The relation between the Baum-Connes Conjecture and the Trace Conjecture

Wolfgang Lück*

Fachbereich Mathematik und Informatik
Westfälische Wilhelms-Universität Münster
Einsteinstr. 62
48149 Münster
Germany

March 22, 2001

Abstract

We prove a version of the L^2 -index Theorem of Atiyah, which uses the universal center-valued trace instead of the standard trace. We construct for G -equivariant K -homology an equivariant Chern character, which is an isomorphism and lives over the ring $\mathbb{Z} \subset \Lambda^G \subset \mathbb{Q}$ obtained from the integers by inverting the orders of all finite subgroups of G . We use these two results to show that the Baum-Connes Conjecture implies the modified Trace Conjecture, which says that the image of the standard trace $K_0(C_r^*(G)) \rightarrow \mathbb{R}$ takes values in Λ^G . The original Trace Conjecture predicted that its image lies in the additive subgroup of $\mathbb{R}$ generated by the inverses of all the orders of the finite subgroups of G , and has been disproved by Roy [13].

Key words: Baum-Connes Conjecture, Trace Conjecture, equivariant Chern character, L^2 -index theorem.

Mathematics subject classification 2000: 19L47, 19K56, 55N91

0. Introduction and statements of results

Throughout this paper let G be a discrete group. The *Baum-Connes Conjecture for G* says that the assembly map

$$\text{asmb}^G : K_0^G(\underline{EG}) \rightarrow K_0(C_r^*(G))$$

from the equivariant K -homology of the classifying space for proper G -actions $\underline{EG}$ to the topological K -theory of the reduced C^* -algebra $C_r^*(G)$ is bijective [3, page 8], [5, Conjecture 3.1]. In connection with this conjecture Baum and Connes [3, page 21] also made the sometimes so called *Trace Conjecture*. It says that the image of the composition

$$K_0(C_r^*(G)) \xrightarrow{i} K_0(\mathcal{N}(G)) \xrightarrow{\text{tr}_{\mathcal{N}(G)}} \mathbb{R}$$

*email: lueck@math.uni-muenster.de, www: <http://www.math.uni-muenster.de/u/lueck>, FAX: 49 251 8338370

is the additive subgroup of $\mathbb{Q}$ generated by all numbers $\frac{1}{|H|}$, where $H \subset G$ runs through all finite subgroups of G . Here $\mathcal{N}(G)$ is the group von Neumann algebra, i the change of rings homomorphism associated to the canonical inclusion $C_r^*(G) \rightarrow \mathcal{N}(G)$ and $\text{tr}_{\mathcal{N}(G)}$ is the map induced by the standard von Neumann trace $\text{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \rightarrow \mathbb{C}$. Roy has constructed a counterexample to the Trace Conjecture in this form in [13] based on her article [14]. She constructs a group Γ , whose finite subgroups are all of order 1 or 3, together with an element in $K_0^G(\underline{EG})$, whose image under $\text{tr}_{\mathcal{N}(\Gamma)} \circ i \circ \text{asmb}$ is $-\frac{1105}{9}$. The point is that $3 \cdot \frac{1105}{9}$ is not an integer. Notice that Roy's example does not imply that the Baum-Connes Conjecture does not hold for Γ . Since the group Γ contains a torsionfree subgroup of index 9 and the Trace Conjecture for torsionfree groups does follow from the Baum-Connes Conjecture, the Baum-Connes Conjecture predicts that the image of $\text{tr}_{\mathcal{N}(\Gamma)} \circ i : K_0(C_r^*(\Gamma)) \rightarrow \mathbb{R}$ is contained in $\{r \in \mathbb{R} \mid 9 \cdot r \in \mathbb{Z}\}$. So one could hope that the following version of the Trace Conjecture is still true. Denote by

$$\Lambda^G := \mathbb{Z} \left[\frac{1}{|\text{Fin}(G)|} \right] \quad (0.1)$$

the ring $\mathbb{Z} \subset \Lambda^G \subset \mathbb{Q}$ obtained from $\mathbb{Z}$ by inverting all the orders $|H|$ of finite subgroups of G . For Roy's group Γ this is $\{m \cdot 3^{-n} \mid m, n \in \mathbb{Z}, n \geq 0\}$ and obviously contains $-\frac{1105}{9}$.

Conjecture 0.2 (Modified Trace Conjecture for a group G) *The image of the composition*

$$K_0(C_r^*(G)) \xrightarrow{i} K_0(\mathcal{N}(G)) \xrightarrow{\text{tr}_{\mathcal{N}(G)}} \mathbb{R}$$

is contained in Λ^G .

The motivation for this paper is to prove

Theorem 0.3 *The image of the composition*

$$\Lambda^G \otimes_{\mathbb{Z}} K_0^G(\underline{EG}) \xrightarrow{\text{id} \otimes \text{asmb}^G} \Lambda^G \otimes_{\mathbb{Z}} K_0(C_r^*(G)) \xrightarrow{i} \Lambda^G \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G)) \xrightarrow{\text{tr}_{\mathcal{N}(G)}} \mathbb{R}$$

is Λ^G .

In particular the modified Trace Conjecture 0.2 holds for G , if the assembly map $\text{asmb}^G : K_0^G(\underline{EG}) \rightarrow K_0(C_r^(G))$ appearing in the Baum-Connes Conjecture is surjective.*

In order to prove Theorem 0.3 (actually a generalization of it in Theorem 0.8), we will prove a slight generalization of Atiyah's L^2 -Index Theorem and construct an equivariant Chern character for equivariant K -homology of proper G -CW-complexes, which is bijective and defined after applying $\Lambda^G \otimes_{\mathbb{Z}} -$.

Let M be a closed Riemannian manifold and $D^* = (D^*, d^*)$ be an elliptic complex of differential operators of order 1 on M . Denote by $\text{index}(D^*) \in \mathbb{Z}$ its index. Let $\overline{M} \rightarrow M$ be a G -covering. Then one can lift D^* to an elliptic G -equivariant complex $\overline{D}^*$. Using the trace $\text{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \rightarrow \mathbb{C}$ Atiyah [1] defines its L^2 -index $\text{index}_{\mathcal{N}(G)}(\overline{D}^*) \in \mathbb{R}$ and shows

$$\text{index}(D^*) = \text{index}_{\mathcal{N}(G)}(\overline{D}^*).$$

The L^2 -index theorem of Atiyah implies that the composition

$$K_0^G(EG) \xrightarrow{\text{asmb}^G} K_0(C_r^*(G)) \xrightarrow{i} K_0(\mathcal{N}(G)) \xrightarrow{\text{tr}_{\mathcal{N}(G)}} \mathbb{R}$$

agrees with the composition

$$K_0^G(EG) \xrightarrow{\text{ind}_{G \rightarrow \{1\}}} K_0(BG) \xrightarrow{K_0(\text{pr})} K_0(*) \xrightarrow{\text{asmb}^{\{1\}}} K_0(C_r^*(\{1\})) \xrightarrow{\text{dimc}} \mathbb{Z} \hookrightarrow \mathbb{R}.$$

Since for a torsionfree group G the spaces EG and $\underline{E}G$ agree, the Baum-Connes Conjecture for a torsionfree group G does imply that the image of $K_0(C_r^*(G)) \xrightarrow{i} K_0(\mathcal{N}(G)) \xrightarrow{\text{tr}_{\mathcal{N}(G)}} \mathbb{R}$ is $\mathbb{Z}$ [3, Corollary 1 on page 21]. Instead of using the standard von Neumann trace $\text{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \rightarrow \mathbb{C}$, one can use the universal center-valued trace $\text{tr}_{\mathcal{N}(G)}^u : \mathcal{N}(G) \rightarrow \mathcal{Z}(\mathcal{N}(G))$ to define an index

$$\text{index}_{\mathcal{N}(G)}^u(\overline{D}^*) \in \mathcal{Z}(\mathcal{N}(G)),$$

which takes values in the center $\mathcal{Z}(\mathcal{N}(G))$ of the group von Neumann algebra $\mathcal{N}(G)$. Thus we get additional information, namely, for any element $g \in G$, whose conjugacy class (g) is finite, we get a complex number. However, it turns out that the value at classes (g) with $g \neq 1$ is zero and that the value at (1) is the index of D^* . Namely, we will show in Section 1

Theorem 0.4 *Under the conditions above we get in $\mathcal{Z}(\mathcal{N}(G))$*

$$\text{index}_{\mathcal{N}(G)}^u(\overline{D}^*) = \text{index}(D^*) \cdot 1_{\mathcal{N}(G)}.$$

As an illustration we discuss the special case, where G is finite, M is an oriented closed $4k$ -dimensional manifold with free orientation preserving G -action and D^* is the signature operator. Then Theorem 0.4 reduces to the well-known statement that the equivariant signature

$$\text{sign}^G(M) := [H_{2k}(M)^+] - [H_{2k}(M)^-] \in \text{Rep}_{\mathbb{C}}(G)$$

is equal to $\text{sign}(G \backslash M) \cdot [\mathbb{C}G]$ for $\text{sign}(G \backslash M) \in \mathbb{Z}$ the (ordinary) signature of $G \backslash M$. We mention that this implies $\text{sign}(M) = |G| \cdot \text{sign}(G \backslash M)$. Theorem 0.4 is a special case of Theorem 5.4 but we will need it in the proof of Theorem 5.4 and therefore will have to prove it first.

The second ingredient is a variation of the equivariant Chern character of [11] for equivariant K -homology of proper G -CW-complexes. Recall that proper means that all isotropy groups are finite. The construction in [11] works for equivariant homology theories with a Mackey structure on the coefficient system in general, but requires to invert all primes. The construction we will give here works after applying $\Lambda^G \otimes_{\mathbb{Z}} ?$ and has a different source.

Denote for a proper G -CW-complex X by $\mathcal{F}(X)$ the set of all subgroups $H \subset G$, for which $X^H \neq \emptyset$, and by

$$\Lambda^G(X) := \mathbb{Z} \left[\frac{1}{\mathcal{F}(X)} \right] \quad (0.5)$$

the ring $\mathbb{Z} \subset \Lambda^G(X) \subset \Lambda^G$ obtained from $\mathbb{Z}$ by inverting the orders of all subgroups $H \in \mathcal{F}(X)$. Denote by

$$J^G \quad \text{resp.} \quad J^G(X) \quad (0.6)$$

the set of conjugacy classes (C) of finite cyclic subgroups $C \subset G$ resp. the subset $J^G(X) \subset J^G$ of conjugacy classes (C) of finite cyclic subgroups $C \subset G$, for which X^C is non-empty. Obviously $\Lambda^G = \Lambda^G(\underline{E}G)$ and $J^G = J^G(\underline{E}G)$ since $\underline{E}G$ is characterized up to G -homotopy by the property that $\underline{E}G^H$ is contractible (and hence non-empty) for finite $H \subset G$ and empty for infinite $H \subset G$. Let $C \subset G$ be a finite cyclic subgroup. Let $C_G C$ be the centralizer and $N_G C$ be the normalizer of $C \subset G$. Let $W_G C$ be the quotient $N_G C / C_G C$. We will construct an idempotent $\theta_C \in \Lambda^C \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{Q}}(C)$ which acts on $\Lambda^C \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C)$. We will see in Lemma 3.4 (b) that the cokernel of

$$\oplus_{D \subset C, D \neq C} \text{ind}_D^C : \oplus_{D \subset C, D \neq C} \mathbb{Z} \left[\frac{1}{|C|} \right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(D) \rightarrow \mathbb{Z} \left[\frac{1}{|C|} \right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C)$$

is isomorphic to the image of the idempotent endomorphism

$$\theta_C : \mathbb{Z} \left[\frac{1}{|C|} \right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) \rightarrow \mathbb{Z} \left[\frac{1}{|C|} \right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C).$$

After introducing and proving some preliminary results about modules over a category and representation theory of finite groups in Sections 2 and 3, we will prove in Section 4

Theorem 0.7 *Let X be a proper G -CW-complex. Put $\Lambda = \Lambda^G(X)$ and $J = J^G(X)$. Then there is for $p = 0, 1$ a natural isomorphism called equivariant Chern character*

$$\begin{aligned} \text{ch}_p^G(X) : \oplus_{(C) \in J} \Lambda \otimes_{\mathbb{Z}} K_p(C_G C \setminus X^C) \otimes_{\Lambda[W_G C]} \text{im}(\theta_C : \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) \rightarrow \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C)) \\ \xrightarrow{\cong} \Lambda \otimes_{\mathbb{Z}} K_p^G(X). \end{aligned}$$

Notice that the equivariant Chern character of Theorem 0.7 reduces to the obvious isomorphism $K_0(G \setminus X) \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(\{1\}) \xrightarrow{\cong} K_0^G(X)$, if G acts freely on X . In the special case, where G is finite, X is the one-point-space $\{*\}$ and $p = 0$, the equivariant Chern character reduces to an isomorphism

$$\begin{aligned} \oplus_{(C) \in J^G} \mathbb{Z} \left[\frac{1}{|G|} \right] \otimes_{\mathbb{Z}[\frac{1}{|G|}][W_G C]} \text{im} \left(\theta_C : \mathbb{Z} \left[\frac{1}{|G|} \right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) \rightarrow \mathbb{Z} \left[\frac{1}{|G|} \right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) \right) \\ \xrightarrow{\cong} \mathbb{Z} \left[\frac{1}{|G|} \right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(G). \end{aligned}$$

This is a strong version of the well-known theorem of Artin that the map induced by induction

$$\oplus_{(C) \in J^G} \mathbb{Q} \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(G)$$

is surjective for any finite group G . Theorem 0.7 gives a computation of $\Lambda^G \otimes K_0^G(\underline{E}G)$, namely

$$\begin{aligned} \oplus_{(C) \in J^G} \Lambda^G \otimes_{\mathbb{Z}} K_p(B(C_G C)) \otimes_{\Lambda^G[W_G C]} \text{im}(\theta_C : \Lambda^G \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) \rightarrow \Lambda^G \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C)) \\ \xrightarrow{\cong} \Lambda^G \otimes_{\mathbb{Z}} K_p^G(\underline{E}G). \end{aligned}$$

Another construction of an equivariant Chern character using completely different methods can be found in [4]. However, it works only after applying $\mathbb{C} \otimes_{\mathbb{Z}} -$ and therefore cannot be used for our purposes here.

In Theorem 5.4 we will identify the composition of the Chern character of Theorem 0.7 with the map

$$\Lambda^G \otimes_{\mathbb{Z}} K_0^G(\underline{E}G) \xrightarrow{\text{id} \otimes \text{asmb}^G} \Lambda^G \otimes_{\mathbb{Z}} K_0(C_r^*(G)) \xrightarrow{\text{id} \otimes i} \Lambda^G \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G))$$

with an easier to understand and to calculate homomorphism, whose image is obvious from its definition. This will immediately imply

Theorem 0.8 *Let Λ^G resp. J^G be the ring resp. set introduced in (0.1) resp. (0.6). Then the image of the composition*

$$\Lambda^G \otimes_{\mathbb{Z}} K_0^G(\underline{E}G) \xrightarrow{\text{id} \otimes_{\mathbb{Z}} \text{asmb}^G} \Lambda^G \otimes_{\mathbb{Z}} K_0(C_r^*(G)) \xrightarrow{\text{id} \otimes_{\mathbb{Z}} i} \Lambda^G \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G))$$

is the image of the map given by induction

$$\oplus_{(C) \in J^G} \text{id} \otimes \text{ind}_C^G : \oplus_{(C) \in J^G} \Lambda^G \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) \rightarrow \Lambda^G \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G)).$$

Now Theorem 0.3 follows from Theorem 0.8.

The change of rings and K-theory map $l : K_0(\mathbb{C}G) \rightarrow K_0(C_r^*(G))$ from the algebraic K_0 -group of the complex group ring $\mathbb{C}G$ to the topological K_0 -group of $C_r^*(G)$ is in general far from being surjective. There is some evidence that it is injective after applying $\Lambda \otimes_{\mathbb{Z}} ?$ (see [11, Theorem 0.1]). Theorem 0.8 gives some evidence for the conjecture that the image of $\Lambda^G \otimes_{\mathbb{Z}} K_0(C_r^*(G)) \xrightarrow{\text{id} \otimes_{\mathbb{Z}} i} \Lambda^G \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G))$ agrees with the image of the composition $\Lambda^G \otimes_{\mathbb{Z}} K_0(\mathbb{C}G) \xrightarrow{l} \Lambda^G \otimes_{\mathbb{Z}} K_0(C_r^*(G)) \xrightarrow{\text{id} \otimes_{\mathbb{Z}} i} \Lambda^G \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G))$. The paper is organized as follows

1. The L^2 -index theorem
 2. Modules over a category
 3. Some representation theory for finite groups
 4. The construction of the Chern character
 5. The Baum-Connes Conjecture and the Trace Conjecture
- References

The author wants to thank the Max-Planck-Institute for Mathematics in Bonn for the hospitality during his stay in January and February 2001, when parts of the paper were written.

1. The L^2 -index theorem

In this section we prove a slight generalization of the L^2 -index theorem of Atiyah [1]. Let $\overline{M}$ be a Riemannian manifold (without boundary) together with a cocompact free proper action of G by isometries. In other words, $M = G \backslash \overline{M}$ is a closed Riemannian manifold, the projection $p : \overline{M} \rightarrow M$ is a G -covering and $\overline{M}$ is equipped with the Riemannian metric induced by the one of M . Let $D^* = (D^*, d^*)$ be an elliptic complex of differential operators $d^p : D^p \rightarrow D^{p+1}$ of order 1 acting on the space of sections $D^p = C^\infty(E^p)$ of vector bundles $E^p \rightarrow M$. Define $\overline{E}^p$ by $p^* E^p$ and $\overline{D}^p$ by $L^2 C^\infty(\overline{E}^p)$. Then G -acts on $\overline{E}^p$ and $\overline{D}^p$. Since differential operators are local operators, there is a unique lift of each operator d^p to a G -equivariant differential operator $\widehat{d}^p : C^\infty(\overline{E}^p) \rightarrow C^\infty(\overline{E}^{p+1})$. We obtain an elliptic G -complex $(C^\infty(\overline{E}^*), \widehat{d}^*)$. Let $\overline{d}^p : \overline{D}^p \rightarrow \overline{D}^{p+1}$ be the minimal closure of $\widehat{d}^p$ which is the same as its maximal closure [1, Proposition 3.1].

Since D^* is elliptic, each cohomology module $H^p(D^*) := \ker(d^p) / \text{im}(d^{p-1})$ is a finitely generated $\mathbb{C}$ -module. Hence we can define the *index* of the elliptic complex D^* by

$$\text{index}(D^*) := \sum_{p \geq 0} \dim_{\mathbb{C}}(H^p(D^*)) \in \mathbb{Z}. \quad (1.1)$$

Next we want to define an analogous invariant for the lifted complex $\overline{D}^*$. The group von Neumann algebra $\mathcal{N}(G)$ of G is the $*$ -algebra $\mathcal{B}(l^2(G))^G$ of all bounded G -equivariant operators $l^2(G) \rightarrow l^2(G)$, where we equip $l^2(G)$ with the obvious left G -action. Let

$$\text{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \rightarrow \mathbb{C} \quad (1.2)$$

be the *standard von Neumann trace*, which sends $f \in \mathcal{N}(G) = \mathcal{B}(l^2(G))^G$ to $\langle f(e), e \rangle_{l^2(G)}$, where e denotes the element in $l^2(G)$ given by the unit element in $G \subset l^2(G)$. Denote by $\mathcal{Z}(\mathcal{N}(G))$ the center of $\mathcal{N}(G)$. There is the universal center-valued trace [7, Theorem 7.1.12 on page 462, Proposition 7.4.5 on page 483, Theorem 8.2.8 on page 517, Proposition 8.3.10 on page 525, Theorem 8.4.3 on page 532]

$$\text{tr}_{\mathcal{N}(G)}^u : \mathcal{N}(G) \rightarrow \mathcal{Z}(\mathcal{N}(G)) \quad (1.3)$$

which is uniquely determined by the following two properties:

- (a) tr^u is a trace with values in the center, i.e. tr^u is $\mathbb{C}$ -linear, for $a \in \mathcal{N}(G)$ with $a \geq 0$ we have $\text{tr}^u(a) \geq 0$ and $\text{tr}^u(ab) = \text{tr}^u(ba)$ for all $a, b \in \mathcal{N}(G)$;
- (b) $\text{tr}^u(a) = a$ for all $a \in Z(\mathcal{N}(G))$.

The map tr^u has the following further properties:

- (c) tr^u is faithful;
- (d) tr^u is normal. Equivalently, tr^u is continuous with respect to the ultraweak topology on $\mathcal{N}(G)$;
- (e) $\|\text{tr}^u(a)\| \leq \|a\|$ for $a \in \mathcal{N}(G)$;
- (f) $\text{tr}^u(ab) = a \text{tr}^u(b)$ for all $a \in Z(\mathcal{N}(G))$ and $b \in \mathcal{N}(G)$;
- (g) Let p and q be projections in $\mathcal{N}(G)$. Then p and q are equivalent, i.e. $p = vv^*$ and $q = v^*v$, if and only if $\text{tr}^u(p) = \text{tr}^u(q)$;
- (h) Any linear functional $f : \mathcal{N}(G) \rightarrow \mathbb{C}$, which is continuous with respect to the norm topology on $\mathcal{N}(G)$ and which is central, i.e. $f(ab) = f(ba)$ for all $a, b \in \mathcal{N}(G)$, factorizes as

$$\mathcal{N}(G) \xrightarrow{\text{tr}^u} Z(\mathcal{N}(G)) \xrightarrow{f|_{Z(\mathcal{N}(G))}} \mathbb{C}.$$

In particular $\text{tr}_{\mathcal{N}(G)} \circ \text{tr}_{\mathcal{N}(G)}^u = \text{tr}_{\mathcal{N}(G)}$.

A Hilbert $\mathcal{N}(G)$ -module V is a Hilbert space V together with a G -action by isometries such that there exists a Hilbert space H and a G -equivariant projection $p : H \otimes l^2(G) \rightarrow H \otimes l^2(G)$ with the property that V and $\text{im}(p)$ are isometrically G -linearly isomorphic. Here $H \otimes l^2(G)$ is the tensor product of Hilbert spaces and G acts trivially on H and on $l^2(G)$ by the obvious left multiplication. Notice that p is not part of the structure, only its existence is required. We call V *finitely generated* if H can be chosen to be finite-dimensional.

Our main examples of Hilbert $\mathcal{N}(G)$ -modules are the Hilbert spaces $\overline{D}^p$ which are isometrically G -isomorphic to $L^2(C^\infty(E^p)) \otimes l^2(G)$. This can be seen using a fundamental domain $\mathcal{F}$ for the G -action on $\overline{M}$ which is from a measure theory point of view the same as M . A morphism $f : V \rightarrow W$ of Hilbert $\mathcal{N}(G)$ -modules is a densely defined closed G -equivariant operator. The differentials $\overline{d}^p$ are morphisms of Hilbert $\mathcal{N}(G)$ -modules.

Let $f : V \rightarrow V$ be a *morphism of Hilbert $\mathcal{N}(G)$ -modules* which is positive. Choose a G -projection $p : H \otimes l^2(G) \rightarrow H \otimes l^2(G)$ and an isometric invertible G -equivariant operator $u : \text{im}(p) \rightarrow V$. Let $\{b_i \mid i \in I\}$ be a Hilbert basis for H . Let $\overline{f}$ be the composition

$$H \otimes l^2(G) \xrightarrow{p} \text{im}(p) \xrightarrow{u} V \xrightarrow{f} V \xrightarrow{u^{-1}} \text{im}(p) \hookrightarrow H \otimes l^2(G).$$

Define the *von Neumann trace* of $f : V \rightarrow V$ by

$$\text{tr}_{\mathcal{N}(G)}(f) := \sum_{i \in I} \langle \overline{f}(b_i \otimes e), b_i \otimes e \rangle_{H \otimes l^2(G)} \in [0, \infty]. \quad (1.4)$$

This is indeed independent of the choice of p , u and the Hilbert basis $\{b_i \mid i \in I\}$. If V is finitely generated, then $\text{tr}_{\mathcal{N}(G)}(f) < \infty$ is always true. Define the von Neumann dimension of a Hilbert $\mathcal{N}(G)$ -module V by

$$\dim_{\mathcal{N}(G)}(V) := \text{tr}_{\mathcal{N}(G)}(\text{id} : V \rightarrow V) \in [0, \infty]. \quad (1.5)$$

If V is a finitely generated Hilbert $\mathcal{N}(G)$ -module, we define the *universal center-valued von Neumann dimension*

$$\dim_{\mathcal{N}(G)}^u(V) := \text{tr}_{\mathcal{N}(G)}^u(\text{id} : V \rightarrow V) \in \mathcal{Z}(\mathcal{N}(G)) \quad (1.6)$$

analogously to $\dim_{\mathcal{N}(G)}(V)$ replacing $\text{tr}_{\mathcal{N}(G)}$ by $\text{tr}_{\mathcal{N}(G)}^u$. Given a finitely generated Hilbert $\mathcal{N}(G)$ -module V , we have $\text{tr}_{\mathcal{N}(G)}(\dim_{\mathcal{N}(G)}^u(V)) = \dim_{\mathcal{N}(G)}(V)$.

Define the L^2 -cohomology $H_{(2)}^p(\overline{D}^*)$ to be $\ker(\overline{d}^p)/\text{clos}(\text{im}(\overline{d}^{p-1}))$, where $\text{clos}(\text{im}(\overline{d}^{p-1}))$ is the closure of the image of $\overline{d}^{p-1}$. Define the p -th Laplacian by $\overline{\Delta}_p = (\overline{d}^p)^*\overline{d}^p + \overline{d}^{p-1}(\overline{d}^{p-1})^*$. By the L^2 -Hodge-deRham Theorem we get a G -equivariant isometric isomorphism $\ker(\overline{\Delta}_p) \xrightarrow{\cong} H_{(2)}^p(\overline{D}^*)$. Thus $H_{(2)}^p(\overline{D}^*)$ inherits the structure of a Hilbert $\mathcal{N}(G)$ -module. Moreover, it turns out to be a finitely generated Hilbert $\mathcal{N}(G)$ -module. This can be deduced from the results of [12], where an index already over $C_r^*(G)$ is defined and the problem of getting finitely generated modules over $C_r^*(G)$ is treated. Namely, one can deduce from [12] after passing to the group von Neumann algebra, that there are finitely generated Hilbert $\mathcal{N}(G)$ -modules U_1, U_2, V_1 and V_2 and Hilbert $\mathcal{N}(G)$ -modules W_1 and W_2 together with a morphism $v : V_1 \rightarrow V_2$ and isomorphisms of Hilbert $\mathcal{N}(G)$ -modules $w : W_1 \xrightarrow{\cong} W_2$, $u_1 : \overline{D}^p \oplus U_1 \xrightarrow{\cong} V_1 \oplus W_1$ and $u_2 : \overline{D}^p \oplus U_2 \xrightarrow{\cong} V_2 \oplus W_2$ such that $u_2 \circ (\Delta_p \oplus 0) = (v \oplus w) \circ u_1$. Obviously the kernel of v and hence the kernel of $\overline{\Delta}_p$ are finitely generated Hilbert $\mathcal{N}(G)$ -modules.

Define the *center-valued L^2 -index* and the *L^2 -index*

$$\text{index}_{\mathcal{N}(G)}^u(\overline{D}^*) := \sum_{p \geq 0} \dim_{\mathcal{N}(G)}^u(H_{(2)}^p(\overline{D}^*)) \in \mathcal{Z}(\mathcal{N}(G)); \quad (1.7)$$

$$\text{index}_{\mathcal{N}(G)}(\overline{D}^*) := \sum_{p \geq 0} \dim_{\mathcal{N}(G)}(H_{(2)}^p(\overline{D}^*)) \in \mathbb{R}. \quad (1.8)$$

The rest of this section is devoted to the proof of Theorem 0.4

Notation 1.9 Denote by $\text{con}(G)_{cf}$ the set of conjugacy classes (g) of elements $g \in G$ such that the set (g) is finite, or, equivalently, the centralizer $C_g(g) = \{g' \in G \mid g'g = gg'\}$ has finite index in G . For $c \in \text{con}(G)_{cf}$ let N_c be the element $\sum_{g \in c} g \in \mathbb{C}G$. In the sequel L_c resp. L_g denotes left multiplication with N_c resp. g for $c \in \text{con}(G)_{cf}$ resp. $g \in G$.

Notice for the sequel that $N_c \in \mathcal{Z}(\mathcal{N}(G))$ and L_c is G -equivariant and commutes with all G -operators.

Lemma 1.10 Consider $a \in \mathcal{Z}(\mathcal{N}(G))$. Then we have $a = 0$ if and only if $\text{tr}_{\mathcal{N}(G)}(N_c a) = 0$ holds for any $c \in \text{con}_{cf}(G)$.

Proof : Consider $a \in \mathcal{N}(G) = \mathcal{B}(l^2(G))^G$ which belongs to $\mathcal{Z}(\mathcal{N}(G))$. Write $a(e) = \sum_{g \in G} \lambda_g \cdot g \in l^2(G)$. Since $aR_g = R_g a$ holds for $g \in G$ and $R_g : l^2(G) \rightarrow l^2(G)$ given by right multiplication with $g \in G$, we get $\lambda_g = \lambda_{hgh^{-1}}$ for $g, h \in G$. This implies that $\lambda_g = 0$ if the conjugacy class (g) is infinite. One easily checks for an element g with finite (g)

$$\lambda_g = \text{tr}_{\mathcal{N}(G)}(N_{(g^{-1})} a). \quad \blacksquare$$

Lemma 1.11 *We get under the conditions above.*

$$\mathrm{tr}_{\mathcal{N}(G)} \left(\mathrm{index}_{\mathcal{N}(G)}^u(\overline{D}^*) \right) = \mathrm{index}(D^*).$$

Proof : The L^2 -index theorem of Atiyah [1, (1.1)] says

$$\mathrm{index}_{\mathcal{N}(G)}(\overline{D}^*) = \mathrm{index}(D^*).$$

We have

$$\begin{aligned} \mathrm{tr}_{\mathcal{N}(G)} \left(\mathrm{index}_{\mathcal{N}(G)}^u(\overline{D}^*) \right) &= \mathrm{tr}_{\mathcal{N}(G)} \left(\sum_{p \geq 0} (-1)^p \dim_{\mathcal{N}(G)}^u(H_{(2)}^p(\overline{D}^*)) \right) \\ &= \sum_{p \geq 0} (-1)^p \mathrm{tr}_{\mathcal{N}(G)} \left(\dim_{\mathcal{N}(G)}^u(H_{(2)}^p(\overline{D}^*)) \right) \\ &= \sum_{p \geq 0} (-1)^p \dim_{\mathcal{N}(G)} \left(H_{(2)}^p(\overline{D}^*) \right) \\ &= \mathrm{index}_{\mathcal{N}(G)}(\overline{D}^*). \quad \blacksquare \end{aligned}$$

Next we want to prove

Lemma 1.12 *Consider an element $c \in \mathrm{con}(G)_{cf}$ with $c \neq (1)$. Then*

$$\mathrm{tr}_{\mathcal{N}(G)} \left(N_c \cdot \mathrm{index}_{\mathcal{N}(G)}^u(\overline{D}^*) \right) = 0.$$

Proof : In the sequel we denote by $\overline{\mathrm{pr}}_p : \overline{D}^p \rightarrow \overline{D}^p$ the projection onto the kernel of the p -th Laplacian $\overline{\Delta}_p = (\overline{d}^p)^* \overline{d}^p + \overline{d}^{p-1} (\overline{d}^{p-1})^*$. By the L^2 -Hodge-deRham Theorem we get a G -equivariant isometric isomorphism $\mathrm{im}(\overline{\mathrm{pr}}_p) \xrightarrow{\cong} H_{(2)}^p(\overline{D}^*)$. This implies

$$\begin{aligned} &\mathrm{tr}_{\mathcal{N}(G)} \left(N_c \cdot \mathrm{index}_{\mathcal{N}(G)}^u(\overline{D}^*) \right) \\ &= \sum_{p \geq 0} (-1)^p \cdot \mathrm{tr}_{\mathcal{N}(G)} \left(N_c \cdot \mathrm{tr}_{\mathcal{N}(G)}^u \left(\mathrm{id} : H_{(2)}^p(\overline{D}^*) \rightarrow H_{(2)}^p(\overline{D}^*) \right) \right) \\ &= \sum_{p \geq 0} (-1)^p \cdot \mathrm{tr}_{\mathcal{N}(G)} \left(L_c : H_{(2)}^p(\overline{D}^*) \rightarrow H_{(2)}^p(\overline{D}^*) \right) \\ &= \sum_{p \geq 0} (-1)^p \cdot \mathrm{tr}_{\mathcal{N}(G)} \left(L_c \circ \overline{\mathrm{pr}}_p : \overline{D}^p \rightarrow \overline{D}^p \right). \end{aligned} \tag{1.13}$$

The operator $e^{-t\overline{\Delta}_p} : \overline{D}^p \rightarrow \overline{D}^p$ is a bounded G -equivariant operator and has a smooth kernel $e^{-t\overline{\Delta}_p}(\overline{x}, \overline{y}) : \overline{E}_{\overline{x}}^p \rightarrow \overline{E}_{\overline{y}}^p$ for $\overline{x}, \overline{y} \in \overline{M}$. Thus $e^{-t\overline{\Delta}_p}(\omega)$ applied to a section ω is given at $\overline{y} \in \overline{M}$ by $\int_{\overline{M}} e^{-t\overline{\Delta}_p}(\overline{x}, \overline{y})(\omega(\overline{x})) d\mathrm{vol}_{\overline{x}}$. The operator $L_c \circ e^{-t\overline{\Delta}_p}$ is also a bounded G -equivariant operator and has a smooth kernel $(L_c \circ e^{-t\overline{\Delta}_p})(\overline{x}, \overline{y})$ satisfying

$$(L_c \circ e^{-t\overline{\Delta}_p})(\overline{x}, \overline{y}) = \sum_{g \in c} L_g \circ e^{-t\overline{\Delta}_p}(\overline{x}, g^{-1}\overline{y}).$$

If $\mathcal{F}$ is a fundamental domain for the G -action, then [1, Proposition 4.6].

$$\begin{aligned} \mathrm{tr}_{\mathcal{N}(G)}(L_c \circ e^{-t\overline{\Delta}_p}) &= \int_{\mathcal{F}} \mathrm{tr}_{\mathbb{C}} \left((L_c \circ e^{-t\overline{\Delta}_p})(\overline{x}, \overline{x}) \right) d\mathrm{vol}_{\overline{x}}; \\ &= \sum_{g \in c} \int_{\mathcal{F}} \mathrm{tr}_{\mathbb{C}} \left(L_g \circ e^{-t\overline{\Delta}_p}(\overline{x}, g^{-1}\overline{x}) \right) d\mathrm{vol}_{\overline{x}}. \end{aligned} \quad (1.14)$$

where $\mathrm{tr}_{\mathbb{C}}$ is the trace of an endomorphism of a finite-dimensional complex vector space. Since M is compact, we can find $\epsilon > 0$ such that the distance of $\overline{x}$ and $g\overline{x}$ is bounded from below by ϵ for all $\overline{x} \in \overline{M}$ and $g \in c$. We have

$$\lim_{t \rightarrow 0} \sup \left\{ \|e^{-t\overline{\Delta}_p}(\overline{x}, g^{-1}\overline{x})\| \mid \overline{x} \in \mathcal{F} \right\} = 0, \quad (1.15)$$

where $\|e^{-t\overline{\Delta}_p}(\overline{x}, g^{-1}\overline{x})\|$ is the operator norm of the linear map $e^{-t\overline{\Delta}_p}(\overline{x}, g^{-1}\overline{x})$ of finite-dimensional Hilbert spaces. This follows from the finite propagation speed method of [6]. There only the standard Laplacian on 0-forms is treated, but the proof presented there carries over to the Laplacian $\overline{\Delta}_p$ associated to the lift $\overline{D}^*$ to the G -covering $\overline{M}$ of an elliptic complex D^* of differential operators of order 1 on a closed Riemannian manifold M in any dimension p . The point is that $\overline{M}$ has bounded geometry, $\overline{\Delta}_p$ is essentially selfadjoint and positive so that $\sqrt{\overline{\Delta}_p}$ makes sense, and $\frac{\partial^2}{\partial t^2} + \overline{\Delta}_p$ is strictly hyperbolic. Now one applies the results of [6, Section 1] and uses the estimate in [9, page 475], where the special case of D^* being the deRham complex is treated.

Since

$$\left| \mathrm{tr}_{\mathbb{C}} \left(L_g \circ e^{-t\overline{\Delta}_p}(\overline{x}, g^{-1}\overline{x}) \right) \right| \leq \dim_{\mathbb{C}}(E^p) \cdot \|e^{-t\overline{\Delta}_p}(\overline{x}, g^{-1}\overline{x})\|$$

and $\mathcal{F}$ is compact, we conclude from (1.14) and (1.15)

$$\lim_{t \rightarrow 0} \mathrm{tr}_{\mathcal{N}(G)}(L_c \circ e^{-t\overline{\Delta}_p}) = 0. \quad (1.16)$$

Since the trace $\mathrm{tr}_{\mathcal{N}(G)}$ is ultraweakly continuous and $\lim_{t \rightarrow \infty} e^{-t\overline{\Delta}_p} = \overline{\mathrm{pr}}_p$ in the weak topology, we get

$$\lim_{t \rightarrow \infty} \mathrm{tr}_{\mathcal{N}(G)}(L_c \circ e^{-t\overline{\Delta}_p}) = \mathrm{tr}_{\mathcal{N}(G)}(L_c \circ \overline{\mathrm{pr}}_p). \quad (1.17)$$

We conclude from (1.13) and (1.17)

$$\mathrm{tr}_{\mathcal{N}(G)} \left(N_c \cdot \mathrm{index}_{\mathcal{N}(G)}^u(\overline{D}^*) \right) = \lim_{t \rightarrow \infty} \sum_{p \geq 0} (-1)^p \cdot \mathrm{tr}_{\mathcal{N}(G)} \left(L_c \circ e^{-t\overline{\Delta}_p} \right). \quad (1.18)$$

We have

$$\begin{aligned}
& \frac{d}{dt} \sum_{p \geq 0} (-1)^p \cdot \text{tr}_{\mathcal{N}(G)} \left(L_c \circ e^{-t\overline{\Delta}_p} : \overline{D}^p \rightarrow \overline{D}^p \right) \\
&= \sum_{p \geq 0} (-1)^p \cdot \text{tr}_{\mathcal{N}(G)} \left(L_c \circ \frac{d}{dt} e^{-t\overline{\Delta}_p} \right) \\
&= \sum_{p \geq 0} (-1)^p \cdot \text{tr}_{\mathcal{N}(G)} \left(L_c \circ (-\overline{\Delta}_p) \circ e^{-t\overline{\Delta}_p} \right) \\
&= - \sum_{p \geq 0} (-1)^p \cdot \text{tr}_{\mathcal{N}(G)} \left(L_c \circ \overline{d}^{p-1} \circ (\overline{d}^{p-1})^* \circ e^{-t\overline{\Delta}_p} \right) \\
&\quad - \sum_{p \geq 0} (-1)^p \cdot \text{tr}_{\mathcal{N}(G)} \left(L_c \circ (\overline{d}^p)^* \circ \overline{d}^p \circ e^{-t\overline{\Delta}_p} \right) \\
&= - \sum_{p \geq 0} (-1)^p \cdot \text{tr}_{\mathcal{N}(G)} \left(L_c \circ \overline{d}^{p-1} \circ (\overline{d}^{p-1})^* \circ e^{-\frac{t}{2}\overline{\Delta}_p} \circ e^{-\frac{t}{2}\overline{\Delta}_p} \right) \\
&\quad - \sum_{p \geq 0} (-1)^p \cdot \text{tr}_{\mathcal{N}(G)} \left(L_c \circ (\overline{d}^p)^* \circ e^{-t\overline{\Delta}_{p+1}} \circ \overline{d}^p \right) \\
&= - \sum_{p \geq 0} (-1)^p \cdot \text{tr}_{\mathcal{N}(G)} \left(L_c \circ e^{-\frac{t}{2}\overline{\Delta}_p} \circ \overline{d}^{p-1} \circ (\overline{d}^{p-1})^* \circ e^{-\frac{t}{2}\overline{\Delta}_p} \right) \\
&\quad + \sum_{p \geq 0} (-1)^p \cdot \text{tr}_{\mathcal{N}(G)} \left(L_c \circ (\overline{d}^{p-1})^* \circ e^{-t\overline{\Delta}_p} \circ \overline{d}^{p-1} \right) \\
&= - \sum_{p \geq 0} (-1)^p \cdot \text{tr}_{\mathcal{N}(G)} \left(L_c \circ (\overline{d}^{p-1})^* \circ e^{-\frac{t}{2}\overline{\Delta}_p} \circ e^{-\frac{t}{2}\overline{\Delta}_p} \circ \overline{d}^{p-1} \right) \\
&\quad + \sum_{p \geq 0} (-1)^p \cdot \text{tr}_{\mathcal{N}(G)} \left(L_c \circ (\overline{d}^{p-1})^* \circ e^{-t\overline{\Delta}_p} \circ \overline{d}^{p-1} \right) \\
&= - \sum_{p \geq 0} (-1)^p \cdot \text{tr}_{\mathcal{N}(G)} \left(L_c \circ (\overline{d}^{p-1})^* \circ e^{-t\overline{\Delta}_p} \circ \overline{d}^{p-1} \right) \\
&\quad + \sum_{p \geq 0} (-1)^p \cdot \text{tr}_{\mathcal{N}(G)} \left(L_c \circ (\overline{d}^{p-1})^* \circ e^{-t\overline{\Delta}_p} \circ \overline{d}^{p-1} \right) \\
&= 0. \tag{1.19}
\end{aligned}$$

Here are some justifications for the calculation above. Recall that L_c is a bounded G -operator and commutes with any G -equivariant operator. We can commute $\text{tr}_{\mathcal{N}(G)}$ and $\frac{d}{dt}$ since $\text{tr}_{\mathcal{N}(G)}$ is ultraweakly continuous. We conclude $e^{-t\overline{\Delta}_{p+1}} \circ \overline{d}^p = \overline{d}^p \circ e^{-t\overline{\Delta}_p}$ from the fact that $\overline{\Delta}_{p+1} \circ \overline{d}^p = \overline{d}^p \circ \overline{\Delta}_p$ holds on $C^\infty(\overline{E}^{p-1})$. We have used at several places the typical trace relation $\text{tr}_{\mathcal{N}(G)}(AB) = \text{tr}_{\mathcal{N}(G)}(BA)$ which is in each case justified by [1, section 4]. In order to be able to apply this trace relation we have splitted $e^{-t\overline{\Delta}_p}$ into $e^{-\frac{t}{2}\overline{\Delta}_p} \circ e^{-\frac{t}{2}\overline{\Delta}_p}$ in the calculation above.

Hence $\sum_{p \geq 0} (-1)^p \cdot \text{tr}_{\mathcal{N}(G)} \left(L_c \circ e^{-t\overline{\Delta}_p} : \overline{D}^p \rightarrow \overline{D}^p \right)$ is independent of t and we conclude from (1.18)

$$\text{tr}_{\mathcal{N}(G)} \left(N_c \cdot \text{index}_{\mathcal{N}(G)}^u(\overline{D}^*) \right) = \lim_{t \rightarrow 0} \sum_{p \geq 0} (-1)^p \cdot \text{tr}_{\mathcal{N}(G)} \left(L_c \circ e^{-t\overline{\Delta}_p} : \overline{D}^p \rightarrow \overline{D}^p \right). \tag{1.20}$$

Now Lemma 1.12 follows from (1.16) (1.20). $\blacksquare$

Finally Theorem 0.4 follows from Lemma 1.10, Lemma 1.11 and Lemma 1.12.

2. Modules over a category

In this section we recall some facts about modules over the category $\mathbf{Sub} = \mathbf{Sub}(G; \mathcal{F}(X))$ for a proper G -CW-complex X as far as needed here. For more information about modules over a category we refer to [10].

Let $\mathbf{Sub} := \mathbf{Sub}(G; \mathcal{F}(X))$ be the following category. Objects are the elements of the set $\mathcal{F}(X)$ of subgroups $H \subset G$, for which $X^H \neq \emptyset$. For two finite subgroups H and K in $\mathcal{F}(X)$ denote by $\text{conhom}_G(H, K)$ the set of group homomorphisms $f : H \rightarrow K$, for which there exists an element $g \in G$ with $gHg^{-1} \subset K$ such that f is given by conjugation with g , i.e. $f = c(g) : H \rightarrow K$, $h \mapsto ghg^{-1}$. Notice that $c(g) = c(g')$ holds for two elements $g, g' \in G$ with $gHg^{-1} \subset K$ and $g'H(g')^{-1} \subset K$ if and only if $g^{-1}g'$ lies in the centralizer $C_G H = \{g \in G \mid gh = hg \text{ for all } h \in H\}$ of H in G . The group of inner automorphisms of K acts on $\text{conhom}_G(H, K)$ from the left by composition. Define the set of morphisms $\text{mor}_{\mathbf{Sub}}(H, K)$ by $\text{Inn}(K) \backslash \text{conhom}_G(H, K)$. Let $N_G H$ be the normalizer $\{g \in G \mid gHg^{-1} = H\}$ of H . Define $H \cdot C_G H = \{h \cdot g \mid h \in H, g \in C_G H\}$. This is a normal subgroup of $N_G H$ and we define $W_G H := N_G H / (H \cdot C_G H)$. One easily checks that $W_G H$ is a finite group and that there is an isomorphism from $W_G H$ to $\text{aut}_{\mathbf{Sub}}(H)$ which sends $g(H \cdot C_G H) \in W_G H$ to the automorphism of H represented by $c(g) : H \rightarrow H$. Notice that there is a morphism from H to K if and only if H is subconjugated to K . There is an isomorphism from H to K if and only if H and K are conjugated. The category $\mathbf{Sub}$ is a so called EI-category, i.e. any endomorphism in $\mathbf{Sub}$ is an isomorphism.

Let R be a commutative associative ring with unit. A *covariant resp. contravariant $R\mathbf{Sub}$ -module* M is a covariant resp. contravariant functor from $\mathbf{Sub}$ to the category of R -modules. Morphisms are natural transformations. The structure of an abelian category on the category of R -modules carries over to the category of $R\mathbf{Sub}$ -modules. In particular the notion of a projective $R\mathbf{Sub}$ -module is defined. Given a contravariant $R\mathbf{Sub}$ -module M and a covariant $R\mathbf{Sub}$ -module N , one can define a R -module, their *tensor product over $\mathbf{Sub}$*

$$M \otimes_{R\mathbf{Sub}} N = \bigoplus_{H \in \mathcal{F}(X)} M(H) \otimes_R N(H) / \sim,$$

where $\sim$ is the typical tensor relation $mf \otimes n = m \otimes fn$, i.e. for each morphism $f : H \rightarrow K$ in $\mathbf{Sub}$, $m \in M(K)$ and $n \in N(H)$ we introduce the relation $M(f)(m) \otimes n - m \otimes N(f)(n) = 0$.

Given a left $R[W_G H]$ -module N for $H \in \mathcal{F}(X)$, define a covariant $R\mathbf{Sub}$ -module $E_H M$ by

$$(E_H M)(K) := R \text{mor}_{\mathbf{Sub}}(H, K) \otimes_{R[W_G H]} N \quad \text{for } K \subset G, |K| < \infty, \quad (2.1)$$

where $R \text{mor}_{\mathbf{Sub}}(H, K)$ is the free R -module generated by the set $\text{mor}_{\mathbf{Sub}}(H, K)$. Given a covariant $R\mathbf{Sub}$ -module M and $H \in \mathcal{F}(X)$, define $M(H)_s$ to be the left R -submodule of $M(H)$, which is spanned by the images of all R -maps $M(f) : M(K) \rightarrow M(H)$, where f runs through all morphisms $f : K \rightarrow H$ in $\mathbf{Sub}$, which have H as target and are not isomorphisms. Obviously $M(H)_s$ is an $R[W_G H]$ -submodule of $M(H)$. Define a left $R[W_G H]$ -module $S_H M$ by

$$S_H M := M(H) / M(H)_s. \quad (2.2)$$

Both functors E_H and S_H respect direct sums and the property finitely generated and the property projective. Given a left $R[W_G H]$ -module M , $S_K \circ E_H M$ is M , if $H = K$ and is 0, if H and K are not conjugated in G .

Let M be a covariant $R\mathbf{Sub}$ -module. We want to check whether it is projective or not. A necessary (but not sufficient) condition is that $S_H M$ is a projective $R[W_G H]$ -module. Assume that $S_H M$ is $R[W_G H]$ -projective for all objects H in $\mathbf{Sub}$. We can choose a $R[W_G H]$ -splitting $\sigma_H : S_H M \rightarrow M(H)$ of the canonical projection $M(H) \rightarrow S_H M = M(H) / M(H)_s$. For a finite subgroup $H \subset G$ define the morphism of covariant $R\mathbf{Sub}$ -modules

$$i_H M : E_H(M(H)) \rightarrow M$$

by $(i_H M)(K)((f : H \rightarrow K) \otimes_{R[W_G H]} m) = M(f)(m)$. We obtain after a choice of representatives $H \in (H)$ for any conjugacy class (H) of subgroups $H \in \mathcal{F}(X)$ a morphism of covariant $R\text{Sub}$ -modules

$$T : \oplus_{(H), H \in \mathcal{F}(X)} E_H S_H M \xrightarrow{\oplus_{(H), H \in \mathcal{F}(X)} E_H(\sigma_H)} \oplus_{(H), H \in \mathcal{F}(X)} E_H(M(H)) \xrightarrow{\oplus_{(H), H \in \mathcal{F}(X)} i_H M} M. \quad (2.3)$$

We get as a special case of [11, Theorem 2.11]

Theorem 2.4 *The morphism T is always surjective. It is bijective if and only if M is a projective $R\text{Sub}$ -module.*

3. Some representation theory for finite groups

Denote for a finite group H by $\text{Rep}_{\mathbb{Q}}(H)$ resp. $\text{Rep}_{\mathbb{C}}(H)$ the ring of finite dimensional H -representations over the field $\mathbb{Q}$ resp. $\mathbb{C}$. Recall for the sequel that these are finitely generated free abelian groups. Given an inclusion of finite groups $H \subset G$, we denote by $\text{ind}_H^G : \text{Rep}_{\mathbb{Q}}(H) \rightarrow \text{Rep}_{\mathbb{Q}}(G)$ and $\text{res}_G^H : \text{Rep}_{\mathbb{Q}}(G) \rightarrow \text{Rep}_{\mathbb{Q}}(H)$ the induction and restriction homomorphism and similar for $R \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{Q}}$, $\text{Rep}_{\mathbb{C}}$ and $R \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}$ for a commutative ring R with $\mathbb{Z} \subset R$. Let $\text{con}_{\mathbb{Q}}(H)$ be the set of $\mathbb{Q}$ -conjugacy classes of elements in H , where h and h' are called $\mathbb{Q}$ -conjugated if the cyclic subgroups $\langle h \rangle$ and $\langle h' \rangle$ are conjugated in G . Let $\text{con}(G)$ be the set of conjugacy classes of elements in G . Denote by $\text{class}_{\mathbb{Q}}(H)$ resp. $\text{class}_{\mathbb{C}}(H)$ the rational resp. complex vector space of functions $\text{con}_{\mathbb{Q}}(H) \rightarrow \mathbb{Q}$ resp. $\text{con}(G) \rightarrow \mathbb{C}$. Character theory yields isomorphisms [15, page 68 and Theorem 29 on page 102]

$$\begin{aligned} \chi_{\mathbb{Q}} : \mathbb{Q} \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{Q}}(H) &\xrightarrow{\cong} \text{class}_{\mathbb{Q}}(H); \\ \chi_{\mathbb{C}} : \mathbb{C} \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(H) &\xrightarrow{\cong} \text{class}_{\mathbb{C}}(H). \end{aligned}$$

For a finite cyclic group C denote by $\theta_C \in \mathbb{Q} \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{Q}}(C)$ the element whose character $\chi_{\mathbb{Q}}(\theta_C)$ sends $c \in C$ to 1, if c generates C , and to 0 otherwise.

Let $C \subset H$ be a cyclic subgroup of the finite group H . Then we get for $h \in H$

$$\frac{1}{[H : C]} \cdot \chi_{\mathbb{Q}}(\text{ind}_C^H \theta_C)(h) = \frac{1}{[H : C]} \cdot \frac{1}{|C|} \cdot \sum_{l \in H, l^{-1}hl \in C} \chi_{\mathbb{Q}}(\theta_C)(l^{-1}hl) = \frac{1}{|H|} \cdot \sum_{l \in H, \langle l^{-1}hl \rangle = C} 1.$$

Denote by $[\mathbb{Q}] \in \text{Rep}_{\mathbb{Q}}(H)$ the class of the trivial H -representation $\mathbb{Q}$. Notice that $\chi_{\mathbb{Q}}([\mathbb{Q}])$ is the constant function with values 1. We get in $\mathbb{Q} \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{Q}}(H)$

$$1 \otimes_{\mathbb{Z}} [\mathbb{Q}] = \sum_{C \subset H, C \text{ cyclic}} \frac{1}{[H : C]} \cdot \text{ind}_C^H \theta_C, \quad (3.1)$$

since for any $l \in H$ and $h \in H$ there is precisely one cyclic subgroup $C \subset H$ with $C = \langle l^{-1}hl \rangle$ and $\chi_{\mathbb{Q}}$ is bijective. In particular we get for a finite cyclic group C in $\mathbb{Q} \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{Q}}(C)$

$$\theta_C = 1 \otimes_{\mathbb{Z}} [\mathbb{Q}] - \sum_{D \subset C, D \neq C} \frac{1}{[C : D]} \cdot \text{ind}_D^C \theta_D. \quad (3.2)$$

Now one easily checks by induction over the order of the finite cyclic subgroup C that the element θ_C satisfies

$$\theta_C \in \mathbb{Z} \left[\frac{1}{|C|} \right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{Q}}(C). \quad (3.3)$$

Obviously θ_C is an idempotent in $\mathbb{Z} \left[\frac{1}{|C|} \right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{Q}}(C)$. By the obvious change of rings homomorphism, $\mathbb{Z} \left[\frac{1}{|C|} \right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C)$ becomes a $\mathbb{Z} \left[\frac{1}{|C|} \right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{Q}}(C)$ -module. Hence multiplication with θ_C defines an idempotent endomorphism

$$\theta_C : \mathbb{Z} \left[\frac{1}{|C|} \right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) \rightarrow \mathbb{Z} \left[\frac{1}{|C|} \right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C).$$

It is natural with respect to group automorphisms of C , since θ_C is invariant under group automorphisms of C .

Lemma 3.4 (a) For a finite group H the map

$$\oplus_{C \subset H, C \text{ cyclic}} \text{ind}_C^H : \oplus_{C \subset H, C \text{ cyclic}} \mathbb{Z} \left[\frac{1}{|H|} \right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) \rightarrow \mathbb{Z} \left[\frac{1}{|H|} \right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(H)$$

is surjective;

(b) Let C be a finite cyclic group. Then the image resp. cokernel of

$$\oplus_{D \subset C, D \neq C} \text{ind}_D^C : \oplus_{D \subset C, D \neq C} \mathbb{Z} \left[\frac{1}{|C|} \right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(D) \rightarrow \mathbb{Z} \left[\frac{1}{|C|} \right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C)$$

is equal resp. isomorphic to the kernel resp. image of the idempotent endomorphism

$$\theta_C : \mathbb{Z} \left[\frac{1}{|C|} \right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) \rightarrow \mathbb{Z} \left[\frac{1}{|C|} \right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C);$$

(c) Let C be a finite cyclic group. The image of the idempotent endomorphism

$$\theta_C : \mathbb{Z} \left[\frac{1}{|C|} \right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) \rightarrow \mathbb{Z} \left[\frac{1}{|C|} \right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C);$$

is a projective $\mathbb{Z} \left[\frac{1}{|C|} \right] [\text{aut}(C)]$ -module, where the $\text{aut}(C)$ -operation comes from the obvious $\text{aut}(C)$ -operation on C and induction.

Proof : (a) follows from the following calculation for $x \in \mathbb{Z} \left[\frac{1}{|H|} \right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(H)$ based on (3.1)

$$x = (1 \otimes_{\mathbb{Z}} [\mathbb{Q}]) \cdot x = \left(\sum_{C \subset H, C \text{ cyclic}} \frac{1}{[H:C]} \cdot \text{ind}_C^H \theta_C \right) \cdot x = \sum_{C \subset H, C \text{ cyclic}} \frac{1}{[H:C]} \cdot \text{ind}_C^H (\theta_C \cdot \text{res}_H^C x).$$

(b) follows from the following two calculations based on (3.2) for $x \in \mathbb{Z} \left[\frac{1}{|C|} \right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(H)$

$$\begin{aligned} x - \theta_C \cdot x &= (1 \otimes [\mathbb{Q}] - \theta_C) \cdot x \\ &= \left(\sum_{D \subset C, D \neq C} \frac{1}{[C:D]} \cdot \text{ind}_D^C \theta_D \right) \cdot x \\ &= \sum_{D \subset C, D \neq C} \frac{1}{[C:D]} \cdot \text{ind}_D^C (\theta_D \cdot \text{res}_C^D x) \end{aligned}$$

and for $D \subset C, D \neq C$ and $y \in \mathbb{Z} \left[\frac{1}{|C|} \right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(D)$

$$\theta_C \cdot \text{ind}_D^C y = \text{ind}_D^C (\text{res}_C^D \theta_C \cdot y) = \text{ind}_D^C (0 \cdot y) = 0.$$

(c) Put $\Lambda = \mathbb{Z} \left[\frac{1}{|C|} \right]$. Let C_p be the p -Sylow subgroup of C for a prime p . There are canonical isomorphisms

$$\begin{aligned} C &\cong \prod_p C_p; \\ \text{aut}(C) &\cong \prod_p \text{aut}(C_p); \\ P : \otimes_p \text{Rep}_{\mathbb{C}}(C_p) &\cong \text{Rep}_{\mathbb{C}}(C), \end{aligned}$$

where p runs through the prime numbers dividing $|C|$. The isomorphism P assigns to $\otimes_p [V_p]$ for C_p -representations V_p the class of the C -representation $\otimes_p V_p$ with the factorwise action of $\text{aut}(C) = \prod_p \text{aut}(C_p)$. The following diagram commutes

$$\begin{array}{ccc} \otimes_p \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C_p) & \xrightarrow{P} & \text{Rep}_{\mathbb{C}}(C) \\ \otimes_p \theta_{C_p} \downarrow & & \downarrow \theta_C \\ \otimes_p \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C_p) & \xrightarrow{P} & \text{Rep}_{\mathbb{C}}(C) \end{array}$$

Thus we obtain an isomorphism of $\Lambda[\text{aut}(C)]$ -modules

$$\otimes_p \text{im}(\theta_{C_p}) \xrightarrow{\cong} \text{im}(\theta_C),$$

where $\text{aut}(C) = \prod_p \text{aut}(C_p)$ acts factorwise on the source. Hence the claim for C follows if we know it for C_p for all primes p . Therefore it remains to treat the case $C = \mathbb{Z}/p^n$ for some prime number p and positive integer n . Notice that then $\Lambda = \mathbb{Z} \left[\frac{1}{p} \right]$.

In the sequel we abbreviate $A(n) = \text{aut}(\mathbb{Z}/p^n)$. This is isomorphic to multiplicative group of units $\mathbb{Z}/p^{n \times}$ in $\mathbb{Z}/p^n$ and hence an abelian group of order $p^{n-1} \cdot (p-1)$. Denote by $A(n)_p$ the p -Sylow subgroup and by $A(n)'_p$ the subgroup $\{a \in A(n) \mid a^{p-1} = 1\}$ which is cyclic of order $(p-1)$. We get a canonical isomorphism

$$A(n) \cong A(n)_p \times A(n)'_p$$

Notice that $\mathbb{Z}/p^n$ has precisely one subgroup of order p^m for $0 \leq m \leq n$ which will be denoted by $\mathbb{Z}/p^m$. These subgroups are characteristic and hence restriction to these subgroups yields homomorphisms $A(n) \rightarrow A(n-1) \rightarrow \dots \rightarrow A(1)$. They induce epimorphisms $A(n)_p \rightarrow A(m-1)_p$ and isomorphisms $A(m)'_p \xrightarrow{\cong} A(m-1)'_p$. Using these isomorphisms we will identify

$$A(n)'_p = A(n-1)'_p = \dots = A(1)'_p = \mathbb{Z}/p^{\times}.$$

Thus we get canonical decompositions

$$A(n) = A(n)_p \times \mathbb{Z}/p^{\times}.$$

Let M be a $\Lambda[A(n)]$ -module. Let $\text{res } M$ be the $\Lambda[\mathbb{Z}/p^{\times}]$ -module obtained by restriction. The following maps are $\Lambda[A(n)]$ -homomorphisms

$$\begin{aligned} q : \Lambda[A(n)_p] \otimes_{\Lambda} \text{res } M &\rightarrow M, & a \otimes m &\mapsto am; \\ s : M &\rightarrow \Lambda[A(n)_p] \otimes_{\Lambda} \text{res } M, & m &\mapsto \frac{1}{|A(n)_p|} \cdot \sum_{a \in A(n)_p} a \otimes a^{-1}m, \end{aligned}$$

where $A(n) = A(n)_p \times \mathbb{Z}/p^\times$ acts factorwise on $\Lambda[A(n)_p] \otimes_\Lambda \text{res } M$. They satisfy $q \circ s = \text{id}$. Obviously $\Lambda[A(n)_p] \otimes_\Lambda \text{res } M$ is $\Lambda[A(n)]$ -projective if $\text{res } M$ is $\Lambda[\mathbb{Z}/p^\times]$ -projective. This shows that M is $\Lambda[A(n)]$ -projective if its restriction $\text{res } M$ to a $\Lambda[\mathbb{Z}/p^\times]$ -module is projective. Therefore it suffices to show that $\text{im}(\theta_C)$ is $\Lambda[\mathbb{Z}/p^\times]$ -projective.

The composition of the induction homomorphism $\text{Rep}_\mathbb{C}(\mathbb{Z}/p^{n-1}) \rightarrow \text{Rep}_\mathbb{C}(\mathbb{Z}/p^n)$ with the restriction homomorphism $\text{Rep}_\mathbb{C}(\mathbb{Z}/p^n) \rightarrow \text{Rep}_\mathbb{C}(\mathbb{Z}/p^{n-1})$ is $p \cdot \text{id} : \text{Rep}_\mathbb{C}(\mathbb{Z}/p^{n-1}) \rightarrow \text{Rep}_\mathbb{C}(\mathbb{Z}/p^{n-1})$. We conclude from Lemma 3.4 (b) that the $\Lambda[\mathbb{Z}/p^\times]$ -module $\text{im}(\theta_C)$ is isomorphic to the kernel of the surjective restriction homomorphism $\text{res} : \Lambda \otimes_\mathbb{Z} \text{Rep}_\mathbb{C}(\mathbb{Z}/p^n) \rightarrow \Lambda \otimes_\mathbb{Z} \text{Rep}_\mathbb{C}(\mathbb{Z}/p^{n-1})$. Hence there is an exact sequence of $\Lambda[\mathbb{Z}/p^\times]$ -modules

$$0 \rightarrow \text{im}(\theta_C) \rightarrow \Lambda \otimes_\mathbb{Z} \text{Rep}_\mathbb{C}(\mathbb{Z}/p^n) \rightarrow \Lambda \otimes_\mathbb{Z} \text{Rep}_\mathbb{C}(\mathbb{Z}/p^{n-1}) \rightarrow 0.$$

It induces an exact sequence of $\Lambda[\mathbb{Z}/p^\times]$ -modules

$$\begin{aligned} 0 \rightarrow \text{im}(\theta_C) &\rightarrow \ker(\Lambda \otimes_\mathbb{Z} \text{Rep}_\mathbb{C}(\mathbb{Z}/p^n) \rightarrow \Lambda \otimes_\mathbb{Z} \text{Rep}_\mathbb{C}(\{1\})) \\ &\rightarrow \ker(\Lambda \otimes_\mathbb{Z} \text{Rep}_\mathbb{C}(\mathbb{Z}/p^{n-1}) \rightarrow \Lambda \otimes_\mathbb{Z} \text{Rep}_\mathbb{C}(\{1\})) \rightarrow 0. \end{aligned}$$

Hence it suffices to show that the $\Lambda[\mathbb{Z}/p^\times]$ -module $\ker(\Lambda \otimes_\mathbb{Z} \text{Rep}_\mathbb{C}(\mathbb{Z}/p^m) \rightarrow \Lambda \otimes_\mathbb{Z} \text{Rep}_\mathbb{C}(\{1\}))$ is projective for $m = 1, 2, \dots, n$.

Recall that $\mathbb{Z}/p^\times$ is a subgroup of $A(m) = \text{aut}(\mathbb{Z}/p^m)$ and thus acts on $\mathbb{Z}/p^m - \{\bar{0}\}$ in the obvious way. Denote for $k \in \mathbb{Z}$ by $\mathbb{C}_k$ the one-dimensional $\mathbb{Z}/p^m$ -representation for which $\bar{b} \in \mathbb{Z}/p^m$ acts by multiplication with $\exp(2\pi i k b)$. We obtain a $\Lambda[\mathbb{Z}/p^\times]$ -homomorphism

$$Q : \Lambda[\mathbb{Z}/p^m - \{0\}] \rightarrow \ker(\Lambda \otimes_\mathbb{Z} \text{Rep}_\mathbb{C}(\mathbb{Z}/p^m) \rightarrow \Lambda \otimes_\mathbb{Z} \text{Rep}_\mathbb{C}(\{1\}))$$

by sending $\bar{k}$ to $[\mathbb{C}_k] - \frac{1}{p^m} \cdot [\mathbb{C}[\mathbb{Z}/p^m]]$. This is the composition of the inclusion $\Lambda[\mathbb{Z}/p^m - \bar{0}] \rightarrow \Lambda[\mathbb{Z}/p^m]$, the isomorphism $\Lambda[\mathbb{Z}/p^m] \rightarrow \Lambda \otimes_\mathbb{Z} \text{Rep}(\mathbb{Z}/p^m)$ sending $\bar{k}$ to $[\mathbb{C}_k]$ and the split epimorphism $\Lambda \otimes_\mathbb{Z} \text{Rep}(\mathbb{Z}/p^m) \rightarrow \ker(\Lambda \otimes_\mathbb{Z} \text{Rep}_\mathbb{C}(\mathbb{Z}/p^m) \rightarrow \Lambda \otimes_\mathbb{Z} \text{Rep}_\mathbb{C}(\{1\}))$ sending $[V]$ to $[V] - \frac{\dim(V)}{p^m} \cdot [\mathbb{C}[\mathbb{Z}/p^m]]$. One easily checks that Q is an isomorphism of $\Lambda[\mathbb{Z}/p^\times]$ -modules. Hence it remains to show that $\mathbb{Z}/p^\times$ acts freely on $\mathbb{Z}/p^m - \{\bar{0}\}$ because then $\Lambda[\mathbb{Z}/p^m - \{0\}]$ is a free $\Lambda[\mathbb{Z}/p^\times]$ -module.

Consider $x \in \mathbb{Z}/p^m$ with $x \neq \bar{0}$. We have to show for $a \in \mathbb{Z}/p^\times = A(m)_p' \subset A(m)$ that $a(x) = x$ implies $a = \text{id}$. Since x is non-zero, x generates a cyclic subgroup $\mathbb{Z}/p^l$ for some $l \in \{1, 2, \dots, m\}$. Then $a \in A(m)$ restricted to $A(l)$ is an automorphism $\mathbb{Z}/p^l \rightarrow \mathbb{Z}/p^l$ which sends a generator to itself. Hence this automorphism of $\mathbb{Z}/p^l$ is the identity. This implies that a is the identity in $A(l)_p' = \mathbb{Z}/p^\times$. This finishes the proof of Lemma 3.4. $\blacksquare$

The next result is analogous to [11, Lemma 7.4] but we have to go through its proof again because here we want to invert only the orders of finite subgroups of G , whereas in [11] we have considered everything over $\mathbb{Q}$.

Theorem 3.5 *Let G be a group and $\Lambda = \Lambda^G(X)$ as defined in (0.5). Consider the covariant Λ Sub-module $\Lambda \otimes_\mathbb{Z} \text{Rep}_\mathbb{C}(?)$ which sends a finite subgroup group $H \subset G$ to $\Lambda \otimes_\mathbb{Z} \text{Rep}_\mathbb{C}(H)$. Then*

(a) *$S_H \Lambda \otimes_\mathbb{Z} \text{Rep}_\mathbb{C}(?)$ is trivial if the finite subgroup $H \subset G$ is not cyclic.*

For a finite cyclic subgroup $C \subset G$, the $\Lambda[W_G C]$ -module $S_C \Lambda \otimes_\mathbb{Z} \text{Rep}_\mathbb{C}(?)$ is isomorphic to the image of the idempotent $\Lambda[W_G C]$ -homomorphism

$$\theta_C : \Lambda \otimes_\mathbb{Z} \text{Rep}_\mathbb{C}(C) \rightarrow \Lambda \otimes_\mathbb{Z} \text{Rep}_\mathbb{C}(C).$$

The isomorphism is given by the composition of the obvious inclusion $\text{im}(\theta_C) \rightarrow \text{Rep}_\mathbb{C}(C)$ with the obvious projection $\text{Rep}_\mathbb{C}(C) \rightarrow S_C \Lambda \otimes_\mathbb{Z} \text{Rep}_\mathbb{C}(?)$;

(b) $\Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(?)$ is a projective ΛSub -module;

(c) Let M be a contravariant ΛSub -module. There is a natural isomorphism of Λ -modules

$$\begin{aligned} \bigoplus_{(C), C \text{ cyclic}, C \in \mathcal{F}(X)} M(C) \otimes_{\Lambda[W_G C]} \text{im}(\theta_C : \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) \rightarrow \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C)) \\ \cong M \otimes_{\Lambda \text{Sub}} \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(?); \end{aligned}$$

(d) $\Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(H)$ is a flat ΛSub -module, i.e. for an exact sequence $0 \rightarrow M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow 0$ of contravariant ΛSub -modules the induced sequence of R -modules $0 \rightarrow M_0 \otimes_{\Lambda \text{Sub}} \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(?) \rightarrow M_1 \otimes_{\Lambda \text{Sub}} \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(?) \rightarrow M_2 \otimes_{\Lambda \text{Sub}} \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(?) \rightarrow 0$ is exact.

Proof : (a) We conclude from Lemma 3.4 (a) that $S_H \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(?)$ is trivial if H is not cyclic. If $H = C$ for a finite cyclic subgroup $C \subset G$, the assertion follows from Lemma 3.4 (b).

(b) Notice that $N_G H / C_G H$ is a subgroup of $\text{aut}(H)$ and all $W_G H$ -operations are induced by the obvious $\text{aut}(H)$ -operations. We conclude from Lemma 3.4 (c) and assertion (a) that $S_H \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(?)$ is a projective $\Lambda[W_G H]$ -module for all $H \in \mathcal{F}(X)$. Because of Theorem 2.4 it suffices to show for the morphism T defined in (2.3) that $T(K)$ is injective for any given element $K \in \mathcal{F}(X)$.

Consider an element u in the kernel of $T(K)$. Put $J(H) = \text{mor}_{\text{Sub}}(H, K) / (W_G H)$ for $H \in \mathcal{F}(X)$ and put $I = \{(H) \mid H \in \mathcal{F}(X)\}$. Choose for any $(H) \in I$ a representative $H \in (H)$. Then fix for any element $\bar{f} \in J(H)$ a representative $f : H \rightarrow K$ in $\text{mor}_{\text{Sub}}(H, K)$. For the remainder of the proof of assertion (b) we abbreviate $L(?) := \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(?)$. We can find elements $x_{H,f} \in S_H L$ for $(H) \in I$ and $\bar{f} \in J(H)$ such that only finitely many of the $x_{H,f}$ -s are different from zero and u can be written as

$$u = \sum_{(H) \in I} \sum_{\bar{f} \in J(H)} (f : H \rightarrow K) \otimes_{\Lambda[W_G H]} x_{H,f}.$$

We want to show that all elements $x_{H,f}$ are zero. Suppose that this is not the case. Let (H_0) be maximal among those elements $(H) \in I$ for which there is $\bar{f} \in J(H)$ with $x_{H,f} \neq 0$, i.e. if for $(H) \in I$ the element $x_{H,f}$ is different from zero for some morphism $f : H \rightarrow K$ in Sub and there is a morphism $H_0 \rightarrow H$ in Sub , then $(H_0) = (H)$. In the sequel we choose for any of the morphisms $f : H \rightarrow K$ in Sub a group homomorphism denoted in the same way $f : H \rightarrow K$ representing it. Recall that $f : H \rightarrow K$ is given by conjugation with an appropriate element $g \in G$. Fix $f_0 : H_0 \rightarrow K$ with $x_{H_0,f_0} \neq 0$. We claim that the composition

$$A : \bigoplus_{(H) \in I} E_H \circ S_H(L(K)) \xrightarrow{T(K)} L(K) \xrightarrow{\text{res}_K^{\text{im}(f_0)}} L(\text{im}(f_0)) \xrightarrow{\text{ind}_{f_0^{-1} : \text{im}(f_0) \rightarrow H_0}} L(H_0) \xrightarrow{\text{pr}_{H_0}} S_{H_0} L$$

maps u to $m \cdot x_{H_0,f_0}$ for some integer $m > 0$ which is invertible in Λ . This would lead to a contradiction because of $T(K)(u) = 0$ and $x_{H_0,f_0} \neq 0$.

Consider $(H) \in I$ and $\bar{f} \in J(H)$. It suffices to show that $A((f : H \rightarrow K) \otimes_{\Lambda[W_G H]} x_{H,f})$ is $[K \cap N_G \text{im}(f_0) : \text{im}(f_0)] \cdot x_{H,f}$ if $(H) = (H_0)$ and $\bar{f} = \bar{f}_0$, and is zero otherwise. One easily checks that $A((f : H \rightarrow K) \otimes_{\Lambda[W_G H]} x_{H,f})$ is the image of $x_{H,f}$ under the composition

$$\begin{aligned} a(H, f) : S_H L \xrightarrow{\sigma_H} L(H) \xrightarrow{\text{ind}_{f : H \rightarrow \text{im}(f)}} L(\text{im}(f)) \xrightarrow{\text{ind}_{\text{im}(f)}^K} L(K) \xrightarrow{\text{res}_K^{\text{im}(f_0)}} L(\text{im}(f_0)) \\ \xrightarrow{\text{ind}_{f_0^{-1} : \text{im}(f_0) \rightarrow H_0}} L(H_0) \xrightarrow{\text{pr}_{H_0}} S_{H_0} L. \end{aligned}$$

The Double Coset formula implies

$$\text{res}_K^{\text{im}(f_0)} \circ \text{ind}_{\text{im}(f)}^K = \sum_{k \in \text{im}(f_0) \backslash K / \text{im}(f)} \text{ind}_{c(k) : \text{im}(f) \cap k^{-1} \text{im}(f_0) k \rightarrow \text{im}(f_0)} \circ \text{res}_{\text{im}(f)}^{\text{im}(f) \cap k^{-1} \text{im}(f_0) k}.$$

The composition $\text{pr}_{H_0} \circ \text{ind}_{f_0^{-1} : \text{im}(f_0) \rightarrow H_0} \circ \text{ind}_{c(k) : \text{im}(f) \cap k^{-1} \text{im}(f_0)k \rightarrow \text{im}(f_0)}$ is trivial, if $c(k) : \text{im}(f) \cap k^{-1} \text{im}(f_0)k \rightarrow \text{im}(f_0)$ is not an isomorphism. Suppose that $c(k) : \text{im}(f) \cap k^{-1} \text{im}(f_0)k \rightarrow \text{im}(f_0)$ is an isomorphism. Then $k^{-1} \text{im}(f_0)k \subset \text{im}(f)$. Since H_0 has been chosen maximal among the H for which $x_{H,f} \neq 0$ for some morphism $f : H \rightarrow K$, this implies $x_{H,f} = 0$ or that $k^{-1} \text{im}(f_0)k = \text{im}(f)$. Suppose $k^{-1} \text{im}(f_0)k = \text{im}(f)$. Then $(H) = (H_0)$ which implies $H = H_0$. Moreover, the homomorphisms in Sub represented by f_0 and f agree. Hence the group homomorphisms f_0 and f agree themselves and we get $k \in N_G \text{im}(f_0) \cap K$. This implies that $a(H, f) = [K \cap N_G \text{im}(f_0) : \text{im}(f_0)] \cdot \text{id}$ if $(H) = (H_0)$ and $\bar{f} = \bar{f}_0$, and that otherwise $a(H, f) = 0$ or $x_{H,f} = 0$ holds. Hence the map T is injective.

(c) follows from assertion (a) and the bijectivity of the isomorphism T defined in (2.3) because there is a natural isomorphism

$$M \otimes_{\Lambda \text{Sub}} E_H S_H \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(?) \xrightarrow{\cong} M(H) \otimes_{\Lambda[W_G H]} S_H \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(?). \quad (3.6)$$

Now (d) follows from (c) and the fact that the $\Lambda[W_G H]$ -module $S_H \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(?) \cong \text{im}(\theta_C)$ is projective. This finishes the proof of Theorem 3.5. $\blacksquare$

4. The construction of the Chern character

In this section we want to prove Theorem 0.7. There are similarities with the construction in [11]. The main difference is that here we want to give a construction, where we only have to invert the orders of elements in $\mathcal{F}(X)$, whereas in [11] we have worked over the rationals. In [11] we have used the Hurewicz homomorphism from stable homotopy to singular homology, which is only an isomorphism after inverting all primes. We will use the multiplicative structure of K_*^G instead and work with a different source for the equivariant Chern character, which allows us to invert only the orders of finite subgroups of G .

In the sequel we denote by $K_p^G(X)$ the equivariant K -homology of a proper G - CW -complex X . It is defined by $\text{colim}_{Y \subset X} K K_G^p(C_0(Y), \mathbb{C})$, where Y runs over all cocompact G -subcomplexes of X and $K K_G^p(C_0(Y), \mathbb{C})$ denotes equivariant KK -theory of the G - C^* -algebra $C_0(X)$ of continuous functions $X \rightarrow \mathbb{C}$, which vanish at infinity, and the C^* -algebra $\mathbb{C}$ with the trivial G -action. Given a homomorphism $\phi : H \rightarrow G$ of groups and a proper H - CW -complex, then $\text{ind}_{\phi} X := G \times_{\phi} X$ is a proper G - CW -complex and there is an induction homomorphism

$$\text{ind}_{\phi} : K_0^H(X) \rightarrow K_0^G(\text{ind}_{\phi} X).$$

If the kernel of ϕ acts freely on X , then ind_{ϕ} is bijective. In particular we get for a proper G - CW -complex X a homomorphism

$$K_p^G(X) \xrightarrow{\text{ind}_{G \rightarrow \{1\}}} K_p(G \backslash X),$$

which is bijective if G acts freely on X . There is an external product

$$\mu : K_p^G(X) \times K_q^{G'}(X') \rightarrow K_{p+q}^{G \times G'}(X \times X')$$

for groups G and G' , a proper G - CW -complex X and a proper G' - CW -complex X' . External products and induction are compatible. For more information about equivariant K -homology and KK -theory we refer to [8] and in particular for the induction homomorphisms to [16].

Let X be a proper G - CW -complex. We have introduced the ring $\Lambda = \Lambda^G(X)$ in (0.5). We want to construct for $H \in \mathcal{F}(X)$ and $p = 0, 1$ a Λ -homomorphism

$$\underline{\text{ch}}_p^G(X)(H) : \Lambda \otimes_{\mathbb{Z}} K_p(C_G H \backslash X^H) \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(H) \rightarrow \Lambda \otimes_{\mathbb{Z}} K_p^G(X), \quad (4.1)$$

where $K_p(C_G H \backslash X^H)$ is the (non-equivariant) K-homology of the CW -complex $C_G H \backslash X^H$. The map will be defined by the following composition

$$\begin{aligned}
& \Lambda \otimes_{\mathbb{Z}} K_p(C_G H \backslash X^H) \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(H) \\
& \downarrow \text{id} \otimes_{\mathbb{Z}} K_p(\text{pr}_1; R) \otimes_{\mathbb{Z}} \text{id} \cong \\
& \Lambda \otimes_{\mathbb{Z}} K_p(EG \times_{C_G H} X^H) \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(H) \\
& \downarrow \text{id} \otimes_{\mathbb{Z}} \text{ind}_{C_G H \rightarrow \{1\}} \otimes j \cong \\
& \Lambda \otimes_{\mathbb{Z}} K_p^{C_G H}(EG \times X^H) \otimes_{\mathbb{Z}} K_0^H(*) \\
& \downarrow \mu \\
& \Lambda \otimes_{\mathbb{Z}} K_p^{C_G H \times H}(EG \times X^H) \\
& \downarrow \text{ind}_{m_H} \cong \\
& \Lambda \otimes_{\mathbb{Z}} K_p^G(\text{ind}_{m_H} EG \times X^H) \\
& \downarrow \text{id} \otimes_{\mathbb{Z}} K_p^G(\text{ind}_{m_H} \text{pr}_2) \\
& \Lambda \otimes_{\mathbb{Z}} K_p^G(\text{ind}_{m_H} X^H) \\
& \downarrow \text{id} \otimes_{\mathbb{Z}} K_p^G(v_H) \\
& K_p^G(X)
\end{aligned}$$

Some explanations are in order. We have a left $C_G H$ -action on $EG \times X^H$ by $g(e, x) = (ge, gx)$ for $g \in C_G H$, $e \in EG$ and $x \in X^H$. It extends to a $C_G H \times H$ -action by letting the factor H acting trivially. The map $\text{pr}_1 : EG \times_{C_G H} X^H \rightarrow C_G H \backslash X^H$ is the canonical projection. It induces an isomorphism

$$\Lambda \otimes_{\mathbb{Z}} K_p(\text{pr}_1; R) : \Lambda \otimes_{\mathbb{Z}} K_p(EG \times_{C_G H} X^H) \xrightarrow{\cong} \Lambda \otimes_{\mathbb{Z}} K_p(C_G H \backslash X^H)$$

since each isotropy group of the $C_G H$ -space X^H is finite and for any finite group L the projection induces an isomorphism $\Lambda \otimes_{\mathbb{Z}} H_p(BL) \xrightarrow{\cong} \Lambda \otimes_{\mathbb{Z}} H_p(*)$ and hence by the Atiyah-Hirzebruch spectral sequence an isomorphism $\Lambda \otimes_{\mathbb{Z}} K_p(BL) \xrightarrow{\cong} \Lambda \otimes_{\mathbb{Z}} K_p(*)$ for all p . The isomorphism $j : K_0^H(*) \xrightarrow{\cong} \text{Rep}_{\mathbb{C}}(H)$ is the canonical isomorphism. The group homomorphism $m_H : C_G H \times H \rightarrow G$ sends (g, h) to gh . We denote by $\text{pr}_2 : EG \times X^H \rightarrow X^H$ the canonical projection. The G -map $v_H : \text{ind}_{m_H} X^H = G \times_{m_H} X^H \rightarrow X$ sends (g, x) to gx .

Notice that we obtain a contravariant **Sub**-module $K_0(C_G ? \backslash X^?)$ by assigning to a finite subgroup $H \subset G$ the Λ -module $\Lambda \otimes_{\mathbb{Z}} K_p(C_G H \backslash X^H)$. We have already introduced the covariant Λ -module $\Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(?)$. Analogously to [11] one checks that the various maps $\text{ch}_p^G(X)(H)$ defined above induce a map of Λ -modules

$$\text{ch}_p^G(X) : \Lambda \otimes_{\mathbb{Z}} K_p(C_G ? \backslash X^?) \otimes_{\Lambda \text{Sub}} \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(?) \rightarrow \Lambda \otimes_{\mathbb{Z}} K_p^G(X). \quad (4.2)$$

Notice that for $L \in \mathcal{F}(X)$ and $X = G/L$ the ΛSub -module $K_0(C_G ? \backslash (G/L)^?)$ is isomorphic to the ΛSub -module $\Lambda \text{mor}_{\text{Sub}}(? , L)$, which sends a finite subgroup $H \subset G$ to the free Λ -module with base $\text{mor}_{\text{Sub}}(H, K)$. By the Yoneda Lemma one obtains a canonical isomorphism

$$\Lambda \otimes_{\mathbb{Z}} K_p(C_G ? \backslash (G/L)^?) \otimes_{\Lambda \text{Sub}} \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(?) \xrightarrow{\cong} \text{Rep}_{\mathbb{C}}(L).$$

One easily checks that under this identification $\text{ch}_0^G(G/L)$ becomes the canonical identification of $\text{Rep}_{\mathbb{C}}(L)$ with $K_0^G(G/L)$. Notice that $K_1(C_G ? \backslash (G/L)^?)$ and $K_1^G(G/L)$ are both trivial. Hence

$\text{ch}_p^G(G/L)$ is bijective for all $L \in \mathcal{F}(X)$ and $p = 0, 1$. Because of Theorem 3.5 (d) the source of ch_*^G is an equivariant homology theory on proper G -CW-complexes Y with $\mathcal{F}(Y) \subset \mathcal{F}(X)$. One easily checks that ch_*^G is compatible with the Mayer-Vietoris sequences. By induction over the number of equivariant cells and the Five-Lemma $\text{ch}_p^G(Y)$ is bijective for any finite proper G -CW-complex Y with $\mathcal{F}(Y) \subset \mathcal{F}(X)$. Notice that $K_p^G(Y)$ is the colimit $\text{colim}_{Z \subset Y} K_p^G(Z)$, where Z runs through all finite G -CW-subcomplexes Z of Y . The analogous statement holds for the source of ch_*^G . Hence $\text{ch}_p^G(Y)$ is bijective for all proper G -CW-complexes Y with $\mathcal{F}(Y) \subset \mathcal{F}(X)$ and $p = 0, 1$. Now Theorem 0.7 follows from Theorem 3.5 (c). ■

5. The Baum-Connes Conjecture and the Trace Conjecture

In the sequel we denote for a proper G -CW-complex X by

$$\text{asmb}^G : K_0^G(X) \rightarrow K_0(C_r^*(G)) \quad (5.1)$$

the assembly map which essentially assigns to an element in $K_0^G(X)$ represented by an equivariant Kasparov cycle its index. Given a homomorphism $\phi : H \rightarrow G$ of groups with finite kernel, there is an induction homomorphism $\text{ind}_\phi : K_p(C_r^*(H)) \rightarrow K_p(C_r^*(G))$ such that the following diagram commutes [16, Theorem 1]

$$\begin{array}{ccc} K_0^H(X) & \xrightarrow{\text{asmb}^H} & K_0(C_r^*(H)) \\ \text{ind}_\phi \downarrow & & \downarrow \text{ind}_\phi \\ K_0^G(\text{ind}_\phi X) & \xrightarrow{\text{asmb}^G} & K_0(C_r^*(G)) \end{array}$$

These induction homomorphisms, the assembly maps and the change of rings homomorphisms associated to the passage from $C_r^*(G)$ to $\mathcal{N}(G)$ are compatible with the external products

$$\begin{aligned} \mu : K_p^G(X) \times K_q^{G'}(X') &\rightarrow K_{p+q}^{G \times G'}(X \times X'); \\ \mu : K_p(C_r^*(G)) \times K_q(C_r^*(G')) &\rightarrow K_{p+q}(C_r^*(G \times G')); \\ \mu : K_p(\mathcal{N}(G)) \times K_q(\mathcal{N}(G')) &\rightarrow K_{p+q}(\mathcal{N}(G \times G')) \end{aligned}$$

for groups G and G' , a proper G -CW-complex X and a proper G' -CW-complex X' . We will use in the sequel the elementary fact that for any G -map $f : X \rightarrow Y$ of proper G -CW-complexes the composition $K_0^G(X) \xrightarrow{K_0^G(f)} K_0^G(Y) \xrightarrow{\text{asmb}^G} K_0(C_r^*(G))$ is $\text{asmb}^G : K_0^G(X) \rightarrow K_0(C_r^*(G))$. In the sequel the letter i denotes change of rings homomorphism for the canonical map $C_r^*(G) \rightarrow \mathcal{N}(G)$.

Let X be a proper G -CW-complex. We have introduced $J = J^G(X)$ in (0.6). Define the homomorphism

$$\begin{aligned} \xi_1 : \oplus_{(C) \in J} \Lambda \otimes_{\mathbb{Z}} K_0(C_G C \setminus X^C) \otimes_{\Lambda[W_G C]} \text{im}(\theta_C : \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) \rightarrow \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C)) \\ \rightarrow K_0(\mathcal{N}(G)) \end{aligned} \quad (5.2)$$

by the composition of the equivariant Chern character of Theorem 0.7

$$\begin{aligned} \text{ch}_0^G(X) : \oplus_{(C) \in J} \Lambda \otimes_{\mathbb{Z}} K_0(C_G C \setminus X^C) \otimes_{\Lambda[W_G C]} \text{im}(\theta_C : \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) \rightarrow \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C)) \\ \xrightarrow{\cong} \Lambda \otimes_{\mathbb{Z}} K_0^G(X), \end{aligned}$$

the assembly map

$$\text{id} \otimes \text{asmb}^G : \Lambda \otimes_{\mathbb{Z}} K_0^G(X) \rightarrow \Lambda \otimes_{\mathbb{Z}} K_0(C_r^*(G))$$

and the change of rings homomorphism

$$\text{id} \otimes i : \Lambda \otimes_{\mathbb{Z}} K_0(C_r^*(G)) \rightarrow \Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G)).$$

This is the homomorphism which we want to understand. In particular we are interested in its image. We will identify it with a second easier to compute homomorphism

$$\begin{aligned} \xi_2 : \bigoplus_{(C) \in J} \Lambda \otimes_{\mathbb{Z}} K_0(C_G C \backslash X^C) \otimes_{\Lambda[W_G C]} \text{im}(\theta_C : \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) \rightarrow \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C)) \\ \rightarrow K_0(\mathcal{N}(G)), \end{aligned} \quad (5.3)$$

which is defined as follows. Let $l : \text{im}(\theta_C) \rightarrow \text{Rep}_{\mathbb{C}}(C)$ be the inclusion. Let $K_0(\text{pr}) : K_0(C_G C \backslash X^C) \rightarrow K_0(*)$ be induced by the projection from $C_G C \backslash X^C$ to the one-point space $*$. We obtain a map

$$(i \circ \text{asmb}^{\{1\}} \circ K_0(\text{pr})) \otimes l : K_0(C_G C \backslash X^C) \otimes \text{im}(\theta_C) \rightarrow K_0(\mathcal{N}(\{1\})) \otimes \text{Rep}_{\mathbb{C}}(C).$$

Define

$$\alpha : K_0(\mathcal{N}(\{1\})) \otimes \text{Rep}_{\mathbb{C}}(C) \rightarrow \text{Rep}_{\mathbb{C}}(C) \quad [U] \otimes [W] \mapsto \dim_{\mathbb{C}}(U) \cdot [W].$$

Notice that α is essentially given by the external product and $K_0(\mathcal{N}(H)) = \text{Rep}_{\mathbb{C}}(H)$ holds by definition for any finite group H . Induction yields a map

$$\text{ind}_C^G : K_0(\mathcal{N}(C)) \rightarrow K_0(\mathcal{N}(G)).$$

The composition of these three maps above induces for any finite cyclic subgroup $C \subset G$ a homomorphism

$$\xi_2(C) : \Lambda \otimes_{\mathbb{Z}} K_0(C_G C \backslash X^C) \otimes_{\Lambda[W_G C]} \text{im}(\theta_C : \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) \rightarrow \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C)) \rightarrow K_0(\mathcal{N}(G)).$$

Define ξ_2 to be the direct sum $\bigoplus_{(C) \in J} \xi_2(C)$ after the choice of a representative $C \in (C)$ for each $(C) \in J$.

Theorem 5.4 *Let X be a proper G -CW-complex. Then the maps ξ_1 of (5.2) and ξ_2 of (5.3) agree.*

Proof : In the sequel maps denoted by the letter μ will be given by external products and pr denotes the projection from a space to the one-point space $*$. Fix a cyclic subgroup $C \in \mathcal{F}(X)$. Notice that the homomorphism $m_C : C_G C \times C \rightarrow G \quad (g, c) \mapsto gc$ has a finite kernel so that induction is defined also on the level of the reduced group C^* -algebra and the group von Neumann algebra. Denote by $\nu : \Lambda \otimes_{\mathbb{Z}} K_0^{C_G C \times C}(EG \times X^C) \rightarrow K_0^G(X)$ the composition of the maps $\text{id} \otimes K_0^G(v_C)$, $\text{id} \otimes K_0^G(\text{ind}_{m_C} \text{pr}_2)$ and ind_{m_C} appearing in the definition of $\underline{\text{ch}}_0(X)(C)$. Then the following diagram commutes

$$\begin{array}{ccc} \Lambda \otimes_{\mathbb{Z}} K_0^{C_G C}(EG \times X^C) \otimes_{\mathbb{Z}} K^C(*) & \xrightarrow{\text{id} \otimes i \circ \text{asmb}^{C_G C} \otimes j} & \Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(C_G C)) \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) \\ \mu \downarrow & & \mu \downarrow \\ \Lambda \otimes_{\mathbb{Z}} K_0^{C_G C \times C}(EG \times X^C) & \xrightarrow{\text{id} \otimes i \circ \text{asmb}^{C_G C \times C}} & \Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(C_G C \times C)) \\ \nu \downarrow & & \text{ind}_{m_C} \downarrow \\ \Lambda \otimes_{\mathbb{Z}} K_0^G(X) & \xrightarrow{\text{id} \otimes i \circ \text{asmb}^G} & \Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G)) \end{array}$$

For any group G the map induced by the center-valued von Neumann dimension

$$\dim_{\mathcal{N}(G)}^u : K_0(\mathcal{N}(G)) \rightarrow \mathcal{Z}(\mathcal{N}(G))$$

is injective. Given a CW -complex Z and an element $\eta \in K_0(Z)$, there is a closed manifold M with a map $f : M \rightarrow BG$ and an elliptic complex D^* of differential operators of order 1 over M such that $K_0(f) : K_0(M) \rightarrow K_0(Z)$ maps the class $[D^*] \in K_0(M)$ to η [2]. In the case $Z = BG$ the composition

$$K_0(M) \xrightarrow{K_0(f)} K_0(BG) \xrightarrow{(\text{ind}_{G \rightarrow \{1\}})^{-1}} K_0^G(EG) \xrightarrow{\text{asmb}^G} K_0(C_r^*(G)) \xrightarrow{i} K_0(\mathcal{N}(G)) \xrightarrow{\dim_{\mathcal{N}(G)}^u} \mathcal{Z}(\mathcal{N}(G))$$

resp. the composition

$$K_0(M) \xrightarrow{K_0(\text{pr})} K_0(*) \xrightarrow{\text{asmb}^{\{1\}}} K_0(C^*(\{1\})) \xrightarrow{i} K_0(\mathcal{N}(\{1\})) \xrightarrow{\text{ind}_{\{1\}}^G} K_0(\mathcal{N}(G)) \xrightarrow{\dim_{\mathcal{N}(G)}^u} \mathcal{Z}(\mathcal{N}(G))$$

maps $[D^*]$ to the element $\text{index}_{\mathcal{N}(G)}^u(\overline{D}^*)$ resp. $\text{index}(D^*) \cdot 1_{\mathcal{N}(G)}$, where $\text{index}_{\mathcal{N}(G)}^u(\overline{D}^*)$ resp. $\text{index}(D^*)$ has been defined in (1.7) resp. (1.1). We conclude from Theorem 0.4 and the injectivity of the map $\dim_{\mathcal{N}(G)}^u$ of (5.5) that the following diagram commutes

$$\begin{array}{ccc} K_0^G(EG) & \xrightarrow{\text{ioasmb}^G} & K_0(\mathcal{N}(G)) \\ K_0(\text{pr}) \circ \text{ind}_{G \rightarrow \{1\}}^{-1} \downarrow & & \uparrow \text{ind}_{\{1\}}^G \\ K_0(*) & \xrightarrow{\text{ioasmb}^{\{1\}}} & K_0(\mathcal{N}(\{1\})) \end{array}$$

Since there is a $C_G C$ -map $EG \times X^C \rightarrow EC_G C$, we conclude from the diagram above applied to the case $G = C_G C$ that the following diagram commutes

$$\begin{array}{ccc} \Lambda \otimes_{\mathbb{Z}} K_0^{C_G C}(EG \times X^C) \otimes_{\mathbb{Z}} K^C(*) & \xrightarrow{\text{id} \otimes \text{ioasmb}^{C_G C} \otimes j} & \Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(C_G C)) \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) \\ \text{id} \otimes (K_0(\text{pr}) \circ \text{ind}_{C_G C \rightarrow \{1\}}) \otimes j \downarrow & & \uparrow \text{id} \otimes \text{ind}_{\{1\}}^{C_G C} \otimes \text{id} \\ \Lambda \otimes_{\mathbb{Z}} K_0(*) \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) & \xrightarrow{\text{id} \otimes \text{ioasmb}^{\{1\}} \otimes \text{id}} & \Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(\{1\})) \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) \end{array}$$

The composition

$$K_0(\mathcal{N}(\{1\})) \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) \xrightarrow{\text{ind}_{\{1\}}^{C_G C} \otimes \text{id}} K_0(\mathcal{N}(C_G C)) \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) \xrightarrow{\mu} K_0(\mathcal{N}(C_G C \times C)) \xrightarrow{\text{ind}_{m_C}} K_0(\mathcal{N}(G))$$

agrees with the composition

$$K_0(\mathcal{N}(\{1\})) \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) \xrightarrow{\alpha} \text{Rep}_{\mathbb{C}}(C) = K_0(\mathcal{N}(C)) \xrightarrow{\text{ind}_C^G} K_0(\mathcal{N}(G)).$$

We conclude that the following diagram commutes for any cyclic subgroup $C \in \mathcal{F}(X)$

$$\begin{array}{ccc} \Lambda \otimes_{\mathbb{Z}} K_0^{C_G C}(EG \times X^C) \otimes_{\mathbb{Z}} K^C(*) & \xrightarrow{(\text{id} \otimes \text{ioasmb}^{C_G C}) \circ \nu \circ \mu} & \Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G)) \\ \text{id} \otimes (\text{ioasmb}^{\{1\}} \circ K_0(\text{pr}) \circ \text{ind}_{C_G C \rightarrow \{1\}}) \otimes j \downarrow & & \uparrow \text{id} \otimes \text{ind}_C^G \\ \Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(\{1\})) \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) & \xrightarrow{\text{id} \otimes \alpha} & \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) \end{array}$$

Hence the following diagram commutes for any cyclic subgroup $C \in \mathcal{F}(X)$

$$\begin{array}{ccc} \Lambda \otimes_{\mathbb{Z}} K_p(C_G C \setminus X^C) \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) & \xrightarrow{\text{id} \otimes (\alpha \circ (\text{ioasmb}^{\{1\}} \circ K_0(\text{pr})) \otimes \text{id})} & \Lambda \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) \\ \text{id} \otimes \underline{\text{ch}}_0^G(X)(C) \downarrow & & \downarrow \text{id} \otimes \text{ind}_C^G \\ \Lambda \otimes_{\mathbb{Z}} K_0^G(X) & \xrightarrow{\text{id} \otimes (\text{ioasmb}^G)} & \Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G)) \end{array}$$

Now Theorem 5.4 (and hence also Theorem 0.8) follow. $\blacksquare$

References

- [1] **Atiyah, M.:** “*Elliptic operators, discrete groups and von Neumann algebras*”, Astérisque 32, 43–72 (1976).
- [2] **Baum, P. and Douglas, R.G.:** “*K-homology and index theory*”, in “Operator algebras and applications”, editor: Kadison, R.V., Proc. Symp. Pure Math. 38 (Part 1), AMS, Providence, 117–173 (1982).
- [3] **Baum, P. and Connes, A.:** “*Geometric K-theory for Lie groups and foliations*”, Brown-University/IHES-preprint 1982, appeared in L’Enseignement Mathématique 46, 3–42 (2000).
- [4] **Baum, P. and Connes, A.:** “*Chern character for discrete groups*”, in: Matsumoto, Miyutami, and Morita (eds.): “*A fête of topology; dedicated to Tamura*”, 163–232, Academic Press (1988).
- [5] **Baum, P., Connes, A., and Higson, N.:** “*Classifying space for proper actions and K-theory of group C^* -algebras*”, in: Doran, R.S. (ed.): “ *C^* -algebras*”, Contemporary Mathematics 167, 241–291 (1994).
- [6] **Cheeger, J., Gromov, M., and Taylor, M.:** “*Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds*”, Journal of Differential Geometry 17, 15–53 (1982).
- [7] **Kadison, R.V. and Ringrose, J.R.:** “*Fundamentals of the theory of operator algebras, volume II: Advanced theory*”, Pure and Applied Mathematics, Academic Press (1986).
- [8] **Kasparov, G.G.:** “*Equivariant KK-theory and the Novikov conjecture*”, Inventiones Mathematicae 91, 147–201 (1988).
- [9] **Lott, J.:** “*Heat kernels on covering spaces and topological invariants*”, Journal of Differential Geometry 35, 471–510 (1992).
- [10] **Lück, W.:** “*Transformation groups and algebraic K-theory*”, Lecture Notes in Mathematics 1408 (1989).
- [11] **Lück, W.:** “*Chern characters for proper equivariant homology theories and applications to K- and L-theory*”, Preprintreihe SFB 478 — Geometrische Strukturen in der Mathematik, Heft 104, Münster (2000).
- [12] **Mishchenko, A.S. and Fomenko, A.T.:** “*The index of elliptic operators over C^* -algebras*”, Math. USSR Izv. 15, 87–112 (1980).
- [13] **Roy, R.:** “*The trace conjecture – a counterexample*”, K-theory 17, 209–213 (1999).
- [14] **Roy, R.:** “*A counterexample to questions on the integrality property of virtual signature*”, Topology and its applications 100, 177–185 (2000).
- [15] **Serre, J.-P.:** “*Linear representations of finite groups*”, Springer-Verlag (1977).
- [16] **Valette, A.:** “*On the Baum-Connes assembly map*”, appendix to “Introduction to the BC conjecture”, to appear as ETHZ lecture notes, Birkhäuser.