THE MOTIVIC DGA

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Abstract. The main goal of this paper is to associate to each smooth quasi-projective scheme X over any field an E^{∞} differential graded algebra whose cohomology groups are the higher Chow groups of X. Modulo torsion, this provides a strictly commutative differential graded algebra. Such a construction is currently known only for the case X itself is a field. Needless to say there are several applications of this result, some of which are considered here: for example, we are able to construct a category of relative mixed Tate-motives associated to any such scheme under the hypothesis that the DGA we obtain is connected. We also show that this holds for all smooth linear projective varieties over a field k, if the Beilinson-Soulé vanishing condition holds for the field. We also establish certain cohomology operations on mod-p motivic cohomology by this procedure.

1. Introduction

The main result of this paper is the construction of an E^{∞} differential graded algebra associated to any smooth quasi-projective scheme over a field k whose cohomology will be the integral motivic cohomology associated to X. Such a result was claimed in [K-M], Part II, section 6; however certain non-trivial difficulties make their arguments valid only for the case $X = Spec \quad k$. Their difficulties may be ultimately traced back to inherent difficulties with the higher cycle complex. These are overcome, following some suggestions of Spencer Bloch, by making use of the motivic complexes. We provide an explicit construction of an operad that lives in the $\mathbb{A}^1$-local category of simplicial presheaves (on the big Zariski or Nisnevich site of schemes of finite type over a given field k.) We show that the (graded) motivic complex may be replaced upto quasi-isomorphism by an object which is an algebra over this operad. The $\mathbb{A}^1$-local structure plays a major role in producing this quasi-isomorphism. The passage from the E^{∞} DGA to a strict DGA on tensoring with $\mathbb{Q}$ is carried out by a standard procedure.

Here is an outline of the paper. We begin section 2 by establishing the correct framework for the rest of the paper. This is followed by the construction of the operad we call *the cubical endomorphism operad* in the next section. Relevant material on simplicial presheaves and operads are recalled here from the appendix. We prove that this operad is in fact an E^{∞}-operad. The fourth section discusses the motivic complexes; we consider cubical versions as these are better suited for our purposes. (In fact, the cubical version of the graded motivic complex already forms a strictly associative differential graded algebra; however it is only homotopy commutative.) We show that the graded motivic complex may be replaced by a quasi-isomorphic algebra over the cubical endomorphism operad. Since the above operad is E^{∞}, we are able to produce a strictly commutative differential graded algebra on tensoring the above algebra with $\mathbb{Q}$. *This is the motivic DGA.* We also show that this differential graded algebra is quasi-isomorphic to the graded cycle complex (tensored with $\mathbb{Q}$) for all smooth quasi-projective varieties over a field.

The next two sections consider applications of the theory developed so far. We show in section 4 that, as an immediate corollary, we obtain certain cohomology operations in mod-p motivic cohomology. The last section shows how to define a category of relative mixed Tate motives from a Hopf-algebra constructed from the motivic DGA. This is followed by a brief discussion of l-adic realization functors following the work of Bloch and Bloch-Kriz. (See [Bl-3] and [Bl-K].) An appendix collects together several results on localization of model categories, discussed for example in [Hirsch] as well as relevant details on operads. The decision to work cubically (as opposed to simplicially) seems to simplify some of our constructions: for example, the construction of the operad seems simpler this way and moreover it makes the differential graded algebra we obtain strictly associative (integrally).

We would like to thank Spencer Bloch, Patrick Brosnan, Zig Fiedorowicz, Eric Friedlander, Peter May and Bertrand Toen for several helpful discussions/correspondence. As should be clear, our paper depends on the fundamental works of Bloch, Morel, Suslin and Voevodsky on motives as well as on fundamental results on operads due to May. Needless to say we have freely adopted many arguments available in the literature and also used the forthcoming book [Hirsch] as a reference for model categories and localization techniques.

Date: March 1, 2001.

2. Notation and Terminology

Let $\mathfrak{S}$ denote a site and let $Presh(\mathfrak{S})$ denote a category of presheaves on the site $\mathfrak{S}$ which has the following properties: (i) it is complete and co-complete (i.e. closed under all small limits and colimits) and (ii) has a unital symmetric monoidal structure defined by a bi-functor $\otimes : Presh(\mathfrak{S}) \times Presh(\mathfrak{S}) \rightarrow Presh(\mathfrak{S})$. The unit for the monoidal structure will be denoted u. The main examples of this frame-work will be the following: the category of simplicial presheaves on $\mathfrak{S}$ (which will be denoted $Simpl.Presh(\mathfrak{S})$), the category of simplicial abelian presheaves on $\mathfrak{S}$ (which will be denoted $Simpl.Ab.Presh(\mathfrak{S})$ and the category of all complexes in the latter category. This will be denoted $C(Simpl.Ab.Presh(\mathfrak{S}))$. In the first (second) case, the monoidal structure is the product of two simplicial presheaves (the tensor product of two simplicial abelian presheaves, respectively). In the last case, the monoidal structure is defined by the tensor product of two complexes. We may also consider the following two symmetric monoidal categories for $Presh(\mathfrak{S})$: $Chain(Ab.Presh(\mathfrak{S}))$ $(Co-chain(Ab.Presh(\mathfrak{S})))$ = the category of all chain complexes (co-chain complexes, respectively) of abelian presheaves on the site $\mathfrak{S}$ with the monoidal structure given by the tensor product of complexes. A chain-complex (co-chain complex will mean a complex with differentials of degree -1 ($+1$, respectively). (We will pass from a chain-complex to a co-chain complex by re-indexing: i.e. if C_* is a chain complex we let C^* denote the associated co-chain complex defined by $C^n = C_{-n}$.) We let the constant presheaf associated to the singleton set $\{1\}$ define the unit for the product of two simplicial presheaves; clearly the constant presheaf $\mathbb{Z}$ defines a unit for the tensor product of two simplicial abelian presheaves and two abelian presheaves.

For the most part, the only sites we consider in this paper will be the following ones. Let S denote a fixed Noetherian base scheme (often a field k of arbitrary characteristic $p \geq 0$). Let $(smt.schemes/S)$ denote the category of all smooth schemes of finite type over k. This category is skeletally small and therefore we may in fact assume it is small. This category will be provided with either the (big) Zariski topology or the Nisnevich topology: the former site (the latter site) will be denoted $(smt.schemes/S)_{Zar}$ ($(smt.schemes/S)_{Nis}$, respectively). The site $\mathfrak{S}$ will usually denote either one of these sites. In sections 4 through 6, we will need to assume the base scheme is the spectrum of a field, whereas this is not needed in section 3.

A map $f : P \rightarrow P'$ of chain complexes of abelian presheaves on the site $\mathfrak{S}$ will be called a *quasi-isomorphism* if it induces a quasi-isomorphism of the associated complexes of sheaves.

One may observe that $Simpl.Presh(\mathfrak{S})$ and $(Simpl.Presh(\mathfrak{S}))^I$ (for any small category I) are simplicial categories. The corresponding bi-functor $Simpl.Presh(\mathfrak{S})^{op} \times Simpl.Presh(\mathfrak{S}) \rightarrow (simplicial \quad sets)$ $((Simpl.Presh(\mathfrak{S})^I)^{op} \times (Simpl.Presh(\mathfrak{S}))^I \rightarrow (simplicial \quad sets))$ will be denoted Map. There is an internal hom in the category $Simpl.Presh(\mathfrak{S})$ which will be denoted $\mathcal{H}om$; this has the following properties.

$$\begin{aligned} Hom_{Simpl.Presh(\mathfrak{S})}(R, \mathcal{H}om(P,Q)) &\cong Hom_{Simpl.Presh(\mathfrak{S})}(R \times P, Q), \\ Map(R, \mathcal{H}om(P,Q)) &\cong Map(R \times P, Q), \\ Map(U, P) &= \Gamma(U, P) \quad and \\ \Gamma(U, \mathcal{H}om(P,Q)) &= Map(P_{|U}, Q_{|U}) \end{aligned} \tag{2.0.1}$$

for R, P, $Q \varepsilon Simpl.Presh(\mathfrak{S})$, $U \varepsilon \mathfrak{S}$. Any presheaf of sets may be viewed as a presheaf of simplicial sets in the obvious manner. If P, Q are two presheaves of sets, $\mathcal{H}om(P,Q)$ identifies with the presheaf-hom of the presheaves P and Q.

3. The cubical endomorphism operad in the $\mathbb{A}^1$-local category

We will begin by defining cubical objects and discuss the relation between cubical objects and chain complexes in abelian categories.

Definition 3.1. Let $\square$ denote the following category. The objects of $\square$ are the sets $\{0,1\}^n$, $n \geq 0$, where $\{0,1\}^0 =$ the singleton set $\{0\}$ and the superscript denotes the cartesian product n-times, for $n > 0$. The morphisms in the category $\square$ between $\{0,1\}^n$ and $\{0,1\}^m$ will be defined to be all maps of the underlying sets. For each $c = 0, 1$, and $1 \leq i \leq n$, we have *injective* maps $\delta^i_c : \{0,1\}^{n-1} \rightarrow \{0,1\}^n$ that place c in the i-th place. For each $0 \leq i \leq n$, we also have a *surjective* map $\sigma^i_c : \{0,1\}^n \rightarrow \{0,1\}^{n-1}$ which is the projection onto all but the i-th factor. One may observe the following relations:

$$d^j_c \circ d^i_{c'} = d^i_{c'} \circ d^{j-1}_c, \quad i < j \quad and \quad for \quad all \quad c, \quad c'. \tag{3.0.2}$$

For each integer $k \geq 1$, let $\square^k$ denote the category whose objects are k-tuples $(b_1, ..., b_k)$ of objects of $\square$. A morphism from a k-tuple $(b_1, ..., b_k)$ to a k-tuple $(b'_1, ..., b'_k)$ is a tuple of morphisms $(f_1, ..., f_k)$, where $f_i : b_i \rightarrow b'_i$ is a morphism of $\square$. If $\mathcal{C}$ is a category, a cubical object (co-cubical object) in $\mathcal{C}$ will be defined to be a contravariant functor (covariant functor, respectively) $K : \square \rightarrow \mathcal{C}$. In this case $d_i^c = K(d_c^i)$, for each i and $c = 0, 1$. The category of cubical (co-cubical) objects in $\mathcal{C}$ will be denoted $\square^{op}(\mathcal{C})$ ($\square(\mathcal{C})$, respectively). Similarly a multi-cubical object of degree k (a multi co-cubical object of degree k) in the category $\mathcal{C}$ will be a contra-variant functor $K : \square^k \rightarrow \mathcal{C}$ (a covariant functor $K : \square^k \rightarrow \mathcal{C}$, respectively).

If $\mathbf{A}$ is an abelian category, one defines a functor

$$NC : (cubical \quad objects \quad in \quad \mathbf{A}) \rightarrow (chain \quad complexes \quad in \quad \mathbf{A}) \tag{3.0.3}$$

by $NC(K)_n = K_n$, $n \geq 0$ and $\delta : NC(K)_n \rightarrow NC(K)_{n-1}$ is defined by $\Sigma_i(-1)^i(d_i^0 - d_i^1)$. The relations in (3.0.2) show that this defines a chain complex (trivial in negative degrees). Given a multi-cubical object K of degree k, the diagonal $\Delta(K)$ will be a cubical object and $NC(\Delta K)$ will be a chain-complex. Similarly, given a multi-co-cubical object K of degree k in the abelian category $\mathbf{A}$, $NC(\Delta K)$ will be a co-chain complex.

Example 3.2. Now we consider a key example of co-cubical objects in the category of schemes. Let S denote a fixed base-scheme and let $\mathbb{A}^n$ denote the corresponding affine space over S. Let

$$\begin{tikzcd}\mathcal{A} : S \arrow[r, shift left=2] \arrow[r] & \mathbb{A}^1 \arrow[l, shift left=2] & \cdots & \mathbb{A}^n \ldots\end{tikzcd} \tag{3.0.4}$$

where the face map d_c^i sends $\mathbb{A}^n$ into $\mathbb{A}^{n+1}$ as the i-th face consisting of tuples $(x_1, ..., x_{i-1}, c, x_i, ..., x_n)$.

3.1. **The tensor structures.** Observe that the category, $\square^{op}(\mathbf{A})$, of cubical objects in an abelian category $\mathbf{A}$ with a monoidal structure $\otimes$, inherits a monoidal structure given as follows. Let K, K' denote two cubical objects in $\mathbf{A}$. Now $K \otimes K'$ will denote the diagonal of the double cubical object $\{K^n \otimes K'^m | n, m\}$. One may similarly define a pairing

$$\otimes : (\square^{op})^k(\mathbf{A}) \times (\square^{op})^l(\mathbf{A}) \rightarrow \square^{op}(\mathbf{A}) \tag{3.1.1}$$

This is defined by taking the diagonal of the multi-cubical object obtained by taking the degree-wise product $\otimes$. Observe that the category of chain complexes (co-chain complexes) in $\mathbf{A}$ inherits a tensor structure given by the product of two complexes: $(K \otimes L)_n = \underset{p+q=n}{\oplus} K_p \otimes L_q$ ($(K \otimes L)^n = \underset{p+q=n}{\oplus} K^p \otimes L^q$, respectively).

Proposition 3.3. *Let* $\mathbf{A}$ *denote an abelian category with a monoidal structure. Then the functor* NC *is compatible with pairings.*

Proof. We skip this direct verification. (See for example [H-Sch] p. 257.) $\square$

Remark 3.4. Let $\mathfrak{S}$ denote a site and let $Simpl.Presh(\mathfrak{S})$ denote the category of all simplicial presheaves on the site $\mathfrak{S}$. For each object X in the site $\mathcal{C}$, we define a simplicial presheaf h_X as the functor $Y \rightarrow Hom_{\mathfrak{S}}(Y, X)$ represented by X. This defines a faithful functor $h : \mathcal{C} \rightarrow Simpl.Presh(\mathcal{C})$. (One may verify that h_X is in fact a sheaf for each X in the site $\mathfrak{S}$.)

The category $Simpl.Ab.Presh(\mathfrak{S})$ is an abelian category with a tensor structure defined by the tensor product of simplicial abelian groups. Therefore the above discussion applies to $Simpl.Ab.Presh(\mathfrak{S})$.

3.2. **The cubical endomorphism operad.** In this section we will first introduce an operad, which will be an approximation to the *the cubical endomorphism operad*, that will play a key role in the paper. The operad we define here, will, in general, fail to be an E^{∞}-operad or even an acyclic operad; however it will act on the motivic complexes. (This will then be replaced by the cubical endomorphism operad which will be E^{∞}.)

3.2.1. It may be worthwhile recalling the properties of the internal hom functor $\mathcal{H}om$ from section 2. Recall in particular that, if P' and P are the obvious constant simplicial presheaves associated to presheaves of sets, then $\mathcal{H}om(P', P)$ is also the constant simplicial presheaf associated to a presheaf of sets. In particular, this applies to the sheaves of sets h_X , namely the functor represented by the object X of the site $\mathfrak{S}$.

Definition 3.5. Let $\mathfrak{S}$ denote either the big Zariski or Nisnevich site of schemes of finite type over a given base scheme S. We define a bi-functor

$$\mathcal{M}ap_{cube} : (Simpl.Presh(\mathfrak{S}))^{\square^k} \times (Simpl.Presh(\mathfrak{S}))^{\square^k} \rightarrow (Simpl.Presh(\mathfrak{S}))^{\square^{op}} \tag{3.2.2}$$

Given P' and $P \varepsilon (Simpl.Presh(\mathfrak{S}))^{\square^k}$, we let

$\mathcal{M}ap_{cube}(P', P)$ denote the cubical object defined as follows:

$$\mathcal{M}ap_{cube}(P', P)_n = \mathcal{H}om(P' \times h_{\mathbb{A}^n}, P) \tag{3.2.3}$$

The multi-co-cubical structure on $P' \times h_{\mathbb{A}^n}$ is induced from the one on P'. As n varies among all the non-negative integers, one obtains a cubical object in $(Simpl.Presh(\mathfrak{S}))$.

Next observe that $\mathcal{A}$ (as in (3.0.4)) defines a co-cubical object of schemes and hence of simplicial presheaves. Therefore $\mathcal{A}^{\overset{k}{\otimes}}$ (defined by $(\mathcal{A}^{\overset{k}{\otimes}})^{(n_1,\ldots,n_k)} = \mathbb{A}^{n_1} \times \cdots \mathbb{A}^{n_k}$, with $\Sigma n_i = n$) and with the induced structure maps forms a multi-co-cubical object of degree k in $(Simpl.Presh(\mathfrak{S}))$. We let $\mathcal{E}nd_{\mathcal{A}}(k) = \mathcal{M}ap_{cube}(\mathcal{A}^{\overset{k}{\otimes}}, \mathcal{A}^{\overset{k}{\otimes}})$. Observe that this is a cubical object of $(Simpl.Presh(\mathfrak{S}))$. We define $\mathcal{E}nd_{\mathcal{A}}(0) = \{1\}$ = the obvious constant sheaf and define an augmentation $\eta : \mathcal{E}nd_{\mathcal{A}}(0) \rightarrow \mathcal{E}nd_{\mathcal{A}}(1)$ by sending 1 to the identity morphism $\mathcal{A} \rightarrow \mathcal{A}$.

One of the main result of this section is the following theorem.

Theorem 3.6. *The collection $\{\mathcal{E}nd_{\mathcal{A}}(k)|k\}$ with the action of the symmetric group Σ_k induced by the permutation action on the factors of the target $\mathcal{A}^{\overset{k}{\otimes}}$ in $\mathcal{M}ap_{cube}(\mathcal{A}^{\overset{k}{\otimes}}, \mathcal{A}^{\overset{k}{\otimes}})$ and with the above augmentation is a commutative operad in $Simpl.Presh(\mathfrak{S})$).*

Proof. We will verify the axioms systematically. Observe first that the action of the symmetric group is not free and $\mathcal{E}nd_{\mathcal{A}}(k)$ is not necessarily acyclic, so that $\{\mathcal{E}nd_{\mathcal{A}}(k)|k\}$ is not an E^{∞}-operad. We define the map $\gamma_k : \mathcal{E}nd_{\mathcal{A}}(k) \times \mathcal{E}nd_{\mathcal{A}}(j_1) \times ... \mathcal{E}nd_{\mathcal{A}}(j_k) \rightarrow \mathcal{E}nd_{\mathcal{A}}(j)$ in (7.4.1) (in the appendix) as follows. Let $f_\alpha : \mathcal{A}^{\overset{j_\alpha}{\otimes}} \times \mathbb{A}^{l_{j_\alpha}} \rightarrow \mathcal{A}^{\overset{j_\alpha}{\otimes}}$ denote elements of $\mathcal{O}(j_\alpha)_{l_{j_\alpha}}$, $\alpha = 1, ..., k$ and let $g_k : \mathcal{A}^{\overset{k}{\otimes}} \times \mathbb{A}^{l_k} \rightarrow \mathcal{A}^{\overset{k}{\otimes}}$ denote an element of $\mathcal{O}(k)_{l_k}$. For each α and $\beta = 1, ..., j_\alpha$, let m_α be a non-negative integer so that the component of f_α of multi-degree $(m_1^\alpha, ..., m_{j_\alpha}^\alpha)$ is represented by $\mathbb{A}^{m_1^\alpha} \times ... \times \mathbb{A}^{m_{j_\alpha}^\alpha} \times \mathbb{A}^{l_{j_\alpha}} \rightarrow \mathbb{A}^{m_1^\alpha} \times ... \times \mathbb{A}^{m_{j_\alpha}^\alpha}$.

Now consider the product $\underset{\alpha}{\Pi} f_\alpha : \mathcal{A}^{\overset{j_1}{\otimes}} \times \cdots \mathcal{A}^{\overset{j_k}{\otimes}} \times \mathbb{A}^{l_{j_1}} \cdots \times \mathbb{A}^{l_{j_k}} \rightarrow \mathcal{A}^{\overset{j_1}{\otimes}} \times \cdots \mathcal{A}^{\overset{j_k}{\otimes}}$. Taking the product of the corresponding components, we obtain the map

$$\mathbb{A}^{\Sigma_\beta m_\beta^{j_1}} \times ... \times \mathbb{A}^{\Sigma_\beta m_\beta^{j_k}} \times \mathbb{A}^{l_{j_1}} \times ... \times \mathbb{A}^{l_{j_k}} \rightarrow (\mathbb{A}^{m_1^1} \times ... \times \mathbb{A}^{m_{j_1}^1}) \times ... \times (\mathbb{A}^{m_1^k} \times ... \times \mathbb{A}^{m_{j_k}^k}) \tag{3.2.4}$$

We see that the map g defines a map $g_{(j_\alpha)} : \mathbb{A}^{\Sigma_\beta m_\beta^{j_1}} \times \cdots \times \mathbb{A}^{\Sigma_\beta m_\beta^{j_k}} \times \mathbb{A}^{l_k} \rightarrow \mathbb{A}^{\Sigma_\beta m_\beta^{j_1}} \times \cdots \times \mathbb{A}^{\Sigma_\beta m_\beta^{j_k}}$. Therefore we may pre-compose the above component of $\underset{\alpha}{\Pi} f_\alpha$ with the map $g_{(j_\alpha)} \times id : \mathbb{A}^{\Sigma_\beta m_\beta^{j_1}} \times \cdots \times \mathbb{A}^{\Sigma_\beta m_\beta^{j_k}} \times \mathbb{A}^{l_k} \times \mathbb{A}^{l_{j_1}} \times ... \times \mathbb{A}^{l_{j_k}} \rightarrow \mathbb{A}^{\Sigma_\beta m_\beta^{j_1}} \times \cdots \times \mathbb{A}^{\Sigma_\beta m_\beta^{j_k}} \times \mathbb{A}^{l_{j_1}} \times ... \times \mathbb{A}^{l_{j_k}}$. We let this composition define the map γ. In view of the observations in (3.2.1), this suffices to define the map γ as a map of simplicial presheaves. Observe that γ maps $\mathcal{E}nd_{\mathcal{A}}(k)_{l_k} \times \mathcal{E}nd_{\mathcal{A}}(j_1)_{l_{j_1}} \times \cdots \times \mathcal{E}nd_{\mathcal{A}}(j_k)_{l_k}$ to $\mathcal{E}nd_{\mathcal{A}}(j)_{l_k + \Sigma_i l_{j_i}}$ and that it is compatible with the structure maps of the cubical objects $\mathcal{E}nd_{\mathcal{A}}(q)$ involved.

One may now verify that the diagram (7.4.1) follows from the associativity of composition of maps; the remaining identities may be verified readily thereby proving the theorem. ☐

Remark 3.7. In order to obtain an acyclic operad this way, one needs to work systematically in the $\mathbb{A}^1$-local theory similar to that of [M-V] and show that the above operad may be functorially replaced by an operad that is acyclic. We will therefore digress to consider several results on $\mathbb{A}^1$-homotopies. A detailed discussion on localization of simplicial presheaves may be found in the appendix.

3.3. The $\mathbb{A}^1$-local categories of simplicial presheaves.

Let $\mathfrak{S}$ denote the big Zariski or the Nisnevich site of all schemes of finite type over a given base scheme S. Let $\mathbb{A}^1$ denote the affine space of dimension 1 over the base-scheme S; we will identify $\mathbb{A}^1$ with the corresponding representable sheaf of sets $h_{\mathbb{A}^1}$. This is an *interval* in the sense of [M-V]; i.e. one has maps $\mu : \mathbb{A}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$ (given by the additive group structure on $\mathbb{A}^1$), and $i_0, i_1 : S \rightarrow \mathbb{A}^1$ which are the closed immersions corresponding to the points 0 and 1. (All products of schemes we consider will be fibered products over S.) Moreover, one obtains the following relations if $p : \mathbb{A}^1 \rightarrow S$ is the canonical projection:

$$\begin{gathered} \mu(i_0 \times id) = \mu(id \times i_0) = i_0 \circ p \\ \mu(i_1 \times id) = \mu(id \times i_1) = id \\ the \quad morphism i_0 \sqcup i_1 : S \sqcup S \rightarrow \mathbb{A}^1 \quad is \quad a \quad monomorphism \end{gathered} \tag{3.3.1}$$

One may extend μ to maps $(\mathbb{A}^1)^n \times \mathbb{A}^1 \rightarrow (\mathbb{A}^1)^n$ in the obvious manner, for each $n \geq 1$, by using the map $\mu : \mathbb{A}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$ on the last factor in $(\mathbb{A}^1)^n$. These maps also will be denoted μ.

Definition 3.8. Let $f, g : P \rightarrow P'$ denote two maps of simplicial presheaves. An *elementary* $\mathbb{A}^1$*- homotopy* from f to g is a morphism $H : P \times \mathbb{A}^1 \rightarrow P'$ so that $H \circ i_0 = f$ and $H \circ i_1 = g$. An elementary *simplicial homotopy* is defined similarly with $\mathbb{A}^1$ replaced by the constant simplicial presheaf $\Delta[1]$. Two morphisms $f, g : P \rightarrow P'$

are $\mathbb{A}^1$-homotopic (simplicially homotopic) if they can be connected by an elementary $\mathbb{A}^1$-homotopy (elementary simplicial homotopy, respectively).

Next, we recall the functor $Sing_*$ defined in [M-V]. We begin with the cosimplicial object $\Delta_{\mathbb{A}^1} = \{(\mathbb{A}^1)^n|n\}$ whose structure maps are defined as follows. Let $f : [n] = (0, \cdots, n) \rightarrow [m] = (0, \cdots, m)$ be a morphism in the category Δ. Define a morphism of sets $\phi(f) : \{1, \cdots, m\} \rightarrow \{0, \cdots, n+1\}$ setting

$$\begin{aligned} \phi(f)(i) &= min \quad \{l \varepsilon \{0, \cdots, n\} | f(l) \geq i\}, \quad if \quad this \quad set \quad is \quad not \quad empty, \\ &= n+1, \quad otherwise \end{aligned} \tag{3.3.2}$$

Denote by $pr_k : (\mathbb{A}^1)^m \rightarrow \mathbb{A}^1$ ($pr_{\phi(f)(k)} : (\mathbb{A}^1)^n \rightarrow \mathbb{A}^1$) the projection to the k-th factor (the factor $\phi(f)(k)$, respectively) and by $p : (\mathbb{A}^1)^n \rightarrow S$. Then $\Delta_{\mathbb{A}^1}(f) : (\mathbb{A}^1)^n \rightarrow (\mathbb{A}^1)^m$ is defined by

$$\begin{aligned} pr_k \circ \Delta_{\mathbb{A}^1}(f) &= pr_{\phi(f)(k)}, \quad if \quad \phi(f)(k) \varepsilon \{1, ..., n\} \\ &= i_0 \circ p, \quad if \quad \phi(f)(k) = n+1 \\ &= i_1 \circ p, \quad if \quad \phi(f)(k) = 0 \end{aligned} \tag{3.3.3}$$

Given a simplicial presheaf P, we let $Sing_*(P) =$ the diagonal of the bisimplicial presheaf $\{\mathcal{H}om((\mathbb{A}^1)^m, P_n)|n, m\}$.

Proposition 3.9. *(See [M-V] Proposition (2.3.4).) Let $f, g : P \rightarrow P'$ be two maps so that there exists an elementary $\mathbb{A}^1$-homotopy from f to g. Then there exists an elementary simplicial homotopy between $Sing_*(f)$ and $Sing_*(g)$*

Proof. Since $Sing_*$ commutes with products it suffices to show that the morphisms $Sing(i_0)$, $Sing(i_1) : S \rightarrow Sing_*(S) \rightarrow Sing_*(\mathbb{A}^1)$ are related by an elementary simplicial homotopy. The required homotopy is given by the morphism $Sing_*(S) \times \Delta[1]_1 \cong \Delta[1]_1 \rightarrow Sing_1(\mathbb{A}^1) = \mathcal{H}om(\mathbb{A}^1, \mathbb{A}^1)$ that sends the generator $i_1 \varepsilon \Delta[1]_1$ to the element of $\mathcal{H}om(\mathbb{A}^1, \mathbb{A}^1)$ which corresponds to the identity map of $\mathbb{A}^1$. ☐

Recall that each $\mathcal{E}nd_{\mathcal{A}}(k)$ is a cubical object in the category of simplicial presheaves. Therefore $\mathbb{Z} \circ Sing_*$ applied to this produces a cubical object in the category of simplicial abelian presheaves. On applying the functor NC to this object, one obtains a chain complex in the category of simplicial abelian presheaves. Finally one applies the normalization functor as in 7.3 to produce a double complex of abelian presheaves, whose total complex will be denoted $\mathcal{C}'_{\mathcal{A}}(k)$. One may take the tensor product of this with $N(\mathbb{Z}(E\Sigma_k))$ to obtain an E^{∞}-operad.

Definition 3.10. We let $\mathcal{C}''_{\mathcal{A}}(k) = \mathcal{C}'_{\mathcal{A}}(k) \otimes N(\mathbb{Z}(E\Sigma_k))$. Observe that this is a chain-complex trivial in negative degrees. We let $\mathcal{C}_{\mathcal{A}}(k)$ denote the associated co-chain complex.

Corollary 3.11. *Each $\mathbb{Z} \circ Sing_*(\mathcal{E}nd_{\mathcal{A}}(k))_n$ (where n denotes the cubical degree) is an acyclic complex of abelian presheaves. The collection $\{\mathcal{C}_{\mathcal{A}}(k))|k\}$ forms an E^{∞}- operad in the category $Co-chain(Ab.Presh(\mathfrak{S}))$.*

Proof. In view of Proposition 3.9, it suffices to define an elementary $\mathbb{A}^1$ homotopy between the identity map id and the trivial map $*$ of $\mathcal{E}nd(\overset{k}{\mathcal{A}^{\otimes}})_n$. Here the trivial map $*$ sends all of $\mathbb{A}^{m_1} \times \cdots \mathbb{A}^{m_k} \times \mathbb{A}^n$ to the origin in $\mathbb{A}^{m_1} \times \cdots \mathbb{A}^{m_k}$. (By Proposition 3.9, such an elementary $\mathbb{A}^1$-null homotopy will induce a simplicial null homotopy on applying the functor $Sing_*$, which in turn, will induce a chain null homotopy of the corresponding complex $\mathcal{C}'_{\mathcal{A}}(k)$.) Such a homotopy may be defined by first defining $H : \mathbb{A}^{m_1} \times \cdots \mathbb{A}^{m_k} \times \mathbb{A}^1 \rightarrow \mathbb{A}^{m_1} \times \cdots \mathbb{A}^{m_k}$ by $H(t_{m_1}, ..., t_{m_k}, t) = (\mu(t_{m_1}, t), \cdots, \mu(t_{m_k}, t))$. Now we let $\mathcal{H} : \mathcal{E}nd_{\mathcal{A}}(k) \times \mathbb{A}^1 \rightarrow \mathcal{E}nd_{\mathcal{A}}(k)$ be defined by $\mathcal{H}(f, t) = H(\quad, t) \circ f$. Then $\mathcal{H} \circ i_0 = *$ while $\mathcal{H} \circ i_1 = id$. The observation that $Sing_*$ commutes with products, shows that one may apply $Sing_*$ to the diagrams in (7.4.1), (7.4.2) and (7.4.3) for the operad $\{\mathcal{E}nd_{\mathcal{A}}(k)|k \geq 0\}$ and obtain corresponding diagrams involving $Sing_*$ applied to the individual terms. This shows that $\{Sing_*(\mathcal{E}nd_{\mathcal{A}}(k)|k\}$ forms an operad in the category $(Simpl.Presh)^{\square^{op}}$. Since the free abelian group functor has the property that given two simplicial presheaves P and P', there is a natural map $\mathbb{Z}(P) \otimes \mathbb{Z}(P') \rightarrow \mathbb{Z}(P \times P')$ and the functors NC and the normalization functor both commute with tensor products, one may see readily that $\{\mathcal{C}_{\mathcal{A}}(k)|k\}$ forms an operad in $Co-chain(Ab.Presh(\mathfrak{S}))$. ☐

Definition 3.12. (The cubical endomorphism operad) We call the operad $\{\mathcal{C}_{\mathcal{A}}(k)|k\}$ the *cubical endomorphism operad.*

4. The Motivic complexes

4.1. Throughout the rest of the paper, we will assume the base scheme S is th spectrum of a field k. Given an abelian presheaf P on $(smt.schemes/k)$, we extend P to a cubical presheaf (i.e. a presheaf of cubical abelian groups) as is done for the simplicial case in [F-S]: i.e. we let $C_*(P)$ denote the cubical presheaf defined by $\Gamma(U, C_n(P)) = \Gamma(U \times \mathbb{A}^n, P)$. This cubical abelian presheaf may be replaced by the corresponding chain complex (i.e. with a differential of degree -1): this complex will be denoted $NC(C_*(P))$. (Now we may re-index $NC(C_*(P))^i = NC(C_*(P))_{-i}$ to obtain a co-chain complex of presheaves that is trivial in positive degrees.)

4.2. Given X, $Y \varepsilon (smt.schemes/k)$, we define $Cor(Y, X)$ be the free abelian group generated by closed integral subschemes $Z \subseteq Y \times X$ that are finite and dominant over each component of Y. One now defines a category $SmCor/k$, whose objects are the smooth schemes in $(smt.schemes/k)$ and where morphisms between two given schemes X and Y is defined by $Hom_{SmCor/k}(Y, X) = Cor(Y, X)$. Sending a map $f : Y \to X$ of schemes to its graph defines a functor $(smt.schemes/k) \to SmCor/k$.

Given a fixed $X \varepsilon (smt.schemes/k)$, one defines a presheaf with transfers, $\mathbb{Z}_{tr}(X)$, by $\Gamma(U, \mathbb{Z}_{tr}(X)) = Cor(U, X)$. It is observed in [F-S-V], that this defines a *sheaf with transfer* on $(smt.schemes/k)_{Nis}$ and $(smt.schemes/k)_{Zar}$.

4.3. The motivic complexes $\mathbb{Z}(n)$.

Definition 4.1. We let $\underline{\mathbb{Z}}_X^{cub}(n)$ =the restriction of $C^*(\mathbb{Z}_{tr}(\mathbb{A}^n))[-2n]$ to the Zariski site of X. We also let $\mathbb{Z}_X^{cub}(n) = \Gamma(X, \underline{\mathbb{Z}}_X^{cub}(n))$. We define $\underline{\mathbb{Z}}_X^{cub} = \underset{n}{\oplus} \underline{\mathbb{Z}}_X^{cub}(n)$ and $\mathbb{Z}_X^{cub} = \Gamma(X, \underline{\mathbb{Z}}_X^{cub})$.

Next, we consider the following results.

Proposition 4.2. $\underset{n}{\oplus} NC(C_*(\mathbb{Z}_{tr}(\mathbb{A}^n)))$ *is a presheaf of strictly associative differential graded algebras.*

Proof. Since each $C_*(\mathbb{Z}_{tr}(\mathbb{A}^n))$ is a cubical object in $C(Ab.Presh(\mathfrak{S}))$, the functor NC applied to it produces a chain-complex. One may observe that the pairing $C_*(\mathbb{Z}_{tr}(\mathbb{A}^n)) \otimes C_*(\mathbb{Z}_{tr}(\mathbb{A}^m) \to C_*(\mathbb{Z}_{tr}(\mathbb{A}^{n+m})$ is given by taking an external product followed by pull-back by the diagonal $\Delta : X \to X \times X$. Therefore this is strictly associative. The functor NC is compatible with tensor-product pairings; this proves the proposition. ☐

Remark 4.3. Nevertheless, observe that the above pairing is not commutative, but only commutative upto homotopy. One of the main advantages of working in the cubical setting is that one obtains the above proposition; in the simplicial setting, one can only obtain a homotopy associative product. The remaining results of this section may be interpreted as proving this pairing is, in fact, coherently homotopy commutative.

Let $U : Simpl.ab.Presh(\mathfrak{S}) \to Simpl.Presh(\mathfrak{S})$ denote the underlying functor. (Recall that U is right adjoint to the free abelian group functor $\mathbb{Z}$.)

Proposition 4.4. $U(\underset{n}{\oplus} C_*(\mathbb{Z}_{tr}(\mathbb{A}^n)))$ *is an algebra over the cubical endomorphism operad* $\{\mathcal{E}nd_{\mathcal{A}}(k) | k\}$ *in the category* $Simpl.Presh(\mathfrak{S})$. *The structure maps* $\theta_k : \mathcal{E}nd_{\mathcal{A}}(k) \times U((\mathbb{Z}^{cub})^{\overset{k}{\times}}) \to U(\mathbb{Z}^{cub})$, $k \geq 1$ *are maps of multi-cubical objects.*

Proof. We define maps $\theta_k : \mathcal{E}nd_{\mathcal{A}}(k) \times U((\mathbb{Z}^{cub})^{\overset{k}{\times}}) \to U(\mathbb{Z}^{cub})$ as follows. Let $f \varepsilon End_{\mathcal{A}}(k)_{l_1}$ $(Z \varepsilon \Gamma(X^{\overset{k}{\times}}, U(\underset{n}{\oplus} C_*(\mathbb{Z}_{tr}(\mathbb{A}^n))))_{l_2}^{\overset{k}{\times}})$ be represented by $f : \mathcal{A} \times \cdots \mathcal{A} \times \mathbb{A}^{l_1} \to \mathcal{A} \times \cdots \mathcal{A}$ $(Z \varepsilon Cor(X^{\overset{k}{\times}} \times \mathbb{A}^{m_1} \cdots \times \mathbb{A}^{m_k}, \mathbb{A}^n)$ with $Z = \underset{i}{\Pi} Z_i$, $Z_i \varepsilon Cor(X \times \mathbb{A}^{m_i}, \mathbb{A}^n)$, respectively.) Here $\Sigma m_i = l_2$.) Now f induces a map $f_{m_1,\ldots,m_k} : \mathbb{A}^{m_1} \times \ldots \times \mathbb{A}^{m_k} \times \mathbb{A}^{l_1} \to \mathbb{A}^{m_1} \times \ldots \times \mathbb{A}^{m_k}$. Therefore $f_{m_1,..,m_k} \times id : \mathbb{A}^{m_1} \times \ldots \times \mathbb{A}^{m_k} \times \mathbb{A}^{l_1} \times \mathbb{A}^n \to \mathbb{A}^{m_1} \times \ldots \times \mathbb{A}^{m_k} \times \mathbb{A}^n$ and $(f_{m_1,..,m_k} \times id)^*(Z)$ represents a class in $Cor(X^{\overset{k}{\times}} \times \mathbb{A}^{m_1} \cdots \times \mathbb{A}^{m_k} \times \mathbb{A}^{l_1}, \mathbb{A}^n)$. We let

$$\theta(f, Z) = \Delta^*((f_{m_1,\ldots,m_k} \times id)^*(Z)) \tag{4.3.1}$$

where $\Delta : X \to X^{\overset{k}{\times}}$ is the diagonal. Observe that now $\theta(f, Z)$ represents a class in $\Gamma(X, U(\underset{n}{\oplus} C_*(\mathbb{Z}_{tr}(\mathbb{A}^n)))_{l_2+L_1})$. In view of (3.2.1), this defines the map θ. Recall that $\mathcal{E}nd_{\mathcal{A}}(k)$ and $U(\underset{n}{\oplus} C_*(\mathbb{Z}_{tr}(\mathbb{A}^n)))$ are both cubical objects, while $U(\underset{n}{\oplus} C_*(\mathbb{Z}_{tr}(\mathbb{A}^n)))^{\overset{k}{\times}}$ is a k-cubical object in the category $Simpl.Presh(\mathfrak{S})$. One may verify that the pairing θ is compatible with all the structure maps of the above cubical and multi-cubical objects. One may also verify that this action satisfies all the required conditions as in (7.4.4) and (7.4.5). ☐

We will replace the algebra $U(\underset{n}{\oplus}C_*(\mathbb{Z}_{tr}(\mathbb{A}^n)))$ by the algebra $Sing_*(U(\underset{n}{\oplus}C_*(\mathbb{Z}_{tr}(\mathbb{A}^n))))$. Clearly the latter is an algebra over the operad $\{Sing_*(\mathcal{E}nd_{\mathcal{A}}(k))|k\}$. Let

$$(4.3.2) \qquad Sing_*(\theta) : Sing_*(\mathcal{E}nd_{\mathcal{A}}(k)) \times Sing_*(U(\underset{n}{\oplus}C_*(\mathbb{Z}_{tr}(\mathbb{A}^n)))^{\overset{k}{\times}}) \rightarrow Sing_*(U(\underset{n}{\oplus}C_*(\mathbb{Z}_{tr}(\mathbb{A}^n))))$$

denote the map induced by θ. This is $\mathbb{Z}$-multi-linear in the last arguments for each fixed value of the first argument. Therefore we obtain an induced pairing

$$(4.3.3) \qquad \mathbb{Z}(Sing_*(\theta)) : \mathbb{Z}(Sing_*(\mathcal{E}nd_{\mathcal{A}}(k))) \otimes (Sing_*(\underset{n}{\oplus}C_*(\mathbb{Z}_{tr}(\mathbb{A}^n))))^{\overset{k}{\otimes}} \rightarrow Sing_*(\underset{n}{\oplus}C_*(\mathbb{Z}_{tr}(\mathbb{A}^n)))$$

between multi-cubical objects in the category $Simpl.ab.Presh(\mathfrak{S})$. One applies the functor $N \circ NC$ followed by the total complex functor to obtain corresponding pairings of chain complexes of abelian presheaves. It follows that $Tot(N \circ NC(Sing_*(\underset{n}{\oplus}C_*(\mathbb{Z}_{tr}(\mathbb{A}^n)))))$ is an algebra over the operad $\{\mathcal{C}'_{\mathcal{A}}(k)|k\}$ and hence over the E^{∞}-operad $\{\mathcal{C}''_{\mathcal{A}}(k)|k\}$.

Definition 4.5. Observe that $NCSing_*(C_*(\mathbb{Z}_{tr}(\mathbb{A}^n)))$ is a chain-complex of simplicial abelian presheaves. We let $\underline{\mathbb{Z}}_X^{mot}(n)[2n]$ be the co-chain complex of abelian presheaves obtained as follows. We first take $Tot(N \circ NC(Sing_*(C_*(\mathbb{Z}_{tr}(\mathbb{A}^n)))))$, where N denotes the normalization functor as in 7.3. This is a chain complex trivial in negative degrees. We re-index this chain complex as usual to obtain a co-chain complex trivial in positive degrees. This will define $\underline{\mathbb{Z}}_X^{mot}(n)[2n]$. We let $\underline{\mathbb{Z}}_X^{mot} = \underset{n}{\oplus}\underline{\mathbb{Z}}_X^{mot}(n)$, $\mathbb{Z}_X^{mot}(n) = \Gamma(X, \underline{\mathbb{Z}}_X^{mot}(n))$ and $\mathbb{Z}_X^{mot} = \underset{n}{\oplus}\mathbb{Z}_X^{mot}(n)$.

The above discussion proves the following theorem.

Theorem 4.6. *$\underline{\mathbb{Z}}_X^{mot}$ is an E^{∞}-algebra over the E^{∞}-operad $\{\mathcal{C}_{\mathcal{A}}(k)|k\}$ in the category $Co-chain(Ab.Presh(\mathfrak{S}))$.*

Next, we proceed to show that the natural map $NC(C_*(\mathbb{Z}_{tr}(\mathbb{A}^n))) \rightarrow Tot(N \circ NCSing_*((C_*(\mathbb{Z}_{tr}(\mathbb{A}^n)))))$ induces a quasi-isomorphism. This will follow from the observation that both the above presheaves are $\mathbb{A}^1$-local (see 4.8 and 7.2.1 in the appendix) and that they are $\mathbb{A}^1$-weakly-equivalent in the $\mathbb{A}^1$-local category of simplicial presheaves sheaves defined as follows.

Theorem 4.7. *(Morel-Voevodsky) (See [M-V] Theorem 2.2.7) Let $\mathfrak{S}$ denote the big Nisnevich or Zariski site of all schemes of finite type over a Noetherian base-scheme. Then $Simpl.Presh(\mathfrak{S})$ and $Simpl.Sh(\mathfrak{S})$ are cellular left proper simplicial model categories and the left Bousfield localization of these with respect to $\mathfrak{A} = \{h_{\mathbb{A}^1} \rightarrow h_S\}$ exist.*

The $\mathbb{A}^1$-local category of simplicial presheaves on $\mathfrak{S}$ is the left Bousfield localization of $Simpl.Presh(\mathfrak{S})$ with respect to $\mathfrak{A}$. (See the appendix for more details on this construction.) This will be denoted $Simpl.Presh(\mathfrak{S})_{\mathfrak{A}}$. Observe by the above theorem and by Proposition 7.5, that this is a left-proper and cellular model category which is also simplicial.

Proposition 4.8. *Let $\mathfrak{S}$ denote the big Zariski site of schemes of finite type over a fixed Noetherian scheme S. Let $\mathfrak{A} = \{h_{\mathbb{A}^1} \rightarrow h_S\}$. Let $P \varepsilon Simpl.Presh(\mathfrak{S})$ be such that $\Gamma(U, P)$ is a fibrant simplicial set for every U in $\mathfrak{S}$ and so that P has cohomological descent on the Zariski site of every scheme $X \varepsilon \mathfrak{S}$. Then P is $\mathfrak{A}$-local if and only if the map $\mathcal{H}om(S, P) \rightarrow \mathcal{H}om(\mathbb{A}^1, P)$ induced by $p : \mathbb{A}^1 \rightarrow S$ is a weak-equivalence.*

Proof. Recall the internal hom in $Simpl.Presh(\mathfrak{S})$ is defined by the bi-functor $\mathcal{H}om$. The proof follows readily from the definition 7.2.1, the relation between the above $\mathcal{H}om$ and the bi-functor Map along with the observation that the maps $\mathcal{H}om(S, P) \rightarrow \mathcal{H}om(S, \mathcal{G}P)$ and $\mathcal{H}om(\mathbb{A}^1, P) \rightarrow \mathcal{H}om(\mathbb{A}^1, \mathcal{G}P)$ are weak-equivalences. (Here $\mathcal{G}P$ is the Godement resolution on the Zariski site; observe that $P \rightarrow \mathcal{G}P$ is a fibrant approximation to P in the model structure defined in Proposition 7.1.) ☐

Proposition 4.9. *Let $\mathfrak{S}$ denote the big Zariski site of all schemes of finite type over a Noetherian base scheme S. Let $P \varepsilon Simpl.Ab.Presh(\mathfrak{S})$ have the Mayer-Vietoris property on the Zariski site of a given scheme X. Then the presheaf $Sing_*(P)$ also has the Mayer-Vietoris property on the Zariski site of X.*

Proof. The key observation is that P is a simplicial *abelian* presheaf. Now the diagram in 7.2(ii) is a fibration sequence if and only if it is a short exact sequence of simplicial abelian groups. Recall $Sing_*(P) =$ the diagonal of the bi-simplicial abelian presheaf $\{\mathcal{H}om((\mathbb{A}^1)^m, P_n)|n, m\}$. Observe that, for each fixed m, and each fixed W in the site $\mathfrak{S}$, $\Gamma(W, \mathcal{H}om((\mathbb{A}^1)^m, P)) = Map(W, \mathcal{H}om((\mathbb{A}^1)^m, P)) = Map(W \times (\mathbb{A}^1)^m, P) = \Gamma(W \times (\mathbb{A}^1)^m, P)$. (These follow from the observation that for a simplicial presheaf Q, $\Gamma(W, Q)_n = Hom_{Simpl.Presh(\mathfrak{S})}(W \times \Delta[n], Q) = Map(W, Q)_n$.) Therefore, the Mayer-Vietoris property for the presheaf P implies the Mayer-Vietoris property for each of the presheaves $\mathcal{H}om((\mathbb{A}^1)^m, P)$, $m \geq 0$. i.e. for each fixed $m \geq 0$, and U, V open in the Zariski site of a given scheme X, the sequence (of simplicial abelian groups)

$$\Gamma(U \cup V, \mathcal{H}om((\mathbb{A}^1)^m, P)) \to \Gamma(U, \mathcal{H}om((\mathbb{A}^1)^m, P)) \times \Gamma(V, \mathcal{H}om((\mathbb{A}^1)^m, P)) \to \Gamma(U \cap V, \mathcal{H}om((\mathbb{A}^1)^m, P))$$

is exact. On varying m and then taking the diagonal, one still obtains a short-exact sequence. ☐

Proposition 4.10. *The presheaves $DN(NC(C_*(\mathbb{Z}_{tr}(\mathbb{A}^n))))$ and $DN(Tot(N \circ NCSing_*(C_*(\mathbb{Z}_{tr}(\mathbb{A}^n)))))$ are both $\mathbb{A}^1$-local. (Here DN denotes the de-normalization functor sending a chain complex that is trivial in negative degrees to a simplicial abelian object as in 7.3.)*

Proof. It suffices to verify the criterion in Proposition 4.8 for the presheaf $DN(NC(C_*(\mathbb{Z}_{tr}(\mathbb{A}^n))))$: observe this is quasi-isomorphic to the higher cycle complex on any quasi-projective scheme over k and therefore has cohomological descent on the Zariski site of any quasi-projective scheme over k. Therefore, the homotopy property of $DN \circ (NC(C_*(\mathbb{Z}_{tr}(\mathbb{A}^n))))$ implies that it is $\mathbb{A}^1$-local. Recall that $Sing_*(C_*(\mathbb{Z}_{tr}(\mathbb{A}^n)))$ is the presheaf defined by $\Gamma(U, Sing_*(C_*(\mathbb{Z}_{tr}(\mathbb{A}^n)))) = \Delta\{\Gamma(U, \mathcal{H}om((\mathbb{A}^1)^m, C_k(\mathbb{Z}_{tr}(\mathbb{A}^n))))|k, m\} = \Delta\{Map(U \times (\mathbb{A}^1)^m, C_k(\mathbb{Z}_{tr}(\mathbb{A}^n))))|k, m\}$. Similarly $\Gamma(U \times \mathbb{A}^1, Sing_*(C_*(\mathbb{Z}_{tr}(\mathbb{A}^n)))) = \Delta\{Map(U \times \mathbb{A}^1 \times (\mathbb{A}^1)^m, C_k(\mathbb{Z}_{tr}(\mathbb{A}^n))))|k, m\}$. Therefore, the homotopy property of $NC(C_*(\mathbb{Z}_{tr}(\mathbb{A}^n)))$ implies that of $Tot(N(Sing_*(NC(C_*(\mathbb{Z}_{tr}(\mathbb{A}^n))))))$. It follows from Proposition 4.9 in the appendix that the last complex also has the Mayer-Vietoris property on the Zariski site of a given Noetherian scheme and hence cohomological descent on the Zariski site of any Noetherian scheme. Therefore $DN(Tot(N \circ NCSing_*(C_*(\mathbb{Z}_{tr}(\mathbb{A}^n)))))$ is also $\mathbb{A}^1$-local. ☐

Corollary 4.11. *The natural map $NC(C_*(\mathbb{Z}_{tr}(\mathbb{A}^n)))) \to Tot(N \circ NCSing_*(C_*(\mathbb{Z}_{tr}(\mathbb{A}^n)))))$ is a quasi-isomorphism. Moreover, this induces a quasi-isomorphism on taking sections over any smooth quasi-projective scheme over k.*

Proof. By de-normalizing the above chain-complexes (which are trivial in negative degrees), one obtains a map of simplicial abelian presheaves. It is clear that the resulting map is an $\mathbb{A}^1$-weak-equivalence. (See for example, [M-V] (2.3.2).) Since both the presheaves are $\mathbb{A}^1$-local, it follows from theorem 7.9 (in the appendix) that this is also a weak-equivalence. Since both are simplicial abelian presheaves, a weak-equivalence is equivalent to a stalk-wise quasi-isomorphism of the associated normalized chain complexes. This proves the first statement. The second statement now follows from the observation that both complexes of presheaves have cohomological descent on the Zariski site of any smooth quasi-projective scheme over k. ☐

Next we obtain the following corollary.

Corollary 4.12. *$\underline{\mathbb{Q}}_X^{mot} = \underline{\mathbb{Z}}_X^{mot} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is a presheaf of commutative differential graded algebras. If $\underline{\mathbb{Q}}_X^{mot}(n) = \underline{\mathbb{Z}}_X^{mot}(n) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, $\underline{\mathbb{Q}}_X^{mot}(n)$ is quasi-isomorphic to $\underline{\mathbb{Z}}_X^{cub}(n) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. For any smooth quasi-projective scheme X, $\mathbb{Q}_X^{mot}(n)$ is quasi-isomorphic to $\mathbb{Z}_X^{cub}(n) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$*

Proof. It follows from [K-M] Corollary (1.5), Part II that any E^∞-differential graded algebra tensored with $\mathbb{Q}$ (or any field of characteristic 0) may be replaced by a quasi-isomorphic *commutative* differential graded algebra. Therefore Theorem 4.6 provides the required algebra structure. The quasi-isomorphism in the second statement now follows from corollary (4.11). ☐

Definition 4.13. If X is a smooth quasi-projective scheme over k, $\mathbb{Q}_X^{mot} = \Gamma(X, \underline{\mathbb{Q}}_X^{mot})$ will be called the *motivic DGA* associated to X.

5. Operations in motivic cohomology

The results of this section follow by well-known results on algebras over E^∞-operads. The main observation is that the structure of an algebra over an E^∞-operad induces various cohomology operations on the cohomology of the algebra. However, the operations we obtain are only some of the cohomology operations used, for example, by Voevodsky. We thank Patrick Brosnan for explaining this to us. Let X denote a smooth quasi-projective scheme over a field k. Let p denote a prime. Let $A^\bullet = \mathbb{Z}_X^{mot} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}/p.\mathbb{Z}$ denote the mod-p motivic complex. Recall this is an E^∞-algebra over the E^∞-operad defined in the last section. Observe that A is now a bi-graded E^∞-differential graded algebra, so that $A^\bullet = \underset{r}{\oplus} A^\bullet(r)$. Moreover $H^i_{\mathcal{M}}(X; \mathbb{Z}/p(r)) = H^i(A^\bullet(r)) = CH^r(X, 2r - i; \mathbb{Z}/p)$

Theorem 5.1. *If $p = 2$, there exist operations $Sq^s = P^s : H^q_{\mathcal{M}}(X, t) \to H^{q+s}_{\mathcal{M}}(X, 2t)$ and for each $0 \leq s \leq q$.*

If $p > 2$, there exist operations $P^s : H^q_{\mathcal{M}}(X, t) \to H^{qp-q+s}_{\mathcal{M}}(X, tp)$ for each $0 \leq s \leq q/2$.

These operations satisfy the following properties:

Let $x \varepsilon H^q_{\mathcal{M}}(X,t)$*. If* $p = 2$, $P^q(x) = x^2$*; if* $p > 2$, $P^s(x) = x^p$ *if* $q = 2s$*. Moreover we obtain the following relations:*

Cartan formula: $P^s(x \otimes y) = \underset{i+j=s}{\Sigma} P^i(x) \otimes P^j(y)$ *and if* $p > 2$, $\beta P^{s+1}(x \otimes y) = \underset{i+j=s}{\Sigma} (\beta P^{i+1}(x) \otimes P^j(y) + (-1)^{degx} P^i(x) \otimes \beta P^{j+1}(y))$*, where* β *is the mod-p Bockstein homomorphism.*

Adem relations: (See [May-2] p. 183.) If either

$p \geq 2$*,* $a < pb$ *and* $\epsilon = 0, 1$*,*

if $p > 2$*,* $\epsilon = 0$ *or*

if $p = 2$*, then one has*

$$\beta^\epsilon P^a P^b = \Sigma_i (-1)^{a+i} (a - pi, (p-1)b - a + i - 1) \beta P^{a+b-i} P^i \tag{5.0.4}$$

If either

$p > 2$*,* $a \leq pb$ *and* $\epsilon = 0, 1$*, then one has*

$$\begin{aligned} \beta^\epsilon P^a \beta P^b = (1-\epsilon)\Sigma_i (-1)^{a+i} (a - pi, (p-1)b - a + i - 1) \beta P^{a+b-i} P^i \\ - \Sigma_i (-1)^{a+i} (a - pi - 1, (p-1)b - a + i) \beta^\epsilon P^{a+b-i} \beta P^i \end{aligned} \tag{5.0.5}$$

where, $\beta^0 P^s = P^s$ *and* $\beta^1 P^s = \beta P^s$*.*

Proof. The proof follows from [May-2] section 5; observe that the discussion in [May-2] is valid for all algebras over E^∞-operads. ☐

6. Relative mixed Tate motives for smooth schemes over a field k

The results of this section generalize the constructions of [Bl-3], [Bl-K] and [K-M] for the category of mixed Tate motives over a field. We fix a smooth quasi-projective scheme X over a field k. We let $A = \mathbb{Q}^{mot}_X$. We may assume therefore that A is bi-graded via k-modules $A(r)$, where $q \varepsilon \mathbb{Z}$ and $r \geq 0$. Let $\mathbb{D}_A$ denote the derived category of cohomologically bounded below A-modules, i.e. differential bi-graded A-modules $M = \underset{q,r}{\oplus} M^q(r)$ where $M^q(r)$ may be non-zero for any pair of integers so that $\mathcal{H}^q(M)(r) = 0$ for all sufficiently small q. One may define a functor $Q : \mathbb{D}_A \to D(\mathbb{Q} - vector \quad spaces)$ by $Q(M) = \mathbb{Q} \overset{L}{\underset{A}{\otimes}} M$. Here $D(\mathbb{Q} - vector \quad spaces)$ denotes the derived category of bounded below complexes of $\mathbb{Q}$-vector spaces. Observe that this category has a natural t-structure, the heart of which is given by the complexes that have cohomology trivial in all degrees except 0. We let $\mathcal{H}_A$ denote the full sub-category of $\mathbb{D}_A$ consisting of complexes K so that $\mathcal{H}^q(Q(K)) = 0$ for all $q \neq 0$. Let $\mathcal{FH}_A$ denote the full sub-category of $\mathcal{H}_A$ consisting of complexes K so that $\mathcal{H}^0(Q(K))$ is a finite dimensional $\mathbb{Q}$-vector space. We will make the following assumption throughout:

6.0.6. *the DGA* A *is connected in the following sense:* $H^i(A)(r) = 0$ *for* $i < 0$*,* $H^0(A)(r) = 0$ *if* $r \neq 0$ *and* $H^0(A)(0) = \mathbb{Q}$*.*

Now we obtain the following theorem as in [K-M].

Theorem 6.1. *The triangulated category* $\mathbb{D}_A$ *admits a* t*-structure whose heart is* $\mathcal{H}_A$*. Moreover* $\mathcal{FH}_A$ *is a graded neutral Tannakian category over* $\mathbb{Q}$ *with fiber functor* $w = \mathcal{H}^0 \circ Q$*.*

Proof. The proof is essentially in [K-M] Theorem (1.1), Part IV. (The key idea here is to use the theory of minimal models.) ☐

One may apply the bar construction (see [K-M] p. 76) to the algebra A: we will denote this by $\bar{B}A$. Let IA denote the augmentation ideal of A. We let $\chi_A = H^0(\bar{B}A)$. This is a commutative Hopf-algebra and as in [K-M] p. 76 is a polynomial algebra with its k-module of indecomposable elements a co-Lie algebra which is denoted γ_A. Now we obtain the following result.

Theorem 6.2. *(See [K-M] p. 77) Assume the hypothesis (6.0.6). Then the following categories are equivalent:*

(i) The heart $\mathcal{H}_A$ *of* $\mathbb{D}_A$

(ii) The category of generalized nilpotent representations of the co-Lie algebra γ_A

(iii) The category of co-modules over the Hopf-algebra χ_A

(iv) The category $\mathcal{T}_A$ of generalized nilpotent twisting matrices in A

The full sub-categories of finite dimensional objects in the categories (i), (ii) and (iii) and of finite matrices in the category (iv) are also equivalent.

Definition 6.3. (Linear schemes over k) (i) A scheme over $Spec \quad k$ is 0-linear if it is either empty or isomorphic to any affine space $\mathbb{A}^n_{Spec \quad k}$.

(ii) Let $n > 0$ be an integer. A scheme Z, over $Spec \quad k$, is n-linear, if there exists a triple (U, X, Y) of schemes over $Spec \quad k$ so that $Y \subseteq X$ is an S-closed immersion with U its complement, Y and one of the schemes U or X is $(n-1)$-linear and Z is the other member in $\{U, X\}$. We say Z is linear if it is n-linear for some $n \geq 0$.

(iii) Any reduced scheme X of finite type over $Spec \quad k$ will be called a variety. Linear varieties over k are varieties over $Spec \quad k$ that are linear schemes.

Example 6.4. The following are common examples of linear varieties. In these examples we fix a base field k and consider only varieties over k.

- All toric varieties
- All spherical varieties (A variety X is spherical if there exists a reductive group G acting on X so that there exists a Borel subgroup having a dense orbit.)
- Any variety on which a connected solvable group acts with finitely many orbits. (For example projective spaces and flag varieties.)
- Any variety that has a stratification into strata each of which is the product of a torus with an affine space.

Corollary 6.5. *Let X denote a smooth projective linear variety over a field k, for instance any one of the schemes appearing in the examples above which is also projective and smooth. Assume that the Beilinson-Soulé conjecture holds for the rational motivic cohomology of $Spec \quad k$, i.e. $H^i_{\mathcal{M}}(Spec \quad k; \mathbb{Q}(r)) = 0$ if $i < 0$, $H^0_{\mathcal{M}}(Spec \quad k; \mathbb{Q}(r)) = 0$ if $r \neq 0$ and $H^0_{\mathcal{M}}(Spec \quad k; \mathbb{Q}(0)) \cong \mathbb{Q}$. Then the conclusions of theorem (6.2) hold for X.*

Proof. It suffices to show that the DGA, A appearing in the theorem is connected. For this we recall that a strong Kunneth decomposition holds for the class of the diagonal Δ in $CH^*(X \times X)$. (See [J-1] or [Tot].) i.e.

$$\Delta = \Sigma_i \alpha_i \times \beta_i = \Sigma_i p_1^*(\alpha_i) \circ p_2^*(\beta_i) \tag{6.0.7}$$

where $p_i : X \times X \rightarrow X$ is the projection to the i-factor and $\circ$ denotes the intersection product. Now we proceed to show that any class $x \varepsilon CH^*(X, n)$ may be written as a linear combination

$$x = \Sigma_i \alpha_i \circ p_{1*}(p_2^*(\beta_i \circ x)) = \Sigma_i \alpha_i \circ {p'_2}^*({p'_1}_*(\beta_i \circ x)) \tag{6.0.8}$$

Here $p'_i : X \rightarrow Spec \quad k$ is the obvious projection. To obtain (6.0.8), first observe that $x = p_{1*}([\Delta] \circ p_2^*(x))$. By the projection formula and the observation that the class $[\Delta] = \Delta_*(1)$, $1 = [X] \varepsilon CH^*(X;)$, we obtain equality of the classes $[\Delta] \circ p_2^*(x) = \Delta_*(\Delta^*(p_2^*(x)))$. Therefore $p_{1*}([\Delta] \circ p_2^*(x)) = p_{1*}(\Delta_*(\Delta^*(p_2^*(x)))) = (p_1 \circ \Delta)_*((p_2 \circ \Delta)^*(x)) = x$. Now substitute the formula for $[\Delta]$ from (6.0.7) and use the projection formula to obtain the first equality in (6.0.8). The equality of this with the right-hand-side follows by flat-base-change. Observe that $\alpha_i \varepsilon CH^*(X, 0)$. Therefore, the hypothesis that $H^i_{\mathcal{M}}(Spec \quad k; \mathbb{Q}(r)) = 0$ for $i < 0$ shows readily that $H^q_{\mathcal{M}}(X; \mathbb{Q}(r)) = CH^{2r-q}(X, r) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} = 0$ for $q < 0$. The hypothesis that $H^0_{\mathcal{M}}(Spec \quad k; \mathbb{Q}(r)) = 0$ for $r \neq 0$ implies that $H^0_{\mathcal{M}}(X; \mathbb{Q}(r)) = 0$ also for $r \neq 0$. Now the hypothesis that $H^0_{\mathcal{M}}(Spec \quad k; \mathbb{Q}(0)) = \mathbb{Q}$ implies $H^0_{\mathcal{M}}(X; \mathbb{Q}(0)) = \mathbb{Q}(0)$. i.e. We have verified that the DGA A associated to the motivic complex of X is connected in the sense of theorem 6.2. ☐

The DGA A has a 1-minimal model, $i : A < 1 > \rightarrow A$. (Recall a connected DGA B is said to be minimal if it is a free commutative algebra with decomposable differential : $d(B) \subseteq (I(B))^2$ where $I(B)$ is the augmentation ideal of B. $B < 1 >$ is the sub-DGA of B generated by the elements of degree ≤ 1 and their differentials. The 1-minimal model of a DGA A is a composite map $B < 1 > \subseteq B \rightarrow A$ with the last map a quasi-isomorphism and with B minimal.) The map i induces an isomorphism on H^1 and is injective on H^2. We say A is a $K(\pi, 1)$ if i is a quasi-isomorphism.

Theorem 6.6. *(See [K-M] p. 77.) The derived category of bounded below chain complexes in $\mathcal{H}_A$ is equivalent to the derived category $\mathbb{D}_{A<1>}$.*

Definition 6.7. (The category of relative mixed Tate motives over X.) Let χ_X^{mot} denote the Hopf algebra $H^0(\bar{B}A)$. The category of (rational) relative mixed Tate motives over X, denoted $\mathcal{MTF}(X)$, to be the category of finite dimensional co-modules over χ_X^{mot}.

Theorem 6.8. *If the DGA A is connected (in the sense of 6.0.6), $\mathcal{MTF}(X)$ is equivalent to the category $\mathcal{FH}_A$. In particular, this holds for all regular quasi-projective linear varieties over k assuming the Beilinson-Soulé conjecture (see above) holds for the motivic cohomology of Spec k.*

Let $\mathbb{Q}(r)$ be the copy of $\mathbb{Q}$ concentrated in bi-degree $(0, r)$ and regarded as a representation of γ_A in the obvious manner.

Corollary 6.9. *If A is a $K(\pi, 1)$, then $Ext^q_{\mathcal{MTF}(X)}(\mathbb{Q}, \mathbb{Q}(r)) = Ext^q_{\mathcal{H}_A}(\mathbb{Q}, \mathbb{Q}(r)) \cong H^q(A(r)) = H^q_{\mathcal{M}}(X, r) = CH^r(X, 2r - q; \mathbb{Q})$*

6.1. ***l*-adic realization.** We will sketch an outline of the corresponding l-adic realization. First, following the approach in [BGSV] and [BMS], one defines a Hopf algebra χ_X^{et} so that the category of co-modules over it is equivalent to the category of mixed Tate l-adic representations of the (algebraic) fundamental group $\pi_1(X)$. (The elements of the Hopf algebra χ_{et}^X are *framed* mixed Tate l-adic representations.) Using the cycle map and a variant of the bar construction, one applies the constructions of [Bl-K] to produce the l-adic realization functor. The relevant arguments are entirely similar to those in [Bl-3], [Bl-K] and are therefore omitted.

7. Appendix

7.1. **Localization of simplicial presheaves (after Hirschorn and Morel-Voevodsky).** We begin with the following two results, which are, by now, well-known in the literature. (See for example, [Jar].)

Proposition 7.1. *Let $\mathfrak{S}$ denote a small site with enough points. Then the following structure defines the structure of a simplicial model category on $Simpl.Presh(\mathfrak{S})$ (or on the category $Simpl.Sh(\mathfrak{S})$ of simplicial sheaves on the site $\mathfrak{S}$):*

- *cofibrations are monomorphisms of presheaves*
- *a map $f : P' \to P$ is a weak-equivalence if it induces a weak-equivalence of the simplicial sets forming the stalks*
- *fibrations are defined by the right lifting property with respect to maps that are trivial cofibrations (i.e. cofibrations and weak-equivalences)*

Moreover the following additional properties hold:

- *this model category structure is cofibrantly generated where the generating cofibrations I (generating trivial cofibrations J) is the set of cofibrations (trivial cofibrations, respectively) $i : U \to V$ so that there exists a large enough cardinal number α with the cardinality of each $V_n(X)$ smaller than α, $X \varepsilon \mathfrak{S}$, $n \geq 0$.*
- *Let $\mathcal{G}P = \underset{\Delta}{holim}\{G^nP|n\}$ denote the Godement resolution (as in [J-2]). Then $\mathcal{G}Ex^{\infty}P$ is a fibrant object in $Simpl.Presh(\mathfrak{S})$ so that the natural map $P \to \mathcal{G}Ex^{\infty}P$ is a weak-equivalence. (Here Ex^{∞} is a functor that produces a functorial fibrant approximation to a simplicial set.)*

Definition 7.2. (i) We say a presheaf P has cohomological descent on the site $\mathfrak{S}$, if for each U in $\mathfrak{S}$, the natural map $\Gamma(U, P) \to \underset{\Delta}{\text{holim}}\{\Gamma(U, \mathcal{G}^nP)|n\}$ is a weak-equivalence.

(ii) Let $\mathfrak{S}$ denote the Zariski site of a Noetherian scheme X. We say that a presheaf P has the Mayer-Vietoris property on X, if for any two Zariski open sub-schemes U, V of X, the diagram

$$\Gamma(U \cup V, P) \to \Gamma(U, P) \times \Gamma(V, P) \to \Gamma(U \cap V, P)$$

is a fibration sequence of simplicial sets.

Proposition 7.3. *Suppose X is a Noetherian scheme and P is a simplicial presheaf on the Zariski site of X having the Mayer-Vietoris property. Then P has cohomological descent on the Zariski site of X.*

For the rest of this section we will assume that $\mathfrak{S}$ is either the big Nisnevich or Zariski site of all schemes of finite type over a given base scheme S. In order to consider the $\mathbb{A}^1$-homotopy theory on $Simpl.Presh(\mathfrak{S})$, we will first recall a few basic facts on localization of model categories, from [Hirsch].

Definition 7.4. (See [Hirsch] chapters 12 and 15.) (i) A model category is *left proper* if every pushout of a weak-equivalence along a cofibration is also a weak-equivalence.

(ii) A *cellular* model category is a cofibrantly generated model category for which there exists a set I (J) of generating cofibrations (generating trivial cofibrations, respectively) so that both the domains and the codomains of the elements of I are compact, the domains of the elements of J are small relative to I and the cofibrations are effective monomorphisms.

Proposition 7.5. *(See [Hirsch] chapters 12 and 14.) (i) Every pushout of a weak-equivalence between cofibrant objects is a weak-equivalence in any model category. (ii) If $\mathcal{M}$ is a model category where every object is cofibrant, $\mathcal{M}$ is a left proper model category. In particular, the categories, $Simpl.Presh(\mathfrak{S})$ and $Simpl.Sh(\mathfrak{S})$ on any small site $\mathfrak{S}$ are left-proper as well as cellular.*

Proof. All the assertions are clear, except the one claiming the categories $Simpl.Presh(\mathfrak{S})$ and $Simpl.Sh(\mathfrak{S})$ on any small site $\mathfrak{S}$ are cellular. To see this, recall that the generating cofibrations I (generating trivial cofibrations J) are those cofibrations $i : U \to V$ as in Proposition 7.1. Therefore the cellularity of the above categories is also clear. (One may readily show that these cofibrations are in fact effective monomorphisms.) ☐

7.2. **Left Bousfield localization of simplicial model categories.** Let $\mathcal{M}$ denote a simplicial model category and let $Map : \mathcal{M}^{op} \times \mathcal{M} \to (simplicial \quad sets)$ denote the bi-functor defined by the simplicial structure. Assume that every object of $\mathcal{M}$ is cofibrant.

Definition 7.6. Let $\mathcal{S}$ denote a set of morphisms in $\mathcal{M}$. An object W of $\mathcal{M}$ is *$\mathcal{S}$-local* if W is fibrant and for every $s : S_1 \to S_2$ in $\mathcal{S}$ and any Z in $\mathcal{M}$, the induced map

$$Map(Z \times S_2, W) \to Map(Z \times S_1, W) \tag{7.2.1}$$

is a weak-equivalence. A morphism $f : X \to Y$ in $\mathcal{M}$ is an *$\mathcal{S}$-weak-equivalence*, if for any $\mathcal{S}$-local object S_0 of $\mathcal{M}$, the map $Hom_{H\mathcal{M}}(Y, S_0) \to Hom_{H\mathcal{M}}(X, S_0)$ induced by f is an isomorphism, where $H\mathcal{M}$ denotes the homotopy category associated to $\mathcal{M}$. The $\mathcal{S}$-cofibrations are defined to be the same as the cofibrations of $\mathcal{M}$ and the $\mathcal{S}$-fibrations are defined by right-lifting property with respect to all maps that are cofibrations and $\mathcal{S}$-weak-equivalences.

The *left Bousfield localization* of $\mathcal{M}$ with respect to $\mathcal{S}$ is a model category structure on the same category underlying $\mathcal{M}$ where the cofibrations (fibrations, weak-equivalences) are the $\mathcal{S}$-cofibrations ($\mathcal{S}$-fibrations, $\mathcal{S}$-weak-equivalences, respectively).

Remark 7.7. One may verify readily that W is $\mathcal{S}$-local if and only if W is fibrant and for every map $s : S_1 \to S_2$, the induced map $s^* : Map(S_2, W) \to Map(S_1, W)$ is a weak-equivalence.

Theorem 7.8. *(See [Hirsch] chapter 4.) Let $\mathcal{M}$ denote a cellular left proper simplicial model category and $\mathcal{S}$ a set of morphisms in $\mathcal{M}$. Then the left Bousfield localization of $\mathcal{M}$ with respect to $\mathcal{S}$ exists. If $L_S(\mathcal{M})$ denotes this localization, then this is a left proper cellular simplicial model category.*

Theorem 7.9. *($\mathcal{S}$-local Whitehead theorem)(See [Hirsch] (3.3.7).) Assume the situation of the above theorem. Let $f : P \to P'$ denote a map that is an $\mathcal{S}$-weak-equivalence with P and P' both $\mathcal{S}$-local. Then f is a weak-equivalence.*

7.3. **The Dold-Puppe correspondence.** Let $\mathbf{A}$ denote an abelian category; a chain complex K in $\mathbf{A}$ will denote a sequence $K^i \varepsilon \mathbf{A}$ provided with maps $d : K_i \to K_{i-1}$ so that $d^2 = 0$. Let $C_0(\mathbf{A})$ denote the category of chain complexes in $\mathbf{A}$ that are trivial in negative degrees. One defines the de-normalizing functor:$DN : C_0(\mathbf{A}) \to$ (*simplicial objects in* $\mathbf{A}$) as in [Ill] pp. 8-9. DN will be inverse to the functor N : (*simplicial objects in* $\mathbf{A}$) $\to C_0(\mathbf{A})$ defined by $(NK)^n = \underset{i\neq 0}{\rightarrow} + coker(d_i : K_n \to K_{n-1})$ with $\delta : (NK)_n \to (NK)_{n-1}$ induced by d_0.

We will define the Godement resolution for chain complexes of abelian sheaves the following way. Given a chain complex, P, of abelian presheaves on the site $\mathfrak{S}$, we let $\mathcal{G}(P) = N(\underset{\Delta}{\text{holim}}\{G^n(DN(P))|n\})$. (One may observe that $\underset{\Delta}{\text{holim}}\{G^n(DN(P))|n\}$ is a simplicial abelian presheaf so that one may apply the normalization to it to produce a chain-complex.)

7.4. **Operads.** Basic definitions of operads and algebras over operads for topological spaces may be found in [May-1]. The same definitions apply with minor modifications to the unital symmetric monoidal category $Presh(\mathfrak{S})$ as in section 1. Recall the main result of section 3 is the construction of an E^∞-operad in the category $Co-chain(Ab.Presh(\mathfrak{S}))$ by first constructing an operad in the category $Simpl.Presh(\mathfrak{S})$. The following detailed definition of operads in the general setting of $Presh(\mathfrak{S})$ is provided so as to clarify the details of this construction. (Recall $Presh(\mathfrak{S})$ is assumed to be provided with the structure of a unital symmetric monoidal structure as in section 1.)

An *associative operad* (or simply operad) $\mathcal{O}$ in $Presh(\mathfrak{S})$ is given by a sequence $\{\mathcal{O}(k)|k \geq 0\}$ of objects in $Presh(\mathfrak{S})$ along with the following data:

for every integer $k \geq 1$ and every sequence $(j_1, ..., j_k)$ of non-negative integers so that $\Sigma_{l=1}^{l=k} j_l = j$ there is given a map $\gamma_k : \mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes ...\mathcal{O}(j_k) \to \mathcal{O}(j)$ so that the following associativity diagrams commute, where $\Sigma_{l=1}^{l=k} j_l = j$ and $\Sigma_{t=1}^{t=j} i_t = i$; we set $g_s = j_1 + ... + j_s$ and $h_s = i_{g_{s-1}+1} + ... + i_{g_s}$ for $1 \leq s \leq k$:

$$\begin{tikzcd}
\mathcal{O}(k) \otimes (\overset{k}{\underset{s=1}{\otimes}} \mathcal{O}(j_s)) \otimes (\overset{j}{\underset{r=1}{\otimes}} \mathcal{O}(i_r)) \arrow[r, "\gamma\otimes id"'] \arrow[dd, "shuffle"'] & \mathcal{O}(j) \otimes (\overset{j}{\underset{r=1}{\otimes}} \mathcal{O}(i_r)) \arrow[d, "\gamma"'] \\
 & \mathcal{O}(i) \\
\mathcal{O}(k) \otimes (\overset{k}{\underset{s=1}{\otimes}} \mathcal{O}(j_s)) \otimes (\overset{j_s}{\underset{q=1}{\otimes}} \mathcal{O}(i_{g_{s-1}+q})) \arrow[r, "id\otimes(\otimes_s \gamma)"] & \mathcal{O}(k) \otimes (\overset{k}{\underset{s=1}{\otimes}} \mathcal{O}(h_s)) \arrow[u, "\gamma"']
\end{tikzcd} \tag{7.4.1}$$

In addition one is provided with a unit map $\eta : u \to \mathcal{O}(1)$ so that the diagrams

$$\begin{tikzcd}
\mathcal{O}(k) \otimes (u^{\overset{k}{\otimes}}) \arrow[r, "\cong"'] \arrow[d, "id\otimes\eta^k"'] & \mathcal{O}(k) \\
\mathcal{O}(k) \otimes \mathcal{O}(1)^{\overset{k}{\otimes}} \arrow[ru, "\gamma"'] &
\end{tikzcd}
\text{ and }
\begin{tikzcd}
u \otimes \mathcal{O}(j) \arrow[r, "\cong"'] \arrow[d, "\eta\otimes id"'] & \mathcal{O}(j) \\
\mathcal{O}(1) \otimes \mathcal{O}(j) \arrow[ru, "\gamma"'] &
\end{tikzcd}$$

commute.

A *commutative operad* is an operad as above provided with an action by the symmetric group Σ_k on each $\mathcal{O}(k)$ so that the diagrams

(7.4.2)
$$\begin{tikzcd}
\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \cdots \mathcal{O}(j_k) \arrow[r, "\sigma\otimes\sigma^{-1}"'] \arrow[d, "\gamma"'] & \mathcal{O}(k) \otimes \mathcal{O}(j_{\sigma(1)}) \cdots \mathcal{O}(j_{\sigma(k)}) \arrow[d, "\gamma"'] \\
\mathcal{O}(j) \arrow[r, "\sigma(j_{\sigma(1)},\cdots,j_{\sigma(k)})"] & \mathcal{O}(j)
\end{tikzcd}$$

and

(7.4.3)
$$\begin{tikzcd}
\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \cdots \mathcal{O}(j_k) \arrow[r, "id\otimes\tau_1\otimes\cdots\otimes\tau_k"'] \arrow[d, "\gamma"'] & \mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \cdots \mathcal{O}(j_k) \arrow[d, "\gamma"'] \\
\mathcal{O}(j) \arrow[r, "\tau_1\oplus\cdots\oplus\tau_k"] & \mathcal{O}(j)
\end{tikzcd}$$

commute. Here $\sigma\varepsilon\Sigma_k$, $\tau_s\varepsilon\Sigma_{j_s}$, the permutation $\sigma(j_{\sigma(1)}, ..., j_{\sigma(k)})$ permutes the k-blocks of letters of length $j_{\sigma(1)} \cdots j_{\sigma(k)}$ as σ permutes k letters and $\tau_1 \oplus \cdots \tau_k\varepsilon\Sigma_j$ is the block sum.

For the remaining statements, we will assume that the category $Presh(\mathfrak{S})$ has a model structure, so that it makes sense to consider objects that are *acyclic*, i.e. weakly-equivalent to the unit object u. An operad is an A^{∞}-operad (or *acyclic* operad) if *each* $\mathcal{O}(k)$ *is acyclic.* It is an E^{∞}-operad, if in addition, it is commutative and *the given action of* Σ_k *on* $\mathcal{O}(k)$ *is free.*

Remark 7.10. The free abelian group functor $\mathbb{Z} : (Simpl.Presh(\mathfrak{S})) \to (Simpl.ab.Presh(\mathfrak{S}))$ will provide a means of constructing operads in the category of simplicial abelian presheaves by starting with an operad of simplicial presheaves. This has the following property: given two simplicial presheaves P and P', there exists a natural map $\mathbb{Z}(P) \otimes \mathbb{Z}(P') \to \mathbb{Z}(P \times P')$. i.e. the functor $\mathbb{Z} : Simpl.Presh(\mathfrak{S}) \to Simpl.Ab.Presh(\mathfrak{S})$ is compatible with the tensor structures.

Definition 7.11. An algebra $\mathcal{A}$ over an operad $\mathcal{O}$ is an object in $Presh(\mathfrak{S})$ provided with maps $\theta : \mathcal{O}(j) \otimes \mathcal{A}^{\overset{j}{\otimes}} \to \mathcal{A}$ for all $j \geq 0$ that are associative and unital in the sense that the following diagrams commute:

(7.4.4)
$$\begin{tikzcd}
\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes ... \otimes \mathcal{O}(j_k) \otimes \mathcal{A}^{\overset{j}{\otimes}} \arrow[r, "\gamma\otimes id"] \arrow[dd, "shuffle"'] & \mathcal{O}(j) \otimes \mathcal{A}^{\overset{j}{\otimes}} \arrow[d, "\theta"'] \\
& A \\
\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes A^{\overset{j_1}{\otimes}} \otimes ... \otimes \mathcal{O}(j_k) \otimes \mathcal{A}^{\overset{j_k}{\otimes}} \arrow[r, "id\otimes\theta^k"] & \mathcal{O}(k) \otimes \mathcal{A}^{\overset{k}{\otimes}} \arrow[u, "\theta"']
\end{tikzcd}$$

and

(7.4.5)
$$\begin{tikzcd}
u \otimes \mathcal{A} \arrow[r, "\simeq"'] \arrow[d, "\eta\otimes id"'] & \mathcal{A} \\
\mathcal{O}(1) \otimes \mathcal{A} \arrow[ru, "\theta"'] &
\end{tikzcd}$$

If the operad is A^∞, we will refer to the algebra $\mathcal{A}$ as an A^∞-algebra. If $\mathcal{A}$ is an algebra over an operad $\mathcal{O}$ as above one defines a left $\mathcal{A}$-module M to be an object in $Presh(\mathfrak{S})$ provided with maps $\lambda : \mathcal{O}(j) \otimes \mathcal{A}^{j-1} \otimes M \to M$ satisfying similar associativity and unital conditions. Right-modules are defined similarly.

A commutative algebra over a commutative operad $\mathcal{O}$ is an A^∞ algebra over the operad $\mathcal{O}$ so that the following diagrams commute:

$$\begin{tikzcd}
\mathcal{O}(j) \otimes \mathcal{A}^{\overset{j}{\otimes}} \arrow[rr, "\sigma\otimes\sigma^{-1}"'] \arrow[dr, "\theta"'] & & \mathcal{O}(j) \otimes \mathcal{A}^{\overset{j}{\otimes}} \arrow[dl, "\theta"] \\
& \mathcal{A} &
\end{tikzcd}$$

If, in addition, the operad is E^∞, we will refer to the algebra $\mathcal{A}$ as an E^∞-algebra.

One may now observe the following. For each integer $n \geq 0$, let $u[\Sigma_n] = \underset{\Sigma_n}{\sqcup} u$ denote the sum of u indexed by the symmetric group Σ_n. One may define the structure of a monoid on $u[\Sigma_n]$ as follows:

let u_g denote the copy of u indexed by $g\varepsilon\Sigma_n$. Now we map $u_g \otimes u_h$ to $u_{g.h}$ by the given map $\mu : u \otimes u \to u$.

If $\mathcal{O}$ is a commutative operad in $Presh(\mathfrak{S})$, one may now observe that each $\mathcal{O}(k)$ is a right-module over the monoid $u[\Sigma_k]$. (Observe that $\mathcal{O}(k) \otimes u[\Sigma_k] \cong \underset{g\varepsilon\Sigma_k}{\oplus} \mathcal{O}(k) \otimes u_g$. We map $\mathcal{O}(k) \otimes u_g$ to $\mathcal{O}(k) \otimes u$ by the map $g \otimes id$. Now apply the given map $\mathcal{O}(k) \otimes u \to \mathcal{O}(k)$.)

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