ON THE EXPONENT OF THE NK_0 -GROUPS OF VIRTUALLY INFINITE CYCLIC GROUPS

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ABSTRACT. It is known that the K -theory of a large class of groups can be computed from the K -theory of their virtually infinite cyclic subgroups. On the other hand, Nil-groups appear to be the obstacle in calculations involving the K -theory of the latter. The main difficulty in the calculation of Nil-groups is that they are infinitely generated when they do not vanish. We develop methods for computing the exponent of NK_0 -groups that appear in the calculation of the K_0 -groups of virtually infinite cyclic groups.

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1. INTRODUCTION

The results of Farrell-Jones ([8]) have shown that the computation of the lower K -theory of a large class of groups depends on our ability to calculate the K -theory of virtually infinite cyclic groups. Virtually infinite cyclic groups are two ended groups and they are of two types:

- (*Type (1)*): Groups that admit an epimorphism, with kernel a finite group, to the infinite cyclic group $\mathbb{Z}$.
- (*Type (2)*): Groups that admit an epimorphism, with finite kernel, to the infinite dihedral group D_∞ .

The groups of the first type have the form $\Gamma = H \rtimes_\alpha \mathbb{Z}$ with H a finite group. In this case, the twisted Bass–Heller–Swan formula can be used for the calculation of the K -theory of Γ . The Nil-groups that appear are the twisted versions of the classical Nil-groups introduced by Bass ([1], [7]). In the case that α is the identity, the exponents of the corresponding NK_0 -groups were computed in [4]. We will generalize their methods to the twisted NK_0 -groups.

The algebraic structure of the groups in the second class is more complicated. A group Γ in the second class has the form $\Gamma = G_0 *_H G_1$ where the groups G_i , $i = 0, 1$, and H are finite and $[G_i : H] = 2$. In this case, the Nil-groups involved are Waldhausen’s Nil-groups. We will use the special structure of Γ to extend the methods of [10] and [11] to this case. Again, our methods heavily depend on the methods in [4].

Our interest in the exponent of the Nil-groups of virtually infinite cyclic groups arises from the fact that Nil-groups measure the difference between controlled and non-controlled phenomena. In many instances that difference is reflected on phenomena that are completely topological in nature ([5]). In a forthcoming paper ([6]) we will use the results on the exponent of the Nil-groups to compute the exponent of equivariant topological K -groups.

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We outline a construction for a positive integer n . Let $n = p_1^{k_1} p_2^{k_2} \dots p_s^{k_s}$ be the decomposition of n into prime numbers. Choose ℓ_i , $i = 1, \dots, s$, integers such that $p_i^{\ell_i} \geq k_i n$ for all i and set $n' = p_1^{\ell_1} p_2^{\ell_2} \dots p_s^{\ell_s}$. The main result of the paper can be summarized as follows: Let $\Gamma = H \rtimes_{\alpha} \mathbb{Z}$ or $\Gamma = G_0 *_H \Gamma_1$ where H is a finite group of order n and, in the second case, $[G_0 : H] = [\Gamma_1 : H] = 2$. We call H the *base* group of the virtually infinite cyclic group.

Theorem (Main Theorem). *The NK_0 -groups that appear in the splitting of the virtually infinite cyclic groups with base a finite group of order n have exponent n' .*

The method of the proof follows closely the ideas developed in [4]. But the methods really go back to [3] and [9]. The main calculational tool is that for any finite group H of order n and an automorphism $\alpha : H \rightarrow H$, there is a pull-back diagram

$$\begin{array}{ccc} \mathbb{Z}H & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow \\ \mathbb{Z}H/n\mathcal{M} & \longrightarrow & \mathcal{M}/n\mathcal{M} \end{array}$$

where $\mathcal{M}$ is a hereditary order such that

- (1) $\mathbb{Z}H \subset \mathcal{M} \subset \mathbf{Q}H$.
- (2) $\mathcal{M}$ is α invariant.
- (3) $n\mathcal{M} \subset \mathbb{Z}H$.

For the NK_0 -groups of Type (1) groups, the calculations of [4] can be generalized in the twisted setting using the α -invariant order $\mathcal{M}$. In the type (2) case, we use the description of the lower Waldhausen's Nil-groups as the reduced K -groups of an augmented ring ([11], [10]). The special structure of the groups in Type (2) allows us to express the augmented ring as a subring of the ring of the 2×2 -matrices over a twisted polynomial ring. Methods similar to the ones in [4] can be applied to complete the proof in this case.

2. TWISTED NIL-GROUPS

This section is basically a modification of the argument given in [4] on the exponent of NK_0 -groups.

Let H be a finite group of order n and $\alpha : H \rightarrow H$ a group automorphism. We write $R = \mathbb{Z}H$. We denote by α the automorphism induced on R by α . Our objective is to calculate the exponent of the twisted Nil-group $NK_0(R, \alpha)$. This group is defined as the kernel of the augmentation map

$$\varepsilon : K_0(R_{\alpha}[t]) \rightarrow K_0(R).$$

The group $NK_0(R, \alpha)$ appears in the twisted Bass–Heller–Swan formula given in [7] which calculates the K -theory of the twisted Laurent ring:

$$K_0(R) \xrightarrow{1-\alpha_*} K_0(R) \rightarrow K_0(R[t, t^{-1}])/\mathcal{N} \rightarrow K_{-1}(R) \xrightarrow{1-\alpha_*} K_{-1}(R)$$

where $\mathcal{N} = NK_0(R, \alpha) \oplus NK_0(R, \alpha^{-1})$. Actually the formula in [7] is given for K_1 -groups only but its extension to lower K -theory can be achieved using the techniques in [14], §11.

Remark 2.1. In [7], Theorem 25, it is shown that if a ring R is regular and α is an automorphism of R , then the twisted polynomial and Laurent extensions of R are regular rings. In particular, in this case all the NK -groups vanish ([13], p. 114).

The main tool for adjusting the proof in [4] to our case is the existence of a hereditary order $\mathcal{M}$ with the properties stated in the introduction. Another important ingredient is the twisted version of Proposition 2.12 in [1]. Recall that a ring S is called *quasi-regular* if there is a two sided nilpotent ideal J of S such that S/J is regular. We adjust this classical definition to our setting.

Definition 2.2. Let R be a ring and α is an automorphism of R . Then R is called α -quasi-regular if there is a two sided nilpotent ideal which is α invariant such that R/J is regular.

Remark 2.3. Let R be an Artinian ring and α an automorphism of R . Then the Jacobson radical J of R is a two-sided nilpotent ideal and it is α -invariant. Thus R is α -quasi-regular for each automorphism α .

Lemma 2.4. Let R be a ring and $\alpha : R \rightarrow R$ a ring automorphism. If R is α -quasi-regular then

$$NK_i(R, \alpha) = 0, \quad \text{for } i \leq 0.$$

Proof. We will show the result for $i = 0$. The other cases follows from standard methods of lower K -theory. There is exists an α -invariant two-sided ideal J of R such that R/J is regular. Then the ring $(R/J)_\alpha[t]$ is also regular. Consider the commutative diagram induced by ring projections

$$\begin{array}{ccccc} K_0(R_\alpha[t]) & \longrightarrow & K_0((R/J)_\alpha[t]) & \xrightarrow{\cong} & K_0(R_\alpha[t]/J_\alpha[t]) \\ \downarrow & & \downarrow & & \downarrow \\ K_0(R) & \longrightarrow & K_0(R/J) & \xrightarrow{=} & K_0(R/J) \end{array}$$

The composite horizontal maps are isomorphisms because $J_\alpha[t]$ and J are nilpotent ideals. The middle vertical map is an isomorphism because R/J is regular (Remark 2.1). Thus the first vertical map is also an isomorphism. The result follows. $\square$

We can now outline the argument given in [4]. The number n' is as in the introduction.

Theorem 2.5. With the above notation, the group $NK_0(R, \alpha)$ has exponent n' .

Proof. Choose an α -invariant hereditary order $\mathcal{M}$ as in the introduction. There are pull-back diagrams

$$\begin{array}{ccccc} R & \longrightarrow & \mathcal{M} & & R_\alpha[t] & \longrightarrow & \mathcal{M}_\alpha[t] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ R/n\mathcal{M} & \longrightarrow & \mathcal{M}/n\mathcal{M} & & (R/n\mathcal{M})_\alpha[t] & \longrightarrow & (\mathcal{M}/n\mathcal{M})_\alpha[t] \end{array}$$

(we still denote by α the automorphism induced by α on all the rings involved) which induces a commutative diagram of exact sequences

$$\begin{array}{ccccccc} K_1((\mathcal{M}/n\mathcal{M}_\alpha)[t]) & \longrightarrow & K_0(R_\alpha[t]) & \longrightarrow & K_1((R/n\mathcal{M})_\alpha[t]) \oplus K_1(\mathcal{M}_\alpha[t]) \\ \downarrow & & \downarrow & & \downarrow \\ K_1(\mathcal{M}/n\mathcal{M}) & \longrightarrow & K_0(R) & \longrightarrow & K_1(R/n\mathcal{M}) \oplus K_1(\mathcal{M}) \end{array}$$

Taking kernels of the vertical maps results to an exact sequence

$$\mathrm{NK}_1(\mathcal{M}/n\mathcal{M}, \alpha) \xrightarrow{\partial} \mathrm{NK}_0(R, \alpha) \rightarrow \mathrm{NK}_0(R/n\mathcal{M}, \alpha) \oplus \mathrm{NK}_0(\mathcal{M}, \alpha).$$

Since $\mathcal{M}$ is regular, $\mathrm{NK}_0(\mathcal{M}, \alpha) = 0$. Since $R/n\mathcal{M}$ is finite, it is artinian. By Remark 2.3, $R/n\mathcal{M}$ is α -quasi-regular and by Lemma 2.4, $\mathrm{NK}_0(R/n\mathcal{M}, \alpha) = 0$. Thus the map ∂ is onto. So it is enough to compute the exponents of a set of generators of $\mathrm{NK}_1(\mathcal{M}/n\mathcal{M})$.

As in [4], Corollary 3.6, the map

$$1 + tJ_\alpha[t] \rightarrow \mathrm{NK}_1(\mathcal{M}, \alpha)$$

is onto. But the elements of the group $1 + tJ_\alpha[t]$ have exponent n' ([4], p. 10). The result follows. $\square$

Remark 2.6. Using the arguments in [4], Proposition 3.7, the elements of the form $B = \{1 + jt^s : j \in J, n \in \mathbb{N}\}$ generate the group $1 + tJ_\alpha[t]$. There is an action of the multiplicative monoid $\mathbb{N}$ on B by $n \cdot (1 + jt^s) = 1 + jt^{sn}$, which induces an action on $1 + tJ_\alpha[t]$. Under this action $1 + tJ_\alpha[t]$ becomes a $\mathbb{Z}[\mathbb{N}]$ -module with basis $\mathcal{B} = \{1 + jt : j \in J\}$. That makes $\mathrm{NK}_0(R, \alpha)$ a finitely generated $\mathbb{Z}[\mathbb{N}]$ -module.

3. WALDHAUSEN'S NIL-GROUPS

For the calculation of Waldhausen's Nil-groups we will use the interpretation given in [5], [10] and [11]. We will review briefly the construction. Let $\mathcal{T}$ be the category of triples $\mathbf{R} = (R; B_0, B_1)$, where B_i , $i = 0, 1$ are two R -bimodules. A morphism in $\mathcal{T}$ is a triple

$$(\phi, f_0, f_1) : (R; B_0, B_1) \rightarrow (S, C_0, C_1)$$

where $\phi : R \rightarrow S$ is a ring homomorphism and $f_i : B_i \otimes_R S \rightarrow C_i$, $i = 0, 1$, are $R - S$ -bimodule homomorphisms. There is a functor

$$\rho : \mathcal{T} \rightarrow \text{Rings}, \quad \rho(\mathbf{R}) = R_\rho = \begin{pmatrix} T_R(B_1 \otimes_R B_0) & B_1 \otimes_R T_R(B_0 \otimes_R B_1) \\ B_0 \otimes_R T_R(B_1 \otimes_R B_0) & T_R(B_0 \otimes_R B_1) \end{pmatrix}$$

with multiplication given as matrix multiplication and on each entry by concatenation. The ring R_ρ is constructed as follows: In [5], there is a construction of a twisted polynomial extension category associated to $\mathbf{R}$. The resulting category admits a basic object $b = (R, R)$ i.e. for each v in the category there is an object u such that $u \oplus v$ is isomorphic to b^k ([2], p. 197). The ring R_ρ is the endomorphism ring of the basic object in the twisted polynomial extension ([11], Proposition 2.3). The basic result is that the lower Waldhausen's Nil-groups associated to the triple $\mathbf{R}$ can be computed using the K -theory of the ring R_ρ . More specifically, there is a natural augmentation map

$$R_\rho \xrightarrow{\varepsilon} \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$$

We write $\mathrm{NK}_i(\mathbf{R})$ for the kernel of the map induced by ε on K_i -groups. Then there is a natural isomorphism ([5], [11])

$$\mathrm{NK}_i(\mathbf{R}) \cong \widetilde{\mathrm{Nil}}_{i-1}^W(\mathbf{R}), \quad \text{for } i \leq 1.$$

Thus vanishing results of Waldhausen's Nil-groups can be applied to the NK-groups.

We now formulate the Nil-groups of interest. Let Γ be a virtually infinite cyclic group of type (2). Then Γ admits a decomposition as an amalgamated free product of finite groups

$$\Gamma = G_0 *_H G_1, \quad [G_0 : H] = [G_1 : H] = 2.$$

Then the Waldhausen's Nil-groups are defined using the triple $(\mathbb{Z}H; \mathbb{Z}[G_0 - H], \mathbb{Z}[G_1 - H])$. Choosing right coset representatives, we see that the $\mathbb{Z}H$ -bimodules $\mathbb{Z}[G_i - H]$, $i = 0, 1$, are isomorphic to $\mathbb{Z}H$ as left $\mathbb{Z}$ -modules, but the right action is twisted by an automorphism $\mathbb{Z}H$, induced by an automorphism of H .

We can now formulate the general setup. Let R be a ring with one and $\alpha : R \rightarrow R$ a ring automorphism. We denote by R^α the $R - R$ -bimodule which is R as a left R -module but the right multiplication is given by: $x.r = x\alpha(r)$. Our objective is to study the lower Waldhausen's Nil-groups of the form $\widetilde{Nil}_*^W(R; R^\alpha, R^\beta)$ where α and β are automorphisms of R . From now on, $\otimes$ means $\otimes_R$.

We start with a basic algebraic observation.

Lemma 3.1. *With the above notation, there is an R -bimodule isomorphism*

$$i_{\alpha, \beta} : R^\alpha \otimes R^\beta \rightarrow R^{\alpha\beta}.$$

Proof. The isomorphism is given by

$$i_{\alpha, \beta}(x \otimes y) = x\alpha(y).$$

□

Proposition 3.2. *There is an isomorphism*

$$\iota : \widetilde{Nil}_i^W(R; R^\alpha, R^\beta) \rightarrow \widetilde{Nil}_i^W(R; R^{\alpha\beta}, R), \quad \text{for all } i \in \mathbb{Z}.$$

Proof. We write $\mathbf{R} = (R; R^\alpha, R^\beta)$. The Waldhausen's Nil-groups are defined using an exact category $\mathcal{Nil}(\mathbf{R})$ with objects quadruples $(P, Q; p, q)$, where P and Q are finitely generated projective right R -modules and

$$p : P \rightarrow Q \otimes R^\alpha, \quad q : Q \rightarrow P \otimes R^\beta$$

is a pair of R -homomorphisms such that the compositions

$$\begin{array}{c} P \xrightarrow{p} Q \otimes R^\alpha \xrightarrow{q \otimes 1} P \otimes R^\beta \otimes R^\alpha \dots \\ Q \xrightarrow{q} P \otimes R^\beta \xrightarrow{p \otimes 1} P \otimes R^\alpha \otimes R^\beta \dots \end{array} \quad (*)$$

are zero after finitely many steps. Morphisms are morphisms on the modules that are compatible with the maps. Exact sequences are defined the obvious way. Notice that there is a forgetful functor $\phi : \mathcal{Nil}(\mathbf{R}) \rightarrow \mathcal{P}_R \times \mathcal{P}_R$, where $\mathcal{P}_R$ is the category of finitely generated projective right R -modules. Then

$$\widetilde{Nil}_i^W(\mathbf{R}) = \ker(K_i(\mathcal{Nil}(\mathbf{R})) \xrightarrow{\phi_i} K_i(\mathcal{P}_R \times \mathcal{P}_R)).$$

Set $\mathbf{R}' = (R; R^{\alpha\beta}, R)$. We will define an equivalence of categories

$$\iota : \mathcal{Nil}(\mathbf{R}) \rightarrow \mathcal{Nil}(\mathbf{R}').$$

On objects, we define

$$\iota(P, Q; p, q) = (P \otimes R^\beta, Q; (1_Q \otimes i_{\alpha, \beta}) \circ (p \otimes 1_{R^\beta}), q)$$

(for every right R -module M we will always identify $M \otimes R$ with M in the natural way). If the sequence of maps, as in (*), starts with $P \otimes R^\beta$, then the resulting composition of maps is equal to the original composition tensored with the identity on R^β . If the sequence starts with Q the resulting composition is equal to the original. In both cases it will be eventually zero. Thus the resulting quadruple is an object of $\mathcal{N}il(\mathbf{R}')$. On morphisms ι is defined by

$$\iota(f, g) = (f \otimes 1_{R^\beta}, g).$$

The functor ι is exact because R^β is a free (and thus flat) left R -module.

We can now describe the inverse of ι . That will be an exact functor

$$\iota' : \mathcal{N}il(\mathbf{R}') \rightarrow \mathcal{N}il(\mathbf{R}).$$

For a quadruple $(P', Q'; p', q')$ with

$$p' : P' \rightarrow Q' \otimes R^{\alpha\beta}, \quad q' : Q' \rightarrow P'$$

we define

$$\iota'(P', Q'; p', q') = (P' \otimes R^{\beta-1}, Q'; (1_Q \otimes i_{\alpha\beta, \beta-1}) \circ (p' \otimes 1_{R^{\beta-1}}), (1_{P'} \otimes i_{\beta-1, \beta} \circ q')).$$

For the same reasons as ι , ι' defines an exact functor on the $\mathcal{N}il$ -categories.

It is immediate to check that ι and ι' define an equivalence between the two categories. Thus ι induces an isomorphism on the K -groups of the $\mathcal{N}il$ -categories. The result follows. $\square$

Corollary 3.3. *There is an isomorphism*

$$NK_i(R; R^\alpha, R^\beta) \cong NK_i(R; R^{\alpha\beta}, R) \quad \text{for } i \leq 1.$$

Proof. The lower Waldhausen's Nil-groups and the lower NK-groups are isomorphic. $\square$

Remark 3.4. Actually the proof of Proposition 3.2 shows that there are isomorphisms for all $i \in \mathbb{Z}$:

$$\widetilde{\text{Nil}}_i^W(R; R^\alpha, R^\beta) \cong \widetilde{\text{Nil}}_i^W(R; R^{\alpha\beta}, R) \cong \widetilde{\text{Nil}}_i^W(R; R, R^{\alpha\beta}).$$

Because of Corollary 3.3, we will concentrate on triples of the form $\mathbf{R} = (R; R^\alpha, R)$ where α is an automorphism of R . In this case, we will first try to give a more manageable description of the ring R_ρ . If we set $B_0 = R^\alpha$, $B_1 = R$, then the ring R_ρ takes the form

$$R_\rho = \begin{pmatrix} T_R(R \otimes_R R^\alpha) & R \otimes_R T_R(R^\alpha \otimes_R R) \\ R^\alpha \otimes_R T_R(R \otimes_R R^\alpha) & T_R(R^\alpha \otimes_R R) \end{pmatrix} = \begin{pmatrix} T_R(R \otimes_R R^\alpha) & T_R(R \otimes_R R^\alpha) \otimes_R R \\ T_R(R^\alpha \otimes_R R) \otimes_R R^\alpha & T_R(R^\alpha \otimes_R R) \end{pmatrix}$$

We will identify R_ρ with a subring of the ring of 2×2 -matrices over a twisted polynomial ring. The automorphism α induces an automorphism on the polynomial ring $R[s]$, denoted α , with $\alpha(s) = s$. Let $R_\alpha^{(2)}[s, t] = (R[s])_\alpha[t]$. Then $R_\alpha^{(2)}[s, t]$ is the polynomial ring on two commuting variables which is isomorphic to $R[s, t]$ as a left R -module. The right action of R on the variables s and t is given by:

$$s.r = rs, \quad t.r = \alpha(r)t, \quad \text{for all } r \in R$$

(the superscript (2) in the notation suggests that the twist, by α , occurs in the second variable). We will describe R_ρ as a subring of $M_2(R_\alpha^{(2)}[s, t])$. Set

$$R_{\alpha, d}^{(2)}[s, t] = \{p \in R_\alpha^{(2)}[s, t] : \text{degree}_s(p) = \text{degree}_t(p)\}$$

for the subring of $R_\alpha^{(2)}[s, t]$ generated by all the monomials which have equal degrees with respect to the variables.

Proposition 3.5. *There is a ring isomorphism*

$$\mu : R_\rho \rightarrow R'_\rho = \begin{pmatrix} R_{\alpha, d}^{(2)}[s, t] & R_{\alpha, d}^{(2)}[s, t]s \\ R_{\alpha, d}^{(2)}[s, t]t & R_{\alpha, d}^{(2)}[s, t] \end{pmatrix}$$

Proof. We describe how to define μ on each entry of R_ρ . It is enough to show how to define μ on each component of an entry of R_ρ . We write 1 for the identity element of R and 1_α for the generator of R_α corresponding to 1. We map 1 to s , 1_α to t and tensor products of 1 and 1_α to corresponding products of s and t . Now every element in a component of an entry of R_ρ can be represented in the form rp , where $r \in R$ and p is a tensor product of 1's and 1_α 's. We define $\mu(rp) = r\mu(p)$. We will show that μ is a ring homomorphism. We write $(1 \otimes 1_\alpha)^\kappa (1_\alpha \otimes 1)$ for the tensor product of κ -copies of $1 \otimes 1_\alpha$ ($1_\alpha \otimes 1$). We will check only that μ is a ring homomorphism on monomials. Actually we will check the first entry of the matrices. The general case follows similarly. Let a and b be two elements in R_ρ . Then a and b have the form

$$\begin{aligned} a &= \begin{pmatrix} r_{1,1}(1 \otimes 1_\alpha)^{\kappa_{1,1}} & r_{1,2}(1 \otimes 1_\alpha)^{\kappa_{1,2}} \otimes 1 \\ r_{2,1}(1_\alpha \otimes 1)^{\kappa_{2,1}} \otimes 1_\alpha & r_{2,2}(1_\alpha \otimes 1)^{\kappa_{2,2}} \end{pmatrix}, \\ b &= \begin{pmatrix} r'_{1,1}(1 \otimes 1_\alpha)^{\kappa'_{1,1}} & r'_{1,2}(1 \otimes 1_\alpha)^{\kappa'_{1,2}} \otimes 1 \\ r'_{2,1}(1_\alpha \otimes 1)^{\kappa'_{2,1}} \otimes 1_\alpha & r'_{2,2}(1_\alpha \otimes 1)^{\kappa'_{2,2}} \end{pmatrix} \end{aligned}$$

Their images are

$$\begin{aligned} \mu(a) &= \begin{pmatrix} r_{1,1}(st)^{\kappa_{1,1}} & r_{1,2}s^{\kappa_{1,2}+1}t^{\kappa_{1,2}} \\ r_{2,1}s^{\kappa_{2,1}}t^{\kappa_{2,1}+1} & r_{2,2}(st)^{\kappa_{2,2}} \end{pmatrix} \\ \mu(b) &= \begin{pmatrix} r'_{1,1}(st)^{\kappa'_{1,1}} & r'_{1,2}s^{\kappa'_{1,2}+1}t^{\kappa'_{1,2}} \\ r'_{2,1}s^{\kappa'_{2,1}}t^{\kappa'_{2,1}+1} & r'_{2,2}(st)^{\kappa'_{2,2}} \end{pmatrix} \end{aligned}$$

We compute the entry at $(1, 1)$ of the product ab in R_ρ .

$$\begin{aligned} p_{1,1} &= [r_{1,1}(1 \otimes 1_\alpha)^{\kappa_{1,1}}][r'_{1,1}(1 \otimes 1_\alpha)^{\kappa'_{1,1}}] + [r_{1,2}(1 \otimes 1_\alpha)^{\kappa_{1,2}} \otimes 1][r'_{2,1}(1_\alpha \otimes 1)^{\kappa'_{2,1}} \otimes 1_\alpha] = \\ &= r_{1,1}\alpha^{\kappa_{1,1}}(r'_{1,1})(1 \otimes 1_\alpha)^{\kappa_{1,1}+\kappa'_{1,1}} + r_{1,2}\alpha^{\kappa_{1,2}}(r'_{2,1})(1 \otimes 1_\alpha)^{\kappa_{1,2}+\kappa'_{2,1}+1} \end{aligned}$$

Thus the $(1, 1)$ entry of $\mu(ab)$ will be

$$\mu(p_{1,1}) = r_{1,2}\alpha^{\kappa_{1,2}}(r'_{2,1})(st)^{\kappa_{1,1}+\kappa'_{1,1}} + r_{1,2}\alpha^{\kappa_{1,2}}(r'_{2,1})(st)^{\kappa_{1,2}+\kappa'_{2,1}+1}.$$

Now we compute the $(1, 1)$ -entry of $\mu(a)\mu(b)$:

$$\begin{aligned} r_{1,1}(st)^{\kappa_{1,1}}r'_{1,1}(st)^{\kappa'_{1,1}} + r_{1,2}s^{\kappa_{1,2}+1}t^{\kappa_{1,2}}r'_{2,1}s^{\kappa'_{2,1}}t^{\kappa'_{2,1}+1} = \\ r_{1,2}\alpha^{\kappa_{1,2}}(r'_{2,1})(st)^{\kappa_{1,1}+\kappa'_{1,1}} + r_{1,2}\alpha^{\kappa_{1,2}}(r'_{2,1})(st)^{\kappa_{1,2}+\kappa'_{2,1}+1} = \mu(p_{1,1}). \end{aligned}$$

The inverse of μ is constructed in the obvious way. $\square$

There is a natural augmentation map

$$\varepsilon' : R'_\rho \rightarrow \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$$

such that $\mu \circ \varepsilon = \varepsilon'$ where ε is the corresponding augmentation map on R_ρ . The following result is a consequence of Proposition 3.5.

Corollary 3.6. *The groups $NK_i(\mathbf{R})$, $i \leq 1$, are isomorphic to the kernel of the map induced by the augmentation ε' :*

$$NK_i(\mathbf{R}) = \ker(K_i(R'_\rho) \xrightarrow{\varepsilon'_i} K_i(R \times R)), \quad i \leq 1.$$

Proposition 3.7. *If R is regular then $NK_i(R; R^\alpha, R) = 0$ for all $i \leq 1$. In particular, the augmentation maps ε_i and ε'_i induce an isomorphisms on lower K -groups.*

Proof. For $i \leq 1$, the groups $NK_i(R; R^\alpha, R)$ are isomorphic to Waldhausen's groups $\widetilde{\text{Nil}}_{i-1}^W(R; R^\alpha, R)$ which vanish for R regular ([15], [16]). The result follows. $\square$

Let J be a two-sided ideal of R which is invariant under the automorphism α . Then the projection $R \rightarrow R/J$ induces a morphism of triples

$$(R; R^\alpha, R) \rightarrow (R/J; (R/J)^\alpha, R/J)$$

and thus a ring homomorphism

$$\chi_J : R'_\rho \rightarrow (R/J)'_\rho.$$

Lemma 3.8. *Using the above notation,*

$$(R/J)'_\rho \cong R'_\rho / J'_\rho$$

where J'_ρ is the two sided ideal of R'_ρ generated by $J \times J$ i.e.

$$J'_\rho = \left\langle \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \right\rangle$$

Proof. The proof is the same as in [11], Proposition 2.10. $\square$

We also denote by $\chi_J : R'_\rho \rightarrow R'_\rho / J'_\rho$ the natural projection homomorphism.

Remark 3.9. With the setting as in Lemma 3.8,

$$J'_\rho = \begin{pmatrix} J_{\alpha,d}^{(2)}[s, t] & J_{\alpha,d}^{(2)}[s, t]s \\ J_{\alpha,d}^{(2)}[s, t]t & J_{\alpha,d}^{(2)}[s, t] \end{pmatrix}$$

In particular, if J is nilpotent J'_ρ is also nilpotent.

We are interested in determining the properties of the map that is induced in K -theory by χ_J .

Proposition 3.10. *Let J be a nilpotent two-sided ideal of R which is invariant under α . Then the ring homomorphism χ_J , described above, induces an isomorphism in K_i -groups, for $i \leq 0$.*

Proof. Using Lemma 3.8, the map χ_J can be described as a quotient map

$$\chi_J : R'_\rho \rightarrow R'_\rho / J'_\rho.$$

But the ideal J'_ρ is a nilpotent ideal of R'_ρ (Remark 3.9). The result follows from [1], Proposition 2.12 (also [11], Proposition 2.11). $\square$

Corollary 3.11. *Let R be an Artinian ring and J its Jacobson radical. Then the quotient map χ_J induces isomorphism on K_i -groups for $i \leq 0$.*

Proof. Since R is Artinian, J is nilpotent. Also, J is invariant of every automorphism of R . The result follows from Proposition 3.10. $\square$

Corollary 3.12. *Let R be an α -quasi-regular ring. Then $NK_i(R; R^\alpha, R) = 0$, for all $i \leq 0$.*

Proof. Let J be a two-sided, α -invariant nilpotent ideal of R such that the ring R/J is regular. By Proposition 3.10, the map

$$(\chi_J)_i : K_i(R'_\rho) \rightarrow K_i((R/J)'_\rho)$$

is an isomorphism for $i \leq 0$. There is a commutative diagram

$$\begin{array}{ccc} K_i(R'_\rho) & \xrightarrow{(\chi_J)_i} & K_i((R/J)'_\rho) \\ \varepsilon'_i \downarrow & & \downarrow (\varepsilon'_J)_i \\ K_i(R \times R) & \longrightarrow & K_i(R/J \times R/J) \end{array}$$

where the vertical maps are induced by the augmentations. Since R/J is regular, ε'_J is an isomorphism (Proposition 3.7). Also, since J is nilpotent the bottom map is an isomorphism. Thus ε'_i is an isomorphism and $NK_i(R; R^\alpha, R) = 0$ for $i \leq 0$. $\square$

Now we specialize to the case that $R = \mathbb{Z}H$, the group ring of a finite group H of order n . Let α be an automorphism of R induced by an automorphism of H . Choose a hereditary order $\mathcal{M}$ as in the introduction. Then we can define triples in $\mathcal{T}$,

$$\begin{aligned} \mathbf{M} &= (\mathcal{M}; \mathcal{M}^\alpha, \mathcal{M}), \\ \mathbf{R}/n\mathbf{M} &= (R/n\mathcal{M}; (R/n\mathcal{M})^\alpha, R/n\mathcal{M}), \\ \mathbf{M}/n\mathbf{M} &= (\mathcal{M}/n\mathcal{M}; (\mathcal{M}/n\mathcal{M})^\alpha, \mathcal{M}/n\mathcal{M}). \end{aligned}$$

(we still use α to denote the homomorphism induced by α on the other rings involved in the calculation). The triples determine matrix rings (Proposition 3.5):

$$\begin{aligned} \mathcal{M}'_\rho &\cong \mathcal{M}_\rho && \text{corresponds to } (\mathcal{M}; \mathcal{M}^\alpha, \mathcal{M}) \\ (R/n\mathcal{M})'_\rho &\cong (R/n\mathcal{M})_\rho && \text{corresponds to } (R/n\mathcal{M}; (R/n\mathcal{M})^\alpha, R/n\mathcal{M}) \\ (\mathcal{M}/n\mathcal{M})'_\rho &\cong (\mathcal{M}/n\mathcal{M})_\rho && \text{corresponds to } (\mathcal{M}/n\mathcal{M}; (\mathcal{M}/n\mathcal{M})^\alpha, \mathcal{M}/n\mathcal{M}) \end{aligned}$$

There is a commutative diagram of matrix rings

$$\begin{array}{ccc} R'_\rho & \xrightarrow{i} & \mathcal{M}'_\rho \\ q \downarrow & & \downarrow p \\ (R/n\mathcal{M})'_\rho & \xrightarrow{j} & (\mathcal{M}/n\mathcal{M})'_\rho \end{array} \quad (**)$$

where i and j are induced by the inclusions and p and q by the projections. Actually, the maps in the diagram are induced by the maps of triples but this is not of use to us.

Proposition 3.13. *The diagram $(**)$ is a pull-back square of rings.*

Proof. By construction, the sequence

$$0 \rightarrow R'_\rho \xrightarrow{(i, -q)} \mathcal{M}'_\rho \oplus (R/n\mathcal{M})'_\rho \xrightarrow{(p, j)} (\mathcal{M}/n\mathcal{M})'_\rho \rightarrow 0.$$

is exact sequence of R'_ρ -bimodules because it is exact on each entry of the matrix representation of the rings. The result follows. $\square$

Remark 3.14. There are also the classical pull-back diagrams

$$\begin{array}{ccc} R & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow \\ R/n\mathcal{M} & \longrightarrow & \mathcal{M}/n\mathcal{M} \end{array} \quad \begin{array}{ccc} R \times R & \longrightarrow & \mathcal{M} \times \mathcal{M} \\ \downarrow & & \downarrow \\ R/n\mathcal{M} \times R/n\mathcal{M} & \longrightarrow & \mathcal{M}/n\mathcal{M} \end{array}$$

The result of Proposition 3.13 allows us to use Mayer-Vietoris type exact sequences for the calculation of the Nil-groups.

Proposition 3.15. *There is an epimorphism*

$$\partial : NK_1(\mathcal{M}/n\mathcal{M}; (\mathcal{M}/n\mathcal{M})^\alpha, \mathcal{M}/n\mathcal{M}) \rightarrow NK_0(R; R^\alpha, R).$$

Proof. Combining the Mayer-Vietoris sequences for the diagram $(**)$ and the second diagram in Remark 3.14, we obtain a commutative diagram of exact sequences

$$\begin{array}{ccccc} K_1((\mathcal{M}/n\mathcal{M})'_\rho) & \longrightarrow & K_0(R'_\rho) & \longrightarrow & K_0((R/n\mathcal{M})'_\rho) \oplus K_0(\mathcal{M}'_\rho) \\ \downarrow & & \downarrow & & \downarrow \\ K_1(\mathcal{M}/n\mathcal{M} \times \mathcal{M}/n\mathcal{M}) & \longrightarrow & K_0(R \times R) & \longrightarrow & K_0(R/n\mathcal{M} \times R/n\mathcal{M}) \oplus K_0(\mathcal{M} \times \mathcal{M}) \end{array}$$

where the vertical maps are introduced by the augmentations. Using Corollary 3.6, we see that taking the kernels of the vertical maps induces an exact sequence

$$NK_1(\mathbf{M}) \xrightarrow{\partial} NK_0(\mathbf{R}) \rightarrow NK_0(\mathbf{R}/n\mathbf{M}) \oplus NK_0(\mathbf{M})$$

But the ring $R/n\mathcal{M}$ is Artinian and thus α -quasi-regular (Remark 2.3, Corollary 3.12) and $\mathcal{M}$ is regular. Thus the two NK-groups at the right end of the sequence vanish. The result follows. $\square$

The rest of the calculation of the exponent of the Nil-groups follows closely the methods used in [4] and [11]. We will outline the method of the proof. Remember that $R = \mathbb{Z}H$ and the order of the finite group H is n , and n' is as in the introduction.

Theorem 3.16. *The group $NK_0(R; R^\alpha, R)$ has exponent n' i.e.*

$$n'NK_0(R; R^\alpha, R) = 0.$$

Proof. By Proposition 3.15, we need to calculate the exponent of the generators of the group $NK_1(\mathcal{M}/n\mathcal{M}; (\mathcal{M}/n\mathcal{M})^\alpha, \mathcal{M}/n\mathcal{M})$. The method of proof of Proposition 4.9 in [11] applies here because J'_ρ is a nilpotent ideal (Remark 3.9). Let I_ρ be the augmentation ideal

$$I_\rho = \ker(\varepsilon' : (\mathcal{M}/n\mathcal{M})'_\rho \rightarrow \mathcal{M}/n\mathcal{M} \times \mathcal{M}/n\mathcal{M}).$$

Let $\tilde{J}_\rho = J'_\rho \cap I_\rho$. Then in Proposition 4.9 in [11], it was shown that there is an epimorphism

$$1 + \tilde{J}_\rho \rightarrow NK_1(\mathbf{M}/n\mathbf{M}).$$

Thus a set of generators of $NK_1(\mathbf{M}/n\mathbf{M})$ can be defined by the image of a set of generators of $1 + \tilde{J}_\rho$. Again, interpreting the result in Proposition 4.9 in [11], a set of generators of $1 + \tilde{J}_\rho$ is given by element of the form

$$\begin{pmatrix} 1 + p_{1,1} & 0 \\ 0 & 1 + p_{2,2} \end{pmatrix}, \quad \begin{pmatrix} 1 & p_{1,2} \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ p_{2,1} & 1 \end{pmatrix}$$

where $p_{1,1}$ and $p_{2,2}$ are in $J_{\alpha,d}^{(2)}[s, t]$, $p_{1,2} \in J_{\alpha,d}^{(2)}[s, t]s$, and $p_{2,1} \in J_{\alpha,d}^{(2)}[s, t]t$. The exponents of these elements is n' ([11], Theorem 5.1). $\square$

Then it is immediate that the same result applies to Waldhausen's Nil-groups.

Theorem 3.17. *The Waldhausen's NK_0 -groups that appear in the decomposition of a virtually infinite cyclic group with base H of order n have exponent n' .*

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