THE FILTERED GESTEN-WITT RESOLUTION FOR REGULAR SCHEMES

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Introduction

Using the work of Grothendieck on the foundations of the theory of algebraic schemes, M. Knebusch ([Kn1]) introduced the theory of quadratic forms over a scheme $(X, \mathcal{O}_X)$. The central object is the Witt group $W(X; \mathcal{L})$, the Grothendieck group on isomorphism classes of symmetric, $\mathcal{O}_X$-bilinear maps

$$\mathcal{E} \times \mathcal{E} \to \mathcal{L},$$

where $\mathcal{E}$ and $\mathcal{L}$ are locally free $\mathcal{O}_X$-modules and $\mathcal{L}$ is rank one, modulo the subgroup generated by lagrangians (see Definition (0.1) below). When $X = \operatorname{Spec} R$ is affine, and $L = R$, this definition spcializes to the the usual $W(R)$. In 1978 he proposed ([Kn2]) the following basic

Conjecture. *Let A be a regular domain and K its field of fractions. Then the natural map*

$$W(A) \to W(K)$$

is injective.

More concretely, and in case A is local, the conjecture says that an invertible symmetric $2n$-by-$2n$ matrix with entries in A is contragredient over A to one with a zero upper left n-by-n block if and only if this is so over K. The conjecture is easily seen to be true if $\dim A = 1$, and was proved by M. Karoubi ([K]) in case $A = k[x_1, \ldots, x_n]$ and by T. Craven, A. Rosenberg and R. Ware ([CRW]) in case A is a complete regular local ring. By quite different methods from these and from each other, M. Ojanguren ([Oj2]) and the author ([P3, Theorem A(i)]) proved the Knebusch Conjecture when $\dim A \leq 3$ and gave a counterexample ([P3, 4.6]) to the conjecture when A is the coordinate ring of the real 4-sphere. A general explanation of this counterexample in terms of the Chow group of zero-cycles of Spec A was given in [P3, Theorem B], together with a consequent affirmation of the conjecture for all regular local rings A of dimension ≤ 4. Ojanguren and the author proved the conjecture independently in case A is regular local and essentially of finite type over a field ([Oj1], [P4, Theorem B]). F. Fernández-Carmena globalized some of the techniques of [P3] and [P4] to prove the Knebusch Conjecture in case A is replaced by a regular scheme X of dimension 2 and K is its function field ([FFC]). He concluded from this and from ideas of J. L. Colliot-Thèléne and J.-J. Sansuc ([CTS]) that $W(X)$ is a birational invariant of X.

The study of the map $W(A) \to W(K)$ in [P3] used localization sequences and leads naturally to other results, including a Chow group description of the skew-symmetric analogue $W^{-1}(A)$ of $W(A)$ ($\dim A = 2$) and a connection between this group and the Serre-Horrocks construction of symmetric resolutions of a codimension 2 ideal in a 2-dimensional regular ring. The description of $W^{-1}(A)$ was later refined in [OPS] and [BOj].

Another by-product of the point of view of localization was a proof of the Gersten Conjecture (or resolution, in its global form) for Witt groups ([P4]), a result which had already been proved by D. Quillen in K-theory.

The object of this paper is to sharpen the techniques of [P4] to prove a *filtered* global version of the Gersten resolution for Witt groups, Theorem 0.12 below and Theorem 5.4 in the body of the paper. This, together with the Milnor Conjecture recently proved by V. Voevodsky, can be used to verify a conjecture in [P2, Conjecture C] which relates the Gersten resolution to the Bloch-Ogus resolution ([BO]) and to construct a spectral sequence (Theorem 0.13 and Theorem 5.5) which can be used to strengthen some of the applications from [P3] mentioned above and to extend the study of resolutions initiated in [P3].

To describe these recent results and their applications in more detail it is necessary to generalize some of the ideas from [P3] to a global setting.

Let $(X, \mathcal{O}_X)$ be a locally Cohen-Macaulay (CM) scheme and $X^{(i)}$, its points of codimension i. If $\mathcal{C}$ is a canonical sheaf for X, there is an injective resolution

$$0 \to \mathcal{C} \to \mathcal{E}^0(\mathcal{C}) \xrightarrow{d^0} \mathcal{E}^1(\mathcal{C}) \xrightarrow{d^1} \cdots \to \mathcal{E}^{n-1}(\mathcal{C}) \xrightarrow{d^{n-1}} \mathcal{E}^n(\mathcal{C}) \to 0$$

where $\mathcal{E}^i(\mathcal{C}) \cong \coprod_{x \in X^{(i)}} i_* E(k(x))$, $E(k(x))$ is the injective hull of the residue class field of x, viewed as a constant sheaf on $\bar{x}$ and $i : \bar{x} \to X$ is the inclusion. If X is Gorensein, then $\mathcal{C}$ is a canonical sheaf for X if and only if it is locally free. Set $\mathcal{V}^i(\mathcal{C}) := \ker d^i$. Unless otherwise noted, $\mathcal{C}$ is fixed and will often be omitted from the notation.

0.1 Definition. *Let i and p be integers, $i \geq 0$, $0 \leq p \leq n$. Then $\mathcal{S}_i^p(X)$ denotes the category of $\mathcal{O}_X$-modules $\mathcal{M}$ such that*

a. cod $\mathcal{M} = p$*, and*
b. *depth* $_{\mathcal{O}_{X,x}} \mathcal{M}_x \geq \inf\{i, \dim_{\mathcal{O}_{X,x}} \mathcal{M}_x\}$*, for all $x \in X$,.*

together with the zero module.

Elements of $\mathcal{S}_i^p(X)$ coincide with those satisfying property (S_i) in [EGA, 5.7.2], but with the extra support condition (a). If $i \geq n - p$ then $\mathcal{S}_i^p(X)$ is the category of codimension p locally Cohen-Macaulay $\mathcal{O}_X$-modules, denoted $\mathcal{CM}^p(X)$. If X is regular, then $\mathcal{S}_n^0(X)$ coincides with the category of locally free $\mathcal{O}_X$-modules. On the other hand, $\mathcal{S}_0^n(X)$ is the category of $\mathcal{O}_X$-modules of finite length.

0.2 Definition. *Let $\mathcal{M} \in \mathcal{S}_i^p(X)$ and $\phi : \mathcal{M} \times \mathcal{M} \to \mathcal{V}^p(\mathcal{C})$ be a symmetric, $\mathcal{O}_X$-bilinear form, such that $\phi_x : \mathcal{M}_x \times \mathcal{M}_x \to \mathcal{V}^p(\mathcal{C})_x$ is nonsingular if* cod $x \leq p + i$*. Then the pair $(\mathcal{M}, \phi)$ is called a* symmetric form *in $\mathcal{S}_i^n(X)$ and the collection of such is denoted $Q(\mathcal{S}_i^p(X); \mathcal{C})$. $(\mathcal{M}, \phi) \in Q(\mathcal{S}_i^p(X); \mathcal{C})$ is called a* lagrangian *if there is $\mathcal{N} \subseteq \mathcal{M}$ where $\mathcal{N}$ and $\mathcal{M}/\mathcal{N} \in \mathcal{S}_i^p(X)$, and the induced pairing $\mathcal{N} \times (\mathcal{M}/\mathcal{N}) \to \mathcal{V}^p(\mathcal{C})$ is nonsingular (i.e.,* *both adjoints are bijective) at all x such that* cod $x \le p+i$. *$\mathcal{N}$ is called a* sublagrangian. *When X is regular one sets*

$$Q(X;\mathcal{C}) := Q(\mathcal{S}_n^0(X);\mathcal{C})$$

the category of symmetric forms on locally free $\mathcal{O}_X$-modules, with values in $\mathcal{C}$.

One may add isomorphism classes of objects in $Q(\mathcal{S}_i^p(X);\mathcal{C})$ by "orthogonal sum" and obtain an abelian semigroup (with the obvious zero element). The *Witt group in the category* $\mathcal{S}_i^p(X;\mathcal{C})$ is the corresponding Grothendieck group modulo the subgroup generated by lagrangians and is denoted

$$W(\mathcal{S}_i^p(X);\mathcal{C}). \tag{0.3}$$

If X is regular, then

$$W(X;\mathcal{C}) := W(\mathcal{S}_n^0(X);\mathcal{C}).$$

At the opposite extreme, for each p, $0 \le p \le n$, set

$$W(\mathcal{F}^p(X);\mathcal{C}) := \coprod_{x \in X^{(p)}} W(\mathcal{S}_0^p(X_x);\mathcal{C}_x), \tag{0.4}$$

the Witt group of forms on finite length $\mathcal{O}_{X,x}$-modules, cod $x = p$. When $X = \operatorname{Spec} R$, R is a CM ring and C is a canonical module for R, we write

$$W(\mathcal{S}_i^p(R);C) = W(\mathcal{S}_i^p(X);\mathcal{C}).$$

The groups $W(\mathcal{S}_i^p(X);\mathcal{C})$ are less exotic than they might appear: if $(R,\mathfrak{m})$ is a local CM ring with canonical module C, its elements can be represented by symmetric bilinear forms $\phi : M \times M \to \Lambda^p N(I) \otimes C$, where M satisfies Grothendieck's condition (S_i), $I \subset \mathfrak{m}$ is a codimension p ideal in R such that $IM = 0$ and $N(I) := \operatorname{Hom}(I/I^2, R/I)$ is the dual of its "conormal module." More exactly, there is a canonical isomorphism

$$\lim_{I \subset R} W(\mathcal{S}_i^0(R/I);\Lambda^p N(I) \otimes C) \xrightarrow{\cong} W(\mathcal{S}_i^p(R);C) \tag{0.5}$$

where the direct limit is taken over all codimension p local complete intersection ideals $I \subset \mathfrak{m}$. Notice that $\Lambda^p N(I) \otimes C$ is a canonical module for R/I and that no reference is made to the injective resolution on the left side of this isomorphism.

The following was proved in [P4] in case X is affine and the canonical sheaf is trivial.

0.6 Theorem. *([P4]) Let $(X, \mathcal{O}_X)$ be a CM scheme and $\mathcal{C}$ a canonical sheaf. Then for each $p \ge 0$ there is an exact sequence*

$$0 \to W(\mathcal{S}_1^p(X);\mathcal{C}) \xrightarrow{\mathcal{K}^p} W(\mathcal{F}^p(X);\mathcal{C}) \xrightarrow{\mathcal{L}^p} W(\mathcal{S}_1^{p+1}(X);\mathcal{C}) \tag{0.7}$$

Hence there is a complex

(0.8)

$$0 \to W(\mathcal{S}_1^0(X);\mathcal{C}) \to W(\mathcal{F}^0(X);\mathcal{C}) \xrightarrow{\partial^0} \dots \xrightarrow{\partial^{p-1}} W(\mathcal{F}^p(X);\mathcal{C}) \xrightarrow{\partial^p} \dots \xrightarrow{\partial^{n-1}} W(\mathcal{F}^n(X);\mathcal{C}) \to 0$$

where $\partial^p := \mathcal{K}^{p+1} \circ \mathcal{L}^p$.

If $X = \operatorname{Spec} R$, where R is regular local ring, essentially of finite type over a field of characteristic $\neq 2$, then (0.7) is short exact and (0.8) is exact.

An argument from [P3] shows that the middle term in the exact sequence (0.7) can be simplified considerably:

0.9 Proposition. *(Dévissage, [P3, Theorem 2.2]) Let $(X, \mathcal{O}_X)$ be a regular scheme and $\mathcal{C}$ a canonical sheaf. Then there is a canonical isomorphism*

$$\coprod_{\mathrm{cod}\ x=p} W(k(x); \Lambda^p N(\mathfrak{m}_x) \otimes \mathcal{C}_x) \xrightarrow{\cong} W(\mathcal{F}^p(X); \mathcal{C}),$$

where $k(x)$ denotes the fraction field at x and $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is the maximal ideal.

For instance, let X be, in addition, of finite type over a field k finite over its prime field or of characteristic zero, and let $\mathcal{C} = \Omega^n_{X/k}$. There is a canonical isomorphism

$$\Lambda^p N(\mathfrak{m}_x) \otimes \mathcal{C} = \Lambda^p N(\mathfrak{m}_x) \otimes \Omega^n_{X/k} \xrightarrow{\cong} \Omega^{n-p}_{k(x)/k}$$

for each $x \in X$ of codimension p. Then (0.8) is the complex studied in [Sc] (although exactness was not proved there). Note that in this case, or indeed for any canonical sheaf $\mathcal{C}$ on a regular scheme X, one may choose an isomorphism

$$\Lambda^p N(\mathfrak{m}_x) \otimes \mathcal{C}_x \cong k(x)$$

hence an isomorphism

$$W(k(x); \Lambda^p N(\mathfrak{m}_x) \otimes \mathcal{C}_x) \cong W(k(x)), \tag{0.10}$$

but no *natural* isomorphism. This is clerly shown by Proposition 0.15 below.

The filtered version of Theorem (0.6) is the basis of the main results of this paper. The non-trivial assertion in it is the surjectivity of $\mathcal{L}^p$. The proof of this (Theorem 4.1 below) follows Quillen's proof of the corresponding K-theory statement, but also needs the trace functions constructed in [SS]. The ideas in this extra ingredient are quite simple in concept, if not in excecution (in the proof of 4.1), so we have given a proof of a well-known theorem of Milnor, to illustrate their use.

We now pass to the global versions of (0.6) and (0.9). Let X be a regular scheme and $\mathcal{C}$, a canonical sheaf. Define the unramified Witt sheaf $\mathcal{W}(X; \mathcal{C})$ for $U \subseteq X$ open by $\mathcal{W}(X;\mathcal{C})(U) := \ker\{W(K; \mathcal{C}_\xi) \xrightarrow{\partial^0} \coprod_{x \in U^{(1)}} W(k(x))\}$, where xi is the generic point of X and K is its field of rational functions. For the regular schemes finite over a field of characteristic $\neq 2$, Ojanguren and Panin ([OjP]) have recently proved the *Purity Theorem*: the stalk of $\mathcal{W}(X;\mathcal{C})$ at $x \in X$ is $W(\mathcal{O}_{X,x}; \mathcal{C}_x)$. Then (0.8) may be sheafified to:

0.11 Corollary. *Let X/k be regular scheme of finite type over a field k, char $k \neq 2$, let K be the function field of X and let $\mathcal{C}$ be a locally free sheaf. Then there is a flasque resolution of $\mathcal{W}(X;\mathcal{C})$,*

$$\mathcal{W}(X;\mathcal{C}) \rightarrowtail i_{\xi *} W(K; \mathcal{C}_\xi) \xrightarrow{\partial^0} \coprod_{\mathrm{cod}\ x=1} i_{x*} W(k(x); N(\mathfrak{m}_x) \otimes \mathcal{C}) \xrightarrow{\partial^1} \cdots$$
$$\xrightarrow{\partial^{n-1}} \coprod_{\mathrm{cod}\ x=n} i_{x*} W(k(x); \Lambda^n N(\mathfrak{m}_x) \otimes \mathcal{C}) \to 0$$

where $W(k(x); \Lambda^p N(\mathfrak{m}_x) \otimes \mathcal{C})$ *is viewed as a constant sheaf on* $\bar{x}$ *and* $i : \bar{x} \to X$ *is the inclusion.*

Now define a decreasing filtration $\{\mathcal{I}_k(X; \mathcal{C})\}$ of $\mathcal{W}(X; \mathcal{C})$ generalizing the standard filtration $\{I^k(F)\}$ of the Witt group $W(F)$ of a field F:

$$\mathcal{I}_k(X; \mathcal{C})(U) := \mathcal{W}(X; \mathcal{C})(U) \cap I^k(K)$$

Then the filtered, global version of (0.8) is:

0.12 Theorem. *Let* X *be regular scheme of finite type over a field* k*, char* $k \neq 2$*, and let* $\mathcal{C}$ *be a locally free sheaf. Then the differentials in the Gersten resolution induce a flasque resolution of* $\mathcal{I}_k(X; \mathcal{C})$ *for each* $k \geq 0$*,*

$$\mathcal{I}_k(X; \mathcal{C}) \rightarrowtail i_{\xi *} I^k(K; \mathcal{C}_\xi) \xrightarrow{\partial^0} \coprod_{\text{cod } x = 1} i_{x*} I^{k-1}(k(x); N(\mathfrak{m}_x) \otimes \mathcal{C}) \xrightarrow{\partial^1} \cdots$$
$$\xrightarrow{\partial^{n-1}} \coprod_{\text{cod } x = n} i_{x*} I^{k-n}(k(x); \Lambda^n N(\mathfrak{m}_x) \otimes \mathcal{C}) \to 0,$$

where $I^s(k(x); \Lambda^p N(\mathfrak{m}_x) \otimes \mathcal{C}) := W(k(x); \Lambda^p N(\mathfrak{m}_x) \otimes \mathcal{C})$ *if* $s < 0$.

Using Voevodsky's theorem affirming the Milnor Conjecture, one may now show that the quotient of these resolutions for any pair k and $k+1$ is the Bloch-Ogus resolution ([BO]) of the (Zariski) sheaf $\mathcal{H}^k_{ét}$ of mod 2 étale cohomology groups, $\{U \mapsto H^k_{ét}(U; \mu_2)\}$. As a consequence. one gets:

0.13 Corollary. *There is a spectral sequence* $\{E_r^{s,t}\}$ *converging to the associated graded groups of a filtration* $\{F^s\}$ *of* $H^*(X; \mathcal{W}(X; \mathcal{C}))$ *such that* $\deg d_r = (r, 1-r)$*,*

$$E_1^{s,t} = H^{s+t}(X; \mathcal{H}^s_{ét}) \Longrightarrow E_\infty^{s,t} = Gr_F H^{s+t}(X; \mathcal{W}(X; \mathcal{C})),$$

$E_1^{s,t} = 0$ *unless* $t \leq 0$ *and* $s + t \geq 0$*, and* $E_1^{k,0} = A^k(X) \otimes \mathbb{Z}/2\mathbb{Z}$.

0.14 Corollary. *If* $\dim X = n$ *and* k *is algebraically closed, then* $H^n(X; \mathcal{W}) \cong \operatorname{cok}\{d_1^{n-1,0} : A^{n-1}(X) \otimes \mathbb{Z}/2\mathbb{Z} \to A^n(X) \otimes \mathbb{Z}/2\mathbb{Z}\}$

As outlined below, Corollaries (0.13) and (0.14) sharpen calculations done in [P3] and elsewhere. For instance, the question posed in [PaS, p. 30] asks whether the edge homomorphism in the spectral sequence

$$\coprod_{k \geq 0} I_k(X; \mathcal{C})/I_{k+1}(X; \mathcal{C}) \to \coprod_{k \geq 0} H^k(X; \mathcal{H}^k_{ét})$$

is an isomorphism and the positive results there prove that the relevant differentials in the spectral sequence are trivial.

Note that the E_1 terms of the spectral sequence are independent of $\mathcal{C}$. However, the differentials are not, as is shown by the following calculation, the second and third equalities of which are proved in [Sc].

0.15 Proposition. *If $X = \mathbb{P}^n$ and $\mathcal{C} = \mathcal{O}(m)$, then the spectral sequence in Theorem (0.13) satisfies $E_1^{s,t} = 0$ if $t \neq 0$, $E_1^{s,0} = \mathbb{Z}/2\mathbb{Z}$ for $0 \leq s \leq n$ and $d_1^{s,0}$ is non-trivial if and only if $s + m$ is odd. Consequently, the edge homomorphism induces an isomorphism*

$$\coprod_{k \geq 0} I_k(\mathbb{P}^n; \mathcal{O}(m))/I_{k+1}(\mathbb{P}^n; \mathcal{O}(m)) \cong \coprod_{k \geq 0} H^0(\mathbb{P}^n; \mathcal{H}^k_{\acute{e}t}),$$

$$H^i(\mathbb{P}^n; \mathcal{W}(\mathbb{P}^n; \mathcal{O}(m)) = 0, \quad 0 < i < n,$$

and if m is even,

$$H^0(\mathbb{P}^n; \mathcal{W}(\mathbb{P}^n; \mathcal{O}(m)) = \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad H^n(\mathbb{P}^n_{\mathbb{C}}; \mathcal{W}) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

while if m is odd

$$H^0(\mathbb{P}^n; \mathcal{W}(\mathbb{P}^n; \mathcal{O}(m)) = 0 \quad \text{and} \quad H^n(\mathbb{P}^n_{\mathbb{C}}; \mathcal{W}) = \begin{cases} 0, & n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z}, & n \text{ even} \end{cases}$$

Here are some other prospective applications of ideas in this paper in contexts where the groups $H^*(X; \mathcal{W}(X; \mathcal{C}))$ appear naturally and the spectral sequence may be applied.

Following an exposition of Ferrand [F], a connection was made in [P3] between Gorenstein ideals in a regular ring A and quadratic forms over A. In a more global context (also following Ferrand) the discussion from [P3] can be summarized as follows. Let X be a regular scheme and $C \subset X$ a codimension c Gorenstein subscheme. Then the canonical sheaf $\omega_C := \mathcal{E}xt^c_{\mathcal{O}_X}(\mathcal{O}_C, \mathcal{K}_X)$ is locally free; if it extends to an invertible sheaf $\mathcal{S}$ on X then it is said to be *subcanonical*, and there is an elment

$$(\mathcal{O}_C, \mu) \in Q(\mathcal{S}^c_n(X); \mathcal{K}_X \mathcal{S}^{-1})$$

(*i.e.*, $\mathcal{O}_C$ supports a nonsingular form with values in $\mathcal{V}^c(\mathcal{K}_X \mathcal{S}^{-1})$). Set $\mathcal{L} = \mathcal{K}_X \mathcal{S}^{-1}$. If $H^c(X; \mathcal{L}) = 0$, $C \subset X$ is then *strongly* subcanonical, and one gets an element $\epsilon \in Ext^c(\mathcal{I}_C, \mathcal{L})$. Now let $c = 2$. Then ϵ yields an exact sequence of sheaves

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{I}_C \to 0$$

where $\mathcal{E}$ is locally free of rank two, there is a canonical isomorphism $\sigma : \Lambda^2 \mathcal{E} \xrightarrow{\cong} \mathcal{L}$ and the composition $\mathcal{E} \times \mathcal{E} \to \Lambda^2 \mathcal{E} \xrightarrow{\sigma} \mathcal{L}$ is a nonsingular skew-symmetric form, an element of what we denote $Q^{-1}(X; \mathcal{L})$. (Compare Definition (0.2).) Geometrically, C is thus the zero locus of the section, given by the composition $\mathcal{E} \to \mathcal{I}_C \hookrightarrow \mathcal{O}_X$, of the bundle corresponding to $\mathcal{E}$. The function $Q(\mathcal{S}^2_n(X); \mathcal{L}) \xrightarrow{SH} Q^{-1}(X; \mathcal{L})$ was called the *Serre-Horrocks construction* in [P3].

0.16 Proposition. *[P3, 4.3] The diagram*

$$\begin{array}{ccc} Q(\mathcal{S}^2_n(X);\mathcal{L}) & \xrightarrow{SH} & Q^{-1}(X;\mathcal{L}) \\ \downarrow & & \downarrow \\ W(\mathcal{S}^2_n(X);\mathcal{L}) & \longrightarrow & W^{-1}(X;\mathcal{L}) \end{array}$$

commutes where the verticals pass to the equivalence class in the apprpriate Witt group and the bottom vertical is a composition of maps in the localization sequences of [P3, Theorem 2.1].

This result was used in [P3] to estimate the size of $W^{-1}(X)$ if $\dim X = 2$; improvements were obtained in [BOj], [OPS] and [Ku]. The definitive result in this direction should follow from the spectral sequence (0.13) and the following:

0.17 Corollary. *Let* $\dim X \leq 3$. *Then the bottom horizontal in (0.16) induces an isomorphism*

$$H^2(X;\mathcal{W}(X;\mathcal{L})) \xrightarrow{\cong} W^{-1}(X;\mathcal{L})$$

In [W], [EPW1] and [EPW2], the authors study a question raised by C. Okonek: are strongly subcanonical codimension 3 subschemes Z of a (usually regular) scheme X cut out by the Pfaffians of a skew-symmetric map of locally free sheaves? This question was motivated by the local theorem of [BE] which says that the skew-symmetric map moreover appears in the middle of a minimal resolution of $\mathcal{O}_Z$. Such Z are called *Pfaffian* and are conversely subcanonical. In [W], C. Walter showed that if $X = \mathbb{P}^n_k$, Z is Pfaffian if and only if it is subcanonical and a certain element of $W^1_0(k)$ vanishes. In [EPW1], the authors recast Okonek's question, giving a general construction of a locally free resolution of $\mathcal{O}_Z$ which they call "symmetrically quasi-isomorphic". Using this, we will now show how the machinery described above may be used to generalize some of Walter's results to apply to any regular scheme X, finite over a field of characteristic $\neq 2$, in place of $\mathbb{P}^n_k$. To describe the ideas, some definitions and other constructions from [EPW1] will now be recalled.

First, to the data given by a strongly subcanonical codimension 3 scheme $Z \subset X$ is assigned a triple $(\mathcal{M}, q; \mathcal{E}, \mathcal{F})$, where $\mathcal{E}$ and $\mathcal{F}$ are sublagrangians of $(\mathcal{M}, q)$, an element of $Q(X;\mathcal{L})$ ([EPW1,(6.1)]). Conversely, given such a triple, one gets ([EPW1,(3.1)]) a subcanonical codimension 3 scheme.

Now the triples $(\mathcal{M}, q; \mathcal{E}, \mathcal{F})$ are *(symmetric) formations*, in the sense of [FFC], and, in the affine case, played a major role in [P1]-[P3]. One may define, in the spirit of the Definition (0.2) above, formations and associated Witt groups ([FFC]),

$$F(\mathcal{S}^p_i(X);\mathcal{L}) \qquad \text{and} \qquad W_1(\mathcal{S}^p_i(X);\mathcal{L}).$$

(The subscript "1" on $W_1(\mathcal{S}^p_i(X);\mathcal{L})$ is meant to suggest K_1.) As in the codimension two case above, the main datum of a subcanonical Z is an element

$$\eta : \mathcal{O}_Z \times \mathcal{O}_Z \to \mathcal{V}^3(\mathcal{L})$$

of $Q(\mathcal{S}^3_n(X);\mathcal{L})$.

So, roughly speaking, there are correspondences

$$(*)\qquad \Big\{(\mathcal{O}_Z,\eta)\in Q(\mathcal{S}^3_n(X);\mathcal{L})\Big\}\longleftrightarrow\Big\{(\mathcal{M},q;\mathcal{E},\mathcal{F})\in F^1(\mathcal{S}_{n,0})=F(X;\mathcal{L})\Big\}$$

which is strongly suggestive of the top of the diagram in Proposition (0.16). Indeed, we have:

0,23 Proposition. *There is a commutative diagram*

$$\begin{array}{ccc} Q(\mathcal{S}^3_n(X);\mathcal{L}) & \xrightarrow{[EPW]} & F(X;\mathcal{L}) \\ \downarrow & & \downarrow \\ W(\mathcal{S}^3_n(X);\mathcal{L}) & \xrightarrow{R} & W_1(X;\mathcal{L}) \end{array}$$

where R is a composition of maps in the localization sequences of [P3,2.1], and Z is Pfaffian if and only if there is strongly subcanonical data (including a line bundle $\mathcal{L}$ and η as above) such that $R([\mathcal{O}_Z,\eta])=0$.

The idea of proof is that the relations defining $W_1(X)$ do not change the degeneracy locus Z of $(\mathcal{M},q;\mathcal{E},\mathcal{F})$: they are essentially the alternating homotopies of [EPW1, §5] and the modification of §4 leading to split bundles, which are, respectively, isomorphism of formations and stabilization to a split formation in the language of [FFC, Appendix]); and, when X is affine, $[(\mathcal{M},q;\mathcal{E},\mathcal{F})]=0$ in $W_1(X;\mathcal{L})$ if and only if stabilization and homotopy bring it to the form ([EPW1, §4]) required for it to give rise to a Pfaffian subscheme ([P3, 1.21]).

The following result has a proof similar to that of Proposition (0.17). Together with Theorem (0.13), it gives substance to Proposition (0.18) and should allow one to generalize the computations in §7 of [EPW1].

0.19 Proposition. *If X is a 3-dimensional regular scheme, finite over a field k of characteristic $\neq 2$, then $H^3(X;\mathcal{W}(X;\mathcal{L})\cong W_1(X;\mathcal{L})$.*

This is consistent with the examples in §7 in [EPW1]: in the special case where $X=\mathbb{P}^3_{\mathbb{C}}$, Propositions (0.15), (0.18) and (0.19) imply that there is at most a single $\mathbb{Z}/2\mathbb{Z}$-obstruction to the subcanonical data on the left side of the EPW correspondence giving rise to a Pfaffian structure on Z. The claim is that this is the same as the element of $W(\mathbb{C})=\mathbb{Z}/2\mathbb{Z}$ appearing in [EPW1, 7.1]; and that even when k is not algebraically closed, the element of $W^0_1(k)$ seen in [EPW1, 7.1] corresponds to the element of $H^3(X;\mathcal{W})$ gotten from Proposition (0.19) and Corollary (0.14).

It is likely that one may generalize the result of [CC] in a similar way. There is an exact sequence, analogous to [P3, 2.1]

$$W(K)\to W_0(\mathcal{CM}_1(X);\mathcal{L})\to W^{-1}_1(X;\mathcal{L})$$

where K is the function field of X and $W^{-1}_1(X;\mathcal{L})$ is an abelian group of equvalence classes of formations analogous to $W_1(X;\mathcal{L})$, but using *skew*-symmetric forms. Then a quadratic sheaf $(\mathcal{F}, \sigma)$, in the sense of [CC, Def. 0.2], evidently gives rise to an element of $W_0(\mathcal{CM}_1(X); \mathcal{L})$ which, by the construction of the exact sequence, is in the image of $W_0^1(K)$ if and only if $(\mathcal{F}, \sigma)$ admits a symmetric resolution as in [CC, (0.3.1)]. The obstruction to the resolution thus lies in $W_1^{-1}(X; \mathcal{L})$, and one shows that if $\dim X \leq 3$, then

$$W_1^{-1}(X; \mathcal{L}) \cong H^1(X; \mathcal{W})$$

Thus, as in the discussion following Corollary 12, [CC, Theorem (0,3)] would be a consequence of Proposition (0.15) and would be generalized to regular 3-dimensional X in place of $\mathbb{P}^3$; and one would get a generalization of [EPW, Theorem (9.1)].

Recently, P. Balmer and C. Walter, using the derived category methods of [Ba], have independently developed the applications to Pfaffian schemes sketched above (in preparation). They have also constructed a "Gersten-Witt spectral sequence" (different from (0.13) above!) in [BaW] and applied it to purity questions for the Witt group.

Suggestions of Spencer Bloch and David Eisenbud have been crucial in formulating the approach taken here. Most important, however, was the observation of J. Barge in [Bar] that the natural places for the values of a quadratic form on torsion modules over a ring R are the terms of the injective resolution of R.

§1. Quadratic forms on Cohen-Macaulay schemes

In this section $(X, \mathcal{O}_X)$ (resp. R) will be a CM (Cohen-Macaulay) scheme (resp. ring) such that $\dim \mathcal{O}_{X,x} = n$ (resp. $\dim R_{\mathfrak{m}} = n$) for all closed points $x \in X$ (resp. maximal ideals $\mathfrak{m} \subset R$). $(X_x, \mathcal{O}_{X,x})$ denotes the scheme Spec $\mathcal{O}_{X,x}$. In [P2]-[P4] we considered forms with values in the terms of the injective resolution of a Gorenstein ring. However, it is both more natural, useful and no less difficult to develop the theory of forms with values in the terms of the injective resolution of a canonical sheaf of a CM scheme. The following facts and more can be found in [BH].

A *canonical sheaf* for X is a coherent sheaf $\mathcal{C}$ such that $\mathrm{Ext}^i_{\mathcal{O}_{X,x}}(k(x), \mathcal{C}_x) = \delta_{in}$ for all closed points $x \in X$. Equivalently, $\mathcal{C}$ has a minimal injective resolution of the form

$$0 \to \mathcal{C} \to \mathcal{E}^0(\mathcal{C}) \xrightarrow{d^0} \mathcal{E}^1(\mathcal{C}) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \mathcal{E}^n(\mathcal{C}) \xrightarrow{d^n} 0 \tag{1.1}$$

where each $\mathcal{E}^p(\mathcal{C})$ decomposes into a direct sum of sheaves supported at $x \in X^{(p)}$:

$$\mathcal{E}^p(\mathcal{C}) = \coprod_{x \in X^{(p)}} i_{x*}\mathcal{E}^p(\mathcal{C})_x \tag{1.2}$$

where $i_x : \overline{x} \to X$ is the inclusion and

$$\mathcal{E}^p(\mathcal{C})_x \cong E_{\mathcal{O}_{X,x}}(k(x)) \tag{1.3}$$

and $E_R(M)$ denotes the injective hull of the R-module M. A canonical sheaf is always a maximal CM sheaf: depth $\mathcal{C}_x = \dim \mathcal{O}_{X,x}$ for all $x \in X$.

If R is local, a canonical module is well-defined up to isomorphism, and, since for any canonical module C the standard homomorphism $R \to \mathrm{End}(R)$ is bijective, is esentially unique. In general, $\mathcal{C}$ is unique up to tensor product with with a locally free sheaf of rank one. X is *Gorenstein* if and only if $\mathcal{C}$ is locally free of rank one. In particular, R is local Gorenstein if and only if the canonical module is isomorphic to R. Not every local CM ring admits a canonical module; it does so if and only if it the image of a local Gorenstein ring, which is how canonical modules will sometimes arise in this paper. Somewhat more generally, if $\phi : R \to S$ is a local homomorphism of local CM rings such that S is a finitely-generated R-module, and C is the canonical module for R, then

$$\mathrm{Ext}^t_R(S, C) \tag{1.4}$$

is the canonical module for S.

Now referring to (1.1), set

$$\mathcal{V}^p(\mathcal{C}) := \ker d^p,\ 0 \le p \le n \tag{1.5}$$

Because of (1.2) and (1.3), the minimal resolution of a canonical sheaf $\mathcal{C}$ is unique up to isomorphism; moreover, cod $\mathcal{E}^p(\mathcal{C}) = p$, for all p, so that $\mathcal{V}^{p+1}(\mathcal{C})_x \subseteq \mathcal{E}^{p+1}(\mathcal{C})_x = 0$, for $x \in X^{(p)}$. Hence,

$$\mathcal{V}^p(\mathcal{C})_x \cong \mathcal{E}^p(\mathcal{C})_x,\ \ x \in X^{(p)} \tag{1.6}$$

For instance, (1.2) and (1.6) imply that $\mathcal{C} \to \mathcal{E}^0(\mathcal{C})$ induces

$$\coprod_{x\in X^{(0)}} i_{x*}\mathcal{C}_x \xrightarrow{\cong} \coprod_{x\in X^{(0)}} i_{x*}\mathcal{E}^0(\mathcal{C})_x = \mathcal{E}^0(\mathcal{C}) \tag{1.7}$$

so the canonical map $\mathcal{C} \hookrightarrow \coprod i_{x*}\mathcal{C}_x$ is an injective hull for $\mathcal{C}$; let $\mathcal{C}^1$ denote its cokernel. Then (1.7) induces an isomorphism $\mathcal{C}^1 \xrightarrow{\cong} \mathcal{V}^1(\mathcal{C})$, from which we get

$$\coprod_{x\in X^{(1)}} i_{x*}\mathcal{C}^1_x \xrightarrow{\cong} \coprod_{x\in X^{(1)}} i_{x*}\mathcal{V}^1(\mathcal{C})_x = \mathcal{E}^1(\mathcal{C}) \tag{1.8}$$

so that $\mathcal{C}^1 \to \coprod_{x\in X^{(1)}} i_{x*}\mathcal{C}^1_x$ is an injective hull for $\mathcal{C}$. One may now continue in this way to build a *canonical* injective resolution of $\mathcal{C}$.

Because the localization sequences we construct will depend on this resolution, we make the

1.9 Convention. *Given a canonical sheaf $\mathcal{C}$ for a CM scheme X, the injective resolution*

$$\mathcal{C} \to \mathcal{E}^{\cdot}(\mathcal{C})$$

will be the canonical injective resolution constructed above.

In [P3], $\mathcal{CM}^p(R)$ denoted the category of codimension p CM R-modules; here $\mathcal{CM}^p(X)$ denotes the category of codimension p coherent CM sheaves. The following larger category plays a central role in this paper.

1.10 Definition. *Let i and p be integers, $i \geq 0$, $0 \leq p \leq n$. Then $\mathcal{S}^p_i(X)$ denotes the the category of coherent sheaves $\mathcal{M}$ on X such that*

a. cod $\mathcal{M} = p$, *and*
b. *depth* $\mathcal{M}_x \geq \inf\{i, \dim \mathcal{M}_x\}$, *for all* $x \in X^{(p)}$,

together with the zero module.

The category of sheaves with codimension $\geq p$, denoted $\mathrm{M_p}$, was used in the derivation of the localization sequence for K-theory in [Q], while condition (1.1b) is property (S_i) in [EGA IV, 5.7.2]. Evidently, $\mathcal{CM}^p(X) \subseteq \mathcal{S}^p_{i+1}(X) \subseteq \mathcal{S}^p_i(X)$ for all i, and, since $\dim \mathcal{M}_x =$ cod $x-p$ if $x \in \operatorname{Supp}\mathcal{M}$, $\mathcal{CM}^p(X) = \mathcal{S}^p_i(X)$ if $i \geq n-p$. In particular, $\mathcal{S}^p_i(X_x) = \mathcal{CM}^p(X_x)$ if $i \geq$ cod $x - p$. On the other hand, if $\mathcal{I} \subset \mathcal{O}_X$ is an ideal sheaf which is a codimension p local complete intersection, Y is the corresponding subscheme and $p \leq r$, then there is an inclusion

$$\mathcal{S}^{r-p}_i(Y) \subseteq \mathcal{S}^r_i(X). \tag{1.11}$$

whose image consists of sheaves $\mathcal{M} \in \mathcal{S}^r_i(X)$ with $\mathcal{I} \subseteq \operatorname{Ann}\mathcal{M}$. When $p = n$ the elements of $\mathcal{S}^p_i(X)$ are simply modules of finite length, and we denote (for any i)

$$\mathcal{F}(X) := \mathcal{S}^n_i(X) \tag{1.12}$$

When $i = 1$ or 2, we will relate ((1.17) and (1.19) below) elements of $\mathcal{S}_i^p(X)$ to modules satisfying torsionless and reflexivity conditions generalizing those in case $p = 0$ ([BH, 1.4.1]).

Our interest in $\mathcal{S}_i^p(X)$ stems from the lack of useful existence theorems for finitely-generated CM modules (objects in $\mathcal{CM}^p(R)$). This difficulty was avoided in [P3] by assuming dim $R \leq 4$; its appearance in this paper will be discussed in (3.) below. However, a functorial method for constructing elements of $\mathcal{S}_2^p(R)$ was given by Hochster ([Ho, p.18]). The proof in different language is reproduced here.

1.13 Proposition. *Let R be a local CM ring with canonical module C, and let $M \in \mathcal{S}_0^p(R)$. Then $Hom(M, V^p(C)) \in \mathcal{S}_2^p(R)$.*

Proof: Let $M\hat{} = \mathrm{Hom}\,(M, V^p(C))$. Since $\mathrm{Ass}\, M\hat{} = \mathrm{Ass}\, V^p(C) \cap \mathrm{Supp}\, M$, cod $M = p$. The proof will show that if s (resp., s, t) is part of a s.o.p. (system of parameters) of M, then s (resp., s, t) is $M\hat{}$-regular (resp., is an $M\hat{}$-regular sequence). It follows from [B, 2.2] that $(M\hat{})_{\mathfrak{p}} = (M_{\mathfrak{p}})\hat{}$ for all primes $\mathfrak{p}$, so this suffices to place $M\hat{} \in \mathcal{S}_1^p$.

Let s be part of a s.o.p. Then $\mathrm{cod}\,(M/sM) = p + 1$, so $\mathrm{Hom}\,(M/sM, V^p(C)) = 0$, Consider

$$_sM := \ker\{M \to M[\tfrac{1}{s}]\}.$$

Clearly, cod $_sM > p$, so $(_sM)\hat{} = 0$. Thus applying (-)$\hat{}$ to the short exact sequence

$$_sM \rightarrowtail M \twoheadrightarrow M/_sM,$$

we obtain

$$(M/_sM)\hat{} \cong M\hat{}. \tag{1.14}$$

But s is not a zero-divisor on $M/_sM$, so from the short exact sequence

$$M/_sM \overset{\cdot s}{\rightarrowtail} M/_sM \twoheadrightarrow N := \mathrm{cok}\ s,$$

we get

$$(M/_sM)\hat{} \overset{\cdot s}{\rightarrowtail} (M/_sM)\hat{} \tag{1.15}$$

since cod $N > p$. This together with (1.14) shows s is a non-zero-divisor on $M\hat{}$.

Next suppose $\{s, t\}$ is part of a s.o.p. for M. Let $M' = M/_sM$. Extending (1.15) to the right, we have an exact sequence

$$0 \to (M')\hat{} \overset{s}{\to} (M')\hat{} \to \mathrm{Ext}^1(M'/sM', V^p(C)).$$

Hence $M\hat{}/sM\hat{} \cong (M')\hat{}/s(M')\hat{} \rightarrowtail \mathrm{Ext}^1(M'/sM', V^p(C))$, so it suffices to show t is a non-zerodivisor on $\mathrm{Ext}^1(M'/sM', V^p(C))$, which is isomorphic to $\mathrm{Hom}\,(M'/sM', V^{p+1}(C))$, since $E^p(C)$ is injective, and cod $M'/sM' > p$, so that $\mathrm{Hom}\,(M'/sM', E^p(C)) = 0$. But t is part of a s.o.p. for M'/sM', so the first part of the argument shows t is a non-zerodivisor on $\mathrm{Hom}\,(M'/sM', V^{p+1}(C))$.

The properties of the $V^k(R)$ developed in [P3, 1.3-1.6], in the case where R is Gorenstein and $C = R$, continue to hold if X is a CM scheme and $\mathcal{V}^k(\mathcal{C})$ replaces $V^k(R)$. In particular, every CM sheaf $\mathcal{M}$ which is of dimension n at closed points is reflexive with respect to a(ny) canonical sheaf $\mathcal{M}$: the canonical homomorphism

$$\kappa : \mathcal{M} \to \mathcal{H}om(\mathcal{H}om(\mathcal{M}, \mathcal{C}), \mathcal{C})$$

is bijective. Moreover, because this condition is necessary to a reasonable theory of nonsingular quadratic forms on CM sheaves over a CM scheme, the canonical module is inevitable: if X is CM and one replaces $\mathcal{C}$ with $\mathcal{O}_X$, κ is not generally bijective. More generally, a coherent sheaf $\mathcal{M} \in \mathcal{S}_0^p(X)$ is called *p-reflexive with respect to* $\mathcal{C}$ if the natural map

$$\kappa : \mathcal{M} \to \mathcal{H}om(\mathcal{H}om(\mathcal{M}, \mathcal{V}^p(\mathcal{C})), \mathcal{V}^p(\mathcal{C})) \tag{1.16}$$

is an isomorphism. Since $\operatorname{Ass} \mathcal{H}om(\mathcal{M}, \mathcal{V}^p(\mathcal{C})) = \operatorname{Supp} \mathcal{M} \cap \operatorname{Ass} \mathcal{V}^p(\mathcal{C})$, p-reflexive sheaves lie in $\mathcal{S}_0^p(X)$. Since $\mathcal{V}^0(\mathcal{C}) = \mathcal{C}$, 0-reflexive sheaves are reflexive with respect to $\mathcal{C}$. The following generalizes their well-known characterization ([BH, 1.4.1b]).

1.17 Proposition. *Let X be a CM scheme and $\mathcal{C}$ a canonical sheaf. Let $\mathcal{M} \in \mathcal{S}_0^p(X)$. Then $\mathcal{M}$ is p-reflexive if and only if $\mathcal{M} \in \mathcal{S}_2^p(X)$.*

Proof: It suffices to take X=Spec R, where R is local CM.

Suppose M is reflexive. Construct an exact sequence

$$P \to Q \to M\hat{} := \operatorname{Hom}(M, V^p(C))$$

where P, $Q \in \mathcal{CM}^p(R)$ (see the beginning of the proof of [P3, 1.6]). Dualize it to

$$M \cong M\hat{}\hat{} \rightarrowtail Q\hat{} \xrightarrow{f\hat{}} P\hat{} \to \operatorname{cok} f\hat{} := C$$

By [P3, 1.6a], $Q\hat{}$, $P\hat{} \in \mathcal{CM}^p(R) \subseteq \mathcal{S}_i^p(R)$, for all i. Since $\dim P\hat{}_\mathfrak{q} \geq \dim C_\mathfrak{q}$ for all primes $\mathfrak{q}$, $\operatorname{depth} P\hat{}_\mathfrak{q} \geq \operatorname{depth} C_\mathfrak{q}$. We may suppose cod $\mathfrak{q} > p$ in verifying that $M \in \mathcal{S}_2^p(R)$, $i \geq 2$. The behavior of depth in short exact sequences is used several times below and comes from [E, p. 451].

Suppose first that $\operatorname{depth} C_\mathfrak{q} < \operatorname{depth} P_\mathfrak{q}$. Then $\operatorname{depth}(\operatorname{im} f\hat{})_\mathfrak{q} = \operatorname{depth} C_\mathfrak{q} + 1$; there are two possibilities here. First, if $\operatorname{depth}(\operatorname{im} f\hat{})_\mathfrak{q} < \operatorname{depth} Q_\mathfrak{q}$ then $\operatorname{depth} M_\mathfrak{q} = \operatorname{depth}(\operatorname{im} f\hat{})_\mathfrak{q} + 1 = \operatorname{depth} C_\mathfrak{q} + 2 \geq 2$. Second, if $\operatorname{depth}(\operatorname{im} f\hat{})_\mathfrak{q} = \operatorname{depth} C_\mathfrak{q}$, then $\operatorname{depth} M_\mathfrak{q} \geq \operatorname{depth} Q_\mathfrak{q} = \operatorname{cod} \mathfrak{q} - p$.

Next suppose $\operatorname{depth} C_\mathfrak{q} = \operatorname{depth} P_\mathfrak{q}$. Then $\operatorname{depth}(\operatorname{im} f\hat{})_\mathfrak{q} \geq \operatorname{cod} \mathfrak{q} - p$, which must be equality by [P3, 1.2(ii)]. Again by [P3, 1.2], $\operatorname{depth} M_\mathfrak{q} = \operatorname{depth} Q_\mathfrak{q} = \operatorname{cod} \mathfrak{q} - p$.

Hence $\operatorname{depth} M_\mathfrak{q} \geq \inf(2, \operatorname{cod} \mathfrak{q} - p)$ for all $\mathfrak{q}$ of codimension $\geq p + 1$.

Suppose conversely that $M \in \mathcal{S}_2^p(R)$. First, M is unmixed: if $\mathfrak{q} \in \operatorname{Ass} M$, then $\operatorname{depth} M_\mathfrak{q} = 0$ and by assumption this is only possible if cod $\mathfrak{q} = p$. On the other hand, at each codimension p prime $\mathfrak{q} \in \operatorname{Ass} M$, M is reflexive by [P3, 1.6a], since $M_\mathfrak{q}$ has finite length. But $(M\hat{}\hat{})_\mathfrak{q} = (M_\mathfrak{q})\hat{}\hat{}$, so the canonical map $\kappa : M \to M\hat{}\hat{}$ can have no codimension p primes in its kernel. Thus, κ is injective; let C denote its cokernel.

By assumption, $M_\mathfrak{q} \in \mathcal{CM}^{\operatorname{cod} \mathfrak{q} - p}(R_\mathfrak{q})$ if cod $\mathfrak{q} \leq p + 2$. By [P3, 1.6a] $M_\mathfrak{q} \cong (M\hat{}\hat{})_\mathfrak{q}$ for such $\mathfrak{q}$, so if $\mathfrak{p} \in \operatorname{Ass} C$, then cod $\mathfrak{p} > p + 2$ and $\operatorname{depth} C_\mathfrak{p} = 0$. But by assumption, $\operatorname{depth} M_\mathfrak{p} \geq 2$, and by Hochster's argument (1.13), $\operatorname{depth}(M\hat{}\hat{})_\mathfrak{p} \geq 2$. By [E, p. 451] this is impossible.

1.18 Corollary. *If $\mathcal{M}$, $\mathcal{N} \in \mathcal{S}^p_i(X)$, $i \geq 2$, and $f : \mathcal{M} \to \mathcal{N}$ is an isomorphism at all points of codimension $\leq p+1$, then f is an isomorphism.*

Proof It suffices to work locally.

First, f is injective because M is unmixed and f is an isomorphism in codimension p. Let $Q = \operatorname{cok} f$ and dualize the exact sequence $M \xrightarrow{f} N \twoheadrightarrow Q$ to

$$Q^{\wedge} \rightarrowtail N^{\wedge} \xrightarrow{f^{\wedge}} M^{\wedge} \to \operatorname{Ext}(Q, V^p(C)),$$

where $(-)^{\wedge} = \operatorname{Hom}(-, V^p(C))$. By assumption cod $Q \geq p+2$, so $Q^{\wedge} = 0$. The short exact sequence $V^p(C) \rightarrowtail E^p(C) \twoheadrightarrow V^{p+1}(C)$ gives rise to an exact sequence

$$\operatorname{Hom}(Q, V^{p+1}(C)) \to \operatorname{Ext}(Q, V^p(C)) \to \operatorname{Ext}(Q, E^p(C))$$

in which the first term vanishes because cod $Q \geq p+2$ and the third does because $E^{p+1}(C)$ is injective. Thus, $f^{\wedge}$ is an isomorphism. However, N and M are reflexive by (1.17), so $f = f^{\wedge\wedge}$ is an isomorphism too.

Here is a corollary of (the easier parts of) the proof of (1.17); it characterizes the elements of $\mathcal{S}_{1,p}(X)$.

1.19 Corollary. *Let $\mathcal{M} \in \mathcal{S}^p_0(X)$. Then the following are equivalent:*

a. *The natural map $\kappa : \mathcal{M} \to \mathcal{H}om(\mathcal{H}om(\mathcal{M}, \mathcal{V}^p(\mathcal{C})), \mathcal{V}^p(\mathcal{C}))$ is injective*
b. $\mathcal{M} \in \mathcal{S}^p_1(X)$
c. cod $x = p$ *for all* $x \in Ass\,\mathcal{M}$.

We can now introduce forms with values in the terms of the injective resolution.

1.20 Definition. *Let $\mathcal{C}$ be a canonical sheaf for the CM scheme X and let $\mathcal{C} \to E^{\cdot}(\mathcal{C})$ be the injective resolution. Let $\mathcal{M} \in \mathcal{S}^p_i(X)$ and $\phi : \mathcal{M} \times \mathcal{M} \to \mathcal{V}^p(\mathcal{C})$ be a symmetric, $\mathcal{O}_X$-bilinear form, such that $\phi_x : \mathcal{M}_x \times \mathcal{M}_x \to \mathcal{V}^p(\mathcal{C})_x = V^p(\mathcal{C}_x)$ is nonsingular (the adjoints are isomorphisms) if* cod $x \leq p+i$. *Then the pair $(\mathcal{M}, \phi)$ is called a $\mathcal{V}^p(\mathcal{C})$-*valued *symmetric $\mathcal{O}_X$-bilinear* form in *$\mathcal{S}^p_i(X)$ and the collection of such is denoted $Q(\mathcal{S}^p_i(X); \mathcal{C})$.*

Observe that there are inclusions $Q(\mathcal{S}^{p+1}_i(X); \mathcal{C}) \subseteq Q(\mathcal{S}^p_i(X); \mathcal{C})$. Thus a form $(\mathcal{M}, \phi)$ in $Q(\mathcal{S}^p_i(X); \mathcal{C})$ is required to be nonsingular in those codimensions where $\mathcal{M}$ is CM. The next result shows that $(\mathcal{M}, \phi)$ is nonsingular in the usual sense if and only if $\mathcal{M}$ is reflexive.

1.21 Proposition. Let $(\mathcal{M}, \phi) \in Q(\mathcal{S}^p_i(X); \mathcal{C})$ for some $i \geq 1$. Then ϕ is nonsingular if and only if $\mathcal{M} \in \mathcal{S}^p_2(X)$.

Proof: We have a commutative diagram

(1.21a)

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\operatorname{Ad}\phi} & \mathcal{M}^{\wedge} \\ \kappa\downarrow & & \downarrow = \\ \mathcal{M}^{\wedge\wedge} & \xrightarrow{(\operatorname{Ad}\phi)^{\wedge}} & \mathcal{M}^{\wedge} \end{array}$$

where $\mathcal{M}\hat{} = \mathcal{H}om(\mathcal{M}, \mathcal{V}^p(\mathcal{C}))$ and κ is the canonical map. Since $(\mathcal{M}, \phi) \in Q(\mathcal{S}_i^p(X); \mathcal{C})$, $\operatorname{Ad}\phi$ is an isomorphism at all codimension p primes. Hence $\operatorname{Ad}\phi$ is injective.

Let $\mathcal{N} = \operatorname{cok} \operatorname{Ad}\phi$; $\operatorname{cod} \mathcal{N} \geq p+1$ because $\operatorname{Ad}\phi$ is an isomorphism at all codimension p primes. Dualizing the short exact sequence $\mathcal{M} \rightarrowtail \mathcal{M}\hat{} \twoheadrightarrow \mathcal{N}$ gives the exact sequence

$$0 \longrightarrow \mathcal{N}\hat{} \longrightarrow \mathcal{M}\hat{}\,\hat{} \xrightarrow{(\operatorname{Ad}\phi)\hat{}} \mathcal{M}\hat{} \longrightarrow \mathcal{E}xt^1(\mathcal{N}, \mathcal{V}^p(\mathcal{C})),$$

Since $\mathcal{E}^p(\mathcal{C})$ is injective and $\operatorname{cod} \mathcal{N} \geq p+1$, $\mathcal{E}xt^1(\mathcal{N}, \mathcal{V}^p) \cong \operatorname{Hom}(\mathcal{N}, \mathcal{V}^{p+1})$. Since $(\mathcal{M}, \phi) \in Q(\mathcal{S}_1^p(X); \mathcal{C})$, $\operatorname{cod} \mathcal{N} \geq p+2$, so $\mathcal{H}om(\mathcal{N}, \mathcal{V}^p(\mathcal{C})) = 0 = \mathcal{H}om(\mathcal{N}, \mathcal{V}^{p+1}(\mathcal{C}))$. Hence $(\operatorname{Ad}\phi)\hat{}$ is an isomorphism. Thus, $\operatorname{Ad}\phi$ is an isomorphism if and only if κ is; and by (1.17), this is equivalent to $\mathcal{M} \in \mathcal{S}_2^p(X)$

In the rest of this chapter, we work locally, so that R is a CM local ring. The next result shows that each $V^p(C)$ is the union of submodules isomorphic to $C/(x_1, \ldots, x_p)C$, where $\{x_1, \ldots, x_p\}$ is an R-sequence on C.

1.22 Proposition. a. *Let C be a canonical module for the local CM ring R, and let $C \to E^{\cdot}(C)$ be the injective resolution (1.9). Let $x_1, \ldots x_\ell$ be a C-sequence, $\ell \geq 1$, let $I_p := (x_1, \ldots, x_p)$, $C^p = C/I_pC$, $1 \leq p \leq \ell$ and let $C^0 = C$. Then there are injections, depending on $x_1, \ldots x_\ell$,*

$$i^p : C^p \longrightarrow V^p, \;\; im\ i^p = (0 : I_p)_{V^p(C)} \tag{1.23}$$

for $0 \leq p \leq \ell$, such that $i^0 : C^0 = C \to V^0 = C$ is the identity, the diagram

$$\begin{array}{ccccc} C^p & \longrightarrow & x_{p+1}^{-1}C^p & \xrightarrow{q^p} & C^{p+1} \\ \downarrow{\scriptstyle i^p} & & \downarrow{\scriptstyle x_{p+1}^{-1}i^p} & & \downarrow{\scriptstyle i^{p+1}} \\ V^p(C) & \xrightarrow{e^p} & E^p(C) & \xrightarrow{d^p} & V^{p+1}(C) \end{array} \tag{1.24}$$

commutes, where $x_{p+1}^{-1}C^p \subset C^p[\frac{1}{x_{p+1}}]$, $i^{p+1} \circ \kappa^{p+1} = \overline{x_{p+1}^{-1}i^p} : x_{p+1}^{-1}C^p/C^p \to V^{p+1}(C)$ is the injection induced by the commutativity of the left square, $\kappa^{p+1} : x_{p+1}^{-1}C^p/C^p \to C^p/(x_{p+1}) =: C^{p+1}$ is the isomorphism sending the class of $x_{p+1}^{-1}c$ to the class of c and

$$q^p = \{x_{p+1}^{-1}C^p \to x_{p+1}^{-1}C^p/C^p \xrightarrow{\kappa^{p+1}} C^{p+1}\}$$

b. *Let $\operatorname{cod} \mathfrak{p} = p$. Then $i_{\mathfrak{p}}^p : C_{\mathfrak{p}}^p \to V_{\mathfrak{p}}^p = E(k(\mathfrak{p})) := E_{R_{\mathfrak{p}}}(k(\mathfrak{p}))$ restricts to an isomorphism*

$$(0 : \mathfrak{p})_{C_{\mathfrak{p}}^p} \xrightarrow{\cong} (0 : \mathfrak{p})_{E(k(\mathfrak{p}))}$$

and both are isomorphic to $k(\mathfrak{p}))$ as $R_{\mathfrak{p}}$-modules (non-canonically).

Proof To begin we remark that if $\mathfrak{p}$ is a prime ideal and $x \notin \mathfrak{p}$, then x acts isomorphically on the $\mathfrak{p}$-primary summand $E(k(\mathfrak{p}))$ of $E^p(C)$ since it is $\mathfrak{p}$-local; this fact gives meaning to the middle vertical map $x_{p+1}^{-1}i^p$ in the diagram above and will also be used below. We proceed by induction to prove part a. In the commutative diagram

$$
\begin{array}{ccccc}
C & \longrightarrow & x_1^{-1}C & \longrightarrow & C/(x_1)C \\
{\scriptstyle =}\big\downarrow & & \big\downarrow & & {\scriptstyle i^1}\big\downarrow \\
C & \xrightarrow{e^0} & E^0(C) & \xrightarrow{d^0} & V^1(C)
\end{array}
\tag{1.25}
$$

the middle vertical sends $x_1^{-1}m$ to $x_1^{-1}e^0(m)$ (x_1 is not a zero-divisor, so is not in any codimension zero prime), and i^1 is the induced map. It is trivial to check that i^1 is injective and has image $(0:x_1)_{V^1(C)}$.

Suppose we have constructed $i^p : C^p \hookrightarrow V^p(C)$ with image $(0:I_p)_{V^p(C)}$. We aim to construct a commutative diagram of short exact sequences (horizontally)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C^p & \longrightarrow & x_{p+1}^{-1}C^p & \xrightarrow{q^p} & C^{p+1} & \longrightarrow & 0 \\
& & {\scriptstyle i^p}\big\downarrow{\scriptstyle \cong} & & \big\downarrow & & {\scriptstyle j^{p+1}}\big\downarrow & & \\
0 & \longrightarrow & (0:I_p)_{V^p(C)} & \longrightarrow & (0:I_p)_{E^p(C)} & \longrightarrow & (0:I_p)_{V^{p+1}(C)} & \longrightarrow & 0 \\
& & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & V^p(C) & \xrightarrow{e^p} & E^p(C) & \xrightarrow{d^p} & V^{p+1}(C) & \longrightarrow & 0
\end{array}
\tag{1.26}
$$

First the horizontal sequences are short exact: this is clear for the upper and lower ones and exactness of the middle one follows from [B, 2.5]; the lower verticals are inclusions; $(0:I_p)_{E^p(C)}$ is injective by [B, 2.5] again, is the sum of its $\mathfrak{p}$-primary components for $\mathfrak{p} \in \operatorname{Ass} M$ and is therefore acted on isomorphically by x_{p+1} (since x_{p+1} is a nonzerodivisor on C^p); thus we get the upper middle vertical, and j^{p+1} is the induced map. One may now repeat the argument given in the case $p=1$ and applied to the first two rows to see that j^{p+1} has image $(0:x_{p+1})_{(0:I_p)_{V^{p+1}(C)}} = (0:I_{p+1})_{V^{p+1}(C)}$. This completes the proof of part a.

To prove b., note that $i^p_{\mathfrak{p}}((0:\mathfrak{p})_{C^p_{\mathfrak{p}}} \subseteq (0:\mathfrak{p})_{V^p_{\mathfrak{p}}}$ and that $(0:\mathfrak{p})_{C^p_{\mathfrak{p}}}$ and $(0:\mathfrak{p})_{V^p_{\mathfrak{p}}} = (0:\mathfrak{p})_{E^p_{\mathfrak{p}}}$ are each isomorphic to $k(\mathfrak{p})$ as $R_{\mathfrak{p}}$-modules ([BH, 3.3.13, 3.3.1]). Since $i^p_{\mathfrak{p}}$ is injective the conclusion follows.

In the notation of the proposition, Grothendieck's fundamental local isomorphism has the form ([AK, p.13])

$$
\operatorname{Ext}^p_R(R/I_p, C) \xrightarrow{\cong} \operatorname{Hom}_R(\Lambda^p(I_p/I_p^2), C^p)
\tag{1.27}
$$

From the injective resolution $C \to E^{\cdot}(C)$ and the fact that $\operatorname{Hom}(R/I_p, E^k(C)) = 0$ if $k < p$, one gets an isomorphism

$$
\operatorname{Ext}^p_R(R/I_p, C) \xrightarrow{\cong} \operatorname{Hom}_R(R/I_p, V^p(C)) = (0:I_p)_{V^p(C)}
$$

Since I_p is generated by a regular sequence, I_p/I_p^2 is R/I_p-free of rank p, so that $\Lambda^p(I_p/I_p^2) \cong R/I_p$. Hence there is a (non-canonical) isomorphism

$$\mathrm{Hom}_R(\Lambda^p(I_p/I_p^2), C^p) \cong C^p$$

Putting these isomorphisms together, we get isomorphisms

$$C^p \cong (0 : I_p)_{V^p(C)}, \; 0 \le p \le \ell$$

just as in Proposition (1.23). However, to prove our main results in §4 we will need not only these isomorphisms, but also the relations between them expressed in Diagram (1.24).

What the fundamental local isomorphism (1.27) does suggest is a way to make the imbeddings i^p of Proposition (1.23) independent of the choice of generators of I_p.

1.28 Proposition. *In the notation of Proposition (1.23) define for each p, $0 \le p \le \ell$,*

$$\epsilon : \mathit{Hom}_R(\Lambda^p(I_p/I_p^2), C^p) \xrightarrow{\cong} C^p, \quad f \mapsto f(\overline{x_1} \wedge \cdots \wedge \overline{x_p})$$

where $\overline{x_i}$ is the class of x_i in I_p/I_p^2. Then the composition

$$j^p := i^p \circ \epsilon : \mathit{Hom}_R(\Lambda^p(I_p/I_p^2), C^p) \hookrightarrow V^p(C)$$

is independent of the choice of generators for I_p.

Proof Let $(x_1, \ldots, x_p) = (y_1, \ldots, y_p) = I_p$. Then if $y_i = \sum c_{ij} x_j$, the matrix (c_{ij}) is invertible. Since R is local, $GL_n(R)$ is generated by elementary matrices and diagonal matrices ([BK, V.9.2]). Let $i^p_{\{x\}}$ and $i^p_{\{y\}}$ denote the imbeddings constructed in (1.23), corresponding to the generating sets $\{x_1, \ldots, x_p\}$ and $\{y_1, \ldots, y_p\}$ for I_p. If we prove first that

a. $y_i = x_{\sigma(i)}$, $\sigma \in S_p \Rightarrow i^p_{\{y\}} = (-1)^\sigma i^p_{\{x\}}$

then it will suffice to show

b. $y_2 = x_2 + r x_1$, $r \in R$, and $y_i = x_i$, $i \ne 2 \Rightarrow i^p_{\{y\}} = i^p_{\{x\}}$, and

c. $y_1 = u x_1$, $u \in R^\times$, and $y_i = x_i$, $i \ne 1 \Rightarrow i^p_{\{y\}} = u i^p_{\{x\}}$.

We prove *a.* in case $\sigma = (12)$, the case of general transpositions being similar. Consider the following diagram, which concatenates copies of Diagram (1.24) where $p = 1$ and 2:

(1.29)

$$\begin{array}{ccccccccc}
C & \longrightarrow & x_1^{-1}C & \longrightarrow & C/x_1C & \longrightarrow & x_2^{-1}(C/x_1C) & \longrightarrow & C/(x_1,x_2)C \\
{\scriptstyle i^0}\big\downarrow{\scriptstyle =} & & {\scriptstyle x_1^{-1}}\big\downarrow & & {\scriptstyle i^1}\big\downarrow & & {\scriptstyle x_2^{-1}(i^1)}\big\downarrow & & {\scriptstyle i^2}\big\downarrow \\
C & \xrightarrow{e^0} & E^0(C) & \xrightarrow{d^0} & E^0(C)/C & \xrightarrow{e^1} & \coprod_{\mathrm{cod}\ \mathfrak{p}=1} E^0(C)/C_{\mathfrak{p}} & \xrightarrow{d^1} & V^2(C)
\end{array}$$

We have, for $c \in C$, $i^1(\overline{c}) = d^0(x_1^{-1} \cdot e^0(c))$, which is sent by e^1 to the sum of its localizations

$$e^1 d^0(x_1^{-1} \cdot e^0(c)) = \sum_{\mathfrak{p}} d^0(x_1^{-1} \cdot e^0(c))_{\mathfrak{p}} \in \coprod_{\substack{\mathrm{cod}\ \mathfrak{p}=1 \\ x \in \mathfrak{p}}} E^0(C)/C_{\mathfrak{p}}$$

Hence, by definition of i^2,

$$i^2_x(\bar{\bar{c}}) = d^1(x_2^{-1}\sum_{\mathfrak{p}} d^0(x_1^{-1}\cdot e^0(c))_{\mathfrak{p}}),\ \bar{\bar{c}} \in C/(x_1,x_2)C$$

Similarly,

$$e^1 d^0(x_2^{-1}\cdot e^0(c)) = \sum_{\mathfrak{q}} d^0(x_2^{-1}\cdot e^0(c))_{\mathfrak{q}} \in \coprod_{\substack{\text{cod }\mathfrak{q}=1\\ x\in\mathfrak{q}}} E^0(C)/C_{\mathfrak{q}}$$

and

$$i^2_y(\bar{\bar{c}}) = d^1(x_1^{-1}\sum_{\mathfrak{q}} d^0(x_2^{-1}\cdot e^0(c))_{\mathfrak{q}}),\ \bar{\bar{c}} \in C/(x_1,x_2)C$$

However,

$$e^1 d^0(x_1^{-1}x_2^{-1}\cdot e^0(c)) = \sum_{\mathfrak{p}} d^0(x_1^{-1}\cdot e^0(c))_{\mathfrak{p}} + \sum_{\mathfrak{q}} d^0(x_2^{-1}\cdot e^0(c))_{\mathfrak{q}}$$

so

$$i^2_x(\bar{\bar{c}}) + i^2_y(\bar{\bar{c}}) = 0$$

as required in *a.*

To prove *b.*, refer to (1.29). We claim

$$x_2^{-1}\cdot e^1 d^0(x_1^{-1}\cdot e^0(c)) = (x_2 + r x_1)^{-1}\cdot e^1 d^0(x_1^{-1}\cdot e^0(c))$$

in $\coprod_{\text{cod }\mathfrak{p}=1} E^0(C)/C_{\mathfrak{p}}$. This holds because $d^0(x_1^{-1}\cdot e^0(c)) = i^1(\bar{c})$, $\bar{c} \in C/(x_1)C$ and im $i^1 = (0 : x^1)_{E^0(C)/C}$, so that the multiplications by x_2 and $x_2 + r x_2$ on $e^1 d^0(x_1^{-1}\cdot e^0(c))$ are the same. This proves *b.*

Part *c.* is easy and is left to the reader.

This proposition has a corollary which clarifies a troublesome point. Call an ideal $I \subset R$ *regular* if it is generated by a regular sequence. Then if $J \subseteq I$ are regular ideals, there are natural maps

(1.30) $$C/JC \to C/IC,\ \text{and } (0 : I)_{V^p(C)} \to (0 : J)_{V^p(C)}$$

so that the maps i^p of Proposition (1.23) cannot be functorial.

However, it is easy to show that, for regular ideals $J \subseteq I$ of codimension p, there is a natural homomorphism

$$\rho_{I,J} : \text{Hom}\,(\Lambda^p(I/I^2), C/IC) \to \text{Hom}\,(\Lambda^p(J/J^2), C/JC)$$

and that the maps j^p of (1.28) *are* functorial. To state the result note that we have isomorphisms

$$\text{Hom}\,(\Lambda^p(I/I^2), C/IC) \xrightarrow{\cong} \text{Hom}\,(\Lambda^p(I/I^2), R/I)\otimes C \xrightarrow{\cong} \Lambda^p\big(\text{Hom}\,(I/I^2, R/I)\big)\otimes C$$

Set

(1.31) $$N(I) := \text{Hom}\,(I/I^2, R/I),$$

the *normal bundle of the regular ideal* I.

1.32 Corollary. *Let $J \subset I$ be codimension p regular ideals in the CM ring R. Let C be a canonical module for R and let j_J^p and j_I^p be the imbeddings of (1.28) corresponding to J and I. Then there is a commutative diagram*

$$\begin{array}{ccc} \Lambda^p N(I) \otimes C & \xrightarrow[\cong]{j_I^p} & (0:I)_{V^p(C)} \\ \rho_{I,J}\downarrow & & \downarrow \\ \Lambda^p N(J) \otimes C & \xrightarrow[\cong]{j_J^p} & (0:J)_{V^p(C)} \end{array} \tag{1.33}$$

where the right vertical is the inclusion. Hence if $\Lambda^p N(I) \otimes C$ is identified with its image in $V^p(C)$, then

$$V^p(C) = \bigcup_I \Lambda^p N(I) \otimes C$$

Since we will not need this result in this paper we will not prove it here.

Proposition (1.23) has another corollary. Let $\mathfrak{p} \subset R$ be a prime ideal of R which is a generic complete intersection: there is a regular ideal I such that $I_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$. Then the last proposition gives us a canonical injection

$$j_\mathfrak{p}^p : \Lambda^p N(\mathfrak{p})_\mathfrak{p} \otimes C \hookrightarrow V^p(C)_\mathfrak{p} = E^p(C)_\mathfrak{p} \tag{1.34}$$

whose image is $(0:\mathfrak{p})_{E^p(C)_\mathfrak{p}}$. The following is an immediate corollary of Proposition (1.28).

1.35 Corollary. *Suppose that the prime $\mathfrak{p} \subset R$ is a generic complete intersection of codimension p. Then there is a canonical isomorphism*

$$\nu(\mathfrak{p}) : \Lambda^p N(\mathfrak{p})_\mathfrak{p} \otimes C \xrightarrow{\cong} (0:\mathfrak{p})_{E^p(C)_\mathfrak{p}} \tag{1.36}$$

We now connect the last few results with quadratic forms. The following is an immediate cosequence of (1.22).

1.37 Proposition. a. *Let $(M, \phi) \in Q(\mathcal{S}_i^r(R); C), i \geq 0$, let $\{x_1, \ldots, x_p\}$ be a C-sequence in $Ann\, M$ and set $C^p := C/(x_1, \ldots, x_p)C$, $R^p := R/(x_1, \ldots, x_p)$. Then there is a unique symmetric R^p-bilinear form*

$$\psi : M \times M \to C^p$$

such that $(M, \psi) \in Q(\mathcal{S}_i^{r-p}(R^p); C^p)$ and

$$i^p \psi = \phi$$

b. *If, in addition, $\mathfrak{p}M = 0$ where $\operatorname{cod} \mathfrak{p} = p$, then there is a unique nonsingular symmetric bilinear form*

$$\psi : M \times M \to (0:\mathfrak{p})_{C^p}$$

such that

$$i^p \psi = \phi$$

We now see that the inclusion $\mathcal{S}_i^0(R/I) \hookrightarrow \mathcal{S}_i^p(R)$ of (1.11) induces an inclusion

$$\kappa_I : Q(\mathcal{S}_i^0(R/I); \Lambda^p N(I) \otimes C) \hookrightarrow Q(\mathcal{S}_i^p(R); C) \tag{1.38}$$

and, if $J \subseteq I$, $\rho_{I,J}$ (see (1.30)) induces

$$Q(\mathcal{S}_i^0(R/I); \Lambda^p N(I) \otimes C) \hookrightarrow Q(\mathcal{S}_i^0(R/J); \Lambda^p N(J) \otimes C) \tag{1.39}$$

Notice that the elements of $Q(\mathcal{S}_i^0(R/I); \Lambda^p N(I) \otimes C)$ are "ordinary" symmetric R/I-bilinear forms $M \times M \to \Lambda^p N(I) \otimes C$, *i.e.* not taking values in, or even making reference to, an injective resolution. Since for any $M \in \mathcal{S}_i^p(R)$ there is a regular sequence $\{x_1, \ldots, x_p\} \subset \operatorname{Ann} M$, (1.32) implies that every element of $Q(\mathcal{S}_i^p(R); C)$ can be represented this way:

1.40 Corollary. *If κ_I identifies $Q(\mathcal{S}_i^0(R/I); \Lambda^p N(I) \otimes C)$ with its image in $Q(\mathcal{S}_i^p(R); C)$, then*

$$Q(\mathcal{S}_i^p(R); C) = \bigcup_I Q(\mathcal{S}_i^0(R/I); \Lambda^p N(I) \otimes C)$$

where the union is over all codimension p regular ideals in R.

Let $(R, \mathfrak{m})$ be local CM with canonical module C. Then there is an inclusion of the category of nonsingular symmetric $k(\mathfrak{m})$-bilinear forms with values in $(0 : \mathfrak{m})_{E^n(C)}$,

$$Q(k(\mathfrak{m}); (0 : \mathfrak{m})_{E^n(C)}) \subseteq Q(\mathcal{F}(R); C)$$

whose image consists of forms $\phi : M \times M \to E^n(C)$ such that $\mathfrak{m}M = 0$. From (1.35) and (1.37b) we get:

1.41 Corollary. *If R is regular local, then the isomorphism $\nu(\mathfrak{m})$ of (1.36) induces*

$$Q(k(\mathfrak{m}); \Lambda^n N(\mathfrak{m}) \otimes C) \xrightarrow{\cong} Q(k(\mathfrak{m}); (0 : \mathfrak{m})_{E^n(C)}) \tag{1.42}$$

In one case the left side of (1.42) assumes a form depending on $\mathfrak{m}$ only. Namely, let R be smooth and essentially of finite type over a field k, so that $\Omega_{R/k}$ is R-free of rank n. If $\mathfrak{p} \subset R$ is a codimension p prime and $k(\mathfrak{p})$ is separable over k, then the conormal sequence ([E, p. 397])

$$0 \to \mathfrak{p}_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}^2 \to \Omega_{R_\mathfrak{p}/k} \otimes k(\mathfrak{p}) \to \Omega_{k(\mathfrak{p})/k} \to 0$$

is split exact, from which we get an isomorphism

$$\Lambda^p N(\mathfrak{p}) \otimes \Omega^n_{R_\mathfrak{p}/k} \otimes k(\mathfrak{p}) \xrightarrow{\cong} \Omega^{n-p}_{k(\mathfrak{p})/k}$$

1.43 Corollary. *For R and $\mathfrak{p}$ as above, $\operatorname{char} k = 0$, there is an inclusion of categories*

$$Q(k(\mathfrak{p}); \Omega^{n-p}_{k(\mathfrak{p})/k}) \hookrightarrow Q(\mathcal{F}(R_\mathfrak{p}); \Omega^n_{R_\mathfrak{p}/k})$$

whose image consists of forms $\phi : F \times F \to E^p(\Omega^n_{R/k})$ where $\mathfrak{p}F = 0$.

§2. Quillen's normalization lemma and the trace map of Scheja and Storch.

We begin by quoting a theorem of Quillen ([Q, (5.12)]).

2.1 Proposition. *Let T be a smooth finitely-generated algebra of dimension n over a field k, let $t \in T$ be a non-zero-divisor and let F be a finite subset of Spec T. Then there exist elements $x_1, ..., x_{n-1}$ of T, algebraically independent over k, such that*

(1) *T/tT is finite over $k[x_1, \ldots, x_{n-1}]$*
(2) *T is smooth of relative dimension one over $k[x_1, \ldots, x_{n-1}]$ at points of F.*

Now suppose we are given a local ring $(R, \mathfrak{m}) = (T_\mathfrak{q}, \mathfrak{q}T_\mathfrak{q})$, where T is a finitely-generated k-algebra; R is then said to be *essentially of finite type over a field.* Let T satisfy the hypotheses of (2.1), with $F = \{\mathfrak{q}\}$ and $t \in \mathfrak{q}$. Then set $A = k[x_1, \ldots, x_{r-1}]_\mathfrak{p}$ and $\mathfrak{n} := \mathfrak{p}A$, where $\mathfrak{p}$ is the restriction of $\mathfrak{q}$ to $k[x_1, \ldots, x_{r-1}]$, and set $B = R/(t)$, $\overline{\mathfrak{m}} := \mathfrak{m}/(t)$. Evidently, $(A, \mathfrak{n}) \to (R, \mathfrak{m})$ is local, as is $(R, \mathfrak{m}) \to (B, \overline{\mathfrak{m}})$. The maximal ideal $\mathfrak{n}$ of A is clearly the inverse image of $\overline{\mathfrak{m}}$, so $(A, \mathfrak{n}) \to (B, \overline{\mathfrak{m}})$ is finite and local. As B is local, Cohen-Macaulay and $\dim A = \dim B$, [Se, IV, Prop. 22] shows B is free, hence flat over A. Finally, the remarks of [Q, top of p. 126] show that in the following, $\ker \rho$ is principal.

2.2 Definition-Proposition. *Let $(R, \mathfrak{m})$ be an n-dimensional regular local ring, the localization of a smooth, finitely-generated algebra T over a field k. Let $t \in \mathfrak{m}, t \neq 0$, be an element of T. Then there is an $(n-1)$-dimensional regular local subring $(A, \mathfrak{n})$ of R so that the inclusion $(A, \mathfrak{n}) \to (R, \mathfrak{m})$ is smooth of relative dimension one and the composition with $\pi : (R, \mathfrak{m}) \to (B, \bar{\mathfrak{m}}) := (R/(t), \mathfrak{m}/(t))$ is finite, flat, and local. In the diagram*

(2.3)
$$\begin{array}{ccc} S := R \otimes_A B & \longleftarrow & R \\ {\scriptstyle\rho}\downarrow & \swarrow{\scriptstyle\pi} & \uparrow \\ B & \longleftarrow & A \end{array}$$

the top horizontal is given by $r \mapsto r \otimes 1$, $\rho(r \otimes b) := \pi(r)b$, the inner triangles commute, ρ splits the natural map $B \to R \otimes_A B$ and $\ker \rho \subset S$ is a principal ideal.

2.4 Proposition. *Assume the hypotheses of (2.2). Let $M \in \mathcal{S}_i^{p+1}(R)$, $p \geq 0$, and let $t \in Ann(M)$. Then $R \otimes_A M \in \mathcal{S}_{i+1}^p(R)$.*

Proof: Observe first that M is finitely-generated as an A-module: it is clearly so as a B-module and B is finite over A.

By [EGA, §0, 16.1.9], $\dim_A M = \dim_B M$, which is clearly $\dim_R M$. Thus, by [EGA, §IV, 6.1.2], we have $\dim_R R \otimes_A M = \dim_R M + 1$. Since $M \in \mathcal{S}_i^{p+1}(R)$, it follows that cod ${}_R R \otimes_A M = p$. It remains to prove that $R \otimes_A M$ satisfies the second condition in Definition (1.10).

For this, let $\mathfrak{p} \subset R$ be prime and $(R \otimes_A M)_\mathfrak{p} \neq 0$. Let $\mathfrak{q} = A \cap \mathfrak{p}$. By [EGA, §IV, 5.7.9.2], there is a prime $\mathfrak{r} \subset B$ lying over $\mathfrak{q}$ such that

$$\text{depth}_{B_\mathfrak{r}} M_\mathfrak{r} = \text{depth}_{A_\mathfrak{q}} M_\mathfrak{q}$$

Let $\tilde{\mathfrak{p}} = q^{-1}(\mathfrak{r})$. Then $M_{\tilde{\mathfrak{p}}} = M_{\mathfrak{r}}$ and so clearly

$$\text{depth}_{\,R_{\tilde{\mathfrak{p}}}} M_{\tilde{\mathfrak{p}}} = \text{depth}_{\,B_{\mathfrak{r}}} M_{\mathfrak{r}}$$

Now $\tilde{\mathfrak{p}} \cap A = \mathfrak{q}$. Let $\text{cod}_{\,R}\mathfrak{p} = k$. Since $A \to R$ is smooth of codimension one, $\text{cod}_{\,A}\mathfrak{q} = k$ or $k+1$. Since B is finite over A, $\text{cod}_{\,B}\mathfrak{r} = \text{cod}_{\,A}\mathfrak{q}$; and $\text{cod}_{\,R}\tilde{\mathfrak{p}} = \text{cod}_{\,B}\mathfrak{r} + 1$, since $\tilde{\mathfrak{p}} = q^{-1}(\mathfrak{r})$. Hence

$$\text{cod}_{\,R}\tilde{\mathfrak{p}} \geq k = \text{cod}_{\,R}\mathfrak{p}$$

Now we have

$$\dim M_{\tilde{\mathfrak{p}}} = \text{cod}_{\,R}\tilde{\mathfrak{p}} - (p+1) \quad \text{and} \quad \dim(R \otimes_A M)_{\mathfrak{p}} = \text{cod}_{\,R}\mathfrak{p} - p$$

and so

$$\dim M_{\tilde{\mathfrak{p}}} \geq \dim(R \otimes_A M)\mathfrak{p} - 1$$

Thus, using [-, §IV, 6.3,2],

$$\begin{aligned} \text{depth}_{\,R_{\mathfrak{p}}}(R \otimes_A M)_{\mathfrak{p}} &= \text{depth}_{\,A_{\mathfrak{q}}} M_{\mathfrak{q}} \\ &= \text{depth}_{\,R_{\tilde{\mathfrak{p}}}} M_{\tilde{\mathfrak{p}}} \, + 1 \\ &\geq \inf(i, \dim_{R_{\tilde{\mathfrak{p}}}} M_{\tilde{\mathfrak{p}}}) \, + 1 \\ &\geq \inf(i+1, \dim(R \otimes_A M)_{\mathfrak{p}}) \end{aligned}$$

2.5 Corollary. *In the setting of (2.4), there is an R-regular sequence $\{t, t_1 \ldots, t_p\} \subseteq Ann_R(M)$ such that $\{t_1, \ldots, t_p\} \subseteq A \subseteq R$. Further, t is a non-zero-divisor on $R \otimes_A M$.*

Proof: We will use the fact that $\text{cod}\, I = \text{grade}\, I$ for ideals I in R, A or B (each is Cohen-Macaulay). Since $\text{cod}_{\,R}M = p+1$, $\text{ht}_{\,B}M = p$, so there is a B-regular sequence $\{s_1, \ldots s_p\} \subseteq \text{Ann}_B(M)$. From the going-down property of $A \to B$ (it is finite and flat) and the fact that $\text{Ann}_{\,B}(M) \cap A = \text{Ann}_{\,A}(M)$, we get $\text{grade}_{\,B}(M) = \text{grade}_{\,A}(M)$, so there is an A-regular sequence $\{t_1 \ldots, t_p\} \subseteq \text{Ann}_A(M)$. By the proof of [EGA, 6.3.1], $\{t_1, \ldots, t_p\}$ is B-regular, so $\{t, t_1, \ldots, t_p\}$ is R-regular. If t were a zero-divisor on $R \otimes_A M$ it would be contained in some $\mathfrak{q}$, $\mathfrak{q} \in \text{Ass}_R(R \otimes_A M)$. Since $\{t_1, ..., t_p\} \subseteq A, \{t_1, \ldots, t_p\} \subseteq \text{Ann}_R(R \otimes_A M)$, so $\{t_1, \ldots, t_p\} \subseteq \mathfrak{q}$ also. Thus, $\mathfrak{q}$ would contain the R-sequence $\{t, t_1, ..., t_p\}$ which would mean $\text{cod}\, \mathfrak{q} \geq p+1$, whereas (1.19) implies that all associated primes of $R \otimes_A M \in \mathcal{S}^p_{i+1}(R)$ have codimension p.

As a consequence of this corollary, we tensor Diagram (2.3) over A with $A^p := A/(t_1, \ldots, t_p)$ to get

(2.6)
$$\begin{array}{ccc} S^p := R^p \otimes_{A^p} R^{p+1} & \longleftarrow & R^p \\ \rho \big\downarrow & \overset{\pi}{\swarrow} & \big\uparrow \\ R^{p+1} & \longleftarrow & A^p \end{array}$$

where $R^p := B/(t_1, \ldots, t_p)B = R/(t_1, \ldots, t_p)R$ and $R^{p+1} := R/(t, t_1, \ldots, t_p)R$. We will use this diagram in conjunction with (1.37) and (2.4) to prove the main technical result of this paper in §4.

We next construct a "trace" homomorphism $\xi : B \to A$ using the data in the diagram (2.3) above. The trace homomorphism we need is essentially constructed in [SS], but in a slightly different context. We will (mostly) use the notation of that paper to state the next theorem, so that we may refer to [SS] for some of the details of the proof.

2.7 Theorem. *Let R be an A-algebra, $t \in R$ a non-zerodivisor, $\pi : R \to R/(t) =: B$ the quotient map and define $\zeta : R \otimes_A B \to B$ by $\zeta(r \otimes b) = \pi(r)b$. Suppose that*

i. the composition $A \to R \xrightarrow{\pi} B$ makes B a finitely generated free A-module, and

ii. $\ker \zeta$ is a principal ideal.

Let $t \otimes 1 = sx$ in $R \otimes_A B$, where $\ker \zeta = (x)$. Then there is a B-isomorphism

$$\Xi : B \to Hom_A(B, A)$$

such that if $\xi := \Xi(1)$, then the composition

$$p : B \longrightarrow R \otimes_A B \xrightarrow{\cdot s} R \otimes_A B \xrightarrow{R \otimes_A \xi} R \otimes_A A = R$$

splits the quotient map π.

Before giving the proof we present an example in which ξ can be computed explicitly.

2.8 Example [SS, pp. 182-183] Let $R = A[X]$ and $t = \sum_{k=1}^{n} a_k X^k$ a monic polynomial. Then $B := A[X]/(t)$ is free of dimension n over A, with basis $\{\overline{1}, \overline{X}, \ldots, \overline{X}^{n-1}\}$ and $\ker \zeta = (X \otimes 1 - 1 \otimes \overline{X})$. Let $t \otimes 1 = (\sum_{k=0}^{n-1} X^k \otimes c_k)(X \otimes 1 - 1 \otimes \overline{X})$, where $c_k \in B$. Then one calculates recursively to find that

$$c_k = \sum_{i=1}^{n-k} a_{k+i} \overline{X}^{i-1}$$

and, in particular, $c_{n-1} = 1$. It is shown in [SS, *loc. cit.*] (see (2.13) below) that in $B \otimes_A B$

$$\sum \overline{X}^k \otimes c_k = (\pi \otimes 1)(\sum X^k \otimes c_k) = \sum \overline{X}^k \otimes (\overline{X}^k)'$$

where $\{\overline{1}', (\overline{X})', \ldots, (\overline{X}^{n-1})'\}$ is the A-basis of B dual to $\{\overline{1}, \overline{X}, \ldots, \overline{X}^{n-1}\}$ with respect to ξ. Hence, $(\overline{X}^{n-1})' = c_{n-1} = 1$.

On the other hand, by its definition, ξ is characterized by the equation

$$\sum \xi(\overline{X}^k)(\overline{X}^k)' = 1$$

in B. Hence

$$\xi = (\overline{X}^{n-1})^* \in \mathrm{Hom}\,(B, A)$$

is the functional dual to $\overline{X}^{n-1}$. In particular, ξ is not the usual trace, $\mathrm{Tr} : B \to A$ associated the natural map $B \xrightarrow{\cong} \mathrm{End}_B(B) \subset \mathrm{End}_A(B)$. For instance, $\xi(1) = 0$, while $\mathrm{Tr}(1)=n$. In fact, $\mathrm{Tr}(b) = \xi(\zeta(s)b)$, for all $b \in B$ ([SS, 4.2]). Also, ξ is the trace used in [Sch, p.215] to prove the exactness of

$$W(K(X)) \longrightarrow \coprod_{(t)\ \mathrm{prime}, \infty} W(K[X]/(t)) \xrightarrow{\coprod \xi_*} W(K)$$

Proof: Since t is a non-zerodivisor in R and $R \otimes_A B$ is R-free, x is a non-zerodivisor in $R \otimes_A B$. Let $\bar{x}, \bar{s} \in B \otimes_A B$ be the images of x, s under $\pi \otimes B : R \otimes_A B \to B \otimes_A B$. It is now easy to verify:

2.9.1. *[SS, (1.2)]* $(\bar{s}) = Ann_{B\otimes B}(\bar{x})$ *and* $(\bar{x}) = Ann_{B\otimes B}(\bar{s})$.

Next let

$$\kappa : B \otimes_A B \to \mathrm{Hom}_A(\mathrm{Hom}_A(B, A), B)$$

be the A-isomorphism defined by $(b_1 \otimes b_2) = (\phi \mapsto \phi(b_1)b_2)$. $B \otimes_A B$ is a B-module in two ways: $b \cdot b_1 \otimes b_2 = bb_1 \otimes b_2$ and $b \circ b_1 \otimes b_2 = b_1 \otimes bb_2$, Since $\ker(m : B \otimes_A B \to B)$ is generated as a $B \otimes_A B$-ideal by$\{b \otimes 1 - 1 \otimes b \,|\, b \in B\}$, $\mathrm{Ann}_{B \underset{A}{\otimes} B}(\ker m)$ is the largest $B \otimes_A B$-submodule on which the two B-module structures coincide. Similarly, $\mathrm{Hom}_A(\mathrm{Hom}_A(B, A), B)$ has two B-module structures, $b \cdot \Phi(\phi) := \Phi(b\phi)$ and $b \circ \Phi(\phi) := b\Phi(\phi)$. The largest submodule on which they coincide is clearly $\mathrm{Hom}_B(\mathrm{Hom}_A(B, A), B)$. Since $\kappa(b \cdot \beta) = b \cdot \kappa(\beta)$ and $\kappa(b \circ \beta) = b \circ \kappa(\beta)$, for $\beta \in B \otimes_A B$, we conclude:

2.9.2. *[SS, (3.2)].* κ *induces a* B*-isomorphism*

$$Ann_{B\otimes_A B}(\ker m) \xrightarrow{\cong} Hom_B(Hom_A(B, A), B)$$

Now $\ker m = (\bar{x})$, since $\ker \zeta = (x)$; so $\mathrm{Ann}_{B\otimes_A B}(\ker m) = (\bar{s})$ by (2.7). Thus $\kappa(\bar{s})$ is a B-basis of $\mathrm{Hom}_B(\mathrm{Hom}_A(B, A), B)$. Since $\mathrm{Hom}_A(B, A)$ is a free B-module ([SS, 1.4)]), it must be that

(2.10) Proposition. *[SS, (3.3)]* $\kappa(\bar{s}) : Hom_A(B, A) \to B$ *is a* B*-isomorphism.*

Set

(2.11) $$\Xi := \kappa(\bar{s})^{-1} \quad \xi := \Xi(1)$$

The fact that $\Xi : B \to \mathrm{Hom}_A(B, A)$ is a B-isomorphism is equivalent to the fact that the symmetric A-bilinear form

(2.12) $$B \times B \to A$$
$$(b_1, b_2) \to \xi(b_1 b_2)$$

is nonsingular. If $\{b_1, \ldots, b_n\}$ is an A-basis of B and $\{b'_1, \ldots, b'_n\}$ is the dual basis with respect to this form (i.e., $\xi(b_i b'_j) = 0$, if $i \neq j; = 1$ if $i = j$) then we claim

(2.13) $$\bar{s} = \sum b_i \otimes b'_i \text{ and } \sum \xi(b_i) b'_i = 1.$$

Indeed, since $\{b_i\}$ is an A-basis of B, $\bar{s} = \sum b_i \otimes \beta_i$ for some $\beta \in B$. It then follows from the definition (-.-) of ξ that

$$\sum \xi(b_i)\beta_i = 1$$

As we remarked above, $\sum bb_i \otimes \beta_i = \sum b_i \otimes b\beta_i$, for all $b \in B$. Applying κ to this relation gives $\sum \phi(bb_i)\beta_i = \sum \phi(b_i)b\beta_i$ for all $\phi \in \mathrm{Hom}\,(B, A)$. In particular, we have for each j,

$$\beta_j = \sum \xi(b'_j b_i)\beta_i = \sum \xi(b_i) b'_j \beta_i = b'_j$$

To prove the final assertion of the theorem, it suffices to show $\pi(R \otimes \xi)(s(1 \otimes b_j)) = b_j$, $j = 1, \ldots, n$, since ξ is A-linear. $R \otimes_A B$ is R-free with basis $\{1 \otimes b'_1, \ldots, 1 \otimes b'_n\}$ and $(\pi \otimes B)(s) = \sum b_i \otimes b'_i$ by (2.13), so $s = \sum r_i \otimes b'_i$, where $\pi(r_i) = b_i$. Thus, we have $(R \otimes \xi)(s(1 \otimes b_j)) = \sum r_i \xi(b'_i b_j) = r_j$, so $\pi(R \otimes \xi)(s(1 \otimes b_j)) = \pi(r_j) = b_j$.

For later use we must put this trace construction into more general contexts, first where the trace is a homomorphism of canonical modules, and the second, where the maps $i^p : C^p \to V^p$ of Theorem (1.22) will also be trace maps.

2.14 Corollary. *Suppose, in addition to the hypotheses made in Theorem (2.7), that* A, R *and* B *are CM with canonical modules* C_A, C_R *and* $C_B := C_R/tC_R$. *Suppose given an isomorphism* $w : R \otimes_A C_A \xrightarrow{\cong} C_R$. *Then* w *induces an isomorphism* $C_B \xrightarrow{\cong} B \otimes_A C_A$ *and there is a* B*-isomorphism*

$$\Xi : B \xrightarrow{\cong} Hom_A(C_B, C_A)$$

such that if $\xi := \Xi(1) : C_B \to C_A$, *then*

$$C_B \to R \otimes_A C_B \xrightarrow{\cdot s} R \otimes_A C_B \xrightarrow{R\otimes_A \xi} R \otimes_A C_A \xrightarrow{w} C_R \to C_R/tC_R = C_B$$

is the identity.

Proof Tensor the composition in (2.17) with $\otimes_A C_A$.

Now we extend the notion of trace map to the injective resolution.

2.15 Definition. *Let* $\phi : R \to S$ *be a homomorphism of CM rings such that* S *is a finitely-generated* R*-module. Let* $c := \dim R - \dim S$ *and let* C_R *and* C_S *be canonical modules for* R *and* S. *Then a* nonsingular trace homomorphism *is an* R*-homomorphism of complexes*

$$\Xi : E^{\cdot}(C_S) \to E^{\cdot}(C_R)[c]$$

such that the induced S*-homomorphism of complexes*

$$E^{\cdot}(C_S) \xrightarrow{\cong} Hom_R(S, E^{\cdot}(C_R)[c])$$
$$e \longrightarrow \{s \longrightarrow \Xi(se)\}$$

is an isomorphism.

2.16 Proposition. *For any finitely-generated* S*-module* M, Ξ *induces an* S*-isomorphism of complexes*

$$\Xi_* : Hom_S(M, E^{\cdot}(C_S)) \xrightarrow{\cong} Hom_R(M, E^{\cdot}(C_R)[c])$$

and hence for each p *an* S*-isomorphism,*

$$\xi^p_* : Hom_S(M, V^p(C_S)) \xrightarrow{\cong} Hom_R(M, V^{p+c}(C_R))$$

where $\xi^p : V^p(C_S) \to V^{p+c}(C_R)$ *is the* R*-homomorphism induced by* Ξ.

Proof For $M = S$ this is part of the definition of Ξ. The standard argument (apply $\mathrm{Hom}_S(-, V^p(C_S)$ and $\mathrm{Hom}_R(-, V^{p+c}(C_R))$ to a finite presentation of M) completes the proof.

It was just observed that a nonsingular trace homomorphism Ξ induces an R-homomorphism

$$\xi := \xi^0 : C_S \to V^c(C_R)$$

such that

$$C_S \longrightarrow \mathrm{Hom}_R(S, V^c(C_R))$$
$$t \longrightarrow \{s \longrightarrow \xi(st)\}$$

is an isomorphism of S-modules. We have seen two examples of such ξ: the homomorphism

$$i^c : C_S \to V^c(C_R)$$

where $S = R/(x_1, \dots, x_c)$, $\{x_1, \dots, x_c\}$ is R-regular and $C_S := C_R/(x_1, \dots, x_c)C_R$ [BH]; and the Scheja-Storch trace map of (2.14), where $c = 0$. Since for any CM ring S, $C_S = E^\cdot(C_S)$ in the derived category and $E^\cdot(C_S)$ is minimal, one expects conversely that ξ will give rise to a unique trace homomorphism Ξ. This is the content of the next result.

2.16 Theorem. *Let $\phi : R \to S$, c, C_R and C_S be as in Definition (2.15). Let*

$$\xi : C_S \to V^c(C_R)$$

be an R-homomorphism such that the induced homomorphism of S-modules

$$C_S \to Hom_R(S, V^c(C_R))$$
$$t \to \{s \to \xi(st)\}$$

is an isomorphism. Then there is a unique nonsingular trace homomorphism

$$\Xi : E^\cdot(C_S) \to E^\cdot(C_R)[c]$$

inducing ξ.

Proof Let $\mathfrak{p}$ be a codimension p prime of R, $\ker\phi \subseteq \mathfrak{p}$ and let M be an R-module with cod $M \geq p$. Then there is a natural isomorphism of R-modules

$$R_\mathfrak{p} \otimes_R M \xrightarrow{\cong} \coprod_{\phi^{-1}(\mathfrak{q})=\mathfrak{p}} S_\mathfrak{q} \otimes_S M$$
$$r \otimes m \to \phi(r) \otimes m$$

Thus, there is a natural R-isomorphism

$$\nu : \coprod_{\substack{\text{cod } \mathfrak{p}=p+c \\ \ker\phi\subseteq\mathfrak{p}}} R_\mathfrak{p} \otimes_R M \xrightarrow{\cong} \coprod_{\text{cod } \mathfrak{q}=p} S_\mathfrak{q} \otimes_S M$$

The construction of Ξ may now be done inductively. To begin define Ξ^0 to be the composition

$$\Xi^0 : \coprod_{\text{cod } \mathfrak{q}=0} S_\mathfrak{q}\otimes_S C_S \xrightarrow{\nu^{-1}} \coprod_{\substack{\text{cod } \mathfrak{p}=c \\ \ker\phi\subseteq\mathfrak{p}}} R_\mathfrak{p}\otimes_R C_S \subseteq \coprod_{\text{cod } \mathfrak{p}=c} R_\mathfrak{p}\otimes_R C_S \xrightarrow{\coprod R_\mathfrak{p}\otimes\xi} \coprod_{\text{cod } \mathfrak{p}=c} R_\mathfrak{p}\otimes_R V^c(C_R)$$

It is easy to check that the following diagram commutes:

$$\begin{array}{ccc} C_S & \hookrightarrow & \coprod_{\text{cod } \mathfrak{q}=0} S_\mathfrak{q} \otimes_S C_S =: E^0(C_S) \\ \downarrow \xi & & \downarrow \Xi^0 \\ V^c(C_R) & \hookrightarrow & \coprod_{\text{cod } \mathfrak{p}=c} R_\mathfrak{p} \otimes_R V^c(C_R) =: E^c(C_R) \end{array}$$

We get the equalities from the definition in (1.9) of the canonical injective resolution.

Next we take $\xi^1 : V^1(C_S) \to V^{c+1}(C_R)$ to be the induced map on the cokernels of the horizontal maps and extend it to a map

$$\Xi^1 : E^1(C_S) := \coprod_{\text{cod } \mathfrak{q}=1} S_\mathfrak{q} \otimes_S V^1(C_S) \to \coprod_{\text{cod } \mathfrak{p}=c+1} R_\mathfrak{p} \otimes_R V^{c+1}(C_R) =: E^{c+1}(C_R)$$

as above. Continuing in this way completes the construction of Ξ.

2.17 Theorem. *Let $\phi : R \to S$ be a homomorphism of CM rings such that S is a finitely-generated R-module. Let $c := \dim R - \dim S$ and let C_R and C_S be canonical modules for R and S. Let*

$$\Xi : E^\cdot(C_S) \to E^\cdot(C_R)[c]$$

be a nonsingular trace homomorphism. Let $\mathfrak{q} \subset S$ be prime and let $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$. Then for each p, $0 \le p \le \dim S$, Ξ^p induces a surjection of $k(\mathfrak{p})$-modules

$$\Xi^p : (0 : \mathfrak{q})_{E^p(C_S)} \longrightarrow (0 : \mathfrak{p})_{E^{p+c}(C_R)}$$

which is an isomorphism if ϕ is surjective.

Proof ϕ induces $\phi_\mathfrak{p} : R_\mathfrak{p} \longrightarrow S_\mathfrak{p} := S \otimes_R R_\mathfrak{p}$. $S_\mathfrak{p}$ is semi-local with maximal ideals corresponding to the primes $\mathfrak{q} \subset S$ over $\mathfrak{p}$. Ξ^p localizes to

$$\Xi^p_\mathfrak{p} : E^p(C_S)_\mathfrak{p} \longrightarrow E^{p+c}(C_R)_\mathfrak{p}$$

and hence restricts to

$$(0 : \mathfrak{p}S)_{E^p(C_S)_\mathfrak{p}} \longrightarrow (0 : \mathfrak{p})_{E^{p+c}(C_R)_\mathfrak{p}}$$

Now

$$E^p(C_S)_\mathfrak{p} \cong \coprod_{\mathfrak{p}=\phi^{-1}(\mathfrak{q}_i)} E^p(C_S)_{\mathfrak{q}_i} \tag{2.18}$$

the splitting of $E^p(C_S)_\mathfrak{p}$ into its primary components. With respect to this splitting, $\Xi^p_\mathfrak{p}$ splits as

$$\coprod (0 : \mathfrak{Q}_i)_{E^p(C_S)_{\mathfrak{q}_i}} \longrightarrow (0 : \mathfrak{p})_{E^{p+c}(C_R)_\mathfrak{p}} \tag{2.19}$$

where $\mathfrak{Q}_i$ are the S-primary components of $\mathfrak{p}S$. . Since $\mathfrak{q}_i \subseteq \mathfrak{Q}_i$, we can restrict further to

$$\coprod (0 : \mathfrak{q}_i)_{E^p(C_S)_{\mathfrak{q}_i}} \longrightarrow (0 : \mathfrak{p})_{E^{p+c}(C_R)_\mathfrak{p}} \tag{2.20}$$

In case ϕ is surjective, there is exactly one prime $\mathfrak{q}$ over $\mathfrak{p}$ and ϕ induces $k(\mathfrak{p}) \xrightarrow{\cong} k(\mathfrak{q})$, so (2.20) is an isomorphism as soon as it is nonzero (it is a $k(\mathfrak{p})$-linear map of one-dimensional $k(\mathfrak{p})$ vector spaces). This non-triviality is proved next and without the assumption that ϕ is surjective.

Note that each component of (2.20) can be identified with the map

$$\mathrm{Hom}_{S_{\mathfrak{q}_i}}(k(\mathfrak{q}_i), E^p(C_S)_{\mathfrak{q}_i}) \to \mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), E^{p+c}(C_R)_\mathfrak{p})$$

which is induced by the injection $k(\mathfrak{p}) \hookrightarrow k(\mathfrak{q}_i)$, (2.18) and $\Xi^p_\mathfrak{p}$. Hence it factors as

$$\mathrm{Hom}_{S_{\mathfrak{q}_i}}(k(\mathfrak{q}_i), E^p(C_S)_{\mathfrak{q}_i}) \xrightarrow{\Xi^p_{\mathfrak{p}\,*}} \mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{q}_i), E^{p+c}(C_R)_\mathfrak{p}) \to \mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), E^{p+c}(C_R)_\mathfrak{p}).$$

The first map is an isomorphism by (2.16), and the second is surjective because its cokernel injects to

$$\mathrm{Ext}^1_{R_\mathfrak{p}}(k(\mathfrak{q}_i)/k(\mathfrak{p}), E^{p+c}(C_R)_\mathfrak{p}) = 0$$

This completes the proof.

We need not only a change of rings result for the value groups of our forms, but also for the modules on which they are defined:

2.21 Theorem. *Let $\phi : R \to S$ be a homomorphism of CM rings such that S is a finitely-generated R-module. Let $c := \dim R - \dim S$. Then for each p, $0 \le p \le \dim R$ and $i \ge 0$, there is a functor*

$$\mathcal{S}^p_i(S) \to \mathcal{S}^{p+c}_i(R)$$

induced by regarding an S-module as an R-module.

Proof We give a proof valid when $i \le 2$ and there is a nonsingular trace homomorphism $\Xi : E^{\cdot}(C_S) \to E^{\cdot}(C_R)[c]$, where C_S and C_R are canonical modules for S and R. This is the only case we will need in this paper. Let $M \in \mathcal{S}^p_0(S)$. Consider the composition

$$\begin{aligned} M \to & \mathrm{Hom}_S(\mathrm{Hom}_S(M, V^p(C_S)), V^p(C_S)) \\ \xrightarrow{\cong} & \mathrm{Hom}_R(\mathrm{Hom}_S(M, V^p(C_S)), V^{p+c}(C_R)) \\ \xrightarrow{\cong} & \mathrm{Hom}_R(\mathrm{Hom}_R(M, V^{p+c}(C_R)), V^{p+c}(C_R)) \end{aligned}$$

where the first map is the canonical S-homomorphism of M to its $V^p(C_S)$ double dual and the second and third are R-isomorphisms induced by $\xi : V^p(C_S) \to V^{p+c}(C_R)$ (2.16). The composition is the canonical R-homomorphism of M to its V^{p+c} double dual. The conclusion follows from (1.17) and (1.19).

§3. Witt groups in the categories $\mathcal{S}_i^p$.

In this section X will denote a CM scheme of dimension n and $\mathcal{C}$, a canonical sheaf. Recall from (1.20) the definition of $Q(\mathcal{S}_i^p(X);\mathcal{C})$. We will define and study Witt groups $W(\mathcal{S}_i^p(X);\mathcal{C})$.

Let $\mathcal{N} \in \mathcal{S}_i^p(X), \mathcal{N}^{\wedge} := \mathcal{H}om(\mathcal{N}, \mathcal{V}^p(\mathcal{C}))$. The *(symmetric) hyperbolic form* $(\mathcal{N} \oplus \mathcal{N}^{\wedge}, \phi_h)$ is defined by requiring that $\phi_h | \mathcal{N} \times \mathcal{N} \equiv 0 \equiv \phi_h | \mathcal{N}^{\wedge} \times \mathcal{N}^{\wedge}$, that $\phi_h | \mathcal{N}^{\wedge} \times \mathcal{N}$ be the natural pairing, and that ϕ_h be symmetric. Since $\mathcal{N}_x \in \mathcal{CM}^j(X_x)$ for cod $x = p + j$ and $j \leq i, (\mathcal{N} \oplus \mathcal{N}^{\wedge}, \phi_h) \in Q(\mathcal{S}_i^p(X);\mathcal{C})$ by [P3,1.6]. More generally:

3.1 Definition. $(\mathcal{M}, \phi) \in Q(\mathcal{S}_i^p(X);\mathcal{C})$ *is called a* lagrangian *if there is* $\mathcal{N} \subseteq \mathcal{M}$ *where* $\mathcal{N}$ *and* $\mathcal{M}/\mathcal{N} \in \mathcal{S}_i^p(X)$, *and the induced pairing* $\mathcal{N} \times (\mathcal{M}/\mathcal{N}) \to \mathcal{V}^p(\mathcal{C})$ *is nonsingular (i.e., both adjoints are bijective) at all* x *such that* cod $x \leq p + i$. $\mathcal{N}$ *is called a* sublagrangian.

One may add isomorphism classes of objects in $Q(\mathcal{S}_i^p(X);\mathcal{C})$ by "orthogonal sum" (denoted $\perp$) and obtain an abelian semigroup (with the obvious zero element). The corresponding Grothendieck group modulo the subgroup generated by lagrangians is denoted

$$W(\mathcal{S}_i^p(X);\mathcal{C}). \tag{3.2}$$

3.3 Proposition. *Let* $(\mathcal{M}, \phi)$ *and* $(\mathcal{M}', \phi') \in Q(\mathcal{S}_i^p(X);\mathcal{C})$ *and* $f : \mathcal{M} \to \mathcal{M}'$ *be an isomorphism at all* x *such that* cod $x \leq p+i$. *Suppose that* $\phi(m_1, m_2) = \phi'(fm_1, fm_2)$ *for all* $m_1, m_2 \in \mathcal{M}$. *Then*

$$[\mathcal{M}, \phi] = [\mathcal{M}', \phi'] \quad \textit{in} \quad W(\mathcal{S}_i^p(X);\mathcal{C}).$$

Proof: Consider the short exact sequence

$$\mathcal{M} \overset{(1,f)}{\rightarrowtail} \mathcal{M} \oplus \mathcal{M}' \overset{\pi_2}{\twoheadrightarrow} \mathcal{M}'.$$

Observe that $\operatorname{im}(1, f)$ is a sublagrangian of $(\mathcal{M}, \phi) \perp (\mathcal{M}', -\phi')$.

3.4 Corollary. *Let* $(\mathcal{M}, \phi) \in Q(\mathcal{S}_1^p(X);\mathcal{C})$. *Then there is* $(\mathcal{M}', \phi') \in Q(\mathcal{S}_2^p(X);\mathcal{C})$ *satisfying the hypotheses of (3.3) where* $i = 1$. *In particular, the inclusion* $Q(\mathcal{S}_2^p(X);\mathcal{C}) \subseteq Q(\mathcal{S}_1^p(X);\mathcal{C})$ *induces a surjection*

$$W(\mathcal{S}_2^p(X);\mathcal{C}) \twoheadrightarrow W(\mathcal{S}_1^p(X);\mathcal{C}).$$

Proof: Let $(\mathcal{M}, \phi) \in Q(\mathcal{S}_1^p(X);\mathcal{C})$. Consider the diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\operatorname{Ad}\phi} & \mathcal{M}^{\wedge} \\ {\scriptstyle\kappa}\downarrow & & \downarrow{\scriptstyle =} \\ \mathcal{M}^{\wedge\wedge} & \xrightarrow{(\operatorname{Ad}\phi)^{\wedge}} & \mathcal{M}^{\wedge} \end{array} \tag{3.5}$$

where κ is the canonical map. By (1.17), $\mathcal{M}\hat{}\in \mathcal{S}_2^p(X)$ so $\mathcal{M}\hat{}\cong (\mathcal{M}\hat{}\hat{})\hat{}$. Let $\phi\hat{}\hat{}: \mathcal{M}\hat{}\hat{}\times \mathcal{M}\hat{}\hat{}\rightarrow \mathcal{V}^p(\mathcal{C})$ be the $\mathcal{O}_X$-bilinear form corresponding to $(\mathrm{Ad}\phi)\hat{}: \mathcal{M}\hat{}\hat{}\rightarrow \mathcal{M}\hat{}= (\mathcal{M}\hat{}\hat{})\hat{}$. It is easily seen that $\phi\hat{}\hat{}$ is symmetric; it is nonsingular by (1.21). Now take $f = \kappa$ and $(\mathcal{M}', \phi') = (\mathcal{M}\hat{}\hat{}, \phi\hat{}\hat{})$ in (3.3).

3.6 Given $(\mathcal{M}, \phi) \in Q(\mathcal{S}_i^p(X); \mathcal{C})$ we obtain by localization an element $(\mathcal{M}_x, \phi_x) \in Q(\mathcal{F}(X_x); \mathcal{C}_x)$, for each $x \in \mathrm{Ass}\,(\mathcal{M})$. (Recall from (1.2) that $\mathcal{F}(X_x)$ is the category of finite length $\mathcal{O}_{X,x}$-modules.) This process induces a homomorphism,

$$\mathcal{K}^p : W(\mathcal{S}_i^p(X); \mathcal{C}) \rightarrow W(\mathcal{F}^p(X); \mathcal{C}) \tag{3.7}$$

where, to simplify notation, we have written

$$W(\mathcal{F}^p(X); \mathcal{C}) := \coprod_{\mathrm{cod}\ x=p} W(\mathcal{F}(X_x); \mathcal{C}_x). \tag{3.8}$$

3.9 Proposition. *When $i = 1, \mathcal{K}^p$ is injective.*

Proof: We are given $(\mathcal{M}, \phi) \in Q(\mathcal{S}_i^p(X); \mathcal{C})$, a Lagrangian at each $x \in X^{(p)}$. We must show $\mathcal{M}$ has a sublagrangian. Consider the collection $\mathbb{L}$ of $\mathcal{L} \subseteq \mathcal{M}$ such that $\mathcal{L} \subseteq \mathcal{L}^\perp$ and $\mathcal{L}_x$ is a sublagrangian of $(\mathcal{M}_x, \phi_x)$ for each x in $\mathrm{Ass}\,(\mathcal{M})$. This collection is non-empty: for if $\mathcal{K}_x$ is a sublagrangian of $\mathcal{M}_x$ and $i : \mathcal{M} \rightarrow \oplus\mathcal{M}_x$ is the canonical imbedding, then $i^{-1}(\oplus\mathcal{K}_x)$ is such an $\mathcal{L}$. Since X is Noetherian and $\mathcal{M}$ is finitely-generated, $\mathbb{L}$ has a maximal element, $\mathcal{N}$. Then $\mathcal{N}$ must equal $\mathcal{N}^\perp$. It suffices to prove this locally. Let $x \in N^\perp - N, n_1, n_2 \in N$ and $a_1, a_2 \in R$. Then

$$\phi(n_1 + a_1x, n_2 + a_2x) = a_1a_2\phi(x, x).$$

We claim that $\phi(x, x) = 0$. If so, $(N + Rx) \subseteq (N + Rx)^\perp$, contradicting the maximality of N and showing $N = N^\perp$.

3.10 Lemma. $(N^\perp)_\mathfrak{p} = (N_\mathfrak{p})^\perp$ *for each* $\mathfrak{p} \in$ *Ass* (M).

Proof: $N^\perp = \ker\{N \rightarrow M\hat{}\}$ where $n \mapsto (m \mapsto \phi(n, m))$. Since $(M\hat{})_\mathfrak{p} \cong (M_\mathfrak{p})\hat{}$, the result follows.

By this lemma, at each $\mathfrak{p}$, $x \in (N^\perp)_\mathfrak{p} = (N_\mathfrak{p})^\perp = N_\mathfrak{p}$. Hence $\phi_\mathfrak{p}(x, x) = 0$ in $E^p(C_\mathfrak{p})$, so $V^p(C) \subseteq \coprod E^p(C)_\mathfrak{p}) = E^p(C)$ implies $\phi(x, x) = 0$.

Now consider the commutative diagram

$$\begin{array}{ccc} & N & \\ \kappa & & \alpha \\ N\hat{}\hat{} & \xrightarrow{\beta\hat{}} & (M/N)\hat{} \end{array}$$

where $\alpha : N \rightarrow (M/N)\hat{}$ and $\beta : M/N \rightarrow N\hat{}$ are the adjoints of bilinear pairings induced from ϕ. Since Ad ϕ is injective, α is injective, which in turn means κ is injective. By (1.19), $N \in \mathcal{S}_1^p(R)$. Since $N = N^\perp$ we conclude β is injective, which implies M/N is unmixed (because $N\hat{}$ is). By (1.19), $M/N \in \mathcal{S}_1^p(R)$. Since ϕ is nonsingular in codimension $\leq p + 1, \alpha$ is surjective there. Similarly, β is nonsingular in codimension $\leq p + 1$, so N is a sublagrangian of (M, ϕ).

3.11 Definition.. *Let $p \geq 0$ be an integer and $\mathcal{F} = \coprod i_{x_i *} F_i$, where $x_1, \ldots, x_n \in X^{(p)}$, $i_{x_i} : \overline{x_i} \to X$ is the inclusion and $F_i \in \mathcal{F}(X_{x_i})$ is viewed a a constant sheaf on $\overline{x_i}$. A* lattice *$\mathcal{L} \subseteq \mathcal{F}$ is an object in $\mathcal{L} \in \mathcal{S}_2^p(X)$ such that $\mathcal{L}_{x_i} = F_i$ for all i. If $(F_i, \tau_i) \in Q(\mathcal{F}(X_{x_i}); C_{x_i})$ and $\tau := \coprod i_{x_i *} \tau_i$, then $\mathcal{L} \subseteq \mathcal{F}$ is called an* integral lattice in $(\mathcal{F}, \tau)$ *if it is a lattice and $\tau(\mathcal{L} \times \mathcal{L}) \subseteq \mathcal{V}^p(\mathcal{C}) (\subseteq \mathcal{E}^p(\mathcal{C}) = \coprod_{\text{cod } x = p} \mathcal{E}^p(\mathcal{C}_x))$.*

3.12 Proposition. *Let $(F_i, \tau_i) \in Q(X_{x_i}; C_{x_i})$ where* cod *$x_i = p, i = 1, \ldots, n$, and let $(\mathcal{F}, \tau) = \coprod i_{x_i *}(F_i, \tau_i)$. Then there exists an integral lattice $\mathcal{L}$ in $(\mathcal{F}, \tau)$.*

Proof: We may assume $n = 1$; let $x = x_1$. Choose a coherent sheaf $\mathcal{L} \subseteq \mathcal{F}$ such that $\mathcal{L}_x = F$. By (1.19), $\mathcal{L} \in \mathcal{S}_1^p(X)$, and the canonical map $\kappa : \mathcal{L} \to \mathcal{L}^{\frown\frown}$is injective, where $\mathcal{L}^{\frown}= \mathcal{H}om(\mathcal{L}, \mathcal{V}^p(\mathcal{C}))$. Since κ_x is bijective, we have

$$\mathcal{L}_x^{\frown\frown} \xrightarrow{\kappa_x^{-1}} \mathcal{L}_x \hookrightarrow F$$

from which we get an injective map $\mathcal{L}^{\frown\frown}\to \mathcal{F}$ (i_{x*} and i^{-1} are adjoint functors), where $\mathcal{L}^{\frown\frown}\in \mathcal{S}_2^p(X)$ by (1.17). Hence there is a lattice in $\mathcal{F}$.

To see that there is an integral lattice, consider the composition

$$\mathcal{L} \times \mathcal{L} \hookrightarrow \mathcal{F} \times \mathcal{F} \xrightarrow{\tau} \mathcal{E}^p(\mathcal{C}) \xrightarrow{d^p} \mathcal{E}^{p+1}(\mathcal{C}) \cong \coprod_{y \in X^{(p+1)}} i_{y*} \mathcal{E}^{p+1}(\mathcal{C})_y$$

Again we assume $\mathcal{F} = i_{x*}F$, where F is a finite length $\mathcal{O}_{X,x}$-module. Since $\mathcal{L}$ is coherent, the image of the composition lies in a finite sum $\coprod i_{y_k *} \mathcal{E}^{p+1}(\mathcal{C})_{y_k}$, $y_k \in \overline{x}$, $k = 1, \ldots, m$, and the image in each summand $i_{y_k *} \mathcal{E}^{p+1}(\mathcal{C})_{y_k}$ has the form $i_{y_k *} I_k$, where I_k is $\mathcal{O}_{X,y_k}$-finitely-generated. For each $y_k \in \overline{x}$, there is an inclusion of rings $\mathcal{O}_{X,y_k} \subset \mathcal{O}_{X,x}$ and a codimension one prime $\mathfrak{p}_k \subset \mathcal{O}_{X,y_k}$, such that $(\mathcal{O}_{X,y_k})_{\mathfrak{p}_k} = \mathcal{O}_{X,x}$ and $\mathfrak{p}_k \mathcal{O}_{X,y_k} = \mathfrak{m}_x$. By [SV, 4.23] and the fact that I_k is finitely-generated, we may choose $r_k \in \mathfrak{m}_{y_k} - \mathfrak{p}_k$ such that $r_k I_k = 0$, $k = 1, \ldots, m$. Let $r := \prod r_k$ in $\mathcal{O}_{X,x}$. Then $r \notin \mathfrak{m}_x$, so multiplication by r induces an isomorphism

$$i_* r : \mathcal{F} := i_* F \xrightarrow{\cong} i_* F =: \mathcal{F}$$

Set $\mathcal{L}_r := i_* r(\mathcal{L}) \subset \mathcal{F}$. Then $\mathcal{L}_r \cong \mathcal{L} \in \mathcal{S}_2^p(X)$, and by construction $d^p \tau(\mathcal{L}_r, \mathcal{L}_r) = 0$, so $\mathcal{L}_r$ is an integral lattice in $(\mathcal{F}, \tau)$. This completes the proof of (3.12).

3.13 Remark Let $X = \operatorname{Spec} R$. The most natural analogy with the ideas of [P1] would require that we actually be able to construct a lattice $L \in \mathcal{CM}^p(R)$, i.e., of homological dimension $n - p$ if R is regular local. Indeed in [P3], this is precisely what was done to construct localization sequences permitting a proof of the injectivity of $W(R) \to W(K)$, where K is the fraction field of R, in case dim $R \leq 4$ and R is *any* regular local ring. However, such a construction is essentially equivalent to the construction of small Cohen-Macaulay modules (see [Ho, Conjecture E''''], about which very little is known for regular local rings R essentially of finite type over a field, if $p \geq 3$. When R is an arbitrary regular local ring, counterexamples to the existence of such $L \in \mathcal{CM}^p(R)$ are known.

3.14 We are now in a position to define the map

$$\mathcal{L}^p : W(\mathcal{F}^p(X); \mathcal{C}) \to W(\mathcal{S}_1^{p+1}(X); \mathcal{C}).$$

Let $(F_i, \tau_i) \in Q(X_{x_i}; \mathcal{C}_{x_i})$, where cod $x_i = p$, $i = 1, \ldots, n$. Set $\mathcal{F} = \coprod i_{x_i *} \mathcal{F}_i$, $\tau = \coprod i_{x_i *} \tau_i :$ $\mathcal{F} \times \mathcal{F} \to \mathcal{E}^p$.

3.15 Definition. *Let $\mathcal{L}$ be an integral lattice in $(\mathcal{F}, \tau)$. The* dual lattice $\mathcal{L}_\tau^*$ *is defined for each affine open $U \subset X$ by*

$$\mathcal{L}_\tau^*(U) = \{f \in \mathcal{F}(U) | \tau(f, \mathcal{L}(U)) \subseteq \mathcal{V}^p(\mathcal{C})(U)\}.$$

Equivalently, $\mathcal{L}_\tau^*$ is the inverse image of $\mathcal{H}om(\mathcal{L}, \mathcal{V}^p(\mathcal{C})) \subseteq \mathcal{H}om(\mathcal{L}, \mathcal{E}^p(\mathcal{C}))$ under the map $\operatorname{Ad}\tau : \mathcal{F} \to \mathcal{H}om(\mathcal{L}, \mathcal{E}^p(\mathcal{C})) = \mathcal{H}om(\mathcal{F}, \mathcal{E}^p(\mathcal{C}))$.

Since $\mathcal{L}$ is integral in $(\mathcal{F}, \tau)$, $\mathcal{L} \subseteq \mathcal{L}_\tau^*$. Since τ is nonsingular, there is an isomorphism

$$\begin{aligned} \mathcal{L}_\tau^* &\xrightarrow{\cong} \mathcal{H}om(\mathcal{L}, \mathcal{V}^p(\mathcal{C})) \\ \ell^* &\mapsto \{\ell \mapsto \tau(\ell^*, \ell)\} \end{aligned} \tag{3.16}$$

In fact, given $\lambda : \mathcal{L} \to \mathcal{V}^p(\mathcal{C})$ on any affine open $U \subseteq X$, compose with $\mathcal{V}^p(\mathcal{C}) \hookrightarrow \mathcal{E}^p(\mathcal{C})$ and use the nonsingularity of τ to get a unique $\ell^* \mapsto \lambda$; or use the equivalent definition of $\mathcal{L}_\tau^*$ above

We now write $\mathcal{L}^*$ for $\mathcal{L}_\tau^*$. Using the fact that $\mathcal{L}_{x_i} = \mathcal{L}_{x_i}^* = \mathcal{F}$ for each i, it follows from [Ka, Ex. 14, p. 103] that $\mathcal{L}^*/\mathcal{L} \in \mathcal{S}_1^{p+1}(X)$. Set

$$\mathcal{M} = \mathcal{L}^*/\mathcal{L} \tag{3.17.1}$$

and define $\phi : \mathcal{M} \times \mathcal{M} \to \mathcal{V}^{p+1}(\mathcal{C})$ on each affine open $U \subseteq X$ by

$$\phi(j\ell_1, j\ell_2) = d_p\tau(\ell_1, \ell_2) \tag{3.17.2}$$

where $j : \mathcal{L}^* \to \mathcal{L}^*/\mathcal{L} = \mathcal{M}$ is the canonical map. To see that $(\mathcal{M}, \phi) \in Q(\mathcal{S}_1^{p+1}(X); \mathcal{C})$, we must show ϕ is nonsingular at all points of codimension $p+1$ and $p+2$. Affinely, let $\mathfrak{q}$ be any prime of codimension $p+1$ or $p+2$, $\mathfrak{q} \in \operatorname{Supp} M$. We may localize the above constructions at $\mathfrak{q}$, permitting the assumptions: L, $L^* \in \mathcal{CM}^p(R)$ and $M \in \mathcal{CM}^{p+1}(R)$.

By [P3, 1.6] there is a short exact sequence $L^{*\frown} \overset{\alpha^\frown}{\rightarrowtail} L^\frown \twoheadrightarrow M^\sim$, where α is the inclusion $(-)^\frown = \operatorname{Hom}(-, V^p(C))$ and $(-)^\sim = \operatorname{Hom}(-, V^{p+1}(C))$ Following [P1, (3.8)], $M^\sim$ is identified with $\operatorname{cok}(\alpha^\frown)$ by

$$\{\text{class of } f : L \to V^p(C) \text{ in } \operatorname{cok}(\alpha^\frown)\} \leftrightarrow \{e : M \to V^{p+1}(C)\} \tag{3.17.2}$$

if $ej = d_p f^*$, where $f^* : L^* \to E^p(C)$ is the extension of f defined by $f^*\alpha = if$ and $i : V^p(C) \to E^p(C)$ is the inclusion. Let $e : M \to V^{p+1}(C)$ be given. Since τ is nonsingular, there is a unique $n_0 \in F$ such that $\tau(n_0, n) = f^*(n)$, for all $n \in L^*$. Since $f^*\alpha(n) = if(n) \in V^p(C)$ for all $n \in L, n_0$ must be in L^*, the dual lattice. Thus, for any $n \in L^*, \phi(jn_0, jn) = d_p\tau(n_0, n) = d_p f^*(n) = e(j(n))$. This proves Ad ϕ is surjective at primes of codimension $p+1$ and $p+2$; it is injective by the definition of L^*.

We have thus shown that $(\mathcal{M}, \phi) \in Q(\mathcal{S}_1^{p+1}(X); \mathcal{C})$ and we set

$$\mathcal{L}^p[\mathcal{F}, \tau] = [\mathcal{M}, \phi] \tag{3.18}$$

where $[\cdot]$ denotes the corresponding element in the appropriate Witt group. We must now show $[\mathcal{M}, \phi]$ is independent of the choices made in its construction.

Any two integral lattices in $\mathcal{F}$ have a common integral sublattice, the double dual of their intersection. Indeed suppose $\mathcal{L}_1,\ \mathcal{L}_2 \in \mathcal{S}_2^p(X)$ are two integral lattices in $(\mathcal{F}, \tau)$. We may assume as we did above that $\operatorname{Supp} \mathcal{F} = \overline{x}$, where $\operatorname{cod} x = p$. Set $\mathcal{L} := \mathcal{L}_1 \cap \mathcal{L}_2 \subseteq \mathcal{L}_i$. Then $\mathcal{L}_x = \mathcal{F}_x$, and the double dual of $\mathcal{L} \subseteq \mathcal{L}_i$ is an inclusion

$$(\mathcal{L}_\tau^*)_\tau^* \subseteq ((\mathcal{L}_i)_\tau^*)_\tau^* = \mathcal{L}_i, \quad i = 1,\, 2$$

where $(\mathcal{L}_\tau^*)_\tau^* \cong \mathcal{L}^{\wedge\wedge} \in \mathcal{S}_2^p$ by (3.16) and (1.13). Hence to show independence of $[\mathcal{M}, \phi] \in W(\mathcal{S}_1^{p+1}(X); \mathcal{C})$ from the choice of lattice it suffices to show that if $\mathcal{I} \subseteq \mathcal{L}$ are integral lattices (so that $\mathcal{L}^* \subseteq \mathcal{I}^*$), then

$$[\mathcal{I}^*/\mathcal{I}, \psi] = [\mathcal{L}^*/\mathcal{L}, \phi], \tag{3.19}$$

with obvious notation.

It is clear that $\mathcal{K} := \mathcal{L}/\mathcal{I}$ is totally isotropic (meaning $\mathcal{K} \subseteq \mathcal{K}^\perp$) in $(\mathcal{I}^*/\mathcal{I}, \psi)$, and that $\mathcal{K} \in \mathcal{S}_1^{p+1}(X)$. Further, $\mathcal{K}^\perp = \mathcal{L}^*/\mathcal{I}$ by definition of dual lattice, and under the isomorphism

$$\mathcal{K}^\perp/\mathcal{K} = (\mathcal{L}^*/\mathcal{I})/(\mathcal{L}/\mathcal{I}) \cong \mathcal{L}^*/\mathcal{L}$$

the form induced on $\mathcal{K}^\perp/\mathcal{K}$ by ψ corresponds to ϕ on $\mathcal{L}^*/\mathcal{L}$. Hence it suffices to show:

3.20 Lemma. *Let $(\mathcal{T}, \psi) \in Q(\mathcal{S}_i^p(X); \mathcal{C})$ and suppose there is $\mathcal{K} \subseteq \mathcal{T}, \mathcal{K} \subseteq \mathcal{K}^\perp$, and $\mathcal{K} \in \mathcal{S}_i^p(X)$. Then if the naturally induced form $(\mathcal{K}^\perp/\mathcal{K}, \phi)$ is in $Q(\mathcal{S}_i^p(X); \mathcal{C})$,*

$$[\mathcal{T}, \psi] = [\mathcal{K}^\perp/\mathcal{K}, \phi] \text{ in } W(\mathcal{S}_i^p(X); \mathcal{C}).$$

Proof: Let $\mathcal{S} = \mathcal{K}^\perp/\mathcal{K}$. Since $[\mathcal{S}, \phi] + [\mathcal{S}, -\phi] = 0$ in $W(\mathcal{S}_i^p(X); \mathcal{C})$, it suffices to show $[\mathcal{T}, \psi] + [\mathcal{S}, -\phi] = 0$. For this take $\mathcal{N} = \mathcal{K}$ and $(\mathcal{M}, \gamma) = (\mathcal{S}, \phi) \perp (\mathcal{S}, -\phi)$ in the following.

3.21. *Suppose given $(\mathcal{M}, \gamma) \in Q(\mathcal{S}_i^p(X); \mathcal{C})$ and $\mathcal{N} \subseteq \mathcal{M}$, such that $\mathcal{N} \subseteq \mathcal{N}^\perp$ and $\mathcal{N} \in \mathcal{S}_i^p(X)$ and so that the induced forms*

$$\beta : \mathcal{M}/\mathcal{N}^\perp \times \mathcal{N} \to V_p$$
$$\delta : \mathcal{N}^\perp/\mathcal{N} \times \mathcal{N}^\perp/\mathcal{N} \to V_p$$

are nonsingular at all primes of codimension $\leq p+1$. Then if $(\mathcal{N}^\perp/\mathcal{N}, \delta) \in Q(\mathcal{S}_i^p(X); \mathcal{C})$ is a lagrangian so is $(\mathcal{M}, \gamma)$.

Proof: Let $\mathcal{P}$ be a sublagrangian for $(\mathcal{N}^\perp/\mathcal{N}, \delta)$ and let $\mathcal{J} \subseteq \mathcal{N}^\perp$ be the inverse image of $\mathcal{P}$ under the quotient map $\mathcal{N}^\perp \to \mathcal{N}^\perp/\mathcal{N}$. Clearly $\mathcal{J} \subseteq \mathcal{J}^\perp$ and

$$\mathcal{N}^\perp/\mathcal{J} \cong (\mathcal{N}^\perp/\mathcal{N})/\mathcal{P}.$$

The sequence of injections $\mathcal{J} \rightarrowtail \mathcal{N}^\perp \rightarrowtail \mathcal{M}$ gives rise to a short exact sequence of cokernels

$$(\mathcal{N}^\perp/\mathcal{N})/\mathcal{P} \overset{i}{\rightarrowtail} \mathcal{M}/\mathcal{J} \twoheadrightarrow \mathcal{M}/\mathcal{N}^\perp.$$

Applying $\mathcal{H}om(-,\mathcal{V}_p(\mathcal{C}))$ to this gives the bottom exact sequence in

$$\begin{array}{ccccc} \mathcal{N} & \rightarrowtail & \mathcal{J} & \twoheadrightarrow & \mathcal{P} \\ \downarrow \mathrm{Ad}\beta & & \downarrow & & \downarrow \\ (\mathcal{M}/\mathcal{N}^\perp)\hat{} & \rightarrowtail & (\mathcal{M}/\mathcal{J})\hat{} & \xrightarrow{\hat{i}} & ((\mathcal{N}^\perp/\mathcal{N})/\mathcal{P})\hat{} \end{array}$$

where the unlabelled verticals are induced from γ and δ, using $\mathcal{J} \subseteq \mathcal{J}^\perp$ and $\mathcal{P} \subseteq \mathcal{P}^\perp$. By assumption the left and right verticals are isomorphisms and $i\hat{}$ is surjective at all points of codimension $\leq p+1$; $\mathcal{J} \to (\mathcal{M}/\mathcal{J})\hat{}$ is therefore an isomorphism at all such points. Since $\mathcal{N}$ and $\mathcal{P} \in \mathcal{S}_1^p(X), \mathcal{J} \in \mathcal{S}_1^p(X)$. From this one easily concludes $\mathcal{M}/\mathcal{J} \in \mathcal{S}_1^p(X)$ and $\mathcal{M}/\mathcal{J} \to \mathcal{J}\hat{}$ is an isomorphism in codimension $\leq p+1$. Hence $(\mathcal{M},\gamma)$ is a lagrangian.

Since $\mathcal{L}^p$ is additive, it remains to show that if $(\mathcal{F},\tau)$ is a lagrangian, so is $(\mathcal{M},\phi)$ in (3.18). By what we have already done, we may assume $(F,\tau) \in Q(\mathcal{F}(X_x);\mathcal{C}_x)$, $\operatorname{cod} x = p$, where (F,τ) is a lagrangian. Choose an integral lattice $\mathcal{L} \subseteq \mathcal{F}$. Let $\mathcal{K}$ be a sublagrangian of $(\mathcal{F},\tau)$ and set $\mathcal{I} = \operatorname{im}\{\mathcal{L} \to \mathcal{F} \to \mathcal{F}/\mathcal{K}\}$. Then $\operatorname{Ass}(\mathcal{I}) = \{x\}$, so $\mathcal{I} \in \mathcal{S}_1^p(X)$; and since $\mathcal{L} \in \mathcal{S}_2^p(X)$, $\mathcal{N} := \ker\{\mathcal{L} \to \mathcal{I}\} \in \mathcal{S}_2^p(X)$. $\mathcal{N} \subseteq \mathcal{N}^\perp$ since $\mathcal{N} \subseteq \mathcal{K}$, so we get a commutative diagram of short exact sequences,

$$\begin{array}{ccccc} \mathcal{N} & \rightarrowtail & \mathcal{L} & \twoheadrightarrow & \mathcal{L}/\mathcal{N} = \mathcal{I} \\ \downarrow & & \downarrow & & \downarrow \\ (\mathcal{L}/\mathcal{N})\hat{} & \rightarrowtail & \mathcal{L}^* & \twoheadrightarrow & \mathcal{N}\hat{} \end{array}$$

where we have identified $\mathcal{L}^*$ with $\mathcal{L}\hat{}$ using (3.16) and the bottom sequence is the dual of the top one. At any point y of codimension $\leq p+1$, the top sequence is in $\mathcal{S}_1^p(X_y) = \mathcal{CM}^p(X_y)$, so at such y the bottom sequence is exact, and the left vertical is the dual of the right. Thus, taking cokernels of the vertical maps gives an exact sequence which becomes short exact at point of codimension $p+1$,

$$\mathcal{S} \to \mathcal{L}^*/\mathcal{L} \to \mathcal{T}$$

where $\phi|\mathcal{S} \times \mathcal{S} \equiv 0$, and $\mathcal{T} \cong \mathcal{H}om(\mathcal{S},\mathcal{V}^{p+1}(\mathcal{C}))$ at primes of codimension $p+1$. Hence $(\mathcal{L}^*/\mathcal{L},\phi)$ is a lagrangian at all codimension $p+1$ primes. By (3.9) $[\mathcal{L}^*/\mathcal{L},\phi] = 0$ in $W(\mathcal{S}_1^{p+1}(X);\mathcal{C})$. This completes the proof that $\mathcal{L}^p$ is well-defined.

As a corollary of the construction of $\mathcal{L}^p$ we have:

3.22 Proposition. *Let* $y \in \overline{x} \subseteq X$, $\operatorname{cod} y = p+1$, $\operatorname{cod} x = p$. *Then*

$$\begin{array}{ccc} & \mathcal{L}^p(X) & W(\mathcal{S}_1^{p+1}(X);\mathcal{C}) \\ W(\mathcal{F}(X_x);\mathcal{C}_x) & & \downarrow i_y^* \\ & \mathcal{L}^p(X_y) & W(\mathcal{F}(X_y);\mathcal{C}_y) \end{array}$$

commutes.

The following is one of the main results of this paper.

3.23 Proposition. *The sequence*

$$W(\mathcal{S}_1^p(X);\mathcal{C}) \xrightarrow{\mathcal{K}^p} W(\mathcal{F}^p(X);\mathcal{C}) \xrightarrow{\mathcal{L}^p} W(\mathcal{S}_1^{p+1}(X);\mathcal{C}),$$

where $W(\mathcal{F}^p(X);\mathcal{C}) := \coprod_{\mathrm{cod}\ x=p} W(\mathcal{F}(X_x);\mathcal{C}_x)$*, is exact for all* $p \geq 0$*.*

Proof: Since $W(\mathcal{S}_2^p(X);\mathcal{C}) \to W(\mathcal{S}_1^p(X);\mathcal{C})$ is surjective ((3.4)), to show $\mathcal{L}^p\mathcal{K}^p = 0$ we may replace $W(\mathcal{S}_1^p(X);\mathcal{C})$ with $W(\mathcal{S}_2^p(X);\mathcal{C})$. In this case, it is immediate from the constructions.

So now suppose $(\mathcal{F},\tau)$ is given, where $\mathcal{F} = \coprod i_{x_i*}F_i, \tau = \coprod i_{x_i*}\tau_i, (F_i,\tau_i) \in Q(X_{x_i};\mathcal{C}_{x_i})$ and $\mathcal{L}^p[\mathcal{F},\tau] = 0$. This means there is an integral lattice $\mathcal{L} \subseteq \mathcal{F}$ with dual lattice $\mathcal{L}^* \supseteq \mathcal{L}, \mathcal{M} := \mathcal{L}^*/\mathcal{L}$, and lagrangians $(\mathcal{S}_i,\psi_i) \in Q(\mathcal{S}_1^{p+1}(X);\mathcal{C})$ such that

$$(3.24) \qquad (\mathcal{M},\phi) \perp (\mathcal{S}_1,\psi_1) \cong (\mathcal{S}_2,\psi_2)$$

where $(\mathcal{M},\phi) \in \mathcal{S}_1^{p+1}(X)$ is the form constructed in (3.18).

Suppose first that $(\mathcal{M},\phi)$ is a lagrangian and let $\mathcal{K} \subseteq \mathcal{M}$ be a sublagrangian. If $p : \mathcal{L}^* \to \mathcal{M}$ is the quotient map, let $\mathcal{I} = p^{-1}(\mathcal{K})$. Since $\mathcal{K} \subseteq \mathcal{K}^\perp$, $\mathcal{I}$ is an integral lattice,

$$\mathcal{L} \subseteq \mathcal{I} \subseteq \mathcal{I}^* \subseteq \mathcal{M}^*,$$

and $\mathcal{I}_y = \mathcal{I}_y^*$ for each y of codimension $p+1$. Hence $\tau|\mathcal{I}\times\mathcal{I} : \mathcal{I}\times\mathcal{I} \to \mathcal{V}^p(\mathcal{C})$ is nonsingular in codimension $\leq p+1$ and $\mathcal{K}^p[\mathcal{I},\tau|\mathcal{I}\times\mathcal{I}] = [\mathcal{F},\tau]$.

To reduce to the case where $(\mathcal{M},\phi)$ is a lagrangian we argue as follows. First we may assume $\mathcal{S}_1 \in \mathcal{S}_2^{p+1}(X)$, for the process described in (3.4) clearly takes lagrangians to lagrangians (in $Q(\mathcal{S}_1^{p+1}(X);\mathcal{C})$). Next, after adding $(\mathcal{S}_1,-\psi_1)$ to both sides of (3.24), we may assume $(\mathcal{S}_1,\psi_1)$ is a hyperbolic form in $Q(\mathcal{S}_2^{p+1}(X);\mathcal{C})$, say $\mathcal{S}_1 = \mathcal{T} \oplus \mathcal{T}^\sim$ with $\mathcal{T}^\sim := \mathcal{H}om(\mathcal{T},\mathcal{V}^{p+1}(\mathcal{C}))$ and $\mathcal{T} = \mathcal{T}^\perp$.

Choose a surjection $\mathcal{J} \twoheadrightarrow \mathcal{T}$, where $\mathcal{J} \in \mathcal{CM}^p(X)$ (see [P3, 1.6]); then $\mathcal{K} := \ker\{\mathcal{J} \twoheadrightarrow \mathcal{T}\} \in \mathcal{S}_2^p(X)$ by [Ka, Ex. 14, p. 103]. Exactly as in [P3, *loc. cit.*], we conclude there is a short exact sequence

$$\mathcal{J}^\wedge \overset{\alpha^\wedge}{\rightarrowtail} \mathcal{K}^\wedge \twoheadrightarrow \mathcal{T}^\sim,$$

where $(-)^\wedge = \mathcal{H}om(-,\mathcal{V}^p(\mathcal{C}))$ and $(-)^\sim = \mathcal{H}om(-,\mathcal{V}^{p+1}(\mathcal{C}))$, and $\alpha : \mathcal{K} \to \mathcal{J}$ is the inclusion. We thus have a short exact sequence

$$\mathcal{K} \oplus \mathcal{J}^\wedge \overset{\alpha\oplus\alpha^\wedge}{\rightarrowtail} \mathcal{J} \oplus \mathcal{K}^\wedge \twoheadrightarrow \mathcal{T} \oplus \mathcal{T}^\sim.$$

Define an $\mathcal{O}_X$-bilinear symmetric form σ on $\mathcal{I} := \mathcal{K} \oplus \mathcal{J}^\wedge$ by $\sigma|\mathcal{K}\times\mathcal{K} \equiv 0 \equiv \sigma|\mathcal{J}^\wedge\times\mathcal{J}^\wedge$ and $\sigma(f,k) = f(\alpha(k))$, where $f \in \mathcal{J}^\wedge$, $k \in \mathcal{K}$. Then the following are routinely verified:

- $(\mathcal{I}_x,\sigma_x) \in Q(\mathcal{F}(X_x);\mathcal{C}_x)$, for all $x \in \mathrm{Ass}\,(\mathcal{I})$,
- $(\mathcal{I}_x,\sigma_x)$ is a lagrangian,
- setting $\mathcal{G} = \coprod_{codx=p} i_{x*}\mathcal{I}_x$ and $\sigma = \coprod_{codx=p} i_{x*}\sigma_x, \mathcal{I} \subseteq G$ is an integral lattice in (G,σ), and
- using the lattice $\mathcal{I}$ in the construction of $\mathcal{L}^p[G,\tau]$ produces $(\mathcal{S}_1,\psi_1)$

Hence adding $(\mathcal{G},\sigma)$ to $(\mathcal{F},\tau)$ does not change the class of $(\mathcal{F},\tau)$ in $\coprod W(\mathcal{F}(X_x);\mathcal{C}_x)$ and allows us to assume $(\mathcal{M},\phi)$ is a lagrangian $(=(\mathcal{S}_2,\psi_2))$ in (3.24). This completes the proof of (3.23).

For each $p \geq 0$, let

$$\partial^p : W(\mathcal{F}^p(X);\mathcal{C}) \to W(\mathcal{F}^{p+1}(X);\mathcal{C})$$

be the composition

$$W(\mathcal{F}^p(X);\mathcal{C}) \xrightarrow{\mathcal{L}^p} W(\mathcal{S}_i^{p+1}(X);\mathcal{C}) \xrightarrow{\mathcal{K}^{p+1}} W(\mathcal{F}^{p+1}(X);\mathcal{C}) \tag{3.25}$$

It follows from the last proposition that $\partial^{p+1}\circ\partial^p = 0$.

3.26 Definition. *Let X be a CM scheme with canonical sheaf $\mathcal{C}$. The complex of flasque sheaves on X*

$$\mathbb{W}^{\cdot}(X;\mathcal{C}) := \{\mathcal{W}(\mathcal{F}^0(X);\mathcal{C}) \xrightarrow{\partial^0} \mathcal{W}(\mathcal{F}^1(X);\mathcal{C}) \xrightarrow{\partial^1} \cdots \xrightarrow{\partial^{n-1}} \mathcal{W}(\mathcal{F}^n(X);\mathcal{C}) \to 0\}$$

where $\mathcal{W}(\mathcal{F}^p(X);\mathcal{C}) := \coprod_{\text{cod } x=p} i_{x*}W(\mathcal{F}(X_x);C_x)$, is called the Gersten-Witt complex of $(X,\mathcal{C})$.

For any field F and one-dimensional F-vector space V, $W(F;V) := W(\mathcal{F}(F);V)$ is the Witt group of all nonsingular symmetric F-bilinear forms $F^m \times F^m \to V$, $m \geq 0$.

Let $x \in X^{(p)}$ and let $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ be the maximal ideal. According to [P3, 2.2], the inclusion of categories

$$Q(\mathcal{F}(k(x));(0:\mathfrak{m}_x)_{E^p(\mathcal{C}_x)}) \subseteq Q(\mathcal{F}(X_x);\mathcal{C}_x)$$

induces a canonical isomorphism

$$W(k(x);(0:\mathfrak{m}_x)_{E^p(\mathcal{C}_x)}) = W(\mathcal{F}(k(x));(0:\mathfrak{m}_x)_{E^p(\mathcal{C}_x)}) \xrightarrow{\cong} W(\mathcal{F}(X_x);\mathcal{C}_x) \tag{3.27}$$

If now X is regular, then (1.41) implies that for each $x \in X$ there is a *canonical* isomorphism

$$W(k(x);\Lambda^p N(\mathfrak{m}_x)\otimes \mathcal{C}_x) \xrightarrow{\cong} W(k(x);(0:\mathfrak{m}_x)_{E^p(\mathcal{C}_x)}),$$

so we have canonical isomorphisms of the terms of the Gersten-Witt complex

$$\mathcal{W}(\mathcal{F}^p(X);\mathcal{C}) := \coprod_{\text{cod } x=p} i_{x*}W(\mathcal{F}(X_x);\mathcal{C}_x) \cong \coprod_{\text{cod } x=p} i_{x*}W(k(x);\Lambda^p N(\mathfrak{m}_x)\otimes\mathcal{C}) \tag{3.28}$$

The main theorem of this paper is that if R is regular local, then a filtered version of $\mathbb{W}^{\cdot}(X;\mathcal{C})$ is exact. We describe this filtration next.

Again assume $x \in X^{(p)}$. Any isomorphism (see (1.22b)) $c : k(x) \xrightarrow{\cong} (0:\mathfrak{m}_x)_{E^p(\mathcal{C})_x}$ induces an isomorphism

$$\gamma : W(k(x)) \xrightarrow{\cong} W(k(x);(0:\mathfrak{m}_x)_{E^p(\mathcal{C})_x})$$

Let $\{I^\ell(k(x))\}_{\ell\geq 0}$ denote the fundamental filtration ([Sc, p. 156]) of $W(k(x))$. Although γ depends on C,

$$I^\ell(k(x);(0:\mathfrak{m}_x)_{E^p(\mathcal{C})_x}) := \gamma(I^\ell(k(x)))$$

does not. For given c_1, $c_2 : k(x) \xrightarrow{\cong} (0:\mathfrak{m}_x)_{E^p(\mathcal{C})_x}$, there is $k^* \in k(x)$ such that $c_1(kk^*) = c_2(k)$ for all $k \in k(x)$. But $\cdot k^* : k(x) \to k(x)$ induces an automorphism of $W(k(x))$ which preserves the fundamental filtration. In fact, if $[\phi] \in I^\ell(k(x))$, then

$$[k^*\phi] = [< k^* >]\cdot[\phi] = [< k^*, -1 >]\cdot[\phi] + [\phi], \tag{3.29}$$

where $\cdot$ denotes product in $W(k(x))$.

3.30 Proposition-Definition. *For each $x \in X^{(p)}$, there is a canonical filtration $\{I^\ell(\mathcal{F}(X_x);\mathcal{C}_x)\}_{\ell\geq 0}$ of $I^0(\mathcal{F}(X_x);\mathcal{C}_x) := W(\mathcal{F}(X_x);\mathcal{C}_x)$, which dévissage and any isomorphism $k(x) \xrightarrow{\cong} (0 : \mathfrak{m}_x)_{E^p(\mathcal{C}_x)}$ identifies with the fundamental filtration $\{I^\ell(k(x))\}_{\ell\geq 0}$ of $W(k(x))$. The resulting filtration*

$$\{\mathcal{I}^\ell(\mathcal{F}^p(X);\mathcal{C})\}_{\ell\geq 0} := \coprod_{\mathrm{cod}\ x=p} i_{x*}\{I^\ell(\mathcal{F}(X_x);C_x)\}_{\ell\geq 0}$$

is called the fundamental filtration of $\mathcal{W}(\mathcal{F}^p(X);\mathcal{C})$. *We make the convention*

$$\mathcal{I}^\ell(\mathcal{F}^p(X);\mathcal{C}) := \mathcal{W}(\mathcal{F}^p(X);\mathcal{C}),\ \ \text{if } \ell < 0$$

We now work toward the proof that the differentials in the Gersten-Witt complex preserve the fundamental filtration. To do this we may work locally, so for the rest of §3 we assume that $X = \operatorname{Spec} R$.

3.31 Theorem. *Let $\phi : R \to S$ be a homomorphism of CM rings such that S is a finitely-generated R-module. Let $c := \dim R - \dim S$ and let C_R and C_S be canonical modules for R and S. Let*

$$\Xi : E^\cdot(C_S) \to E^\cdot(C_R)[c]$$

be a nonsingular trace homomorphism. Then for each p, $0 \leq p \leq \dim S$, there is a commutative diagram of exact sequences

$$(3.32)\qquad \begin{array}{ccccc} W(\mathcal{S}_1^p(S);C_S) & \xrightarrow{\mathcal{K}^p(S;C_S)} & W(\mathcal{F}^p(S);C_S) & \xrightarrow{\mathcal{L}^p(S;C_S)} & W(\mathcal{S}_1^{p+1}(S);C_S) \\ \downarrow{\scriptstyle W(\xi^p)} & & \downarrow{\scriptstyle W(\Xi^p)} & & \downarrow{\scriptstyle W(\xi^{p+1})} \\ W(\mathcal{S}_1^{p+c}(R);C_R) & \xrightarrow{\mathcal{K}^{p+c}(R;C_R)} & W(\mathcal{F}^{p+c}(R);C_R) & \xrightarrow{\mathcal{L}^{p+c}(R;C_R)} & W(\mathcal{S}_1^{p+c+1}(R);C_R) \end{array}$$

where $\Xi^p : E^p(C_S) \to E^{p+c}(C_R)$ is Ξ in degree p and $\xi^p : V^p(C_S) \to V^{p+c}(C_R)$ is its restriction to $V^p(C_S)$.

Hence Ξ induces a map of Gersten complexes

$$W^\cdot(\Xi^\cdot) : W^\cdot(S;C_S) \to W^\cdot(R;C_R)[c].$$

Proof Given $\{M \times M \to V^p(C_S)\} \in Q(\mathcal{S}_1^p(S);C_S)$ (resp., $\{F \times F \to E^p(C_S)\} \in Q(\mathcal{F}^p(S);C_S)$), its composition with $\xi^p : V^p(C_S) \to V^{p+c}(C_R)$ (resp., $\Xi^p : E^p(C_S) \to E^{p+c}(C_R)$) gives an element of $Q(\mathcal{S}_1^{p+c}(R);C_R)$ (resp., $Q(\mathcal{F}^{p+c}(R);C_R)$), by Proposition (2.16). This defines the vertical maps in the diagram. The horizontal ones are from (3.23), and the commutativity of the left square is clear.

Let $\mathfrak{q} \subset S$ be a codimension p prime, let F be a finite length $S_\mathfrak{q}$-module and let $\tau : F \times F \to E^p(C_S)$ be nonsingular. Let $L \subseteq F$, $L \in \mathcal{S}_2^p(S)$, be an integral lattice in (F,τ). Then by (2.21) $L \in \mathcal{S}_2^{p+c}(R)$ and is evidently an integral lattice in $(F, \Xi^p \circ \tau)$. There is an inclusion of dual lattices fitting into the commutative diagram

$$\begin{array}{ccc} L^*_\tau & \longrightarrow & L^*_{\Xi^p\circ\tau} \\ \downarrow & & \downarrow \\ \operatorname{Hom}_S(L, V^p(C_S)) & \xrightarrow{\xi^p_*} & \operatorname{Hom}_R(L, V^{p+c}(C_R)) \end{array}$$

where the verticals are the natural isomorphisms of (3.16) and ξ^p_* is an isomorphism by (2.16). Hence $L^*_\tau = L^*_{\Xi^p\circ\tau}$.

Now the commutative diagram

$$\begin{array}{ccccc} L^*_\tau \times L^*_\tau & \longrightarrow & E^p(C_S) & \xrightarrow{d^p} & V^{p+1}(C_S) \\ \downarrow & & \downarrow{\scriptstyle \Xi^p} & & \downarrow{\scriptstyle \xi^{p+1}} \\ L^*_{\Xi^p\circ\tau} \times L^*_{\Xi^p\circ\tau} & \longrightarrow & E^{p+c}(C_R) & \xrightarrow{d^{p+c}} & V^{p+c+1}(C_R) \end{array}$$

and the definition of $\mathcal{L}^p$ and of $\mathcal{L}^{p+c}$ show that the right square in (3.32) commutes.

The first half of another of the main results of this paper is the assertion that the differentials of the Gersten complex preserve the fundamental filtration, in the following sense.

3.33 Theorem. *Let R be CM, C_R a canonical module for R. Then the differentials of the Gersen-Witt complex preserve the fundamental filtration: for each p, $0 \le p \le \dim R$ and all $\ell \ge 1$,*

$$\partial^p\big(I^\ell(\mathcal{F}^p(R);C)\big) \subseteq I^{\ell-1}(\mathcal{F}^{p+1}(R);C) \tag{3.34}$$

Proof To begin we show that we may assume R is local and $p+1=n$. Given any codimension $p+1$ prime $\mathfrak{q}_0$ there is a commutative diagram with verticals induced by $\otimes R_{\mathfrak{q}_0}$:

$$\begin{array}{ccccc} W(\mathcal{F}^p(R);C) & \xrightarrow{\mathcal{L}^p(R)} & W(\mathcal{S}^{p+1}_i(R);C) & \xrightarrow{\mathcal{K}^{p+1}(R)} & W(\mathcal{F}^{p+1}(R)) \\ \downarrow & & \downarrow & & \downarrow \\ W(\mathcal{F}^p(R_{\mathfrak{q}_0});C_{\mathfrak{q}_0}) & \xrightarrow{\mathcal{L}^p(R_{\mathfrak{q}_0})} & W(\mathcal{S}^{p+1}_i(R_{\mathfrak{q}_0});C_{\mathfrak{q}_0}) & \xrightarrow{=} & W(\mathcal{F}^{p+1}(R_{\mathfrak{q}_0});C_{\mathfrak{q}_0}) \end{array}$$

where the right vertical is the projection

$$\coprod_{\text{cod }\mathfrak{q}=p+1} W(\mathcal{F}(R_\mathfrak{q});C_\mathfrak{q}) \to W(\mathcal{F}(R_{\mathfrak{q}_0};C_{\mathfrak{q}_0})$$

and the left vertical is the projection

$$\coprod_{\text{cod }\mathfrak{p}=p} W(\mathcal{F}(R_\mathfrak{p});C_\mathfrak{p}) \to \coprod_{\substack{\text{cod }\mathfrak{p}=p\\ \mathfrak{p}\subset\mathfrak{q}_0}} W(\mathcal{F}(R_\mathfrak{p});C_\mathfrak{p})$$

It is obvious that the right square commutes; commutativity of the left square is (3.22). Hence if $\partial(R_{\mathfrak{q}_0})\big(I^\ell(\mathcal{F}^p(R_{\mathfrak{q}_0});C_{\mathfrak{q}_0})\big) \subseteq I^{\ell-1}(\mathcal{F}^{p+1}(R_{\mathfrak{q}_0});C_{\mathfrak{q}_0})$, then $\partial(R)\big(I^\ell(\mathcal{F}^p(R);C)\big) \subseteq I^{\ell-1}(\mathcal{F}^{p+1}(R);C)$.

So we now assume $(R,\mathfrak{m})$ is local CM of dimension n, and we are to show (3.34) for $p+1=n$. Let $\mathfrak{p}_0$ be a codimension $n-1$ prime and consider the composition

$$\phi : R \to R/\mathfrak{p}_0 \to S$$

where $R/\mathfrak{p}_0 \to S$ is the integral closure. Then S is semi-local Dedekind and $C_S := \mathrm{Ext}^{n-1}_R(S, C_R) = \mathrm{Hom}_R(S, V^{n-1}(C_R))$ is a canonical module for S (1.4). The canonical map $\xi : C_S \to V^{n-1}(C_R)$ induces a map of complexes

$$\Xi : E^{\cdot}(C_S) \to E^{\cdot}(C_R)[n-1]$$

and hence a commutative diagram

$$\begin{array}{ccc} W(\mathcal{F}^0(S); C_S) & \xrightarrow{\partial^0(S)=\mathcal{L}^0(S;C_S)} & W(\mathcal{F}^1(S); C_S) \\ {\scriptstyle W(\Xi^0)}\downarrow & & {\scriptstyle W(\Xi^1)}\downarrow \\ W(\mathcal{F}^{n-1}(R); C_R) & \xrightarrow{\partial^{n-1}(R)=\mathcal{L}^{n-1}(R;C_R)} & W(\mathcal{F}^n(R); C_R) \end{array}$$

which equals

$$\begin{array}{ccc} W(K(S); C_S) & \xrightarrow{\partial^0(S)=\mathcal{L}^0(S)} & \coprod_{\mathfrak{n} \text{ maximal}} W(k(\mathfrak{n})) \\ {\scriptstyle W(\Xi^0)}\downarrow & & {\scriptstyle W(\Xi^1)}\downarrow \\ \coprod_{\text{cod } \mathfrak{p}=n-1} W(\mathcal{F}(R_{\mathfrak{p}}); C_{R_{\mathfrak{p}}}) & \xrightarrow{\partial^{n-1}(R)=\mathcal{L}^{n-1}(R)} & W(k(\mathfrak{m})) \end{array} \tag{3.36}$$

where the left vertical maps isomorphically to $W(\mathcal{F}(R_{\mathfrak{p}_0}); C_{R_{\mathfrak{p}_0}}) = W(k(\mathfrak{p}_0))$ and the right vertical is a sum of trace maps, which, according to [Ar] and (2.17), preserve the fundamental filtration. This completes the proof of (3.33).

3.37 Corollary. *Let X be a CM scheme and $\mathcal{C}$, a canonical sheaf. Then for each $\ell \geq 0$, there is a subcomplex of the Gersten-Witt complex $\mathbb{W}^{\cdot}(X;\mathcal{C})$,*

$$\mathbb{I}^{\cdot,\ell}(X;\mathcal{C}) := \{\mathcal{I}^{\ell}(\mathcal{F}^0(X);\mathcal{C}) \xrightarrow{\partial^{0,\ell}} \ldots \mathcal{I}^{\ell-p}(\mathcal{F}^p(X);\mathcal{C}) \xrightarrow{\partial^{p,\ell-p}} \mathcal{I}^{\ell-p-1}(\mathcal{F}^{p+1}(X);\mathcal{C}) \xrightarrow{\partial^{p+1,\ell-p-1}} \ldots \xrightarrow{\partial^{n-1,\ell-n+1}} \mathcal{I}^{\ell-n}(\mathcal{F}^n(X);\mathcal{C})\},$$

where $\mathcal{I}^j(\mathcal{F}^q(X);\mathcal{C}) := \mathcal{W}(\mathcal{F}^q(X);\mathcal{C})$ if $j < 0$, and $\mathbb{I}^{\cdot,0}(X;\mathcal{C}) = \mathbb{W}^{\cdot}(X;\mathcal{C})$.

§4. The main technical result.

This section is devoted to the proof of the following theorem. An easy consequence will be the surjectivity of $\mathcal{L}^p$ in (3.23).

4.1 Theorem. *Let R be a smooth local ring, essentially of finite type over a field k of characteristic $\neq 2$. Let C be a canonical module for R. Let $(M,\phi) \in Q(\mathcal{S}_i^{p+1}(R);C), i \geq 1, p \geq 0$. Then there exist $(F_i,\tau_i) \in Q(\mathcal{F}(R_{\mathfrak{q}_i});C_{\mathfrak{q}_i}), \operatorname{cod} \mathfrak{q}_i = p, i = 1,\ldots n$, an integral lattice $L \in \mathcal{S}_{i+1}^p(R)$ in $(F,\tau) := (\amalg F_i, \amalg\tau_i)$ and a surjection $j_\tau : L_\tau^* \to M$, where L_τ^* is the dual lattice, such that*

i. *j_τ induces an isomorphism $L_\tau^*/L \cong M$, and*
ii. *for all $\ell_1,\ell_2 \in L_\tau^*$,*

$$d^p\tau(\ell_1,\ell_2) = \phi(j_\tau\ell_1, j_\tau\ell_2)$$

where $d^p : E^p(C) \to V^{p+1}(C)$ is the differential in the injective resolution $E^\cdot(C)$ of (1.9).

Proof: The proof has four steps. In Steps 1 and 2 the modules (resp., values of the forms involved) are lifted back from R (resp., the injective resolution) to a complete intersection quotients R^{p+1} of R (resp., C^{p+1} of C) using (1.22); in Step 3 dual lattices are replaced by dual modules using (3.16); and in Step 4 Quillen's variant of normalization (2.1) is applied to get a lattice and form ($(S^p \otimes_{R^{p+1}} M, \Psi)$ below) over a finite extension of R and then the Scheja-Storch trace map (2.14) is used to construct the data required in (4.1) over R. All of the ideas in the proof of (4.1) appear in much simpler form in the proof of Milnor's theorem presented in the Appendix to this chapter.

Step 1 is to use (1.22) to realize (M,ϕ) as an element of $Q(\mathcal{S}_i^0(R');C')$, for some complete intersection quotients R' of R and C' of C. Indeed, since $(M,\phi) \in Q(\mathcal{S}_i^{p+1}(R);C)$ there is by (1.22) an R-regular sequence $\{t,t_1,\ldots,t_p\} \subseteq \operatorname{Ann}(M)$ and a symmetric R-bilinear form

$$\phi' : M \times M \to C^{p+1},\ \ (M,\phi') \in Q(\mathcal{S}_i^0(R^{p+1});C^{p+1}) \tag{4.2}$$

where $R^{p+1} = R/(t,t_1,\ldots,t_p)$ and $C^{p+1} = C/(t,t_1,\ldots,t_p)C$, such that

$$i^{p+1}\phi' = \phi, \tag{4.3}$$

where $i^{p+1} : C^{p+1} \to V^{p+1}(C)$ is the injection in (1.23).

Step 2 is to verify that it is sufficent to find

i. $L \in \mathcal{S}_{i+1}^p(R)$, where $\{t_1,\ldots,t_p\} \in \operatorname{Ann}(L)$ and t is L-regular,
ii. a symmetric R-bilinear form $\Phi' : L\times L \to C^p$, such that $\Phi'[\frac{1}{t}] : L[\frac{1}{t}]\times L[\frac{1}{t}] \to C^p[\frac{1}{t}]$ is nonsingular, where $R^p = R/(t_1,\ldots,t_p)$ and $C^p = C/(t_1,\ldots,t_p)C$and
iii. a surjection $j_{\Phi'[\frac{1}{t}]} : L^*_{\Phi'[\frac{1}{t}]} \to M$, where $L^*_{\Phi'[\frac{1}{t}]} := \{\ell \in L[\frac{1}{t}] | \Phi'[\frac{1}{t}](\ell,L) \subset R^p\} \subset L[\frac{1}{t}]$ is the dual lattice, such that

a. $j_{\Phi'[\frac{1}{t}]}$ induces an isomorphism $L^*_{\Phi'[\frac{1}{t}]}/L \xrightarrow{\cong} M$ and

b.

$$q^p\Phi'[\tfrac{1}{t}](\ell_1,\ell_2) = \phi'(j_{\Phi'[\frac{1}{t}]}\ell_1, j_{\Phi'[\frac{1}{t}]}\ell_2)$$

for all $\ell_1, \ell_2 \in L^*_{\Phi'[\frac{1}{t}]}$, where $q^p : t^{-1}C^p \to C^{p+1}$ is the map in (1.24). Note that since $tM = 0$, *iiia.* implies that $L^*_{\Phi'[\frac{1}{t}]} \subseteq t^{-1}L$; thus, in *iiib.*, $q^p\Phi'[\frac{1}{t}](\ell_1, \ell_2)$ makes sense.

To begin this verification, set

$$\Phi := i^p\Phi' : L \times L \to V^p(C). \tag{4.4}$$

If $\mathfrak{Q} = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_r\} = \operatorname{Ass}(L)$, then $\Phi_{\mathfrak{Q}} := \Phi \otimes R_{\mathfrak{Q}} : L_{\mathfrak{Q}} \times L_{\mathfrak{Q}} \to \coprod_i E^p(C)_{\mathfrak{q}_i}$ is nonsingular, since $t \notin \cup\mathfrak{q}_i$. Further, it is easily seen that the localization

$$\lambda_{\mathfrak{Q}} : L[\tfrac{1}{t}] \to L_{\mathfrak{Q}}$$

preserves the dual lattices:

$$\lambda_{\mathfrak{Q}}(L^*_{\Phi'[\frac{1}{t}]}) \subseteq L^*_{\Phi_{\mathfrak{Q}}}.$$

Thus there is a commutative diagram

$$\begin{array}{ccc} L^*_{\Phi'[\frac{1}{t}]} & \xrightarrow{\lambda_{\mathfrak{Q}}} & L^*_{\Phi_{\mathfrak{Q}}} \\ \cong\big\downarrow & & \big\downarrow\cong \\ \operatorname{Hom}(L, C^p) & \xrightarrow{i^p_*} & \operatorname{Hom}(L, V^p(C)) \end{array}$$

in which the verticals are the isomorphisms given in (3.16), and i^p_* is an isomorphism because $(t_1, \cdots, t_p) \subseteq \operatorname{Ann} L$ and $\operatorname{im} i^p = (0 : (t_1, \cdots, t_p))_{V^p(C)}$. Hence

$$\lambda_{\mathfrak{Q}} : L^*_{\Phi'[\frac{1}{t}]} \xrightarrow{\cong} L^*_{\Phi_{\mathfrak{Q}}}.$$

Also, $\Phi_{\mathfrak{Q}}$ extends $\Phi'[\frac{1}{t}]$ in the sense that for all ℓ_1, $\ell_2 \in L^*_{\Phi'[\frac{1}{t}]}$,

$$(t^{-1}i^p)\Phi'[\tfrac{1}{t}](\ell_1, \ell_2) = \Phi_{\mathfrak{Q}}(\lambda_{\mathfrak{Q}}\ell_1, \lambda_{\mathfrak{Q}}\ell_2)$$

We now check the condition (4.1ii). To simplify notation, set $\lambda_{\mathfrak{Q}}\ell_i = \ell_{i,\mathfrak{Q}}$. Then we have

$$d^p\Phi_{\mathfrak{Q}}(\ell_{1,\mathfrak{Q}}, \ell_{2,\mathfrak{Q}}) = d^p(t^{-1}i^p)\Phi'[\tfrac{1}{t}](\ell_1, \ell_2) = i^{p+1}q^p\Phi'[\tfrac{1}{t}](\ell_1, \ell_2) =$$
$$i^{p+1}\phi'(j_{\Phi'[\frac{1}{t}]}\ell_1, j_{\Phi'[\frac{1}{t}]}\ell_2) = \phi(j_{\Phi_{\mathfrak{Q}}}\ell_{1,\mathfrak{Q}}, j_{\Phi_{\mathfrak{Q}}}\ell_{2,\mathfrak{Q}})$$

Taking $\tau = \Phi_Q$ and $F = L_Q$ thus completes Step 2.

Step 3 is to replace the dual lattice $L^*_{\Phi'[\frac{1}{t}]}$ in Step 2 with $\operatorname{Hom}(L, C^p)$ using (3.16). Namely we want to show that it is sufficient to produce

i. $L \in \mathcal{S}^p_{i+1}(R)$, where $\{t_1, \ldots, t_p\} \in \operatorname{Ann} L$ and t is L-regular,

ii. a symmetric R-bilinear form $\Phi' : L \times L \to C^p$, such that

$$\Phi'[\tfrac{1}{t}] : L[\tfrac{1}{t}] \times L[\tfrac{1}{t}] \to C^p[\tfrac{1}{t}]$$

is nonsingular, and

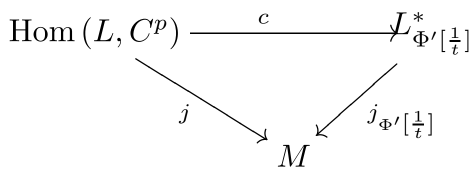
iii. a surjection $j : \operatorname{Hom}(L, C^p) \to M$, such that *a.* j induces an isomorphism $\mathrm{Hom}\,(L, C^p)/\operatorname{im}\,\mathrm{Ad}\,\Phi' \xrightarrow{\cong} M$ and

b.

$$\pi\lambda_1((\mathrm{Ad}\,\Phi')^{-1}(t\lambda_2)) = \phi'(j\lambda_1, j\lambda_2)$$

for all $\lambda_1, \lambda_2 \in \mathrm{Hom}\,(L, C^p)$, where $\pi : C^p \to C^p/(t) =: C^{p+1}$ is the projection, so that $\pi \cdot t : t^{-1}C^p \to C^{p+1}$ is the map q^p in (1.24).

Define $j_{\Phi'[\frac{1}{t}]}$ by commutativity of the diagram



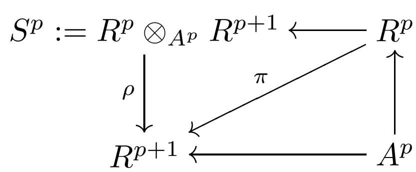
where the isomorphism c is from (3.16). We can now compute as follows. For any $\ell_1, \ell_2 \in L^*_{\Phi'[\frac{1}{t}]}$, there are unique $\lambda_1,\ \lambda_2 \in \mathrm{Hom}\,(L, C^p)$ such that $c\lambda = \ell_i$ and so

$$q^p\Phi'[\tfrac{1}{t}](\ell_1, \ell_2) = (\pi \cdot t)\Phi'[\tfrac{1}{t}](c\lambda_1, c\lambda_2) = \pi\lambda_1((\mathrm{Ad}\,\Phi')^{-1}(t\lambda_2)$$

$$= \phi'(j\lambda_1, j\lambda_2) = \phi'(j_{\Phi'[\frac{1}{t}]}c\lambda_1, j_{\Phi'[\frac{1}{t}]}c\lambda_2) = \phi'(j_{\Phi'[\frac{1}{t}]}\ell_1, j_{\Phi'[\frac{1}{t}]}\ell_2)$$

as required in Step 2, *iiib.* This completes Step 3.

Step 4 is to produce the data of Step 3, *i.*-*iii.* This will complete the proof of (4.1). Recall from (2.6) the diagram



in which the two inner triangles commute, $\ker\rho =: (x)$, $\ker\pi = (t)$ and $x = st$ in S^p. Now choose an isomorphism $w : R^p \to C^p$, so that we may apply (2.14), with $C_A = A$. In particular we have a trace homomorphism $\xi : C^{p+1} \to A^p$, with the property stated in *loc. cit.*

Since S^p is flat over R^{p+1}, the sequence

$$S^p \otimes_{R^{p+1}} M \xrightarrow{x \otimes \mathrm{Ad}\ \phi'} S^p \otimes_{R^{p+1}} \mathrm{Hom}\,_{R^p}(M, C^{p+1}) \xrightarrow{\rho\otimes(\mathrm{Ad}\ \phi')^{-1}} R^{p+1} \otimes_{R^{p+1}} M \cong M$$

is exact, and, since M is finitely presented, there is a canonical isomorphism

$$K : S^p \otimes_{R^{p+1}} \mathrm{Hom}\,_{R^p}(M, C^{p+1}) \xrightarrow{\cong} \mathrm{Hom}\,_{S^p}(S^p \otimes_{R^{p+1}} M, S^p \otimes_{R^{p+1}} C^{p+1}).$$

Hence we obtain a symmetric S^p-bilinear form

$$\Psi : (S^p \otimes_{R^{p+1}} M) \times (S^p \otimes_{R^{p+1}} M) \to S^p \otimes_{R^{p+1}} C^{p+1}$$

such that $\mathrm{Ad}\,\Psi = K \circ (x \otimes \mathrm{Ad}\,\phi')$ and, for $m_1, m_2 \in M$ and $s_1, s_2 \in S^p$ and with respect to the canonical isomorphism of R^p-modules $S^p \otimes_{R^{p+1}} C^{p+1} = R^p \otimes_{B^p} C^{p+1}$,

$$\Psi(s_1 \otimes_{R^{p+1}} m_1, s_2 \otimes_{R^{p+1}} m_2) = xs_1s_2(1 \otimes_{A^p} \phi'(m_1, m_2)) \in R^p \otimes_{A^p} C^{p+1}$$

Similarly, $S^p \otimes_{R^{p+1}} M = R^p \otimes_{A^p} M$ as R^p-modules. The Scheja-Storch trace $\xi : C^{p+1} \to A^p$, tensored up to an R^p-homomorphism,

$$\eta := R^p \otimes_{A^p} \xi : R^p \otimes_{A^p} C^{p+1} \to R^p \otimes_{A^p} A^p = R^p \underset{\cong}{\overset{w}{\longrightarrow}} C^p$$

induces by (2.16) an isomorphism of R^p-modules

$$\eta_* : \mathrm{Hom}_{S^p}(S^p \otimes_{R^{p+1}} M, S^p \otimes_{R^{p+1}} C^{p+1}) \overset{\cong}{\longrightarrow} \mathrm{Hom}_{R^p}(R^p \otimes_{A^p} M, C^p).$$

So finally we define the symmetric R^p-bilinear form Φ', the lattice $L \in \mathcal{S}^p_{i+1}$ and the surjection $j : \mathrm{Hom}\,(L, C^p) \to M$ required in Step 3 as follows:

$$\Phi' := R^p \otimes_{A^p} \eta \circ \Psi, \quad L := R^p \otimes_{A^p} M, \quad j := (\rho \otimes (\mathrm{Ad}\,\phi')^{-1}) \circ K^{-1} \circ ((R^p \otimes \eta)_*)^{-1}$$

Let $\mu \in \mathrm{Hom}\,(M, C^{p+1})$. Then $\eta \circ (S^p \otimes_{R^{p+1}} \mu \in \mathrm{Hom}\,(L, C^p)$. an easy calculation shows that if $\mu = \mathrm{Ad}\,\phi'(m)$, where $m \in M$, then

$$j(\eta \circ (S^p \otimes_{R^{p+1}} \mu)) = m \tag{4.5}$$

and

$$\mathrm{Ad}\,\Phi'(s \cdot (1 \otimes m)) = t\eta \circ (S^p \otimes_{R^{p+1}} \mu) \tag{4.6}$$

Using this, we verify *iiib.* in Step 3 as follows. For $\lambda_i = \eta \circ (S^p \otimes_{R^{p+1}} \mu_i) \in \mathrm{Hom}\,(L, C^p)$, where $\mu_i \in \mathrm{Hom}\,(M, C^{p+1})$, we compute using (2.14):

$$\pi\lambda_1((\mathrm{Ad}\,\Psi')^{-1}(t\lambda_2)) = \pi[\eta \circ (S^p \otimes_{R^{p+1}} \mu_1)(s \cdot (1 \otimes m_2))]$$

$$= \pi[\eta \circ (\cdot 1 \otimes \phi'(m_1, m_2))] = \phi'(m_1, m_2) = \phi'(j\lambda_1, j\lambda_2)$$

An immediate conseqence of Theorem (4.1) and (3.17.1), (3.17.2) and (3.18) is:

4.7 Corollary. *Let R be a smooth local ring, essentially of finite type over a field, and let C be the canonical module. Then for each $p \geq 0$, the homomorphism of (3.18)*

$$\mathcal{L}^p : W(\mathcal{F}^p(R); C) \to W(\mathcal{S}^{p+1}_1(R); C)$$

is surjective.

Recall that we have shown in (3.33) that the differentials in the Gersten complex preserve the fundamental filtration. The next result will imply that, in case R is smooth, the filtered complexes are also exact.

4.8 Theorem. *Keep the hypotheses of Theorem (4.1) and the notation of Step 4 in its proof:* $(M, \phi) \in Q(S_i^{p+1}(R); C)$, $(t, t_1, \ldots, t_p) \subseteq Ann\, M$, $C^{p+1} := C/(t, t_1, \ldots, t_p)C$ *and* $\phi = i^{p+1} \circ \phi'$, *where* $\phi' : M \times M \to C^{p+1}$ *and* $i^{p+1} : C^{p+1} \to V^{p+1}(C)$ *are constructed in (1.37) and (1.22). Let* $\Phi(x)$ *be the composition*

$$R^p \otimes_{A^p} M \times R^p \otimes_{A^p} M \xrightarrow{R^p \otimes \phi'} R^p \otimes_{A^p} C^{p+1} \xrightarrow{\cdot x} R^p \otimes_{A^p} C^{p+1} \xrightarrow{\eta := w(R^p \otimes \xi)} C^p \xrightarrow{i^p} V^p(C)$$

where $x \in R^p \otimes_{A^p} R^{p+1}$ *is a non-zero-divisor. Suppose that* $(M_{\mathfrak{q}_i}, \phi_{\mathfrak{q}_i}) \in I^{k+1}(\mathcal{F}(R_{\mathfrak{q}_i}); C_{\mathfrak{q}_i})$, *the (k+1)-st stage of the fundamental filtration (3.30), for all* $\mathfrak{q}_i \in Ass_{R^{p+1}}(M)$. *Then if* x, $y \in R^p \otimes_{B^p} R^{p+1}$ *are non-zero-divisors,*

$$(\Phi(x) \perp \Phi(y))_{\mathfrak{P}} \in I^k(\mathcal{F}(R_{\mathfrak{P}}); C_{\mathfrak{P}})$$

for all $\mathfrak{P} \in Ass_{R^p}(R^p \otimes_{B^p} M)$.

Proof For any R^{p+1}-module M and $\mathfrak{P} \in \mathrm{Spec}(R^p)$, we have

$$(R^p \otimes_{A^p} M)_{\mathfrak{P}} = \coprod R^p_{\mathfrak{P}} \otimes_{A^p_{\mathfrak{p}}} M_{\mathfrak{q}_i}$$

where $\mathfrak{p} = \mathfrak{P} \cap A^p$ and $\mathfrak{q}_1, \mathfrak{q}_2, \ldots$ are the primes of R^{p+1} over $\mathfrak{p}$. This implies that in $Q(\mathcal{F}(R^p_{\mathfrak{P}}); C^p_{\mathfrak{P}})$

$$\Phi(x) = \coprod \Phi(x_i)$$

where $\Phi(x_i)$ is the composition

$$R^p_{\mathfrak{P}} \otimes_{A^p_{\mathfrak{p}}} M_{\mathfrak{q}_i} \times R^p_{\mathfrak{P}} \otimes_{A^p_{\mathfrak{p}}} M_{\mathfrak{q}_i} \xrightarrow{R^p_{\mathfrak{P}} \otimes {\phi'}_{\mathfrak{q}_i}} R^p_{\mathfrak{P}} \otimes_{A^p_{\mathfrak{p}}} C^{p+1}_{\mathfrak{q}_i} \xrightarrow{\cdot x_i} R^p_{\mathfrak{P}} \otimes_{A^p_{\mathfrak{p}}} C^{p+1}_{\mathfrak{q}_i} \xrightarrow{R^p_{\mathfrak{P}} \otimes \xi_{\mathfrak{q}_i}} C^p_{\mathfrak{P}} \xrightarrow{i^p_{\mathfrak{P}}} E^p(C_{\mathfrak{P}})$$

and x_i is the i-th component of x with respect to the decomposition

$$(R^p \otimes_{A^p} R^{p+1})_{\mathfrak{P}} = \coprod R^p_{\mathfrak{P}} \otimes_{A^p_{\mathfrak{p}}} R^{p+1}_{\mathfrak{q}_i}$$

and $\xi_{\mathfrak{q}_i}$ is the restriction of $\xi_{\mathfrak{p}}$ to the i-th component of $C^{p+1} = \coprod C^{p+1}_{\mathfrak{q}_i}$. Note that

$$\xi_{\mathfrak{q}_i}((0 : \mathfrak{q}_i)_{C^{p+1}_{\mathfrak{q}_i}} \subseteq (0 : \mathfrak{p})_{A^p_{\mathfrak{p}}}$$

since $\mathfrak{p} \subseteq \mathfrak{q}_i$.

Now by Dévissage there is $(N_i, \psi_i) \in Q(\mathcal{F}(R^{p+1}_{\mathfrak{q}_i}); C^{p+1}_{\mathfrak{q}_i})$ such that $\mathfrak{q}_i N = 0$ and $[M_{\mathfrak{q}_i}, \phi_{\mathfrak{q}_i}] = [N_i, \psi_i]$ in $W(\mathcal{F}(R^{p+1}_{\mathfrak{q}_i}); C^{p+1}_{\mathfrak{q}_i})$. Let $\psi'_i : N_i \times N_i \to C^{p+1}$ be such that $\psi_i = i^{p+1} \circ \psi'_i$ (1.37). Then in $W(\mathcal{F}(R^p_{\mathfrak{P}}); C^p_{\mathfrak{P}})$

$$[R^p_{\mathfrak{P}} \otimes_{A^p_{\mathfrak{p}}} M_{\mathfrak{q}_i}, \Phi(x_i)] = [R^p_{\mathfrak{P}} \otimes_{A^p_{\mathfrak{p}}} N_i, \Psi(x_i)]$$

where $\Psi(x_i)$ is the composition (all tensor products over $A^p_{\mathfrak{p}}$)

$$R^p_{\mathfrak{P}} \otimes N_i \times R^p_{\mathfrak{P}} \otimes N_i \xrightarrow{R^p_{\mathfrak{P}} \otimes \psi_i} R^p_{\mathfrak{P}} \otimes (0 : \mathfrak{q}_i) \xrightarrow{\cdot x_i} R^p_{\mathfrak{P}} \otimes (0 : \mathfrak{q}_i) \xrightarrow{R^p_{\mathfrak{P}} \otimes \xi_{\mathfrak{q}_i}} R^p_{\mathfrak{P}} \otimes (0 : \mathfrak{p}) = (0 : \mathfrak{P})_{R^p_{\mathfrak{P}}} \xrightarrow{i^p_{\mathfrak{P}}} E(R^p_{\mathfrak{P}})$$

Now $R^p_{\mathfrak{P}} \otimes N_i \times R^p_{\mathfrak{P}} \otimes N_i \xrightarrow{R^p_{\mathfrak{P}}\otimes\psi_i} R^p_{\mathfrak{P}} \otimes (0 : \mathfrak{q}_i)$ is a nonsingular $R^p_{\mathfrak{P}} \otimes_{B'_{\mathfrak{p}}} k(\mathfrak{q}_i)$ bilinear form. The multiplication $\cdot x_i : R^p_{\mathfrak{P}} \otimes (0 : \mathfrak{q}_i) \to R^p_{\mathfrak{P}} \otimes (0 : \mathfrak{q}_i)$ is the same as that by $\bar{x}_i :=$ the image of x_i in $R^p_{\mathfrak{P}} \otimes_{B'_{\mathfrak{p}}} k(\mathfrak{q}_i)$. And $R^p_{\mathfrak{P}} \otimes_{A^p_{\mathfrak{p}}} k(\mathfrak{q}_i) \cong k(\mathfrak{P}) \otimes_{k(\mathfrak{p})} k(\mathfrak{q}_i)$ as rings.

Hence,

$$(R^p_{\mathfrak{P}} \otimes_{A^p_{\mathfrak{p}}} N_i, x_i \cdot (R^p_{\mathfrak{P}} \otimes_{A^p_{\mathfrak{p}}} \psi_i)) \cong (k(\mathfrak{P}) \otimes_{k(\mathfrak{p})} N_i, \bar{x}_i \cdot (k(\mathfrak{P}) \otimes_{k(\mathfrak{p})} \psi_i))$$

Next we claim that there is a splitting as rings

$$k(\mathfrak{P}) \otimes_{k(\mathfrak{p})} k(\mathfrak{q}_i) \cong \prod K_{i,j}$$

where each $K_{i,j}$ is a field extension of both $k(\mathfrak{p})$ and $k(\mathfrak{q}_i)$. Hence

$$k(\mathfrak{P}) \otimes_{k(\mathfrak{p})} (0 : \mathfrak{q}_i) \cong \prod V_{i,j}$$

where $V_{i,j} = K_{i,j} \otimes_{k(\mathfrak{q}_i)} (0 : \mathfrak{q}_i)$. Putting all this together, we see that for each i the first part of $\Psi(x_i)$,

$$R^p_{\mathfrak{P}} \otimes_{A^p_{\mathfrak{p}}} N_i \times R^p_{\mathfrak{P}} \otimes_{A^p_{\mathfrak{p}}} N_i \xrightarrow{\cdot x_i \circ (R^p_{\mathfrak{P}} \otimes \psi_i)} R^p_{\mathfrak{P}} \otimes_{A^p_{\mathfrak{p}}} (0 : \mathfrak{q}_i),$$

splits orthogonally (because of the *ring* decomposition above) as a sum of $V_{i,j}$-vaued nonsingular $K_{i,j}$-bilinear forms

$$\lambda_{i,j} : N_{i,j} \times N_{i,j} \to V_{i,j}$$

Further, the $k(\mathfrak{P})$-linear trace

$$k(\mathfrak{P}) \otimes_{k(\mathfrak{p})} (0 : \mathfrak{q}_i) \xrightarrow{k(\mathfrak{P})\otimes\xi_i} (0 : \mathfrak{P})_{C^p_{\mathfrak{P}}}$$

decomposes as

$$\prod V_{i,j} \xrightarrow{\prod \xi_{i,j}} (0 : \mathfrak{P})_{C^p_{\mathfrak{P}}}$$

where each $\xi_{i,j} : V_{i,j} \to (0 : \mathfrak{P})_{C^p_{\mathfrak{P}}}$ is $k(\mathfrak{P})$-linear (each $K_{i,j}$ is an extension of $k(\mathfrak{P})$) and surjective (2.17).

We are now done, for in $W(k(\mathfrak{P}); (0 : \mathfrak{P}))$ we have

$$\Psi(x_i) \perp \Psi(y_i) = \perp_j (\xi_{i,j})_*[(x_i, y_i) \otimes [N_{i,j}, \lambda_{i,j})]$$

where $(\xi_{i,j})_* : W(K_{i,j}; V_{i,j}) \to W(k(\mathfrak{P}); (0 : \mathfrak{P}))$ and Arason shows in [Ar] that

$$(\xi_{i,j})_*(I^{k+1}(K_{i,j}; V_{i,j})) \subseteq I^{k+1}(k(\mathfrak{P}); (0 : \mathfrak{P}))$$

4.9 Corollary. *Keep the notation of Theorem (4.8). If* $[M,\phi] \in W(\mathcal{S}_1^{p+1}(R);C)$ *and* $\mathcal{K}^p[m,\phi] \in I^k(\mathcal{F}^{p+1}(R);C)$*, then there is* $[F,\tau] \in I^{k+1}(\mathcal{F}^p(R);C)$ *such that*

$$\mathcal{L}^p[F,\tau] = [M\phi].$$

Proof In Theorem (4.8) take x to be the element x in Step 4 of the proof of Theorem (4.1), and take $y = 1$. Then according to (4.1), $\mathcal{L}^p(\coprod \Phi(x)_{\mathfrak{P}}) = [M,\phi]$ and clearly $\mathcal{L}^p(\coprod \Phi(y)_{\mathfrak{P}}) = 0$.

To end this section we repeat the observation of [Q, §5] that, roughly speaking, a regular local ring R, which is essentially of finite type over a field, is the direct limit of smooth subrings. This will permit us to extend the results of §4 to such R.

Let T be a finitely-generated algebra over k and let $\mathfrak{q} \subset T$ be a prime such that $T_{\mathfrak{q}} = R$. Then there is a subfield $k_f \subseteq k$, finitely generated over the prime field k_0, a finitely-generated algebra T_f over k_f, and a prime $\mathfrak{p} \subseteq T_f$ such that $T_f \otimes_{k_f} k = T$ and $T\mathfrak{p} = \mathfrak{q}$. Clearly $(T_f)_{\mathfrak{p}}$ is regular local. Let $\{k_\alpha\}$ be the partially ordered set of subfields of k which are finitely-generated extensions of k_f. Let $T_\alpha = T_f \otimes_{k_f} k_\alpha$, a finitely generated algebra over k_α and $\mathfrak{q}_\alpha := T_\alpha \mathfrak{p}$. Then $\mathfrak{q}_\alpha$ is prime and $R_\alpha := (T_\alpha)_{\mathfrak{q}_\alpha}$ is regular local. Since $k = \lim_{\rightarrow} k_\alpha$,

$$T = \lim_{\rightarrow} T_\alpha, \qquad R = \lim_{\rightarrow} R_\alpha$$

and the connecting maps $f_{\alpha\beta} : R_\alpha \to R_\beta$ are flat local homomorphisms. If C is a canonical module for R, then $C \cong R$, so there is obviously canonical module C_α for R_α such that $R \otimes_{R_\alpha} C_\alpha = C$.

4.10 Proposition. *Let* $p \geq 0$ *and let* $x \in W(\mathcal{S}_i^p(R);C)$*. Then there is an* α*, a homomorphism*

$$t_\alpha : W(\mathcal{S}_i^p(R_\alpha);C_\alpha) \to W(\mathcal{S}_i^p(R);C),$$

and an element $x_\alpha \in W(\mathcal{S}_i^p(R_\alpha);C_\alpha)$ *such that*

$$t_\alpha(x_\alpha) = x.$$

Furthermore, $R_\alpha = T_{\mathfrak{p}}$ *where* T *is a smooth, finite type* k_0*-algebra,* $\mathfrak{p} \in$ *Spec* T *and* k_0 *is the prime field.*

Proof: Let $x \in W(\mathcal{S}_i^p(R);C)$ be represented by $\phi : M \times M \to V^p(C)$. By [EGA, 5.13.7.1] there is an R_α-module M_α such that

$$R \otimes_{R_\alpha} M_\alpha \cong M.$$

Since $R_\alpha \to R$ is flat and local, and (clearly) $\dim_{R_\alpha} M_\alpha = \dim_R M$, [EGA, 6.3.3] shows

$$M_\alpha \in \mathcal{S}_i^p(R_\alpha).$$

To complete the lift of $\phi : M \times M \to V^p(C)$ back to R_α, we first show

4.11 Lemma. $R \otimes_{R_\alpha} Hom_{R_\alpha}(M_\alpha, V^p(C_\alpha)) \xrightarrow{\cong} Hom_R(M, V^p(C))$.

Proof: Since M_α is finitely-presented, it suffices to prove

$$R \otimes_{R_\alpha} V^p(C_\alpha) \xrightarrow{\cong} V^p(C). \tag{4.12}$$

First, $R \otimes E^p(C_\alpha) \xrightarrow{\cong} (\lim_\to R_\beta) \otimes E^p(C_\alpha) \xrightarrow{\cong} \lim_\to (R_\beta \otimes E^p(C_\alpha))$, where $\lim_\to$ is over all $\beta \geq \alpha$. We claim there is an isomorphism of R_β-complexes

$$R_\beta \otimes E^*(C_\alpha) \xrightarrow{\cong} E^*(C_\beta), \quad \beta \geq \alpha. \tag{4.13}$$

For this, recall that the separable extension k_β of k_α is a purely transcendental extension followed by a separable algebraic one. If the purely transcendental part is trivial, then (4.13) follows from [5, VI 5.3]. Moreover, the arguments of [HaRD, loc. cit.] plus [La, Cor. 2, p. 67] take care of the transcendental part of k_β/k_α.

Now in [Q, p. 123] it is shown that, given any prime $\mathfrak{p} \subset R$, there is α and a prime $\mathfrak{q} \subset R_\alpha$ such that the extension of $\mathfrak{q}$ in R is $\mathfrak{p}$. It follows easily from the arguments of [HaRD, VI.5.3] that there is an isomorphism

$$R \otimes_{R_\alpha} E_{C_\alpha}(R_\alpha/\mathfrak{q}) \cong E_R(C/\mathfrak{p}). \tag{4.14}$$

Since $E^p(C) = \coprod E(C/\mathfrak{p})$, where the sum is over all codimension p primes, (4.13) and (4.14) prove that

$$R \otimes_{R_\alpha} E^p(C_\alpha) \cong E^p(C). \tag{4.15}$$

This in turn proves (4.12).

Next, it is easy to modify the argument of [EGA, 5.13.7.1] to find $\phi_\alpha : M \times M \to V^p(C_\alpha)$ such that $R \otimes \phi_\alpha = \phi$. Since R is faithfully flat over R_α, ϕ_α is nonsingular.

It remains to prove the last statement. Simplifying notation (and hoping not to create notational confusion), we replace R_α, T_α and k_α by R, T and k in the rest of the proof. By construction, $R = T_\mathfrak{q}$ is regular local, where T is algebra-finitely generated over the finitely generated extension k of the prime field k_0. It follows that there is a finitely generated k_0-algebra B and a multiplicative subset $S \subset B$ such that $B_S = T$. Then $\mathfrak{q} = \mathfrak{r}_S$ for some prime $\mathfrak{r} \subseteq B$ and $B_\mathfrak{p} = T_\mathfrak{q} = R$. Thus, since k_0 is perfect, $B_\mathfrak{p}$ is smooth over k_0; further, there is $s \in B - \mathfrak{p}$ such that $B[\frac{1}{s}]$ is smooth. $B[\frac{1}{s}]$ is the desired k_0-algebra. The proof of (4.10) is complete.

Appendix: Using the Scheja-Storch trace

To illustrate the main ideas in the rather complicated proof of Theorem 4.1, we give a proof (which we believe is new) of a well-known theorem of Milnor. All of the essential points in the proof of Theorem 4.1 are present in the proof of Theorem A.3 below.

Let A be a Dedekind domain and K, its field of fractions. Recall the definition of the map (sum of "second residue homomorphisms")

$$\partial : W(K) \to \coprod_{\substack{\mathfrak{p} \text{ prime} \\ \mathfrak{p}\neq 0}} W(k(\mathfrak{p}))$$

where $k(\mathfrak{p}) := A_{\mathfrak{p}}/\mathfrak{p}A\mathfrak{p}$. For each nonzero prime ideal $\mathfrak{p} \subset A$, choose a generator ("uniformizer") π of $\mathfrak{p}A_{\mathfrak{p}}$. Now fix such a $\mathfrak{p}$. Given a nonsingular symmetric K-bilinear form $\Phi : V \times V \to K$, one may represent it in $W(K)$ by a diagonal matrix

$$< u_1\pi^{-1}, \dots, u_m\pi^{-1}, u_{m+1}, \dots, u_n >,$$

where $u_i \in A_{\mathfrak{p}}^{\times}$. Then

$$\partial_{\mathfrak{p}}[V, \Phi] := \begin{cases} [< \overline{u_1}, \dots, \overline{u_m} >], \ m \geq 1 \\ 0, \ m = 0 \end{cases}$$

where [-] denotes Witt class and $\overline{u_i} \in k(\mathfrak{p})$ is the class of u_i, and we set

$$\partial := \coprod \partial_{\mathfrak{p}} \tag{A.1}$$

It is straightforward to show this is well-defined and, in view of (0.9), agrees with ∂^0 in (0.8) (from the Introduction to this paper).

We must extend the definition of ∂ slightly in order (among other things) not to have to diagonalize (V, Φ). (This extension is already present in [Sch, p. 204] and is needed in much greater generality in [P3] and below.) Let $L \subset V$ be an *integral lattice* for Φ, a finitely-generated A-submodule such that $KL = V$ and $\Phi(L \times L) \subseteq A$. Let $L^* := \{v \in V | \Phi(L, v) \in A\}$, the *dual (to L) lattice*. Suppose that $\mathfrak{p}L_{\mathfrak{p}}^* \subseteq L_{\mathfrak{p}}$. Then $L_{\mathfrak{p}}^*/L_{\mathfrak{p}}$ is a finitely-generated $k(\mathfrak{p})$-vector space and the map

$$\phi(\mathfrak{p}) : L_{\mathfrak{p}}^*/L_{\mathfrak{p}} \times L_{\mathfrak{p}}^*/L_{\mathfrak{p}} \to \mathfrak{p}^{-1}A_{\mathfrak{p}}/A_{\mathfrak{p}}, \ \ (\overline{v_1}, \overline{v_2}) \mapsto \Phi(v_1, v_2) \pmod{A_{\mathfrak{p}}}$$

is a well-defined $k(\mathfrak{p})$-bilinear form. Now using the uniformizer π, one defines an isomorphism

$$\kappa(\pi) : \mathfrak{p}^{-1}A_{\mathfrak{p}}/A_{\mathfrak{p}} \xrightarrow{\cong} k(\mathfrak{p}), \ \ \overline{\pi^{-1}a} \mapsto \overline{a},$$

hence a $k(\mathfrak{p})$-bilinear form

$$\kappa(\pi) \circ \phi(\mathfrak{p}) : L_{\mathfrak{p}}^*/L_{\mathfrak{p}} \times L_{\mathfrak{p}}^*/L_{\mathfrak{p}} \to k(\mathfrak{p})$$

which is easily verified to agree with $\partial_{\mathfrak{p}}[V, \Phi]$ in $W(k(\mathfrak{p}))$. We call this the *lattice definition* of ∂. Finally, if we replace the assumption $\mathfrak{p}L_{\mathfrak{p}}^* \subseteq L_{\mathfrak{p}}$ by the stronger condition, $\mathfrak{p}L^* \subseteq L$, then $\partial_{\mathfrak{q}}[V, \Phi] = 0$ if $\mathfrak{q} \neq \mathfrak{p}$.

Next recall ([Sch, p.47]) that, given any nonsingular K-bilinear form (V, Φ) and any nonzero L-linear map $s : K \to L$, where K/L is a finite field extension, there is a nonsingular L-bilinear form $s_*(V, \Phi) := (V, s \circ \Phi)$.

Let $E = F[X]/(p(X))$ be a simple extension of fields, and let $x \in E$ be the class of X. Suppose $\deg p = d$. Let $s_p : E \to F$ be the F-linear ("trace") map such that $s_p(x^{d-1}) = 1$ and $s_p(x^i) = 0$, $i = 0, \dots, d-2$. (This is the trace used in Scharlau's reciprocity theorem for $\mathbb{P}^1$ [Sch, p.215].) Then Scheja and Storch show (see (2.8) below) that if $s_p[X] : E[X] \to F[X]$ is the $F[X]$-linear extension of s_p fixing X, and $p(X) = (X - x)r(X)$ in $E[X]$, then for any $e \in E$,

$$e \equiv s_p[X](r(X)e) \pmod{p(X)} \tag{A.2}$$

We can now give a simple proof of the following theorem of Milnor.

A.3 Theorem. *Let F be a field of characteristic $\neq 2$. Then*

$$\partial : W(F(X)) \to \coprod_{\substack{p(X) \text{ irreducible,} \\ \text{monic}}} W(F[X]/(p(X))$$

is surjective. Indeed, if $[< e >] \in W(F[X]/(p(x)))$ denotes the class of the unary form $< e >$, then

$$\partial\,[s_p(X)_* < e(X - x) >] = [< e >]$$

where $E(X)$ is regarded as an $F(X)$ vector space, and $s_p(X) : E(X) \to F(X)$ is the $F(X)$-linear extension of s_p fixing X.

Proof: Clearly $L := E[X] \subset E(X)$ is an integral lattice for the unary form $< e(X-x) >$ over $E(X)$, and $L^* := (X - x)^{-1}E[X]$ is its dual lattice. Evidently, the $F[X]$-lattice dual to L with respect to $s_p(X)_* < e(X - x) >$ is contained in L^*. It turns out that the dual lattices are identical. (This follows from the facts that $s_p[X]$ induces an isomorphism

$$\operatorname{Hom}_{E[X]}(E[X], E[X]) \xrightarrow{\cong} \operatorname{Hom}_{F[X]}(E[X], F[X]), \tag{A.4}$$

and the respective dual lattices can be identified with the left and right sides.)

Now there are canonical isomorphisms of $F[X]/(p(X))$-vector spaces,

$$L^*/L \cong (X - x)^{-1}E[X]/E[X] \cong E[X]/(X - x) \cong F[X]/(p(X)).$$

Hence, according to the lattice definition of ∂, $\partial\,[s_p(X)_* < e(X - x) >]$ is represented by the composition

$$\begin{aligned} F[X]/(p(X)) \times F[X]/(p(X)) &\xrightarrow{\cong} (X - x)^{-1}E[X]/E[X] \times (X - x)^{-1}E[X]/E[X] \\ &\xrightarrow{\overline{<e(X-x)>}} (X - x)^{-1}E[X]/E[X] \xrightarrow{\overline{s_p(X)}} p(X)^{-1}F[X]/F[X] \xrightarrow{\kappa(p)} F[X]/(p(X)). \end{aligned} \tag{A.5}$$

Notice that the $F(X)$-linearity of $s_p(X)$ implies that $s_p(X)\big((X-x)^{-1}E[X]\big) \subseteq p(X)^{-1}F[X]$, so the map $\overline{s_p(X)}$ in the composition (A.5) makes sense: for any $g(x) \in E[X]$,

$$s_p(X)\big\{(X - x)^{-1}g(X)\big\} = s_p(X)\big\{p(X)^{-1}r(X)g(X)\big\} = p(X)^{-1}s_p(X)\big\{r(X)g(X)\big\}.$$

Moreover, in the composition (A.5) we have

$$\begin{aligned} (\bar{1}, \bar{1}) \mapsto (\overline{(X - x)^{-1}}, \overline{(X - x)^{-1}}) \mapsto (X - x)^{-2}e(X - x)\overline{s_p(X)((X - x)^{-1}e)} \\ = \overline{p(X)^{-1}s_p(X)(r(X)e)} \mapsto \overline{s_p(X)(r(X)e)} = e \end{aligned}$$

where the last equality is (A.2), and so the proof is complete.

§5. Filtered exactness of the Gersten-Witt complex

5.1 Theorem Let R be a regular local ring, essentially of finite type over a field k of characteristic $\neq 2$, and let C be a canonical module for R. Then the exact sequence of (3.23)

$$0 \to W(\mathcal{S}_1^p(R);C) \xrightarrow{\mathcal{K}^p} W(\mathcal{F}^p(R;C) \xrightarrow{\mathcal{L}^p} W(\mathcal{S}_1^{p+1}(R);C) \to 0$$

is short exact for each $p \geq 0$.

Proof: According to (3.9) and (3.22), it remains to prove $\mathcal{L}^p$ is surjective. If R is, in addition, smooth over k, this is Corollary (4.7). In general, given $x \in W(\mathcal{S}_1^{p+1}(R);C)$, there is by (4.10) a smooth $\tilde{R}$, a flat local homomorphism $f : \tilde{R} \to R$, and element $y \in W(\mathcal{S}_1^{p+1}(\tilde{R});\tilde{C})$ such that $f_* y = x$. It is readily verified that there is a commutative diagram (cf. (4.13)-(4.15))

$$\begin{array}{ccc} W(\mathcal{F}^p(\tilde{R});\tilde{C}) & \xrightarrow{\mathcal{L}^p(\tilde{R})} & W(\mathcal{S}_1^{p+1}(\tilde{R});\tilde{C}) \\ \downarrow & & \downarrow f_* \\ W(\mathcal{F}^p(R);C) & \xrightarrow{\mathcal{L}^p(R)} & W(\mathcal{S}_1^{p+1}(R);C) \end{array}$$

from which the surjectivity of $\mathcal{L}^p(R)$ follows.

We can now prove the local form of the first main theorem of this paper.

5.2 Theorem. *Let $(R,\mathfrak{m})$ be regular local ring, essentially of finite type over a field of characteristic $\neq 2$ and with fraction field K. Let C be a canonical module for R. Then the Gersten-Witt complex $W^\cdot(R;C)$ of (R,C)*

$$W(K;K\otimes C) \xrightarrow{\partial^0} \cdots \xrightarrow{\partial^{p-1}} \coprod_{\text{cod }\mathfrak{p}=p} W(k(\mathfrak{p});\Lambda^p N(\mathfrak{p})\otimes C) \xrightarrow{\partial^p} \cdots \xrightarrow{\partial^{n-1}} W(k(\mathfrak{m});\Lambda^n N(\mathfrak{m})\otimes C) \to 0$$

is exact and $\ker \partial^0 = W(\mathcal{S}_1^0(R);C)$.

Proof: Concatenate the sequences of (5.1) for all p, and use (3.28).

More generally, the following filtered version is also exact, as follows immediately from (3.37) and (4.9):

5.3 Theorem. *With the same hypotheses as above, the filtered Gersten-Witt complex $I^{\cdot,\ell}(R;C)$*

$$I^\ell(K;K\otimes C) \xrightarrow{\partial^0} \cdots \xrightarrow{\partial^{p-1}} \coprod_{\text{cod }\mathfrak{p}=p} I^{\ell-p}(k(\mathfrak{p});\Lambda^p N(\mathfrak{p})\otimes C) \xrightarrow{\partial^p} \cdots \xrightarrow{\partial^{n-1}} I^{\ell-n}(k(\mathfrak{m});\Lambda^n N(\mathfrak{m})\otimes C) \to 0$$

is exact for each $\ell \geq 0$, where $I^j(k(\mathfrak{p});\Lambda^p N(\mathfrak{p})\otimes C) = W(k(\mathfrak{p});\Lambda^p N(\mathfrak{p})\otimes C)$ if $j < 0$.

Recall from (3.26) and (3.37) the complexes of sheaves $\mathbb{W}^\cdot(X;\mathcal{C})$ and $\mathbb{I}^{\cdot,\ell}(X;\mathcal{C})$. The space of global sections of $\ker \partial^0$ in $\mathbb{W}^\cdot(X;\mathcal{C})$ (resp. of $\ker \partial^{0,\ell}$ in $\mathbb{I}^{\cdot,\ell}(X;\mathcal{C})$) is the much-studied *unramified Witt group* $W_{\text{nr}}(X)$ of X (resp., the fundamental filtration of the unramified Witt group). Indeed, if $x \in X$, let $\mathfrak{m}_x$ denote maximal ideal of $\mathcal{O}_{X,x}$, $N(x) =$

$\operatorname{Hom}(\mathfrak{m}_x/\mathfrak{m}_x^2; k(x))$ and let ξ denote the generic point of X. Then $\mathcal{W}_{\mathrm{nr}}(X;\mathcal{C})$ is the sheaf of abelian groups on X defined, for $U \subseteq X$ open, by

$$\mathcal{W}_{\mathrm{nr}}(X;\mathcal{C})(U) := \ker\bigl\{W(k(\xi);\mathcal{C}_\xi) \xrightarrow{\partial^0} \coprod_{x\in U^{(1)}} W(k(x);N(x)\otimes\mathcal{C}_x)\bigr\}$$

When $\mathcal{C} = \mathcal{O}_X$ and $U = \operatorname{Spec} R$, then this is the unramified Witt group $W_{\mathrm{nr}}(R)$ of R, and, as the proof of (3.9) shows, is canonically isomorphic to $W(\mathcal{S}_1^0(R);C)$, where $C = \mathcal{C}(U)$. More generally, set

$$\mathcal{I}^\ell_{\mathrm{nr}}(X;\mathcal{C})(U) := \ker\bigl\{I^\ell(k(\xi);\mathcal{C}_\xi) \xrightarrow{\partial^{0,\ell}} \coprod_{x\in U^{(1)}} I^{\ell-1}(k(x);N(x)\otimes\mathcal{C}_x)\bigr\}$$

and $I^\ell_{\mathrm{nr}}(R) = \mathcal{I}^\ell_{\mathrm{nr}}(X;\mathcal{C})(U)$ when $U = \operatorname{Spec} R$.

The following global version of Theorem 5.3 is now immediate from the definitions:

5.4 Theorem. *Let X be a regular scheme of finite type over a field of characteristic $\neq 2$. Let $\mathcal{C}$ be a rank one locally free sheaf on X. Then the complexes of sheaves $\mathbb{I}^{\cdot,\ell}(X;\mathcal{C})$, $\ell \geq 0$,*

$$\mathbb{I}^{\cdot,\ell}(X;\mathcal{C}) = \left\{\mathcal{I}^\ell_{\mathrm{nr}}(X;\mathcal{C}) \xrightarrow{\partial^0} i_{\xi*}I^\ell(K;\mathcal{C}_\xi) \xrightarrow{\partial^1} \dots \xrightarrow{\partial^{\ell-n}} \coprod_{x\in X^{(n)}} i_{x*}I^{\ell-n}(k(x);N(x)\otimes\mathcal{C}_x) \to 0\right\}$$

are exact.

Notice that for each $\ell \geq 0$, the associated graded complex $\mathbb{I}^{\cdot,\ell}(X;\mathcal{C})/\mathbb{I}^{\cdot,\ell+1}(X;\mathcal{C})$ is a flasque resolution of $\mathcal{I}^\ell_{\mathrm{nr}}(X;\mathcal{C})/\mathcal{I}^{\ell+1}_{\mathrm{nr}}(X;\mathcal{C})$; moreover, the terms

$$\coprod_{x\in X^{(p)}} i_{x*}\bigl\{I^{\ell-p}(k(x);\Lambda^pN(\mathfrak{m}_x)\otimes\mathcal{C}_x)/I^{\ell-p+1}(k(x);\Lambda^pN(\mathfrak{m}_x)\otimes\mathcal{C}_x)\bigr\}$$

in $\mathbb{I}^{\cdot,\ell}(X;\mathcal{C})/\mathbb{I}^{\cdot,\ell+1}(X;\mathcal{C})$ are canonically isomorphic to the terms

$$\coprod_{x\in X^{(p)}} i_{x*}H^{\ell-p}_{et}(k(x),\mu_2)$$

in the Bloch-Ogus resolution of the sheaf $\mathcal{H}^\ell(X,\mu_2)$. This is because the calculation (3.29) shows that any isomorphism $c : k(x) \to \Lambda^pN(\mathfrak{m}_x)\otimes\mathcal{C}_x$ induces a canonical isomorphism

$$I^{\ell-p}(k(x))/I^{\ell-p+1}(k(x)) \xrightarrow{\cong} I^{\ell-p}(k(x);\Lambda^pN(\mathfrak{m}_x)\otimes\mathcal{C}_x)/I^{\ell-p+1}(k(x);\Lambda^pN(\mathfrak{m}_x)\otimes\mathcal{C}_x)$$

and Voevodsky has shown that the left side is canonically isomorphic to $H^{\ell-p}_{et}(k(x),\mu_2)$. Hence for each $\ell \geq 0$ we have a flasque resolution $\mathbb{H}^{\ell,\cdot}(X;\mu_2)$ of $\mathcal{I}^\ell_{\mathrm{nr}}(X;\mathcal{C})/\mathcal{I}^{\ell+1}_{\mathrm{nr}}(X;\mathcal{C})$, where

$$\mathbb{H}^{\ell,p} := \coprod_{x\in X^{(p)}} i_{x*}H^{\ell-p}_{et}(k(x);\mu_2).$$

It is now immediate from [PaS, 1.4] that there is a canonical isomorphism

$$\mathcal{I}^{\ell}_{\rm nr}(X;\mathcal{C})/\mathcal{I}^{\ell+1}_{\rm nr}(X;\mathcal{C}) \xrightarrow{\cong} \mathcal{H}^{\ell}(X,\mu_2)$$

so that the the associated graded quotients

$$\mathbb{H}^{\ell,\cdot}(X;\mu_2) = \mathbb{I}^{\cdot,\ell}(X;\mathcal{C})/\mathbb{I}^{\cdot,\ell+1}(X;\mathcal{C})$$

are flasque resolutions of $\mathcal{H}^{\ell}(X,\mu_2)$. In fact, this is the Bloch-Ogus resolution, but we do not need this fact.

We now invoke the standard construction of the spectral sequence of a filtered complex to get:

5.5 Theorem. *There is a spectral sequence* $\{E_r^{s,t}\}$ *converging to (the associated graded groups of a filtration of)* $H^*(X;\mathcal{W}(X;\mathcal{C}))$ *such that*

a. $\deg d_r = (r, 1-r)$,

b. $E_1^{s,t} = H^{s+t}(X;\mathcal{H}^s_{ét})$,

c. $E_1^{s,t} = 0$ *unless* $t \leq 0$ *and* $s+t \geq 0$,

d. $E_1^{s,0} = A^s(X)\otimes \mathbb{Z}/2\mathbb{Z}$ *and*

e.

$$E_\infty^{s,t} = \frac{\mathrm{im}\big[H^{s+t}(X;\mathcal{I}^s(X;\mathcal{C})) \to H^{s+t}(X;\mathcal{W}(X;\mathcal{C}))\big]}{\mathrm{im}\big[H^{s+t}(X;\mathcal{I}^{s+1}(X;\mathcal{C})) \to H^{s+t}(X;\mathcal{W}(X;\mathcal{C}))\big]}.$$

References

[AK] A. Altman and S. Kleiman, *Introduction to Grothendieck Duality Theory*, Springer LNM **146** (1970).

[Ar] J. Kr. Arason, *Cohomologische Invarianten quadratischer Formen*, J. Algebra **36** (1975), 448-491.

[B] H. Bass, *On the ubiquity of Gorenstein rings*, Math. Zeitschrift **82** (1963), 8-28.

[BK] H. Bass, *Algebraic K-Theory*, W. A. Benjamin, 1968.

[Ba] P. Balmer, *Derived Witt groups of a scheme*, J. Pure App. Alg. **141** (1999), 101-129.

[BaW] P. Balmer and C. Walter, *A Gersten-Witt spectral sequence for regular schemes*, `http://www.math.uiuc.edu/K-theory/0388/`.

[Bar] J. Barge, *Formes Hermitiennes sur les* $\mathbb{Z}[X,X^{-1}]$*-modules*, Bull. Soc. Math. France **48** (1976), 7-10.

[BH] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge U. Press, 1993.

[BOj] J. Barge and M. Ojanguren, *Fibrés algébriques sur une surface réelle*, Comm. Math. Helv. **62** (1987), 616-629.

[BO] S. Bloch and A. Ogus, *Gersten's conjecture and the homology of schemes*, Ann. Sci. ENS **7** (1974), 181-201.

[Ca] F. Catanese, *Homological algebra and algebraic surfaces*, Proc. Symp. Pure Math. **62** (1997), 3-56.

[CC] G. Casnati and F. Catanese, *Even sets of nodes are bundle symmetric*, J. Diff. Geom. **47** (1997), 237-256.

[CTS] J.-L. Colliot-Thélène and J.-J. Sansuc, *Fibrés quadratiques et composantes réeles connexes*, Math. Ann. **244** (1979), 105-134.

[CRW] T. Craven, A. Rosenberg and R. Ware, *The map of the Witt ring of a domain to the Witt ring of its field of fractions*, Proc. AMS **51** (1975), 25-30.

[E] D. Eisenbud, *Commutative Algebra*, vol. Graduate Texts in Math., Springer Verlag, 1994.

[EGA] A. Grothendieck, *Éléments de Géometrie Algébriques*, Publ. Math. IHES **16, 20, 24** (1965), 5–231.

[EPW1] D. Eisenbud, S. Popescu and C. Walter, *Lagrangian subbundles and codimension 3 subcanonical subschemes*, preprint.

[EPW2] D. Eisenbud, S. Popescu and C. Walter, *Enriques surfaces and other non-Pfaffian subcanonical subschemes of codimension 3*, preprint.

[F] D. Ferrand, *Construction des fibrés de rang deux*, Astérisque **36**, 149-169.

[FFC] F. Fernández-Carmena, *On the injectivity of the map of the Witt group of a scheme into the Witt group of its function field*, Math. Ann. **277** (1987), 453-468.

[HaRD] R. Hartshorne, *Residues and Duality*, Springer Lecture Notes in Math. **20**.

[Ho] M. Hochster, *Cohen-Macaulay modules*, Springer LNM **311**, 120-153.

[Ka] I. Kaplansky, *Commutative Rings*, U. of Chicago Press (1974).

[K] M. Karoubi, *Localisation de formes quadratiques II*, Ann. scient.Èc. Norm. Sup. **8** (1975), 99-155.

[Kn1] M. Knebusch, *Grothendieck und Wittringe von nicht ausgearteten symmetrischen bilinearformen*, Sitzungbericht Heidelberger Akad. Wiss **3** (1969/70), 89-157.

[Kn2] M. Knebusch, *Symmetric bilinear over algebraic varieties*, Queens Papers in Pure and Applied Mathematics **46** (1977), 103-284.

[Ku] W. Kucharz, *Symmetric bilinear forms on real affine surfaces*, Glasgow Math. J. **31** (1989), 195-198.

[La] S. Lang, *Introduction to Algebraic Geometry*, Interscience (1958).

[Oj1] M. Ojanguren, *Quadratic forms over regular rings*, J. Ind. Math. Soc. **44** (1980), 109-116.

[Oj2] M. Ojanguren, *A splitting theorem for quadratic forms*, Comm. Math. Helv. **57** (1982), 145-157.

[OjP] M. Ojanguren and I. Panin, *A purity theorem for the Witt group*, Ann. Scient. Ëc. Norm. Sup. **32** (1999), 71-86.

[OPS] M. Ojanguren, R. Parimala and R. Sridharan, *Symplectic bundles over affine surfaces*, Comm. Math. Helv. **61** (1986), 491-500.

[PaS] R. Parimala and R. Sridharan, *Graded Witt ring and unramified cohomology*, K-Theory **6** (1992), 29-44.

[P1] W. Pardon, *The exact sequence of a localization for Witt groups*, SLN **551** (1976), 336-379.

[P2] W. Pardon, *A "Gersten Conjecture" for Witt groups*, SLN **967** (1982), 300-315.

[P3] W. Pardon, *A relation between Witt groups and zero-cycles in a regular ring*, SLN **1046** (1984), 261-328.

[P4] W. Pardon, *A relation between the Witt group of a regular local ring and the Witt groups of its residue class fields*, preprint (1984).

[Q] D. Quillen, *Higher algebraic K-theory I*, SLN **341** (1973), 85-147.

[Sc] M. Schmid, *Wittringhomologie*, Dissertation, Uni. Regensburg (1998).

[Sch] W. Scharlau, *Quadratic and Hermitian Forms*, Grundlehren der Math. Wissenschaften (1985), Springer Verlag.

[Se] J.-P. Serre, *Algèbra Locale, Multiplicités*, Springer Lecture Notes in Math. **11** (1957).

[SS] G. Scheja and U. Storch, *Über Spurfunktionen bei vollständigen Durchschnitten*, J. Reine Angew. Math. **278/279** (1975), 174-190.

[SV] D. W. Sharpe and P. Vamós, *Injective Modules* (1972), Cambridge U. Press.

[W] C. Walter, *Pfaffian subschemes*, J. Alg. Geom. **5** (1996), 671-704.

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