^{n-1}}$ we observe that its motivic cohomology is actually bigger than the one of $\mathbf{E}_{q^{t-1}}$:

Lemma 2.3. *The cohomology long exact sequence associated to*

$$(2.2.7) \quad \Sigma^{2(q^n-1), q^n-1} \mathbf{E}_{q^{n-1}} \xrightarrow{a_{q^{n-1}}} \mathbf{E}_{q^{n-1}} \xrightarrow{\eta_{q^n}} \mathbf{E}_{q^n}$$

splits in short exact sequences

$$(2.2.8) \quad 0 \longleftarrow H^{*,*}(\mathbf{E}_{q^{n-1}}; \mathbb{Z}/q) \xleftarrow{\eta_{q^n}^*} H^{*,*}(\mathbf{E}_{q^n}; \mathbb{Z}/q) \xleftarrow{\delta} H^{*+1,*}(\Sigma^{2(q^n-1), q^n-1} \mathbf{E}_{q^{n-1}}; \mathbb{Z}/q) \longleftarrow 0$$

Proof. As before we have the following maps dual one to the other

$$(2.2.9) \quad \begin{aligned} H^{*,*}((\mathbf{P}^1)^{\wedge q^n-1}; \mathbb{Z}/q) &\xleftarrow{(a_{q^{n-1}})^*} H^{*,*}(\mathbf{E}_{q^{n-1}}; \mathbb{Z}/q) \\ H_{*,*}((\mathbf{P}^1)^{\wedge q^n-1}; \mathbb{Z}/q) &\xrightarrow{(a_{q^{n-1}})_*} H_{*,*}(\mathbf{E}_{q^{n-1}}; \mathbb{Z}/q) \end{aligned}$$

The Hurewicz homomorphism is defined by the image of the generator of

$$H_{*,*}((\mathbf{P}^1)^{\wedge q^n-1}; \mathbb{Z})$$

through $(a_{q^{n-1}})_*$ which we know being 0, when composed with the reduction map modulo \mathbb{Z}/q , hence also its dual map in cohomology is zero; therefore the short exact sequences in this case are of the kind stated in the Lemma. \square

A precise description of the action of the motivic Steenrod algebra on $H^{*,*}(\mathbf{E}_{q^n}, \mathbb{Z}/q)$ can be given. In particular

Lemma 2.4. *Let τ_{q^n} be the bottom cohomology class of \mathbf{E}_{q^n} ; then $Q_n(\tau_{q^n}) \in H^{*,*}(\mathbf{E}_{q^n}, \mathbb{Z}/q)$ is nonzero, hence it must be $\delta(\Sigma^{2(q^n-1), q^n-1} \tau_{q^{n-1}})$.*

In other words, going from $\mathbf{E}_{q^{n-1}}$ to \mathbf{E}_{q^n} we have added the operation Q_n on τ_{q^n} .

Proof. Voevodsky in [9], Lemma 3.6, has proved that any lifting e of the Thom class $\mathbf{MGl} \rightarrow \mathbf{H}_{\mathbb{Z}/q}$ to Φ_1 (the fiber of $\mathbf{H}_{\mathbb{Z}/q} \xrightarrow{Q_n} \Sigma^{|\mathbf{Q}_n|} \mathbf{H}_{\mathbb{Z}/q}$) sends the homotopy class $a_{q^{n-1}}$ to a nonzero homotopy class of Φ_1 . This is equivalent to showing that $Q_n \tau \neq 0$

in the cone of $\Sigma^{|a_{q^n-1}|} \mathbf{MGI} \xrightarrow{a_{q^n-1}} \mathbf{MGI}$ as one can easily see from the commutative diagram

(2.2.10)

$$\begin{array}{ccccccc}
 \cdots & \Sigma^{-1,0} \mathbf{H}_{\mathbb{Z}/q} & \xrightarrow{\Sigma^{-1,0} Q_n} & \Sigma^{|a_{q^n-1}|} \mathbf{H}_{\mathbb{Z}/q} & \longrightarrow & \Phi_1 & \longrightarrow & \mathbf{H}_{\mathbb{Z}/q} & \xrightarrow{Q_n} & \Sigma^{|Q_n|} \mathbf{H}_{\mathbb{Z}/q} \\
 & \uparrow \Sigma^{-1,0} \tau & \nearrow \Sigma^{-1,0} (Q_n \tau) & & & \uparrow e & & \uparrow \tau & \nearrow Q_n(\tau) & \\
 \cdots & \Sigma^{-1,0} \text{cone}(a_{q^n-1}) & \longrightarrow & (\mathbf{P}^1)^{\wedge 2(q^n-1)} & \xrightarrow{a_{q^n-1}} & \mathbf{MGI} & \longrightarrow & \text{cone}(a_{q^n-1}) & \cdots
 \end{array}$$

Using the map $\mathbf{MGI} \rightarrow \mathbf{E}_{q^n-1}$ we can right away produce a map $f : \text{cone}(a_{q^n-1}) \rightarrow \mathbf{E}_{q^n}$ as fill in map between two exact triangles. Consequently, we get $f^*(Q_n \tau_{q^n}) = Q_n f^*(\tau_{q^n}) = Q_n \tau \neq 0$ proving the Lemma. \square

Corollary 2.1. *The class $Q_n \tau_{q^n}$ generates a free copy of $H^{*,*}$.*

Proof. By induction, assume that $H^{*,*}(\mathbf{E}_{q^n-1}; \mathbb{Z}/q)$ is $H^{*,*}$ free; if $a Q_n \tau_{q^n-1} = b_1 \beta_1 + b_2 \beta_2 + \cdots + b_k \beta_k$ with $b_i \in H^{*,*}$ and β_i some classes of the cohomology of \mathbf{E}_{q^n-1} then not all of the β_i are in the image of δ or else the equality $a \Sigma^{2(q^n-1), q^n-1} \tau_{q^n} = b_1 \beta_1 + b_2 \beta_2 + \cdots + b_k \beta_k$ would hold in $H^{*,*}(\Sigma^{2(q^n-1), q^n-1} \mathbf{E}_{q^n-1}; \mathbb{Z}/q)$, contraddicting the inductive assumption. On the other hand, applying $\eta_{q^n}^*$ to that very same equality we would get $0 = b_{i_1} \beta_{i_1} + b_{i_2} \beta_{i_2} + \cdots + b_{i_h} \beta_{i_h}$ for those β_{i_j} that are not in the image of δ , contraddicting again the inductive assumption. \square

In conclusion, after having taken the colimit of the diagram 2.2.1, the cohomology is going to be as stated by the theorem and the cone of $\Sigma^{2(q^t-1), q^t-1} k'(t) \xrightarrow{a_{q^t-1}} k'(t)$

will have the motivic Steenrod algebra as cohomology ring. This finishes the proof of the theorem.

□

Remark: Let's denote the direct limit of 2.2.1 by $\frac{\mathbf{MGI}}{(q, a_i/i \neq q^t - 1)}$. Using the same techniques as in the proof of 2.1 one can show that

$$(2.2.11) \quad H^{*,*}\left(\frac{\mathbf{MGI}}{(q, a_i/i \neq q^k - 1, k = 1, 2, \dots)}, \mathbb{Z}/q\right) = \frac{\mathcal{A}^{**}}{\mathcal{A}^{**}(Q_0, \dots, Q_t, \dots)}$$

and

$$(2.2.12) \quad H^{*,*}\left(\frac{\mathbf{MGI}}{(q, a_i/i \neq q^k - 1, k = 1, 2, \dots, n)}, \mathbb{Z}/q\right) = \frac{\mathcal{A}^{**}}{\mathcal{A}^{**}(Q_0, \dots, Q_n)}$$

Therefore these two spectra could be taken as a model for the Brown-Peterson spectrum \mathbf{BP}' and $\mathbf{BP}'\langle n \rangle$, respectively.

We have now all the information needed to construct a tower for $k'(t)$. Consider first the following diagram in which we denote $q^t - 1 = d$ and a_{q^t-1} with v_t for brevity:

$$(2.2.13) \quad \begin{array}{ccccc} & & & \Phi'_1 & \\ & & & \uparrow & \text{---} \exists f_1 \text{---} \\ & & & k'(t) & \longrightarrow C_0 \\ \Sigma^{2d,d}k'(t) & \xrightarrow{\cdot v_t} & & & \\ & & & \uparrow \cdot v_t^2 & \\ & & & \Sigma^{4d,2d}k'(t) & \\ & & & \uparrow \cdot v_t & \\ & & & \Sigma^{2d,d}k'(t) & \end{array}$$

the multiplication by v_t is the one by the **MGI**-module structure of $k'(t)$:

$$(\mathbf{P}^1)^d \wedge k'(t) \xrightarrow{v_t \wedge id} \mathbf{MGI} \wedge k'(t) \xrightarrow{m} k'(t)$$

and both the sequences are exact triangles. From the cohomology long exact sequence associated to the exact triangle $\Sigma^{4d,2d}k'(t) \xrightarrow{v_t^2} k'(t) \rightarrow \Phi'_1$ we can get that $H^{*,*}(\Phi'_1, \mathbb{Z}/q) \cong \frac{\mathcal{A}^{**}}{\mathcal{A}^{**}Q_t}x \oplus \frac{\mathcal{A}^{**}}{\mathcal{A}^{**}Q_t}y'_1$ as left \mathcal{A}^{**} -module, with $x \in H^{0,0}(\Phi'_1, \mathbb{Z}/q)$ being the class hitting the bottom class of $k'(t)$, and $y'_1 \in H^{4d+1,2d}(\Phi'_1, \mathbb{Z}/q)$ is the image of the bottom class of $\Sigma^{4d+1,2d}k'(t)^1$; this follows since v_t^i all induce the zero map in cohomology, being the cohomology an \mathcal{A}^{**} -cyclic module. Turning now to the cone C_{-1} of f_1 , by observing the long exact sequence

$$(2.2.14) \quad \cdots \longleftarrow H^{*,*}(\Phi'_1, \mathbb{Z}/q) \longleftarrow \mathcal{A}^{**} \longleftarrow H^{*,*}(C_{-1}, \mathbb{Z}/q) \longleftarrow \cdots$$

we can see that there will definitely be a bottom class $\alpha \in H^{2d+1,d}(C_{-1}, \mathbb{Z}/q)$ going to hit $Q_t \iota$ (ι is the bottom class of \mathcal{A}^{**}) which generates an \mathcal{A}^{**} -cyclic module isomorphic to $\frac{\mathcal{A}^{**}}{\mathcal{A}^{**}Q_t}$ and then another class $\gamma \in H^{2d+2,d}(C_{-1}, \mathbb{Z}/q)$, image of y'_1 ; no other generators over \mathcal{A}^{**} are present. There is also the relation $Q_t \alpha = \gamma$ which can be obtained by realizing the exact triangle in topology. The fact that there are no other relations follows immediately from

Lemma 2.5. *The Margolis homology group $HM(\mathcal{A}^{**}, Q_t)$ is zero.*

$^1y'_1$ generates a cyclic module isomorphic to $\Sigma^{4d+1,2d} \frac{\mathcal{A}^{**}}{\mathcal{A}^{**}Q_t}$ just because $H^{*,*}(\Sigma^{4d+1,2d}k'(t), \mathbb{Z}/q) = \Sigma^{4d+1,2d} \frac{\mathcal{A}^{**}}{\mathcal{A}^{**}Q_t}$.

Proof. Let $I = (i_1, i_2, \dots)$ and $aQ_I(r_1, \dots)$ be a generic element expressed in the Milnor basis. We have

$$Q_t \cdot (aQ_I(r_1, \dots)) = Q_t \cdot^A aQ_I(r_1, \dots) \pm aQ_t Q_I(r_1, \dots).$$

Therefore left multiplication by Q_t is the differential of a double complex. Its homology is $HM(H^{*,*}, Q_t) \otimes_{\mathbb{Z}/q} HM(\mathcal{A}_{top}, Q_t)$ which must be zero because it is well known that $HM(\mathcal{A}_{top}, Q_t) = 0$. \square

All this combined together yields $H^{*,*}(C_{-1}, \mathbb{Z}/q) \cong \Sigma^{2d+1, d} \mathcal{A}^{**}$.

In the same way we can argue on the diagram

$$(2.2.15) \quad \begin{array}{ccccc} & & \Phi'_2 & & \\ & & \uparrow & \searrow \exists f_2 & \\ \Sigma^{4d, 2d} k'(t) & \xrightarrow{\cdot v_t^2} & k'(t) & \longrightarrow & \Phi'_1 \\ & \swarrow \cdot v_t & \uparrow \cdot v_t^3 & & \\ & & \Sigma^{6d, 4d} k'(t) & & \end{array}$$

and conclude with an exact triangle $\Phi'_2 \xrightarrow{f_2} \Phi'_1 \rightarrow C_1$ with $H^{*,*}(C_1, \mathbb{Z}/q) \cong \Sigma^{2d+1, d} \mathcal{A}^{**}$. The tower $\dots \rightarrow \Phi'_2 \xrightarrow{f_2} \Phi'_1 \xrightarrow{f_1} C_0$ is the tower we will need for $k'(t)$.

Define now the following exact triangle $\Phi_1 \rightarrow \mathbf{H}_{\mathbb{Z}/q} \xrightarrow{Q_t} \Sigma^{2d+1, d} \mathbf{H}_{\mathbb{Z}/q}$ so that we have the commutative diagram

$$(2.2.16) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & \Phi_1 & \longrightarrow & \mathbf{H}_{\mathbb{Z}/q} & \longrightarrow & \Sigma^{2d+1,d}\mathbf{H}_{\mathbb{Z}/q} \longrightarrow \cdots \\ & & \uparrow \text{\scriptsize $\exists i_1$} & & \uparrow 1 & & \uparrow \Sigma^{2d+1,d}1 \\ \cdots & \longrightarrow & \Phi'_1 & \longrightarrow & C_0 & \xrightarrow{f_0} & C_{-1} \longrightarrow \cdots \end{array}$$

and i_1 is the fill in map; it will induce a cohomology isomorphism because the other two maps do. We can let $y'_1 = i^*y_1$ and define a new exact triangle $\Phi_2 \rightarrow \Phi_1 \xrightarrow{y_1} \Sigma^{4d+1,2d}\mathbf{H}_{\mathbb{Z}/q}$; it can be nested in a diagram similar to the previous one which yields a map $i_2 : \Phi'_2 \rightarrow \Phi_2$ inducing isomorphism in motivic cohomology, so that y_2 can be defined likewise and continue in this fashion. The colimit of this new tower will be denoted by $k(t)$ and called the algebraic Morava K-theory for the positive integer t , at the prime q .

What we will actually use is a slightly different version of this tower for $k(t)$: infact we want to have a series of exact triangles of the kind $\Sigma^{2d,d}\Phi_{r-1} \rightarrow \Phi_r \rightarrow \mathbf{H}_{\mathbb{Z}/q}$. In the next diagram define F_i to be the $F_{i-1} \times_{\Phi_{i-1}} \Phi_i$:

$$(2.2.17) \quad \begin{array}{ccccccc} \cdots & \xrightarrow{p_2} & \Phi_2 & \xrightarrow{p_1} & \Phi_1 & \xrightarrow{p_0} & \mathbf{H}_{\mathbb{Z}/q} \\ & & \uparrow & & \uparrow & & \uparrow u \\ \cdots & \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & \text{Spec } k \end{array}$$

and the map u is the composition $\text{Spec } k \rightarrow \text{Spec } k_+ = \mathbf{S}^0 \rightarrow \mathbf{H}_{\mathbb{Z}/q}$. Notice that F_i is non canonically isomorphic to $\Sigma^{2d,d}\Phi_{i-1}$ for all i because the cone of p_0 is $\Sigma^{2d+1,d}\mathbf{H}_{\mathbb{Z}/q}$, so F_1 is the homotopy fiber of $\text{Spec } k \rightarrow \Sigma^{2d+1,d}\mathbf{H}_{\mathbb{Z}/q}$ i.e. canonically

(up to homotopy) $\Sigma^{2d,d}\mathbf{H}_{\mathbb{Z}/q}$; then by induction on the stage of the tower we get what stated.

CHAPTER 3

The theorem in the general form

Fix an odd prime q , an embedding of the base field $k \subset \mathbb{C}$, smooth projective varieties X and Y of dimension dr with $d = q^t - 1$ and a morphism $f : Y \rightarrow X$. In this chapter, $[X]$ will stand for the algebraic cobordism class of X in $\mathbf{MGI}_{*,*}$. Let $e : \mathbf{MGI} \rightarrow \Phi_r$ any map to the r -th stage of the $\mathbf{H}_{\mathbb{Z}/q}$ -resolution of the Morava K-theory at the prime q . Using the fibration $\Sigma^{2d,d}\Phi_{r-1} \xrightarrow{\phi} \Phi_r \xrightarrow{\psi} \mathbf{H}_{\mathbb{Z}/q}$ we get the following commutative diagram:

(3.3.1)

$$\begin{array}{ccccc}
 & & \pi_{2dr,dr}(\mathbf{MGI} \wedge Y_+) & & \\
 & & \downarrow e_* & & \\
 \pi_{2dr,dr}(\Sigma^{2d,d}\Phi_{r-1} \wedge Y_+) & \xrightarrow{\phi_*} & \pi_{2dr,dr}(\Phi_r \wedge Y_+) & \xrightarrow{\psi_*} & \pi_{2dr,dr}(\mathbf{H}_{\mathbb{Z}/q} \wedge Y_+) = CH_{dr}(Y) \otimes \mathbb{Z}/q = \mathbb{Z}/q[Y]_{\mathbf{H}_{\mathbb{Z}/q}} \\
 \downarrow f_* & & \downarrow f_* & & \downarrow \deg f \\
 \pi_{2dr,dr}(\Sigma^{2d,d}\Phi_{r-1} \wedge X_+) & \xrightarrow{\phi_*} & \pi_{2dr,dr}(\Phi_r \wedge X_+) & \xrightarrow{\psi_*} & \pi_{2dr,dr}(\mathbf{H}_{\mathbb{Z}/q} \wedge X_+) = CH_{dr}(X) \otimes \mathbb{Z}/q = \mathbb{Z}/q[X]_{\mathbf{H}_{\mathbb{Z}/q}} \\
 \downarrow p_X & & \downarrow p_X & & \\
 \pi_{2d(r-1),d(r-1)}(\Phi_{r-1}) & \xrightarrow[\cong]{\phi_*} & \pi_{2dr,dr}(\Phi_r) = \mathbb{Z}/q & &
 \end{array}$$

where p_X is the map induced in homotopy by the projection on the first factor $\Phi_r \wedge X_+ \rightarrow \Phi_r$.

Proposition 3.1. *For each smooth, projective scheme W of dimension n we have:*

- (a) a class $[W]_{\mathbf{MGI}} \in \mathbf{MGI}_{2n,n}(W_+)$ whose topological realization is the fundamental \mathbf{MU} -homology class $[W(\mathbb{C})]_{\mathbf{MU}}$;
- (b) a class $[W] \in \mathbf{MGI}_{2n,n}(S^0)$ whose topological realization is the cobordism class $[W(\mathbb{C})]$;
- (c) the map $\mathbf{MGI}_{*,*}(W_+) \xrightarrow{p_1} \mathbf{MGI}_{*,*}(S^0)$ induced by $W_+ \rightarrow S^0$ sends $[W]_{\mathbf{MGI}}$ to $[W]$.

Remark 3.1. *Using this proposition we can define a fundamental class $[W]_{\mathbf{E}}$ for certain spectra \mathbf{E} . Let \mathbf{E} be a ring spectrum equipped with a Thom class, i.e. a stable map $\tau : \mathbf{MGI} \rightarrow \mathbf{E}$ such that $u^*(\tau) = 1 \in \mathbf{E}^{0,0}(\mathbf{S}^0)$ where $u : \mathbf{S}^0 \rightarrow \mathbf{MGI}$ is the unit. Then define $[W]_{\mathbf{E}} = \tau_*([W]_{\mathbf{MGI}})$. Notice that this definition is compatible with the topological environment.*

Proof. As for the second and third part of the Proposition, once we have $[W]_{\mathbf{MGI}}$, we can define $[W]$ to be the image of $[W]_{\mathbf{MGI}}$ under p_1 . Infact, it suffices to prove the first part for $W = \mathbf{P}^N$ for every positive integer N : since X is projective, there is a closed embedding $W \xrightarrow{i} \mathbf{P}^N$ for some N . Let ν_W be the normal bundle of this closed embedding; then in [5] it is shown that $\mathbf{P}^N/(\mathbf{P}^N - i(W)) \cong Th(\nu_W)$, hence there is a projection $p : \mathbf{P}^N \rightarrow Th(\nu_W)$. This map induces

$$\mathbf{MGI}_{*,*}(\mathbf{P}_+^N) \xrightarrow{p_*} \mathbf{MGI}_{*,*}(Th(\nu_W)_+)$$

and we call $\alpha_N = p_*([\mathbf{P}^N]_{\mathbf{MGI}})$. There is a well known pairing called cap product with α_N :

$$(3.3.2) \quad \mathbf{MGI}_{2N,N}(Th(\nu_W)_+) \otimes \mathbf{MGI}^{2(N-n),N-n}(Th(\nu_W)_+) \xrightarrow{\alpha_N \cap -} \mathbf{MGI}_{2n,n}(W_+).$$

we will define $[W]_{\mathbf{MGI}}$ to be $\alpha_N \cap \tau_{\nu_W}$, where $\tau_{\nu_W} : Th(\nu_W) \rightarrow Th(\eta_{N-n})$ is the Thom class of ν_W i.e. the \mathbf{MGI} cohomology class induced by the classifying map

$$(3.3.3) \quad \begin{array}{ccc} \nu_W & \longrightarrow & \eta_{N-n} \\ \downarrow & & \downarrow \\ W & \longrightarrow & G_{N-n} \end{array}$$

η_{N-n} being the universal $N - n$ -plane bundle. To prove the first statement of the Proposition for \mathbf{P}^N , we need to use the following

Lemma 3.1. *Let \mathbf{E} be a ring spectra with multiplication m . If $\mathbf{E}^{*,*}(\mathbf{P}_+^N)$ is a free and finitely generated $\mathbf{E}^{*,*}(S^0)$ -module with basis b_0, b_1, \dots, b_N in bidegree $(2i, i)$, then $\mathbf{E} \wedge \mathbf{P}_+^N$ splits as $\bigvee_{b_i} \mathbf{E} \wedge (\mathbf{P}^1)^{\wedge i}$.*

Proof. There is an exact triangle $\mathbf{P}^{N-1} \hookrightarrow \mathbf{P}^N \rightarrow (\mathbf{P}^1)^{\wedge N}$. We will denote with ϕ_i the composition

$$(3.3.4) \quad \mathbf{E} \wedge \mathbf{P}^N \xrightarrow{1 \wedge b_i} \mathbf{E} \wedge \mathbf{E} \wedge (\mathbf{P}^1)^{\wedge i} \xrightarrow{m \wedge 1} \mathbf{E} \wedge (\mathbf{P}^1)^{\wedge i}.$$

Let also ϕ^N be the wedge of the ϕ_i for all the $i = 0, 1, \dots, N$. Then we have the following commutative diagram

$$(3.3.5) \quad \begin{array}{ccccc} \mathbf{E} \wedge \mathbf{P}^{N-1} & \longrightarrow & \mathbf{E} \wedge \mathbf{P}^N & \longrightarrow & \mathbf{E} \wedge (\mathbf{P}^1)^{\wedge N} \\ \cong \downarrow \phi^{N-1} & & \downarrow \phi^N & & \parallel \\ \bigvee_{i=0}^{N-1} \mathbf{E} \wedge (\mathbf{P}^1)^{\wedge i} & \longrightarrow & \bigvee_{i=0}^N \mathbf{E} \wedge (\mathbf{P}^1)^{\wedge i} & \longrightarrow & \mathbf{E} \wedge (\mathbf{P}^1)^{\wedge N} \end{array}$$

in which the left isomorphism holds by induction, and the assumption on the freeness of the \mathbf{E} -cohomology of \mathbf{P}^N implies the commutativity of the right square. \square

The fact that \mathbf{MGI} satisfies the assumptions of the Lemma is proved in [4], therefore we can conclude that $\mathbf{MGI} \wedge \mathbf{P}_+^N \cong \bigvee_{i=0}^N \mathbf{MGI} \wedge (\mathbf{P}^1)^{\wedge i}$. It follows that a map $(\mathbf{P}^1)^{\wedge N} \rightarrow \mathbf{MGI} \wedge \mathbf{P}_+^N$ is an element of a direct sum of the kind

$$\bigoplus_{i=0}^N \pi_{2N,N}(\mathbf{MGI} \wedge (\mathbf{P}^1)^{\wedge i}) = \bigoplus_{i=0}^N \pi_{2(N-i),(N-i)}(\mathbf{MGI}).$$

Recall that the canonical formal group law of \mathbf{MGI} gives a canonical embedding of the Lazard ring $L_{*,*} \hookrightarrow \pi_{*,*}(\mathbf{MGI})$, therefore letting $[\mathbf{P}^N]_{\mathbf{MGI}} = \sum_{i=0}^N u_i$ where the homotopy classes $u_i \in L_{2i,i}$ are described by $[\mathbf{P}^N(\mathbb{C})]_{\mathbf{MU}} = \sum_{i=0}^N u_i$ gives a definition compatible with the topological one. \square

Remark 3.1 implies at once the following

Corollary 3.1. *Let $e_r : \mathbf{MGI} \rightarrow \Phi_r$ be defined inductively as a lifting of e_{r-1} :*

$$(3.3.6) \quad \begin{array}{ccccc} \Phi_r & \longrightarrow & \Phi_{r-1} & \longrightarrow & \Sigma^{2dr+1, dr} \mathbf{H}_{\mathbb{Z}/q} \\ & \swarrow e_r & \uparrow e_{r-1} & \searrow 0 & \\ & & \mathbf{MGI} & & \end{array}$$

and $e_0 : \mathbf{MGI} \rightarrow \mathbf{H}_{\mathbb{Z}/q}$ is the Thom class. Then $(\psi_* \circ (e_r)_*)[X]_{\mathbf{MGI}} = [X]_{\mathbf{H}_{\mathbb{Z}/q}}$, and the same is true for Y .

An immediate, fundamental consequence is the following congruence: we let $I_r(X)$ to be the image of $p_X \circ \phi_*$ and $t_r(W) = p_W((e_r)_*([W]_{\mathbf{MGI}})) \in \pi_{2dr, dr}(\phi_r) \cong \mathbb{Z}/q$ in 3.3.1;

$$(3.3.7) \quad t_r(Y) \equiv \deg(f)t_r(X) \pmod{I_r(X)}$$

The purpose of the remaining part of this chapter is to find sufficient conditions for $t_r(Y)$ and $t_r(X)$ to be nonzero. We have the commutative diagram:

$$(3.3.8) \quad \begin{array}{ccc} \pi_{2dr, dr}(\mathbf{MGI} \wedge Y_+) & \xrightarrow{(e_r)_*} & \pi_{2dr, dr}(\Phi_r \wedge Y_+) \\ \downarrow f_* & & \downarrow f_* \\ \pi_{2dr, dr}(\mathbf{MGI} \wedge X_+) & \longrightarrow & \pi_{2dr, dr}(\Phi_r \wedge X_+) \\ \downarrow p_X & & \downarrow p_X \\ \pi_{2dr, dr}(\mathbf{MGI}) & \xrightarrow{(e_r)_*} & \pi_{2dr, dr}(\Phi_r) \end{array}$$

which suggests us that if we had an \mathbf{MGI} -orientation e_r of Φ_r such that $(e_r)_*[Y]$ is nonzero in $\pi_{2dr, dr}(\Phi_r)$ ($[Y]$ here stands for the algebraic cobordism class of Y), then automatically $p_X(f_*(\alpha_Y)) \neq 0$ because of the part (c) of 3.1.

Definition 3.1. Let v_t denote any homotopy class in $\pi_{2d,d}(\mathbf{MGl})$ such that there exists a lifting e_1 of a map representing the Thom class in $\tau \in H^{0,0}(\mathbf{MGl}, \mathbb{Z}/q)$ with the property that $(e_1)_*v_t$ is a non zero homotopy class in $\pi_{2d,d}(\Phi_1)$.

For example, we can take v_t to be the algebraic cobordism class represented by a v_t -variety (see next chapter for the definition), as shown by Voevodsky in [9], Lemma 3.6.

Proposition 3.2. Any morphism $\mathbf{MGl} \xrightarrow{e_r} \Phi_r$ such that the composition $\mathbf{MGl} \xrightarrow{e_r} \Phi_r \rightarrow \Phi_{r-1} \rightarrow \cdots \rightarrow \Phi_1 \rightarrow \mathbf{H}_{\mathbb{Z}/q}$ is nonzero, has the property that $(e_r)_*(v_t^r) \neq 0$.

Proof. The topological realization of e_r , which we will denote by e_r^{top} , is going to be nonzero when composed down to $\mathbf{H}_{\mathbb{Z}/q}$ in the tower. By the previous definition of v_t , v_t^r will map to its topological counterpart so it is easy to see that we only have to prove the Proposition in the topological case. The base of the induction amounts to proving that for any lifting e of the Thom class, e_*v_t is nonzero. The difference Δ between two liftings e' and e'' factors through $\Sigma^{2d}\mathbf{H}_{\mathbb{Z}/q}$:

$$(3.3.9) \quad \begin{array}{ccccccc} & & \Sigma^{2d}\mathbf{H}_{\mathbb{Z}/q} & \longrightarrow & \Phi_1 & & \\ & \nearrow & \uparrow \Delta & \nearrow e & \downarrow & & \\ \mathbf{S}^{2d} & \xrightarrow{v_t} & \mathbf{MU} & \xrightarrow{\tau} & \mathbf{H}_{\mathbb{Z}/q} & \xrightarrow{Q_t} & \Sigma^{2d+1}\mathbf{H}_{\mathbb{Z}/q} \end{array}$$

Since the image of v_t by the Hurewicz map is zero, v_t^* is zero in cohomology, hence the composition $\mathbf{S}^{2d} \xrightarrow{v_t} \mathbf{MU} \xrightarrow{\Delta} \Sigma^{2d}\mathbf{H}_{\mathbb{Z}/q}$ is null homotopic. Let now e_r be as in the

statement of the Proposition; we have the commutative diagram

$$(3.3.10) \quad \begin{array}{ccccc} \Sigma^{2d}\Phi_{r-1} & \longrightarrow & \Phi_r & \longrightarrow & \mathbf{H}_{\mathbb{Z}/q} \\ & & \uparrow e_r & \nearrow 0 & \\ f \uparrow \vdots & & & & \\ \Sigma^{2d}\mathbf{MU} & \xrightarrow{v_t} & \mathbf{MU} & & \end{array}$$

where $f_*(v_t^{r-1})$ is nonzero by inductive assumption and it must go to a nonzero homotopy class of Φ_r which, by commutativity, is the same as $(e_r)_*((v_t)_*(v_t^{r-1})) = (e_r)_*(v_t^r)$. \square

CHAPTER 4

The higher degree formula

In this section we are going to specialize the formula 3.3.7 to the case of smooth, projective algebraic varieties with the space of \mathbb{C} -rational points having the same cobordism class of a product of smooth, irreducible, projective (v_t, q) -varieties X_i of dimension $d = q^t - 1$. As before $[X]$ will denote the algebraic cobordism class of X , $[X(\mathbb{C})]$ the topological cobordism class of the \mathbb{C} -rational points of X and $[X]_{\mathbf{E}}$ will be the \mathbf{E} -fundamental class of X for a ring spectrum \mathbf{E} .

Let me recall the following (see [9]):

Definition 4.1. *A complex algebraic variety X is called a (v_t, q) -variety if*

- (a) $\dim(X) = q^t - 1$;
- (b) *all characteristic numbers of $X(\mathbb{C})$ are divisible by q , and*
- (c) $\langle s_{q^t-1}, [X(\mathbb{C})]_{\mathbf{MU}} \rangle \not\equiv 0 \pmod{q^2}$.

As a reminder, the classes s_n are defined as the cohomology classes in $H^{2n}(BG_m, \mathbb{Z})$, the cohomology of the classifying space for the infinite Grassmanian of m -planes in \mathbb{C}^∞ with $m \geq n$, corresponding to $t_1^n + t_2^n + \cdots + t_m^n$, if we identify the Chern classes c_i of the universal m -plane bundle with $\sigma_i(t_1, \cdots, t_m)$, the elementary symmetric polynomials in the variables $\{t_1, \cdots, t_m\}$. This is usually being done by noticing that

the map $\mathbb{P}^\infty \times \cdots \times \mathbb{P}^\infty \rightarrow BG_m$ that classifies the bundle $\mu = \mu_1 \times \cdots \times \mu_m$, with μ_i being the universal plane bundle of the i -th copy of \mathbb{P}^∞ , induces an injection in cohomology sending the total Chern class of the universal m -plane bundle to the total Chern class of μ which is $\prod_{i=1}^m (1 + t_i)$, and t_i is the first Chern class of μ_i .

Voevodsky has shown in [9] that each of such X_i can be taken as a model for a class representing $v_t \in \pi_{2d,d}(\mathbf{MGI})$ as defined in 3.1; therefore if X is cobordant to the product of r copies of X_i , it is a model for $v_t^r \in \pi_{2dr,dr}(\mathbf{MGI})$. Proposition 3.2 implies that there exists an \mathbf{MGI} -orientation of Φ_{r-1} which we called e_r , that sends $[X]$ to a nonzero homotopy class of Φ_r or, using the same notation as in 3.3.7, $p_X(f_*(\alpha_X)) \neq 0 \in \pi_{2dr,dr}(\Phi_r) = \mathbb{Z}/q$. Therefore we have the following theorem, which is a consequence of 3.3.7 and which shows how assumptions on cobordism classes of the \mathbb{C} -rational points of the varieties involved can affect the degree of a map between them and vice versa:

Theorem 4.1. *Let X and Y be smooth, projective algebraic varieties of dimension $r(q^t - 1)$ for an odd prime number q and positive integers r and t ; let also $f : Y \rightarrow X$ be any map between them. If f is not dominant, then $(e_r)_*([Y]) \in I_r(X)$. If we assume that $I_r(X) = 0$ and f dominant, then,*

- (a) $\deg(f) \pmod q$ is a (topological) cobordism invariant in the source Y ;
- (b) if $[Y(\mathbb{C})] = v_t^r$, we have $\deg(f) \not\equiv 0 \pmod q$ and $[X] \equiv n[Y] \pmod{\ker(e_r)_*}$ with $n \in \mathbb{Z}$ and $n \equiv \deg(f) \pmod q$ in $\pi_{2r(q^t-1),r(q^t-1)}(\mathbf{MGI})$; in particular $[X(\mathbb{C})]$ is not divisible by q in $\pi_{2r(q^t-1)}(\mathbf{MU})$;

- (c) if $[Y(\mathbb{C})] = [X(\mathbb{C})] = v_t^r$, then $\deg(f) \equiv 1 \pmod{q}$;
- (d) if $[X(\mathbb{C})] = v_t^r$ and $\deg(f) \not\equiv 0 \pmod{q}$ then $[Y] \equiv m[X] \pmod{\ker(e_r)_*}$ with $m \in \mathbb{Z}$ and $m \equiv \deg(f) \pmod{q}$; in particular $[Y(\mathbb{C})]$ is not divisible by q .

Remark 4.1. In the previous theorem, the condition $[W(\mathbb{C})] = v_t^r$ with $W = Y, X$, can be replaced by $[W(\mathbb{C})] = v_t^r \pmod{\ker(e_r)_*}$ throughout, without changing the conclusions.

The next task is to give conditions on X so that $I_r(X) = 0$. Recall the definition of $I_r(X)$ from 3.3.1: it is the image of either of the two compositions in the commutative square

$$(4.4.1) \quad \begin{array}{ccc} \pi_{2dr,dr}(\Sigma^{2d,d}\Phi_{r-1} \wedge X) & \xrightarrow{\phi_*} & \pi_{2dr,dr}(\Phi_r \wedge X) \\ \downarrow p_X & & \downarrow p_X \\ \pi_{2dr,dr}(\Sigma^{2d,d}\Phi_{r-1}) = \pi_{0,0}(\Phi_{r-1}) & \xrightarrow{\cong} & \pi_{2dr,dr}(\Phi_r) = \mathbb{Z}/q \end{array}$$

Remark: For $r = 1$, we have

$$(4.4.2) \quad \begin{array}{ccc} \pi_{2d,d}(\Sigma^{2d,d}\mathbf{H}_{\mathbb{Z}/q} \wedge X) \cong CH_0(X) \otimes \mathbb{Z}/q & \xrightarrow{p_X} & \\ & & \\ & \longrightarrow & \pi_{2d,d}(\Sigma^{2d,d}\mathbf{H}_{\mathbb{Z}/q}) \cong CH_0(\text{Spec } k) \otimes \mathbb{Z}/q \cong \mathbb{Z}/q \end{array}$$

and the map p_X can be seen as $g \otimes 1$, where $g : CH_0(X) \rightarrow CH_0(\text{Spec } k) \cong \mathbb{Z}$ is the map induced by the canonical map $X \rightarrow \text{Spec } k$: it takes a closed point $\text{Spec } L \rightarrow X$ to the positive integer $[L; k]$.

Notation: Denote with $J(X)$ the ideal in \mathbb{Z} generated by the image of g . More in general, if $f : W \rightarrow \text{Spec } k$ is a proper scheme, and $W_E \xrightarrow{f} \text{Spec } E$ is the base changed scheme by $\text{Spec } E \rightarrow \text{Spec } k$, E being a field extension of k , $J(W_E)$ will be the ideal in \mathbb{Z} generated by the direct image homomorphism $f_* : CH_0(W_E) \rightarrow CH_0(\text{Spec } E) \cong \mathbb{Z}$.

We can now formulate the following result:

Proposition 4.1. *Let X_1, \dots, X_r be smooth, irreducible, projective varieties of dimension d , $X = X_1 \times \dots \times X_r$; assume that $J((X_i)_{K(X_1 \times \dots \times X_{i-1})}) \subset q\mathbb{Z}$ for every $i = 1, \dots, r$.¹ Then $I_r(X) = 0$.*

Proof. The first step is to show that $(p_{X_1})_* \circ (p_{X_2})_* = 0$ in the diagram below, that is $I_2(X_1 \times X_2) = 0$

(4.4.3)

$$\begin{array}{ccccc}
\pi_{0,0}(\mathbf{H}_{\mathbb{Z}/q} \wedge X_1 \times X_2) & \xrightarrow{\phi_*} & \pi_{2d,d}(\Phi_1 \wedge X_1 \times X_2) & \xrightarrow{\psi_*} & \pi_{2d,d}(\mathbf{H}_{\mathbb{Z}/q} \wedge X_1 \times X_2) = CH_d(X_1 \times X_2) \otimes \mathbb{Z}/q \\
\downarrow (p_{X_2})_* & & \downarrow (p_{X_2})_* & & \downarrow (p_{X_2})_* \\
\pi_{0,0}(\mathbf{H}_{\mathbb{Z}/q} \wedge X_1) & \xrightarrow{\phi_*} & \pi_{2d,d}(\Phi_1 \wedge X_1) & \xrightarrow{\psi_*} & \pi_{2d,d}(\mathbf{H}_{\mathbb{Z}/q} \wedge X_1) = CH_d(X_1) \otimes \mathbb{Z}/q = \mathbb{Z}/q[X_1] \\
\downarrow (p_{X_1})_* & & \downarrow (p_{X_1})_* & & \\
CH_0(\text{Spec } k) \otimes \mathbb{Z}/q & \xrightarrow{\cong} & \pi_{2d,d}(\Phi_1) = \mathbb{Z}/q & &
\end{array}$$

By diagram chasing, it is enough to prove that,

- (a) $(p_{X_2})_* : CH_d(X_1 \times X_2) \otimes \mathbb{Z}/q \rightarrow CH_d(X_1) \otimes \mathbb{Z}/q$ is the zero map and also
- (b) $(p_{X_1})_* : CH_0(X_1) \otimes \mathbb{Z}/q \rightarrow CH_0(\text{Spec } k) \otimes \mathbb{Z}/q$ is the zero map.

¹here $K(X_1 \times \dots \times X_{i-1})$ denotes the function field of $X_1 \times \dots \times X_{i-1}$ and $(X_i)_{K(X_1 \times \dots \times X_{i-1})}$ is the base changed scheme by the canonical map $\text{Spec } K(X_1 \times \dots \times X_{i-1}) \rightarrow \text{Spec } k$.

The second follows immediately by the assumption $J(X_1) \subset q\mathbb{Z}$. To prove the first, we fix a cycle α of dimension d in $X_1 \times X_2$; $(p_{X_2})_*\alpha$ is of the kind mX_1 , where m is $[K(\alpha); K(X_1)]$. Let $\widehat{p_{X_2}}$ be the base change of p_{X_2} over the generic point $\text{Spec } K(X_1)$ of X_1

$$(4.4.4) \quad \begin{array}{ccccc} & & (X_1 \times X_2)_{K(X_1)} & \longrightarrow & X_1 \times X_2 & \longrightarrow & X_2 \\ & \swarrow & \downarrow & & \downarrow & & \downarrow \\ & \text{can} & \widehat{p_{X_2}} & & p_{X_2} & & \\ (X_2)_{K(X_1)} & & \downarrow & & \downarrow & & \downarrow \\ & \searrow & \text{Spec } K(X_1) & \longrightarrow & X_1 & \longrightarrow & \text{Spec } k \end{array}$$

$\widehat{p_{X_2}}$ induces a map between the Chow groups $CH_0((X_1 \times X_2)_{K(X_1)})$ and $CH_0(K(X_1))$. The ideal generated by the image of $CH_0((X_1 \times X_2)_{K(X_1)}) \xrightarrow{(\widehat{p_{X_2}})^*} CH_0(K(X_1))$ is canonically contained in $J((X_2)_{K(X_1)})$, because of the factorization of the diagram, and the latter is in $q\mathbb{Z}$ by assumption; therefore m must be divisible by q because $mK(X_1)$ is the image through $(\widehat{p_{X_2}})_*$ of the 0-cycle $\alpha_{K(X_1)} \in CH_0((X_1 \times X_2)_{K(X_1)})$.²

For the last case, the diagram becomes

$$(4.4.5) \quad \begin{array}{ccccc} \pi_{2d(r-2), d(r-2)}(\Phi_{r-2} \wedge X_r) & \xrightarrow{\phi_*} & \pi_{2d(r-1), d(r-1)}(\Phi_{r-1} \wedge X_r) & \xrightarrow{\psi_*} & CH_{d(r-1)}(X_r) \otimes \mathbb{Z}/q \\ \downarrow (p_{X_r})_* & & \downarrow (p_{X_r})_* & & \downarrow (p_{X_r})_* \\ \pi_{2d(r-2), d(r-2)}(\Phi_{r-2} \wedge X_{r-1}) & \xrightarrow{\phi_*} & \pi_{2d(r-1), d(r-1)}(\Phi_{r-1} \wedge X_r) & \xrightarrow{\psi_*} & CH_{d(r-1)}(X_{r-1}) \otimes \mathbb{Z}/q \\ \downarrow (p_{X_{r-1}})_* & & \downarrow (p_{X_{r-1}})_* & & \\ \pi_{2d(r-2), d(r-2)}(\Phi_{r-2}) = I_{r-1}(X_{r-1}) & \xrightarrow{\cong} & \pi_{2d(r-1), d(r-1)}(\Phi_{r-1}) = \mathbb{Z}/q & & \end{array}$$

²here it is used that m is $[K(\alpha); K(X_1)]$.

where we have used the abbreviation $X_{\underline{r}}$ for $X_1 \times \cdots \times X_r$ and $X_{\underline{r-1}}$ for $X_1 \times \cdots \times X_{r-1}$

We have to check that the following maps are zero:

- (a) $p_{X_r*} : CH_{d(r-1)}(X_{\underline{r}}) \otimes \mathbb{Z}/q \rightarrow CH_{d(r-1)}(X_{\underline{r-1}}) \otimes \mathbb{Z}/q$
- (b) $p_{X_{r-1}*} : \pi_{2d(r-2), d(r-2)}(\Phi_{r-2} \wedge X_{\underline{r-1}}) \rightarrow \pi_{2d(r-2), d(r-2)}(\Phi_{r-2})$.

The second map is zero by induction. Take β to be a $d(r-1)$ -cycle in $X_{\underline{r}}$; its direct image $p_{X_r*}(\beta)$ will be mW with W a $d(r-1)$ -cycle in $X_{\underline{r-1}}$ and m , as before, is $[K(\beta); K(W)]$. From

$$(4.4.6) \quad \begin{array}{ccccc} \beta_{K(W)} & \longrightarrow & \beta_{K(X_{\underline{r-1}})} & \longrightarrow & \beta \\ \downarrow p_{K(W)} & & \downarrow p_{K(X_{\underline{r-1}})} & & \downarrow p_{X_r} \\ \text{Spec } K(W) & \longrightarrow & \text{Spec } K(X_{\underline{r-1}}) & \longrightarrow & W \end{array}$$

and the fact that $p_{K(W)}$ is finite of degree m , we conclude that also $p_{K(X_{\underline{r-1}})}$ must be finite of the same degree and the direct image $p_{K(X_{\underline{r-1}})*}(\beta_{K(X_{\underline{r-1}})}) = mK(X_{\underline{r-1}}) \in CH_0(K(X_{\underline{r-1}}))$. And now, as for the case $r = 2$, we have the diagram

$$(4.4.7) \quad \begin{array}{ccc} & (X_{\underline{r}})_{K(X_{\underline{r-1}})} & \longrightarrow & X_{\underline{r}} \\ & \swarrow \text{can} & & \downarrow p_{X_r} \\ (X_{\underline{r}})_{K(X_{\underline{r-1}})} & & & \text{Spec } K(X_{\underline{r-1}}) \longrightarrow X_{\underline{r-1}} \\ & \searrow & \downarrow p_{K(X_{\underline{r-1}})} & \\ & & & \end{array}$$

with $\beta_{K(X_{\underline{r-1}})}$ that is a 0-cycle in $(X_{\underline{r}})_{K(X_{\underline{r-1}})}$ on which we have no assumption; but the crucial fact is, once again, that $p_{K(X_{\underline{r-1}})}$ factors through $(X_{\underline{r}})_{K(X_{\underline{r-1}})}$, hence

$p_{K(\underline{X}_{r-1})_*}(\beta_{K(\underline{X}_{r-1})}) \in J((X_r)_{K(\underline{X}_{r-1})}) \subset q\mathbb{Z}$ so q divides m and this finishes the proof of the proposition. \square

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