

Push-forwards in oriented cohomology theories of algebraic varieties

I. Panin and A. Smirnov

Abstract

Notions of a cohomology theory and an oriented cohomology theory for algebraic varieties are introduced, push-forwards for projective morphisms are constructed and a Riemann-Roch theorem for ring operations between oriented theories is proved. An explicit formula for the Todd genus related to a ring operation is given. The theory is illustrated by motivic cohomology, algebraic K-theory, algebraic cobordism theory and by other examples.

1 Introduction

Let k be a field and consider the category of pairs (X, U) with a smooth variety X over k and its open subset U . By a cohomology theory we mean a contravariant functor A from this category to the category of abelian groups endowed with a functor transformation $\partial : A(U) \rightarrow A(X, U)$ and satisfying the localization, Nisnevich excision and homotopy invariance properties (2.0.1).

An oriented cohomology theory is a ring cohomology theory A (3.1.1) together with a rule assigning to each variety X and to each vector bundle E/X a two-sided $A(X)$ -module isomorphism $A(X) \rightarrow A(E, E - X)$ satisfying certain natural properties (3.1.1) and called Thom isomorphisms.

Any oriented cohomology theory is endowed with Chern classes for line bundles and therefore with higher Chern classes. Vice versa a ring theory endowed with Chern classes can be oriented in such a way that the corresponding Chern classes coincide with the given ones.

Any oriented cohomology theory is endowed with push-forwards for projective morphisms satisfying certain natural properties (4.1.3). Vice versa a ring cohomology theory endowed with push-forwards (4.1.2) can be oriented in such a way that the corresponding push-forwards coincide with the given ones.

For well-known theories (like K-theory, étale cohomology theory, motivic cohomology theory) one can use the usual Chern classes and in order to get an orientation such that the mentioned push-forwards coincide with the classical ones. Another example of an oriented cohomology theory is algebraic cobordism theory [V1] which can be oriented by choosing the tautological class as the Thom class. If $k = \mathbb{C}$ and the theory is the complex cobordism theory restricted to the category of pairs of algebraic varieties then

the Chern classes corresponding to the orientation coincide with the Conner-Floyd classes [CF] and the corresponding push-forwards coincide with the push-forwards in the complex cobordism theory [Qu1].

With a given ring operation $\varphi : A \rightarrow B$ (2.3.3) between oriented theories we associate an inverse Todd series $itd_\varphi(t) \in \bar{B}[[t]]$ where $\bar{B} = B\langle pt \rangle$ is the coefficient ring of the theory B. This gives rise to an inverse Todd genus $itd_\varphi(E)$ of a vector bundle E . The Riemann-Roch theorem (5.1.3) states that for a closed imbedding $i : Y \rightarrow X$ and for each $a \in A(Y)$ one has the following relation in $B(X)$

$$i_B(\varphi(a) \cup itd_\varphi(N)) = \varphi(i_A(a)),$$

where N is the normal bundle to Y and i_A (resp. i_B) is the push-forward for i corresponding to the theory A (resp. B) and to its orientation.

In the case when the inverse Todd series $itd_\varphi(t)$ is invertible in $\bar{B}[[t]]$ we define the Todd series $td_\varphi(t)$ as the multiplicative inverse to $itd_\varphi(t)$. This gives rise to a Todd genus $td_\varphi(E)$ of any vector bundle E . The Riemann-Roch theorem (5.1.4) states that for a projective morphism $f : Y \rightarrow X$ and for each $a \in A(Y)$ one has the following relation in $B(X)$

$$f_B(\varphi(a) \cup td_\varphi(T_Y)) = \varphi(f_A(a)) \cup td_\varphi(T_X).$$

Note that in the case when φ is the Chern character from K-theory to Chow ring these formulas coincide with the classical ones. In the case when φ is the Adams operation ψ^r from K-theory to itself the first of these formulas coincides with the classical ones as well.

Now take $k = \mathbb{R}$ and consider the theory A sending (X, U) to $H^*(X(\mathbb{R}), U(\mathbb{R}); \mathbb{Z}/2)$ (the singular cohomology of topological spaces). It turns out that the first Stiefel-Whitney class satisfies the axioms of Chern classes for line bundles (3.2.1) in this case and thus it determines the corresponding orientation of A . The total Steenrod square Sq is a ring operation from A to itself. The inverse Todd genus $itd(E)$ coincides in this case with the total Stiefel-Whitney class $w(E(\mathbb{R}))$ of the vector bundle $E(\mathbb{R})$. Thus the Riemann-Roch theorem for a closed imbedding $i : Y \hookrightarrow X$ states that the following relation holds

$$i_*(w(N(\mathbb{R}))) = Sq([Y])$$

where N is the normal bundle to Y . This is the known Wu formula.

Other examples are discussed in the main text below.

The Riemann-Roch theorems (5.1.3) and (5.1.4) are inspired by a Riemann-Roch theorem from [Dy]. However the explicit definitions of the inverse Todd genus (5.1.1) and of the Todd genus (5.1.2) possibly are not mentioned even in the topological context.

The subject of (3.3) and (3.7) was inspired by Morel's lectures (Muenster, June, 1999).

We thank A. Suslin for his interest to the subject. His remark on possible existence of push-forwards in general situation was the origin of our work on the subject.

1.1 Terminology and notation

Let k be a field. The term "variety" is used in this text for an arbitrary quasi-projective variety over k (in particular, for a singular variety). We fix the following notation:

- $\mathcal{A}b$ - the category of abelian groups;
- $\mathcal{S}m$ - the category of smooth varieties;
 $\mathcal{S}m\mathcal{O}p$ - the category of pairs (X, U) with smooth X and open U in X . Morphisms are morphisms of pairs.

We identify the category $\mathcal{S}m$ with a full subcategory of $\mathcal{S}m\mathcal{O}p$ assigning to a variety X the pair (X, \emptyset) ;

- $pt = \text{Spec}(k)$;
 For a smooth X and an effective divisor $D \subset X$ we write $L(D)$ for a line bundle over X whose sheaf of sections is the sheaf $\mathcal{L}_X(D)$ (see [Har, Ch.II,§6, 6.13]).
 $\mathbf{P}(V) = \text{Proj}(S^*(V^\vee))$ -the space of lines in a finite dimensional k -vector space V ;
 $L_V = \mathcal{O}_V(-1)$ -the tautological line bundle over $\mathbf{P}(V)$;
- $\mathbf{P}(E)$ - the space of lines in a vector bundle E ;
 $L_E = \mathcal{O}_E(-1)$ - the tautological line bundle on $\mathbf{P}(E)$;
 E^0 - the complement to the zero section of E ;
 E^\vee - the vector bundle dual to E ;
 $z : X \rightarrow E$ - the zero section of a vector bundle E ;
- For a contravariant functor A on $\mathcal{S}m$ set

$$A(\mathbf{P}^\infty) = \varprojlim A(\mathbf{P}(V)), \quad (1)$$

where the projective system is induced by all the finite dimensional vector subspaces $V \hookrightarrow k^\infty$.

For such V and $a \in A(\mathbf{P}^\infty)$ denote by a_V the canonical image of a in $A(\mathbf{P}(V))$.

In particular, taking the Picard group Pic as A one gets the definition of $\text{Pic}(\mathbf{P}^\infty)$.

Similarly set

$$A(\mathbf{P}^\infty \times \mathbf{P}^\infty) = \varprojlim A(\mathbf{P}(V) \times \mathbf{P}(W))$$

where the projective system is induced by all the finite dimensional subspaces $V, W \subset k^\infty$.

Moreover in the case when A is defined on $\mathcal{S}m\mathcal{O}p$ set

$$A(L_\infty, L_\infty^0) = \varprojlim A(L_V, L_V^0), \quad (2)$$

where the projective system is induced by all the finite dimensional subspaces $V \subset k^\infty$.

For a finite dimensional $V \subset k^\infty$ and $a \in A(L_\infty, L_\infty^0)$ denote by a_V the canonical image of a in $A(L_V, L_V^0)$.

2 Cohomology theories

2.0.1. Definition. A cohomology theory is a contravariant functor $\mathcal{S}mOp \xrightarrow{A} \mathcal{A}b$ together with a functor morphism $\partial : A(U) \rightarrow A(X, U)$ satisfying the following properties

1. *Localization:* the sequence $A(X) \xrightarrow{j^A} A(U) \xrightarrow{\partial_P} A(X, U) \xrightarrow{i^A} A(X) \xrightarrow{j^A} A(U)$ is exact for each pair $P = (X, U) \in \mathcal{S}mOp$, where $j : U \hookrightarrow X$ and $i : (X, \emptyset) \hookrightarrow (X, U)$ are the natural inclusions;
2. *Excision:* the operator $A(X, U) \rightarrow A(X', U')$ induced by a morphism $e : (X', U') \rightarrow (X, U)$ is an isomorphism, if the morphism e is etale and for $Z = X - U$, $Z' = X' - U'$ one has $e^{-1}(Z) = Z'$ and $e : Z' \rightarrow Z$ is an isomorphism;
3. *Homotopy invariance:* the operator $A(X) \rightarrow A(X \times \mathbf{A}^1)$ induced by the projection $X \times \mathbf{A}^1 \rightarrow X$ is an isomorphism.

The operator ∂_P is called the boundary operator and is written usually as ∂ .

We assume in the text, although it is not necessary, that A is a \mathbb{Z} -graded theory. This means that A is a \mathbb{Z} -graded functor and that ∂ is a graded operator of the degree $(+1)$.

We write also $A_Z(X)$ for $A(X, U)$, where $Z = X - U$, and call the group $A_Z(X)$ cohomology of X with the support on Z . The operator

$$A_Z(X) \xrightarrow{i^A} A(X) \quad (3)$$

is called the support extension operator for the pair (X, U) .

2.1 Examples

Consider a number of examples.

2.1.1. Etale cohomology. Let F be a locally constant torsion sheaf on the etale k -situs and assume that $\text{char}(k)$ is prime to the torsion of F . In this example $A^n(X, U) = H_{\mathbb{Z}}^n(X_{et}, F)$ [Mi, 3.1] and ∂ is defined in [Mi, 3.1.25]. The localization property for the pair (A, ∂) is proved in [Mi, 3.1.25], the excision property is proved in [Mi, 3.1.27] and the homotopy invariance is proved in [Mi, 6.4.20].

2.1.2. K -theory. Algebraic K -theory also can be fitted to the definition 2.0.1. To do this use, for instance, K -groups with support $K_n(X \text{ on } Z)$ ($n \geq 0$) of [TT]. So set $A^n(X, U) = K_{-n}(X \text{ on } Z)$, where $Z = X - U$. The definition of ∂ and the exactness of the localization sequence are contained in [TT, Theorem 5.1] (except the surjectivity of the restriction $A^0(X) \rightarrow A^0(U)$). If X is quasi-projective then $K(X \text{ on } X)$ coincides with the Quillen's K -groups $K_n^Q(X)$ by [TT, 3.9, 3.10]. This proves in particular the homotopy invariance $A^n(X)$ for smooth X . The excision property for A follows from [TT, 3.19]. It remains now to check the surjectivity of the restriction $A^0(X) \rightarrow A^0(U)$. Clearly $A^0(X) = K_0^Q(X)$ coincides with the Grothendieck group of the vector bundles on X . Since X is smooth the desired surjectivity follows from [BS, §8, Prop.7]. Thus (A, ∂) satisfies the definition 2.0.1.

2.1.3. Motivic cohomology. Here $A_Z^p(X) = H_Z^p(X, \mathcal{C}) := \text{Hom}_{DM^-(k)}(M_Z(X), \mathcal{C}[p])$ is the motivic cohomology with coefficients in a motivic complex $\mathcal{C} \in DM^-(k)$ [SV], where the motive $M_Z(X)$ with supports on Z is defined in [SV, the text just below the proof of Th.4.8]. The motive $M_Z(X)$ is identified with the complex $C^*(\mathbb{Z}_{tr}(X)/\mathbb{Z}_{tr}(X-Z))$ in the proof of Lemma 4.11 in [SV]. The homotopy invariance property holds by [SV, Prop.4.2]. The excision property is proved in [SV, the proof of Lemma 4.11]. The localization property follows from the exactness of the complex

$$0 \rightarrow \mathbb{Z}_{tr}(X-Z) \rightarrow \mathbb{Z}_{tr}(X) \rightarrow \mathbb{Z}_{tr}(X)/\mathbb{Z}_{tr}(X-Z) \rightarrow 0$$

because the functor C^* takes short exact sequences to exact triangles [SV, Th.1.12].

2.1.4. Representable theories. Here $A^p(X, U) = \bigoplus_q E^{p,q}(X/U)$ where E is a T -spectra [V1]. In particular, in the case $E = MGL$ [V1, Sect.6.3] we obtain the algebraic cobordism theory.

2.2 General properties of cohomology theories

We specify here certain properties of an arbitrary cohomology theory A which are useful below in the text.

2.2.1. The localization property implies that $A_\emptyset(X) = A(X, X) = 0$.

2.2.2. If any two of morphisms $(X, U) \rightarrow (Y, V)$, $X \rightarrow Y$, $U \rightarrow V$, defined by a morphism $f : (X, U) \rightarrow (Y, V)$, induce isomorphisms on the level of A then the third of these morphisms induces an isomorphism on the level of A .

2.2.3. Let $i_r : X_r \hookrightarrow X_1 \amalg X_2$ be the natural inclusion ($r = 1, 2$). Then the induced map $A(X_1 \amalg X_2) \rightarrow A(X_1) \oplus A(X_2)$ is an isomorphism.

2.2.4. Mayer-Vietoris sequence. If X is a union of two open subsets U_1 and U_2 and let Y be a closed subset in X , then there is a long exact sequence depending functorially on X, U_1, U_2 and Y

$$\dots \rightarrow A_Y^n(X) \rightarrow A_{Y_1}^n(U_1) \oplus A_{Y_2}^n(U_2) \rightarrow A_{Y_{12}}^n(U_{12}) \rightarrow A_Y^{n+1}(X) \rightarrow \dots$$

where $U_{12} = U_1 \cap U_2$, $Y_1 = Y \cap U_1$, $Y_2 = Y \cap U_2$, and $Y_{12} = Y \cap U_{12}$. This sequence is called the Mayer-Vietoris sequence.

2.2.5. Let $p : T \rightarrow X$ be an affine bundle (i.e., a torsor under a vector bundle). Then the induced operator $A(X) \rightarrow A(T)$ is an isomorphism. If $s : X \rightarrow T$ is a section then the induced operator $s^A : A(T) \rightarrow A(X)$ is an isomorphism as well.

2.2.6. Deformation to the normal cone. The deformation to the normal cone is a well-known construction (for example, see [Fu]). Since the construction and its property (5) play an important role in what follows we give here some details.

Let $i : Y \hookrightarrow X$ be a closed imbedding of smooth varieties with the normal bundle N . There exists a smooth variety X_t together with a smooth morphism $p_t : X_t \rightarrow \mathbf{A}^1$ and a closed imbedding $i_t : Y \times \mathbf{A}^1 \hookrightarrow X_t$ such that the map $p_t \circ i_t$ coincides with the projection $Y \times \mathbf{A}^1 \rightarrow \mathbf{A}^1$ and

- the fiber of p_t over $1 \in \mathbf{A}^1$ is canonically isomorphic to X and the base change of i_t by means of the imbedding $1 \hookrightarrow \mathbf{A}^1$ coincides with the imbedding $i : Y \hookrightarrow X$;
- the fiber of p_t over $0 \in \mathbf{A}^1$ is canonically isomorphic to N and the base change of i_t by means of the imbedding $0 \hookrightarrow \mathbf{A}^1$ coincides with the zero section $Y \hookrightarrow N$.

Thus we have the diagram

$$(N, N - Y) \hookrightarrow (X_t, X_t - Y \times \mathbf{A}^1) \hookrightarrow (X, X - Y) \quad (4)$$

Here and further we identify a variety with its image under the zero section of any vector bundle over this variety.

Let us recall a construction of X_t , p_t and i_t . Take X'_t to be the blow-up of $X \times \mathbf{A}^1$ with the center $Y \times \{0\}$. Set $X_t = X'_t - \tilde{X}$ where \tilde{X} is the the proper preimage of $X \times \{0\}$ under the blow-up map. Let $\sigma : X_t \rightarrow X \times \mathbf{A}^1$ be the restriction of the blow-up map to X_t and set p_t to be the composition of σ and the projection $X \times \mathbf{A}^1 \rightarrow \mathbf{A}^1$.

The proper preimage of $Y \times \mathbf{A}^1$ under the blow-up map is mapped isomorphically to $Y \times \mathbf{A}^1$ under the blow-up map. Thus the inverse isomorphism gives the desired imbedding $i_t : Y \times \mathbf{A}^1 \hookrightarrow X_t$ (observe that $i_t(Y \times \mathbf{A}^1)$ does not cross \tilde{X}).

We claim that the diagram (4) consists of isomorphisms on the level of A . Namely, the following diagram consists of isomorphisms

$$A_Y(N) \xleftarrow{i_0^A} A_{Y \times \mathbf{A}^1}(X_t) \xrightarrow{i_1^A} A_Y(X). \quad (5)$$

This can be proved by a method, which is similar to a method used in [MV, 3.2.4]. Let us sketch the proof. First the property can be proved immediately for the pair $(X, Z) = (Y \times \mathbf{A}^n, Y \times 0)$. Next one can prove the property for a pair (X, Z) endowed with an etale morphism $q : X \rightarrow \mathbf{A}^n$ satisfying the condition $Z = q^{-1}(\mathbf{A}^m \times 0)$. In this proof one constructs by means of a diagonal trick a diagram $X \xleftarrow{p_1} \tilde{X} \xrightarrow{p_2} Z \times \mathbf{A}^k$ ($k = n - m$) where p_1, p_2 are etale morphisms and the morphisms $p_1^{-1}(Z) \rightarrow Z$ and $p_2^{-1}(Z) \rightarrow Z \times 0$ are isomorphisms. Thus p_1 and p_2 satisfy the excision property (2.0.1) and one can reduce this case to the previous one. Finally general case can be reduced to the previous one by means of a finite covering of X by such open U , that the pairs $(U, U \cap Z)$ satisfy the previous case.

2.2.7. Let X be a smooth variety and let L be a line bundle over X . We claim that the localization sequence (localization property, 2.0.1) for the pair $(\mathbf{P}(E), U)$ where $E = 1 \oplus L$, $U = \mathbf{P}(E) - \mathbf{P}(1)$ gives rise the following exact sequence

$$0 \rightarrow A(\mathbf{P}(E), U) \xrightarrow{i^A} A(\mathbf{P}(E)) \xrightarrow{j^A} A(U) \rightarrow 0. \quad (6)$$

In fact U becomes a line bundle over X by means of the linear projection $q : U \rightarrow \mathbf{P}(L) = X$ (the line bundle is isomorphic to L^\vee) and $p^A \circ (q^A)^{-1}$ splits j^A where $\mathbf{P}(E) \xrightarrow{p} X$ is the natural projection. This implies the surjectivity of j^A and the injectivity of i^A .

2.3 Ring cohomology theories

2.3.1. Definition. One says that a cohomology theory A is a ring cohomology theory if for every $(X, U), (Y, V) \in \mathcal{SmOp}$ it is chosen and fixed a bilinear morphism (of graded groups) called cross-product

$$\times : A^p(X, U) \times A^q(Y, V) \rightarrow A^{p+q}(X \times Y, X \times V \cup U \times Y)$$

which is functorial in both variables and satisfies the following properties

1. associativity: $(a \times b) \times c = a \times (b \times c)$ for $a \in A(X, U)$, $b \in A(Y, V)$, $c \in A(Z, W)$;
2. there is an element $1 \in A(pt)$ such that for any pair $(X, U) \in \mathcal{SmOp}$ and any $a \in A(X, U)$ one has $1 \times a = a = a \times 1$;
3. partial Leibnitz rule: $\partial(a \times b) = \partial(a) \times b$ for $a \in A(U)$, $b \in A(Y)$ and $\partial(c \times d) = (-1)^n c \times \partial(d)$ for $c \in A^n(X)$, $d \in A(V)$.

Given cross-products define cup-products $\cup : A_Z^p(X) \times A_{Z'}^q(X) \rightarrow A_{Z \cap Z'}^{p+q}(X)$ by

$$a \cup b = \Delta^A(a \times b), \quad (7)$$

where $\Delta : (X, U \cup V) \hookrightarrow (X \times X, X \times V \cup U \times X)$ is the diagonal. Clearly cup-products thus defined are associative: $(a \cup b) \cup c = a \cup (b \cup c)$; the element $p^A(1) \in A(X)$, (here p is the projection $X \rightarrow pt$) is the unit for the cup-product; and a partial Leibnitz rule holds: $\partial(a \cup b) = \partial(a) \cup b$ and $\partial(c \cup d) = (-1)^n c \cup \partial(d)$ for $a \in A(U)$, $b \in A(X)$, $c \in A^n(X)$, $d \in A(V)$. Finally these cup-products are functorial in both variables.

Given cup-products one can construct cross-products by $a \times b = p_X^A(a) \cup p_Y^A(b)$ for $a \in A(X, U)$ and $b \in A(Y, V)$. Clearly these two constructions are inverse each to other. Thus having products of one kind we have products of the other kind and can use both products in the same time.

2.3.2. Definition. Let A be a ring cohomology theory and Z be a closed subvariety of a smooth variety X . An element $a \in A_Z(X)$ is called $A(X)$ -central if for each $b \in A(X)$ one has $a \cup b = b \cup a$ in $A_Z(X)$. An element $a \in A_Z(X)$ is called universally central if for any morphism $f : \tilde{X} \rightarrow X$ the element $f^A(a) \in A_{f^{-1}(Z)}(\tilde{X})$ is $A(\tilde{X})$ -central. We denote by $A_Z^{uc}(X)$ the set of all the universally central elements in $A_Z(X)$. If $Z = X$ then $A_Z(X) = A(X)$ and $A_Z^{uc}(X)$ is a subring of $A(X)$ denoted by $A^{uc}(X)$.

Note, that if A is a graded-commutative theory, i.e. for any $a \in A_Z^p(X)$ and $b \in A_{Z'}^q(Y)$ one has $a \cup b = (-1)^{pq} b \cup a$ then each even degree element is a universally central element.

In this text it is not taken in account whether a morphism of cohomology theories commutes with the boundary mappings or not. A part of reason is this: it's often much easier to construct such a morphism and the consequences are the same. Thus we prefer to consider the following notion in this text.

2.3.3. Definition. By an operation $\varphi : A|_{\mathcal{Sm}} \rightarrow B|_{\mathcal{Sm}}$, where A and B are cohomology theories, we mean a morphism of the functors. Let A and B be ring cohomology theories. The operation $\varphi : A|_{\mathcal{Sm}} \rightarrow B|_{\mathcal{Sm}}$ is an ring operation if it commutes with the products and takes the unit of A to the unit of B : $\varphi(a) \cup \varphi(b) = \varphi(a \cup b)$ and $\varphi(1_A) = 1_B$.

2.4 Examples

Consider following examples.

2.4.1. Etale cohomology. Let $A_Z^*(X) = \bigoplus_{q=-\infty}^{+\infty} H_Z^*(X, \mu_m^{\otimes q})$ be the etale cohomology theory, where m is an integer prime to $\text{char}(k)$. The cup-products are described in [Mi, Ch.V, §1, 1.17].

2.4.2. K-theory. Let A be the algebraic K -theory from the example 2.1.2. So $A^n(X, U) = K_{-n}(\text{Xon}Z)$, where $Z = X - U$. The products are defined in [TT].

2.4.3. Motivic cohomology. Let $A_Z^p(X) = \bigoplus_{q=0}^{\infty} H_Z^p(X, \mathbb{Z}(q))$ be the motivic cohomology [SV]. The products are defined in [SV, the text just below Lemma 3.3].

2.4.4. Algebraic cobordism theory. To introduce a ring structure on the algebraic cobordism theory (2.1.4) it would be convenient to enrich MGL with a symmetric ring structure [Ja, Sect.4]. For that we construct another T -spectra $\mathbb{M}GL$ which is a commutative symmetric ring spectra by the very construction and which is weekly equivalent to MGL as the T -spectra. Now we are going to describe $\mathbb{M}GL$. A ring structure on the algebraic cobordism theory was introduced as well in [Hu].

Let V be a countable dimension vector space and let $e \in V$ be a non-zero vector (for instance one can take $V = k^\infty$ and $e = (1, 0, \dots)$). Now set $V_n = V \oplus V \oplus \dots \oplus V$ (n summands) and let $e_i^n = (0, \dots, e, \dots, 0) \in V_n$ be a vector with the only non-zero term e on i -th position. For a finite dimensional vector space W and a positive m let $\mathbf{Gr}(m, W)$ be the Grassmannian of m -dimensional subspaces of W and Let $\mathcal{T}(m, W)$ be the tautological rank m vector bundle on $\mathbf{Gr}(m, W)$. For a vector bundle E over a smooth variety X we denote $Th(E)$ its the Thom space $E/(E - X)$ ([V1, p. 422]. In this subsection we identify $\mathcal{S}m$ with the full subcategory of spaces $\mathcal{S}pc$ [V1, p. 421].

Let $G(n) = \cup \mathbf{Gr}(n, W)$ and let $\mathcal{T} = \cup \mathcal{T}(n, W)$ and let $Th(\mathcal{T}(n)) = \cup Th(\mathcal{T}(n, W))$ where W runs over all finite-dimensional vector subspaces of V_n . Set

$$\mathbb{M}GL_n = Th(\mathcal{T}(n)).$$

The subspace of V_n generated by the vectors e_i^n ($i = 1, \dots, n$) gives a distinguish point g_n of the space $G(n)$. The fiber of the vector bundle $\mathcal{T}(n)$ over the point g_n denote \mathbb{A}^n (it is an n -dimensional vector bundle over the point g_n equipped with a free basis $e_1^n, e_2^n, \dots, e_n^n$).

We will identify the space $G(m) \times G(n)$ with a subspace of $G(m+n)$ identifying the point $(\{W_1\}, \{W_2\}) \in G(m) \times G(n)$ with the point $\{W_1 \oplus W_2\} \in G(m+n)$. Observe that the distinguish point (g_m, g_n) is identified with the distinguish point g_{m+n} under the mentioned identification. Observe furthermore that the restriction of the vector bundle $\mathcal{T}(m+n)$ to $G(m) \times G(n)$ coincides with the vector bundle $\mathcal{T}(m) \times \mathcal{T}(n)$. Thus one has the inclusion of spaces (see [V1, p. 422])

$$\mathbb{M}GL_m \wedge \mathbb{M}GL_n \subset \mathbb{M}GL_{m+n}. \quad (8)$$

The inclusion of the fiber \mathbb{A}^m in $\mathcal{T}(m)$ induces an inclusion $Th(\mathbb{A}^m) \subset Th(\mathcal{T}(m)) = \mathbb{MGL}_m$. Composing it with the inclusion (8) one gets the following inclusion

$$Th(\mathbb{A}^m) \wedge \mathbb{MGL}_n \subset \mathbb{MGL}_{m+n}. \quad (9)$$

Consider the action of the permutation group Σ_k on the vector space V_k by the permutation of the summands. This action induces a Σ_k -action on the spaces $G(k)$, $\mathcal{T}(k)$ and $\mathbb{MGL}_k = Th(\mathcal{T}(k))$. As well this action permutes the vectors $e_1^k, e_2^k, \dots, e_k^k$ and thus acts on the fiber \mathbb{A}^k and on the Thom space $Th(\mathbb{A}^k) = Th(\mathbb{A}^1) \wedge Th(\mathbb{A}^1) \wedge \dots \wedge Th(\mathbb{A}^1)$ permuting the coordinates. If we consider the obvious inclusion of groups $\Sigma_m \times \Sigma_n \subset \Sigma_{m+n}$ then the inclusions (8) and (9) are $\Sigma_m \times \Sigma_n$ -equivariant.

Now the family of spaces \mathbb{MGL}_n together with the inclusions (9) forms a symmetric T -spectrum [Ja, Sect.4] which we denote \mathbb{MGL} . The family of inclusions (8) forms a structure of the commutative symmetric ring spectrum on the symmetric spectrum \mathbb{MGL} (see [Ja, Sect.4.3]).

Considered as a T -spectra the spectra \mathbb{MGL} is weakly equivalent to the spectra MGL from [V1, 6.3]. To prove this consider a subspace $V'_n = \langle e_1^n, e_2^n, \dots, e_{n-1}^n \rangle \oplus V$ of the space V_n where the summand V is the very last copy of V in V_n . Let $G'(n) = \cup Gr(n, W)$ and $\mathcal{T}'(n) = \cup \mathcal{T}(n, W)$ where W runs over all finite-dimensional subspace of V'_n . Let $MGL_n = Th(\mathcal{T}'(n))$ be the Thom space of $\mathcal{T}'(n)$. It is exactly the space MGL_n from [V1, 6.3].

The imbedding $i_n : V_n \hookrightarrow V_{n+1}$ onto the last n copies of V in V_{n+1} induces an imbedding $i'_n : V'_n \hookrightarrow V'_{n+1}$ which takes the vector e_i^n to the vector e_{i+1}^{n+1} ($i = 1, 2, \dots, n-1$) and which is the identity on the summand V . These inclusions give rise to a commutative diagram of spaces

$$\begin{array}{ccc} Th(\mathbb{A}^1) \wedge Th(\mathcal{T}'(n)) & \longrightarrow & Th(\mathcal{T}'(n+1)) \\ \downarrow & & \downarrow \\ Th(\mathbb{A}^1) \wedge Th(\mathcal{T}(n)) & \longrightarrow & Th(\mathcal{T}(n+1)) \end{array}$$

Replacing just the notation one gets a commutative diagram of spaces

$$\begin{array}{ccc} Th(\mathbb{A}^1) \wedge MGL_n & \longrightarrow & MGL_{n+1} \\ \downarrow & & \downarrow \\ Th(\mathbb{A}^1) \wedge \mathbb{MGL}_n & \longrightarrow & \mathbb{MGL}_{n+1} \end{array}$$

Now observe that the top horizontal arrow in the last diagram is exactly the assembly map for the T -spectra MGL from [V1, 6.3]. Observe furthermore that the bottom horizontal arrow in this diagram is exactly the assembly map for the symmetric T -spectra \mathbb{MGL} considered just as the T -spectra. Thus the family of inclusions $MGL_n \subset \mathbb{MGL}_n$ induced by the inclusions $V'_n \subset V_n$ is a T -spectra morphism $MGL \rightarrow \mathbb{MGL}$. The individual inclusions of spaces $MGL_n \subset \mathbb{MGL}_n$ is a weak equivalence. Thus the T -spectra morphism $MGL \rightarrow \mathbb{MGL}$ is a weak equivalence as well.

3 Orientations

In this section A is a ring cohomology theory. We introduce three following structures which A can be endowed with: orientations, Chern classes for line bundles and Thom classes for line bundles. We show that there is a natural one-to-one correspondence between these structures (see Theorems 3.2.3, 3.4.1 and 3.4.2).

3.1 Orientations on a ring cohomology theory

Let us recall that for a vector bundle E over a variety X we identify X with $z(X)$ where $z : X \rightarrow E$ is the zero section.

3.1.1. Definition. *An orientation on the theory A is a rule assigning to each smooth variety X , to each its closed subset Z and to each vector bundle E/X an operator*

$$th_Z^E : A_Z(X) \rightarrow A_Z(E)$$

which is a two-sided $A(X)$ -module isomorphism and satisfies the following properties

1. *invariance: for each vector bundle isomorphism $\varphi : E \rightarrow F$ the diagram commutes*

$$\begin{array}{ccc} A_Z(X) & \xrightarrow{th_Z^E} & A_Z(F) \\ id \downarrow & & \downarrow \varphi^A \\ A_Z(X) & \xrightarrow{th_Z^E} & A_Z(E) \end{array}$$

2. *base change: for each morphism $f : (X', X' - Z') \rightarrow (X, X - Z)$ with closed subsets $Z \hookrightarrow X$ and $Z' \hookrightarrow X'$ and for each vector bundle E/X and for its pull-back E' over X' and for the projection $g : E' = E \times_X X' \rightarrow E$ the diagram commutes*

$$\begin{array}{ccc} A_Z(X) & \xrightarrow{th_Z^E} & A_Z(E) \\ f^A \downarrow & & \downarrow g^A \\ A_{Z'}(X') & \xrightarrow{th_{Z'}^{E'}} & A_{Z'}(E') \end{array}$$

3. *for each vector bundles $p : E \rightarrow X$ and $q : F \rightarrow X$ the following diagram commutes*

$$\begin{array}{ccc} A_Z(X) & \xrightarrow{th_Z^E} & A_Z(E) \\ th_Z^E \downarrow & & \downarrow th_Z^{q^*(E)} \\ A_Z(F) & \xrightarrow{th_Z^{p^*(F)}} & A_Z(E \oplus F) \end{array}$$

and both compositions coincides with the operator $th_Z^{E \oplus F}$.

The operators th_Z^E are called *Thom isomorphisms*. The theory A is called *orientable* if there exists an orientation of A . The theory A is called *oriented* if an orientation is chosen and fixed.

3.1.2. Remark. Any orientation on A allows to assign to each vector bundle E/X the class $th(E) = th_X^E(1) \in A_X(E)$ called the *Thom class* for E . The classes satisfy the following properties: $th(E)$ is *universally central*; $\varphi^A(th(F)) = th(E)$ for each vector bundle isomorphism $\varphi : E \rightarrow F$; $f^A(th(E)) = th(f^*(E))$ for each morphism $f : Y \rightarrow X$; the operator $A(X) \rightarrow A_X(E)$, $a \rightarrow th(E) \cup a$ is an isomorphism. In addition the Thom classes satisfy the *multiplicativity property*: for the projections $q_i : E_1 \oplus E_2 \rightarrow E_i$ ($i = 1, 2$) one has

$$q_1^*th(E_1) \cup q_2^*th(E_2) = th(E_1 \oplus E_2) \in A_X(E_1 \oplus E_2) \quad (10)$$

Conversely the Thom classes for vector bundles allow to reconstruct the orientation on A taking $th_Z^E(m) = th(E) \cup m$.

Next we are going to describe a number of data which allow to orient A .

3.2 Chern and Thom classes for line bundles

3.2.1. Definition. One says that A is endowed with *Chern classes for line bundles* if for each smooth X and each line bundle L/X it is chosen and fixed a *universally central element* $c(L) \in A(X)$ satisfying the following properties

1. *functoriality*:

$c(L_1) = c(L_2)$ for isomorphic line bundles L_1 and L_2 ;

$f^A(c(L)) = c(f^*(L))$ for each morphism $f : Y \rightarrow X$;

2. *nondegeneracy*: the operator $(1, \xi) : A(X) \oplus A(X) \rightarrow A(X \times \mathbf{P}^1)$ is an isomorphism where $\xi = c(\mathcal{O}(-1))$ and $\mathcal{O}(-1)$ is the tautological line bundle on \mathbf{P}^1 ;

3. *vanishing*: $c(1) = 0$.

3.2.2. Definition. One says that A is endowed with *Thom classes for line bundles* if for each smooth variety X and each line bundle L/X it is chosen and fixed a *universally central element* $th(L) \in A_X(L)$ satisfying the following properties

1. *functoriality*:

$\varphi^A(th(L_2)) = th(L_1)$ for each isomorphism $\varphi : L_1 \rightarrow L_2$ of line bundles;

$f_L^A(th(L)) = th(L_Y)$ for each morphism $f : Y \rightarrow X$ and each line bundle L/X where $L_Y = L \times_X Y$ is the line bundle over Y and $f_L : L_Y \rightarrow L$ is the projection to L ;

2. *nondegeneracy*: the cup-product $\cup th(1) : A(X) \rightarrow A_X(X \times \mathbf{A}^1)$ is an isomorphism (here X is identified with $X \times \{0\}$).

Now we are going to describe a one-to-one correspondence between endowments of A with Chern classes for line bundles and endowments of A with Thom classes for line bundles.

Assuming that A is endowed with Thom classes for line bundles $L \mapsto th(L)$ endow A with Chern classes for line bundles as follows. For a line bundle L over a smooth X set

$$c(L) = [z^A \circ i^A](th(L)) \in A(X) \quad (11)$$

where $i^A : A_X(L) \rightarrow A(L)$ is the support extension operator (see 2.0.1) and $z^A : A(L) \rightarrow A(X)$ is the operator induced by the zero section $z : X \rightarrow L$. Assuming that A is endowed with Chern classes for line bundles $L \mapsto c(L)$ endow A with Thom classes for line bundles as follows. For a line bundle L over a smooth X set

$$th(L) = e^A(c(\mathcal{O}_E(1) \otimes p^*L)) \in A(L, L^0) = A_X(L) \quad (12)$$

where $E = 1 \oplus L$, $U = \mathbf{P}(E) - \mathbf{P}(1)$, $\mathbf{P}(E) \xrightarrow{p} X$ is the natural projection and $(L, L^0) \xrightarrow{e} (\mathbf{P}(E), U)$ is the natural inclusion. The term $e^A(c(\mathcal{O}_E(1) \otimes p^*L))$ is well defined since $A(\mathbf{P}(E), U)$ is in fact a subgroup in $A(\mathbf{P}(E))$ (see 2.2.7) by means of the support extension operator i^A (see 2.0.1) related to the pair $(\mathbf{P}(E), U)$ and the element $c(\mathcal{O}_E(1) \otimes p^*L)$ lying a priori in $A(\mathbf{P}(E))$ in fact lies in $A(\mathbf{P}(E), U)$. To see this it is sufficient to show that

$$j^A(c(\mathcal{O}_E(1) \otimes p^*L)) = 0 \quad (13)$$

where $j : U \rightarrow \mathbf{P}(E)$ is the natural inclusion. The inclusion $i_L : X = \mathbf{P}(L) \hookrightarrow U$ coincides with the zero section of the line bundle $q : U \rightarrow X$ (see 2.2.7) and (13) follows from the relation $i_L^A \circ j^A(c(\mathcal{O}_E(1) \otimes p^*L)) = 0$ which followed by the fact that the bundle $\mathcal{O}(1) \otimes p^*L$ is trivial being restricted to $\mathbf{P}(L)$ and therefore it trivial being restricted to U .

3.2.3. Theorem. *For any ring cohomology theory A the following assertions hold.*

1. *If A is endowed with Thom classes for line bundles $L \mapsto th(L)$ then the assignment $L \mapsto c(L)$ given by (11) endows A with Chern classes for line bundles;*
2. *If A is endowed with Chern classes for line bundles $L \mapsto c(L)$ then the assignment $L \mapsto th(L)$ given by (12) endows A with Thom classes for line bundles;*
3. *The constructions described in the items 2 and in 1 are inverse of each other.*

In the case when (12) holds for all line bundles (or (11) holds for all line bundles which is the same) we say that the endowment of A with Chern classes for line bundles and the endowment of A with Thom classes for line bundles correspond to each other.

3.3 Chern classes for vector bundles

Now we are going for a ring cohomology theory endowed with Chern classes for line bundles construct higher Chern classes. We use the known Grothendieck's method.

3.3.1. Theorem (Projective Bundle Cohomology). *Let A be a ring cohomology theory endowed with Chern classes for line bundles $L \mapsto c(L)$. Let X be a smooth variety and let E/X be a vector bundle with $\text{rk}E = n$. For $\xi_E = c(\mathcal{O}_E(-1)) \in A(\mathbf{P}(E))$ we have an isomorphism*

$$(1, \xi_E, \dots, \xi_E^{n-1}) : A(X) \oplus A(X) \cdots \oplus A(X) \rightarrow A(\mathbf{P}(E))$$

where (and elsewhere) we denote the operator of \cup -product with a universally central element by the symbol of the element.

Moreover, for trivial E we have $\xi_E^n = 0$. In addition, all the assertions hold if the element $\zeta_E = c(\mathcal{O}_E(1)) \in A(\mathbf{P}(E))$ is used instead of ξ_E .

3.3.2. Corollary. *Let A be a ring cohomology theory which can be endowed with Chern classes for line bundles. For each smooth X and for each vector bundle E/X the sequence*

$$0 \rightarrow A(\mathbf{P}(F), U) \xrightarrow{i^A} A(\mathbf{P}(F)) \xrightarrow{j^A} A(U) \rightarrow 0 \quad (14)$$

is exact, where $F = 1 \oplus E$, $U = \mathbf{P}(F) - \mathbf{P}(1)$, $j : U \hookrightarrow \mathbf{P}(F)$ is the natural inclusion and i^A is the support extension map for the pair $(\mathbf{P}(F), U)$.

Proof. Note that U becomes a line bundle over $\mathbf{P}(E)$ by means of the linear projection $q : U \rightarrow \mathbf{P}(E)$ (in fact we get the bundle $\mathcal{O}_E(1)$) and the inclusion $\mathbf{P}(E) \hookrightarrow U$ coincides with its zero section. Thus we have the isomorphism $q^A : A(\mathbf{P}(E)) \rightarrow A(U)$.

To check that (14) is an exact sequence it is sufficient to check that $q^A(\xi_E) \in j^A(A(\mathbf{P}(F)))$ since by (3.3.1) $A(\mathbf{P}(E))$ is generated over $A(X)$ by ξ_E . However $q^A(\xi_E) = j^A(c(\mathcal{O}_F(-1)))$ since the restriction of $\mathcal{O}_F(-1)$ to $\mathbf{P}(E)$ is $\mathcal{O}_E(-1)$. \square

3.3.3. Theorem. *Let A be endowed with Chern classes for line bundles $L \mapsto c(L)$. Then A can be uniquely endowed with Chern classes of vector bundles, i.e. there is a rule assigning to each smooth variety X and to each vector bundle E/X classes $c_i(E) \in A(X)$ ($i = 0, 1, \dots$) which are universally central and satisfy the following properties*

1. $c_0(E) = 1$;
 $c_1(L) = c(L)$;
2. *functoriality:*
 $c_i(E) = c_i(E')$ for isomorphic vector bundles E and E' ;
 $f^A(c_i(E)) = c_i(f^*(E))$ for each morphism $f : Y \rightarrow X$;
3. *Cartan formula:* $c_r(E) = c_0(E_1) \cup c_r(E_2) + \cdots + c_r(E_1) \cup c_0(E_2)$ for each short exact sequence $0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$ of vector bundles;

Moreover $c_i(E)$ are nilpotent for $i > 0$ and $c_m(E) = 0$ for $m > \text{rk}(E)$.

Sketch of proof. Let X be a smooth variety and E/X be a vector bundle with $\text{rk}E = n$. Set $\xi = c(\mathcal{O}_E(-1))$. By (3.3.1) there are unique elements $c_i(E) \in A(X)$ such that

$$\xi^n - c_1(E)\xi^{n-1} + \cdots + (-1)^n c_n(E) = 0. \quad (15)$$

Set $c_0(E) = 1$ and $c_m(E) = 0$ if $m > n$. The classes $c_i(E)$ satisfy the theorem. \square

3.4 Orienting a theory

In this section A is a ring cohomology theory. The following two theorem shows how one can construct orientations using Chern classes (or Thom classes) for line bundles and how one can construct Chern classes (or Thom classes) using an orientation.

3.4.1. Theorem. *Given Chern classes $L \mapsto c(L)$ for line bundles (or the corresponding by 3.2.3 Thom classes $L \mapsto th(L)$ for line bundles) there exists an orientation $(X, Z, E) \mapsto th_Z^E$ on A such that the following properties hold*

1. *for each smooth variety X and each line bundle L/X one has $th(L) = th_X^L(1)$;*
2. *for each smooth X and each line bundle L/X one has $z^A \circ i^A \circ th_X^L(a) = c(L) \cup a$ where $a \in A(X)$ is any element, $i^A : A_X(L) \rightarrow A(L)$ is the support extension operator for the pair $(L, L - X)$, $z : X \rightarrow L$ is the zero section.*

Moreover the required orientation is uniquely determined both by the property (1) and by the property (2).

3.4.2. Theorem. *If $(X, Z, E) \mapsto th_Z^E$ is an orientation on A then the assignment $L \mapsto z^A \circ i^A \circ th_X^L(1)$ endows A with Chern classes for line bundles, the assignment $L \mapsto th_X^L(1)$ endows A with Thom classes for line bundles and so constructed Chern and Thom classes for line bundles correspond to each other.*

Moreover the construction of an orientation by means of Chern (or Thom) classes for line bundles given by Theorem 3.4.1 and the construction of Chern and Thom classes for line bundles by means of an orientation are inverse of each other.

Sketch of proof of Theorem 3.4.1. It is enough (see 3.1.2) to construct appropriate Thom classes for vector bundles. Let E/X be such a bundle with $\text{rk} E = n$. We are going to construct $th(E) \in A_X(E)$. To do this identify $A_X(E)$ with $A(\mathbf{P}(F), U)$, where $F = E \oplus 1$ and $U = \mathbf{P}(F) - \mathbf{P}(1)$, by means of the open imbedding $(E, X) \rightarrow (\mathbf{P}(F), U)$ which satisfy the excision property (see 2.0.1). Thus we have to construct an element in $A(\mathbf{P}(F), U)$ which will be in the sketch also denoted by $th(E)$. Since we have the exact sequence (see 3.3.2)

$$0 \rightarrow A(\mathbf{P}(F), U) \xrightarrow{i^A} A(\mathbf{P}(F)) \xrightarrow{j^A} A(U) \rightarrow 0$$

it is enough to construct $th(E) \in A(\mathbf{P}(F))$ and check that $j^A(th(E)) = 0$. Set

$$th(E) = c_n(\mathcal{O}_F(1) \otimes p^*E) \in A(\mathbf{P}(F)) \tag{16}$$

where c_n is the n -th Chern class (see 3.3.3) and $p : \mathbf{P}(F) \rightarrow X$ is the natural projection.

To check that $j^A(th(E)) = 0$ identify $A(U)$ with $A(\mathbf{P}(E))$ by means of the linear projection $q : U \rightarrow \mathbf{P}(E)$ which makes U a line bundle over $\mathbf{P}(E)$ where the zero section z of this bundle coincides with the natural inclusion $\mathbf{P}(E) \rightarrow U$ (use 2.2.5). After the identification it is enough to check that $c_n(\mathcal{O}_F(1) \otimes p^*E)$ vanishes being restricted to $\mathbf{P}(E)$. To prove this consider the canonical exact sequence of vector bundles on $\mathbf{P}(E)$

$$0 \rightarrow \mathcal{O}_E(-1) \rightarrow p_E^*(E) \rightarrow Q \rightarrow 0$$

where $p_E : \mathbf{P}(E) \rightarrow X$ is the natural projection. Twisting this sequence with $\mathcal{O}_E(1)$ one gets the exact sequence

$$0 \rightarrow \mathcal{O} \rightarrow p_E^*(E) \otimes \mathcal{O}_E(1) \rightarrow Q \otimes \mathcal{O}_E(1) \rightarrow 0$$

Now the relation $j^A(th(E)) = 0$ is implied by the Cartan formula (see 3.3.3, 3) and by the fact that $c_n(Q \otimes \mathcal{O}_E(1)) = 0$ because $n > \text{rk}(Q \otimes \mathcal{O}_E(1))$ (see 3.3.3). \square

3.5 Orienting a representable theory.

Here we give a technics which allows to orient a theory A represented by a ring T -spectra [V1] starting with very small set of data, namely only with an element $th \in A_{\mathbf{P}^\infty}(\mathcal{O}(-1))$ or alternatively with an element $ch \in A(\mathbf{P}^\infty)$ satisfying certain properties. This is done in [M] in a slightly different way.

However we prefer to work in a more general context. Namely we do not assume that A is represented by a ring T -spectra, but we assume that there exists a universally central element $\sigma \in A_{\{0\}}(\mathbf{P}^1)$ such that for each smooth X the map

$$\cup \sigma : A(X) \rightarrow A_{\{0\} \times X}(\mathbf{P}^1 \times X), \quad a \mapsto \sigma \cup a \quad (17)$$

is an isomorphism. This is the case for any A represented by a symmetric ring T -spectra (see [Ja]) since the T -suspension $\sigma \in A_{\{0\}}^{2,1}(\mathbf{P}^1)$ of the unite $1 \in A^{0,0}(pt)$ satisfies the required property.

3.5.1. Definition. *A universally central element $ch \in A(\mathbf{P}^\infty)$ is called a Chern element if for a flag $l \subset V \subset k^\infty$ consisting of 1- and 2-dimensional vector subspaces the element ch_l vanishes in $A(\mathbf{P}(l)) = A(pt)$ and the operator $(\cup 1, \cup ch_V) : A(pt) \oplus A(pt) \rightarrow A(\mathbf{P}(V))$ is an isomorphism.*

A universally central element $th \in A(L_\infty, L_\infty^0)$ is called a Thom element if for a 1-dimensional subspace $l \subset k^\infty$ the operator $\cup th_l : A(pt) \rightarrow A_{\mathbf{P}(l)}(\mathcal{O}_l(-1)) = A_{\mathbf{P}(l)}(l)$ is an isomorphism.

3.5.2. Theorem. *Let $ch \in A(\mathbf{P}^\infty)$ be a Chern element. Then the assignment $L \mapsto ch(L)$ (see 6, (27)) endows A with Chern classes for line bundles and $ch(\mathcal{O}_V(-1)) = ch_V$ in $A(\mathbf{P}(V))$. Thus A is oriented (see 3.4.1).*

Let $th \in A_{\mathbf{P}^\infty}(\mathcal{O}(-1))$ be a Thom element. Then the assignment $L \mapsto th(L)$ (see 6, (28)) is well defined by Theorem 6.1.5 and it endows A with Thom classes for line bundles and $th(\mathcal{O}_V(-1)) = th_V$ in $A_{\mathbf{P}(V)}(\mathcal{O}_V(-1))$. Thus A is oriented (see 3.4.1).

3.5.3. Theorem. *One says that a Chern element $ch \in A(\mathbf{P}^\infty)$ and a Thom element $th \in A_{\mathbf{P}^\infty}(\mathcal{O}(-1))$ correspond to each other if $ch_V = (z^A \circ i^A)(th_V) \in A(\mathbf{P}(V))$ for each finite-dimensional subspace V in k^∞ , where $i^A : A_{\mathbf{P}(V)}(\mathcal{O}_V(-1)) \rightarrow A(\mathcal{O}_V(-1))$ is the support extension operator and $z^A : A(\mathcal{O}_V(-1)) \rightarrow A(\mathbf{P}(V))$ is induced by the zero section z .*

Let a Chern element $ch \in A(\mathbf{P}^n)$ and a Thom element $th \in A_{\mathbf{P}^\infty}(\mathcal{O}(-1))$ correspond to each other. Then the Chern classes for line bundles $L \mapsto c(L)$ and the Thom classes

for line bundles $L \mapsto th(L)$ given by (3.5.2) correspond to each other (3.2.3). Thus the orientation defined by the Chern classes for line bundles and the orientation defined by the Thom classes for line bundles coincide (see 3.4.1).

Next we are going to study bigraded structures on A . As before assume that A is a ring cohomology theory satisfying (17).

To each smooth variety X associate an operator $\varepsilon_X : A(X) \rightarrow A(X)$ given by the composition

$$A(X) \xrightarrow{\sigma \circ \sigma} A_X(X \times \mathbf{A}^1 \times \mathbf{A}^1) \xrightarrow{(id_X \times i)^A} A_X(X \times \mathbf{A}^1 \times \mathbf{A}^1) \xrightarrow{\sigma^{-1} \circ \sigma^{-1}} A(X)$$

where $i : \mathbf{A}^1 \times \mathbf{A}^1 \rightarrow \mathbf{A}^1 \times \mathbf{A}^1$ is the transposition $(x, y) \mapsto (y, x)$.

Set $\varepsilon(X) = \varepsilon_X(1) \in A(X)$.

For each map $f : Y \rightarrow X$ of smooth varieties one has $f^*(\varepsilon(X)) = \varepsilon(Y)$. In particular for the projection $p : X \rightarrow pt$ one has the relation $p^*(\varepsilon(pt)) = \varepsilon(X)$. We write ε for $\varepsilon(pt)$. Clearly $\varepsilon^2 = 1$.

Assume moreover that A is a bigraded theory, i.e., for each pair (X, U) in \mathcal{SMOP} one has $A^p(X, U) = \bigoplus_q A^{p,q}(X, U)$, and A is a bigraded functor, and $\partial_{X,U}$ is a bigraded operator of the degree $(1, 0)$. Moreover assume that the cup-product respect this bigrading as well. This means that for elements $\alpha \in A^{p,q}$ and $\beta \in A^{r,s}$ one has $\alpha \cup \beta \in A^{p+r, q+s}$. Assume also that the element σ belongs to the group $A_{\{0\}}^{2,1}(\mathbf{P}^1)$ and that for any $a \in A^{p,q}$ and $b \in A^{r,s}$ one has the relation

$$a \cup b = (-1)^{pr} \varepsilon^{qs} (b \cup a). \quad (18)$$

Clearly $\varepsilon \in A^{0,0}(pt)$ under our assumptions.

A typical example of such a theory A is a theory represented by a commutative symmetric ring T -spectra (in this case (18) is proved in [M]).

3.5.4. Theorem. *Let $th \in A_{\mathbf{P}^\infty}^{2,1}(\mathcal{O}(-1))$ be an element and let l be a 1-dimensional subspace in k^∞ such that $th_l = \sigma \in A_{\mathbf{P}(l)}^{2,1}(\mathcal{O}_l(-1)) = A_{\mathbf{P}(l)}^{2,1}(l)$. Then $\varepsilon = 1 \in A^{0,0}(pt)$ and th is a Thom element (3.5.1). Furthermore A is endowed with Thom classes for line bundles $L \mapsto th(L)$ (3.5.2) and thus A is oriented (see 3.4.1).*

Let $ch \in A_{\mathbf{P}^\infty}^{2,1}(\mathcal{O}(-1))$ be an element and let $l \subset V \subset k^\infty$ be a flag consisting of 1- and 2-dimensional subspaces such that $ch_l = 0$ and $ch_V = -j(\sigma)$ in $A^{2,1}(\mathbf{P}(V))$ where $j : A_{\mathbf{P}(l)}(\mathbf{P}(V)) \rightarrow A(\mathbf{P}(V))$ is the support extension map. Then $\varepsilon = 1 \in A^{0,0}$ and ch is a Chern element (3.5.1). Furthermore the rule $L \mapsto ch(L)$ from (3.5.2) endows A with Chern classes for line bundles and thus A is oriented (see 3.4.1).

3.5.5. Theorem. *Let an element $ch \in A^{2,1}(\mathbf{P}^\infty)$ and an element $th \in A_{\mathbf{P}^\infty}^{2,1}(\mathcal{O}(-1))$ be such that for each finite-dimensional subspace $V \subset k^\infty$ the following relation holds in $A(\mathbf{P}(V))$*

$$ch_V = (z^A \cdot i^A)(th_V),$$

where $i^A : A_{\mathbf{P}(V)}(\mathcal{O}_V(-1)) \rightarrow A(\mathcal{O}_V(-1))$ is the support extension map and where z^A is induced by the zero section $z : \mathbf{P}(V) \rightarrow \mathcal{O}_V(-1)$. Assuming that $th_l = \sigma \in A_{\mathbf{P}(l)}^{2,1}(l)$ for a

1-dimensional subspace l in k^∞ we claim that the element th is a Thom element and the element ch is a Chern element and the orientations on A constructed in (3.5.4) by means of the elements ch and th coincide.

A proof of (3.5.4) is based on the following observation. Let $x_0, x_1, x_2 \in \mathbf{P}^2$ be three different points. Let l_1 (l_2) be the only projective line going through x_0 and x_1 (x_0 and x_2). Let $i : l_1 \hookrightarrow \mathbf{P}^2$ be the inclusion. If there exists $\tilde{\sigma} \in A_{l_2}(\mathbf{P}^2)$ such that $i^A(\tilde{\sigma}) = \sigma$ in $A_{x_0}(l_1) = A_{\mathbf{P}^0}(\mathbf{P}^1)$ then $\varepsilon = 1$ in $A^{0,0}(pt)$.

3.6 Examples

3.6.1. Let A be the algebraic K -theory (2.1.2). The rule $L \rightarrow [1] - [L^\vee]$ endows A with Chern classes for line bundles (the property (4) follows from [Qu2, §8, Th.2.1]) and thus orients A .

It's interesting to observe that the corresponding Chern class c_n of a rank n vector bundle E is exactly the known class $\lambda_{-1}(E^\vee) = [1] - [E^\vee] + [\wedge^2 E^\vee] + \cdots + (-1)^n [\wedge^n E^\vee]$.

3.6.2. Let A be the étale cohomology theory $A_Z^*(X) = \bigoplus_{q=-\infty}^{+\infty} H_Z^*(X, \mu_m^{\otimes q})$, where m is an integer prime to $\text{char}(k)$. Consider the short exact sequence of the étale sheaves $0 \rightarrow \mu_m \rightarrow \mathbb{G} \xrightarrow{\times m} \mathbb{G} \rightarrow 0$ and denote by $\partial : H^1(X, \mathbb{G}_m) \rightarrow H^2(X, \mu_m)$ the boundary map. For a line bundle L over a smooth variety X let $[L] \in H^1(X, \mathbb{G}_m)$ be its isomorphism class. It is known [Mi] that the rule $L \mapsto \partial([L])$ endows A with Chern classes for line bundles. Thus A is oriented.

3.6.3. Let A be the motivic cohomology [SV]: $A_Z^p(X) = \bigoplus_{q=0}^{\infty} H_Z^p(X, \mathbb{Z}(q))$. Recall that $H_{\mathcal{M}}^2(X, \mathbb{Z}(1)) = CH^1(X)$ for a smooth X [SV]. For a line bundle L over a smooth variety X let $D(L) \in CH^1(X)$ be the associated class divisor. The rule $L \mapsto D(L)$ endows A with Chern classes for line bundles in the characteristic zero [SV, Cor. 4.12.1] (now it is known in any characteristic). Thus A is oriented.

3.6.4. Let A be the K -cohomology [Qu2, §7, 5.8]: $A_Z^p(X) = \bigoplus_{q=0}^{\infty} H_Z^p(X, \mathcal{K}_q)$, where \mathcal{K} is the sheaf of K -groups. Recall that the sheaf \mathcal{K}_1 coincides with the sheaf \mathcal{O}^* of invertible functions. For a line bundle L over a smooth variety X let $[L] \in H^1(X, \mathcal{K}_1) = H^1(X, \mathcal{O}^*)$ be the isomorphism class of L . The rule $L \mapsto [L]$ endows A with Chern classes for line bundles [Gi, Th.8.10] and thus orients A .

3.6.5. Let $k = \mathbb{R}$ and let A be the theory $A^p(X, U) = H^p(X(\mathbb{R}), U(\mathbb{R}); \mathbb{Z}/2\mathbb{Z})$. For a line bundle L consider the real line bundle $L(\mathbb{R})$ over the topological space $X(\mathbb{R})$ and set $c_1(L) = w_1(L(\mathbb{R}))$ (the first Stiefel-Whitney class). Since $\mathbf{P}^n(\mathbb{R}) = \mathbb{R}P^n$ is the real projective space the rule $L \mapsto c_1(L)$ endows A with Chern classes for line bundles and thus orients A .

3.6.6. Orienting the algebraic cobordism theory. In this example the notation of 2.4.4 are used.

The identity morphism MGL_1 to itself gives rise in the standard manner to an element $[id_1] \in \text{MGL}^{2,1}(\text{MGL}_1)$. By the very definition $\text{MGL}_1 = \text{Th}(\mathcal{T}(1))$ and $\mathcal{T}(1)$ is the tautological line bundle $\mathcal{O}(-1)$ over the space $G(1) = \mathbf{P}(V) = \mathbf{P}^\infty$. Now set

$$th = [id_1] \in \text{MGL}^{2,1}(\text{MGL}_1) = \text{MGL}_{\mathbf{P}^\infty}^{2,1}(\mathcal{O}(-1)).$$

Consider the fiber \mathbb{A}^1 of $\mathcal{T}(1)$ over the point $g_1 \in \mathbf{P}(V)$.

The restriction of the element th to the Thom space $\text{Th}(\mathbb{A}^1) = \mathbb{A}^1/(\mathbb{A}^1 - \{0\})$ coincides with the T -suspension $\sigma \in \text{MGL}^{2,1}(\text{Th}(\mathbb{A}^1)) = \text{MGL}_{\{0\}}^{2,1}(\mathbb{A}^1)$ of the unite $1 \in \text{MGL}^{0,0}(pt)$. Thus the element th orients the algebraic cobordism theory MGL due to (3.5.4).

3.7 The formal group law F_ω

Let ω be an orientation of A . Thus A is endowed with Chern classes for line bundles which correspond to ω (see 3.4.2 and 3.4.1). Following [Qu1] we associate a formal group low F_ω with ω . This formal group low is defined over the ring \bar{A}^{uc} and gives an expression of the first Chern class of $L_1 \otimes L_2$ in terms of the first Chern classes of line bundles L_1, L_2 .

Using (3.3.1) identify the formal power series in one variable $\bar{A}[[u]]$ with the ring $A(\mathbf{P}^\infty)$ identifying u with $c_1(\mathcal{O}(1)) \in A(\mathbf{P}^\infty)$. The two "projections" $p_i : \mathbf{P}^\infty \times \mathbf{P}^\infty \rightarrow \mathbf{P}^\infty$ induce two pull-back maps $p_i^A : A(\mathbf{P}^\infty) \rightarrow A(\mathbf{P}^\infty \times \mathbf{P}^\infty)$. Using (3.3.1) again identify $A(\mathbf{P}^\infty \times \mathbf{P}^\infty)$ with $\bar{A}[[u_1, u_2]]$ where $u_i = p_i^*(u) = c_1(p_i^*(\mathcal{O}(1)))$. Set

$$F_\omega(u_1, u_2) = c_1(p_1^*(\mathcal{O}(1)) \otimes p_2^*(\mathcal{O}(1))) \in \bar{A}[[u_1, u_2]] \quad (19)$$

Since the first Chern class is a universally central element $F_\omega \in \bar{A}^{uc}[[u_1, u_2]]$

3.7.1. Proposition. *For any $X \in \mathcal{S}m$ and line bundles $L_1/X, L_2/X$ one has the following relation in $A(X)$*

$$c_1(L_1 \otimes L_2) = F_\omega(c_1(L_1), c_1(L_2))$$

Here the right hand side is well-defined since the first Chern classes are universally central and nilpotent (3.3.3).

Sketch of proof. Using (6.1.4) reduce the assertion to the case when $L_i^\vee = f_i^*(\mathcal{O}(-1))$ for a maps $f_i : X \rightarrow \mathbf{P}^N$. Then $L_i = f_i^*(\mathcal{O}(1))$. Let $f = f_1 \times f_2$. The chain of relations

$$c_1(L_1 \otimes L_2) = f^A(c_1(p_1^*(\mathcal{O}(1)) \otimes p_2^*(\mathcal{O}(1)))) = f^A(F_\omega(u_1, u_2)) = F_\omega(c_1(L_1), c_1(L_2)).$$

completes the sketch. □

For instance, if $A = H_{\mathcal{M}}^*(-, \mathbb{Z}(*))$ with the first Chern class c_1^H then $c_1^H(L_1 \otimes L_2) = c_1^H(L_1) + c_1^H(L_2)$; if $A = K$ -theory with the Chern classes for line bundles defined by $L \mapsto [1] - [L^\vee]$ then $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2) - c_1(L_1)c_1(L_2)$. More complicated examples are related to other theories.

3.7.2. Proposition. *The formal power series $F_\omega \in \bar{A}^{uc}[[u_1, u_2]]$ is a commutative formal group law ($[Ha]$) with the "inverse element" $I(u) = c_1(\mathcal{O}(-1)) \in A^{uc}(\mathbf{P}^\infty) = \bar{A}^{uc}[[u]]$.*

Proof. This is implied by the corresponding properties of line bundles by means of (3.7.1). \square

3.7.3. Definition. *The formal group law F_ω is called the formal group law associated with A endowed with the orientation ω .*

4 Push-forwards

Let A be a ring cohomology theory.

4.1 Integrations on a ring cohomology theory

Here we define a notion of an integration on A and prove that orientations on A are in a natural one-to-one correspondence with integrations on A .

Let us recall a notion.

4.1.1. Definition. *Let $i : Y \hookrightarrow X$ be a closed imbedding of smooth varieties. Consider a Cartesian square (in the category of schemes)*

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{i}} & \tilde{X} \\ \tilde{\varphi} \downarrow & & \downarrow \varphi \\ Y & \xrightarrow{i} & X \end{array}$$

consisting of smooth varieties. This square is called transversal if the canonical morphism $\tilde{\varphi}^(N) \rightarrow \tilde{N}$ is an isomorphism (here N and \tilde{N} are the normal bundle to Y in X and to \tilde{Y} in \tilde{X}).*

4.1.2. Definition. *Let A be the ring cohomology theory. An integration on A is a rule assigning to each projective morphism of smooth varieties $f : Y \rightarrow X$ a two-sided $A(X)$ -module operator*

$$f_A : A(Y) \rightarrow A(X)$$

called the push-forward (for f) and satisfying the following properties

1. $(f \circ g)_A = f_A \circ g_A$ for any projective morphisms $Z \xrightarrow{g} Y$ and $Y \xrightarrow{f} X$ of smooth varieties;
2. for the transversal square from (4.1.1) the following diagram commutes

$$\begin{array}{ccc} A(\tilde{Y}) & \xrightarrow{\tilde{i}_A} & A(\tilde{X}) \\ \tilde{\varphi}^A \uparrow & & \uparrow \varphi^A \\ A(Y) & \xrightarrow{i_A} & A(X) \end{array}$$

3. for any projective morphism of smooth varieties $f : Y \rightarrow X$ the following diagram commutes

$$\begin{array}{ccc} A(\mathbf{P}^n \times Y) & \xleftarrow{(id \times f)^A} & A(\mathbf{P}^n \times X) \\ (p_Y)_A \downarrow & & \downarrow (p_X)_A \\ A(Y) & \xleftarrow{f^A} & A(X) \end{array}$$

where $p_Y : \mathbf{P}^n \times Y \rightarrow Y$ and $p_X : \mathbf{P}^n \times X \rightarrow X$ are the natural projections;

4. normalization: for any smooth variety X one has $(id_X)_A = id_{A(X)}$.

The line bundle $L(D)$ used below in the text and associated with an effective divisor D on a smooth variety X is defined in 1.1.

4.1.3. Theorem. *Let A be a ring cohomology theory.*

1. Assuming A endowed with an orientation ω one can uniquely endow A with an integration such that for each smooth variety X and each smooth divisor $i : D \hookrightarrow X$ one has $i_A(1) = c(L(D))$, where $L \mapsto c(L)$ is the Chern classes for line bundles corresponding to ω by Theorem 3.4.2.
2. Assuming A endowed with an integration $f \mapsto f_A$ one can endow A with an orientation ω such that for each smooth variety X and each smooth divisor $i : D \hookrightarrow X$ one has $i_A(1) = c(L(D))$ where $L \mapsto c(L)$ are the Chern classes for line bundles corresponding to ω by Theorem 3.4.2.

Moreover, for a smooth variety X and a line bundle L/X one has $c(L) = z^A(z_A(1)) \in A(X)$.

3. The two correspondences between orientations and integrations described in the previous items are inverse of each other.

In what follows we denote by f_ω the push-forward f_A determined by an orientation ω due to (4.1.3).

4.1.4. Corollary. *Let A be a ring cohomology theory.*

1. Given Chern classes for line bundles $L \mapsto c(L)$ one can uniquely endow A with an integration such that for each smooth variety X and each smooth divisor $i : D \hookrightarrow X$ one has $i_A(1) = c(L(D))$.
2. Given an integration $f \mapsto f_A$ on A one can uniquely endow A with Chern classes for line bundles $L \mapsto c(L)$ such that for each smooth variety X and each smooth divisor $i : D \hookrightarrow X$ one has $i_A(1) = c(L(D))$. Moreover, for a smooth variety X and a line bundle L over X this Chern class is given by $c(L) = z^A(z_A(1)) \in A(X)$.
3. The two correspondences between the endowments of A with Chern classes and integrations described in the previous items are inverse of each other.

4. If an orientation on A and Chern classes $L \mapsto c(L)$ for line bundles correspond to each other as in Theorem 3.4.1 then the integrations on A given by Theorem 4.1.3 and by this Corollary coincide.

Each theory from 3.6 is oriented and thus endowed with the integration given by (4.1.3). For any such theory A the integration coincides with the well-known one as follows from the known properties of push-forwards in these cases. In particular for the algebraic K -theory the push-forwards given by (4.1.3) and the orientation from 3.6 coincides with the push-forwards from [Qu2, §7, 2.7]. Note that the algebraic cobordism theory being oriented (see 3.6.6) is also endowed with an integration due to (4.1.3).

In the following three examples we assume that A is an ring cohomology theory endowed with an orientation ω and thus endowed with the corresponding Chern classes and with the corresponding integration.

4.1.5. Self-intersection formula. Let $i : Y \hookrightarrow X$ be a closed imbedding of smooth varieties of pure codimension d with the normal bundle N . Then for each $\alpha \in A(Y)$ one has the relation in $A(Y)$ (compare with [BS, Prop.12])

$$i^A(i_\omega(\alpha)) = c_d(N) \cup \alpha = \alpha \cup c_d(N).$$

4.1.6. Excess formula (compare with [BS, Lemma 19(c)]). Let $i : Y \hookrightarrow X$ be a closed imbedding of smooth varieties of pure codimension d with the normal bundle N . Let $\sigma : X' \rightarrow X$ be the blow-up with the center Y and let $Y' = \mathbf{P}(N)$ be the projective bundle associated with N and let

$$\begin{array}{ccc} Y' & \xrightarrow{j} & X' \\ g \downarrow & & \downarrow \sigma \\ Y & \xrightarrow{i} & X \end{array}$$

be the standard blow-up Cartesian square. And let E be the factor-bundle $g^*(N)/\mathcal{O}_N(-1)$ on $Y' = \mathbf{P}(N)$. Then for each $\alpha \in A(Y)$ one has the relation in $A(X')$

$$\sigma^A(i_\omega(\alpha)) = j_\omega(c_{d-1}(E) \cup g^A(\alpha)).$$

4.1.7. A formula for the top Chern class (compare with [Gr, Th.2]). Let X be a smooth variety and let E/X be a vector bundle with $\text{rk}E = d$ and let $s : X \rightarrow E$ be a section of E such that the scheme theoretic vanishing locus Y of the section s is a smooth subvariety in X of pure codimension d . Then

$$i_\omega(1) = c_d(E).$$

where $i : Y \hookrightarrow X$ is the natural imbedding.

4.2 Push-forwards for closed imbeddings

Let A be a ring cohomology theory endowed with an orientation ω . Below we give a construction of push-forwards for closed imbeddings. The push-forward to be constructed for a closed imbedding i will be temporarily (until 4.4.1) denoted by i_{gys} . The deformation to the normal cone (see 2.2.6) plays an important role here.

For a closed imbedding $i : Y \hookrightarrow X$ of smooth varieties define an operator

$$i_{th} : A(Y) \rightarrow A_Y(X) \quad (20)$$

as the composition $i_{th} : A(Y) \xrightarrow{th_Y^N} A_Y(N) \xrightarrow{(i_0^A)^{-1}} A_{Y \times \mathbf{A}^1}(X_t) \xrightarrow{i_1^A} A_Y(X)$ where the notation for $N = N_{X/Y}$, i_0^A and i_1^A are taken from (2.2.6) and th_Y^N is the Thom operator corresponding to the orientation ω (3.1.1). The operator i_{th} is an isomorphism (see 3.1.2). Define the Gysin operator

$$i_{gys} : A(Y) \rightarrow A(X) \quad (21)$$

as the composition $i_{gys} : A(Y) \xrightarrow{i_{th}} A_Y(X) \xrightarrow{j^A} A(X)$ where j^A is the support extension operator for the pair $(X, X - Y)$ (see 2.0.1).

The following properties of the Gysin operators can be proved before the theorem (4.4.1) and are useful when proving this theorem.

4.2.1. The Gysin operators commute with a transversal base change, i.e., for a transversal square from (4.1.1) the following diagram commutes

$$\begin{array}{ccc} A(\tilde{Y}) & \xrightarrow{\tilde{i}_{gys}} & A(\tilde{X}) \\ \tilde{\varphi}^A \uparrow & & \varphi^A \uparrow \\ A(Y) & \xrightarrow{i_{gys}} & A(X) \end{array}$$

4.2.2. Let j_1 and j_2 be the natural imbeddings of smooth varieties Y_1 and Y_2 to $Y = Y_1 \amalg Y_2$. For a closed imbedding $i : Y \hookrightarrow X$ one has $i_{gys} = (i_1)_{gys} \circ j_1^A + (i_2)_{gys} \circ j_2^A$ where i_r is the composition $i \circ j_r$.

4.2.3. One has $i_{gys} \circ j_{gys} = (i \circ j)_{gys}$ for closed imbeddings $Z \xrightarrow{j} Y \xrightarrow{i} X$ of smooth varieties.

4.2.4. For a closed imbedding of smooth varieties $i : Y \hookrightarrow X$ the operator $i_{gys} \circ i^A : A(X) \rightarrow A(X)$ coincides with the operator given by the cup-product with the element $i_{gys}(1)$.

4.2.5. Let $z : Y \rightarrow E$ be the zero section of a vector bundle E/Y . Then the operator z_{th} coincides with the Thom operator th_Y^E .

To state the following few properties recall that A is endowed with the Chern classes for line bundles corresponding to the orientation ω (3.4.2) and with the corresponding higher Chern classes (3.3.3).

4.2.6. Let E/Y be a rank n vector bundle and let $s : Y \rightarrow \mathbf{P}(E)$ be a section of the natural projection $p : \mathbf{P}(E) \rightarrow Y$. Consider the natural inclusion $\mathcal{O}_E(-1) \rightarrow p^*(E)$ and let Q be the factor-bundle $p^*(E)/\mathcal{O}_E(-1)$. In $A(\mathbf{P}(E))$ one has $s_{gys}(1) = c_{n-1}(\mathcal{O}_E(1) \otimes p^*s^*Q)$.

4.2.7. For a smooth divisor $i : D \hookrightarrow X$ one has $i_{gys}(1) = c_1(L(D))$ in $A(X)$ (see 1.1 for notation).

4.2.8. Self-intersection formula. For a closed imbedding of smooth varieties $i : Y \hookrightarrow X$ the operator $i^A \circ i_{gys} : A(Y) \rightarrow A(Y)$ coincides with the cup-product by $c_n(N)$ where N is the normal bundle and $n = \text{rk}N$.

4.3 Push-forward for the projection $X \times \mathbf{P}^n \rightarrow X$

Let A be a ring cohomology theory endowed with an orientation ω and therefore endowed with the corresponding Chern classes for line bundles ((3.4.2) and (3.4.1)) and with the corresponding higher Chern classes (3.3.3). Below we give a construction of push-forwards for projections. As above $\bar{A} = A(pt)$ and \bar{A}^{uc} is the subring of all the universally central elements in \bar{A} .

We give two alternative ways to construct push-forwards for the projections. The first way is based on the fact that the cobordism ring MU is the coefficient ring of an universal formal group law. The second way uses no complicated facts and is based on residues theory. The push-forward to be constructed for the projection $p : X \times \mathbf{P}^n \rightarrow X$ will be temporarily (until 4.4.1) denoted by p_{quil} .

4.3.1. MU-approach. Consider the complex cobordism theory $\text{MU}(\ast)$ and the formal group law F_{MU} associated with this theory and its canonical Chern class for line bundles (Conner-Floyd class) [CF]. This law is defined over the ring $\text{MU} = \text{MU}(pt)$. According to a theorem of Quillen ([Qu1, Th.2]) F_{MU} is a universal commutative formal group law in one variable. This means that for any commutative ring R and any commutative formal group law in one variable F over R there exists a unique ring homomorphism $l_F : \text{MU} \rightarrow R$ such that the coefficients of F coincide with the l_F -images of the corresponding coefficients of F_{MU} . For the theory A endowed with the orientation ω denote by

$$l_\omega : \text{MU} \rightarrow \bar{A}^{uc} \tag{22}$$

the homomorphism l_F where $F = F_\omega$ is the formal group law associated with the orientation ω on A (3.7.3) and set

$$[\mathbf{P}^n]_\omega = l_\omega([\mathbb{C}P^n]),$$

where $[\mathbb{C}P^n]$ is the class of $\mathbb{C}P^n$ in MU .

For the projection $p : X \times \mathbf{P}^n \rightarrow X$ define the operator

$$p_{quil} : A(X \times \mathbf{P}^n) \rightarrow A(X) \tag{23}$$

as follows. Identify $A(X \times \mathbf{P}^n)$ with $A(X)[t]/(t^{n+1})$ taking t to the element $\zeta = c_1(\mathcal{O}(1))$ (see 3.3.1), consider the structural morphism $f : X \rightarrow pt$ and set p_{quil} to be the unique

two-sided $A(X)$ -module operator which takes the element t^i to the element $f^A([\mathbf{P}^{n-i}]_\omega) \in A(X)$ for $i = 0, 1, \dots, n$.

The following properties of the operator p_{quilt} can be proved before the theorem (4.4.1) and are useful when proving this theorem.

1. The operator $p_{quilt} : A(X \times \mathbf{P}^n) \rightarrow A(X)$ is a two-sided $A(X)$ -module homomorphism.
2. Let $\varphi : Y \rightarrow X$ be a morphism of smooth varieties and let $\tilde{\varphi}$ be the base change of φ by the natural projection $p_X : X \times \mathbf{P}^n \rightarrow X$. Then one has the relation $(p_Y)_{quilt} \circ \tilde{\varphi}^A = \varphi^A \circ (p_X)_{quilt}$ where $p_Y : Y \times \mathbf{P}^n \rightarrow Y$ is the natural projection.
3. The following diagram commutes

$$\begin{array}{ccc} A(X \times \mathbf{P}^n \times \mathbf{P}^m) & \xrightarrow{(\tilde{p}_n)_{quilt}} & A(X \times \mathbf{P}^m) \\ (\tilde{p}_m)_{quilt} \downarrow & & \downarrow (p_m)_{quilt} \\ A(X \times \mathbf{P}^n) & \xrightarrow{(p_n)_{quilt}} & A(X) \end{array}$$

where $p_n : X \times \mathbf{P}^n \rightarrow X$, $p_m : X \times \mathbf{P}^m \rightarrow X$, $\tilde{p}_n : X \times \mathbf{P}^n \times \mathbf{P}^m \rightarrow X \times \mathbf{P}^m$ and $\tilde{p}_m : X \times \mathbf{P}^n \times \mathbf{P}^m \rightarrow X \times \mathbf{P}^n$ are the natural projections.

4. For a linear imbedding $i : \mathbf{P}^n \rightarrow \mathbf{P}^m$ the following diagram commutes

$$\begin{array}{ccc} A(X \times \mathbf{P}^n) & \xrightarrow{i_{gys}} & A(X \times \mathbf{P}^m) \\ (p_n)_{quilt} \downarrow & & \downarrow (p_m)_{quilt} \\ A(X) & \xrightarrow{id} & A(X) \end{array}$$

where $p_n : X \times \mathbf{P}^n \rightarrow X$ and $p_m : X \times \mathbf{P}^m \rightarrow X$ are the natural projections.

5. For a closed imbedding of smooth varieties $i : Y \hookrightarrow X$ the following diagram commutes

$$\begin{array}{ccc} A(Y \times \mathbf{P}^n) & \xrightarrow{(i \times id)_{gys}} & A(X \times \mathbf{P}^n) \\ (p_Y)_{quilt} \downarrow & & \downarrow (p_X)_{quilt} \\ A(Y) & \xrightarrow{i_{gys}} & A(X) \end{array}$$

where $p_X : X \times \mathbf{P}^n \rightarrow X$ and $p_Y : Y \times \mathbf{P}^n \rightarrow Y$ are the natural projections.

6. For a section $s : X \rightarrow X \times \mathbf{P}^n$ of the projection $p : X \times \mathbf{P}^n \rightarrow X$ one has $p_{quilt} \circ s_{gys} = id_{A(X)}$.

4.3.2. Approach by means of residues. Here we sketch another way to introduce push-forwards for the projections. This way is based on an elementary residues theory and uses no complicated facts. Thus it can be applied in a more general situation.

Let $p : \mathbf{P}(E) \rightarrow X$ be the natural projection where E/X is a vector bundle. Below (see 4.4.1) we will construct push-forwards for any projective morphisms. In particular, the push-forward p_ω for the projection p can be computed as follows (compare with Quillen's formula [Qu1, Th.1]):

$$p_\omega(a) = \text{Res}\left(\frac{h}{q_E}\omega_F\right) \quad (24)$$

where ω_F is the normalized invariant differential on the formal group F related to the theory A and its orientation ω , and $h(t)$ is an arbitrary polynomial with $h(\zeta_E) = a$, and the series $q_E \in A(X)[[t]]$ corresponds to the total Chern class

$$c_{tot}(E \otimes \mathcal{O}(1)) \quad (25)$$

under the identification $A(X)[[t]]$ with $A(X \times \mathbf{P}^\infty)$ which identifies the variable t with the element $c_1(\mathcal{O}(1))$ (here for a vector bundle F we write $c_{tot}(F)$ for the element $c_0(F) + c_1(F) + \dots$).

One gets the alternative definition of the operator p_{quil} by means of the formula (24) for a trivial vector bundle E . In this case the formula (24) shows that the push-forward p_ω takes the element ζ_E^i to the coefficient of $t^{n-i}dt$ in $\omega_F(t)$. The new definition coincides with the previous one by [Qu1, Cor. of Th.1]. Observe that the new definition allows to work easily in a relative situation (i.e. for varieties over a base). Main technical tool for the work with this definition is the invariance of the residues with respect to variable changes allowing a nilpotent shift of the coordinate origin.

One concludes this subsection with the following observation: if $E^\vee = \bigoplus L_i$ is the direct sum of line bundles then one has (compare with [Qu1, Th.1])

$$q_E = \prod_{i=1}^d (t -_F \lambda_i) \quad (26)$$

where $\text{rk}E = d$, $\lambda_i = c_1(L_i)$.

4.4 Push-forwards for projective morphisms

Let A be a ring cohomology theory endowed with an orientation ω and therefore endowed with the corresponding Chern classes for line bundles ((3.4.2) and (3.4.1)) The main result of this section is the following

4.4.1. Theorem. *There exists and unique family of maps $f_A : A(Y) \rightarrow A(X)$, where $f : Y \rightarrow X$ runs over all projective morphisms of smooth quasi-projective varieties, satisfying the following properties*

- 1) $f_A \circ g_A = (f \circ g)_A$ for morphisms $g : Z \rightarrow Y$ and $f : Y \rightarrow X$;
- 2) $i_A = i_{gys}$ for a closed imbedding $i : Y \hookrightarrow X$;

3) $p_A = p_{quil}$ for the projection $p : X \times \mathbf{P}^n \rightarrow X$.

Moreover the family of the maps f_A from this theorem is the integration on A required by Theorem 4.1.3 and Corollary 4.1.4.

Let $f : Y \rightarrow X$ be a projective morphism of smooth varieties. One can present f as a composition of a closed imbedding $i : Y \hookrightarrow X \times \mathbf{P}^n$ and the projection $p : X \times \mathbf{P}^n \rightarrow X$, i.e. $f = p \circ i$. Define now the desired map f_A by the formula

$$f_A = p_{quil} \circ i_{gys}.$$

To prove the theorem it suffices to check that the resulting map f_A does not depend on the particular choice of the decomposition of f and to verify 4.4.1(1), which does not follow directly from well-defines of f_A .

A complete proof of Theorem 4.4.1 will be written in a next preprint. Let us only to stress here that an important point in the proof is the following fact: given a section $s : X \rightarrow X \times \mathbf{P}^n$ of the projection $p : X \times \mathbf{P}^n \rightarrow X$ one has the relation $p_{quil} \circ s_{gys} = id_{A(X)}$.

5 Riemann-Roch theorem

Let A and B be oriented ring cohomology theories endowed with orientations ω_A and ω_B , with the corresponding Chern classes c_i^A and c_i^B and with the corresponding integrations (see Theorem 4.1.3, Corollary 4.1.4 and Theorem 3.3.3).

Let \bar{A}^{uc} be the ring of universally central elements in $A(pt)$ (2.3.2) and let \bar{B}^{uc} be the ring of universally central elements in $B(pt)$. As in 3.7 identify $A(\mathbf{P}^\infty)$ with the formal power series in one variable $\bar{A}[[u]]$ identifying the variable u with $c_1^A(\mathcal{O}(1))$ and identify $B(\mathbf{P}^\infty)$ with $\bar{B}[[t]]$ identifying the variable t with $c_1^B(\mathcal{O}(1))$.

Below in this section we consider ring operations defined on the category $\mathcal{S}m$ rather than on $\mathcal{S}m\mathcal{O}p$ since some interesting facts hold in this case and it is more easy to construct such operations.

5.1 Todd genus

Let $\varphi : A|_{\mathcal{S}m} \rightarrow B|_{\mathcal{S}m}$ be a ring operation (see 2.3.3) which takes universally central elements to the universally central ones. Consider the operation φ on \mathbf{P}^∞ :

$$\varphi_\infty : A(\mathbf{P}^\infty) \rightarrow B(\mathbf{P}^\infty)$$

and define a series $\Phi(t) \in \bar{B}[[t]]$ as follows

$$\Phi(t) = \varphi_\infty(u)$$

where t and u are the variables from the identifications $A(\mathbf{P}^\infty) = \bar{A}[[u]]$ and $B(\mathbf{P}^\infty) = \bar{B}[[t]]$. It's interesting to observe that the series $\Phi(t)$ presents a homomorphism of the

formal group laws $F_{\omega_B} \rightarrow F_{\omega_A} \otimes_{\bar{A}^{uc}} \bar{B}^{uc}$ related to the given orientations of A and of B , but we will not use this observation below in the preprint.

Note that all the coefficients of the formal power series $\Phi(t)$ lie in \bar{B}^{uc} . The series $\Phi(t)$ is divisible by t , because $c_1^A(\mathcal{O}(1))$ vanishes being restricted to a point.

5.1.1. Definition. 1) Define the inverse Todd genus of φ as the formal power series

$$itd_\varphi(t) = \Phi(t)/t \in B^{uc}[[t]]$$

2) Let t_1, t_2, \dots, t_n be independent variables and let $\sigma_1, \sigma_2, \dots, \sigma_n$ be the symmetric polynomial in these variables: $\sigma_1 = t_1 + t_2 + \dots + t_n$, $\sigma_2 = t_1 t_2 + \dots + t_{n-1} t_n$, ... , $\sigma_n = t_1 t_2 \dots t_n$. Set

$$itd_\varphi(\sigma_1, \sigma_2, \dots, \sigma_n) = \prod_{i=1}^n itd_\varphi(t_i) \in B^{uc}[[\sigma_1, \sigma_2, \dots, \sigma_n]]$$

3) the inverse Todd genus of a vector bundle E with $rkE = n$ over a smooth X is the evaluation of the series $itd_\varphi(\sigma_1, \sigma_2, \dots, \sigma_n)$ on the Chern classes of E

$$itd_\varphi(E) = itd_\varphi(c_1^B(E), c_2^B(E), \dots, c_n^B(E)) \in B^{uc}(X)$$

Here the right hand side is well-defined because the Chern classes are universally central and nilpotent (see 3.3.3).

5.1.2. Definition. Assume that the series itd_φ is invertible in $B^{uc}[[t]]$ (this is equivalent to the assertion that its free term is a unit in the ring B^{uc} or, in the other words, that $\Phi(t)$ is a local parameter in $\bar{B}^{uc}[[t]]$).

1) Define the Todd genus of φ as the multiplicative inverse of $itd_\varphi(t)$

$$td_\varphi(t) = t/\Phi(t) \in B^{uc}[[t]];$$

2) Set $td_\varphi(\sigma_1, \sigma_2, \dots, \sigma_n) = \prod_{i=1}^n td_\varphi(t_i) \in B^{uc}[[\sigma_1, \sigma_2, \dots, \sigma_n]];$

3) Define the Todd genus of a vector bundle E with $rkE = n$ over a smooth X by

$$td_\varphi(E) = td_\varphi(c_1^B(E), c_2^B(E), \dots, c_n^B(E)) \in B^{uc}(X)$$

Here the right hand side is well-defined because the Chern classes $c_i(E)$ are universally central and nilpotent (see 3.3.3).

5.1.3. Theorem. Let φ be the ring transformation and let $i : Y \hookrightarrow X$ be a closed imbedding of smooth varieties with the normal bundle N . Then for each element $\alpha \in A(Y)$ one has the relation in $B(X)$

$$i_B(\varphi(\alpha) \cup itd_\varphi(N)) = \varphi(i_A(\alpha)).$$

5.1.4. Theorem. *Let φ be the ring transformation. Suppose that the series $itd_\varphi(t)$ is invertible in $\bar{B}^{uc}[[t]]$. Then for each projective morphism of smooth projective varieties $f : Y \rightarrow X$ and each element $\alpha \in A(Y)$ one has the relation in $B(X)$*

$$f_B(\varphi(\alpha) \cup td_\varphi(T_Y)) = \varphi(f_A(\alpha)) \cup td_\varphi(T_X)$$

where T_Y (resp. T_X) is the tangent bundle to Y (resp. to X).

5.1.5. Corollary. *Under the hypothesis of Theorem 5.1.4 for each smooth projective variety X and each $a \in A(X)$ the following relation holds in $B(pt)$*

$$f_B(\varphi(a) \cup td_\varphi(T_X)) = \varphi(f_A(a))$$

where T_X is the tangent bundle to X and f is the structural morphism $X \rightarrow pt$.

5.1.6. Corollary. *Suppose the ring transformation φ takes the Chern class c_1^A to the Chern class c_1^B (for each smooth variety X and each line bundle L/X one has $\varphi(c_1^A(L)) = c_1^B(L)$). Then φ commutes with the push-forwards in theories A and B . Namely for each projective morphism of smooth varieties $f : Y \rightarrow X$ one has the relation*

$$f_B \circ \varphi_Y = \varphi_X \circ f_A.$$

5.1.7. Corollary. *Let A be a ring cohomology theory endowed with Chern classes c' and c'' for line bundles. For a projective morphism $f : Y \rightarrow X$ of smooth varieties let $f'_A : A(Y) \rightarrow A(X)$ and $f''_A : A(Y) \rightarrow A(X)$ be the push-forwards given by Corollary 4.1.4 and the classes c' , c'' respectively. Let $\varphi : A \rightarrow A$ be the identity transformation. Then the inverse Todd genus itd_φ is invertible in $\bar{A}[[t]]$ and thus the Todd genus $td_\varphi(t)$ is well-defined and for each $\alpha \in A(Y)$ one has the relation in $A(X)$*

$$f''_A(\alpha \cup f^A(itd_\varphi(T_X)) \cup td_\varphi(T_Y)) = f'_A(\alpha).$$

5.1.8. Corollary. *Let φ be the ring transformation. Suppose the inverse Todd genus $itd_\varphi(t)$ is invertible in $\bar{B}[[t]]$. Then for each finite etale morphism $f : Y \rightarrow X$ one has the relation*

$$f_B \circ \varphi_Y = \varphi_X \circ f_A.$$

5.2 Examples and applications

Here we consider ring cohomology theories from 3.6. Thus the push-forwards on these theories given by Theorem 4.1.3 coincides with the well-known ones (see 4.1).

5.2.1. Suppose $k = \mathbb{C}$. Let $A = K$ -theory and let B be the usual singular cohomology with rational coefficients and let φ be the Chern character $ch : A|_{Sm} \rightarrow B|_{Sm}$.

Recall that for a line bundle L over a smooth variety X one has $c_1^A(L) = [1] - [L^{-1}]$ (see 3.6.1) and $c_1^B(L) = D(L) \in H^2(X, \mathbb{Q})$ is the class of the divisor associated with the

line bundle L . Observe that $c_1^B(L^{-1}) = -c_1^B(L)$. Therefore if $t = c^B(\mathcal{O}(1))$ as in the beginning of this Section then one has a chain of relations

$$\varphi(c_1^A(\mathcal{O}(1))) = ch([1] - [\mathcal{O}(-1)]) = e^{c_1^B(1)} - e^{c_1^B(\mathcal{O}(-1))} = 1 - e^{-t}$$

Thus $td_\varphi(t) = t/(1 - e^{-t})$ and the Todd genus (5.1.2) coincides in the case with the classical Todd genus [H, Ch.III].

A similar computation shows that for the Chern character $ch : K\text{-theory} \rightarrow K\text{-cohomology}$ defined in [Gi] the Todd genus is the same series $t/(1 - e^{-t})$.

5.2.2. Let $\Psi^n : K\text{-theory} \rightarrow K\text{-theory}$ be the Adams operation (see [So] or [Gra]) considered just on smooth varieties ($n \neq \pm 1$ and $n \neq 0$). One has a chain of relations

$$\varphi(c_1^K(\mathcal{O}(1))) = \Psi^n([1] - [\mathcal{O}(-1)]) = [1] - [\mathcal{O}(-1)]^n$$

and furthermore $t = c_1^K(\mathcal{O}(1)) = [1] - [\mathcal{O}(-1)]$. Finally one gets

$$itd_\varphi(t) = ([1] - [\mathcal{O}(-1)]^n)/([1] - [\mathcal{O}(-1)]) = [1] + [\mathcal{O}(-1)] + \cdots + [\mathcal{O}(-n+1)]$$

Thus for a line bundle L over a smooth variety X one has

$$itd_\varphi(L) = [1] + [L^{-1}] + \cdots + [L^{-n+1}]$$

This class coincides with the element $\theta(L^{-1})$ but not with the element $\theta(L) = [1] + [L] + \cdots + [L^{n-1}]$ which is used in the Riemann-Roch theorem for Ψ^n proved in [So].

There is no confusion because it is written in [So, p. 519] the following relation $[1] - [L] = j_K(1)$ where $j : Y \rightarrow L$ is the zero section of a line bundle L over a smooth variety Y . But $j_K(1) = [1] - [L^\vee]$ in that case.

Thus the Riemann-Roch formula for the operation Ψ^n holds exactly with the class $itd_\varphi(L)$ above.

5.2.3. Let $\Psi^{-1} : K\text{-theory} \rightarrow K\text{-theory}$ be the Adams operation considered on smooth varieties. It takes a vector bundle E to the class of its dual E^\vee . In this case for a rank n vector bundle E one has

$$td_\varphi(E) = (-1)^{rk(E)}[\wedge^n(E^\vee)]$$

and thus for a projective morphism $f : Y \rightarrow X$ of smooth varieties and for an element $\alpha \in K(Y)$ one has the relation in $K(X)$

$$f_K(\Psi^{-1}\alpha \cdot K_Y) = (-1)^{\dim(Y) - \dim(X)} \Psi^{-1}(f_K(\alpha)) \cdot K_X,$$

where K_Y and K_X are the canonical classes of the varieties Y and X respectively and f_K is the push-forward from [Qu2, §7, 2.7] (see comments just after Corollary 4.1.4).

If $X = pt$ then $K_0(X) = \mathbb{Z}$ and $K_X = 1 \in K(X)$, and for each $a \in K_0(X)$ one has $\Psi^{-1}(a) = a$. Thus the previous relation looks as follows (for a vector bundle E/Y)

$$f_K([K_Y \otimes E^\vee]) = (-1)^{\dim(Y)} f_K([E]).$$

Since for a vector bundle F/Y its push-forward $f_K([F])$ coincides with the Euler characteristic $\chi(F)$ the very last relations follows as well from the Serre duality for the vector bundles over a smooth projective varieties.

5.2.4. (compare with [AH]). Let $k = \mathbb{R}$ and let $A(X, U) = H^*(X(\mathbb{R}), U(\mathbb{R}); \mathbb{Z}/2)$ the usual singular cohomology of the underlying topological pair of real point spaces. For a line bundle L over a smooth variety X consider the first Stiefel-Whitney class $w_1(L(\mathbb{R}))$ of the real vector bundle $L(\mathbb{R})$ over the space $X(\mathbb{R})$ and set $c_1^H(L) = w_1(L(\mathbb{R}))$. Consider the total Steenrod square operation on the $\mathbb{Z}/2$ -cohomology

$$Sq = 1 + Sq^1 + Sq^2 + \dots : H^* \rightarrow H^*$$

Then one has a chain of relations

$$td_\varphi(t) = w_1(\xi)/Sq(w_1(\xi)) = t/(t + t^2) = 1/(1 + t)$$

Thus $itd_\varphi(t) = 1 + t$ and $itd_\varphi(L) = 1 + w_1(L(\mathbb{R}))$ for a line bundle L and

$$itd(E) = w(E(\mathbb{R}))$$

(the total Stiefel-Whitney class of the real vector bundle $E(\mathbb{R})$ over the space $X(\mathbb{R})$). If $i : Y \hookrightarrow X$ is a closed imbedding of smooth varieties with the normal bundle N then for the element $\alpha = 1$ Theorem 5.1.3 states that

$$i_*(w(N(\mathbb{R}))) = Sq([Y])$$

where $[Y] = i_*(1)$ is the class of the variety Y . This is the known Wu formula.

5.2.5. In this example the characteristic of the ground field k is zero. Let $A_{\mathbb{Z}}^p(X) = \bigoplus_{q=0}^{\infty} H_{\mathcal{M}}^p(X, X - Z; \mathbb{Z}/2(q))$ and let $\tau = -1 \in \mu_2 = H_{\mathcal{M}}^0(k, \mathbb{Z}/2(1))$. Let

$$Sq_\tau : H_{\mathcal{M}}^*(-, \mathbb{Z}/2(*)) \rightarrow H_{\mathcal{M}}^*(-, \mathbb{Z}/2(*))$$

be the operation $1 + \tau Sq^1 + \tau Sq^2 + \tau^2 Sq^3 + \tau^2 Sq^4 + \dots$ considered on smooth varieties where Sq^i are the motivic Steenrod squares [V2]. If $-1 \in k^*$ is a square then the operation Sq_τ is a ring transformation on the category $\mathcal{S}m$ [V2]. (The total motivic Steenrod square operation $Sq = 1 + Sq^1 + Sq^2 + \dots$ is not a ring transformation in contrast with the previous example. Even if $-1 \in k^*$ is a square still the operation Sq is not multiplicative.) Now one has a chain of relations

$$itd_\varphi(t) = Sq_\tau(c_1^H(\mathcal{O}(1)))/c_1^H(\mathcal{O}(1)) = (t + \tau t^2)/t = 1 + \tau t$$

Thus $itd(E) = c_\tau^H(E) = 1 + \tau c_1^H(E) + \tau^2 c_2^H(E) + \dots \text{ mod } 2$ and Theorem 5.1.3 states that

$$i_*(c_\tau^H(N)) = Sq_\tau([Y])$$

where $[Y] \in H^{2d}(X, \mathbb{Z}/2(d)) = CH^d(X)$ is the class of Y in the Chow group of X .

One may ask whether the relation $i_*(c_\tau^H(N)) = Sq_\tau([Y])$ holds? The answer is yes, it does hold. The proof is based on a Riemann-Roch theorem for the operation $Sq^{ev} = 1 + Sq^2 + Sq^4 + \dots$. This operation is no more multiplicative, but still a variant of Riemann-Roch theorem holds for it.

5.2.6. Let k and A be as in the previous example and assume -1 is a square in the field k . Certain divisibility property of so called Segre numbers are deduced in this example from the Riemann-Roch theorem for the operation Sq_τ .

Consider n -dimensional smooth projective variety Y and its structural morphism to the point $f : Y \rightarrow pt$. Recall ([Fu, Sect.3.2]) that the Segre classes $s_d(E) \in H_{\mathcal{M}}^{2d}(Y, \mathbb{Z}(d)) = CH^d(Y)$ of a vector bundle E are described by the relations $c_z^H(E) = 1 + c_1^H(E).z + c_2^H(E).z^2 + \dots$ and $c_z^H(E)^{-1} = 1 + s_1(E).z + s_2(E).z^2 + \dots$ where z is a variable. The integer

$$s_n(Y) = deg(s_n(T_Y))$$

is called the Segre number of the variety Y . We claim that it always is divisible by 2.

To prove this claim consider the operation Sq_τ from the previous example. Since $itd(E) = c_\tau \tau(E)$ the Todd genus of Sq_τ is exactly the Segre class $s_\tau(E) = 1 + \tau s_1(E) + \tau^2 s_2(E) + \dots$ modulo 2. Thus one has the following relation in $H_{\mathcal{M}}^0((pt), \mathbb{Z}/2(n))$

$$f_*(s_\tau(T_Y)) = Sq_\tau(f_*(1)) = 0$$

because already $f_*(1) = 0$. Hence one has the relations in $H_{\mathcal{M}}^0((pt), \mathbb{Z}/2(n))$

$$\tau^n f_*(s_n(T_Y)) = f_*(\tau^n s_n(T_Y)) = 0.$$

Since the cup-product with the class τ defines an injection $H_{\mathcal{M}}^0((pt), \mathbb{Z}/2) \rightarrow H_{\mathcal{M}}^0((pt), \mathbb{Z}/2(n))$ one concludes that $f_*(s_n(T_Y))$ is divisible by 2. Thus the Segre number $s_n(Y)$ is divisible by 2.

5.2.7. Let again the field k be of characteristic zero and let p be a prime. Let A be as in the previous example but with \mathbb{Z}/p coefficients instead of $\mathbb{Z}/2$. Consider an operator

$$P_{\mathcal{M}} : H_{\mathcal{M}}^*(-, \mathbb{Z}/p(*)) \rightarrow H_{\mathcal{M}}^*(-, \mathbb{Z}/p(*))$$

where $P_{\mathcal{M}} = 1 + P^1 + P^2 + \dots$ and P^i are the the motivic reduced power operators [V2] considered on smooth varieties.

The operator $P_{\mathcal{M}}$ is a ring transformation [V2] in contrast with the mod 2 case. Its inverse Todd genus is computed as follows

$$itd_\varphi(t) = P_{\mathcal{M}}(c_1^H(\mathcal{O}(1)))/c_1^H(\mathcal{O}(1)) = (t + t^p)/t = 1 + t^{p-1}.$$

Thus for a line bundle L its the inverse Todd genus is the class $1 + c_1^H(L)^{p-1}$ modulo p . Let $itd(\sigma_1, \dots, \sigma_n)$ be the inverse Todd genus let $td(\sigma_1, \dots, \sigma_n)$ be the Todd genus and let the class \bar{b}_i be the i -th homogeneous component of the series $td(\sigma_1, \dots, \sigma_n)$. Now for a rank n vector bundle E over a smooth variety X set

$$\bar{b}_i(E) = \bar{b}_i(c_1^H(E), \dots, c_n^H(E)) \in H_{\mathcal{M}}^{2i}(X, \mathbb{Z}/p(i)) = CH^i(X)/pCH^i(X).$$

Clearly the classes \bar{b}_i vanish if i is not divisible by $p-1$. Finally for a smooth projective variety X of dimension d and its tangent bundle T_X set $\bar{b}_d(X) = deg(\bar{b}_d(T_X)) \in \mathbb{Z}/p\mathbb{Z}$.

We claim that for any smooth projective variety X its number $\bar{b}_d(X)$ always vanishes. In fact by Corollary 5.1.5 one has a chain of relations in $H_{\mathcal{M}}^0(pt, \mathbb{Z}/p)$

$$f_*(\bar{b}_d(T_X)) = Sq_{\mathcal{M}}(f_*(1)) = 0$$

because already $f_*(1) = 0$. It's interesting to compare the computations presented in 5.2.5, 5.2.6 and 5.2.7 with the ones given in [Br].

5.2.8. Let A be as in the previous example. Let $f : Y \rightarrow X$ be a finite etale morphism. Then one has the relation

$$f_* \circ P = P \circ f_*.$$

This is just a particular case of Corollary 5.1.8.

5.2.9. Let $A = B$ and let the transformation $\varphi : A \rightarrow B$ be the identity. Assume $c_1^B = a_0 \cdot c_1^A$ for a unite $a_0 \in A^{uc}$. Then one has $td_{\varphi}(t) = a_0$ and

$$td_{\varphi}(E) = a_0^{rk(E)}$$

Using a slight generalization of the method developed in [PY, §1] one can prove the following result generalizing the Suslin-Gabber rigidity theorem. The proof uses the push-forwards, but it does not use any Riemann-Roch type theorem.

5.2.10. Theorem (Rigidity theorem). *Let A be an oriented cohomology theory and let n be an integer prime to $\text{char}(k)$. Then for a smooth variety X and for a F -rational point $x \in X$ the homomorphisms*

$$x^A : A(X_x^h)/nA(X_x^h) \rightarrow A(k)/nA(k)$$

$$x^A : {}_nA(X_x^h) \rightarrow {}_nA(k)$$

are isomorphisms, where ${}_nM$ is the subgroup of the exponent n elements of a group M .

6 Appendix

In topology the tautological rank r vector bundle over the Grassmannian $\mathbf{Gr}(\infty, r)$ is a universal vector bundle of rank r . Below we present a technic allowing to consider the tautological line bundle $\mathcal{O}(-1)$ over the projective space \mathbf{P}^{∞} as a universal line bundle in a homotopy category. Recall that by an affine bundle we mean a torsor under a vector bundle.

6.1 A homotopy category.

6.1.1. Definition. *A morphism $X \xrightarrow{\pi} Y$ is called a quasi-affine bundle if it can be presented as the composition of several morphisms*

$$X = X^{(1)} \rightarrow X^{(2)} \rightarrow \dots \rightarrow X^{(n)} = Y$$

such that each arrow is an affine bundle.

A base change of a quasi-affine bundle is a quasi-affine bundle and the composition of two quasi-affine bundles is a quasi-affine bundle. If $X_1 \xrightarrow{\pi_1} X$ and $X_2 \xrightarrow{\pi_2} X$ are quasi-affine bundles then the map $X_1 \times_X X_2 \rightarrow X$ is a quasi-affine bundle.

Denote by $\{X, Y\}$ the set of all quasi-morphisms from X to Y , i.e., the set of diagrams

$$X \xleftarrow{\pi} \tilde{X} \xrightarrow{f} Y$$

where $\pi : \tilde{X} \rightarrow X$ is a quasi-affine bundle and f is a usual morphism. Define the composition $(g/\rho) \circ (f/\pi)$ of quasi-morphisms as $(g \cdot f_{\tilde{Y}})/(\pi \cdot \rho_Z)$, where ρ_Z is the base change of ρ by means of f and $f_{\tilde{Y}}$ is the base change of f by means of ρ . The quasi-morphisms form a category. A usual morphism $f : X \rightarrow Y$ one can consider as the quasi-morphism (id, f) .

One says that quasi-morphisms f/π and f'/π' are elementary homotopic if there exists a diagram

$$\begin{array}{ccccc} & & X' & & \\ & \swarrow \pi' & \uparrow \rho' & \searrow f' & \\ X & \xleftarrow{\Pi} & \mathcal{X} & \xrightarrow{\quad} & Y \\ & \swarrow \pi'' & \downarrow \rho'' & \searrow f'' & \\ & & X'' & & \end{array}$$

and a morphism $H : \mathbf{A}^1 \times \mathcal{X} \rightarrow Y$ satisfying the following conditions

1. $\pi' \cdot \rho' = \Pi = \pi'' \cdot \rho''$,
2. Π, ρ and ρ' are quasi-affine bundles,
3. $f' \cdot \rho' = H|_{\{0\} \times \mathcal{X}}$ and $f'' \cdot \rho'' = H|_{\{1\} \times \mathcal{X}}$.

6.1.2. Definition. One says that quasi-morphisms f'/π' and f''/π'' are naively homotopic if there exists a family of quasi-morphisms $f_0/\pi_0, \dots, f_r/\pi_r$ such that $(\pi', f') = (\pi_0, f_0)$, $(\pi'', f'') = (\pi_r, f_r)$, and for each index $i = 0, \dots, r-1$ the quasi-morphisms (π_i/f_i) and (π_{i+1}/f_{i+1}) are elementary homotopic. The set of all naively homotopy classes of quasi-morphisms X to Y denote $[X, Y]$. The naively homotopy class of a quasi-morphism (π/f) denote $[\pi, f]$. The composition of quasi-morphisms induces a composition of their naively homotopy classes.

6.1.3. Definition. For an inclusion $V \subset W \subset k^\infty$ of finite-dimensional subspaces we write $i_{V,W} : \mathbf{P}(V) \subset \mathbf{P}(W)$ for the inclusion of the associated projective spaces. The assignment $f/\pi \mapsto (\pi, i_{V,W} \cdot f)$ gives a map of sets $[X, \mathbf{P}(V)] \rightarrow [X, \mathbf{P}(W)]$ and we set $[X, \mathbf{P}^\infty] = \varinjlim [X, \mathbf{P}(V)]$, where the inductive system is taken over all finite-dimensional subspaces V of k^∞ .

One has $\text{Pic}(\mathbf{P}^\infty) = \mathbb{Z}$ and we denote $[\mathcal{O}(-1)]$ the unique element of $\text{Pic}(\mathbf{P}^\infty)$ which maps to $[\mathcal{O}_V(-1)]$ under the canonical map $\text{Pic}(\mathbf{P}^\infty) \rightarrow \text{Pic}(\mathbf{P}(V))$.

6.1.4. Theorem. *Let X be a smooth variety. Then the assignment $f/\pi \mapsto ((\pi^*)^{-1} \cdot f^*)([L_\infty]) \in \text{Pic}(X)$ induces a well-defined map*

$$cl_X : [X, \mathbf{P}^\infty] \rightarrow \text{Pic}(X).$$

The map cl_X has the following property

1. the map cl_X is a bijection;
2. if $g : Y \rightarrow X$ is a morphism of smooth varieties, then one has $g^* \cdot cl_X = cl_Y \cdot g^!$, where the map $g^! : [X, \mathbf{P}^\infty] \rightarrow [Y, \mathbf{P}^\infty]$ takes $[\pi, f]$ to $[\pi, f] \cdot g$;
3. if $V \subset k^\infty$ is a finite-dimensional subspace and $X = \mathbf{P}(V)$ then $cl_X([id_X, i_{V,V}]) = [\mathcal{O}_V(-1)]$.

Now given a cohomology theory A Theorem 6.1.4 allows to associate with any class $a \in A(\mathbf{P}^\infty)$ (see 1.1) and any line bundle L/X over a smooth variety X a class $a(L) \in A(X)$ as follows. Consider an affine bundle $p : \tilde{X} \rightarrow X$, a finite dimensional vector subspace V of the vector space k^∞ and a map $i : \tilde{X} \rightarrow \mathbf{P}(V)$ such that $p^*(L) = i^*(L_V)$. Set

$$a(L) = [(p^A)^{-1} \circ i^A](a_V) \in A(X). \quad (27)$$

The class $a(L)$ is well-defined by 6.1.4 and satisfies the following properties

1. $g^A(a(L)) = a(g^*(L))$ for any morphism $g : Y \rightarrow X$ of smooth varieties;
2. $a(L_W) = a_W \in A(\mathbf{P}(W))$ for any finite-dimensional vector subspace $W \subset k^\infty$.

Now set $A_{\mathbf{P}^\infty}(L_\infty) = A(L_\infty, L_\infty^0)$ (see 1.1) and take an element $\tau \in A_{\mathbf{P}^\infty}(L_\infty)$. Let L be a line bundle over a smooth variety X . By 6.1.4 there exists an affine bundle $p : \tilde{X} \rightarrow X$ and a finite dimensional vector subspace $V \subset k^\infty$ and a morphism $i : \tilde{X} \rightarrow \mathbf{P}(V)$ such that the line bundles $\tilde{L} = L \times_X \tilde{X}$ and $\tilde{L}_V = L_V \times_{\mathbf{P}(V)} \tilde{X}$ over the variety \tilde{X} are isomorphic. Denote $p_L : \tilde{L} \rightarrow L$ and $i_V : \tilde{L}_V \rightarrow L_V$ the projections, choose a line bundle isomorphism $\varphi : \tilde{L} \rightarrow \tilde{L}_V$ and set

$$\tau(L) = [(p_L^A)^{-1} \circ \varphi^A \circ i_V^A](\tau_V) \in A_X(L). \quad (28)$$

6.1.5. Theorem. *For each line bundle L over a smooth variety X the element $\tau(L)$ depends only on the line bundle L . The assignment $L \mapsto \tau(L)$ has the following properties*

- (1) *invariance:* $\alpha^A(\tau(L)) = \tau(L)$ for each line bundle automorphism $\alpha : L \rightarrow L$;
- (2) *functoriality:* $g_L^A(\tau(L)) = \tau(\tilde{L})$ for each morphism $g : \tilde{X} \rightarrow X$ of smooth varieties and the line bundle $\tilde{L} = \tilde{X} \times_X L$ over \tilde{X} and the projection $g_L : \tilde{L} \rightarrow L$.
- (3) *normalization:* $\tau(L_V) = \tau_V$ in $A_{\mathbf{P}(V)}(L_V)$.

In the rest of this section we introduce a group structure on $[X, \mathbf{P}^\infty]$ and state that the bijection cl_X becomes a group isomorphism. Let V_1 and V_2 be two finite dimensional vector spaces over the field k and let $V_1 \otimes V_2$ be the tensor product of these spaces. Let

$$s_{V_1, V_2} : \mathbf{P}(V_1) \times \mathbf{P}(V_2) \rightarrow \mathbf{P}(V_1 \otimes V_2)$$

be the Segre map which takes the point $(\{l_1\}, \{l_2\})$ to the point $\{l_1 \otimes l_2\}$. Let $\alpha, \beta \in [X, \mathbf{P}^\infty]$ and let (π_1, f_1) and (π_2, f_2) be their representatives, where f_i is a morphism $\mathcal{X}_i \rightarrow \mathbf{P}(V_i)$ and V_i is a finite-dimensional subspace in k^∞ ($i = 1, 2$). Choose a k -linear imbedding $V_1 \otimes V_2 \hookrightarrow k^\infty$ and denote j the induced imbedding of the projective spaces. Now set

$$\alpha + \beta = j \cdot s_{V_1, V_2} \cdot (\alpha \times \beta) \cdot \Delta \in [X, \mathbf{P}^\infty]$$

where $\Delta : X \rightarrow X \times X$ is the diagonal. Let $[\pi, f] \in [X, \mathbf{P}(V)]$ where V is a finite-dimensional subspace in k^∞ . Consider the vector space V^\vee dual to the space V and the canonical pairing $(,) : V \times V^\vee \rightarrow k$ given by $(v, f) = f(v)$. Consider a hypersurface $M_V \hookrightarrow \mathbf{P}(V) \times \mathbf{P}(V^\vee)$ given by the equation $(v, f) = 0$. Denote $p : \mathbf{P}(V) \times \mathbf{P}(V^\vee) - M_V \rightarrow \mathbf{P}(V)$ and $p^\vee : \mathbf{P}(V) \times \mathbf{P}(V^\vee) - M_V \rightarrow \mathbf{P}(V^\vee)$ the projections on $\mathbf{P}(V)$ and $\mathbf{P}(V^\vee)$ respectively. Choose a k -linear isomorphism $V^\vee \rightarrow V$ and let $i : \mathbf{P}(V^\vee) \rightarrow \mathbf{P}(V)$ be the induced isomorphism of the projective spaces. Now set

$$-[\pi, f] = i \cdot [p, p^\vee] \cdot [\pi, f] \in [X, \mathbf{P}^\infty].$$

Let $l \subset k^\infty$ be a 1-dimensional subspace and let $f_l : X \rightarrow \mathbf{P}^\infty$ be the morphism which sends X to the point $\{l\} \in \mathbf{P}^\infty$. Now set

$$[0_X] = [id_X, f_l] \in [X, \mathbf{P}^\infty].$$

6.1.6. Theorem. *The operation $+$ described just above defines an abelian group structure on $[X, \mathbf{P}^\infty]$. The inverse element of the element $[\pi, f]$ is the element $-[\pi, f]$ constructed just above and the zero of this group is the element $[0_X]$ described just above.*

Finally the map $cl_X : [X, \mathbf{P}^\infty] \rightarrow Pic(X)$ is a group isomorphism.

References

- [AH] *M.F. Atiyah, F. Hirzebruch.* Cohomologie-Operationen und charakteristische Klassen, Math.Zeitschr. 77 (1961), 149-187.
- [Br] *P. Brosnan.* Steenrod operations in Chow theory, <http://www.math.uiuc.edu/K-theory/0370/>, 1999.
- [Dy] *E. Dyer.* Relations between cohomology theories. Col. Alg. Topology(1962), 1-10.
- [Ja] *J.F. Jardine.* Motivic symmetric spectra, <http://www.math.uiuc.edu/K-theory/448> .

- [PY] *I. Panin, S. Yagunov.* Rigidity for orientable functors. MPI-preprint, 2000.
- [Qu1] *D. Quillen.* On the formal group laws of unoriented and complex cobordism theory. Bull. AMS 75(1969), 1293-1298.
- [Qu2] *D. Quillen.* Higher algebraic K -theory:I. Lect. Notes Math. 341 (1973), 85-147.
- [V1] *V. Voevodsky.* \mathbf{A}^1 -Homotopy theory. Doc. Math., Extra Vol. ICM 1998(I), 417-442.
- [V2] *V. Voevodsky.* The Milnor Conjecture, MPI-preprint, 1996.
- [M] *F. Morel.* Basic properties of the stable homotopy category of smooth schemes, taken from a Web site approximately in March, 2000.
- [MV] *F. Morel, V. Voevodsky.* Homotopy category of schemes over a base. 1997.
- [H] *F. Hirzebruch.* Topological methods in algebraic geometry. Die Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, b. 131, 1966.
- [BS] *A. Borel, J.P. Serre.* Le theoreme de Riemann-Roch. BSMF, 86 (1958), 97-136.
- [CF] *P.E. Conner, E.E. Floyd.* The relation of cobordism to K -theory. Lect. Notes Math., 28 (1966), Springer-Verlag, Berlin.
- [Mi] *J. S. Milne.* Etale Cohomology. Princeton University Press, 1980.
- [TT] *R. Thomason, T. Throbaugh.* Higher algebraic K-theory of schemes and of derived categories. In The Grothendieck festschrift, 3, 247-436, Birkhauser, Boston, 1990.
- [SV] *A. Suslin, V. Voevodsky.* Bloch-Kato conjecture and motivic cohomology with finite coefficients. The arithmetics and geometry of algebraic cycles, 117-189, NATO Sci.Ser.C Math.Phys.Sci., 548, Kluver Acad.Publ., Dortrecht, 2000.
- [Fu] *W. Fulton.* Intersection theory. Springer-Verlag, 1984.
- [Su] *A. Suslin.* On K-theory of algebraically closed fields. Inv.Math, 73(1983), 241-245.
- [Ha] *M. Hazewinkel.* Formal groups and applications. Academic Press, NY, 1989.
- [Har] *R. Hartshorne.* Algebraic geometry. Graduated texts in Mathematics 52, 1977.
- [Hu] *Po Hu.* S-modules in the category of schemes. <http://www.math.uiuc.edu/K-theory/396> .
- [Gi] *H. Gillet.* Riemann-Roch Theorems for Higher Algebraic K -Theory. Advances in Math. 40 (1981), 203-289.
- [Gr] *A. Grothendieck.* Bull.Soc.Math.France, 86 (1958), 136-154.

- [So] *Ch. Soule* Operations en K -theory algebrique. *Canad. J. Math.* 37 (1985), no.3, 488-550.
- [Gra] *D.Grayson*. Adams operations on higher K -theory. *K-theory* 6 (1992), no.2, 97-111.