Δ-closed classes

Vladimir Voevodsky[1,2]

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1 Introduction

In recent years there appeared several constructions which produce homotopy categories for objects of different types by taking some kind of a "free" homotopy category (e.g. a homotopy category of presheaves) and localizing it with respect to a class of "new" weak equivalences which reflects the specifics

[1]Supported by the NSF grants DMS-97-29992 and DMS-9901219, Sloan Research Fellowship and Veblen Fund

[2]School of Mathematics, Institute for Advanced Study, Princeton NJ, USA. e-mail: vladimir@ias.edu

of the situation. The class of new weak equivalences is usually obtained from a smaller class of "generating" weak equivalences by a saturation procedure of some sort. I am aware of only two basic types of saturation processes. One, which applies only in the additive case, can be traced back to the notion of a thick subcategory introduced by Verdier. It comprises in taking all morphisms whose cones belong to the smallest subcategory which is closed under some elementary operations (e.g. cones and direct summands)and contains cones of the generating weak equivalences. The other one, which goes back to Bousfield, can be used in a more general setting. Its main idea is to use the generating weak equivalences to define a class of local objects and then define new weak equivalences as morphisms which define bijections on Hom-sets with values in these local objects. A systematic study of the second approach in the context of model categories can be found in [6]. Both of these approaches can be used in the context of compactly generated triangulated categories. If the domains and codomains of generating weak equivalences are compact they are known to be equivalent which is one of the main reasons why the localization theory of compactly generated triangulated categories is such a convenient tool.

The main goal of this paper is to develop an analog of the first approach to saturation in the non-additive context. The main advantage of the Verdier saturation for triangulated categories is its simple behavior with respect to functors. Any triangulated functor takes the Verdier saturation of a class of morphisms to the Verdier saturation of the image of this class. In the non-additive case one would like to preserve this property for as wide a class of functors as possible. In particular, examples from the motivic homotopy theory show that it is important to be able to compute the image of the saturation with respect to functors which do not commute with finite coproducts (e.g. the symmetric product functor). On the other hand, since the Bousfield approach to saturation is known to work very well in many important cases, one wants to have a general enough equivalence theorem asserting that the two saturation processes lead to the same result.

In the additive homotopy theory (i.e. homological algebra) one has a choice to work with either triangulated categories or complexes over additive categories. The former context is known (?) to be somewhat more general because of such examples as the stable homotopy category. In my opinion, we do not have any good context for doing general homotopy theory. In particular, I find the formalism of closed model categories to be both too restrictive and not reflecting the structures and properties relevant for concrete computations. Thus, being unable to find a non additive analog of triangulated categories, I tried in this paper to find a non additive analog of complexes over additive categories and work with it. While closed model structures are used in this paper as a tool they do not show up much in either definitions or main results.

Ideally, we would like to consider any category C with finite coproducts instead of an additive category and simplicial objects in C instead of complexes. Unfortunately, our methods require not only finite but also infinite coproducts. As a result, we start with a small category C with finite coproducts, embed it into a large category $[C]$ which has all coproducts such that objects of C form a set of compact generators of $[C]$, and then proceed to do homotopy theory in the category of simplicial objects in $[C]$.

We say that a class of morphisms E in $\Delta^{op}[C]$ is Δ-closed if it contains homotopy equivalences, has the "2 out of 3 property" and the diagonal of a morphism of bisimplicial objects whose rows (or columns) are in E is in E. We say that a morphism is a coprojection if it is of the form $X \to X \coprod Y$. Note that a colimit of a sequence of coprojections exists in any category with countable coproducts. We say that a class of morphisms is $\bar{\Delta}$-closed if it is Δ-closed, closed under coproducts and under sequential colimits of term-wise coprojections. In the first section of the paper we study elementary properties of Δ- and $\bar{\Delta}$-closed classes. Since the existence of compact generators plays here no role we do it not for categories of the form $[C]$ but for all categories with coproducts.

Any small additive category A can be embedded into the abelian category $R(A)$ of contravariant additive functors from it to the category of abelian groups. Similarly, any category with finite coproducts can be embedded into the category of functors from C to $Sets$ which take finite coproducts to products. We call such functors radditive. At the beginning of the second section we prove basic properties of the category of radditive functors $R(C)$. In particular we show that it has all limits and colimits. Besides the example of additive functors on an additive category the examples of categories of radditive functors include categories of presheaves on a small category, categories of pointed presheaves and categories of (commutative, associative) algebras over a ring. The category of simplicial radditive functors can be given a finitely generated simplicial closed model structure which we call the projective closed model structure. The corresponding homotopy category $H(C)$ is called the homotopy category of (simplicial objects in) C.

Having a simplicial closed model structure on $\Delta^{op}R(C)$ we can define, following [6], for any class of morphisms E, the notion of E-local object and the notion of E-local equivalence. The class of E-local equivalences, which always contains E, is the Bousfield saturation of E in our setting. If the domains and codomains of elements of E are cofibrant and reliably compact, the class of E-local equivalences is contained in the $\bar{\Delta}$-closure of the union of E with the class of projective weak equivalences. It is clear that the opposite inclusion does not hold in general even for $E = \emptyset$ because the class of projective weak equivalences of radditive functors need not be closed under finite coproducts. In Section 4 we establish a set of sufficient conditions on C which implies that for any E the class of E-local equivalences is $\bar{\Delta}$-closed. For such categories C and sets of morphisms E whose domains and codomains are cofibrant and reliably compact we conclude that the class of E-local equivalences coincides with the $\bar{\Delta}$-closure of the union of E with the class of projective weak equivalences (see Theorem 4.3.7). This is the first form of our result on the equivalence of two types of saturation processes.

Our main tool is the wrapping functor which takes any simplicial object X to a degeneracy free object $Wr(X)$ together with a natural morphism $Wr(X) \to X$. We show that under some conditions on C the morphisms $Wr(X) \to X$ are projective weak equivalences which implies that the classes of E-local equivalences are Δ-closed. These conditions can be easily checked in the three standard cases of an additive C, C having free coproducts and C having free pointed coproducts.

Since representable functors on an additive category A are projective objects of $R(A)$ and any functor can be covered by a direct sum of representable ones the derived category of $R(A)$ is equivalent to the homotopy category of complexes over $[A]$. We proof a non additive analog of this fact. Using a construction of a simplicial resolution of a radditive functor by objects of $[C]$ we show that the functor $\Delta^{op}[C] \to H(C)$ is the localization with respect to the smallest $\bar{\Delta}$-closed class $cl_{\bar{\Delta}}(\emptyset)$ in $\Delta^{op}[C]$. Combining this result with the results of Section 4 we show that, under appropriate assumptions on C and E, the localization $H(C, E)$ of $H(C)$ with respect to E-local equivalences is also a localization of $\Delta^{op}[C]$ with respect to the $\bar{\Delta}$-closure of E. This is the second form of our result on the equivalence of two types of saturation processes (see Corollary 4.3.8).

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of places in the previous version of the paper which required corrections.

2 Elementary properties of Δ-closed classes

2.1 Basic definitions and elementary lemmas

Let C be a category and $\Delta^{op}C$ the category of simplicial objects over C. Following [11] define a unit homotopy from a morphism $f : A \to B$ to a morphism $g : A \to B$ in $\Delta^{op}C$ as a collection of morphisms $h_i^n : A_n \to B_n$ where $n \geq 0$ and $i = -1, \ldots, n$ satisfying the following conditions:

1. $h_{-1}^n = f_n$, $h_n^n = g_n$ where f_n and g_n are the components of f and g
2. $\partial_i h_j = h_{j-1}\partial_i$ if $i \leq j$, $\partial_i h_j = h_j \partial_i$ if $i > j$
3. $s_i h_j = h_{j+1} s_i$ if $i \leq j$, $s_i h_j = h_j s_i$ if $i > j$.

If C has finite coproducts, K is a finite set and X an object of C we denote by $X \otimes K$ the object $\coprod_K X$. Similarly for a finite simplicial set K and an object X of $\Delta^{op}C$ we denote by $X \otimes K$ the simplicial object with terms $X_n \otimes K_n$. One verifies easily (see [11, Prop. 2.1]) that a homotopy from f to g in the sense of the definition given above is the same as a morphism $h : A \otimes \Delta^1 \to B$ such that $h \circ (Id \otimes \partial_0) = f$ and $h \circ (Id \otimes \partial_1) = g$.

Two morphisms are called homotopic if they can be connected by a chain of unit homotopies (going in either direction). A morphism $f : A \to B$ in $\Delta^{op}C$ is called a homotopy equivalence if there exist a morphism $g : B \to A$ such that the compositions gf and fg are homotopic to the corresponding identity morphisms.

Definition 2.1.1 *Let $\mathcal{C}$ be a category. A class E of morphisms in $\Delta^{op}\mathcal{C}$ is called Δ-closed if the following conditions hold.*

1. *All homotopy equivalences are in E.*
2. *If f and g are morphisms such that the composition gf is defined and two out of three morphisms f, g, gf are in E then the third is in E.*
3. *If $f : B \to B'$ is a morphism of bisimplicial objects over C such that the rows or columns of f are in E then the diagonal morphism $\Delta(f)$ is in E*

If C has finite (resp. all) coproducts we say that a class is $(\Delta, \coprod_{<\infty})$-closed (resp. $(\Delta, \coprod)$-closed) if it is Δ-closed an closed under finite (resp. all) coproducts. We denote the smallest Δ-closed class containing a class E by $cl_\Delta(E)$ and use a similar notation for the $(\Delta, \coprod)$-closed classes.

Definition 2.1.2 *A morphism $e : A \to X$ in a category C is called a coprojection if there exists a morphism $f : Y \to X$ such that $f \coprod e$ is an isomorphism. A morphism $f : A \to X$ in $\Delta^{op}C$ is called a term-wise coprojection if for any $i \geq 0$ the morphism $f_i : A_i \to X_i$ is a coprojection.*

Proposition 2.1.3 *Let C be a category with finite coproducts and an initial object and E a class of morphisms in $\Delta^{op}C$. Denote by $E \coprod Id$ the class of morphisms of the form $f \coprod Id_Y$ for $Y \in ob(C)$. Then one has*

$$cl_{\Delta,\coprod_{<\infty}}(E) = cl_\Delta(E \coprod Id).$$

Proof: Since identities belong to any Δ-closed class the right hand side is contained in the left hand side. On the other hand E is contained in the right hand side and the right hand side is Δ-closed. It remains to show that the right hand side is closed under finite coproducts. For a pair of morphisms $f : X \to X'$, $g : Y \to Y'$ we have

$$f \coprod g = (Id_{X'} \coprod g) \circ (f \coprod Id_Y)$$

and therefore it is sufficient to check that for a morphism f in $cl_\Delta(E \coprod Id)$ and an object X in $\Delta^{op}C$ one has $f \coprod Id_X \in cl_\Delta(E \coprod Id)$. Consider the class H of morphisms f in $cl_\Delta(E \coprod Id)$ for which this holds for any X in $\Delta^{op}C$. For a simplicial object X with terms $X_i \in C$ we can write $f \coprod Id_X$ as the diagonal of a morphism of bisimplicial objects whose rows are of the form $f \coprod Id_{X_i}$. This implies that $E \coprod Id_X \subset H$. On the other hand it is easy to see that H is Δ-closed. Thus $H = cl_\Delta(E \coprod Id)$.

For any morphism $f : B \to A$ and any object C the square

$$\begin{array}{ccc} B & \xrightarrow{e_B} & B \coprod C \\ \downarrow & & \downarrow \\ A & \xrightarrow{e_A} & A \coprod C \end{array} \qquad (1)$$

is a push-forward square. This shows that in a category with finite coproducts there exist push-forwards for all pairs of morphisms (e, f) such that e is a coprojection. Therefore the same is true for pairs of morphisms (e, f) in $\Delta^{op}C$ such that e is a term-wise coprojection.

Definition 2.1.4 *A square is called an elementary exact square if it is isomorphic to the push-forward square for a pair of morphisms* (e, f) *where* e *is a term-wise coprojection.*

For any commutative square Q of the form

$$\begin{array}{ccc} B & \xrightarrow{e_B} & Y \\ u\downarrow & & \downarrow v \\ A & \xrightarrow{e_A} & X \end{array} \tag{2}$$

denote by K_Q the object defined by the elementary exact square

$$\begin{array}{ccc} B\coprod B & \longrightarrow & B\otimes\Delta^1 \\ \downarrow & & \downarrow \\ A\coprod Y & \longrightarrow & K_Q \end{array} \tag{3}$$

and by $p_Q : K_Q \to X$ the obvious morphism. For a morphism $f : X \to X'$ the object K_{Q_f} defined by the square

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & X' \end{array} \tag{4}$$

is called the cylinder of f and denoted by $cyl(f)$.

Lemma 2.1.5 *The morphisms* $X' \to cyl(f)$ *and* $cyl(f) \to X'$ *are mutually inverse homotopy equivalences.*

Proof: The object $cyl(f)$ is defined by the elementary exact square

$$\begin{array}{ccc} X & \xrightarrow{Id\otimes\partial_0} & X\otimes\Delta^1 \\ \downarrow & & \downarrow \\ X' & \longrightarrow & cyl(f) \end{array} \tag{5}$$

The composition $X' \to cyl(f) \to X'$ is the identity. The homotopy from the identity of $cyl(f)$ to the composition $cyl(f) \to X' \to cyl(f)$ is given by the morphism $cyl(f)\otimes\Delta^1 \to cyl(f)$ which equals to the projection $X'\otimes\Delta^1 \to X'$ on $X' \otimes \Delta^1$ and to the morphism $Id \otimes h$ on

$$(X' \otimes \Delta^1) \otimes \Delta^1 = X' \otimes (\Delta^1 \times \Delta^1)$$

where $h : \Delta^1 \times \Delta^1 \to \Delta^1$ is the usual homotopy from the identity to the composition $\Delta^1 \to \Delta^0 \stackrel{\partial_0}{\to} \Delta^1$.

Lemma 2.1.6 *For an elementary exact square Q of the form (2) the morphism $p_Q : K_Q \to X$ belongs to $cl_\Delta(\emptyset)$.*

Proof: The object K_Q is the diagonal of the bisimplicial object whose rows are K_{Q_i} where Q_i is the square formed by the i-th terms of A, B and Y and p_Q is the diagonal of the morphism whose terms are $p_{Q_i} : K_{Q_i} \to X_i$. Therefore, it is sufficient to prove the statement of the lemma for a square in C of the form (1). Since $K_{Q_1 \coprod Q_2} = K_{Q_1} \coprod K_{Q_2}$ and a square of the form (1) is a coproduct of a square of the form (4) and a transpose of such a square our result follows from Lemma 2.1.5.

The object K_Q is isomorphic to the diagonal of a bisimplicial object whose rows are of the form $A \coprod Y \coprod (\coprod_n B)$. This isomorphism is natural in Q which immediately implies the following result.

Lemma 2.1.7 *Let $f = (f_A, f_B, f_Y, f_X) : Q \to Q'$ be a morphism of commutative squares of the form (2) then one has*

$$(K(f) : K_Q \to K_{Q'}) \in cl_{\Delta,\coprod_{<\infty}}(\{f_A, f_B, f_Y\})$$

Lemma 2.1.8 *Let C be a category with finite coproducts. Then for any elementary exact square in $\Delta^{op}C$ of the form (2) one has*

$$e_A \in cl_{\Delta,\coprod_{<\infty}}(\{e_B\})$$

$$v \in cl_{\Delta,\coprod_{<\infty}}(\{u\})$$

Proof: To prove the first inclusion consider the morphism of squares of the form

$$\begin{pmatrix} B & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \end{pmatrix} \longrightarrow \begin{pmatrix} B & \longrightarrow & Y \\ \downarrow & & \downarrow \\ A & \longrightarrow & X \end{pmatrix} \qquad (6)$$

The morphism $e_A : A \to X$ can be represented as the composition

$$A \to K_{Q_0} \to K_Q \to X$$

The first and the third arrows are in $cl_{\Delta,\coprod_{<\infty}}(\emptyset)$ by Lemma 2.1.6. The second one is in $cl_{\Delta,\coprod_{<\infty}}(e_B)$ by Lemma 2.1.7. We conclude that $e_A \in cl_{\Delta,\coprod_{<\infty}}(e_B)$.

To prove the second inclusion one applies the same reasoning to the morphism of squares

$$\begin{pmatrix} B & \longrightarrow & Y \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{pmatrix} \longrightarrow \begin{pmatrix} B & \longrightarrow & Y \\ \downarrow & & \downarrow \\ A & \longrightarrow & X \end{pmatrix}$$

Lemma 2.1.9 *For any commutative square Q of the form (2) the morphism $A \to K_Q$ belongs to $cl_{\Delta,\coprod_{<\infty}}(\{e_B\})$.*

Proof: Follows from Lemma 2.1.5 and Lemma 2.1.7 in the same way as in the proof of Lemma 2.1.8 if one considers the morphism of commutative squares

$$\begin{pmatrix} B & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & X \end{pmatrix} \longrightarrow \begin{pmatrix} B & \longrightarrow & Y \\ \downarrow & & \downarrow \\ A & \longrightarrow & X \end{pmatrix}$$

Lemma 2.1.10 *Consider a commutative diagram of the form*

$$\begin{array}{ccccc} B & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & X & \longrightarrow & T \end{array} \qquad (7)$$

Denote the left square by Q_1, the right square by Q_2 and the big square by Q_3. Consider the canonical morphisms

$$p_1 : K_{Q_1} \to X \quad p_2 : K_{Q_2} \to T \quad p_3 : K_{Q_3} \to T$$

and let E be a $(\Delta, \coprod_{<\infty})$-closed class such that two out of three morphisms p_1, p_2, p_3 are in E. Then the third is in E.

Proof: Let Q_4 be the square

$$\begin{array}{ccc} Y & \longrightarrow & Z \\ \downarrow & & \downarrow \\ K_{Q_1} & \longrightarrow & K_{Q_3} \end{array} \tag{8}$$

One can easily see that it is elementary exact. The identity morphisms $Y \to Y$, $Z \to Z$ and the morphism $p_1 : K_{Q_1} \to X$ define a morphism of squares $f : Q_4 \to Q_2$ and we get a commutative diagram

$$\begin{array}{ccc} K_{Q_4} & \xrightarrow{K(f)} & K_{Q_2} \\ {\scriptstyle p_4}\downarrow & & \downarrow{\scriptstyle p_2} \\ K_{Q_3} & \xrightarrow{p_3} & T \end{array} \tag{9}$$

By Lemma 2.1.6 we have $p_4 \in cl_{\Delta,\coprod_{<\infty}}(\emptyset)$ and by Lemma 2.1.7 we have $K(f) \in cl_{\Delta,\coprod_{<\infty}}(p_1)$. This implies the statement of the lemma.

2.2 The case of a category with small coproducts

If C is a category with countable coproducts and $X_0 \to \ldots \to X_n \to \ldots$ is a sequence of coprojections then the colimit of this sequence exists in C. This allows us to give the following definition.

Definition 2.2.1 *Let C be a category with all small coproducts. A class of morphisms E in $\Delta^{op}C$ is called $\bar{\Delta}$-closed if it is $(\Delta, \coprod)$-closed and for any two sequences of term-wise coprojections $X_i \to X_{i+1}$, $Y_i \to Y_{i+1}$ and a morphism of sequences $f : X_i \to Y_i$ such that $f_i \in E$ for all i one has $colim_i f_i \in E$.*

Lemma 2.2.2 *The class of weak equivalences in $\Delta^{op}Sets$ is $\bar{\Delta}$-closed.*

Proof: The fact that the class of weak equivalences satisfies the first two conditions of Definition 2.1.1 is easy to see. The fact that it satisfies the third one is proved in [7, Lemma 5.3.1 p.129]. The fact that the class of weak equivalences is closed under coproducts and filtered colimits follows immediately from the definition of weak equivalences.

If C is a category with all small coproducts X is an object of C and K is a set we denote by $X \otimes K$ the object $\coprod_K X$. For a simplicial set K we denote by $X \otimes K$ the simplicial object with terms of the form $X \otimes K_i$. Let l_∞ be the simplicial set obtained by gluing together sequentially infinitely many copies of Δ^1.

Lemma 2.2.3 *A class of morphisms E is $\bar{\Delta}$-closed if and only if it is $(\Delta, \coprod)$-closed and for any object U in C the morphism $U \otimes l_\infty \to U$ belongs to E.*

Proof: Let E be a $\bar{\Delta}$-closed class and let l_n be the subset of l_∞ which consists of the first n copies of Δ^1. Then $U \otimes l_\infty$ is the colimit of the sequence of term-wise coprojections $U \otimes l_n \to U \otimes l_{n+1}$ and the morphism $U \otimes l_\infty \to U$ is the corresponding colimit of morphisms $U \otimes l_n \to U$. Since the simplicial sets l_n are contractible these morphisms are in E and therefore the morphism $U \otimes l_\infty \to U$ is in E. This proves the "only if" part of the lemma.

To prove the "if" part consider a class E satisfying the conditions of the lemma and let $(f_n) : (X_n) \to (Y_n)$ be a morphism of sequences of term-wise coprojections such that f_n are in E. Define the telescope $Tel(X_n)$ of the sequence $(j_n : X_n \to X_{n+1})$ by the elementary exact square

$$\begin{array}{ccc} \coprod_n (X_n \coprod X_n) & \longrightarrow & \coprod_n X_n \\ \downarrow & & \downarrow \\ \coprod_n (X_n \otimes \Delta^1) & \longrightarrow & Tel(X_n) \end{array}$$

where the left vertical arrow maps first copy of each X_n to $X_n \otimes \Delta^1$ by $Id \otimes \partial_0$ and the second copy to $X_{n+1} \otimes \Delta^1$ by $j_n \otimes \partial_1$. For any morphism of sequences (f_n) the corresponding morphism of the telescopes is in $cl_{\Delta,\coprod}(\{f_n\})$ by Lemma 2.1.7. Therefore, to show that $colim(f_n)$ is in E it remains to check that the morphism $Tel(X_n) \to colim(X_n)$ is in E. Since E is Δ-closed we may assume that (X_n) is a sequence of coprojections in C of the form $X'_0 \to X'_0 \coprod X'_1 \to X'_0 \coprod X'_1 \coprod X'_2 \to \ldots$. For such a sequence the morphism $Tel(X_n) \to colim(X_n)$ is isomorphic to the morphism $\coprod_i X'_i \otimes l_\infty \to \coprod X'_i$ and therefore belongs to E.

Proposition 2.2.4 *Let $f : K \to K'$ be a weak equivalence of simplicial sets and X an object of C. Then one has*

$$(Id_X \otimes f : X \otimes K \to X \otimes K') \in cl_{\bar{\Delta}}(\emptyset)$$

Proof: Let $Ex = colim_n Ex_n$ be the Kan completion functor. Since f is a weak equivalence the map $Ex(f)$ is a homotopy equivalence and therefore

the same holds for the morphism $Id_X \otimes Ex(f)$. It remains to show that the map
$$X \otimes K \to X \otimes Ex(K)$$
belongs to $cl_{\bar{\Delta}}(\emptyset)$. Since any $\bar{\Delta}$-closed class is closed under countable compositions of coprojections it is sufficient to show that the maps

$$X \otimes Ex_n(K) \to X \otimes Ex_{n+1}(K)$$

belong to $cl_{\bar{\Delta}}(\emptyset)$. By definition of Ex_n (see e.g. [3]) we have a square of the form

$$\begin{array}{ccc} \coprod_{\Lambda^{n,k}\to Ex_n(K)} X \otimes \Lambda^{n,k} & \longrightarrow & X \otimes Ex_n(K) \\ \downarrow & & \downarrow \\ \coprod_{\Lambda^{n,k}\to Ex_n(K)} X \otimes \Delta^n & \longrightarrow & X \otimes Ex_{n+1}(K) \end{array}$$

which is clearly elementary exact. It remains to notice that the left vertical arrow is a homotopy equivalence and thus the right vertical arrow is in $cl_{\Delta,\coprod_{\leq\infty}}(\emptyset)$ by Lemma 2.1.8.

Proposition 2.2.4 and Lemma 2.2.2 imply the following.

Corollary 2.2.5 *The class $cl_{\bar{\Delta}}(\emptyset)$ in $\Delta^{op}Sets$ coincides with the class of weak equivalences.*

Let $\Delta_{\leq n}$ be the full subcategory of Δ which consists of objects $[i]$ for $i \leq n$. For a simplicial object X in C let $d_{\leq n}X$ be the restriction of X to a functor from $\Delta_{\leq n}$ to C.

Definition 2.2.6 *An object X of $\Delta^{op}C$ is said to be of dimension $\leq n$ if for any Y in $\Delta^{op}C$ the map*

$$Hom(X, Y) \to Hom(d_{\leq n}X, d_{\leq n}Y)$$

is bijective. An object is said to be of finite dimension if there exists n such that X is of dimension $\leq n$.

Definition 2.2.7 *Let C be a category with coproducts. An object $X = (X_n)$ of $\Delta^{op}C$ is called reliably compact if the following two conditions hold:*

1. *for any $n \geq 0$ and any sequence of coprojections Y_i the map*

$$colim_i Hom(X_n, Y_i) \to Hom(X_n, colim_i Y_i)$$

   is bijective

2. *X is of finite dimension.*

The following lemmas are straightforward.

Lemma 2.2.8 *Let X be a reliably compact object and $Y_i \to Y_{i+1}$ a sequence of term-wise coprojections in $\Delta^{op}C$. Then the map*

$$colim_i Hom(X, Y_i) \to Hom(X, colim_i Y_i)$$

is bijective.

Lemma 2.2.9 *For a reliably compact object X and a finite simplicial set K of finite dimension the object $X \otimes K$ is reliably compact.*

Lemma 2.2.10 *If in an elementary exact square of the form (2) the objects B, Y and A are reliably compact then X is reliably compact.*

For a pair of objects X, Y in $\Delta^{op}C$ denote by $S(X, Y)$ the simplicial function space whose set of n-simplexes is $Hom(X \otimes \Delta^n, Y)$. One verifies easily that for a simplicial set K one has

$$S(K, S(X, Y)) = S(X \otimes K, Y)$$

Remark 2.2.11 The category $\Delta^{op}C$ with the simplicial function spaces defined above and the functors $X \mapsto X \otimes K$ defined at the beginning of the section satisfy the first two conditions of the definition of a simplicial category given in [5, p.81]. It need not satisfy the third condition since it requires products in C (see also Lemma 3.2.1).

Proposition 2.2.12 *Let C be a category with small coproducts and let E and N be two sets of morphisms between reliably compact objects in $\Delta^{op}C$ such that the elements of N are term-wise coprojections. Then there exists a functor*

$$Ex_{E,N} : \Delta^{op}C \to \Delta^{op}C$$

and a natural transformation $Id \to Ex_{E,N}$ such that the following conditions hold:

1. *for any X in $\Delta^{op}C$ the morphism $X \to Ex_{E,N}(X)$ belongs to $cl_{\bar{\Delta}}(E)$*

2. *for any $f : X \to X'$ from E and any Y the map*

$$S(X', Ex_{E,N}(Y)) \to S(X, Ex_{E,N}(Y))$$

   is a weak equivalence

3. *for any $f : X \to X'$ from N and any Y the map*

$$S(X', Ex_{E,N}(Y)) \to S(X, Ex_{E,N}(Y))$$

   is a Kan fibration.

Proof: For a morphism $f : X \to X'$ denote by $i_f : X \to cyl(f)$ the composition

$$X \stackrel{Id\otimes\partial_1}{\to} X \otimes \Delta^1 \to cyl(f).$$

Note that i_f is a term-wise coprojection. Define functors

$$Ex_n : \Delta^{op}C \to \Delta^{op}C$$

$n \geq 0$ and natural transformations $Ex_n \to Ex_{n+1}$ as follows. Set $Ex_0(X) = X$ and define $Ex_{n+1}(X)$ for even n by the push-forward square

$$\begin{array}{ccc} \coprod_{(f:Y\to Y')\in E, i\geq 0, g\in A(f,i)} (cyl(f) \otimes \partial\Delta^i \coprod_{Y\otimes\partial\Delta^i} Y \otimes \Delta^i) & \longrightarrow & Ex_n(X) \\ \downarrow & & \downarrow \\ \coprod_{(f:Y\to Y')\in E, i\geq 0, g\in A(f,i)} cyl(f) \otimes \Delta^i & \longrightarrow & Ex_{n+1}(X) \end{array} \qquad (10)$$

where

$$A(f,i) = Hom(cyl(f) \otimes \partial\Delta^i \coprod_{Y\otimes\partial\Delta^i} Y \otimes \Delta^i, Ex_n(X))$$

and for odd n by the push forward square

$$\begin{array}{ccc} \coprod_{(f:Y\to Y')\in N, i\geq 0, 0\geq k\geq i, g\in B(f,i)} (Y' \otimes \Lambda^{i,k} \coprod_{Y\otimes\Lambda^{i,k}} Y \otimes \Delta^i) & \longrightarrow & Ex_n(X) \\ \downarrow & & \downarrow \\ \coprod_{(f:Y\to Y')\in E, i\geq 0, 0\geq k\geq i, g\in B(f,i)} Y' \otimes \Delta^i & \longrightarrow & Ex_{n+1}(X) \end{array} \qquad (11)$$

where

$$B(f,i) = Hom(Y' \otimes \Lambda^{i,k} \coprod_{Y \otimes \Lambda^{i,k}} Y \otimes \Delta^i, Ex_n(X))$$

Our construction is well defined, i.e. the required push-forwards exist, because the left vertical arrows in squares of both types are term-wise coprojections (for squares (11) this follows from our assumption that elements of N are term-wise coprojections). Define $Ex_{E,N}$ as $colim_{n\geq 0} Ex_n$. To prove the first condition of the lemma it is sufficient to check that the morphisms $Ex_n \to Ex_{n+1}$ belong to $cl_{\bar{\Delta}}(E)$. Consider first the squares (10). In view of Lemma 2.1.8 it is sufficient to show that the morphism

$$cyl(f) \otimes \partial\Delta^i \coprod_{Y \otimes \partial\Delta^i} Y \otimes \Delta^i \to cyl(f) \otimes \Delta^i$$

belongs to $cl_{\bar{\Delta}}(f)$. This morphism fits into the commutative diagram

$$\begin{array}{ccccc} Y \otimes \partial\Delta^i & \longrightarrow & cyl(f) \otimes \partial\Delta^i & & \\ \downarrow & & \downarrow & & \\ Y \otimes \Delta^i & \longrightarrow & cyl(f) \otimes \partial\Delta^i \coprod_{Y \otimes \partial\Delta^i} Y \otimes \Delta^i & \longrightarrow & cyl(f) \otimes \Delta^i \end{array} \qquad (12)$$

The definition of the morphism $Y \to cyl(f)$ implies easily that it belongs to $cl_{\bar{\Delta}}(f)$. Thus for any simplicial set K the morphism $Y \otimes K \to cyl(f) \otimes K$ belongs to $cl_{\bar{\Delta}}(f)$. In particular the upper horizontal arrow in (12) and the composition of the two lower horizontal ones belong to $cl_{\bar{\Delta}}(f)$. By Lemma 2.1.8 we conclude that the first lower horizontal arrow is in $cl_{\bar{\Delta}}(f)$ and therefore the second one is in $cl_{\bar{\Delta}}(f)$ by the two out of three property.
The morphisms

$$Y' \otimes \Lambda^{i,k} \coprod_{Y \otimes \Lambda^{i,k}} Y \otimes \Delta^i \to Y' \otimes \Delta^i$$

used in squares (11) fit into commutative diagrams

$$\begin{array}{ccccc} Y \otimes \Lambda^{i,k} & \longrightarrow & Y \otimes \Delta^i & & \\ \downarrow & & \downarrow & & \\ Y' \otimes \Lambda^{i,k} & \longrightarrow & Y' \otimes \Lambda^{i,k} \coprod_{Y \otimes \Lambda^{i,k}} Y \otimes \Delta^i & \longrightarrow & Y' \otimes \Delta^i \end{array} \qquad (13)$$

and one argues in a similar way using the fact that the upper horizontal arrow and the composition of the lower horizontal ones are in $cl_{\bar{\Delta}}(\emptyset)$ by Proposition 2.2.4 since the inclusions $\Lambda^{i,k} \to \Delta^i$ are weak equivalences of simplicial sets.

To prove the second condition observe first that for any $f : X \to X'$ the morphism $cyl(f) \to X'$ is a homotopy equivalence by Lemma 2.1.5 and therefore it is sufficient to show that for f in E the map of simplicial sets

$$S(cyl(f), Ex_{E,N}(Y)) \to S(X, Ex_{E,N}(Y))$$

defined by $i_f : X \to cyl(f)$ is a weak equivalence. Since X and X' are reliably compact so is $cyl(f)$ and this map is isomorphic to the map

$$colim_n S(cyl(f), Ex_n(Y)) \to colim_n S(X, Ex_n(Y))$$

Our choice of the squares (10) implies that it has the right lifting property with respect to all embeddings $\partial\Delta^i \to \Delta^i$. Any such map is a trivial fibration and in particular a weak equivalence.

The third condition follows from the fact that, because of our choice of squares (11), for any $f : X \to X'$ in N the map

$$colim_n S(X', Ex_n(Y)) \to colim_n S(X, Ex_n(Y))$$

has the right lifting property with respect to the embeddings $\Lambda^{i,k} \to \Delta^i$.

2.3 Δ-closed classes and functors

Let C and C' be categories. We say that a functor $F : \Delta^{op}C \to \Delta^{op}C'$ is a term-wise functor if $(F(X))_n = F(X_n)$ for some functor $F : C \to C'$. Any term-wise functor takes homotopic morphisms to homotopic morphisms and if we define F on bisimplicial objects setting $(F(X))_{ij} = F(X_{ij})$ we have $F \circ \Delta = \Delta \circ F$. This implies the following result.

Lemma 2.3.1 *Let C, C' be categories and $F : \Delta^{op}C \to \Delta^{op}C'$ a term-wise functor. Then for any class of morphisms E in $\Delta^{op}C$ one has*

$$F(cl_\Delta(E)) \subset cl_\Delta(F(E))$$

If C has coproducts of some kind and F commutes with them then we can extend the statement of Lemma 2.3.1 to $cl_{\Delta,\coprod}(E)$. If F does not commute with coproducts we have instead the following result where $cl_{\Delta,colim}$ denotes the smallest Δ-closed class which is closed under coproducts and filtered colimits.

Proposition 2.3.2 *Let C be a category with an initial object and all (small) coproducts, C' a category with filtered colimits and $F : C \to C'$ a functor such that for any set A of elements of C the morphism*

$$colim_{I\in Fin(A)} F(\coprod_{X\in I} X) \to F(\coprod_{X\in A} X) \qquad (14)$$

where $Fin(A)$ is the partially ordered set of finite subsets of A, is an isomorphism. Let further C_0 be a class of objects of C closed under finite coproducts and such that $cl_{\coprod}(C_0) = C$. Then for any class of morphisms E in $\Delta^{op}C$ one has

$$F(cl_{\bar{\Delta}}(E)) \subset cl_{\Delta,colim}(F(E\coprod Id_{C_0}))$$

where $E\coprod Id_{C_0}$ is the class of morphisms of the form $f\coprod Id_X$ for $f \in E$ and $X \in C_0$.

Proof: It is enough to show that there exists a $\bar{\Delta}$-closed class H which contains E and such that $F(H) \subset cl_{\Delta,colim}(F(E\coprod Id_{C_0}))$. Let H be the class of morphisms f such that for any object X in $\Delta^{op}C$ we have $F(f\coprod Id_X) \in cl_{\Delta,colim}(F(E\coprod Id_{C_0}))$. This class is clearly Δ-closed and it is closed under finite coproducts (see the beginning of the proof of Proposition 2.1.3). Since morphisms of the form (14) are isomorphisms this implies that H is closed under coproducts and colimits of sequences of term-wise coprojections. Using the fact that C_0 is closed under finite coproducts and $cl_{\coprod}(C_0) = C$ we conclude that H contains E.

Corollary 2.3.3 *Let C, C' and $F : C \to C'$ be as in Proposition 2.3.2. Then one has*

$$F(cl_{\bar{\Delta}}(\emptyset)) \subset cl_{\Delta,colim}(\emptyset).$$

3 Homotopy theory of simplicial radditive functors

3.1 Radditive functors

Let C be a category with finite coproducts and the initial object 0. Denote by $R(C)$ the full subcategory of the category of contravariant functors from

C to sets which consists of functors F such that $F(0) = pt$ and for any finite family of objects X_i, $i \in I$ the map

$$F(\coprod_{i \in I} X_i) \to \prod_{i \in I} F(X_i)$$

is bijective. The objects of $R(C)$ will be called radditive functors.

Example 3.1.1 If C is an additive category then $R(C)$ is equivalent to the category of additive contravariant functors from C to abelian groups.

Example 3.1.2 Recall that a presheaf on a small category is a contravariant functor from this category to the category of sets. Let C be a small category and $C^{\amalg_{<\infty}}$ the full subcategory of the category of presheaves on C which consists of finite coproducts of representable presheaves. Then $R(C^{\amalg_{<\infty}})$ is equivalent to the category of presheaves on C.

Example 3.1.3 For an object X of a category C let X_+ be the pointed presheaf on C obtained from the presheaf represented by X by the addition of a disjoint base point. Let $C_+^{\amalg_{<\infty}}$ be the full subcategory of the category of pointed presheaves on C which consists of coproducts of objects of the form X_+. Then $R(C_+^{\amalg_{<\infty}})$ is the category of pointed presheaves on C.

Example 3.1.4 Let A be a commutative ring and C the category of finitely generated free commutative algebras over A. Then $R(C)$ is equivalent to the category of all commutative algebras over A.

Any representable functor is radditive by definition of coproducts. Therefore we have a full embedding $C \to R(C)$ which sends an object to the corresponding representable functor. The following lemma is straightforward.

Lemma 3.1.5 *The functor $C \to R(C)$ commutes with finite coproducts.*

Lemma 3.1.6 *The category $R(C)$ has limits.*

Proof: The limit of a diagram $F : I \to R(C)$ is the same as its limit in the category of contravariant functors with values in sets i.e.

$$(limF(i))(U) = lim(F(i)(U)).$$

Lemma 3.1.7 *Let $F : I \to R(C)$ be a filtered system of radditive functors. Then the colimit $colimF(i)$ of F in the category of functors is radditive and gives a colimit of F in the category of radditive functors.*

Proof: The second statement follows immediately from the first. The first one follows from the fact that filtered colimits of sets commute with finite products.

Lemma 3.1.8 *Let X_α be a collection of objects of C. Denote by the same symbols the radditive functors represented by X_α. Define $\cup X_\alpha$ as the functor given by*

$$U \mapsto colim_{I \subset A} Hom(U, \coprod_{i \in I} X_i)$$

where I run through finite subsets of A. Then $\cup_\alpha X_\alpha$ is radditive and it is a coproduct of X_α in the category of radditive functors.

Proposition 3.1.9 *The inclusion functor*

$$R(C) \to Funct(C^{op}, Sets)$$

has a left adjoint r.

Proof: Let F be a functor which is not necessarily radditive. Consider the diagram in $R(C)$ of the form

$$\cup_{(p:U \to V) \in C} U \otimes F(V) \rightrightarrows \cup_{W \in C} W \otimes F(W)$$

where one arrow maps the summand U corresponding to $(p : U \to V, f \in F(V))$ to the summand corresponding to $(U, F(p)(f))$ by identity and the other one maps it to the summand corresponding to (V, f) by p. Let $r(F)$ be the coequalizer of these two maps in the category of functors. Since finite products of sets commute with coequalizers of reflective pairs this functor is radditive and one verifies easily that for any radditive G one has

$$Hom(F, G) = Hom(r(F), G)$$

Remark 3.1.10 As the following example shows the functor r is not, in general, left exact i.e. it does not commute with finite limits. Therefore, radditive functors can not be thought of as sheaves with respect to some topology on C.

Example 3.1.11 Let C be the category of finitely generated free abelian groups such that $R(C)$ is equivalent to the category of all abelian groups. Let F_2 be the coequalizer of the zero map $\mathbf{Z} \to \mathbf{Z}$ and the multiplication by 2 in the category of all contravariant functors from C to sets (we have $r(F_2) = \mathbf{Z}/2$ but F_2 itself does not take values in abelian groups). Let G be the functor given by the fiber product

$$\begin{array}{ccc} G & \longrightarrow & \mathbf{Z} \\ \downarrow & & \downarrow \\ \mathbf{Z} & \longrightarrow & F_2 \end{array}$$

where $\mathbf{Z} \to F_2$ are the canonical morphisms. Let $G \to \mathbf{Z} \times \mathbf{Z}$ be the obvious embedding. We claim that $r(G) \to r(\mathbf{Z} \times \mathbf{Z}) = \mathbf{Z} \times \mathbf{Z}$ is not a monomorphism. In fact let us show that te abelian group $r(G)$ has torsion. A simple computation shows that G fits into a push-forward square of the form

$$\begin{array}{ccc} \mathbf{Z} & \xrightarrow{\Delta} & \mathbf{Z} \times \mathbf{Z} \\ {\scriptstyle 2}\downarrow & & \downarrow \\ \mathbf{Z} & \longrightarrow & G \end{array}$$

where Δ is the diagonal and 2 is the multiplication by 2. Since r is a left adjoint it commutes with colimits and therefore $r(G)$ is given by a similar push-forward square in the category of abelian groups. An easy computation shows that it has 2-torsion.

Lemma 3.1.12 *The category $R(C)$ has colimits.*

Proof: For a diagram $F : I \to R(C)$ of radditive functors one gets $colimF$ applying the functor r of Proposition 3.1.9 to the colimit of F in the category of functors.

3.2 Projective closed model structure

Let $\Delta^{op}R(C)$ be the category of simplicial objects in $R(C)$ or, equivalently, the category of radditive functors with values in simplicial sets. For X in $\Delta^{op}R(C)$ and a simplicial set K define $\underline{Hom}(K, X)$ as the simplicial radditive functor which takes $U \in C$ to $S(K, X(U))$ (here $S(-,-)$ refers to the simplicial function object in the category of simplicial sets). The following lemma is straightforward.

Lemma 3.2.1 *The category $\Delta^{op}R(C)$ together with the simplicial function space $S(-,-)$ and functors $X \mapsto X \otimes K$, $X \mapsto \underline{Hom}(K, X)$ is a simplicial category in the sense of [5, p.81].*

A morphism of radditive simplicial functors $f : X \to Y$ is called a projective weak equivalence if, for any U in C, the map of simplicial sets $X(U) \to Y(U)$ defined by f is a weak equivalence. We define the homotopy category $H(C)$ of C as the localization of the category of simplicial radditive functors with respect to the class of projective weak equivalences. Our next goal is to construct a finitely generated simplicial closed model structure on the category of simplicial radditive functors with this class of weak equivalences.

Lemma 3.2.2 *The class of projective weak equivalences of simplicial radditive functors is Δ-closed.*

Proof: Follows from Lemma 2.3.1 applied to functors $U \mapsto X(U)$ and Lemma 2.2.2.

In view of Lemma 3.1.7 we also have the following result.

Lemma 3.2.3 *The class of projective weak equivalences of simplicial radditive functors is closed under filtered colimits.*

Proposition 3.2.4 *The class of projective weak equivalences contains the class $cl_{\bar{\Delta}}(\emptyset)$.*

Proof: We have to show that for any elements $f : X \to Y$ in $cl_{\bar{\Delta}}(\emptyset)$ and any U in C the map of simplicial sets $f_U : X(U) \to Y(U)$ defined by f is a weak equivalence. Lemma 3.1.7 implies that the functor $X \mapsto U$ satisfies the conditions of Corollary 2.3.3. Therefore f_U belongs to $cl_{\Delta,colim}(\emptyset)$. Since the class of weak equivalences of simplicial sets is $(\Delta, colim)$-closed we conclude that it is a weak equivalence.

Remark 3.2.5 The class of projective weak equivalences of simplicial radditive functors is not, in general, closed with respect to finite coproducts. In particular it need not be $\bar{\Delta}$-closed. Consider the situation of Example 3.1.4. Recall that the coproducts in the category of commutative algebras are tensor products. Given a weak equivalence of simplicial algebras $f : X \to Y$ and a (simplicial) algebra Z the morphism $f \otimes Id_Z$ need not be a weak equivalence unless Z is flat or X and Y are flat.

Define the classes of projective fibrations and cofibrations in the category of simplicial radditive functors as follows. A morphism $f : X \to Y$ is a projective fibration if for any U in C the map of simplicial sets $X(U) \to Y(U)$ defined by f is a Kan fibration. A morphism is called a projective cofibration if it has the right lifting property with respect to trivial projective fibrations i.e. projective fibrations which are projective weak equivalences. Let I be the set of morphisms of the form

$$U \otimes \partial\Delta^n \to U \otimes \Delta^n$$

where U runs through all objects in C_0 and $n \geq 0$. Let J be the set of morphisms of the form

$$U \otimes \Lambda^{n,k} \to U \otimes \Delta^n$$

where U runs through all objects in C_0, $n \geq 0$ and $k = 0, \ldots, n$. One can easily see that a morphism is a projective fibration if and only if it has the right lifting property with respect to elements of J and it is a trivial projective fibration if and only if it has the right lifting property with respect to elements of I.

Following [7] we say that, for a given class of morphisms E, a morphism is an E-cell if it is a filtered composition of morphisms which are push-forwards of elements of E. We also say that a morphism is a sequential E-cell if it is a countable composition of morphisms which are push-forwards of coproducts of elements of E. One can easily see that any sequential E-cell is an E-cell and that any E-cell is a filtered composition of sequential E-cells. If every element of E is a term-wise coprojection then any sequential E-cell is a term-wise coprojection. The following lemma follows from the standard decomposition techniques together with the obvious observation that domains and codomains of elements of both I and J are reliably compact.

Lemma 3.2.6 *Any morphism f in $\Delta^{op}R(C)$ can be represented as a composition of a sequential J-cell (resp. sequential I-cell) and a projective fibration (resp. a trivial projective fibration).*

Proposition 3.2.7 *The classes of projective weak equivalences, projective fibrations and projective cofibrations form a finitely generated closed model structure on the category of simplicial radditive functors.*

Proof: Let us show that I and J satisfy the conditions of [7, Theorem 2.1.19]. The fifth and the sixth conditions are obvious as well modulo the corresponding fact for simplicial sets. Let us prove the fourth one. We have to show that if f is a J-cell, then f is a projective weak equivalence. The class of projective weak equivalences is closed under filtered compositions because of the corresponding properties of morphisms of simplicial sets and Lemma 3.1.7. Therefore it is sufficient to show that any push-forward of an element of J is a projective weak equivalence. Consider a push-forward square of the form

$$\begin{array}{ccc} A & \longrightarrow & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ X & \longrightarrow & X' \end{array}$$

where f is an element of J. Then f is a term-wise coprojection, this square is elementary exact and by Lemma 2.1.8 we conclude that f' belongs to $cl_{\Delta,\coprod_{<\infty}}(\{f\})$. Since f is a homotopy equivalence we conclude that f' is in $cl_{\Delta,\coprod_{<\infty}}(\emptyset)$ and therefore it is a projective weak equivalence by Proposition 3.2.4.

The standard argument based on the lifting axioms together with Lemma 3.2.6 implies the following result.

Lemma 3.2.8 *A morphism f in $\Delta^{op}R(C)$ is a projective cofibration (resp. a trivial projective cofibration) if and only if it is a retract of a sequential I-cell (resp. J-cell).*

One can easily see that if $f : X \to Y$ is a sequential I-cell then for each n the morphism $f_n : X_n \to Y_n$ is isomorphic to the morphism $X_n \to X_n \coprod X'_n$ where X'_n is a coproduct of representable functors. Combining this observation with Lemma 3.2.8 we get the following result.

Lemma 3.2.9 *A projective cofibration is a retract of a morphism f such that each term of f is of the form $X \to X \coprod X'$ where X' is a coproduct of representable radditive functors.*

Lemma 3.2.10 *Let $f : K \to L$ be a weak equivalence of simplicial sets and $j : B \to A$ a term-wise coprojection. Then the morphism*

$$A \otimes K \to A \otimes K \coprod_{B \otimes K} B \otimes L$$

is a weak equivalence.

Proof: Consider the push-forward square

$$\begin{array}{ccc} B\otimes K & \longrightarrow & B\otimes L \\ \downarrow & & \downarrow \\ A\otimes K & \longrightarrow & X \end{array}$$

Since $B \to A$ is a term-wise coprojection the same holds for $B \otimes K \to A \otimes K$ and therefore by Lemma 2.1.8 we conclude that the morphism $A \otimes K \to X$ belongs to $cl_{\Delta,\coprod_{<\infty}}(\{Id_B \otimes f\})$. By Proposition 2.2.4 we conclude that this morphism is contained in $cl_{\bar{\Delta}}(\emptyset)$ and therefore is a projective weak equivalence by Proposition 3.2.4.

Proposition 3.2.11 *Let $j : B \to A$ be a projective cofibration and $i : K \to L$ a cofibration of simplicial sets. Then the morphism*

$$h(i,j) : H(i,j) = B \otimes L \coprod_{B\otimes K} A \otimes K \to A \otimes L$$

is a cofibration. If i of j is a projective weak equivalence then so is $h(i,j)$.

Proof: Let us show first that $h(i,j)$ is a cofibration. Lemma 3.2.8 implies that it is sufficient to show that the class of morphisms j, such that $h(i,j)$ is a cofibration for all i, contains elements of I, i.e. morphisms of the form $U \otimes \partial\Delta^n \to U \otimes \Delta^n$, and is closed under retracts, coproducts, push-forwards and sequential compositions. For j of the form $U \otimes \partial\Delta^n \to U \otimes \Delta^n$ the morphism $h(i,j)$ is of the form $U \otimes K' \to U \otimes L'$ where $K' \to L'$ is a monomorphism of simplicial sets. Any such morphism is a cofibration because it has the right lifting property for trivial projective fibrations. The fact that our class is closed under all required operations is also easy to check.

Assume that i is a projective weak equivalence. In view of Lemma 3.2.8 it is a retract of a term-wise coprojection and we may assume that it is a term-wise coprojection. Then Lemma 3.2.10 implies easily that $h(i,j)$ is a projective weak equivalence.

Assume that j is a projective weak equivalence. Then j is a projective trivial cofibration and therefore, by Lemma 3.2.8, a retract of a J-cell. For an element of J and any i the morphism $h(i,j)$ is a clearly a projective weak equivalence. Checking again that the class of trivial cofibrations j such that $h(i,j)$ is a trivial cofibration is closed under retracts, push-forwards, coproducts and sequential compositions we conclude that $h(j,i)$ is a projective weak equivalence if j is.

Corollary 3.2.12 *The projective closed model structure is simplicial i.e. it satisfies the axiom SM7 (see e.g. [5, p.89]).*

Proof: We have to show that for a projective cofibration $j : A \to B$ and a projective fibration $q : X \to Y$ the morphism

$$S(B, X) \to S(A, X) \times_{S(A,Y)} S(B, Y)$$

is a Kan fibration which is a weak equivalence if j or q is a weak equivalence. This follows by adjunction from Proposition 3.2.11.

3.3 E-local equivalences

Let $f : X \to X'$ be a morphism in $\Delta^{op}R(C)$. We say that a morphism $\tilde{f} : \tilde{X} \to \tilde{X}'$ is a cofibrant replacement of f if $\tilde{X}$ and $\tilde{X}'$ are cofibrant (in the projective closed model structure) and there is a commutative square of the form

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{X}' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & X' \end{array}$$

where the vertical arrows are projective weak equivalences. The following definitions are particular cases of more general definition given in [6].

Definition 3.3.1 *Let E be a class of morphisms in $\Delta^{op}R(C)$. An object Y of this category is called E-local if it is projectively fibrant and for any element $f : X \to X'$ of E and any cofibrant replacement $\tilde{f} : \tilde{X} \to \tilde{X}'$ of f the map of simplicial sets*

$$S(\tilde{X}', Y) \to S(\tilde{X}, Y)$$

defined by $\tilde{f}$ is a weak equivalence.

Definition 3.3.2 *Let E be a class of morphisms in $\Delta^{op}R(C)$. A morphism $f : X \to X'$ is called a (left) E-local equivalence if for any E-local Y and any cofibrant replacement $\tilde{f} : \tilde{X} \to \tilde{X}'$ of f the map of simplicial sets*

$$S(\tilde{X}', Y) \to S(\tilde{X}, Y)$$

defined by $\tilde{f}$ is a weak equivalence.

We denote the class of left E-local equivalences by $cl_l(E)$. The goal of this and the next few sections is to analize the relations between $cl_l(E)$ and $cl_{\bar{\Delta}}(E)$. The following lemmas summarize some obvious properties of E-local objects and E-local equivalences.

Lemma 3.3.3 *An object Y is E-local if and only if it is fibrant and for any element $f : X \to X'$ of E there exists at least one cofibrant replacement $\tilde{f} : \tilde{X} \to \tilde{X}'$ of f such that the map of simplicial sets*

$$S(\tilde{X}', Y) \to S(\tilde{X}, Y)$$

defined by $\tilde{f}$ is a weak equivalence.

Lemma 3.3.4 *A morphism $f : X \to X'$ belongs to $cl_l(E)$ if and only if for any E-local Y there exists at least one cofibrant replacement $\tilde{f} : \tilde{X} \to \tilde{X}'$ of f such that the map of simplicial sets*

$$S(\tilde{X}', Y) \to S(\tilde{X}, Y)$$

defined by $\tilde{f}$ is a weak equivalence.

Lemma 3.3.5 *An E-equivalence whose domain and codomain are E-local is a projective weak equivalence.*

Lemma 3.3.6 *Let E be a set of morphisms between reliably compact, projectively cofibrant objects in $\Delta^{op}R(C)$. Let N be the set of morphisms of the form $0 \to U$ for $U \in C$ and $Ex_{E,N}$ the corresponding functor of Proposition 2.2.12. Then for any X the object $Ex_{E,N}(X)$ is E-local.*

Proof: Follows immediately from definitions and Proposition 2.2.12.

Proposition 3.3.7 *Let E be as in Lemma 3.3.6 then one has*

$$cl_l(E) \subset cl_{\bar{\Delta}}(E \cup W_{proj})$$

where W_{proj} is the class of projective weak equivalences.

Proof: Follows from Proposition 2.2.12(1), Lemma 3.3.6 and Lemma 3.3.5.

Lemma 3.3.8 *For any class E the class $cl_l(E)$ is closed under sequential colimits.*

Proof: Let $f_n : X_n \to X'_n$, $n \geq 0$ be a morphism of sequences of morphisms in $\Delta^{op}R(C)$. We want to show that $colim f_n$ is contained in $cl_l(\{f_n\}_{n\geq 0})$. Observe first that using the factorization and lifting axioms for the projective closed model structure one can inductively construct a commutative square of morphisms of sequences of the form

$$\begin{array}{ccc} \tilde{X}_n & \xrightarrow{\tilde{f}_n} & \tilde{X}'_n \\ \downarrow & & \downarrow \\ X_n & \xrightarrow{f_n} & X'_n \end{array}$$

where vertical arrows are trivial fibrations, objects $\tilde{X}_n$ and $\tilde{X}'_n$ are cofibrant and morphisms $\tilde{X}_n \to \tilde{X}_{n+1}$ and $\tilde{X}'_n \to \tilde{X}'_{n+1}$ are cofibrations. Since a sequential colimit of weak equivalences is a projective weak equivalence by Lemma 3.2.3 it is sufficient to check that $colim\tilde{f}_n$ is in $cl_l(E)$. This follows from the fact that a morphism of towers of fibrations of Kan simplicial sets which is a term-wise weak equivalence defined a weak equivalence of inverse limits.

Lemma 3.3.9 *Let C be a category such that the projective closed model structure on $\Delta^{op}R(C)$ is left proper. Consider a push-forward square of the form*

$$\begin{array}{ccc} B & \longrightarrow & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A & \longrightarrow & X \end{array}$$

where f is a cofibration and an E-local equivalence. Then g is an E-local equivalence.

Proof: Our assumption that the projective closed model structure is left proper implies that the statement of the lemma holds if $B \to Y$ is a projective weak equivalence. Therefore, since any morphism is a composition of a cofibration and a projective weak equivalence we may assume that $B \to Y$ is a cofibration.

In the case when B and, therefore, A, Y and X are cofibrant the statement follows easily from definitions. For a general B consider a diagram of the form

$$\begin{array}{ccccc} \tilde{B} & \longrightarrow & B & \longrightarrow & Y \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow & & \downarrow \\ \tilde{A} & \longrightarrow & A & \longrightarrow & X \end{array}$$

where the left horizontal arrows are trivial fibrations, the left vertical arrow $\tilde{f}$ is a cofibration and $\tilde{B}$ and, therefore, $\tilde{A}$ are cofibrant. The morphism $Y \to X$ is the composition of two morphisms $Y \to \tilde{A} \coprod_{\tilde{B}} Y$ and $\tilde{A} \coprod_{\tilde{B}} Y \to X$. The first of them is a push-forward of $\tilde{f}$ and therefore is an E-local equivalence by the first part of the proof. Let us show that the second morphism is a projective weak equivalence. This morphism fits into the push-forward square

$$\begin{array}{ccc} \tilde{A} \coprod_{\tilde{B}} B & \longrightarrow & \tilde{A} \coprod_{\tilde{B}} Y \\ {\scriptstyle h}\downarrow & & \downarrow \\ A & \longrightarrow & X \end{array}$$

where the upper horizontal arrow is a push-forward of $B \to Y$ and therefore a cofibration. Since the projective closed model structure is left proper it remains to check that the left vertical arrow h is a weak equivalence. This follows from the fact that $\tilde{A} \to A$ and $\tilde{B} \to B$ are weak equivalences and the left properness assumption.

Lemma 3.3.10 *Let H be a class of morphisms such that every element of H is a cofibration between cofibrant objects and an E-local equivalence. Then any sequential H-cell is a cofibration and an E-local equivalence.*

Proof: A sequential H-cell is a sequential colimit of morphisms which are push-forwards of coproducts of elements of H. In view of Lemma 3.3.8 and Lemma 3.3.9 it remains to check that a coproduct of elements of H is a cofibration and an E-local equivalence. This follows easily from the definitions.

Lemma 3.3.11 *Let $f : X \to X'$ be a projective weak equivalence between cofibrant objects and K be a simplicial set. Then $f \otimes Id_K$ is a projective weak equivalence.*

Proof: The morphism $f \otimes Id_K$ is the diagonal of a morphism of bisimplicial objects whose terms are coproducts of copies of f. A coproduct of copies of f is a filtered colimit of finite coproducts of copies of f. It remains to check that a finite coproduct of projective weak equivalences of cofibrant objects is a projective weak equivalence. This holds in any model category. See e.g. [7, Lemma 5.2.6, p.126].

Let E be a set of morphisms in $\Delta^{op}R(C)$. Denote by i_E be the set of morphisms of the form $X \to cyl(f)$ where $f : X \to X'$ is in E and by H_E be the set of morphisms of the form

$$X' \otimes \partial\Delta^n \coprod_{X\otimes\partial\Delta^n} X \otimes \Delta^n \to X' \otimes \Delta^n$$

where $f : X' \to X$ is in i_E and $n \geq 0$.

Lemma 3.3.12 *Let E be a set of morphisms between cofibrant objects in $\Delta^{op}R(C)$. Then any element of H_E is a cofibration and an E-local equivalence and an object Y is E-local if and only if the morphism $Y \to pt$ has the right lifting property for elements of $J \cup H_E$ (here J is the standard generating set of trivial cofibrations in the projective model structure).*

Proof: The fact that elements of H_E are cofibrations follows from Proposition 3.2.11. The fact that they are E-local equivalences follows from Lemmas 3.3.11 and 3.3.9. An object Y has the right lifting property for $J \cup H_E$ if and only if it is fibrant in the projective model structure and for any f in E the map of simplicial sets

$$S(cyl(f), Y) \to S(X, Y) \tag{15}$$

defined by $i_f : X \to cyl(f)$ is a trivial Kan fibration. In view of Lemma 2.1.5 this implies that Y is E-local. The same lemma shows that if Y is E-local then the morphisms (15) are projective weak equivalences. It is easy to see that for a morphism f with cofibrant domain and codomain the morphism (15) is always a fibration which finishes the proof.

Definition 3.3.13 *Let E be a class of morphisms in $\Delta^{op}R(C)$. A morphism is called an E-local fibration if it has the right lifting property with respect to projective cofibrations which are E-local equivalences.*

Lemma 3.3.14 *Let f be a morphism which is an E-local fibration and an E-local equivalence. Then it is a projective weak equivalence.*

Proof: We have to show that any morphism satisfying the condition of the lemma has the right lifting property with respect to all cofibrations. This follows by "Joyal trick". See [9, p.64].

Proposition 3.3.15 *Let C be a category such that the projective closed model structure on $\Delta^{op}R(C)$ is left proper. Let E be a set of morphisms in $\Delta^{op}R(C)$ such that the domains and codomains of elements of E are cofibrant and reliably compact. Then a morphism $f : Y \to X$ with an E-local codomain is an E-fibration if and only if it has the right lifting property for elements of $J \cup H_E$.*

Proof: The "only if" part follows immediately from the fact that elements of $J \cup H_E$ are cofibrations and E-local equivalences which is proved in Lemma 3.3.12. To prove the "if" part consider a commutative square of the form

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow \\ A & \longrightarrow & X \end{array}$$

where $B \to A$ is a cofibration which is an E-local equivalence, $Y \to X$ has the right lifting property for elements of $J \cup H_E$ and X is E-local In view of Lemma 3.3.9 the push-forward of $B \to A$ with respect to $B \to Y$ is again a cofibration and an E-local equivalence. Therefore it is sufficient to show that for a square of the form

$$\begin{array}{ccc} Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ A & \longrightarrow & X \end{array}$$

where $Y \to A$ is a cofibration and an E-local equivalence, $Y \to X$ has the right lifting property with respect to $J \cup H_E$ and X is E-local there exists a morphism $A \to Y$ making the two triangles commutative. Since the domains and codomains of elements of E are reliably compact the same holds for domains and codomains of elements of $J \cup H_E$ and we can use the standard decomposition method to represent the morphism $A \to X$ as a composition of the form $A \to A' \to X$ where the first arrow is a sequential $J \cup H_E$-cell and the second one has the right lifting property with respect to $J \cup H_E$. In particular, the first morphism is a cofibration and E-local equivalence by Lemma 3.3.10 and therefore the morphism $Y \to A'$ is a cofibration and E-local equivalence. By Lemma 3.3.12 we conclude that both Y and A' are E-local. Therefore the morphism $Y \to A'$ is a projective weak equivalence. Since $Y \to X$ has the right lifting property for J it is a projective fibration and we conclude that there exists a morphism $A' \to Y$ such that the composition $A \to A' \to Y$ solves our lifting problem.

Corollary 3.3.16 *Let C and E be as in Proposition 3.3.15. Then an object is E-fibrant if and only if it is E-local.*

Proof: Follows immediately by combining Proposition 3.3.15 and Lemma 3.3.12.

Remark 3.3.17 The assumption of Proposition 3.3.15 which requires domains and codomains of elements of E to be reliably compact can be replaced with the assumption that they are small with respect to some cardinal. In this case the usual decomposition method should be replaced by a more complicated transfinite version.

The results about E-local equivalences, cofibrations and E-local fibrations proved so far show that, at least under the assumptions of Proposition 3.3.15, these classes satisfy all the axioms of a closed model structure except possibly the one which asserts the existence of a decomposition of any morphism into a cofibration followed by an E-local fibration. One can not immediately use the standard decomposition technique to prove this axiom because we do not have any explicit set of cofibrations which are E-local equivalences sufficient to detect E-local fibrations. This problem occurs in all the cases I know where one tries to localize a closed model structure (e.g. in [10], [6] and [4]). The standard way to deal with it is to show formally that some such set exists and then use large ordinals and transfinite arguments to get the required decomposition. The most general result of this sort is obtained in [6]. In our context it takes the following form.

Proposition 3.3.18 *Let C be a category such that one has:*

1. *for any simplicial radditive functor F and any X in $[C]$ the morphism $F \to X \coprod F$ is an effective monomorphism (i.e. F is the equalizer of the two morphisms from $X \coprod F$ to $F \coprod X \coprod X$)*

2. *the projective closed model structure is left proper*

Let further E be a set of morphisms in $\Delta^{op}R(C)$. Then the classes of E-local equivalences and projective cofibrations define a left proper simplicial closed model structure on $\Delta^{op}R(C)$.

Proof: Lemma 3.2.9 states that every cofibration is a retract of a morphism which is term-wise of the form $F \to F \coprod X$ where X is in $[C]$. In particular, the first condition of the proposition implies that any cofibration is an effective monomorphism. The statement of the proposition follows now from the main theorem of [6].

Remark 3.3.19 The only part of Proposition 3.3.18 which is not contained in the results already proved in this section and does not easily follow from them is the fact that there exists a set of cofibrations which are E-local equivalences sufficient to detect all E-local fibrations. I would like to add that I do not know of any example of the situation where this particular result is used where it can not be replaced by the characterization of E-local fibrations with E-local codomain given in Proposition 3.3.15.

Because of the problems discussed above we do not know whether or not the closed model structure of Proposition 3.3.18 is finitely generated. Fortunately, it turns out that in many applications (see e.g. [8]) the notion of a finitely generated model structure can be replaced by a weaker notion of an almost finitely generated structure defined below.

Definition 3.3.20 *A closed model structure on a category C is called almost finitely generated if there exist a set I of cofibrations with compact domains and codomains and a set J_0 of trivial cofibrations with compact domains and codomains such that the following conditions hold:*

1. *a morphism is a trivial fibration if and only if it has the right lifting property with respect to I*
2. *a morphism with a fibrant codomain is a fibration if and only if it has the right lifting property with respect to J_0.*

Proposition 3.3.21 *Let C be a category satisfying the conditions of Proposition 3.3.18 and E be a set of morphisms between cofibrant reliably compact objects in $\Delta^{op}R(C)$. Then the E-local closed model structure on $\Delta^{op}R(C)$ is almost finitely generated.*

Proof: This follows immediately from Proposition 3.3.15 and Lemma 3.3.12.

3.4 Standard simplicial resolution of a functor

In this section we construct a functor

$$G : R(C) \to \Delta^{op}[C]$$

which sends a radditive functor to its simplicial "resolution" by coproducts of representable functors. We start with a more general construction. Let R be a category with coproducts, C a small subcategory in R and $[C]$ the full subcategory of R which consists of objects isomorphic to coproducts of objects in C. Define a functor

$$G_C : R \to \Delta^{op}[C]$$

as follows. For F in R the object of n-simplexes of $G(F)$ is

$$G(F)_n = \coprod_{X_0 \to \ldots \to X_n} X_0 \otimes Hom(X_n, F) \tag{16}$$

where $X_0 \to \ldots \to X_i$ run through all sequences of morphisms of length i in C. The face morphism $\partial_i : G_n(F) \to G_{n-1}(F)$ maps the summand X_0 indexed by $(X_0 \stackrel{g_0}{\to} \ldots \stackrel{g_{n-1}}{\to} X_n \stackrel{f}{\to} F)$ to:

X_1 indexed by $(X_1 \stackrel{g_1}{\to} \ldots \stackrel{g_{n-1}}{\to} X_n \stackrel{f}{\to} F)$ if $i = 0$

X_0 indexed by $(X_1 \stackrel{g_1}{\to} \ldots \to \widehat{X}_i \to \ldots \stackrel{g_{n-1}}{\to} X_n \stackrel{f}{\to} F)$, where the hat indicates that the corresponding object is omitted and the incoming and outgoing morphisms are composed, if $i = 1, \ldots, n-1$

X_0 indexed by $(X_1 \stackrel{g_1}{\to} \ldots \stackrel{g_{n-2}}{\to} X_{n-1} \stackrel{f \circ g_{n-1}}{\to} F)$ if $i = n$

The degeneracy morphism $s_i : G_n(F) \to G_{n+1}(F)$ for $i = 0, \ldots, n$ maps the summand X_0 indexed by $(X_0 \stackrel{g_0}{\to} \ldots \stackrel{g_{n-1}}{\to} X_n \stackrel{f}{\to} F)$ to the summand X_0 indexed by $(X_0 \stackrel{g_0}{\to} \ldots \to X_i \stackrel{Id}{\to} X_i \to \ldots \stackrel{g_{n-1}}{\to} X_n \stackrel{f}{\to} F)$.

Remark 3.4.1 Our construction is a particular case of the simplicial replacement of a diagram considered in [2, XII.5]. The object $G(F)$ can be identified with the simplicial replacement of the diagram $C/F \to [C]$. In particular for any simplicial object X in D the simplicial set $S(G(F), X)$ can be identified with the homotopy limit of the diagram of simplicial sets $U \mapsto Hom(U, X)$ indexed by $(C/F)^{op}$.

Lemma 3.4.2 *For any Y in C the morphism $G(Y) \to Y$ is a homotopy equivalence.*

Remark 3.4.3 We do not know whether or not this morphism is a homotopy equivalence for Y in $[C]$.

Proof: The terms of the simplicial object $G(Y)$ are given by

$$G_n(Y) = \coprod_{X_0 \to \ldots \to X_n \to Y} X_0. \tag{17}$$

The embedding of Y to $G_n(Y)$ on the summand indexed by $Y \stackrel{Id}{\to} \ldots \stackrel{Id}{\to} Y$ commutes with faces and degeneracies. This gives us a morphism $Y \to G(Y)$ such that the composition $Y \to G(Y) \to Y$ is the identity. Let us show that the composition in the other direction is homotopic to identity of $G(Y)$. We will use the description of simplicial homotopies given at the beginning of Section 2.1. According to it a simplicial homotopy between two maps $f, g : A \to B$ is given by a collection of morphisms

$$h_i^n : A_n \to B_n$$

where $i = -1, \ldots, n$ which satisfy certain relations. For $i = -1$ we set h_i^n on the summand X_0 indexed by $(X_0 \to \ldots \to X_n \to Y)$ to be the obvious map to the summand Y indexed by the sequence of identities. For $i \geq 0$ we set h_i^n on the summand X_0 indexed by $(X_0 \to \ldots \to X_n \to Y)$ to be the identity map to the summand X_0 indexed by $(X_0 \to \ldots \to X_i \to Y \to \ldots \to Y)$ where the map $X_i \to Y$ is the composition of the sequence $X_i \to X_{i+1} \to \ldots \to Y$ and the morphisms between Y's are identities. One verifies by an explicit check that this collection is a homotopy from the composition $G(Y) \to Y \to G(Y)$ to the identity.

Let us consider now the case when C is a category with finite coproducts and $R = R(C)$. We get a functor

$$G : R(C) \to \Delta^{op}[C]$$

which we call the standard simplicial resolution functor.

Remark 3.4.4 The functor G does not commute with finite coproducts. The example of C being the category of finitely generated free abelian groups and $R(C)$ being all abelian groups shows that in general it is not possible to find a functor from $R(C)$ to $\Delta^{op}[C]$ which commutes with finite coproducts and satisfies the conclusion of Lemma 3.4.8.

Lemma 3.4.5 *Let $f : X \to Y$ be a morphism of simplicial objects over $[C]$ which is a projective weak equivalence as a morphism in $\Delta^{op}R(C)$. Then f belongs to $cl_{\bar{\Delta}}(\emptyset)$.*

Proof: Applying Proposition 2.2.12 to $E = \emptyset$ and N being the set of morphisms $0 \to U$ for all U in C we get a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ Ex(X) & \xrightarrow{Ex(f)} & Ex(Y) \end{array}$$

where the vertical arrows are in $cl_{\bar{\Delta}}(\emptyset)$ and the domain and the codomain of the lower horizontal one take objects of C to fibrant simplicial sets. Looking at the construction of $Ex(X)$ from the point of view of the projective closed model structure we see that the vertical morphisms are J-cells and in particular projective weak equivalences. Thus $Ex(f)$ is a projective weak equivalence and from now on we may assume that X and Y takes objects of C to Kan simplicial sets.

Let X_i, Y_i be the terms of X and Y and let C' be the full subcategory of $[C]$ which consists of objects of C and X_i, Y_i. Denote by G' the functor

$$G_{C'} : R(C) \to \Delta^{op}[C'] = \Delta^{op}[C]$$

Lemma 3.4.2 shows that the canonical morphisms $G'(X_i) \to X_i$ and $G'(Y_i) \to Y_i$ are homotopy equivalences. For a simplicial radditive functor let $\Delta G'(-)$ be the diagonal of the bisimplicial object $G'(-)$. We have a commutative square of the form

$$\begin{array}{ccc} \Delta G'(X) & \longrightarrow & \Delta G'(Y) \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array} \qquad (18)$$

where the vertical morphisms are diagonals of morphisms of bisimplicial objects which are row-wise homotopy equivalences and therefore they belong to

$cl_\Delta(\emptyset)$. Consider the upper vertical arrow. It is the diagonal of a morphism whose columns are of the form

$$\coprod_\alpha U_\alpha \otimes X(U_\alpha) \to \coprod_\alpha U_\alpha \otimes Y(U_\alpha) \tag{19}$$

where U_α is in $C \cup \{X_i, Y_i\}$. For U_α in C the map of simplicial sets $X(U_\alpha) \to Y(U_\alpha)$ is a weak equivalence by the assumption on f. The same is true for U_α in $\{X_i, Y_i\}$ since these objects are coproducts of objects in C and a product of weak equivalences of fibrant simplicial sets is a weak equivalence. This implies easily that the morphisms (19) are homotopy equivalences and in particular belong to $cl_\Delta(\emptyset)$.

Combining Proposition 3.2.4 with Lemma 3.4.5 we get the following result.

Proposition 3.4.6 *The class of morphisms in $\Delta^{op}[C]$ which are projective weak equivalences as morphisms of simplicial radditive functors coincides with $cl_{\bar{\Delta}}(\emptyset)$. In particular it is $\bar{\Delta}$-closed.*

Corollary 3.4.7 *For any Y in $[C]$ the morphism $G(Y) \to Y$ is in $cl_{\bar{\Delta}}(\emptyset)$.*

Proof: Lemma 3.1.7 implies that for a filtered system $F(i)$ of radditive functors one has $G(colim(F(i))) = colim G(F(i))$. Since any object of $[C]$ is a filtered colimit of object lying in C we conclude by Lemma 3.4.2 that our morphism is a filtered colimit of homotopy equivalences. Using again Lemma 3.1.7 we conclude that it is a projective weak equivalence. Therefore it belongs to $cl_{\bar{\Delta}}(\emptyset)$ by Lemma 3.4.5.

Lemma 3.4.8 *For any radditive functor F the morphism $G(F) \to F$ is a projective weak equivalence.*

Proof: Let $0 \to Cof(F) \to F$ be the standard factorization of the morphism $0 \to F$ such that the first arrow is a sequential I-cell and the second one has the right lifting property for elements of I i.e. is a trivial projective fibration. Consider the commutative square

$$\begin{array}{ccc} \Delta G(Cof(F)) & \longrightarrow & Cof(F) \\ \downarrow & & \downarrow \\ G(F) & \longrightarrow & F \end{array}$$

The upper horizontal morphism is the diagonal of a morphism which is row-wise of the form $G(T_i) \to T_i$ where T_i are the terms of $Cof(F)$. Since elements of I are term-wise coprojections each term of $Cof(F)$ is a filtered colimit of objects of C. Therefore Lemma 3.4.2 implies that each morphism $G(T_i) \to T_i$ is a filtered colimit of homotopy equivalences and in particular a projective weak equivalence. The left vertical arrow is the diagonal of a morphism which is column-wise of the form $\coprod U \otimes Cof(F)(V) \to \coprod U \otimes F(V)$. Since $Cof(F)(V) \to F(V)$ is a weak equivalence of simplicial sets this morphism belongs to $cl_{\bar{\Delta}}(\emptyset)$ by Proposition 2.2.4. Proposition 3.4.6 implies that it is a projective weak equivalence. We conclude that three out of four arrows in our square are projective weak equivalences and therefore the fourth is a projective weak equivalence.

For a functor $F : C \to D$ denote by $iso(F)$ the class of morphisms f in C such that $f(f)$ is an isomorphism.

Proposition 3.4.9 *Let C be a small category with finite coproducts. Then the functor $\Phi : \Delta^{op}[C] \to H(C)$ is a localization and $iso(\Phi) = cl_{\bar{\Delta}}(\emptyset)$.*

Proof: It is sufficient to show that Φ is a localization. The fact that $iso(\Phi) = cl_{\bar{\Delta}}(\emptyset)$ follows from Lemma 3.4.5. Let $H'(C)$ be the localization of $\Delta^{op}[C]$ with respect to $cl_{\bar{\Delta}}(\emptyset)$. To prove the proposition we will construct two functors

$$[inc] : H'(C) \to H(C)$$

and

$$[\Delta G] : H(C) \to H'(C)$$

together with isomorphisms $[inc][\Delta G] \to Id_{H(C)}$ and $[\Delta G][inc] \to Id_{H'(C)}$. Let inc be the inclusion of $\Delta^{op}[C]$ to $\Delta^{op}R(C)$. Proposition 3.4.6 implies that inc takes elements of $cl_{\bar{\Delta}}(\emptyset)$ to projective weak equivalences and thus defines a functor $[inc]$ between the localized categories. Define

$$\Delta G : \Delta^{op}R(C) \to \Delta^{op}[C]$$

as the functor which takes a simplicial presheaf X to the diagonal of the bisimplicial object $G(X)$.

Lemma 3.4.10 *The functor ΔG takes projective weak equivalences to elements of $cl_{\bar{\Delta}}(\emptyset)$.*

Proof: If $f : X \to X'$ is a projective weak equivalence then the columns of the map of bisimplicial objects $G(X) \to G(X')$ are of the form $\coprod U \otimes f$ where $U \in ob(C)$ and f are weak equivalences of simplicial sets. By Proposition 2.2.4 they belong to $cl_{\bar{\Delta}}(\emptyset)$.

We denote by $[\Delta G]$ the functor $H(C) \to H'(C)$ defined by ΔG according to Lemma 3.4.10. For any simplicial radditive functor X we have an obvious morphism $inc(\Delta G(X)) \to X$ and for any X in $\Delta^{op}[C]$ an obvious morphism $\Delta G(inc(X)) \to X$. These morphisms define natural transformations $[inc][\Delta G] \to Id_{H(C)}$ and $[\Delta G][inc] \to Id_{H'(C)}$. Lemma 3.4.8 and Corollary 3.4.7 imply that they are isomorphisms.

For a class of morphisms E in $\Delta^{op}R(C)$ let $E \cap [C]$ denote the subset of elements of E whose domains and codomains are in $\Delta^{op}[C]$. Lemmas 3.3.6 and 3.3.5 together with Lemma 3.4.5 imply the following version of Proposition 3.3.7.

Proposition 3.4.11 *Let E be as in Lemma 3.3.6 and assume in addition that the domains and codomains of elements of E are in $\Delta^{op}[C]$. Then one has*

$$cl_l(E) \cap [C] \subset cl_{\bar{\Delta}}(E)$$

Denote by $H(C, E)$ the localization of the category $\Delta^{op}R(C)$ with respect to the class of E-local equivalences and let $Phi : \Delta^{op}[C] \to H(C, E)$ be the obvious functor.

Proposition 3.4.12 *Let E be a class of morphisms such that domains and codomains of elements of E are cofibrant and reliably compact. Then the functor Φ is a localization and $iso(\Phi) = cl_l(E) \cap [C]$.*

Proof: The fact that Φ is a localization follows easily from Proposition 3.4.9. Our conditions on E imply that there exists a natural transformation of functors $Id \to Ex$ such that for an object X the morphism $X \to Ex(E)$ is an E-local equivalence and $Ex(X)$ is E-local. This implies that any element of $iso(\Phi)$ is an E-local equivalence.

Lemma 3.4.13 *Let E be a class of morphisms and assume that domains and codomains of elements of E are cofibrant and reliably compact. Then any morphism $f : X \to Y$ in $H(C, E)$ is isomorphic to the image of a morphism in $\Delta^{op}[C]$.*

Proof: Let $\tilde{X} \to X$ be the standard cofibrant replacement of X in the projective model structure and $Y \to Ex(Y)$ an E-local replacement of Y. Then f is isomorphic to the image of a morphism $\tilde{f} : \tilde{X} \to Ex(Y)$ in $\Delta^{op}R(C)$. Applying to $\tilde{f}$ the functor ΔG we get a morphism in $\Delta^{op}[C]$ whose image is isomorphic to f.

Remark 3.4.14 The conditions on the domains and codomains of elements of E used in P roposition 3.4.12 and Lemma 3.4.13 are not necessary. For a more general E one can still prove the same results using the transfinite arguments to produce E-local replacements of objects.

The following lemma is included here as an application of Proposition 3.4.9. Let us say that an object X of $\Delta^{op}C$ is n-connected if for any object Y of C the simplicial set $S(Y, X)$ is n-connected.

Proposition 3.4.15 *Let C be a category with a final object pt and $F : C \to C'$ a functor such that $F(pt)$ is a final object of C'. The the extension of F to $\Delta^{op}C$ takes n-connected objects to n-connected objects.*

Proof: We have to show that for any n-connected object X of $\Delta^{op}C$ and any object U of C' the simplicial set $S(U, F(X))$ is n-connected. Replacing C by the full subcategory generated by the terms of X and C' by the full subcategory generated by the terms of $F(C)$ and U we may assume that both categories are small. As in Example 3.1.2 let $C^{\amalg_{<\infty}}$ be the full subcategory of the category of presheaves on C which consists of finite coproducts of representable presheaves and let $C^{\amalg} = [C^{\amalg_{<\infty}}]$ be the full subcategory which consists of all coproducts of representable presheaves. Let X be an n-connected object of $\Delta^{op}C^{\amalg}$. Then we can find a weakly equivalent to it simplicial presheaf X' such that $X'_i = *$ for $i \leq n$. Applying the canonical simplicial resolution functor G to X' we get a bisimplicial object X'' over $C^{\amalg}$ whose diagonal is weakly equivalent to X. Since the extension of F to a functor $C^{\amalg} \to C'^{\amalg}$ commutes with coproducts it satisfies the condition of Proposition 2.3.2. Proposition 3.4.9 implies now that $F(X)$ is weakly equivalent to $F(diag(X'')) = diag(F(X''))$. The first n rows of X'' are homotopy equivalent to the final object by our choice of X' and Lemma 3.4.2. Therefore the same holds for $F(X'')$. It remains to use the fact that if K is a bisimplicial set whose first n rows are contractible then $diag(K)$ is n-connected (see [,]).

4 Clant morphisms

4.1 Clant morphisms and grainy categories

Definition 4.1.1 *Let C be a category with push-forwards and W a class of morphisms in C. A morphism $f : B \to A$ is called clant with respect to W if for any morphism $B \to Y$ and any morphism $Y \to Z$ in W the morphism $X \to T$, defined by the following push-forward squares, is in W:*

$$\begin{array}{ccccc} B & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & X & \longrightarrow & T \end{array}$$

One may say that a morphism f is pre-clant with respect to W if a push-forward of an element of W by f is in W. Then a morphism is clant if it is universally (with respect to push-forwards) pre-clant.

Example 4.1.2 An object X is called clant if the morphism $0 \to X$, where 0 is an initial object, is clant. One can easily see that X is clant if and only if $W \coprod Id_X \subset W$.

Example 4.1.3 If C is a closed model category with the class of weak equivalences W, then C is left proper if and only if every cofibration is clant.

The following lemma is straightforward.

Lemma 4.1.4 *The composition of two clant morphisms is clant. A push-forward of a clant morphism is clant. A retract of a clant morphism is clant.*

Everywhere below when we consider morphisms in $\Delta^{op}R(C)$ a clant morphism means a morphism clant with respect to the class of projective weak equivalences.

Lemma 4.1.5 *Let $f : B \to A$ be a clant morphism in $\Delta^{op}R(C)$. Then for any push-forward square Q of the form (2) the morphism $K_Q \to X$ is a projective weak equivalence.*

Proof: We have a diagram of push-forward squares of the form

$$\begin{array}{ccccc} B & \longrightarrow & cyl(B \to Y) & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & K_Q & \longrightarrow & X \end{array}$$

By Lemma 2.1.5 the morphism $cyl(B \to Y) \to Y$ is a projective weak equivalence and we conclude that $K_Q \to X$ is a projective weak equivalence.

Definition 4.1.6 *A category C is called grainy if it has finite coproducts (including the initial object) and any object of $R(C)$ is clant in $\Delta^{op}R(C)$.*

Example 4.1.7 The categories of the three "standard" types described in Examples 3.1.1, 3.1.2, 3.1.3 are grainy.

Example 4.1.8 Let C_A be the category of free finitely generated commutative algebras over a commutative ring A (see Example 3.1.4) such that $R(C_A)$ is the category of all commutative algebras over A. A flat algebra over A is clant as an object of $R(C_A)$ but in general an algebra need not to be clant. The category C_A is grainy if A is a field but not if A is a more general ring.

Lemma 4.1.9 *Let C be a grainy category. Then the class of projective weak equivalences in $\Delta^{op}R(C)$ is $\bar{\Delta}$-closed. In particular any object of $\Delta^{op}R(C)$ is clant.*

Proof: By Lemma 3.2.3 the class of projective weak equivalences is closed under filtered colimits and by Lemma 3.2.2 it is Δ-closed. Therefore it is sufficient to check that it is closed under finite coproducts or, equivalently, that for a simplicial radditive functor A and a projective weak equivalence $f : X \to Y$ the morphism $f \coprod Id_A$ is a projective weak equivalence. This morphism is the diagonal of a morphism of bisimplicial objects whose rows are of the form $f \coprod A_i$ where A_i are the terms of X. They are projective weak equivalences by definition of a grainy category and we conclude that $f \coprod Id_A$ is a projective weak equivalence by Lemma 3.2.2.

Lemma 4.1.10 *Let C be a grainy category, $f : X \to X'$ a projective weak equivalence in $\Delta^{op}R(C)$ and K a simplicial set. Then $f \otimes Id_K : X \otimes Id_K \to X' \otimes Id_K$ is a projective weak equivalence.*

Proof: The morphism $f \otimes Id_K$ is the diagonal of a morphism of bisimplicial objects whose rows are coproducts of copies of f. The statement of the lemma follows now from Lemma 3.2.2 and Lemma 4.1.9.

Lemma 4.1.11 *Let C be a grainy category. Then a morphism $f : B \to A$ is clant if and only if for any push-forward square Q of the form (2) the morphism $K_Q \to X$ is a projective weak equivalence.*

Proof: The "only if" part follows from Lemma 4.1.5. To prove the "if" part consider two push-forward squares of the form

$$\begin{array}{ccccc} B & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & X & \longrightarrow & T \end{array}$$

where $Y \to Z$ is a projective weak equivalence. Let Q_1 be the left square, Q_2 the right square and Q_3 the big square. By our condition the morphisms $K_{Q_1} \to X$ and $K_{Q_3} \to T$ are projective weak equivalences. Lemma 2.1.10 together with Lemma 4.1.9 imply that the morphism $X \to T$ belongs to $cl_{\Delta,\coprod}(\{Y \to Z\})$. Since C is grainy the class of projective weak equivalences is $(\Delta, \coprod)$-closed by Lemma 3.2.2 and Lemma 4.1.9 and we conclude that the morphism $X \to T$ is a projective weak equivalence.

Lemma 4.1.12 *Let C be a grainy category and $f : B \to A$ a morphism in $\Delta^{op}R(C)$ such that the terms $f_i : B_i \to A_i$ are clant. Then f is clant.*

Proof: By Lemma 4.1.11 it is sufficient to show that for a push-forward square of the form (2) the morphism $K_Q \to X$ is a projective weak equivalence. This morphism is the diagonal of a morphism of bisimplicial objects whose rows are of the form $K_{Q_i} \to X_i$ where Q_i is the square formed by the i-th terms of A, B, Y, X. If $B_i \to A_i$ are clant then $K_{Q_i} \to X_i$ are projective weak equivalences by Lemma 4.1.5 and we conclude that $K_Q \to X$ is a projective weak equivalence by Lemma 3.2.2.

Proposition 4.1.13 *Let C be a grainy category. Then the projective closed model structure on $\Delta^{op}R(C)$ is left proper.*

Proof: We have to show that a projective cofibration is a clant morphism. Since a retract of a clant morphism is clant, Lemma 3.2.9 implies that it is sufficient to show that a term-wise coprojection is clant. In view of Lemma 4.1.12 it is enough to check that a coprojection in $R(C)$ is clant which follows from the definition of grainy categories.

Lemma 4.1.14 *Let C be a grainy category and E a class of morphisms in $\Delta^{op}[C]$. Then one has*

$$\Delta G(cl_{\bar{\Delta}}(E \cup W_{proj})) \subset cl_{\bar{\Delta}}(E)$$

where on the left hand side E is considered as a class of morphisms in $\Delta^{op}R(C)$ and W_{proj} is the class of projective weak equivalences.

Proof: Denote the class $\Delta G^{-1}(cl_{\bar{\Delta}}(E))$ by H. Corollary 3.4.7 implies that E is contained in H and Lemma 3.4.10 implies that W_{proj} is contained in H. It remains to show that H is $\bar{\Delta}$-closed. It is clearly Δ-closed. Let us show that it is closed under coproducts. Let $f_\alpha : X_\alpha \to Y_\alpha$ be a family of elements of H. We have a commutative square of the form

$$\begin{array}{ccc} \coprod \Delta G(X_\alpha) & \longrightarrow & \coprod \Delta G(Y_\alpha) \\ \downarrow & & \downarrow \\ \Delta G(\coprod X_\alpha) & \longrightarrow & \Delta G(\coprod Y_\alpha) \end{array}$$

which shows that it is sufficient to prove that for any family X_α the morphism

$$\coprod \Delta G(X_\alpha) \to \Delta G(\coprod X_\alpha) \tag{20}$$

is in $cl_{\bar{\Delta}}(\emptyset)$. Both sides of (20) map to $\coprod X_\alpha$. The first of these morphisms is a projective weak equivalence by Lemma 4.1.9 and the assumption that C is grainy. The second one is a projective weak equivalence by Lemma 3.4.8. Therefore, (20) is a projective weak equivalence and since its domain and codomain are in $\Delta^{op}[C]$ we conclude by Lemma 3.4.5 that it is in $cl_{\bar{\Delta}}(\emptyset)$. In view of Lemma 2.2.3 it remains to check that morphisms of the form $\Delta G(F \otimes l_\infty \to F)$ are in $cl_{\bar{\Delta}}(\emptyset)$. This follows from the fact that ΔG commutes with filtered colimits and that $F \otimes l_\infty \to F$ is a filtered colimit of homotopy equivalences.

Proposition 4.1.15 *Let E be a grainy category and E a class of morphisms in $\Delta^{op}R(C)$. Then one has*

$$cl_{\bar{\Delta}}(E \cup W_{proj}) \cap [C] = cl_{\bar{\Delta}}(E)$$

Proof: The right hand side is obviously contained in the left hand side. Let $f : X \to Y$ be a morphism in $\Delta^{op}R(C)$ which belongs to $cl_{\bar{\Delta}}(E \cup W_{proj})$ and whose domain and codomain are in $\Delta^{op}[C]$. Then the square

$$\begin{array}{ccc} \Delta G(X) & \longrightarrow & \Delta G(Y) \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

together with Corollary 3.4.7 and Lemma 4.1.14 imply that f is in $cl_{\bar{\Delta}}(E)$.

4.2 Degeneracy free objects and the wrapping functor

Let Δ_{Mon} be the subcategory of monomorphisms in the standard simplicial category Δ. A contravariant functor from Δ_{Mon} to a category is a "simplicial object with no degeneracies". Let π_* be the obvious forgetful functor from $\Delta^{op}C$ to $\Delta^{op}_{Mon}C$. General argument shows that if C has colimits then π_* has a left adjoint π^*. In fact, since any morphism in Δ has a canonical decomposition into an epimorphism followed by a monomorphism, one needs only finite coproducts to define π^*. For a functor $Z = (Z_i)$ from Δ_{Mon} to C the simplicial object $\pi^*(Z)$ has terms of the form

$$\pi^*(Z)_i = \coprod_{[i]\to[j]} Z_j$$

where $[i] \to [j]$ runs through epimorphisms from $[i]$ to $[j]$ in Δ (see [12, Ex.8.1.5]).

Definition 4.2.1 *Let C be a category with finite coproducts. An object X in $\Delta^{op}R(C)$ is called degeneracy free if it belongs to the image of the functor π^*.*

If $Z = (Z_n)$ is an object of Δ^{op}_{Mon} and $X = \pi^*(Z)$ the corresponding degeneracy free simplicial object we say that X is based on the sequence (Z_n). Since $R(C)$ has colimits, for any simplicial object X we can define the skeletons $sk_n(X)$ in the usual way such that $X = colimsk_n(X)$ and for each n one has a push-forward square of the form

$$\begin{array}{ccc} L_n(X)\otimes\Delta^n \coprod_{L_n(X)\otimes\partial\Delta^n} X_n\otimes\partial\Delta^n & \longrightarrow & sk_{n-1}(X) \\ \downarrow & & \downarrow \\ X_n\otimes\Delta^n & \longrightarrow & sk_n(X) \end{array} \tag{21}$$

where $L_n(X) = (sk_{n-1}(X))_n$ is the n-th latching object of X.

Lemma 4.2.2 *Let X be a degeneracy free object based on (Z_n). Then for any $n \geq 0$ there is a push-forward square of the form*

$$\begin{array}{ccc} Z_n\otimes\partial\Delta^n & \longrightarrow & sk_{n-1}(X) \\ \downarrow & & \downarrow \\ Z_n\otimes\Delta^n & \longrightarrow & sk_n(X) \end{array}$$

Proof: See [5, Cor. 1.14, p.358].

Definition 4.2.3 *An object of* $\Delta^{op}R(C)$ *is called easy if it is a degeneracy free object based on a sequence* (Z_n) *where* Z_n *are in* $[C]$.

Lemma 4.2.4 *Let* X *be an easy object of* $\Delta^{op}R(C)$. *Then one has:*

1. *the terms of* X *are in* $[C]$
2. X *is cofibrant in the projective model structure.*

Proof: It follows from Lemma 4.2.2. For the second statement one uses the fact that for $Z \in [C]$ the morphism $Z \otimes \partial\Delta^n \to Z \otimes \Delta^n$ is a cofibration which is a particular case of Proposition 3.2.11.

We call the functor $\pi^*\pi_*$ the wrapping functor and denote it by Wr. For any simplicial object X the object $Wr(X)$ is degeneracy free and its terms are given by the formula

$$Wr(X)_i = \coprod_{[i]\to[j]} X_j \tag{22}$$

where $[i] \to [j]$ runs through epimorphisms from $[i]$ to $[j]$ in Δ and X_j are the terms of X. If X belongs to $\Delta^{op}[C]$ then $Wr(X)$ is easy and therefore cofibrant. For any X, the adjunction defines a natural morphism $Wr(X) \to X$. The main goal of this section is to formulate a sufficient condition on X implying that this morphism is a projective weak equivalence. The following lemma, which follows immediately from Lemma 4.2.2, is not used anywhere in this paper. It is included here for future reference.

Lemma 4.2.5 *Let* $f : B \to B'$ *be a morphism of bisimplicial objects and* $f_i : B_i \to B'_i$ *be the rows of* f. *Let* $Wr_{col}(B)$ *and* $Wr_{col}(B')$ *be the objects obtained by applying the wrapping functor to simplicial objects in* $\Delta^{op}R(C)$ *formed by* B *and* B' *and* $Wr_{col}(f)$ *be the morphism* $Wr_{col}(B) \to Wr_{col}(B')$ *defined by* f. *Let further* E *be a class of morphisms in* $\Delta^{op}R(C)$ *such that:*

1. $f_i \in E$ *for all* $i \geq 0$
2. E *is closed under sequential colimits of coprojections*
3. *if* f *is in* E *and* K *is a finite simplicial set then* $f \otimes Id_K$ *is in* E

4. *if $f = (f_A, f_B, f_Y, f_X) : Q \to Q'$ is a morphism of elementary exact squares of the form (2) such that $f_A, f_B, f_Y \in E$ then $f_X \in E$.*

Then one has $\Delta(Wr_{col}(f)) \in E$.

Definition 4.2.6 *An object X of $\Delta^{op}R(C)$ is called vertically clant if the morphisms $L_n(X) \to X_n$ are clant.*

Proposition 4.2.7 *For any $n \geq 0$ the canonical morphism of simplicial sets $Wr(\Delta^n) \to \Delta^n$ is a weak equivalence.*

Proof: We need to show that the geometric realization of $Wr(\Delta^n)$ is contractible. Let us show that it is acyclic (i.e. has homology of the poin), simply connected and connected. The fact that it is contractible will follow from the Whitehead theorem. Consider the geometric realization of the one dimensional skeleton of $Wr(X)$ for a simplicial set X. It is a graph which has the same vertices as the geometric realization of X and whose edges are of two types. There is one edge for each edge of the geometric realization of X (i.e. a non-degenerate simplex of X) and also one edge for each degenerate simplex of X which starts and ends in the corresponding vertex. This description implies that for a connected X, $Wr(X)$ is also connected and in order to show that $Wr(\Delta^n)$ is simply connected it is sufficient to check that the loops corresponding to the degenerate 1-simplexes represent the unit element of π_1. Choose a vertex $v = v_{(0)}$ of Δ^n and let $v_{(1)}$ be the corresponding degenerate 1-simplex of Δ^n and $v_{(2)}$ the corresponding degenerate 2-simplex. Then the boundary of the 2-simplex in the geometric realization of $Wr(X)$ corresponding to $v_{(2)}$ is a loop of the form $v_{(1)} \circ v_{(1)}^{-1} \circ v_{(1)}$ which implies that $v_{(1)} \circ v_{(1)}^{-1} \circ v_{(1)} = 1$ in π_1 and therefore $v_{(1)} = 1$ in π_1.

Let us prove now that $Wr(\Delta^n)$ is acyclic. We will show a more general fact namely that for any simplicial set X the morphism $Wr(X) \to X$ defines an isomorphism on homology. Denote by $C(X)$ the complex with terms $C_n(X) = \mathbf{Z}(X_n)$ where $\mathbf{Z}(X_n)$ is the free abelian group generated by X_n and denote by $NC(X)$ the normalized chain complex of the simplicial abelian group $\mathbf{Z}(X)$ such that we have

$$NC_n(X) = \cap_{i=1,\dots,n} ker(\partial_n^i : \mathbf{Z}(X_n) \to \mathbf{Z}(X_{n-1}))$$

It is a well known fact that the inclusion $NC(X) \to C(X)$ is a quasi-isomorphism (see [12, 8.3.8]). We claim that the map $Wr(X)$ induces an isomorphism of complexes $NC(Wr(X)) \to C(X)$. Indeed, the projection $C_n(Wr(X)) \to C_n(X)$ is isomorphic to the projection from $C_n(Wr(X))$ to the quotient with respect to the sum of the images of the degeneracy morphisms. This projection is an isomorphism on NC_n by [12, 8.3.8].

Proposition 4.2.8 *Let C be a grainy category and X a vertically clant object in $\Delta^{op}R(C)$. Then the canonical morphism $Wr(X) \to X$ is a projective weak equivalence.*

Proof: Consider first the case of X of the form $A \otimes \Delta^n$ where A is in $R(C)$. We have $Wr(A \otimes \Delta^n) = A \otimes Wr(\Delta^n)$ and by Proposition 4.2.7 and Proposition 2.2.4 we conclude that the morphisms $Wr(A \otimes \Delta^n) \to A \otimes \Delta^n$ are in $cl_{\bar{\Delta}}(\emptyset)$. By Lemma 4.1.9 we conclude that these morphisms are projective weak equivalences. Our result follows now from Lemma 4.2.9 applied to the natural transformation $Wr \to Id$.

Lemma 4.2.9 *Let C be a grainy category, $F, G : \Delta^{op}R(C) \to \Delta^{op}R(C)$ two functors and $\phi : F \to G$ a natural transformation. Assume that the following conditions hold:*

1. *F and G commute with push-forwards and sequential colimits*
2. *for a morphism $f : X \to Y$ the terms $F(f)_i : F(X)_i \to F(Y)_i$ of the morphism $F(f)$ belong to $cl_{\coprod}(f_i : X_i \to Y_i)$ and the same holds for G*
3. *for A in $R(C)$ and $n \geq 0$ the morphism $F(A \otimes \Delta^n) \to G(A \otimes \Delta^n)$ is a projective weak equivalence.*

Then the morphism $F(X) \to G(X)$ is a projective weak equivalence for any vertically clant X.

Proof: By Lemma 3.2.3 the class of projective weak equivalences in $\Delta^{op}R(C)$ is closed under filtered colimits. Therefore it is sufficient to check inductively that the morphisms $F(sk_nX) \to G(sk_nX)$ are projective weak equivalences. For $n = 0$ this follows directly from our conditions. For $n > 0$ denote by Q_n the push-forward square (21) and consider the morphism $F(Q_n) \to G(Q_n)$ defined by ϕ. The terms of the left vertical arrow of (21) are finite coproducts of morphisms of the form $Id : X_n \to X_n$ and $L_n(X) \to X_n$. By our second condition on F and G we conclude that the terms of the left vertical arrows in $F(Q_n)$ and $G(Q_n)$ are finite coproducts of clant morphisms. Since C is grainy this implies that they are clant and by Lemma 4.1.5 we conclude that the morphisms $K_{F(Q_n)} \to F(sk_n(X))$ and $K_{G(Q_n)} \to G(sk_n(X))$ are projective weak equivalences. Consider the commutative square:

$$\begin{array}{ccc} K_{F(Q_n)} & \longrightarrow & K_{G(Q_n)} \\ \downarrow & & \downarrow \\ F(sk_n(X)) & \longrightarrow & G(sk_n(X)) \end{array}$$

To show that the lower horizontal arrow is a projective weak equivalence it remains to show that the upper horizontal arrow is. By Lemma 2.1.7 and Lemma 4.1.9 it is sufficient to show that the morphisms

$$F(sk_{n-1}(X)) \to G(sk_{n-1}(X))$$

$$F(L_n(X) \otimes \Delta^n \coprod_{L_n(X)\otimes\partial\Delta^n} X_n \otimes \partial\Delta^n) \to G(L_n(X) \otimes \Delta^n \coprod_{L_n(X)\otimes\partial\Delta^n} X_n \otimes \partial\Delta^n)$$

$$F(X_n \otimes \Delta^n) \to G(X_n \otimes \Delta^n)$$

are projective weak equivalences. For the first one it follows from the inductive assumption. For the last one it follows from our condition on ϕ. It remains to check that the second morphism is a projective weak equivalence. Consider the push-forward square Q'_n of the form

$$\begin{array}{ccc} L_n(X) \otimes \partial\Delta^n & \longrightarrow & L_n(X) \otimes \Delta^n \\ \downarrow & & \downarrow \\ X_n \otimes \partial\Delta^n & \longrightarrow & L_n(X) \otimes \Delta^n \coprod_{L_n(X)\otimes\partial\Delta^n} X_n \otimes \partial\Delta^n \end{array}$$

Using the same reasoning as before we see that it is sufficient to check that the morphisms

$$F(L_n(X) \otimes \partial\Delta^n) \to G(L_n(X) \otimes \partial\Delta^n)$$

$$F(X_n \otimes \partial\Delta^n) \to G(X_n \otimes \partial\Delta^n)$$

$$F(L_n(X) \otimes \Delta^n) \to G(L_n(X) \otimes \Delta^n)$$

are projective weak equivalences. For the first two it follows from the inductive assumption since

$$L_n(X) \otimes \partial\Delta^n = sk_{n-1}(L_n(X) \otimes \partial\Delta^n)$$

$$X_n \otimes \partial\Delta^n = sk_{n-1}(X_n \otimes \partial\Delta^n)$$

and for the last one from our assumption on ϕ.

4.3 E-local equivalences in grainy categories

Lemma 4.3.1 *Let C be a grainy category. Then for any class E the class $cl_l(E)$ is closed under coproducts.*

Proof: Follows from the fact that in a grainy category a coproduct of projective weak equivalences is a projective weak equivalence and therefore for cofibrant replacements $\tilde{f}_\alpha$ of f_α the coproduct $\coprod_\alpha \tilde{f}_\alpha$ is a cofibrant replacement of $\coprod_\alpha f_\alpha$.

Lemma 4.3.2 *Let C be a grainy category. Then for any $f : X \to X'$ and a simplicial set K the map $f \otimes Id_K : X \otimes K \to X' \otimes K$ belongs to $cl_l(\{f\})$.*

Proof: For a cofibrant object A and a simplicial set K the object $A \otimes K$ is cofibrant by Proposition 3.2.11. Together with Lemma 4.1.10 this implies that for a cofibrant replacement $\tilde{f}$ of f the morphism $\tilde{f} \otimes Id_K$ is a cofibrant replacement of $f \otimes Id_K$. The statement of the lemma follows immediately from this fact.

Lemma 4.3.3 *Let B be a bisimplicial radditive functor such that the simplicial object in $\Delta^{op}R(C)$ formed by the rows of B is degeneracy free on cofibrant objects Z_n. Then $\Delta(B)$ is cofibrant.*

Proof: Applying Lemma 4.2.2 and using the fact that the diagonal commutes with colimits we conclude that there is a push-forward square of the form

$$\begin{array}{ccc} \Delta(Z_n \otimes \partial\Delta^n) & \longrightarrow & \Delta(sk_{n-1}(B)) \\ \downarrow & & \downarrow \\ \Delta(Z_n \otimes \Delta^n) & \longrightarrow & \Delta(sk_n(B)) \end{array}$$

Since the left vertical arrow in this diagram is isomorphic to $Z_n \otimes \partial\Delta^n \to Z_n \otimes \Delta^n$, where Z_n is now considered as a simplicial object over C, it is a cofibration by Proposition 3.2.11 and we conclude that $\Delta(B) = colim \Delta(sk_n(B))$ is cofibrant.

Lemma 4.3.4 *Let $f : B \to B'$ be a morphism of bisimplicial radditive functors such that the simplicial objects formed by rows of B and B' are degeneracy free on cofibrant objects of $\Delta^{op}R(C)$. Then $\Delta(f) \in cl_l(\{f_i\})$.*

Proof: Let Y be an object local with respect to the rows of f. Lemma 4.2.2 implies that the rows of both B and B' are cofibrant. Therefore, the maps $S(B'_i, Y) \to S(B_i, Y)$ are weak equivalences. The simplicial set $S(\Delta(B), Y)$ (resp. $S(\Delta(B'), Y)$) is the total space of the cosimplicial simplicial set formed by $S(B_i, Y)$'s (resp. $S(B'_i, Y)$'s). Our condition on B and B' implies that these cosimplicial simplicial sets are fibrant in the sense of [2]. Since the total space functor preserves weak equivalences of fibrant cosimplicial simplicial sets ([1]) we conclude that $S(\Delta(B'), Y) \to S(\Delta(B), Y)$ is a weak equivalence. Together with Lemma 4.3.3 this implies that $\Delta(f)$ is in $cl_l(\{f_i\})$.

Lemma 4.3.5 *Let C be a grainy category and $f : B \to B'$ be a morphism of bisimplicial objects in $R(C)$ whose columns are vertically clant and rows are cofibrant. Then $\Delta(f) \in cl_l(\{f_i\}_{i\geq 0})$ where $f_i : B_i \to B'_i$ are the rows of f.*

Proof: Applying the wrapping functor to the columns of B and B' we get a commutative diagram of the form

$$\begin{array}{ccc} Wr_{col}(B) & \longrightarrow & Wr_{col}(B') \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array}$$

where the vertical arrows are column-wise projective weak equivalences by Proposition 4.2.8. In view of Lemma 3.2.2 they define projective weak equivalences of the corresponding diagonal objects. On the other hand the explicit form of the functor Wr given by (22) shows that the rows of $Wr_{col}(f)$ are finite coproducts of rows of f. In particular they are in $cl_l(\{f_i\})$ by Lemma 4.3.1. The simplicial objects formed by rows of $Wr_{col}(B)$ and $Wr_{col}(B')$ are degeneracy free on the rows of B and B' respectively. Our result follows now from Lemma 4.3.4.

Proposition 4.3.6 *Let C be a category such that any simplicial radditive functor on C is vertically clant. Then for any class E the class $cl_l(E)$ is $\bar{\Delta}$-closed.*

Proof: Observe first that any category satisfying the condition of the lemma is grainy. Lemmas 3.3.8 and 4.3.1 imply that $cl_l(E)$ is closed under sequential colimits and coproducts. It remains to show that if $f : B \to B'$ is a morphism of bisimplicial radditive functors with rows $f_: B_i \to B'_i$ then one has $\Delta(f) \in$

$cl_l(\{f_i\})$. Applying the cofibrant replacement functor to the rows of B and B' we get a commutative square of the form

$$\begin{array}{ccc} Cof_{rows}(B) & \longrightarrow & Cof_{rows}(B') \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array}$$

In view of Lemma 3.2.2 the vertical arrows define projective weak equivalences on the diagonal objects. It remains to show that the diagonal of the upper horizontal arrow is in $cl_l(\{f_i\})$ which follows from our assumption on C and Lemma 4.3.5.

Theorem 4.3.7 *Let C be a category such that every simplicial radditive functor on C is vertically clant. Let E be a set of morphisms in $\Delta^{op}[C]$ whose domains and codomains are projectively cofibrant (e.g. easy) and reliably compact. Then one has:*

$$cl_l(E) = cl_{\bar{\Delta}}(E \cup W_{proj}) \tag{23}$$

$$cl_l(E) \cap [C] = cl_{\bar{\Delta}}(E) \tag{24}$$

Proof: Proposition 4.3.6 implies that the right hand side of (23) is contained in the left hand side. The opposite inclusion is proved in Proposition 3.3.7. The second equality follows from the first one and Proposition 4.1.15.

Corollary 4.3.8 *Under the assumptions of Theorem 4.3.7 the functor*

$$\Phi : \Delta^{op}[C] \to H(C, E)$$

is a localization and $iso(\Phi) = cl_{\bar{\Delta}}(E)$.

Proof: It follows from the second equality of the theorem and Proposition 3.4.12.

Remark 4.3.9 One can easily see that the categories of the three standard classes considered in Examples 3.1.1, 3.1.2, 3.1.3 satisfy the conditions of Theorem 4.3.7.

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