SEMI-TOPOLOGICAL K -HOMOLOGY AND THOMASON'S THEOREM

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ABSTRACT. In this paper, we introduce the “semi-topological K -homology” of complex varieties, a theory related to semi-topological K -theory much as connective topological K -homology is related to connective topological K -theory. Our main theorem is that the semi-topological K -homology of a smooth, quasi-projective complex variety Y coincides with the connective topological K -homology of the associated analytic space Y^{an} . From this result, we deduce a pair of results relating semi-topological K -theory with connective topological K -theory. In particular, we prove that the “Bott inverted” semi-topological K -theory of a smooth, projective complex variety X coincides with the topological K -theory of X^{an} . In combination with a result of Friedlander and the author [12, 3.8], this gives a new proof, in the special case of smooth, projective complex varieties, of Thomason’s celebrated theorem that “Bott inverted” algebraic K -theory with $\mathbb{Z}/n$ coefficients coincides with topological K -theory with $\mathbb{Z}/n$ coefficients.

1. INTRODUCTION

In a series of papers ([11], [12], [13]), the author and Eric Friedlander have developed a theory for quasi-projective, complex varieties which sits in a sense half-way between algebraic K -theory and topological K -theory. The theory, written $K_*^{\text{semi}}(X)$, is called “semi-topological K -theory” and is equivalent (at least for weakly normal, projective varieties) to the “holomorphic K -theory” defined and studied by R. Cohen and P. Lima-Filho (cf. [9]). Numerous interesting properties have been established about semi-topological K -theory, many of which provide important foundational results in the subject. One significant such result is that the algebraic and semi-topological K -theories with coefficients in $\mathbb{Z}/n$, for $n > 0$, of a projective, complex variety coincide [12, 3.8].

The goal of this paper is to prove two important results which relate the semi-topological K -theory of a smooth, projective complex variety X with the topological K -theory of the associated smooth closed manifold X^{an} . Namely, we show that upon inverting the Bott element $\beta \in K_2^{\text{semi}}(X)$ in the ring $K_*^{\text{semi}}(X)$, there is a natural isomorphism of graded rings

$$(1) \quad K_*^{\text{semi}}(X)[1/\beta] \cong K_{\text{top}}^{-*}(X^{\text{an}}).$$

Additionally, we show that without inverting the Bott element, the natural map

$$(2) \quad K_q^{\text{semi}}(X) \longrightarrow K_{\text{top}}^{-q}(X^{\text{an}})$$

is a split surjection for q at least twice the (algebraic) dimension of X . The former of these two results was previously established in [12, 4.2], by building on the

rational version of this result asserted in [9, 3.5] and on Thomason’s seminal result concerning Bott-inverted algebraic K -theory with finite coefficients [35, 4.11]. However, the proof given here relies on neither of these sources.

The equivalence (1) and split surjection (2) are actually proven at the level of spectra, in a suitable sense, and thus also hold after taking coefficients in $\mathbb{Z}/n$. In light of the aforementioned result of the Friedlander and the author [12, 3.8] that the algebraic and semi-topological K -theory of a projective complex variety coincide when coefficients in $\mathbb{Z}/n$ are taken, we deduce immediately from (1) and (2) their algebraic analogues: The natural maps induce an isomorphism of graded rings

$$(3) \quad K_*^{\text{alg}}(X; \mathbb{Z}/n)[1/\beta] \cong K_{\text{top}}^{-*}(X^{\text{an}}; \mathbb{Z}/n)$$

and a split surjection

$$(4) \quad K_q^{\text{alg}}(X; \mathbb{Z}/n) \longrightarrow K_{\text{top}}^{-q}(X^{\text{an}}; \mathbb{Z}/n),$$

for $n > 0$ and $q \geq 2 \dim(X)$, whenever X is a smooth, projective complex variety. The isomorphism (3) recovers Thomason’s result [35, 4.11] and the split surjection (4) recovers one of the two main results of [14], both in the special case of smooth, projective complex varieties. Note, however, that the techniques employed here are radically different and arguably more elementary than those used in the original proofs. For example, Thomason’s original result [36] depends on complicated constructions involving the cohomology of spectrum-valued presheaves, and the result of [14] relies on the extremely difficult motivic Atiyah-Hirzebruch spectral sequence (cf. [3], [10], [22]). By contrast, the central point behind the proofs of (1) and (2), which lead to (3) and (4), is a geometric argument establishing what we term the Fundamental Comparison Theorem (cf. Theorem 5.1).

To accomplish the main goals of this paper, we need essentially three major ingredients. The first is the development of bivariant versions of semi-topological K -theory and connective topological K -theory, together with a natural map between these theories. In particular, this includes the development of “semi-topological K -homology”, which plays a role analogous to the connective topological K -homology of CW complexes. The second ingredient is a rigorous development of the standard cohomology and homology operations for semi-topological K -theory which are compatible with the analogous operations for topological K -theory and topological K -homology. In particular, we need the semi-topological K -theory analogues of cap, cup and slant products. Finally, the third and most significant ingredient is the Fundamental Comparison Theorem (cf. Theorem 5.1), which asserts that the semi-topological K -homology of a smooth, projective complex variety is equivalent to the connective topological K -homology of the associated complex manifold. This result should be interpreted as a homological version of the semi-topological analogue of the Quillen-Lichtenbaum conjecture. However, semi-topological K -homology is, in a certain sense, already truncated in low degrees, so that unlike the assertion of the original Quillen-Lichtenbaum conjecture, the result holds in all degrees, not just sufficiently large ones.

These ingredients are developed in the first six sections of the paper. The seventh and final section combines them into proofs of the major results mentioned above. We also include an appendix providing basic background material for the homotopy-theoretic tools used in the body of the paper.

The development of semi-topological K -theory represents a long-term collaboration between the author and Eric Friedlander, and this paper would obviously not be possible without this past work. Beyond that, a good portion of this paper owes a tremendous amount to numerous discussions between the author and Friedlander. It is impossible for the author to list every detail of the current paper which is indebted to such conversations, since there are so many. (For example, the very nature of the definition of semi-topological K -homology was inspired by discussions which occurred over two years ago.) The author must content himself with a blanket acknowledgement and by expressing his profound gratitude to Friedlander for the uncountable ways in which he helped with this paper.

2. DEFINITION OF BIVARIANT K -THEORIES

In this section we define bivariant versions of semi-topological K -theory and topological K -theory. These definitions are more or less obvious generalizations of the definitions of semi-topological K -theory and topological K -theory given in [11].

Definition 2.1. *Let Y be a quasi-projective complex variety. Let G_Y^m be the functor which sends a scheme X to the collection of quotient objects of the form $\mathcal{O}_{X \times Y}^m \twoheadrightarrow M$ such that the following two conditions hold:*

1. *The support of M is finite over X and $\pi_* M$ is a locally free $\mathcal{O}_X$ -sheaf, where $\pi : X \times Y \rightarrow X$ is the canonical projection. We will refer to these conditions more briefly by saying M is finite and flat over X .*
2. *The induced map $\mathcal{O}_X^m \rightarrow \pi_* M$, defined as the composition of $\mathcal{O}_X^m \rightarrow \pi_*(\mathcal{O}_{X \times Y}^m) \rightarrow \pi_* M$, is also surjective.*

Recall that a quotient object is an isomorphism class of surjections, and when necessary we use the notation $[\mathcal{O}_{X \times Y}^m \twoheadrightarrow M]$ to signify that we are talking about the quotient object represented by the displayed surjection. The second condition in Definition 2.1 might seem peculiar at first. It turns out that ultimately it can be dropped without affecting the resulting bivariant theory up to equivalence. However, its inclusion simplifies the construction of certain functorial maps and also causes the functor G_Y^m to be representable by an honest variety, whereas without this condition one would need an ind-variety.

Lemma 2.2. *For any quasi-projective complex variety Y and integer $n \geq 0$, the functor G_Y^n is representable by a quasi-projective complex variety.*

Proof. Note that the condition on a quotient $[\mathcal{O}_{U \times Y}^n \twoheadrightarrow M]$ that the induced map $\mathcal{O}_U^n \twoheadrightarrow \pi_* M$ is surjective forces M to have rank at most n over U . Thus, the functor G_Y^n is a subfunctor of Grothendieck's functor $\coprod_{r=0}^n \text{Quot}_{\mathcal{O}_Y^n/Y/\text{Spec } \mathbb{C}}^r$, where $\text{Quot}_{\mathcal{O}_Y^n/Y/\text{Spec } \mathbb{C}}^r$ is the functor sending U to those quotients $[\mathcal{O}_{U \times Y}^n \twoheadrightarrow M]$ such that M is finite and flat over U and $\pi_* M$ is projective of rank r on U (cf. [18]). Let us write the functor $\coprod_{r \leq n} \text{Quot}_{\mathcal{O}_Y^n/Y/\text{Spec } \mathbb{C}}^r$ more briefly as Q_Y^n . As shown in [18], the functor Q_Y^n is represented by a quasi-projective complex variety, which we write here as $\mathcal{Q}_Y^n$.

To proceed further, we briefly recall the construction of $\mathcal{Q}_Y^n$. First, suppose $Y = X$ is actually a projective variety. Then one chooses a closed embedding $X \subset \mathbb{P}^m$, and for a fixed $\nu \gg 0$, the function on $\mathbb{C}$ -points

$$[\mathcal{O}_X^n \twoheadrightarrow M] \mapsto [\Gamma(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}^n(\nu)) \twoheadrightarrow \Gamma(X, \mathcal{O}_X^n(\nu)) \rightarrow \Gamma(X, M(\nu))]$$

is shown to define the desired closed embedding $\mathcal{Q}_X^n \subset \text{Grass}_e(\mathbb{C}^E)$, for some e and E with $\mathbb{C}^E \cong \Gamma(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}^n(\nu))$. (Here, $\text{Grass}_e(\mathbb{C}^E)$ denotes the projective variety parameterizing dimension e quotients of $\mathbb{C}^E$.) If $Y \subset X$ is an open subscheme of X , with X projective as above, then the condition defining the subfunctor $\mathcal{Q}_Y^n \subset \mathcal{Q}_X^n$ corresponds to an open condition on the variety $\mathcal{Q}_X^n$, so that $\mathcal{Q}_Y^n$ is represented by an open subscheme $\mathcal{Q}_Y^n$ of $\mathcal{Q}_X^n$.

We now show the subfunctor $G_Y^n \subset \mathcal{Q}_Y^n$ is defined by an open condition on the variety $\mathcal{Q}_Y^n$, thereby proving the lemma. Let $Y \subset X \subset \mathbb{P}^m$ be embeddings as above. Given a $\mathbb{C}$ -point $[q : \mathcal{O}_Y^n \twoheadrightarrow M]$ of $\mathcal{Q}_Y^n$, since the support of M is a finite set, there exists a homogeneous form $h \in \Gamma(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(\nu))$ such that the support of M misses the subvariety defined by h . The condition that $\mathbb{C}^n \rightarrow \pi_* M$ be surjective (where $\pi : Y \rightarrow \text{Spec } \mathbb{C}$ is the structure map) is equivalent to the assertion that the composite map

$$\Gamma(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}^n) \xrightarrow{\cdot h} \Gamma(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}^n(\nu)) \longrightarrow \Gamma(\mathbb{P}^m, M) \cong \Gamma(Y, M)$$

be surjective. This surjectivity amounts to the non-vanishing of one of a finite list of determinants, and thus gives a open condition in a neighborhood of $[q] \in \mathcal{Q}_Y^n$. Let $\mathcal{G}_Y^n \subset \mathcal{Q}_Y^n$ be the open subscheme defined locally by this condition. (The definition holds in the neighborhood of $[q]$ consisting of quotients whose supports miss the subvariety defined by h .)

It remains to show $\mathcal{G}_Y^n$ really does represent the desired functor. That is, suppose $U \rightarrow \mathcal{G}_Y^n \subset \mathcal{Q}_Y^n$ is a map of schemes with U of finite type over $\mathbb{C}$. Such a morphism corresponds to a quotient $[p : \mathcal{O}_{U \times Y}^n \twoheadrightarrow M]$, and we need to show $\mathcal{O}_U^n \rightarrow \pi_* M$ is surjective. It suffices to work locally on U so that we may assume U is the spectrum of a local ring essentially of finite type over $\mathbb{C}$. But since M is finite over U , it follows that the support of M contains only finitely many $\mathbb{C}$ -points. Therefore, there exists a homogeneous form h of degree ν whose associated subvariety misses the support of M . It follows that in the commutative diagram

$$\begin{array}{ccc} \Gamma(\mathbb{P}^m, \mathcal{O}_{U \times \mathbb{P}^m}^n(\nu)) & \longrightarrow & \Gamma(\mathbb{P}^m, M(\nu)) \\ \cdot h \uparrow & & \cdot h \uparrow \cong \\ \Gamma(\mathbb{P}^m, \mathcal{O}_{U \times \mathbb{P}^m}^n) & \longrightarrow & \Gamma(\mathbb{P}^m, M) \end{array}$$

the right-hand vertical map is an isomorphism. Moreover, the definition of $\mathcal{G}_Y^n$ implies that the composite map

$$\Gamma(\mathbb{P}^m, \mathcal{O}_{U \times \mathbb{P}^m}^n) \xrightarrow{\cdot h} \Gamma(\mathbb{P}^m, \mathcal{O}_{U \times \mathbb{P}^m}^n(\nu)) \longrightarrow \Gamma(\mathbb{P}^m, M(\nu))$$

of $\mathcal{O}_U$ -modules is surjective upon restriction to the closed point of U . It follows from Nakayama's Lemma that this composite map is surjective, and thus the diagram shows that $\mathcal{O}_U^n \rightarrow \pi_* M$ is surjective. $\square$

We write $\mathcal{G}_Y^n$ for the quasi-projective variety representing the functor G_Y^n . We let G_Y^∞ denote the direct limit of functors

$$G_Y^0 \rightarrow G_Y^1 \rightarrow \dots,$$

where the transition map $G_Y^n \rightarrow G_Y^{n+1}$ is defined using the projection $\mathcal{O}_Y^{n+1} \twoheadrightarrow \mathcal{O}_Y^n$ onto the first n standard basis elements. The natural transformations $G_Y^n \rightarrow G_Y^{n+1}$ correspond to closed immersions of schemes $\mathcal{G}_Y^n \rightarrow \mathcal{G}_Y^{n+1}$, and the functor G_Y^∞ is

represented by the ind-variety

$$\mathcal{G}_Y^\infty = \varinjlim_n \mathcal{G}_Y^n.$$

Lemma 2.3. *Let $f : Y \rightarrow Y'$ be a map of quasi-projective complex varieties. Then there is a natural transformation of functors defined at U by the function*

$$f_* : G_Y^n(U) \rightarrow G_{Y'}^n(U)$$

sending a quotient $[q : \mathcal{O}_{U \times Y}^n \twoheadrightarrow M]$ to $[\mathcal{O}_{U \times Y'}^n \twoheadrightarrow (id \times f)_ M]$. This natural transformation thus defines a morphism of varieties $\mathcal{G}_Y^n \rightarrow \mathcal{G}_{Y'}^n$.*

Proof. Condition (2) of Definition 2.1 ensures that given a surjection $\mathcal{O}_{U \times Y}^n \twoheadrightarrow M$, the induced map $\mathcal{O}_{U \times Y'}^n \rightarrow (id \times f)_* M$ is indeed a surjection. Thus the displayed function is well-defined. The required naturality is obvious. $\square$

Recall from [15] that given two quasi-projective complex varieties U and V , the collection of *continuous algebraic maps* from U to V , written $\text{Mor}(U, V)$, admits the natural structure of a topological space, which we write as $\mathcal{M}\text{or}(U, V)$. If U is weakly normal, the set $\text{Mor}(U, V)$ coincides with $\text{Hom}(U, V)$, the set of morphisms of varieties. More generally, $\text{Mor}(U, V)$ is isomorphic to $\text{Hom}(U^w, V)$, where $U^w \rightarrow U$ is the weak normalization of U . For any U and V , $\mathcal{M}\text{or}(U, V)$ is a compactly generated topological space having the homotopy type of a CW complex [15, 2.5]. More generally, if V is an infinite disjoint union of quasi-projective varieties (or even an ind-variety), we define $\mathcal{M}\text{or}(U, V)$ to be the space obtained by taking the evident direct limit in the category of compactly generated topological spaces.

In order to elucidate the topology of $\mathcal{M}\text{or}(U, V)$, we consider the special case where U and V are both projective complex varieties (and U is weakly normal). In this case, the collection of morphisms from U to V of a fixed degree d has the natural structure of the $\mathbb{C}$ -points of a quasi-projective variety (cf. [11, 4.1]), and thus the collection of all such morphisms is an infinite disjoint union of sets of $\mathbb{C}$ -points of quasi-projective varieties. In this case, the topology on $\mathcal{M}\text{or}(U, V)$ is given as the analytic topology.

Given quasi-projective varieties X and Y , we may thus consider the space $\mathcal{M}\text{or}(X, \mathcal{G}_Y^\infty)$, defined as the inductive limit (in the category of compactly generated spaces) of the spaces $\mathcal{M}\text{or}(X, \mathcal{G}_Y^n)$. We now observe that $\mathcal{M}\text{or}(X, \mathcal{G}_Y^\infty)$ admits the natural action by the classical linear isometries operad $\mathcal{L}$. (Recall that $\mathcal{L}(j)$ is the space of linear isometries from $(\mathbb{C}^\infty)^{\times j}$ into $\mathbb{C}^\infty$.) Given a linear isometry $\alpha : \mathbb{C}^m \hookrightarrow \mathbb{C}^n$ and a quotient $p : \mathcal{O}_{X \times Y}^m \twoheadrightarrow M$, we define $\alpha^*(p)$ to be the quotient given by the composition

$$\mathcal{O}_{X \times Y}^n \xrightarrow{\alpha^*} \mathcal{O}_{X \times Y}^m \xrightarrow{p} M$$

where α^* is map induced by the dual of α . One readily verifies that the pairing $(\alpha, [p]) \mapsto [\alpha^*(p)]$ stabilizes nicely to induce a continuous pairing

$$\mathcal{L}(1) \times \mathcal{M}\text{or}(X, \mathcal{G}_Y^\infty) \rightarrow \mathcal{M}\text{or}(X, \mathcal{G}_Y^\infty),$$

which we also write as $(\alpha, [p]) \mapsto [\alpha^*(p)]$. This map extends to give the family of pairings

$$\mathcal{L}(j) \times \mathcal{M}\text{or}(X, \mathcal{G}_Y^\infty)^{\times j} \rightarrow \mathcal{M}\text{or}(X, \mathcal{G}_Y^\infty), \quad j \geq 0,$$

by the formula

$$(\alpha_1, \dots, \alpha_j, [p_1], \dots, [p_j]) \mapsto [(\alpha_1^*(p_1), \dots, \alpha_n^*(p_n))^t].$$

It is easy to see that the required axioms hold so that the collection of these pairings endow $\mathcal{M}\text{or}(X, \mathcal{G}_Y^\infty)$ with the structure of an $\mathcal{L}$ -space (cf. [25, §1]).

Definition 2.4. Define $\mathcal{K}^{\text{semi}}(X, Y)$ to be the spectrum associated to the $\mathcal{L}$ -space $\mathcal{M}\text{or}(X, \mathcal{G}_Y^\infty)$, as defined in [25, §14].

To clarify the nature of $\mathcal{K}^{\text{semi}}(X, Y)$ somewhat, we remind the reader that the total space $\Omega^\infty \mathcal{K}^{\text{semi}}(X, Y)$ of the spectrum $\mathcal{K}^{\text{semi}}(X, Y)$ is the homotopy-theoretic group completion of the homotopy-commutative H -space $\mathcal{M}\text{or}(X, \mathcal{G}_Y^\infty)$, where latter is endowed with the structure of an H -space by the rule

$$[\mathcal{O}_{X \times Y}^\infty \twoheadrightarrow M] + [\mathcal{O}_{X \times Y}^\infty \twoheadrightarrow N] = [\mathcal{O}_{X \times Y}^\infty \xrightarrow{\theta} \mathcal{O}_{X \times Y}^\infty \oplus \mathcal{O}_{X \times Y}^\infty \twoheadrightarrow M \oplus N].$$

(Here, the isomorphism θ arises by choosing an linear isometric isomorphism $\theta : \mathbb{C}^\infty \oplus \mathbb{C}^\infty \cong \mathbb{C}^\infty$.) In other words, there is a map of homotopy-commutative H -spaces

$$\mathcal{M}\text{or}(X, \mathcal{G}_Y^\infty) \rightarrow \Omega^\infty \mathcal{K}^{\text{semi}}(X, Y)$$

such that (1) the induced map on π_0 gives the group completion of the abelian monoid $\pi_0 \mathcal{M}\text{or}(X, \mathcal{G}_Y^\infty)$ and (2) for any commutative ring of coefficients A , the induced map on homology

$$H_*(\mathcal{M}\text{or}(X, \mathcal{G}_Y^\infty), A) \rightarrow H_*(\Omega^\infty \mathcal{K}^{\text{semi}}(X, Y), A)$$

is given by inverting the action of $\pi_0 \mathcal{M}\text{or}(X, \mathcal{G}_Y^\infty)$ on the source.

Observe that if $X' \rightarrow X$ (resp., $Y \rightarrow Y'$) is a morphism of quasi-projective varieties, then the induced map $\mathcal{M}\text{or}(X, \mathcal{G}_Y^\infty) \rightarrow \mathcal{M}\text{or}(X', \mathcal{G}_Y^\infty)$ given by composition (resp., $\mathcal{M}\text{or}(X, \mathcal{G}_Y^\infty) \rightarrow \mathcal{M}\text{or}(X, \mathcal{G}_{Y'}^\infty)$ given in Lemma 2.3) is compatible with the action of the operad $\mathcal{L}$. Consequently, there are induced maps of spectra

$$\mathcal{K}^{\text{semi}}(X, Y) \rightarrow \mathcal{K}^{\text{semi}}(X', Y), \quad \mathcal{K}^{\text{semi}}(X, Y) \rightarrow \mathcal{K}^{\text{semi}}(X, Y').$$

The following result gives some sense of the significance of $\mathcal{K}^{\text{semi}}(X, Y)$. Let $K_0^{\text{alg}}(X, Y)$ denote the Grothendieck group of the exact category of coherent sheaves on $X \times Y$ which are finite and flat over X . For complex varieties X and Y , an element α of $K_0^{\text{alg}}(X, Y)$ is said to be *algebraically equivalent to zero* if there is a smooth, connected complex curve C with $\mathbb{C}$ -points c_0, c_1 and a class $\gamma \in K_0^{\text{alg}}(X \times C, Y)$ such that $\iota_0^*(\gamma) = 0$ and $\iota_1^*(\gamma) = \alpha$, where $\iota_\epsilon : X \rightarrow X \times C$ is the closed immersion sending x to (x, c_ϵ) , for $\epsilon = 0, 1$. One may readily verify that the set of elements algebraically equivalent to zero form a subgroup of $K_0^{\text{alg}}(X, Y)$ (cf. [16, §10.3]).

Proposition 2.5. *If X is a weakly normal, projective variety and Y is a quasi-projective variety, then the group $K_0^{\text{semi}}(X, Y)$ is naturally isomorphic to the quotient of $K_0^{\text{alg}}(X, Y)$ by the subgroup of elements algebraically equivalent to zero.*

Proof. The group $K_0^{\text{semi}}(X, Y)$ is isomorphic to the group completion of the abelian monoid $\pi_0 \mathcal{M}\text{or}(X, \mathcal{G}_Y^\infty)$ and is thus generated by equivalence classes of quotients of the form $[\mathcal{O}_{X \times Y}^\infty \twoheadrightarrow M]$. As argued in the proof of [11, 2.10], the class of such a quotient depends only on the isomorphism class of the coherent sheaf M and moreover the monoid structure on $\pi_0 \mathcal{M}\text{or}(X, \mathcal{G}_Y^\infty)$ (induced by the H -space structure on $\mathcal{M}\text{or}(X, \mathcal{G}_Y^\infty)$) is given by direct sum of coherent sheaves. Thus $K_0^{\text{semi}}(X, Y)$ is a quotient of the group completion of the abelian monoid $\mathcal{N}$ of isomorphism classes of coherent sheaves M on $X \times Y$ which are finite and flat over X and

generated by global sections. The proof of [11, 2.12] applies to show that the equivalence relation on $\mathcal{N}$ is given by algebraic equivalence and that every element of $K_0^{\text{alg}}(X, Y)/(\text{algebraic equivalence})$ is a difference of the classes of coherent sheaves generated by global sections. The result follows. $\square$

We now define a version of bivariant, connective topological K -theory. For any pointed CW complex T , we let $\mathcal{C}_0(T)$ denote the non-unital topological ring of all continuous complex valued functions which vanish at the base-point of T . The topology on $\mathcal{C}_0(T)$ is given by $\mathcal{C}_0(T) = \underline{\text{Maps}}(T, \mathbb{C})$, where in general $\underline{\text{Maps}}(-, -)$ denotes the space of pointed continuous maps (where the base point of $\mathbb{C}$ is the origin) endowed with the compactly generated compact-open topology. There is a natural involution on $\mathcal{C}_0(T)$ defined by complex conjugation, and when T is compact $\mathcal{C}_0(T)$ is a C^* -algebra. For any $n \geq 0$, we let $\text{Mat}_n(\mathcal{C}_0(T))$ denote the topological ring of n -by- n matrices equipped with the involution given by taking conjugate transposes of matrices. Equivalently, $\text{Mat}_n(\mathcal{C}_0(T))$ can be identified with the space $\underline{\text{Maps}}(T, \text{Mat}_n(\mathbb{C}))$, where $\text{Mat}_n(\mathbb{C})$ is the evident topological ring with involution. By $\text{Mat}_\infty(\mathcal{C}_0(T))$, we mean the direct limit (in the category of compactly generated spaces) of the non-unital topological rings $\text{Mat}_n(\mathcal{C}_0(T))$, $n \geq 0$, where the inclusion $\text{Mat}_n(\mathcal{C}_0(T)) \subset \text{Mat}_{n+1}(\mathcal{C}_0(T))$ is defined by adding a row and a column of all zeroes.

Given a pair of pointed CW complexes W and V , we write $\text{Hom}_*(\mathcal{C}_0(V), \text{Mat}_n(\mathcal{C}_0(W)))$ for the space of continuous $\mathbb{C}$ -algebra homomorphisms which preserve the involution – we refer to elements of $\text{Hom}_*(\mathcal{C}_0(V), \text{Mat}_n(\mathcal{C}_0(W)))$ more briefly as **-maps*. The topology on $\text{Hom}_*(\mathcal{C}_0(V), \text{Mat}_n(\mathcal{C}_0(W)))$ is given as a subspace of $\underline{\text{Maps}}(\mathcal{C}_0(V), \text{Mat}_n(\mathcal{C}_0(W)))$. Note that by adjointness, we have the natural homeomorphism

$$\text{Hom}_*(\mathcal{C}_0(V), \text{Mat}_n(\mathcal{C}_0(W))) \cong \underline{\text{Maps}}(W, \text{Hom}_*(\mathcal{C}_0(V), \text{Mat}_n(\mathbb{C}))).$$

The space $\text{Hom}_*(\mathcal{C}_0(V), \text{Mat}_\infty(\mathcal{C}_0(W)))$ is defined as the evident direct limit:

$$\text{Hom}_*(\mathcal{C}_0(V), \text{Mat}_\infty(\mathcal{C}_0(W))) = \varinjlim_n \text{Hom}_*(\mathcal{C}_0(V), \text{Mat}_n(\mathcal{C}_0(W))).$$

Since CW complexes are paracompact, a standard argument shows that an element of $\text{Hom}_*(\mathcal{C}_0(V), \text{Mat}_n(\mathbb{C}))$ necessarily factors as a surjection $\mathcal{C}_0(V) \rightarrow \mathbb{C} \times \cdots \times \mathbb{C}$, given by evaluation of functions at a finite number of points of V , followed by a (non-unital) C^* -algebra map of the form $\mathbb{C} \times \cdots \times \mathbb{C} \rightarrow \text{Mat}_n(\mathbb{C})$. Thus, a point in $\text{Hom}_*(\mathcal{C}_0(V), \text{Mat}_n(\mathbb{C}))$ is given by a finite (unordered) list $v_1, \dots, v_k$ of points of V together with a list of pairwise orthogonal subspaces $X_1, \dots, X_k$ of $\mathbb{C}^n$. Similarly, a point in $\text{Hom}_*(\mathcal{C}_0(V), \text{Mat}_\infty(\mathbb{C}))$ consists of an unordered finite list $v_1, \dots, v_k$ of points in V together with a list $X_1, \dots, X_k$ of finite dimensional, pairwise orthogonal subspaces of $\mathbb{C}^\infty$. These descriptions of the spaces $\text{Hom}_*(\mathcal{C}_0(V), \text{Mat}_n(\mathbb{C}))$ and $\text{Hom}_*(\mathcal{C}_0(V), \text{Mat}_\infty(\mathbb{C}))$ illustrate the connection between our constructions and those of Segal [31].

In particular, we have the following homeomorphisms which realize the spaces above as direct limits of mapping spaces of C^* -algebras:

$$\begin{aligned} \text{Hom}_*(\mathcal{C}_0(V), \text{Mat}_n(\mathbb{C})) &\cong \varinjlim_D \text{Hom}_*(\mathcal{C}_0(D), \text{Mat}_n(\mathbb{C})) \\ \text{Hom}_*(\mathcal{C}_0(V), \text{Mat}_\infty(\mathbb{C})) &\cong \varinjlim_D \text{Hom}_*(\mathcal{C}_0(D), \text{Mat}_\infty(\mathbb{C})) \end{aligned}$$

where D ranges over all compact pointed subspaces of V . (The spaces in the bottom direct limit are mapping spaces of C^* -algebras too, since a C^* -algebra map of the

form $\mathcal{C}_0(D) \rightarrow \text{Mat}_\infty(\mathbb{C})$ necessarily factors through $\text{Mat}_n(\mathbb{C})$ for $n \gg 0$.) More generally, if W is compact, we have the following isomorphisms involving direct limits of mappings spaces of C^* -algebras:

$$\begin{aligned} \text{Hom}_*(\mathcal{C}_0(V), \text{Mat}_n(\mathcal{C}_0(W))) &\cong \varinjlim_D \text{Hom}_*(\mathcal{C}_0(D), \text{Mat}_n(\mathcal{C}_0(W))) \\ \text{Hom}_*(\mathcal{C}_0(V), \text{Mat}_\infty(\mathcal{C}_0(W))) &\cong \varinjlim_D \text{Hom}_*(\mathcal{C}_0(D), \text{Mat}_\infty(\mathcal{C}_0(W))). \end{aligned}$$

We now observe that there is an action of the linear isometries operad $\mathcal{L}$ on the space $\text{Hom}_*(\mathcal{C}_0(V), \text{Mat}_\infty(\mathcal{C}_0(W)))$, for any pair W, V of pointed CW complexes. Namely, given a linear isometry $\alpha : \mathbb{C}^m \hookrightarrow \mathbb{C}^n$ and a representation $\rho : \mathcal{C}_0(V) \rightarrow \text{Mat}_m(\mathcal{C}_0(W))$, we define ρ^α by $\rho^\alpha(f) = \alpha f \alpha^*$, where α^* is the conjugate transpose of α . One may readily verify that this pairing stabilizes and generalizes to give the family of continuous pairings

$$\mathcal{L}(j) \times \text{Hom}_*(\mathcal{C}_0(V), \text{Mat}_\infty(\mathcal{C}_0(W)))^{\times j} \rightarrow \text{Hom}_*(\mathcal{C}_0(V), \text{Mat}_\infty(\mathcal{C}_0(W))), \quad j \geq 0,$$

defined by sending $(\alpha_1, \dots, \alpha_j, \rho_1, \dots, \rho_j)$ to $(\rho_1^{\alpha_1} + \dots + \rho_j^{\alpha_j})$. This family of pairings satisfies the axioms required for $\text{Hom}_*(\mathcal{C}_0(V), \text{Mat}_\infty(\mathcal{C}_0(W)))$ to be an $\mathcal{L}$ -space (cf. [25, §1]).

The following definition is inspired by a construction of Segal [31]. Indeed, in the case $W = S^0$, the definition given here is essentially equivalent to the definition of K -homology given by Segal.

Definition 2.6. *Given pointed CW complexes W and V , $\mathbf{bu}^c(W, V)$ is the spectrum associated to the $\mathcal{L}$ -space $\text{Hom}_*(\mathcal{C}_0(V), \text{Mat}_\infty(\mathcal{C}_0(W)))$.*

If W (resp., V) is an unpointed space, then $\mathbf{bu}^c(W, V)$ is defined by replacing W with W_+ (resp., V with V_+), where in general T_+ denotes the addition of a disjoint base point to an unpointed space T .

As discussed further in the next section, if W is pointed and connected and V is not connected (for example, if $V = S^0$), the definition of $\mathbf{bu}^c(W, V)$ is rather inadequate, since the space $\text{Hom}_*(\mathcal{C}_0(V), \text{Mat}_\infty(\mathcal{C}_0(W)))$ coincides with $\text{Hom}_*(\mathcal{C}_0(V^\circ), \text{Mat}_\infty(\mathcal{C}_0(W)))$, where V° is the connected component of the base point of V . We will explain later how to accommodate this situation by stabilization – i.e., by taking suspensions of V .

If we specialize by setting $W = V = S^0$, we see that $\mathbf{bu}^c(S^0, S^0)$ is the spectrum associated to the $\mathcal{L}$ -space of non-unital C^* -algebra maps $\mathbb{C} \rightarrow \text{Mat}_\infty(\mathbb{C})$. Such a map is precisely given by specifying a finite dimensional subspace of $\mathbb{C}^\infty$. Thus, $\text{Hom}_*(\mathbb{C}, \text{Mat}_\infty(\mathbb{C}^\infty))$ is equivalent to the $\mathcal{L}$ -space Grass^{an} of [11], and consequently $\mathbf{bu}^c(S^0, S^0)$ is equivalent to the connective spectrum $\mathbf{bu}$ (cf. [11, 6.7]). Thus we have:

Proposition 2.7. *The spectrum $\mathbf{bu}^c(S^0, S^0)$ is homotopy equivalent to the connective spectrum $\mathbf{bu}$ representing connective topological K -theory.*

3. CATEGORICAL MODELS

In this section, we develop equivalent models for $\mathcal{K}^{\text{semi}}(X, Y)$ and $\mathbf{bu}^c(W, V)$ using topological categories. The point of introducing these categorical models is that they are more suitable for the purposes of defining cup product and other cohomological operations. The topological categories we define here lead to the definitions of certain Γ -spaces (cf. [30]) and hence spectra. The multiplicative

structures (that is, the cup product operations) will involve the notion of a Γ -ring (cf. [29], [23]), and also the notion of a *functor with smash product* (cf. [5]). Basic definitions and results concerning Γ -spaces, Γ -rings, and functors with smash product are recalled in the Appendix of this paper.

The design of these models owes itself to several sources, which we pause to mention here since these inspirations might not be apparent as we get into the details of the constructions. The first motivation is Segal's paper [31], in which he describes a version of the connective topological K -homology of CW complexes which is very similar to the spectrum $\mathbf{bu}(pt, V)$ presented below. The second source arises from an idea of Suslin [33], in which he describes a simple proof of Bott periodicity using Γ -spaces. (Suslin's idea is also related to a proof of Bott periodicity due to Harris [19].) The third and most significant source is the paper of Grayson and the author [17], in which a model of bivariant algebraic K -theory is given which is closely analogous to the models presented below.

3.1. Semi-topological model. In this subsection, we define the categorical model for the bivariant theory $\mathcal{K}^{\text{semi}}(X, Y)$. We begin by defining a topological category $\mathcal{A}^{\text{semi}}(X, Y)$, where X and Y are quasi-projective complex varieties. Regarding $\mathcal{A}^{\text{semi}}(X, Y)$ as a bi-functor from varieties to topological categories, we will show that it is covariantly functorial in Y and contravariantly functorial in X .

The space of objects of $\mathcal{A}^{\text{semi}}(X, Y)$ is defined to be $\coprod_n \mathcal{M}\text{or}(X, \mathcal{G}_Y^n)$. Thus, (when X is weakly normal) an object of $\mathcal{A}^{\text{semi}}(X, Y)$ is a quotient of the form $[\mathcal{O}_{X \times Y}^n \twoheadrightarrow N]$, with N finite and flat over X such that the induced map $\mathcal{O}_X^n \rightarrow \pi_* N$ is a surjection. We write $\mathbf{ob}_n \mathcal{A}^{\text{semi}}(X, Y)$ for $\mathcal{M}\text{or}(X, \mathcal{G}_Y^n)$, so that $\mathbf{ob} \mathcal{A}^{\text{semi}}(X, Y) = \coprod_{n \geq 0} \mathbf{ob}_n \mathcal{A}^{\text{semi}}(X, Y)$.

In order to define the morphisms in $\mathcal{A}^{\text{semi}}(X, Y)$, we first introduce some notation. Let $\underline{n}$ denote the set $\{1, \dots, n\}$, for $n \geq 0$. (So $\underline{0}$ is the empty set.) Given an injective (not necessarily order preserving) function $j : \underline{m} \hookrightarrow \underline{n}$, let $\tilde{j} : \mathcal{O}_{X \times Y}^m \hookrightarrow \mathcal{O}_{X \times Y}^n$ be given by the formula $\tilde{j}(e_i) = e_{j(i)}$, where $e_1, e_2, \dots$ denote the standard basis elements. Let $\tilde{j}^* : \mathcal{O}_{X \times Y}^n \hookrightarrow \mathcal{O}_{X \times Y}^m$ be given by the transpose of the matrix defining $\tilde{j}$. Observe that $\tilde{j}^* \circ \tilde{j} = \text{id}$. A morphism from $[p : \mathcal{O}_{X \times Y}^m \twoheadrightarrow M]$ to $[q : \mathcal{O}_{X \times Y}^n \twoheadrightarrow N]$ is by definition an injective map $j : \underline{m} \hookrightarrow \underline{n}$ such that $[q] = [p \circ \tilde{j}^*]$. In other words, such an j determines a morphism from the quotient object $[p]$ to the quotient object $[q]$ if and only if the diagram

$$\begin{array}{ccc} \mathcal{O}_{X \times Y}^m & \xrightarrow{p} & M \\ \tilde{j}^* \uparrow & & \downarrow \cong \\ \mathcal{O}_{X \times Y}^n & \xrightarrow{q} & N \end{array}$$

can be completed with an isomorphism as indicated. Composition in $\mathcal{A}^{\text{semi}}(X, Y)$ is given by composition of injective functions in the evident manner. Observe that the target of a morphism is uniquely determined by the source object and the function j involved in its definition. We thus topologize the collection of all morphisms of $\mathcal{A}^{\text{semi}}(X, Y)$, written $\mathbf{mor} \mathcal{A}^{\text{semi}}(X, Y)$, by identifying it with

$$\coprod_{m, n} \mathbf{ob}_m \mathcal{A}^{\text{semi}}(X, Y) \times I(m, n),$$

where $I(m, n)$ is the discrete space of all injective functions from $\underline{m}$ to $\underline{n}$. One may readily verify that the source, target, composition, and identity maps of $\mathcal{A}^{\text{semi}}(X, Y)$ are continuous, and thus $\mathcal{A}^{\text{semi}}(X, Y)$ is a topological category.

Remark 3.1. *The topological category $\mathcal{A}^{\text{semi}}(X, Y)$ can be equivalently viewed as a functor from a certain discrete category I to Spaces, the category of compactly generated topological spaces. Namely, let I be the subcategory of the category of sets consisting of the objects $\underline{m}$, $m \geq 0$, with morphisms given by injective functions. Since $\text{mor} \mathcal{A}^{\text{semi}}(X, Y)$ is homeomorphic to $\coprod_{m, n} \text{ob}_m \mathcal{A}^{\text{semi}}(X, Y) \times I(m, n)$, we can interpret $\mathcal{A}^{\text{semi}}(X, Y)$ as the functor sending $\underline{m}$ to $\text{ob}_m \mathcal{A}^{\text{semi}}(X, Y)$ and a function $j : \underline{m} \rightarrow \underline{n}$ to the continuous map $\text{ob}_m \mathcal{A}^{\text{semi}}(X, Y) \rightarrow \text{ob}_n \mathcal{A}^{\text{semi}}(X, Y)$ defined by taking targets of morphisms in $\mathcal{A}^{\text{semi}}(X, Y)$. We use the notation $\underline{m} \mapsto \text{ob}_m \mathcal{A}^{\text{semi}}(X, Y)$ to denote this functor.*

The intuition behind the definition of $\mathcal{A}^{\text{semi}}(X, Y)$ is that the objects of this category represent isomorphism classes of coherent sheaves on $X \times Y$ which are finite and flat over X (and which are generated by their global sections) together with a finite list of generators. The morphisms and topologies on the objects of this category allow one to join via a path two different lists of generators for the same coherent sheaf. In particular, π_0 of the nerve of $\mathcal{A}^{\text{semi}}(X, Y)$ is easily seen to be a quotient of the set of isomorphism classes of such coherent sheaves. Momentarily, we will endow $\mathcal{A}^{\text{semi}}(X, Y)$ with the additional structure of a Γ -object, which will permit the construction of Grothendieck group of the category of such sheaves. This Grothendieck group is isomorphic to the quotient of $K_0^{\text{alg}}(X, Y)$ by algebraic equivalence and thus coincides with $\pi_0 \mathcal{K}^{\text{semi}}(X, Y)$.

It is convenient to extend the definition of $\mathcal{A}^{\text{semi}}(X, Y)$ to the case where Y is endowed with a $\mathbb{C}$ -rational base point y_0 . Pointed varieties will be used for purely formal reasons, and will in fact only be employed when the base point of Y forms a disjoint connected component isomorphic to $\text{Spec } \mathbb{C}$. For example, starting with an arbitrary variety Y , we will have need to consider Y_+ , defined as $Y \coprod \text{Spec } \mathbb{C}$, with $\text{Spec } \mathbb{C}$ serving as the base point. Additionally, we will need to interpret an object $n_+ = \{*, 1, \dots, n\}$ of Γ^{op} (see the Appendix) as the variety defined as the disjoint union of $n + 1$ copies of $\text{Spec } \mathbb{C}$, indexed by the elements of n_+ , with the copy corresponding to $*$ serving as the base point. Finally, and most generally, we will have need to consider the pointed variety $n_+ \wedge Y_+$, which is defined as $(Y \coprod \dots \coprod Y)_+$, with n copies of Y indexed by $1, \dots, n \in n_+$.

However, the definition of $\mathcal{A}(X, (Y, y_0))$ makes sense for an arbitrary pointed $\mathbb{C}$ -variety Y . Namely, observe that $\mathcal{A}^{\text{semi}}(X, y_0)$ is naturally a subcategory of $\mathcal{A}^{\text{semi}}(X, Y)$, and moreover there exist no morphisms in $\mathcal{A}^{\text{semi}}(X, Y)$, going in either direction, between an object in $\mathcal{A}^{\text{semi}}(X, y_0)$ and one not in $\mathcal{A}^{\text{semi}}(X, y_0)$. We define $\mathcal{A}^{\text{semi}}(X, (Y, y_0))$ to be the quotient category

$$\mathcal{A}^{\text{semi}}(X, (Y, y_0)) = \mathcal{A}^{\text{semi}}(X, Y) / \mathcal{A}^{\text{semi}}(X, y_0),$$

whose space of objects is given by the quotient space $\text{ob} \mathcal{A}^{\text{semi}}(X, Y) / \text{ob} \mathcal{A}^{\text{semi}}(X, y_0)$. The morphisms of $\mathcal{A}^{\text{semi}}(X, (Y, y_0))$ are identified with the space

$$\coprod_{m, n} \text{ob}_m \mathcal{A}^{\text{semi}}(X, Y) / \text{ob}_m \mathcal{A}^{\text{semi}}(X, y_0) \times I(m, n).$$

Observe that in the case where the base point of Y is a disjoint copy of $\text{Spec } \mathbb{C}$, we may canonically identify $\mathcal{A}^{\text{semi}}(X, (Y, y_0))$ with $\mathcal{A}^{\text{semi}}(X, Y - \{y_0\})$.

The topological category $\mathcal{A}^{\text{semi}}(X, (Y, y_0))$ is covariantly (and continuously) functorial for maps of pointed varieties $(Y, y_0) \rightarrow (Y', y'_0)$ and contravariantly (and continuously) functorial for maps of varieties $X' \rightarrow X$. The covariance is defined by pushing forward coherent sheaves along $X \times Y \rightarrow X \times Y'$. The contravariance is given by pulling back along $X' \times Y \rightarrow X \times Y$.

In particular, fixing varieties X and Y we can form a Γ -object in the category of topological categories via

$$n_+ \mapsto \mathcal{A}^{\text{semi}}(X, n_+ \wedge Y_+),$$

where, as indicated above, we interpret n_+ a pointed variety and define $n_+ \wedge Y_+$ in the obvious way. An object of $\mathcal{A}^{\text{semi}}(X, n_+ \wedge Y_+)$ is by definition a quotient of the form

$$[\mathcal{O}_{X \times Y}^m \amalg \cdots \amalg X \times Y \twoheadrightarrow M]$$

with M finite and flat over $X \amalg \cdots \amalg X$ (n copies of X) and such that $\mathcal{O}_X^m \amalg \cdots \amalg X \rightarrow \pi_* M$ is also surjective. Such a quotient defines an n -tuple of quotients

$$([p_1 : \mathcal{O}_{X \times Y}^m \twoheadrightarrow M_1], \dots, [p_n : \mathcal{O}_{X \times Y}^m \twoheadrightarrow M_n])$$

which are in “general position” in the sense that the map

$$(p_1, \dots, p_m)^t : \mathcal{O}_{X \times Y}^m \rightarrow \bigoplus_i M_i$$

is surjective. This point of view makes clear the connection between $\mathcal{A}^{\text{semi}}(X, Y)$ and the construction of algebraic K -theory presented in [17].

For any X and Y , the nerve of $\mathcal{A}^{\text{semi}}(X, Y)$ (as with any topological category) is a simplicial space, which we write as $d \mapsto \mathcal{N}_d \mathcal{A}^{\text{semi}}(X, Y)$. By $|\mathcal{A}^{\text{semi}}(X, Y)|$ we mean the geometric realization of this simplicial space. Using functoriality in Y , we obtain the Γ -space

$$n_+ \mapsto |\mathcal{A}^{\text{semi}}(X, n_+ \wedge Y_+)|,$$

and thus an associated spectrum $\mathbb{S} \mathcal{A}^{\text{semi}}(X, - \wedge Y_+)$. (See the Appendix for a precise definition of the spectrum associated to a Γ -space.) In Theorem 3.5 below, we prove this spectrum is equivalent to $\mathcal{K}^{\text{semi}}(X, Y)$.

The following re-interpretation of the nerve of $\mathcal{A}^{\text{semi}}(X, Y)$ will prove useful.

Lemma 3.2. *The geometric realization of the nerve of $\mathcal{A}^{\text{semi}}(X, Y)$ is equivalent to the homotopy colimit of the functor from I to Spaces associated to $\mathcal{A}^{\text{semi}}(X, Y)$ (cf. Remark 3.1).*

Proof. This follows from [7, XII.5.2], since the simplicial space associated to the functor $\mathcal{A}^{\text{semi}}(X, Y) : I \rightarrow \text{Spaces}$ coincides with the nerve of the topological category $\mathcal{A}^{\text{semi}}(X, Y)$. $\square$

Recall that the homotopy colimit coincides up to weak homotopy with the ordinary colimit provided the indexing category is directed. However, the category I is not directed, and the homotopy colimit provides the “correct” homotopy-theoretic replacement for the poorly behaved colimit. Thus, using Lemma 3.2 we see that the nerve of $\mathcal{A}^{\text{semi}}(X, Y)$ is, at an intuitive level, the “union” of all the spaces $\mathfrak{ob}_n \mathcal{A}(X, Y)$. We proceed to formalize this intuition, which leads to a comparison of the spectrum associated to $\mathcal{A}^{\text{semi}}(X, Y)$ with $\mathcal{K}^{\text{semi}}(X, Y)$.

To facilitate this comparison, we introduce an “infinite” variant of the category $\mathcal{A}^{\text{semi}}(X, Y)$. Namely, we define $\mathcal{A}_\infty^{\text{semi}}(X, Y)$ to be the topological category whose

space of objects is given by the direct limit (taken in the category of compactly generated topological spaces)

$$\mathbf{ob}_0 \mathcal{A}^{\text{semi}}(X, Y) \rightarrow \mathbf{ob}_1 \mathcal{A}^{\text{semi}}(X, Y) \rightarrow \mathbf{ob}_2 \mathcal{A}^{\text{semi}}(X, Y) \rightarrow \dots$$

The transition maps in this direct limit are defined by precomposition with the canonical surjection $\mathcal{O}_{X \times Y}^{k+1} \twoheadrightarrow \mathcal{O}_{X \times Y}^k$ onto the first k standard basis elements. Thus an object of $\mathcal{A}_{\infty}^{\text{semi}}(X, Y)$ is a quotient $\mathcal{O}_{X \times Y}^{\infty} \twoheadrightarrow M$ which factors through the canonical surjection $\mathcal{O}_{X \times Y}^{\infty} \twoheadrightarrow \mathcal{O}_{X \times Y}^n$ for $n \gg 0$ and such that the induced map $\mathcal{O}_X^{\infty} \rightarrow \pi_* M$ is also a surjection. We also write $\mathbf{ob}_{\infty} \mathcal{A}^{\text{semi}}(X, Y)$ for $\mathbf{ob} \mathcal{A}_{\infty}^{\text{semi}}(X, Y)$.

A morphism in $\mathcal{A}_{\infty}^{\text{semi}}(X, Y)$ from $p : \mathcal{O}_{X \times Y}^{\infty} \twoheadrightarrow M$ to $q : \mathcal{O}_{X \times Y}^{\infty} \twoheadrightarrow N$ is an injective function $j : \mathbb{N} \hookrightarrow \mathbb{N}$ (where $\mathbb{N} = \{1, 2, \dots\}$) such that $[q] = [\tilde{j}^* p]$, where as before $\tilde{j}$ is the evident infinite matrix associated to j and $\tilde{j}^*$ is its transpose. (It may seem unnatural to take the transpose of an infinite matrix, but observe that $\tilde{j}$ is both row and column finite. Moreover, a given object of $\mathcal{A}_{\infty}^{\text{semi}}(X, Y)$ involves only finitely many basis elements of $\mathbb{C}^{\infty}$.) The composition rule for $\mathcal{A}_{\infty}^{\text{semi}}(X, Y)$ is given as before by composing injective functions.

Finally, we also need a sort of union of the categories $\mathcal{A}^{\text{semi}}(X, Y)$ and $\mathcal{A}_{\infty}^{\text{semi}}(X, Y)$. Namely, let $\tilde{\mathcal{A}}^{\text{semi}}(X, Y)$ denote the topological category with objects defined by $\coprod_w \mathbf{ob}_w \mathcal{A}^{\text{semi}}(X, Y)$, where w ranges over the set $\{0, 1, \dots, \infty\}$. For $v, w \in \{0, 1, \dots, \infty\}$, let $I(v, w)$ denote the collection of injective functions from $\underline{v}$ to $\underline{w}$, where we identify $\underline{\infty}$ with $\mathbb{N}$. The space of morphisms of $\tilde{\mathcal{A}}^{\text{semi}}(X, Y)$ is given as

$$\coprod_{v, w \in \{0, 1, \dots, \infty\}} \mathbf{ob}_v \mathcal{A}^{\text{semi}}(X, Y) \times I(v, w),$$

and target map is defined using the transpose $\tilde{j}^*$ of the matrix $\tilde{j}$ associated to an injective function $j : \underline{v} \hookrightarrow \underline{w}$. Composition is given by composition of functions. It is evident that there are natural inclusions of topological categories

$$\mathcal{A}^{\text{semi}}(X, Y) \longrightarrow \tilde{\mathcal{A}}^{\text{semi}}(X, Y) \longleftarrow \mathcal{A}_{\infty}^{\text{semi}}(X, Y).$$

Lemma 3.3. *For any quasi-projective varieties X and Y , the functors*

$$\mathcal{A}^{\text{semi}}(X, Y) \longrightarrow \tilde{\mathcal{A}}^{\text{semi}}(X, Y) \longleftarrow \mathcal{A}_{\infty}^{\text{semi}}(X, Y)$$

induce weak equivalences on geometric realizations of nerves.

Proof. Just as for $\mathcal{A}^{\text{semi}}(X, Y)$, we can interpret the category $\mathcal{A}_{\infty}^{\text{semi}}(X, Y)$ as arising from a functor with target *Spaces*. Namely, let B denote the classifying category of the monoid of all injective functions $\mathbb{N} \hookrightarrow \mathbb{N}$ under composition. Thus, B has one object, ∞ , and $\text{End}_B(\infty)$ is the set of injective functions $\mathbb{N} \hookrightarrow \mathbb{N}$ with composition in B given by composition of functions. The category $\mathcal{A}_{\infty}^{\text{semi}}(X, Y)$ arises from the functor $\mathbf{ob}_{\infty} \mathcal{A}^{\text{semi}}(X, Y) : B \rightarrow \text{Spaces}$ sending the unique object ∞ to $\mathbf{ob}_{\infty} \mathcal{A}^{\text{semi}}(X, Y)$ and a morphism j to the continuous map $\mathbf{ob}_{\infty} \mathcal{A}^{\text{semi}}(X, Y) \rightarrow \mathbf{ob}_{\infty} \mathcal{A}^{\text{semi}}(X, Y)$ determined by the target map for the topological category $\mathcal{A}_{\infty}^{\text{semi}}(X, Y)$. Consequently, from [7, XII.5.2] we have

$$|\mathcal{A}_{\infty}^{\text{semi}}(X, Y)| \sim \text{hocolim}_B \mathbf{ob}_{\infty} \mathcal{A}^{\text{semi}}(X, Y).$$

Similarly, the category $\tilde{\mathcal{A}}^{\text{semi}}(X, Y)$ arises from the evident functor $\mathbf{ob}_{\tilde{I}} \tilde{\mathcal{A}}^{\text{semi}}(X, Y) : \tilde{I} \rightarrow \text{Spaces}$, where $\tilde{I}$ consists of objects $\underline{v}$, $v \in \{0, 1, \dots, \infty\}$ and the morphisms of $\tilde{I}$ are given by injective functions. As above, we have the weak equivalence.

$$|\tilde{\mathcal{A}}^{\text{semi}}(X, Y)| \sim \text{hocolim}_{\tilde{I}} \mathbf{ob}_{\tilde{I}} \tilde{\mathcal{A}}^{\text{semi}}(X, Y).$$

We now observe that the functor $\mathbf{ob}_\infty \mathcal{A}^{\text{semi}}(X, Y)$ is obtained by restriction along $B \subset \tilde{I}$ from the functor $\mathbf{ob}_- \mathcal{A}^{\text{semi}}(X, Y)$ and that the inclusion $B \subset \tilde{I}$ is right cofinal. (A functor is *right cofinal* if it satisfies the dual of the left cofinality condition of [7, XI.9].) Furthermore, the functor $\mathbf{ob}_- \mathcal{A}^{\text{semi}}(X, Y) : \tilde{I} \rightarrow \text{Spaces}$ is readily verified to be the topological left Kan extension of the functor $\mathbf{ob}_- \mathcal{A}^{\text{semi}}(X, Y) : I \rightarrow \text{Spaces}$ along the inclusion $I \subset \tilde{I}$. Therefore, as argued in the proof of [32, 2.3.10], the natural maps

$$\text{hocolim}_I \mathbf{ob}_- \mathcal{A}^{\text{semi}}(X, Y) \rightarrow \text{hocolim}_{\tilde{I}} \mathbf{ob}_- \mathcal{A}^{\text{semi}}(X, Y) \leftarrow \text{hocolim}_B \mathbf{ob}_\infty \mathcal{A}^{\text{semi}}(X, Y)$$

are weak equivalences. $\square$

Lemma 3.4. *For any quasi-projective varieties X and Y , the inclusion of the space of objects of $\mathcal{A}_\infty^{\text{semi}}(X, Y)$ (regarded as a constant simplicial space) into the nerve of this category induces a weak equivalence upon taking geometric realizations:*

$$\mathbf{ob}_\infty \mathcal{A}^{\text{semi}}(X, Y) \xrightarrow{\sim} |\mathcal{A}_\infty^{\text{semi}}(X, Y)|$$

Proof. Let $I(\infty, \infty)$ denote the monoid of injective functions from $\mathbb{N}$ to $\mathbb{N}$. Then the nerve of $\mathcal{A}_\infty^{\text{semi}}(X, Y)$ is given explicitly by

$$d \mapsto \mathbf{ob}_\infty \mathcal{A}_\infty^{\text{semi}}(X, Y) \times I(\infty, \infty)^{\times d}.$$

This nerve projects to the simplicial space

$$d \mapsto I(\infty, \infty)^{\times d},$$

which is the nerve of the category B defined in the proof of Lemma 3.3. This projection is a fibration with fiber $\mathbf{ob}_\infty \mathcal{A}_\infty^{\text{semi}}(X, Y)$, and thus it suffices to show B is contractible. This is proven in [17, Proof of 3.1] (and is easy). $\square$

Theorem 3.5. *For quasi-projective complex varieties X and Y , the spectrum associated to the Γ -space $n_+ \mapsto \mathcal{A}^{\text{semi}}(X, n_+ \wedge Y_+)$ is naturally weakly equivalent to the spectrum $\mathcal{K}^{\text{semi}}(X, Y)$.*

Proof. From the preceding two lemmas, we have the natural weak equivalences of Γ -spaces

$$|\mathcal{A}^{\text{semi}}(X, - \wedge Y_+)| \xrightarrow{\sim} |\tilde{\mathcal{A}}^{\text{semi}}(X, - \wedge Y_+)| \xleftarrow{\sim} \mathbf{ob}_\infty \mathcal{A}^{\text{semi}}(X, - \wedge Y_+),$$

and thus it suffices to show that the spectrum associated to the Γ -space

$$\mathbf{ob}_\infty \mathcal{A}^{\text{semi}}(X, - \wedge Y_+)$$

is naturally weakly equivalent to $\mathcal{K}^{\text{semi}}(X, Y)$. Note that $\mathbf{ob}_\infty \mathcal{A}^{\text{semi}}(X, n_+ \wedge Y_+)$ is the space of n -tuples of quotients

$$([q_i : \mathcal{O}_{X \times Y}^\infty \twoheadrightarrow \mathcal{O}_{X \times Y}^N \twoheadrightarrow M_i])_{i=1, \dots, n},$$

which are in “general position” – i.e., satisfy the condition that $\sum_i q_i : \mathcal{O}_{X \times Y}^\infty \rightarrow \bigoplus M_i$ be surjective. By mimicking the proof of [11, 6.8], the desired equivalence now follows from the “equivalence of infinite loop space machines” established in [26]. $\square$

We now redefine $\mathcal{K}^{\text{semi}}(X, Y)$ using our categorical definition.

Definition 3.6. Let X and Y be quasi-projective complex varieties. Define $\mathcal{K}^{\text{semi}}(X, Y)$ to be the spectrum associated to the Γ -space

$$n_+ \mapsto |\mathcal{A}^{\text{semi}}(X, n_+ \wedge Y_+)|.$$

As before, $K_p^{\text{semi}}(X, Y)$ denotes the p -th homotopy group of the spectrum $\mathcal{K}^{\text{semi}}(X, Y)$.

Lemma 3.7. For quasi-projective varieties X and Y , the Γ -space

$$n_+ \mapsto |\mathcal{A}^{\text{semi}}(X, n_+ \wedge Y)|$$

is special.

Proof. Since $|\mathcal{A}^{\text{semi}}(X, n_+ \wedge Y_+)|$ is homotopy equivalent to $\mathbf{ob}_\infty \mathcal{A}^{\text{semi}}(X, n_+ \wedge Y_+)$, for all $n \geq 0$, it suffice to show $\mathbf{ob}_\infty \mathcal{A}^{\text{semi}}(X, - \wedge Y_+)$ is a special Γ -space. The proof of [17, 2.2] carries over in the obvious manner. $\square$

Using Lemma 3.7 and [30, 1.4], it follows that the total space of the spectrum $\mathcal{K}^{\text{semi}}(X, Y)$ given by

$$\Omega^1 |n \mapsto |\mathcal{A}^{\text{semi}}(X, n_+ \wedge Y)||.$$

We will explore a complete theory of homological and cohomological operations for $\mathcal{K}^{\text{semi}}(X, Y)$ in Section 6. However, in the special case where $Y = \text{Spec } \mathbb{C}$, we may give an simple description of a pairing on the spectrum $\mathcal{K}^{\text{semi}}(X, \text{Spec } \mathbb{C})$ which endows it with the structure of a commutative ring spectrum. This pairing defines the cup product operation on $K_*^{\text{semi}}(X, \text{Spec } \mathbb{C})$, making it into a graded-commutative ring. To accomplish this, we enrich the structure of $|\mathcal{A}^{\text{semi}}(X, -)|$ to be that of a commutative Γ -ring (see the Appendix), and use the fact a commutative Γ -ring determines a commutative ring spectrum (cf. Lemma A.3). As explained more carefully in the Appendix, a Γ -ring is essentially a Γ -space equipped with a product operation and a unit, each of which is given by maps of Γ -spaces. In this situation, the “unit map” is the map of Γ -spaces $\mathbf{1} : S \rightarrow |\mathcal{A}^{\text{semi}}(X, -)|$, (where $S(n_+)$ is defined as the pointed space n_+) given by letting

$$\mathbf{1}_n : n_+ \rightarrow |\mathcal{A}^{\text{semi}}(X, n_+)|,$$

be the map which sends i to the point represented by tuple of quotients

$$(0, \dots, 0, [\mathcal{O}_X \xrightarrow{\cong} \mathcal{O}_X], 0, \dots, 0)$$

(with the non-zero quotient appearing in the i -th position). The “multiplication” map $\mu : |\mathcal{A}^{\text{semi}}(X, -)| \times |\mathcal{A}^{\text{semi}}(X, -)| \rightarrow |\mathcal{A}^{\text{semi}}(X, -)|$ is best defined in slightly greater generality by introducing a natural pairing of topological categories

$$\otimes : \mathcal{A}^{\text{semi}}(X, Y_+) \times \mathcal{A}^{\text{semi}}(X, Z_+) \rightarrow \mathcal{A}^{\text{semi}}(X, (Y \times Z)_+)$$

for arbitrary varieties Y and Z . The pairing $\otimes$ sends a pair of quotients $([\mathcal{O}_{X \times Y}^k \rightarrow M], [\mathcal{O}_{X \times Z}^l \rightarrow N])$ to the quotient associated to the composition

$$\mathcal{O}_{X \times Y \times Z}^{kl} \cong \mathcal{O}_{X \times Y \times Z}^k \otimes_{\mathcal{O}_{X \times Y \times Z}} \mathcal{O}_{X \times Y \times Z}^l \twoheadrightarrow \pi_{X \times Y}^* M \otimes_{\mathcal{O}_{X \times Y \times Z}} \pi_{X \times Z}^* N.$$

Here, the isomorphism is given by “lexicographic ordering” of the bases – that is, $e_i \otimes e_j$ is identified with $e_{l(i-1)+j}$ – and the maps $\pi_{X \times Y}$ and $\pi_{X \times Z}$ are the evident projections. The pairing $\otimes$ is given on morphisms by identifying $\underline{k} \times \underline{l}$ with $\underline{kl}$ also by use of lexicographic ordering. One may readily verify that $\otimes$ defines a natural pairing of topological categories and, moreover, it is an associative pairing in the sense that the two possible compositions

$$\mathcal{A}^{\text{semi}}(X, Y_+) \times \mathcal{A}^{\text{semi}}(X, Z_+) \times \mathcal{A}^{\text{semi}}(X, W_+) \rightarrow \mathcal{A}^{\text{semi}}(X, (Y \times Z \times W)_+)$$

coincide. We thus obtain a natural associative pairing of spaces

$$\otimes : |\mathcal{A}^{\text{semi}}(X, Y_+)| \times |\mathcal{A}^{\text{semi}}(X, Z_+)| \rightarrow |\mathcal{A}^{\text{semi}}(X, (Y \times Z)_+)|$$

by taking geometric realizations of nerves. Specializing to $Y = \underline{m}$ and $Z = \underline{n}$, for $n, m \geq 0$, we obtain the desired pairing μ . The various unital, associativity, and commutativity axioms are readily verified, showing that $\mathbf{1}$ and μ endow $|\mathcal{A}^{\text{semi}}(X, -)|$ with the structure of a commutative Γ -ring. Using Lemma A.3, we immediately obtain:

Proposition 3.8. *For any quasi-projective variety X , the spectrum $\mathcal{K}^{\text{semi}}(X, \text{Spec } \mathbb{C})$ has the natural structure of a commutative ring spectrum.*

3.2. Topological models. We now turn to the “purely topological” analogue of $\mathcal{A}^{\text{semi}}(-, -)$. We work in category of spaces admitting the structure of CW complexes (referred to as “CW complexes”, for short). Our primary interest will be spaces obtained by the analytic realizations of complex varieties. Such spaces are CW complexes by [20].

Let (W, w_0) and (V, v_0) be pointed CW complexes. We define a topological category $\mathcal{A}^{\text{top}}(W, V)$ (the base points are implicit in the notation) as follows. The space of objects of $\mathcal{A}^{\text{top}}(W, V)$ is the disjoint union of spaces $\coprod_{n \geq 0} \mathbf{ob}_n \mathcal{A}^{\text{top}}(W, V)$ where $\mathbf{ob}_n \mathcal{A}^{\text{top}}(W, V) = \text{Hom}_*(\mathcal{C}_0(V), \text{Mat}_n(\mathcal{C}_0(W)))$, as defined in Section 2. A morphism of $\mathcal{A}^{\text{top}}(W, V)$ from $\rho : \mathcal{C}_0(V) \rightarrow \text{Mat}_m(\mathcal{C}_0(W))$ to $\eta : \mathcal{C}_0(V) \rightarrow \text{Mat}_n(\mathcal{C}_0(W))$ is given by an injective function $\underline{m} \hookrightarrow \underline{n}$ such that $\eta(f) = \tilde{j}\rho(f)\tilde{j}^*$, for all $f \in \mathcal{C}_0(V)$. The topology on $\mathbf{mor} \mathcal{A}^{\text{top}}(W, V)$ is given by identifying it with

$$\prod_{m, n} \mathbf{ob}_m \mathcal{A}^{\text{top}}(W, V) \times I(m, n),$$

just as for $\mathbf{mor} \mathcal{A}^{\text{semi}}(X, Y)$. Composition in $\mathcal{A}^{\text{top}}(W, V)$ is given by composition of injective functions as with $\mathcal{A}^{\text{semi}}$.

Often in the applications of $\mathcal{A}^{\text{top}}$ discussed in this paper, one or both of the space W and V are *unpointed*. We view unpointed spaces as forming a subcategory of pointed spaces by identifying an unpointed space X with X_+ , the pointed space obtained by adjoining a disjoint base point. Thus, if W is not pointed, then the notation $\mathcal{A}^{\text{top}}(W, V)$ is interpreted as $\mathcal{A}^{\text{top}}(W_+, V)$, and similarly if V is not pointed. In practice, this abuse of notation should not cause any confusion.

To clarify the motivation for the definition of $\mathcal{A}^{\text{top}}(W, V)$, suppose V and W are both unpointed, compact CW complex. Observe that $\mathcal{C}_0(V_+)$ and $\mathcal{C}_0(W_+)$ are identical to the C^* -algebras $\mathcal{C}(V)$ and $\mathcal{C}(W)$ (viewed as a non-unital rings), and an object of $\mathcal{A}^{\text{top}}(W, V)$ is a (non-unital) C^* -algebra map

$$\rho : \mathcal{C}(V) \rightarrow \text{Mat}_n(\mathcal{C}(W)).$$

Such a map naturally determines a finitely generated projective $\mathcal{C}(W)$ -module equipped with the structure of a $\mathcal{C}(V)$ - $\mathcal{C}(W)$ -bimodule. Namely, the image of 1 under ρ determines a unitary idempotent in $\text{Mat}_n(\mathcal{C}(W))$, and hence a projective $\mathcal{C}(W)$ -module defined by the summand $\text{im } \rho(1)$ of $\mathcal{C}(W)^n$. The action of $\mathcal{C}(V)$ on this module is induced by ρ in the evident manner:

$$s \cdot \rho(1)(v) = \rho(s)(v).$$

Thus an object of $\mathcal{A}^{\text{top}}(W, V)$ defines a $\mathcal{C}(V)$ - $\mathcal{C}(W)$ -bimodule P which is “finite and flat” (i.e., finitely generated and projective) as a $\mathcal{C}(W)$ -module together with a

surjection $\mathcal{C}(W)^n \twoheadrightarrow P$ of $\mathcal{C}(W)$ -modules. Thus, the category $\mathcal{A}^{top}(W, V)$ is closely related to the category $\mathcal{A}^{semi}(X, Y)$ when X and Y are affine varieties.

The construction $\mathcal{A}^{top}(W, V)$ is contravariant with respect to continuous pointed maps $(W', w'_0) \rightarrow (W, w_0)$ and covariant with respect to continuous pointed maps $(V', v'_0) \rightarrow (V, v_0)$, in the obvious ways. Namely, such maps induce $*$ -maps (i.e., continuous, involution preserving $\mathbb{C}$ -algebra homomorphisms) $\text{Mat}_n(\mathcal{C}_0(W')) \rightarrow \text{Mat}_n(\mathcal{C}_0(W))$ and $\mathcal{C}_0(V') \rightarrow \mathcal{C}_0(V)$, which in turn define continuous functors by taking compositions. In particular, we can define a Γ -object in the category of topological categories

$$n_+ \mapsto \mathcal{A}^{top}(W, n_+ \wedge V),$$

where n_+ is identified with the evident discrete pointed space. Taking geometric realizations of nerves, we obtain the Γ -space

$$n_+ \mapsto |\mathcal{A}^{top}(W, n_+ \wedge V)|$$

and thus an associated spectrum. As in Section 2, if V is not connected, then this construction is only reasonable when W is an unpointed CW complex. Indeed, taking W to be pointed and connected and V to be S^0 , we see that the category $\mathcal{A}^{top}(W, n_+)$ is essentially trivial and thus the Γ -space $n_+ \mapsto |\mathcal{A}^{top}(W, n_+)|$ is uninteresting. Eventually, we will extend this construction to define a reasonable spectrum for a connected, pointed CW complex W by replacing V with its suspensions.

In order to identify the spectrum associated to the Γ -space $n_+ \mapsto |\mathcal{A}^{top}(W, n_+ \wedge V)|$, we need to consider the analogue of $\mathcal{A}^{semi}_\infty$. Specifically, let $\mathcal{A}^{top}_\infty(W, V)$ denote the category whose space of objects is

$$\text{ob}_\infty \mathcal{A}^{top}(W, V) = \text{Hom}_*(\mathcal{C}_0(V), \text{Mat}_\infty(\mathcal{C}_0(W))).$$

A morphism in $\mathcal{A}^{top}_\infty(W, V)$ from $\rho : \mathcal{C}_0(V) \rightarrow \text{Mat}_\infty(\mathcal{C}_0(W))$ to $\eta : \mathcal{C}_0(V) \rightarrow \text{Mat}_\infty(\mathcal{C}_0(W))$ is an injection $j : \mathbb{N} \hookrightarrow \mathbb{N}$ such that $\eta(f) = \tilde{j} \cdot \rho(f) \tilde{j}^*$ for all $f \in \mathcal{C}_0(V)$. The space $\text{mor}_\infty \mathcal{A}^{top}(V, W)$ is topologized by identifying it with the space

$$\text{ob}_\infty \mathcal{A}^{top}(W, V) \times I(\infty, \infty)$$

where $I(\infty, \infty)$ is the discrete space of all injections $\mathbb{N} \hookrightarrow \mathbb{N}$. Composition in $\mathcal{A}^{top}_\infty(W, V)$ is given by composition of injective functions.

Similarly, we define $\tilde{\mathcal{A}}^{top}(W, V)$ in a manner analogous to $\tilde{\mathcal{A}}^{semi}$. We omit the details. Just as with $\mathcal{A}^{semi}$ there are natural continuous functors

$$\mathcal{A}^{top}(W, V) \longrightarrow \tilde{\mathcal{A}}^{top}(W, V) \longleftarrow \mathcal{A}^{top}_\infty(W, V)$$

defined in the evident manner.

The following two lemmas and subsequent theorem are the analogues of Lemmas 3.3, 3.4 and Theorem 3.5. Their proofs are virtually identical the those above and are thus omitted.

Lemma 3.9. *For any pointed CW complexes W and V , the natural functors of topological categories*

$$\mathcal{A}^{top}(W, V) \longrightarrow \tilde{\mathcal{A}}^{top}(W, V) \longleftarrow \mathcal{A}^{top}_\infty(W, V)$$

induce a weak equivalences on geometric realizations of nerves.

Lemma 3.10. *For any pointed CW complexes W and V , the inclusion of the space of objects of $\mathcal{A}_\infty^{top}(W, V)$ (regarded as a constant simplicial space) into the nerve of this category induces a weak equivalence*

$$\mathbf{ob}_\infty \mathcal{A}^{top}(W, V) \rightarrow |\mathcal{A}_\infty^{top}(W, V)|.$$

Theorem 3.11. *For any pointed CW complexes W and V , the spectrum associated to the Γ -space $n_+ \mapsto |\mathcal{A}(W, n_+ \wedge V)|$ is naturally weakly homotopy equivalent to $\mathbf{bu}^c(W, V)$.*

In light of Theorem 3.11, we are justified in redefining $\mathbf{bu}^c(W, V)$.

Definition 3.12. *For pointed CW complexes W and V , let $\mathbf{bu}^c(W, V)$ denote the spectrum associated to the Γ -space*

$$n_+ \mapsto |\mathcal{A}^{top}(W, n_+ \wedge V)|.$$

We write $\mathbf{bu}_q^c(W, V)$ for the q -th homotopy group of $\mathbf{bu}^c(W, V)$.

Lemma 3.13. *For pointed CW complexes W and V , the Γ -space $n_+ \mapsto |\mathcal{A}^{top}(W, n_+ \wedge V)|$ is special. Thus, the total space of the spectrum $\mathbf{bu}^c(W, V)$ may be identified with*

$$\Omega^1 |n \mapsto |\mathcal{A}^{top}(W, n_+ \wedge V)||.$$

Proof. As in the proof of Lemma 3.7, it suffices to show $\mathbf{ob}_\infty \mathcal{A}^{\text{semi}}(W, - \wedge V)$ is a special Γ -space, and the proof of [17, 2.2] suffices for this. $\square$

The superscript c in the notation $\mathbf{bu}^c(W, V)$ signifies that this spectrum is connective, for any choice of W, V . In particular, even if we avoid the difficulty posed when W is pointed and connected by assuming W is unpointed, we see that $\mathbf{bu}^c(W_+, S^0)$ cannot coincide with the cohomology theory on unpointed spaces defined by the connective spectrum $\mathbf{bu}$. However, we show in the next section that $\mathbf{bu}^c(W_+, S^0)$ is the connective cover of the spectrum $\mathbf{Maps}(W_+, \mathbf{bu})$.

Consider $|\mathcal{A}^{top}(S^0, -)|$ as a functor from pointed CW complexes to spaces. Observe there is a natural transformation from the identity functor to $|\mathcal{A}^{top}(W, -)|$

$$(5) \quad V \rightarrow |\mathcal{A}^{top}(S^0, V)|$$

induced by sending $v \in V$ to the object $\rho : \mathcal{C}_0(V) \rightarrow \mathbb{C} = \text{Mat}_1(\mathcal{C}_0(S^0))$ of $\mathcal{A}^{top}(S^0, V)$ defined by evaluation at v . Additionally, there is a natural pairing of spaces

$$(6) \quad |\mathcal{A}^{top}(S^0, U)| \wedge |\mathcal{A}^{top}(S^0, V)| \rightarrow |\mathcal{A}^{top}(S^0, U \wedge V)|$$

induced by the pairing of topological categories

$$(7) \quad \otimes : \mathcal{A}^{top}(S^0, U) \times \mathcal{A}^{top}(S^0, V) \rightarrow \mathcal{A}^{top}(S^0, U \wedge V)$$

defined as follows. Given objects $\rho : \mathcal{C}_0(U) \rightarrow \text{Mat}_m(\mathbb{C})$ and $\eta : \mathcal{C}_0(V) \rightarrow \text{Mat}_n(\mathbb{C})$, we have that ρ and η factor as $\mathcal{C}_0(U) \rightarrow \mathbb{C}^k \rightarrow \text{Mat}_m(\mathbb{C})$ and $\mathcal{C}_0(V) \rightarrow \mathbb{C}^l \rightarrow \text{Mat}_n(\mathbb{C})$, where the initial maps are given by evaluation lists of points $u_1, \dots, u_k$ and $v_1, \dots, v_l$. We define $\rho \otimes \eta$ to be the map

$$\rho \otimes \eta : \mathcal{C}_0(U \wedge V) \rightarrow \mathbb{C}^k \otimes \mathbb{C}^l \rightarrow \text{Mat}_m(\mathbb{C}) \otimes \text{Mat}_n(\mathbb{C}) \cong \text{Mat}_{mn}(\mathbb{C}),$$

where $\mathcal{C}_0(U \wedge V) \rightarrow \mathbb{C}^k \otimes \mathbb{C}^l$ sends a function f to $\sum_{i,j} f(u_i, v_j) e_i \otimes e_j$, and the displayed isomorphism is given by the lexicographic indexing convention used throughout this paper – namely, $\mathbb{C}^m \otimes \mathbb{C}^n \cong \mathbb{C}^{mn}$ sends $e_i \otimes e_j$ to $e_{n(i-1)+j}$. The pairing

$\otimes$ is defined on morphisms similarly, by identifying $\underline{m} \wedge \underline{n}$ with $\underline{mn}$ using lexicographic ordering. (In terms of the description of the space $\mathfrak{ob}_n \mathcal{A}^{top}(S^0, V)$ given in Section 2 involving a list of points on V and a list of pairwise orthogonal subspaces of $\mathbb{C}^n$, the pairing $\otimes$ may equivalently be defined on objects by sending the data $\{u_1, \dots, u_k\} \subset U; Y_1, \dots, Y_k \subset \mathbb{C}^m$ and $\{v_1, \dots, v_l\} \in V; X_1, \dots, X_l \subset \mathbb{C}^n$ to the data $\{u_{i,j}\} \subset U \times V; \{Y_i \otimes X_j \subset \mathbb{C}^m \otimes \mathbb{C}^n \cong \mathbb{C}^{mn}\}$.)

Recall that Segal's tensor construction for a Γ -space G and pointed space T results in a new Γ -space, written $T \otimes G$ [30, §3]. See the Appendix for details.

Theorem 3.14. *For any pointed CW complexes V and T , there is a natural homeomorphism of Γ -spaces*

$$T \otimes |\mathcal{A}^{top}(S^0, - \wedge V)| \cong |\mathcal{A}^{top}(S^0, - \wedge T \wedge V)|.$$

In particular, the spectrum $\mathbf{bu}^c(S^0, V)$ may also be described via the sequence of spaces

$$|\mathcal{A}^{top}(S^0, V)|, \quad |\mathcal{A}^{top}(S^0, S^1 \wedge V)|, \quad |\mathcal{A}^{top}(S^0, S^2 \wedge V)|, \quad \dots$$

with bonding maps given as the composition of the natural maps

$$S^1 \wedge |\mathcal{A}^{top}(S^0, S^k \wedge V)| \rightarrow |\mathcal{A}^{top}(S^0, S^1)| \wedge |\mathcal{A}^{top}(S^0, S^k \wedge V)| \rightarrow |\mathcal{A}^{top}(S^0, S^{k+1} \wedge V)|.$$

Proof. It suffices to show there is a natural homeomorphism of spaces (i.e., of Γ -spaces evaluated at 1_+) of the form

$$T \otimes |\mathcal{A}^{top}(S^0, - \wedge V)| \cong |\mathcal{A}^{top}(S^0, T \wedge V)|.$$

(See the Appendix for a definition of $T \otimes G$ where G is any Γ -space and T is any pointed space.) We have the chain of natural homeomorphisms

$$\begin{aligned} T \otimes |\mathcal{A}^{top}(S^0, - \wedge V)| &\cong \coprod_n \frac{T^{\times n} \times \left(\frac{\coprod_d \mathcal{N}_d \mathcal{A}^{top}(S^0, n_+ \wedge V) \times \Delta_{top}^d}{\sim} \right)}{\sim} \\ &\cong \frac{\coprod_{n,d} T^{\times n} \times \mathcal{N}_d \mathcal{A}^{top}(S^0, n_+ \wedge V) \times \Delta_{top}^d}{\sim} \\ &\cong \frac{\coprod_d \left(\frac{\coprod_n T^{\times n} \times \mathcal{N}_d \mathcal{A}^{top}(S^0, n_+ \wedge V)}{\sim} \right) \times \Delta_{top}^d}{\sim} \\ &\cong |d \mapsto T \otimes \mathcal{N}_d \mathcal{A}^{top}(S^0, - \wedge V)|. \end{aligned}$$

It therefore suffices to show $T \otimes \mathcal{N}_d \mathcal{A}^{top}(S^0, - \wedge V)$ is naturally homeomorphic to $\mathcal{N}_d \mathcal{A}^{top}(S^0, T \wedge V)$ for all d, V, T . Using the fact that $\mathcal{N}_d \mathcal{A}^{top}(S^0, V)$ is naturally homeomorphic to

$$\coprod_{k_0, \dots, k_d} \mathfrak{ob}_{k_0} \mathcal{A}^{top}(S^0, V) \times \mathcal{I}(k_0, k_1) \times \dots \times \mathcal{I}(k_{d-1}, k_d),$$

we further reduce the proof to checking that there is a natural homeomorphism

$$T \otimes \mathfrak{ob}_k \mathcal{A}^{top}(S^0, - \wedge V) \cong \mathfrak{ob}_k \mathcal{A}^{top}(S^0, T \wedge V),$$

for all k, V , and T .

We proceed to define a natural map of the form

$$\theta : T \otimes \mathfrak{ob}_k \mathcal{A}^{top}(S^0, - \wedge V) = \frac{\coprod_n T^{\times n} \times \mathfrak{ob}_k \mathcal{A}^{top}(S^0, n_+ \wedge V)}{\sim} \rightarrow \mathfrak{ob}_k \mathcal{A}^{top}(S^0, T \wedge V).$$

Note that a point of the source of θ is given by a pointed map of spaces $t : n_+ \rightarrow T$ and a $*$ -map $\rho : \mathcal{C}_0(n_+ \wedge V) \rightarrow \text{Mat}_k(\mathbb{C})$. Associated to such a pair (t, ρ) we can form the composition of

$$\mathcal{C}_0(T \wedge V) \xrightarrow{(t \wedge \text{id}_V)^*} \mathcal{C}_0(n_+ \wedge V) \xrightarrow{\rho} \text{Mat}_k(\mathbb{C}).$$

It is evident that the map

$$\coprod_n T^n \times \mathfrak{ob}_k \mathcal{A}^{\text{top}}(S^0, n_+ \wedge V) \rightarrow \mathcal{A}^{\text{top}}(S^0, T \wedge V)$$

so defined respects the needed equivalence relation and thus descends to the desired map θ . The bijectivity of θ is easily verified using the fact that for any pointed CW complex U , a $*$ -map $\eta : \mathcal{C}_0(U) \rightarrow \text{Mat}_k(\mathbb{C})$ factors as

$$\mathcal{C}_0(U) \xrightarrow{u^*} \mathcal{C}_0(m_+) \xrightarrow{\bar{\eta}} \text{Mat}_k(\mathbb{C})$$

where $u : m_+ \rightarrow U$ is a pointed map and $\bar{\eta}$ is injective, and such a factorization is unique up to the evident action of Σ_m . Finally, we claim the surjection

$$\coprod_n T^{\times n} \times \mathfrak{ob}_k \mathcal{A}^{\text{top}}(S^0, n_+ \wedge V) \rightarrow \mathfrak{ob}_k \mathcal{A}^{\text{top}}(S^0, T \wedge V)$$

is a closed mapping, and thus θ is a homeomorphism. For observe that both spaces are realized as direct limits (in the category of compactly generated spaces) indexed by all compact pointed subspaces of V and T . Moreover, for any two such compact pointed subspaces $D \subset V$ and $E \subset T$, the square

$$\begin{array}{ccc} \coprod_n E^{\times n} \times \mathfrak{ob}_k \mathcal{A}^{\text{top}}(S^0, n_+ \wedge D) & \longrightarrow & \coprod_n T^{\times n} \times \mathfrak{ob}_k \mathcal{A}^{\text{top}}(S^0, n_+ \wedge V) \\ \downarrow & & \downarrow \\ \mathfrak{ob}_k \mathcal{A}^{\text{top}}(S^0, E \wedge D) & \longrightarrow & \mathfrak{ob}_k \mathcal{A}^{\text{top}}(S^0, T \wedge V) \end{array}$$

is Cartesian. This allows us to reduce to the case where V and T are in fact compact, but then the result is obvious since the space $\mathfrak{ob}_k \mathcal{A}^{\text{top}}(S^0, C)$ is compact if C is. $\square$

Recall that if E is the spectrum consisting of spaces $E_0, E_1, \dots$, and bonding maps $f_i : S^1 \wedge E_i \rightarrow E_{i+1}$, $i \geq 0$, and V is a pointed CW complex, then by $E \wedge V$ we mean the spectrum consisting of the spaces $E_0 \wedge V, E_1 \wedge V, \dots$ and bonding maps $f_i \wedge \text{id} : S^1 \wedge E_i \wedge V \rightarrow E_{i+1} \wedge V$, $i \geq 0$.

Corollary 3.15. *For any pointed CW complex V , the spectrum $\mathbf{bu}^c(S^0, V)$ is naturally weakly equivalent to the spectrum $\mathbf{bu} \wedge V$.*

Proof. As shown in [6, 4.1], for any special Γ -space G and any pointed CW complex T , there is a natural stable weak equivalence of spectra

$$T \wedge \mathbb{S}(G) \sim \mathbb{S}(T \otimes G).$$

The result now follows immediately from Lemma 3.13, Theorem 3.14, and Proposition 2.7. $\square$

Theorem 3.16. *For pointed CW complexes W and V , the spectrum $\mathbf{bu}^c(W, V)$ is naturally weakly equivalent to the spectrum associated to the Γ -space*

$$n_+ \mapsto \underline{\text{Maps}}(W, \mathbf{bu}^c(S^0, n_+ \wedge V))$$

Proof. There is an evident natural map of Γ -spaces

$$|\mathcal{A}^{top}(W, - \wedge V)| \rightarrow \underline{\text{Maps}}(W, |\mathcal{A}^{top}(S^0, - \wedge V)|).$$

The source of this map is a special Γ -space by Lemma 3.13. Observe that the target is as well, since for any special Γ -space G and any pointed CW complex W , the Γ -space $\underline{\text{Maps}}(W, G(-))$ is obviously special. It therefore suffices to show that map

$$|\mathcal{A}^{top}(W, V)| \rightarrow \underline{\text{Maps}}(W, |\mathcal{A}^{top}(S^0, V)|)$$

is a weak equivalence of spaces.

The theorem now follows from Lemma 3.9 and 3.10, since in the diagram

$$\begin{array}{ccccc} |\mathcal{A}^{top}(W, V)| & \longrightarrow & |\tilde{\mathcal{A}}^{top}(W, V)| & \longleftarrow & \mathfrak{ob}_\infty \mathcal{A}^{top}(W, V) \\ \downarrow & & \downarrow & & \downarrow \\ \underline{\text{Maps}}(W, |\mathcal{A}^{top}(S^0, V)|) & \longrightarrow & \underline{\text{Maps}}(W, |\tilde{\mathcal{A}}^{top}(S^0, V)|) & \longleftarrow & \underline{\text{Maps}}(W, \mathfrak{ob}_\infty \mathcal{A}^{top}(S^0, V)) \end{array}$$

all horizontal arrows are weak equivalences and the right-most vertical arrow is a homeomorphism. $\square$

We close this subsection by defining an extension $\mathbf{bu}(W, V)$ of $\mathbf{bu}^c(W, V)$ which is non-connective and by relating $\mathbf{bu}(W, V)$ to the cohomology theory (in the variable W) defined by the spectrum $\mathbf{bu}^c(S^0, V) \sim \mathbf{bu} \wedge V$.

Recall that if E is the spectrum consisting of spaces $E_0, E_1, \dots$ and bonding maps $S^1 \wedge E_i \rightarrow E_{i+1}$, $i \geq 0$, and W is a pointed CW complex, $\underline{\text{Maps}}(W, E)$ is the spectrum consisting of the spaces $\underline{\text{Maps}}(W, E_0), \underline{\text{Maps}}(W, E_1), \dots$ with bonding maps given by the adjoints of the evident maps $\underline{\text{Maps}}(W, E_i) \rightarrow \underline{\text{Maps}}(W, \Omega^1 E_{i+1}) \cong \Omega^1 \underline{\text{Maps}}(W, E_{i+1})$, $i \geq 0$. The cohomology theory E^* represented by the spectrum E is defined by the homotopy groups of the spectrum $\underline{\text{Maps}}(W, E)$:

$$E^p(W) = \pi_{-p} \underline{\text{Maps}}(W, E) = \varinjlim_{k \geq 0} \pi_{k-p} \underline{\text{Maps}}(W, E_k).$$

If E is an Ω -spectrum, the above direct system is constant once $k \geq p$.

In the terminology of the Appendix, for any fixed W , the functor $V \mapsto |\mathcal{A}^{top}(W, V)|$ defines a $\mathcal{W}$ -space and thus an associated spectrum, comprised of the spaces

$$|\mathcal{A}^{top}(W, S^0)|, \quad |\mathcal{A}^{top}(W, S^1)|, \quad |\mathcal{A}^{top}(W, S^2)|, \quad \dots$$

Moreover, starting from the natural transformations (5) and (6) above, we obtain the natural transformation

$$(8) \quad \iota : V \rightarrow |\mathcal{A}^{top}(W, V)|$$

and the natural pairing

$$(9) \quad \mu : |\mathcal{A}^{top}(W, U)| \wedge |\mathcal{A}^{top}(W, V)| \rightarrow |\mathcal{A}^{top}(W, U \wedge V)|,$$

by identifying $|\mathcal{A}^{top}(W, V)|$ with $|\underline{\text{Maps}}(W, \mathcal{A}^{top}(S^0, V))|$. The natural maps (8) and (9) are readily verified to satisfy the required axioms for $|\mathcal{A}(W, -)|$ to be a *functor with smash product* (FSP for short). (See the Appendix for details concerning FSP's.) In particular, for any pointed CW complex V , we may construct a spectrum comprised of the spaces

$$|\mathcal{A}^{top}(W, V)|, \quad |\mathcal{A}^{top}(W, V \wedge S^1)|, \quad |\mathcal{A}^{top}(W, V \wedge S^2)|, \quad \dots$$

and with bonding maps defined by the composition of

$$\begin{array}{ccc} S^1 \wedge \mathcal{A}^{top}(W, S^k \wedge V) & & \mathcal{A}^{top}(W, S^{k+1} \wedge V) \\ \iota \wedge id \downarrow & & \cong \downarrow \\ \mathcal{A}^{top}(W, S^1) \wedge \mathcal{A}^{top}(W, S^k \wedge V) & \xrightarrow{\mu} & \mathcal{A}^{top}(W, S^1 \wedge S^k \wedge V), \quad k \geq 0. \end{array}$$

Theorem 3.17. *For pointed CW complexes W and V , the spectrum defined from the FSP $|\mathcal{A}^{top}(W, -)|$ and the pointed CW complex V (consisting of spaces*

$$|\mathcal{A}^{top}(W, V)|, \quad |\mathcal{A}^{top}(W, S^1 \wedge V)|, \quad |\mathcal{A}^{top}(W, S^2 \wedge V)|, \quad \dots$$

is naturally stably equivalent to the spectrum $\underline{\text{Maps}}(W, \mathbf{bu}^c(S^0, V))$ and thus is naturally stably equivalent to the spectrum $\underline{\text{Maps}}(W, \mathbf{bu} \wedge V)$.

Proof. There is a natural map of spectra defined by the sequence of maps of spaces

$$|\mathcal{A}^{top}(W, S^k \wedge V)| \rightarrow \underline{\text{Maps}}(W, |\mathcal{A}^{top}(S^0, S^k \wedge V)|), \quad k \geq 0,$$

and it suffices to show this map is a weak equivalence for each $k \geq 0$. In other words, it suffices to show

$$|\mathcal{A}^{top}(W, V)| \rightarrow \underline{\text{Maps}}(W, |\mathcal{A}^{top}(S^0, V)|)$$

is a weak equivalence for all pointed CW complexes V . But note that there are also natural maps

$$|\tilde{\mathcal{A}}^{top}(W, V)| \rightarrow \underline{\text{Maps}}(W, |\tilde{\mathcal{A}}^{top}(S^0, V)|)$$

and

$$\mathbf{ob}_\infty \mathcal{A}^{top}(W, V) \rightarrow \underline{\text{Maps}}(W, \mathbf{ob}_\infty \mathcal{A}^{top}(S^0, V)).$$

In light of the pair of natural weak equivalences

$$|\mathcal{A}^{top}(W, V)| \xrightarrow{\sim} |\tilde{\mathcal{A}}^{top}(W, V)| \xleftarrow{\sim} \mathbf{ob}_\infty \mathcal{A}^{top}(W, V)$$

established in Lemmas 3.9 and 3.10, it suffices to show that

$$\mathbf{ob}_\infty \mathcal{A}^{top}(W, V) \rightarrow \underline{\text{Maps}}(W, \mathbf{ob}_\infty \mathcal{A}^{top}(S^0, V))$$

is a weak equivalence. In fact, this map is a homeomorphism. $\square$

Remark 3.18. *Since it helps illustrate the previous theorem and for its sheer aesthetic quality, we give here a different proof that $|\mathcal{A}^{top}(W, S^1)|$ defines a first de-looping of the connected space defining the K -theory of W . This proof is due to Suslin [33] and is also essentially the same idea exploited in [19].*

Recall that $|\mathcal{A}^{top}(W, S^1)|$ is homotopy equivalent to $\mathbf{ob}_\infty \mathcal{A}^{top}(W, S^1)$, which in turn is homeomorphic to $\underline{\text{Maps}}(W, \mathbf{ob}_\infty \mathcal{A}^{top}(S^0, S^1))$. Thus, we seek to prove $\mathbf{ob}_\infty \mathcal{A}^{top}(S^0, S^1)$ is homotopy equivalent to $\underline{U}$, the space of infinite-by-infinite unitary matrices. A point of $\mathbf{ob}_\infty \mathcal{A}^{top}(S^0, S^1)$ is given by a C^ -map $\mathcal{C}_0(S^1) \rightarrow \text{Mat}_\infty(\mathbb{C})$. As seen in Section 2, such a map may be identified with an unordered list of points $u_1, \dots, u_k$ on S^1 and a corresponding list of pairwise orthogonal subspaces $X_1, \dots, X_k$ of $\mathbb{C}^\infty$. If we regard S^1 as the unit circle in the complex plane, we may interpret the u_i 's as complex numbers of norm 1. The data thus uniquely determine a unitary matrix which induces multiplication by u_i on X_i and multiplication by 1 on the orthogonal complement of $X_1 \perp \dots \perp X_k$ in $\mathbb{C}^\infty$. Indeed, one may readily verify that $\mathbf{ob}_\infty \mathcal{A}(S^0, S^1)$ is homeomorphic to the space U .*

Finally, we observe (as Suslin points out in [33]) that the homeomorphism $\mathbf{ob}_\infty \mathcal{A}(S^0, S^1) \cong U$ leads to a proof of Bott periodicity. Namely, we know that $\Omega^1 \mathbf{ob}_\infty \mathcal{A}(S^0, S^1)$ is

homotopy equivalent to $BU \times \mathbb{Z}$, since it is the homotopy-theoretic group completion of the H -space $\coprod_n \text{Grass}_n(\mathbb{C}^\infty)$. Thus we have $\Omega^1 U \sim BU \times \mathbb{Z}$ and consequently $\Omega^2 U \sim U$.

Using Theorem 3.17 as our guide, we extend the definition of a bivariate topological theory to a pair of pointed CW complexes by using functors with smash product.

Definition 3.19. For pointed CW complexes W and V , we write $\mathbf{bu}(W, V)$ for the spectrum

$$|\mathcal{A}^{\text{top}}(W, V)|, |\mathcal{A}^{\text{top}}(W, S^1 \wedge V)|, |\mathcal{A}^{\text{top}}(W, S^2 \wedge V)|, \dots$$

associated to the functor with smash products $|\mathcal{A}^{\text{top}}(W, -)|$ and the pointed CW complex V .

We write $bu_q(W, V)$ for the q -th homotopy group of $\mathbf{bu}(W, V)$ (which can be non-zero even for $q < 0$).

Notice that, in particular, Theorem 3.17 gives a natural isomorphism between the homotopy groups of $\mathbf{bu}(W, S^0)$ and the cohomology groups of W for the cohomology theory defined by the connective spectrum $\mathbf{bu}$. Similarly, the homotopy groups of $\mathbf{bu}(S^0, V)$ are naturally isomorphic to the $\mathbf{bu}$ -homology groups of V . This serves as justification for our use of the notation $\mathbf{bu}(-, -)$.

3.3. Hybrid model. Finally, we will have need for a sort of “mixed” version of the categorical models discussed in the previous two subsections. Namely, given an *unpointed* CW complex W and a quasi-projective complex variety Y , we define a spectrum $\mathcal{K}^{\text{qtop}}(W, Y)$, referred to as the “quasi-topological” K -theory of the pair (W, Y) .

The definition of $\mathcal{K}^{\text{qtop}}(W, Y)$ arises from a topological category $\mathcal{A}^{\text{qtop}}(W, Y)$ where the objects are given by the space $\coprod_n \text{Maps}(W_+, \mathbf{ob}_n \mathcal{A}^{\text{semi}}(\mathbb{C}, Y))$. Let Var^W denote the filtered category for which an object is a continuous map $W \rightarrow U^{\text{an}}$, where U is a quasi-projective complex variety. A morphism in Var^W from $W \rightarrow U^{\text{an}}$ to $W \rightarrow V^{\text{an}}$ is a morphism of varieties $V \rightarrow U$ causing the evident triangle to commute. Then the space of objects of $\mathcal{A}^{\text{qtop}}(W, Y)$ can equivalently be defined as the filtered direct limit

$$\varinjlim_{(W \rightarrow U^{\text{an}}) \in \text{Var}^W} \mathbf{ob} \mathcal{A}^{\text{semi}}(U, Y).$$

(In fact, one can assume the U ’s in this filtered limit are all affine, as shown in the proof of [12, 5.1].) In other words, an object of this category is given an equivalence class of pairs consisting of a continuous map $W \rightarrow U^{\text{an}}$ and a quotient $\mathcal{O}_{U \times Y}^n \twoheadrightarrow M$. Morphisms in $\mathcal{A}^{\text{qtop}}(W, Y)$ are given by injective functions $\underline{n} \hookrightarrow \underline{m}$ just as for $\mathcal{A}^{\text{semi}}(X, Y)$. Indeed, the topological category $\mathcal{A}^{\text{qtop}}(W, Y)$ can be identified with the direct limit of topological categories

$$\varinjlim_{(W \rightarrow U^{\text{an}}) \in \text{Var}^W} \mathcal{A}^{\text{semi}}(U, Y).$$

(Here, the topologies on objects and morphisms are formed by taking the direct limit in the category of compactly generated topological spaces.) All of the properties and definitions for $\mathcal{A}^{\text{semi}}$ thus carry over to $\mathcal{A}^{\text{qtop}}(W, Y)$.

Definition 3.20. For a quasi-projective complex variety Y and an (unpointed) compact CW complex W , we write $\mathcal{K}^{\text{qtop}}(W, Y)$ for the spectrum associated to the Γ -space $n_+ \mapsto |\mathcal{A}^{\text{qtop}}(W, n_+ \wedge Y_+)|$.

We write $K_q^{\text{qtop}}(W, Y)$ for the q -th homotopy groups of $\mathcal{K}^{\text{qtop}}(W, Y)$.

As before, we have the “infinite” variant of $\mathcal{A}^{\text{qtop}}$, the topological category $\mathcal{A}_\infty^{\text{qtop}}(W, Y)$, defined as the direct limit of topological categories

$$\varinjlim_{(W \rightarrow U^{\text{an}}) \in \text{Var}^W} \mathcal{A}_\infty^{\text{semi}}(U, Y).$$

Theorem 3.21. *For a compact CW complex W and quasi-projective variety Y , the spectrum defined by $\mathcal{A}^{\text{qtop}}(W, Y)$ coincides with that defined by the Γ -space $\text{ob}\mathcal{A}_\infty^{\text{qtop}}(W, - \wedge Y_+)$.*

Proof. For all $W \rightarrow U^{\text{an}}$ in Var^W and all quasi-projective varieties Z , we have the natural weak equivalences

$$|\mathcal{A}^{\text{semi}}(U, Z)| \xrightarrow{\sim} |\mathcal{A}_\infty^{\text{semi}}(U, Z)| \xleftarrow{\sim} \text{ob}\mathcal{A}_\infty^{\text{semi}}(U, Z)$$

of spaces established in Lemmas 3.3 and 3.4. The result now follows by taking direct limits and letting Z vary over $n_+ \wedge Y_+$, for $n \geq 0$. $\square$

In Section 5, we will show that in fact the spectrum $\mathcal{K}^{\text{qtop}}(W, Y)$ coincides up to weak equivalence with the spectrum $\mathbf{bu}^c(W, Y^{\text{an}})$, and thus does not represent a “new” theory. However, the category $\mathcal{A}^{\text{qtop}}(W, Y)$ will play a crucial role in allowing us to compare $\mathcal{K}^{\text{semi}}$ and $\mathbf{bu}^c$.

4. GROUP COMPLETIONS VIA MAPPING TELESCOPES

The primary goal of this section is to describe in more explicit terms the infinite loop spaces associated to the spectra $\mathcal{K}^{\text{semi}}(\text{Spec } \mathbb{C}, Y)$ and $\mathbf{bu}^c(S^0, V)$, where Y is a quasi-projective variety and V is a pointed CW complex. In other words, we wish to describe the homotopy-theoretic group completions of the H -spaces $|\mathcal{A}^{\text{semi}}(\text{Spec } \mathbb{C}, Y)|$ and $|\mathcal{A}^{\text{top}}(S^0, V)|$ in a direct fashion. (Recall that each of these spaces is a homotopy commutative H -space since each is the 0-space of a special Γ -space. The homotopy-theoretic group completion of an H -space is defined in [8].)

The description of the homotopy-theoretic group completion of $|\mathcal{A}^{\text{top}}(S^0, V)|$ in terms of a mapping telescope is then applied to prove that the natural map of spectra

$$\mathbf{bu}^c(W, V) \rightarrow \mathbf{bu}(W, V)$$

realizes the source as the connective cover of the target, where W is any unpointed, compact CW complex. (See Theorem 4.4.) Additionally, we use the results of this section in Section 5 to deduce Corollary 5.9 from Theorem 5.1.

We begin by describing the mapping telescopes. Each telescope is associated to a specific endomorphism of the spaces $|\mathcal{A}^{\text{semi}}(\text{Spec } \mathbb{C}, Y)|$ and $|\mathcal{A}^{\text{top}}(S^0, V)|$, which we now define. For the variety Y , choose a finite collection of $\mathbb{C}$ -points $y_1, \dots, y_k$ with one point chosen from each connected component. For the space V , we make the additional assumption that V has only finitely many connected components, and we choose a list of points $v_1, \dots, v_k$ with one point from each connected component other than the component containing the base point. Associated to these choices, we specify an object in each of the categories $\mathcal{A}^{\text{semi}}(\text{Spec } \mathbb{C}, Y)$ and $\mathcal{A}^{\text{top}}(S^0, V)$. For the former, the object is the quotient object $[\mathcal{O}_Y^k \twoheadrightarrow \bigoplus_i \mathcal{O}_{y_i}]$ given by the direct sum of the canonical surjections $\mathcal{O}_Y \twoheadrightarrow \mathcal{O}_{y_i}$, where $\mathcal{O}_{y_i}$ denotes the structure sheaf of one-point integral subvariety associate to the point y_i . For the category $\mathcal{A}^{\text{semi}}(S^0, V)$, let $[\eta : \mathcal{C}_0(V) \rightarrow \text{Mat}_k(\mathbb{C})]$ be the object given by the composition

$$\mathcal{C}_0(V) \xrightarrow{\text{eval}_{y_1, \dots, y_k}} \mathbb{C}^k \twoheadrightarrow \text{Mat}_k(\mathbb{C}),$$

where the first map is the evident evaluation map and the second is the inclusion into the diagonal. We then define an endofunctor

$$\alpha^{semi} : \mathcal{A}^{semi}(\text{Spec } \mathbb{C}, Y) \rightarrow \mathcal{A}^{semi}(\text{Spec } \mathbb{C}, Y)$$

by taking “direct sum” with the object $[\mathcal{O}_Y^n \rightarrow \bigoplus_i \mathcal{O}_{y_i}]$. Specifically, α^{semi} sends an object $[\mathcal{O}_Y^m \rightarrow M]$ to the object

$$[\mathcal{O}_Y^{k+m} = \mathcal{O}_Y^k \oplus \mathcal{O}_Y^m \rightarrow (\bigoplus_i \mathcal{O}_{y_i}) \oplus M].$$

The affect of α^{semi} on morphisms is given in the evident manner by sending $j : \underline{m} \rightarrow \underline{n}$ to $id \oplus j : \underline{k+m} \rightarrow \underline{k+n}$, where $(id \oplus j)(i)$ is defined as i , for $1 \leq i \leq k$, and $j(i)+k$, for $k < i \leq k_m$. Similarly, we define a endofunctor α^{top} on $\mathcal{A}^{top}(S^0, V)$, by sending $\rho : \mathcal{C}_0(V) \rightarrow \text{Mat}_m(\mathbb{C})$ to $\eta \oplus \rho$, defined as the composition of

$$\mathcal{C}_0(V) \rightarrow \text{Mat}_k(\mathbb{C}) \times \text{Mat}_m(\mathbb{C}) \rightarrow \text{Mat}_{k+m}(\mathbb{C}).$$

The affect of α^{top} on morphisms is given just as for α^{semi} .

The two functors α^{semi} and α^{top} induce endomorphisms of spaces

$$\alpha^{semi} : |\mathcal{A}^{semi}(\text{Spec } \mathbb{C}, Y)| \rightarrow |\mathcal{A}^{semi}(\text{Spec } \mathbb{C}, Y)|$$

and

$$\alpha^{top} : |\mathcal{A}^{top}(S^0, V)| \rightarrow |\mathcal{A}^{top}(S^0, V)|$$

(which, by abuse of notation, are written using the same symbols for the underlying functors), and it is these endomorphisms that we use to form the promised mapping telescopes. For brevity, let G_0 denote either of the two pointed spaces $|\mathcal{A}^{semi}(\text{Spec } \mathbb{C}, Y)|$ or $|\mathcal{A}^{top}(S^0, V)|$, and let g_0 denote its base point (corresponding to $[\mathcal{O}_Y^0 \rightarrow 0]$ or $\mathcal{C}_0(V) \rightarrow 0$). Note that G_0 is a homotopy commutative H -space, since it is the 0 space of a special Γ -space. Let α stand for the corresponding self map on G_0 , α^{semi} or α^{top} . Form a sequence of maps of pointed spaces

$$G_0 \xrightarrow{\alpha} G_1 \xrightarrow{\alpha} G_2 \longrightarrow \dots,$$

where G_n is the same space as G_0 , except that its base point is defined to be $g_n = \alpha^{on}(g_0)$. Finally, let $\text{Tel}(G_i, \alpha)$ denote the mapping telescope of this sequence of pointed maps.

Theorem 4.1. *Let Y be any quasi-projective complex variety and V any pointed CW complex with only finitely many connected components. Define spaces G_i and maps $\alpha : G_i \rightarrow G_{i+1}$ as above. Then the natural map*

$$G_0 \rightarrow \text{Tel}(G_i, \alpha)$$

is a homotopy-theoretic group completion of the H -space G_0 .

Proof. Using [8, 1.2], it suffices to show that the natural map

$$G_0 \rightarrow \text{Tel}(G_i, \alpha)$$

is a map of homotopy commutative H -spaces such that the induced map on π_0 is group completion and for any commutative ring of coefficients A the induced map on homology $H_*(-; A)$ is given by localization with respect to the natural action of $\mathbb{Z}[\pi_0(G_0)]$ on $H_*(G_0; A)$.

The proof that $\text{Tel}(G_i, \alpha)$ has the structure of homotopy commutative H -space and that the indicated map is a map of such objects is given (in a slightly different context) by the proof of [11, 3.4]. The condition concerning the induced map on homology with coefficients clearly holds provided the condition on π_0 holds, since

the homology of a mapping telescope is given by the direct limit of the homology groups of the constituent spaces.

To establish the condition involving π_0 , we observe that in the first of the two cases, an element of $\pi_0|\mathcal{A}^{\text{semi}}(\text{Spec } \mathbb{C}, Y)|$ is given by an isomorphism class of coherent sheaves on Y having finite length. For given any two surjections $p : \mathcal{O}_Y^m \twoheadrightarrow M$ and $q : \mathcal{O}^n \twoheadrightarrow M'$ with $M \cong M'$, the maps $\underline{m} \mapsto \underline{m+n}$ and $\underline{n} \mapsto \underline{m+n}$ given by inclusion into the first m and last n elements, respectively, define morphisms

$$[\mathcal{O}_Y^m \twoheadrightarrow M] \rightarrow [\mathcal{O}_Y^{m+n} \twoheadrightarrow M']$$

and

$$[\mathcal{O}_Y^n \twoheadrightarrow M'] \rightarrow [\mathcal{O}_Y^{m+n} \twoheadrightarrow M'],$$

which show that $[\mathcal{O}_Y^m \twoheadrightarrow M]$ and $[\mathcal{O}_Y^n \twoheadrightarrow M']$ belong to the same path component of $|\mathcal{A}^{\text{semi}}(\text{Spec } \mathbb{C}, Y)|$. A similar argument shows that given a short exact sequence

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

of coherent $\mathcal{O}_Y$ -modules of finite length, we have the equation $[M] = [M'] + [M'']$ of associated classes in $\pi_0|\mathcal{A}^{\text{semi}}(\text{Spec } \mathbb{C}, Y)|$. Since a coherent sheaf of finite length admits a finite filtration with quotients of the form $\mathcal{O}_y$, it follows that $\pi_0|\mathcal{A}^{\text{semi}}(\text{Spec } \mathbb{C}, Y)|$ is generated by classes of the form $[\mathcal{O}_y]$. Finally, the topology on $\mathfrak{ab}\mathcal{A}^{\text{semi}}(\text{Spec } \mathbb{C}, Y)$ is such that given a connected variety T and $\mathbb{C}$ -points t_0, t_1 of T , if $\mathcal{M}$ is a coherent sheaf on $T \times Y$ which is finite and flat over T , then $\mathcal{M}|_{\{t_0\} \times Y}$ and $\mathcal{M}|_{\{t_1\} \times Y}$ determine the same class in $\pi_0|\mathcal{A}^{\text{semi}}(\text{Spec } \mathbb{C}, Y)|$. Taking T to be a connected component of Y and taking $\mathcal{M}$ to be the structure sheaf of the graph of the inclusion $T \hookrightarrow Y$, we see that the class of $\mathcal{O}_y$ in $\pi_0|\mathcal{A}^{\text{semi}}(\text{Spec } \mathbb{C}, Y)|$ depends only on the connected component of Y in which y lies. It follows immediately that $\pi_0|\mathcal{A}^{\text{semi}}(\text{Spec } \mathbb{C}, Y)|$ is isomorphic to the free abelian monoid $\bigoplus_i \mathbb{N} \cdot [\mathcal{O}_{y_i}]$. We therefore have that, in this case, the map

$$\pi_0 G_0 \rightarrow \pi_0 \text{Tel}(G_i, \alpha^{\text{semi}})$$

is given by inverting the class $\sum_i [\mathcal{O}_{y_i}]$, and is thus gives the group completion of the abelian monoid $\pi_0 G_0$.

For the case $G_0 = |\mathcal{A}^{\text{top}}(S^0, V)|$, recall that a point of the space G_0 consists of a set of points $w_1, \dots, w_n$ of V together with finite dimensional pairwise orthogonal subspaces $X_1, \dots, X_n$ of $\mathbb{C}^\infty$. Given such a point in G_0 , the connected component containing this point depends only on the dimension of the X_i 's and the location of the w_i 's, since the space of n -tuples of pairwise orthogonal subspaces of $\mathbb{C}^\infty$ having proscribed dimensions is connected. Moreover, keeping the subspaces X_i fixed, we can move the w_i 's freely via paths in V . It follows that $\pi_0 G_0$ is isomorphic to the quotient of the free abelian monoid on the set of connected components of V by the submonoid generated by the component containing the base point. As a set, we have natural bijections $\pi_0 G_0 \cong \pi_0 G_i$ and the map $\alpha : \pi_0 G_i \rightarrow \pi_0 G_{i+1}$, for all i , is given by addition of $\sum_j [v_j] \in \pi_0 G_0$. It follows immediately that the natural map $\pi_0 G_0 \rightarrow \pi_0(\text{Tel}(G_i, \alpha))$ gives the group completion of the abelian monoid $\pi_0 G_0$. $\square$

To illustrate the utility of Theorem 4.1, consider momentarily an arbitrary special Γ -space G , and let W be an unpointed, compact CW complex. Since $G(0_+) \rightarrow \Omega^1 G(S^1) = S^1 \otimes G(-)$ is a homotopy-theoretic group completion, there is a natural map

$$[W_+, G(0_+)]^+ \rightarrow [W_+, \Omega^1 G(S^1)],$$

where the superscript $+$ denote group completion of the indicated abelian monoid. However, this map need not be an isomorphism for an arbitrary G . (For an example of this failure, let G be the special Γ -space whose 0 space is $\coprod_n BGL_n(R)$ for some discrete ring R , as described in [30, §2].) One of consequences of the telescope construction presented in Theorem 4.1 is that this map is in fact an isomorphism for the Γ -spaces under consideration:

Corollary 4.2. *Let W be an unpointed, compact CW complex, let Y be a quasi-projective complex variety, and let V be a pointed CW complex. If G is either of the two Γ -spaces $|\mathcal{A}^{semi}(\text{Spec } \mathbb{C}, - \wedge Y_+)|$ or $|\mathcal{A}^{top}(S^0, - \wedge V)|$, then the natural map*

$$[W_+, G(0_+)]^+ \rightarrow [W_+, \Omega^1 G(S^1)]$$

is an isomorphism.

Proof. Since V is a filtered direct limit of compact CW complexes and both sides of the map commuted with such limits, we may assume V is compact, and thus has only finitely many connected components.

By the theorem, we have a weak homotopy equivalence

$$\Omega^1 G(S^1) \sim \text{Tel}(G_i, \alpha)$$

and thus an isomorphism

$$\varinjlim_i [W_+, G_i] \cong [W_+, \Omega^1 G(S^1)].$$

But since G_i coincides with G_0 as an unpointed space and W is base point free, the monoid $[W_+, G_i]$ is naturally isomorphic to $[W_+, G_0]$. Moreover, the transition maps in the direct limit $\varinjlim_i [W_+, G_i] = \varinjlim_i [W_+, G_0]$ are each given by addition by the same class in $[W_+, G_0]$. Since $[W_+, \Omega^1 G(S^1)]$ is an abelian group, the map

$$[W_+, G_0] \rightarrow \varinjlim_i [W_+, G_i]$$

must give the group completion of the monoid $[W_+, G(0_+)]$. $\square$

As above, associated to a special Γ -space G and an unpointed compact CW complex W , we may form another special Γ -space $\underline{\text{Maps}}(W_+, G(-))$ and thus an associated spectrum comprised of the spaces

$$\underline{\text{Maps}}(W_+, G(0_+)), \quad S^1 \otimes \underline{\text{Maps}}(W_+, G(-)), \quad S^2 \otimes \underline{\text{Maps}}(W_+, G(-)), \quad \dots$$

Alternatively, we may first promote G to an FSP (see the Appendix) via the formula $G(T) = T \otimes G$. Then $\underline{\text{Maps}}(W_+, G(-))$ is again an FSP and we have another spectrum comprised of the spaces

$$\underline{\text{Maps}}(W_+, G(0_+)), \quad \underline{\text{Maps}}(W_+, G(S^1)), \quad \underline{\text{Maps}}(W_+, G(S^2)), \quad \dots$$

Moreover, as explained more carefully in the Appendix, there is a natural map between these spectra, induced by the family of maps of spaces

$$S^q \otimes \underline{\text{Maps}}(W_+, G(-)) \rightarrow \underline{\text{Maps}}(W_+, G(S^q)).$$

For a general special Γ -space G , there is no reason to expect this map to be a weak equivalence of spectra, even if one restricts to connective covers. However, the telescope construction of Theorem 4.1 gives the following result:

Corollary 4.3. *Let W be an unpointed, compact CW complex, let Y be a quasi-projective complex variety, and let V be a pointed CW complex. If G is either of the two Γ -spaces $|\mathcal{A}^{semi}(\text{Spec } \mathbb{C}, -\wedge Y_+)|$ or $|\mathcal{A}^{top}(S^0, -\wedge V)|$, then the natural map from the spectrum associated to the Γ -space $n_+ \mapsto \underline{\text{Maps}}(W_+, G(n_+))$ to the spectrum associated to the FSP defined by $T \mapsto T \otimes \underline{\text{Maps}}(W_+, G(-))$ is an isomorphism on non-negative homotopy groups.*

Proof. In the case $G = |\mathcal{A}^{top}(S^0, -\wedge V)|$, since W is compact, the Γ -space $\underline{\text{Maps}}(W, G(-))$ is the direct limit of Γ -spaces indexed over compact subspaces of V , and thus without loss of generality we may assume V is compact too. It suffices to show the natural map

$$\Omega^k(S^k \otimes \underline{\text{Maps}}(W_+, G(-))) \rightarrow \Omega^k(\underline{\text{Maps}}(W_+, G(S^k)))$$

is a weak equivalence for all $k \geq 1$. To show this, it suffices to show that the natural map of H -spaces

$$(10) \quad \underline{\text{Maps}}(W_+, G(0_+)) \rightarrow \Omega^k(\underline{\text{Maps}}(W_+, G(S^k))) \sim \underline{\text{Maps}}(W_+, \Omega^k G(S^k))$$

is a homotopy-theoretic group completion. Since W is unpointed and there is a natural weak equivalence $\Omega^1 G(S^1) \sim \Omega^k G(S^k)$, the map (10) satisfies the needed condition on π_0 by Corollary 4.2. From Theorem 4.1 and since W is compact, we have a weak homotopy equivalence

$$\text{Tel}(\underline{\text{Maps}}(W_+, G_i), \alpha_*) \sim \underline{\text{Maps}}(W_+, \Omega^k G(S^k)),$$

and thus the needed condition on homology with coefficients follows readily. $\square$

Recall that $\mathbf{bu}^c(W, V)$ is the spectrum associated to the functor $|\mathcal{A}^{top}(W, -\wedge V)|$ regarded as Γ -space, and that $\mathbf{bu}(W, V)$ is the spectrum associated to the functor $|\mathcal{A}^{top}(W, -\wedge V)|$ regarded as an FSP. There is therefore a natural map of spectra

$$(11) \quad \mathbf{bu}^c(W, V) \rightarrow \mathbf{bu}(W, V).$$

Theorem 4.4. *For an unpointed, compact CW complex W and any pointed CW complex V , the natural map*

$$\mathbf{bu}^c(W, V) \rightarrow \mathbf{bu}(W, V)$$

induces an isomorphism on non-negative homotopy groups – that is, the spectrum $\mathbf{bu}^c(W, V)$ is the connective cover of the spectrum $\mathbf{bu}(W, V)$.

Proof. Using Lemmas 3.9 and 3.10, we see that the Γ -space (respectively, FSP) $\underline{\text{Maps}}(W_+, |\mathcal{A}^{top}(S^0, -\wedge V)|)$ is naturally equivalent to the Γ -space $|\mathcal{A}^{top}(W, -\wedge V)|$ (respectively, FSP). The result now follows immediately from Corollary 4.3. $\square$

Remark 4.5. *Theorem 4.4 is clearly false if we allow W to be pointed. For example, if W is any pointed, connected CW complex and $V = S^0$, then, as we observed previously, the Γ -space $|\mathcal{A}^{top}(W, -\wedge S^0)|$ is contractible. (This is because an object of $\mathcal{A}^{top}(W, n_+)$ is given by an n -tuple of vector bundles on W , each of which is required to have rank 0 at the base point of W .) Thus $\mathbf{bu}^c(W, S^0)$ is trivial. By contrast the spectrum $\mathbf{bu}(W, S^0)$ gives the topological K -theory of W .*

5. FUNDAMENTAL COMPARISON THEOREM

If we let $Y = \operatorname{Spec} \mathbb{C}$ in $\mathcal{K}^{\text{semi}}(X, Y)$, it follows from Theorem 3.5 that we recover the semi-topological K -theory of X as defined in [11]. Setting X to be $\operatorname{Spec} \mathbb{C}$ and letting Y vary, we obtain a theory of complex varieties which is covariant (for all maps – not just proper ones). We refer to $\mathcal{K}^{\text{semi}}(\mathbb{C}, Y) = \mathcal{K}^{\text{semi}}(\operatorname{Spec} \mathbb{C}, Y)$ as the “semi-topological K -homology of Y ”. In this section, we establish a natural weak equivalence (in the homotopy category) between $\mathcal{K}^{\text{semi}}(\mathbb{C}, Y)$ and $\mathbf{bu}^c(S^0, Y^{\text{an}}) \sim \mathbf{bu} \wedge Y_+^{\text{an}}$, where Y is any smooth, quasi-projective complex variety. In other words, we show that the semi-topological K -homology of a smooth, quasi-projective complex variety is equivalent to the topological K -homology of the associated analytic space Y^{an} . Unlike the results of the previous sections, which were formal in nature, this “Fundamental Comparison Theorem” establishes an important connection between semi-topological K -theory and topological K -theory. Indeed, it is the key point underlying the major theorems proven in Section 7.

Let Y be a quasi-projective complex variety and Y^{an} its analytic realization. By [20], the space Y^{an} is triangulable and thus admits the structure of a CW complex. An object of the topological category $\mathcal{A}^{\text{top}}(S^0, Y^{\text{an}})$ is represented an unordered list of $\mathbb{C}$ -points $y_1, \dots, y_k$ of Y together with a list of pairwise orthogonal subspaces $V_1, \dots, V_k$ of $\mathbb{C}^n$. From this data, we can define a surjective map of coherent $\mathcal{O}_Y$ -modules $\mathcal{O}_Y^n \rightarrow M$ by taking the composition of

$$\mathcal{O}_Y^n \xrightarrow{(\text{eval}_{y_1, \dots, y_n})} \bigoplus_{i=1}^k \mathbb{C}^n \longrightarrow \bigoplus V_i^*.$$

(Alternatively, given $\rho : \mathcal{C}_0(Y^{\text{an}}) \rightarrow \operatorname{Mat}_n(\mathbb{C})$, it follows that ρ factors as $\mathcal{C}_0(Y^{\text{an}}) \rightarrow \mathcal{C}_0(U^{\text{an}}) \rightarrow \operatorname{Mat}_n(\mathbb{C})$ for $U = \operatorname{Spec} R$ a Zariski open affine subvariety of Y . Thus ρ induces a non-unital $\mathbb{C}$ -algebra map given by the composition of $R \hookrightarrow \mathcal{C}_0(U^{\text{an}}) \rightarrow \operatorname{Mat}_n(\mathbb{C})$, which defines an quotient $\mathbb{C}^n \twoheadrightarrow V$ where V is endowed with the structure of an R -module.) This construction defines a continuous function $\mathbf{ob} \mathcal{A}^{\text{top}}(S^0, Y^{\text{an}}) \rightarrow \mathbf{ob} \mathcal{A}^{\text{semi}}(\mathbb{C}, Y)$, and this function extends to a continuous map

$$\mathbf{mor} \mathcal{A}^{\text{top}}(S^0, Y^{\text{an}}) \rightarrow \mathbf{mor} \mathcal{A}^{\text{semi}}(\mathbb{C}, Y),$$

since the morphisms of each category are defined in the same fashion using the discrete spaces $I(m, n)$, $m, n \geq 0$. One may readily verify that these maps determine a continuous functor

$$\mathcal{A}^{\text{top}}(S^0, Y^{\text{an}}) \rightarrow \mathcal{A}^{\text{semi}}(\mathbb{C}, Y),$$

which is natural in Y . The definition of this functor extends in the evident manner to the case where Y is pointed. In particular, we obtain a map of Γ -spaces

$$(n_+ \mapsto |\mathcal{A}^{\text{top}}(S^0, n_+ \wedge Y_+^{\text{an}})|) \rightarrow (n_+ \mapsto |\mathcal{A}^{\text{semi}}(\mathbb{C}, n_+ \wedge Y_+)|)$$

and thus a map of associated spectra

$$\mathbf{bu}^c(S^0, Y^{\text{an}}) \rightarrow \mathcal{K}^{\text{semi}}(\mathbb{C}, Y).$$

Theorem 5.1. (Fundamental Comparison Theorem) *Let Y be a smooth, quasi-projective complex variety, and let Y^{an} denote the associated analytic space. Then the natural map*

$$\mathbf{bu}^c(S^0, Y^{\text{an}}) \xrightarrow{\sim} \mathcal{K}^{\text{semi}}(\mathbb{C}, Y)$$

defined above is a weak equivalence of spectra. In other words, the connected topological K -homology of Y^{an} and the semi-topological K -homology of Y coincide up to weak equivalence.

Proof. The proof of this theorem occupies most of the remainder of this section, and starts with the following reduction.

Lemma 5.2. *To prove the theorem, it suffices to show*

$$\mathbf{ob}_n \mathcal{A}^{top}(S^0, Y^{an}) \rightarrow \mathbf{ob}_n \mathcal{A}^{semi}(\mathbb{C}, Y)$$

is a homology equivalence for all smooth, quasi-projective varieties Y and for all $n \geq 0$.

Proof. Assume the indicated homology equivalences are established for all n and all smooth varieties Y . Since the morphisms of $\mathcal{A}^{semi}$ and $\mathcal{A}^{top}$ are both given in terms the sets $I(n, m)$, $n, m \geq 0$, we see that the maps of simplicial spaces $\mathcal{N}\mathcal{A}^{top}(S^0, Y^{an}) \rightarrow \mathcal{N}\mathcal{A}^{semi}(\mathbb{C}, Y)$ is a degree-wise homology equivalence. Thus the map

$$|\mathcal{A}^{top}(S^0, m_+ \wedge Y_+^{an})| \rightarrow |\mathcal{A}^{semi}(S^0, m_+ \wedge Y_+)|$$

is a homology equivalence for all m , since the geometric realization of a degree-wise homology equivalence is a homology equivalence and $m_+ \wedge Y_+ = (Y \coprod \cdots \coprod Y)_+$ is a (pointed) smooth variety. But then the map of associated spectra is also a homology equivalence, since the map on 1-spaces is given by the map

$$(12) \quad |m \mapsto |\mathcal{A}^{top}(S^0, m_+ \wedge Y_+^{an})|| \rightarrow |m \mapsto |\mathcal{A}^{semi}(S^0, m_+ \wedge Y_+)||,$$

which is induced from a degree-wise homology equivalence of simplicial spaces. Finally, a homology equivalence of connected infinite loop spaces such as (12) is necessarily a homotopy equivalence. $\square$

To finish the proof of the theorem, we fix $n \geq 0$ and show $\mathbf{ob}_n \mathcal{A}^{top}(S^0, Y^{an}) \rightarrow \mathbf{ob}_n \mathcal{A}^{semi}(\mathbb{C}, Y)$ is a homology equivalence. Recall that the space $\mathbf{ob}_n \mathcal{A}^{semi}(S^0, Y)$ is defined as the geometric realization of $\mathcal{G}_Y^n$, the variety representing the functor G_Y^n of Definition 2.1. For the purposes of abbreviation, we will let $\mathcal{F}_Y^n$ denote the space $\mathbf{ob}_n \mathcal{A}^{top}(S^0, Y^{an}) = \text{Hom}_*(\mathcal{C}_0(Y), \text{Mat}_n(\mathbb{C}))$. Using this notation, we have a continuous map $\mathcal{F}_Y^n \rightarrow (\mathcal{G}_Y^n)^{an}$ which we must show is a homology equivalence.

We begin with a few general observations which serve to clarify the nature of $\mathcal{F}_Y^n$ and $\mathcal{G}_Y^n$. First, note that for $U \subset Y$ an open subvariety, the induced map $\mathcal{G}_U^n \rightarrow \mathcal{G}_Y^n$ is also an open immersion. (This follows readily from the explicit construction of $\mathcal{G}_Y^n$ described in the proof of Lemma 2.2.) Similarly, the natural map $\mathcal{F}_U^n \rightarrow \mathcal{F}_Y^n$ is an open subspace inclusion, a fact which is seen to hold using the description of $\text{Hom}_*(\mathcal{C}_0(Y), \text{Mat}_n(\mathbb{C}))$ given by lists of points on Y^{an} together with pairwise orthogonal subspaces of $\mathbb{C}^n$. When $U = \text{Spec } R$ is affine, $\mathcal{G}_U^n$ is the variety parameterizing equivalence classes of pairs $(\pi : \mathbb{C}^n \twoheadrightarrow V, \rho : R \rightarrow \text{End}_{\mathbb{C}}(V))$ consisting of a surjection π of $\mathbb{C}$ -vector spaces and a $\mathbb{C}$ -algebra map ρ . This pair is equivalent to the pair $(\mathbb{C}^n \twoheadrightarrow W, R \rightarrow \text{End}_{\mathbb{C}}(W))$ if there is a vector space isomorphism $V \cong W$ causing the two evident triangle to commute. Indeed, the scheme $\mathcal{G}_U^n$ represents the functor sending $\text{Spec } A$ to equivalence classes of pairs $(A^n \twoheadrightarrow P, R \rightarrow \text{End}_A(P))$, where P is a projective A -module. Such a pair is equivalent to the pair $(A^n \twoheadrightarrow Q, R \rightarrow \text{End}_A(Q))$ if there is an A -module isomorphism $P \cong Q$ causing the two evident triangles to commute.

In general, the map $\mathcal{F}_Y^n \rightarrow (\mathcal{G}_Y^n)^{an}$ is injective, and we claim it is in fact a subspace inclusion. To show this, it suffices to work locally, and thus we may assume $Y = \text{Spec } R$ is affine, with R a finitely generated $\mathbb{C}$ -algebra. In this case, observe that R is naturally a subring of $\mathcal{C}(Y^{an})$, and the map $\mathcal{F}_Y^n \rightarrow (\mathcal{G}_Y^n)^{an}$ sends a $*$ -map $\rho : \mathcal{C}(Y^{an}) \rightarrow \text{Mat}_n(\mathbb{C})$ to the quotient object represented by the pair $(\mathbb{C}^n \twoheadrightarrow$

$\text{im } \rho(1), R \hookrightarrow \mathcal{C}(Y^{\text{an}}) \rightarrow \text{im } \rho(1)$). Here, the surjection $\mathbb{C}^n \twoheadrightarrow \text{im } \rho(1)$ is orthogonal projection. We claim $(\mathbb{C}^n \twoheadrightarrow V, \rho : R \rightarrow \text{End}_{\mathbb{C}}(V))$ lies in the image of $\mathcal{F}_Y^n \rightarrow (\mathcal{G}_Y^n)^{\text{an}}$ if and only if ρ is a *normal* map. By “normal” we mean the following: Observe that $\text{End}_{\mathbb{C}}(V)$ is naturally a C^* -algebra since V inherits a Hermitian inner product from $\mathbb{C}^n$. We say ρ is *normal* if $\rho(r)$ commutes with $\rho(r)^*$ for all $r \in R$ (i.e., the image of ρ is contained in the set of normal elements of the C^* -algebra $\text{End}_{\mathbb{C}}(V)$). To see that our description of the image of $\mathcal{F}_Y^n \rightarrow (\mathcal{G}_Y^n)^{\text{an}}$ is correct, note that a $*$ -map $\eta : \mathcal{C}(Y^{\text{an}}) \rightarrow \text{Mat}_n(\mathbb{C})$ determines the element $(\mathbb{C}^n \twoheadrightarrow V, \rho : R \rightarrow \text{End}_{\mathbb{C}}(V))$ with ρ normal, since by definition η preserves involutions and $\mathcal{C}(Y^{\text{an}})$ is a commutative ring. Conversely, given $(\mathbb{C}^n \twoheadrightarrow V, \rho : R \rightarrow \text{End}_{\mathbb{C}}(V))$ with ρ normal, let A be the C^* -subalgebra of $\text{End}_{\mathbb{C}}(V)$ generated by $\text{im}(\rho)$ (i.e., A is the smallest $\mathbb{C}$ -subalgebra containing $\text{im}(\rho)$ and closed under involution). Then A is abelian. It follows from the Spectral Theorem (cf. [21, 4.4.3]) that $A \cong \mathbb{C} \times \cdots \times \mathbb{C}$, with k factors, and ρ factors as $R \rightarrow \mathbb{C}^k \rightarrow \text{End}_{\mathbb{C}}(V)$, with the first map being evaluation at k closed points $y_1, \dots, y_k$ of Y and second map being a C^* -algebra map. Thus the element $(\mathbb{C}^n \twoheadrightarrow V, \rho : R \rightarrow \text{End}_{\mathbb{C}}(V))$ of $(\mathcal{G}_Y^n)^{\text{an}}$ is the image of the $*$ -map

$$\mathcal{C}(Y^{\text{an}}) \xrightarrow{\text{eval}_{y_1, \dots, y_k}} \mathbb{C}^k \rightarrow \text{End}_{\mathbb{C}}(V) \hookrightarrow \text{Mat}_n(\mathbb{C}),$$

where the inclusion $\text{End}_{\mathbb{C}}(V) \hookrightarrow \text{Mat}_n(\mathbb{C})$ is given by using the orthogonal splitting of $\mathbb{C}^n \twoheadrightarrow V$.

Let the image of $\mathcal{F}_Y^n \rightarrow (\mathcal{G}_Y^n)^{\text{an}}$ be written $(\mathcal{G}_Y^n)_{\text{norm}}^{\text{an}}$. To show $\mathcal{F}_Y^n$ is homeomorphic to $(\mathcal{G}_Y^n)_{\text{norm}}^{\text{an}}$, it suffices to show these spaces represent the same functor on the category of compact, Hausdorff spaces. (This suffices since both of these spaces are compactly generated.) In other words, it suffices to show that for a compact Hausdorff space T , the function $\text{Maps}(T, \mathcal{F}_Y^n) \rightarrow \text{Maps}(Y, (\mathcal{G}_Y^n)_{\text{norm}}^{\text{an}})$ is surjective. For such a T , a map $T \rightarrow (\mathcal{G}_Y^n)^{\text{an}}$ is given by a pair $(\pi : \mathcal{C}(T)^n \twoheadrightarrow P, \rho : R \rightarrow \text{End}_{\mathcal{C}(T)}(P))$, with P a projective $\mathcal{C}(T)$ -module. Such a map lands in $(\mathcal{G}_Y^n)_{\text{norm}}^{\text{an}}$ if and only if for all $t \in T$ and $r \in R$, the induced endomorphism $\rho(r)|_{P_t} \in \text{End}_{\mathbb{C}}(P_t)$ is normal. Here, P_t denotes the finite dimensional complex vector space given by the fiber of P over $t \in T$. Note that the surjection $\mathcal{C}(T)^n \twoheadrightarrow P$ defines a continuous Hermitian inner product on P (i.e., a Hermitian inner product on each fiber P_t which varies continuously) and thus a C^* -algebra structure for $\text{End}_{\mathcal{C}(T)}(P)$. Thus the condition that $T \rightarrow \mathcal{G}_Y^n$ land in $(\mathcal{G}_Y^n)_{\text{norm}}$ may be expressed by saying the image of ρ is a subset of the set of normal elements of the C^* -algebra $\text{End}_{\mathcal{C}(T)}(P)$. Consequently, if $g : T \rightarrow (\mathcal{G}_Y^n)_{\text{norm}}^{\text{an}}$ is any continuous map, represented by the pair (π, ρ) , then the C^* -subalgebra A generated by $\text{im}(\rho)$ is abelian. Once again, the Spectral Theorem implies that $A \cong \mathcal{C}(E)$, where E is a compact subspace of Y^{an} , and ρ factors as

$$R \xrightarrow{\subset} \mathcal{C}(Y^{\text{an}}) \xrightarrow{\text{restriction}} \mathcal{C}(E) \xrightarrow{\cong} A \xrightarrow{\subset} \text{End}_{\mathcal{C}(T)}(P).$$

Using the orthogonal splitting of $\mathcal{C}(T)^n \twoheadrightarrow P$, we define a $*$ -map given by the composition of

$$\mathcal{C}(Y^{\text{an}}) \xrightarrow{\text{restriction}} \mathcal{C}(E) \rightarrow \text{End}_{\mathcal{C}(T)}(P) \subset \text{Mat}_n(\mathcal{C}(T)).$$

This $*$ -map gives a lifting of g to a map $T \rightarrow \mathcal{F}_Y^n$, and so $\mathcal{F}_Y^n \rightarrow (\mathcal{G}_Y^n)^{\text{an}}$ is a subspace inclusion as claimed.

We have thus proven:

Lemma 5.3. *For any quasi-projective complex variety Y , the natural map $\mathcal{F}_Y^n \rightarrow (\mathcal{G}_Y^n)^{\text{an}}$ is a subspace inclusion. Moreover, if $Y = \text{Spec } R$ where R is a finitely generated $\mathbb{C}$ -algebra, then the space $(\mathcal{G}_Y^n)^{\text{an}}$ is given by equivalence classes of pairs $(\pi : \mathbb{C}^n \rightarrow V, \rho : R \rightarrow \text{End}_{\mathbb{C}}(V))$ and $\mathcal{F}_Y^n$ is the subspace of those pairs for which ρ is a normal map.*

We now define a continuous map

$$\theta = \theta_Y : (\mathcal{G}_Y^n)^{\text{an}} \rightarrow \left(\prod_{k=0}^n S^k Y \right)^{\text{an}},$$

where $S^k Y$ denotes the k th symmetric power of the variety Y . The map θ takes a quotient $\mathcal{O}_Y^n \twoheadrightarrow M$ and sends it to the 0-cycle on Y defined by the support of M . To establish the continuity of θ , we show θ is in fact given by a morphism of varieties

$$\theta : \mathcal{G}_Y^n \rightarrow \prod_{k=0}^n S^k Y.$$

Given an open subscheme $V \subset Y$, the induced maps $\mathcal{G}_V^n \rightarrow \mathcal{G}_Y^n$ and $\prod_{k=0}^n S^k V \rightarrow \prod_{k=0}^n S^k Y$ are open immersions. Moreover, since any finite set of points of Y is contained in an open affine subscheme, it suffices to define compatible maps θ_V for all open affine subschemes $V = \text{Spec } B$ of Y . To define θ_V , it suffices to exhibit a natural transformation of functors (of affine, connected varieties) from the functor

$$\text{Spec } A \mapsto \{[(A \otimes B)^n \twoheadrightarrow P] \mid P \text{ is } A\text{-projective and } A^n \rightarrow P \text{ is onto}\}$$

to the functor

$$\text{Spec } A \mapsto \prod_k \text{Hom}_{\mathbb{C}\text{-algebras}}((B^{\otimes k})^{\Sigma_k}, A).$$

This natural transformation is given as follows: For a quotient object $(A \otimes B)^n \twoheadrightarrow P$, let $\det_A(P) = \Lambda_A^k(P)$, where $k = \text{rank}_A(P)$. Observe that $\det_A(P)$ is naturally a $(B^{\otimes k})^{\Sigma_k}$ - A bi-module and thus there is a natural map of $\mathbb{C}$ -algebras

$$(B^{\otimes k})^{\Sigma_k} \rightarrow \text{End}_A(\det_A(P)) \cong A,$$

where the last isomorphism holds since $\det_A(P)$ is a rank one projective A -module. One may readily verify that the morphisms θ_V so defined patch together to form the desired morphism θ_Y .

The composition of the maps

$$\mathcal{F}_Y^n \xrightarrow{\subset} (\mathcal{G}_Y^n)^{\text{an}} \xrightarrow{\theta} \left(\prod_{k=0}^n S^k Y \right)^{\text{an}} = \prod_{k=0}^n S^k(Y^{\text{an}}),$$

which we will write as θ_{top} , is given more explicitly as follows. A $*$ -map $\rho : \mathcal{C}(Y^{\text{an}}) \rightarrow \text{Mat}_n(\mathbb{C})$ is given by points $y_1, \dots, y_j$ together with pairwise orthogonal subspaces $X_1, \dots, X_j$ of $\mathbb{C}^n$. The map $\theta_{\text{top}} : \mathcal{F}_Y^n \rightarrow \prod_{k=0}^n S^k(Y^{\text{an}})$ takes the point given by $((y_i), (X_i))$ to the point $\sum_i \dim(X_i) y_i$.

The strategy for the rest of the proof is to show the subspace inclusion $\mathcal{F}_Y^n \hookrightarrow (\mathcal{G}_Y^n)^{\text{an}}$ is “locally” a homotopy equivalence in a suitably strong sense. Namely, we will show that given any point $\underline{y} = \sum_i m_i y_i \in \prod_{k=0}^n S^k(Y^{\text{an}})$, there is an arbitrarily small analytic neighborhood V of $\underline{y}$ so that the induced map

$$\theta_{\text{top}}^{-1}(V) \rightarrow \theta^{-1}(V)$$

is a homotopy equivalence. The following lemma shows that this will suffice to prove the theorem.

Lemma 5.4. *Suppose we have a commuting triangle of topological spaces*

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ & \searrow p \quad \swarrow q & \\ & Z & \end{array}$$

having the following property: for all $z \in U \subset Z$ with U open, there is an open V with $z \in V \subset U$ such that the induced map $f|_{p^{-1}(V)} : p^{-1}(V) \rightarrow q^{-1}(V)$ is a homotopy equivalence. Then $f : W \rightarrow X$ is a homology equivalence.

Proof. It suffices to prove f induces an isomorphism on cohomology with coefficients in an arbitrary abelian group A . Call an open subset V of Z “nice” if $f|_{p^{-1}(V)} : p^{-1}(V) \rightarrow q^{-1}(V)$ is a homotopy equivalence. Let $Z = \cup V_\alpha$ be an open cover of Z by nice open subsets; the existence of such a cover follows immediately from the hypothesis. Write $\mathcal{V}^0$ for $\coprod_\alpha V_\alpha$. Also by hypothesis, for each α , there is an open cover $\cup_\beta V_{\alpha,\beta}$ of V_α by nice open subsets. Write $\mathcal{V}^1$ for $\coprod_{\alpha,\beta} V_{\alpha,\beta}$. Continuing in this fashion, we obtain an open hypercovering of Z by nice open subsets (i.e., an augmented simplicial space):

$$\cdots \rightarrow \mathcal{V}^2 \rightarrow \mathcal{V}^1 \rightarrow \mathcal{V}^0 \rightarrow Z.$$

Write $\mathcal{V}_X^i$ and $\mathcal{V}_W^i$ for the pullback of $\mathcal{V}^i$ to each of X and W , so that we have open hypercovers

$$\cdots \rightarrow \mathcal{V}_W^2 \rightarrow \mathcal{V}_W^1 \rightarrow \mathcal{V}_W^0 \rightarrow W$$

and

$$\cdots \rightarrow \mathcal{V}_X^2 \rightarrow \mathcal{V}_X^1 \rightarrow \mathcal{V}_X^0 \rightarrow X,$$

and a map between them. By the niceness condition, for each i we have an isomorphism

$$H^*(\mathcal{V}_X^i, A) \cong H^*(\mathcal{V}_W^i, A).$$

A simple spectral sequence argument completes the proof. $\square$

We now begin with the core of the proof of Theorem 5.1. Let us describe the proof heuristically before proceeding with the rigorous details: We first establish a form of étale descent for the construction $Y \mapsto \mathcal{G}_Y^n$. Since a smooth variety is locally in the étale topology isomorphic to affine space, we reduce the theorem to the case where Y is an étale neighborhood of the origin in affine space. In this case, we prove the theorem by constructing an explicit homotopy. The following lemma provides the first step.

Lemma 5.5. *Suppose $f : U \rightarrow V$ is a étale map of quasi-projective complex varieties. Let $[q : \mathcal{O}_U^n \rightarrow M]$ be a $\mathbb{C}$ -point of $\mathcal{G}_U^n$ such that $\theta([q]) = m_1 u_1 + \cdots + m_\nu u_\nu$ (where $u_i \neq u_j$ for $i \neq j$) has the property that $f(u_i) \neq f(u_j)$ for $i \neq j$. Then the induced morphism of varieties $f_* : \mathcal{G}_U^n \rightarrow \mathcal{G}_V^n$ is étale near $[q]$.*

Proof. Recall that if $U' \subset U$ is a Zariski open subset of U , then the natural morphism $\mathcal{G}_{U'}^n \rightarrow \mathcal{G}_U^n$ is an open immersion of varieties. Consequently, since any finite number of points on a quasi-projective varieties is contained in a Zariski open subset, we may assume that both U and V are affine. In fact, we take the intersection $\cap_{U' \subset U} \mathcal{G}_{U'}^n$, where U' ranges over all Zariski open subsets of U containing the points $u_1, \dots, u_\nu$. Writing R for $\mathcal{O}_{U;u_1,\dots,u_\nu}$, the semilocalization of U at the u_i ’s, this intersection yields the affine scheme $\mathcal{G}_R^n$ representing the functor which sends $\text{Spec } A$

to the collection of quotients $[(A \otimes R)^n \twoheadrightarrow M]$ such that M is a projective A module and $A^n \twoheadrightarrow M$ is surjection. Similarly, let $v_i = f(u_i)$ for all i and form the affine scheme $\cap \mathcal{G}_{V'}^n$, where V' ranges over all Zariski open neighborhoods of $v_1, \dots, v_\nu$ in V . We write the resulting scheme as $\mathcal{G}_S^n$, where $S = \mathcal{O}_{V;v_1,\dots,v_\nu}$. It suffices to show the natural map of affine schemes $\mathcal{G}_R^n \rightarrow \mathcal{G}_S^n$ induced by the natural ring map $S \rightarrow R$ is étale. Note that our hypotheses ensure that $S \rightarrow R$ is an étale map of semilocal rings such that $J_S R = J_R$ and $S/J_S^N \cong R/J_R^N$ for all N , where J_S and J_R are the Jacobson radicals of S and R .

To show $\mathcal{G}_R^n \rightarrow \mathcal{G}_S^n$ is étale, we use Artin's criterion [2, I.1.1], which shows that it suffices to prove the diagram of $\mathbb{C}$ -algebras

$$(13) \quad \begin{array}{ccc} \mathrm{Spec} A/I & \longrightarrow & \mathcal{G}_R^n \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec} A & \longrightarrow & \mathcal{G}_S^n \end{array}$$

can be completed uniquely, where A is a local, Artinian $\mathbb{C}$ -algebra and I is a proper ideal. Recall that a map $\mathrm{Spec} A \rightarrow \mathcal{G}_R^n$ is defined by an equivalence class of pairs $(p : A^n \twoheadrightarrow P, \rho : R \rightarrow \mathrm{End}_A(P))$, where P is a finitely generated projective A -module and ρ is a $\mathbb{C}$ -algebra map. Similarly, a map $\mathrm{Spec} A/I \rightarrow \mathcal{G}_R^n$ is given by a class of pairs $(q : (A/I)^n \twoheadrightarrow Q, \eta : S \rightarrow \mathrm{End}_{A/I}(Q))$, and the commutativity of the above diagram amounts to the existence of an A/I -module isomorphism $Q \cong P/I$ such that the two evident triangles involving $\bar{p} : (A/I)^n \twoheadrightarrow P/I$, $R \rightarrow \mathrm{End}_A(P) \rightarrow \mathrm{End}_{A/I}(P/I)$, q , and η commute.

Since A is local, we can find suitable isomorphisms $P \cong A^r$ and $Q \cong (A/I)^r$, so that we have a commutative square

$$(14) \quad \begin{array}{ccc} S & \longrightarrow & \mathrm{Mat}_r(A) \\ \downarrow & \nearrow & \downarrow \\ R & \longrightarrow & \mathrm{Mat}_r(A/I). \end{array}$$

The desired completion of diagram (13) will exist provided we can find a ring map $R \rightarrow \mathrm{Mat}_r(A)$ completing diagram (14) as indicated. But note that $\mathrm{Mat}_r(A)$ is an Artinian ring and $\mathrm{Mat}_r(A) \rightarrow \mathrm{Mat}_r(A/I)$ is surjective with nilpotent kernel. Thus, the maps $R \rightarrow \mathrm{Mat}_r(A)$ and $S \rightarrow \mathrm{Mat}_r(A/I)$ factor through R/J_R^N and S/J_S^N , respectively, for some $N \gg 0$. But since $R/J_R^N \cong S/J_S^N$, the result now follows immediately. $\square$

Remark 5.6. *The hypothesis $f(u_i) \neq f(u_j)$ for $i \neq j$ is essential in Lemma 5.5. For say $V = \mathrm{Spec} \mathbb{C}$, $U = V \amalg V$, and $U \rightarrow V$ is the obvious étale map. Then the fibers of $\mathcal{G}_U^n \rightarrow \mathcal{G}_V^n$ involve Grassmann varieties, and thus this map is not étale (or even quasi-finite).*

Given any subspace D of the analytic space Y^{an} , we write $(\mathcal{G}_Y^n)_D^{\mathrm{an}}$ for the subspace of those objects $\mathcal{O}_Y^n \twoheadrightarrow M$ such that M is supported on D . More formally, $(\mathcal{G}_Y^n)_D^{\mathrm{an}}$ is the inverse image under θ of the subspace $\coprod_k S^k D \subset \coprod_k S^k Y^{\mathrm{an}}$. In particular, observe that $(\mathcal{G}_Y^n)_D^{\mathrm{an}}$ is an open (resp., closed) subspace of $(\mathcal{G}_Y^n)^{\mathrm{an}}$ provided D is an open (resp., closed) subspace of Y^{an} . Similarly, $(\mathcal{F}_Y^n)_D$ is defined as the inverse image of $\coprod_k S^k D$ under θ_{top} . Observe that $(\mathcal{F}_Y^n)_D$ is more simply defined as the

subspace of those $*$ -maps $\mathcal{C}(Y) \rightarrow \text{Mat}_n(\mathbb{C})$ which factor as $\mathcal{C}(Y) \rightarrow \mathcal{C}(D) \rightarrow \text{Mat}_n(\mathbb{C})$.

Lemma 5.7. *Suppose $f : U \rightarrow V$ is a map of quasi-projective complex varieties and $D \subset U^{\text{an}}$ is an analytic open subset of U^{an} such that $f|_D : D \rightarrow V^{\text{an}}$ is one-to-one and f is étale near every point of D . Then the natural morphism of varieties $f_* : \mathcal{G}_U^n \rightarrow \mathcal{G}_V^n$ induces a homeomorphism*

$$f_* : (\mathcal{G}_U^n)_D^{\text{an}} \xrightarrow{\cong} (\mathcal{G}_V^n)_{f(D)}^{\text{an}}$$

Moreover, this homeomorphism restricts to a homeomorphism

$$f_* : (\mathcal{F}_U^n)_D \xrightarrow{\cong} (\mathcal{F}_V^n)_{f(D)}.$$

Proof. By Lemma 5.5, the map $f_* : \mathcal{G}_U^n \rightarrow \mathcal{G}_V^n$ is étale near each point of $(\mathcal{G}_U^n)_D^{\text{an}}$. In particular, the continuous map

$$f_*|_D : (\mathcal{G}_U^n)_D^{\text{an}} \rightarrow (\mathcal{G}_V^n)_{f(D)}^{\text{an}}$$

of topological spaces is open, and so it suffices to show $f_*|_D$ is bijective. As in the proof of Lemma 5.5, we let R be the semi-local ring of U at a collection of points $u_1, \dots, u_\nu$ of D and we let S be the semi-local ring of V at $v_1, \dots, v_\nu$, where $v_i = f(u_i)$. Since $f|_D : D \rightarrow f(D)$ is bijective, it suffices to establish the bijectivity of the function

$$G_R^n(\mathbb{C}) \rightarrow G_S^n(\mathbb{C}),$$

which sends $[\mathbb{C}^n \twoheadrightarrow V, R \rightarrow \text{End}_{\mathbb{C}}(V)]$ to $[\mathbb{C}^n \twoheadrightarrow V, S \rightarrow \text{End}_{\mathbb{C}}(V)]$ by restriction along the natural map $S \rightarrow R$. As in the proof of Lemma 5.5, this follows immediately from the fact that $\text{End}_{\mathbb{C}}(V)$ is Artinian and $R/J_R^N \cong S/J_S^N$, for all N , where J_R, J_S are the Jacobson radicals of R and S .

To prove the final assertion, note that $f|_D : D \rightarrow f(D)$ is an open map (in the analytic topology), since f is étale, and hence open, in a Zariski open neighborhood of each point of D . Thus, $f_D : D \rightarrow f(D)$ is a homeomorphism of topological spaces. Now merely observe that $(\mathcal{F}_U^n)_D$ and $(\mathcal{F}_V^n)_{f(D)}$ may be identified with the spaces of $*$ -maps of the form $\mathcal{C}(D) \rightarrow \text{Mat}_n(\mathbb{C})$ and $\mathcal{C}(f(D)) \rightarrow \text{Mat}_n(\mathbb{C})$, respectively. $\square$

Lemma 5.8. *The hypothesis of Lemma 5.4 are satisfied for the triangle of spaces*

$$\begin{array}{ccc} \mathcal{F}_Y^n & \xrightarrow{\quad} & (\mathcal{G}_Y^n)^{\text{an}} \\ & \searrow \theta_{\text{top}} \quad \swarrow \theta & \\ & \coprod_{k=0}^n S^k Y^{\text{an}}. & \end{array}$$

Consequently, the map $\mathcal{F}_Y^n \rightarrow (\mathcal{G}_Y^n)^{\text{an}}$ is a homology equivalence.

Proof. Choose a point $m_1 y_1 + \dots + m_\nu y_\nu$ of $\coprod_{k=0}^n S^k Y^{\text{an}}$ and an analytic open neighborhood U of this point. Observe that we can find an affine open subscheme of Y containing the y_i 's, and thus we may as well assume Y is affine. Let $Y_1, \dots, Y_c$ denote the connected components of Y , and assume Y_l has dimension d_l , for each l . Let X be the disjoint union of affine space $\coprod_l \mathbb{A}^{d_l}$, and observe that there exists a morphism $\pi : Y \rightarrow X$ such that $\pi(u_i) \neq \pi(u_j)$, for all $i \neq j$, and π is étale near each u_i . Consequently, we can choose, for each i , an open polydisk neighborhood E_i of $\pi(u_i)$ in the appropriate affine space $\mathbb{A}^{d_{i_i}}$ such that the closures of E_i and E_j are disjoint, for all $i \neq j$, and such that $\pi^{-1}(E_i) \rightarrow E_i$ is a homeomorphism, for all

i. Set $D = \pi^{-1}(E_1) \coprod \cdots \coprod \pi^{-1}(E_\nu)$ and note that we may assume that the E_i 's were chosen sufficiently small so that $D \subset U$.

Then by Lemma 5.7, we have a commutative square of spaces

$$\begin{array}{ccc} (\mathcal{F}_Y^n)_D & \longrightarrow & (\mathcal{G}_Y^n)_D^{\text{an}} \\ \cong \downarrow & & \downarrow \cong \\ (\mathcal{F}_X^n)_{\pi(D)} & \longrightarrow & (\mathcal{G}_X^n)_{\pi(D)}^{\text{an}} \end{array}$$

whose vertical maps are homeomorphisms. It thus suffices to show the bottom map of this diagram is a homotopy equivalence. Recall $\pi(D) = E_1 \coprod \cdots \coprod E_\nu$, where E_i is an open polydisk in $\mathbb{A}^{d_{i_1}}$. For each i , there exist a linear map $p_i : \mathbb{A}^{d_{i_1}} \rightarrow \mathbb{A}^{d_{i_1}}$, such that p_i sends the open unit polydisk of $\mathbb{A}^{d_{i_1}}$ homeomorphically onto E_i . Let $T = \mathbb{A}^{d_{i_1}} \coprod \cdots \coprod \mathbb{A}^{d_{i_\nu}}$ and $C = C_1 \coprod \cdots \coprod C_\nu$ where C_i is the open unit polydisk in $\mathbb{A}^{d_{i_1}}$, and define $p : T \rightarrow X$ to be the map $\coprod_i p_i$. Then Lemma 5.7 applies again to establish that the vertical maps in the commutative square of spaces

$$\begin{array}{ccc} (\mathcal{F}_T^n)_C & \longrightarrow & (\mathcal{G}_T^n)_C^{\text{an}} \\ p_* \downarrow \cong & & p_* \downarrow \cong \\ (\mathcal{F}_X^n)_{\pi(D)} & \longrightarrow & (\mathcal{G}_X^n)_{\pi(D)}^{\text{an}} \end{array}$$

are homeomorphisms. It thus suffices to show $(\mathcal{F}_T^n)_C \rightarrow (\mathcal{G}_T^n)_C^{\text{an}}$ is a homotopy equivalence.

Write $T = \text{Spec } A$, where

$$A = \mathbb{C}[x_{1,1}, \dots, x_{1,d_{i_1}}] \times \cdots \times \mathbb{C}[x_{\nu,1}, \dots, x_{\nu,d_{i_\nu}}].$$

Then a point of $(\mathcal{G}_T^n)_C^{\text{an}}$ is represented by a pair $(\mathbb{C}^n \twoheadrightarrow V, A \rightarrow \text{End}_{\mathbb{C}}(V))$. In other words, such a point is given by specifying a surjection $\mathbb{C}^n \twoheadrightarrow V$, idempotents $\epsilon_1, \dots, \epsilon_\nu$ of $\text{End}_{\mathbb{C}}(V)$, and elements $\theta_{i,j} \in \text{End}_{\mathbb{C}}(V)$, for $i = 1, \dots, \nu$, $j = 1, \dots, d_{i_1}$, such that the following conditions hold:

1. the elements $\epsilon_i, \theta_{i,j}$ of $\text{End}_{\mathbb{C}}(V)$ commute with each other,
2. $\sum_i \epsilon_i = 1$,
3. $\epsilon_i \epsilon_j = 0$ for $i \neq j$,
4. $\theta_{i,j} \epsilon_i = \theta_{i,j}$, for all i, j .

Moreover, such a point belongs to $(\mathcal{G}_T^n)_C^{\text{an}}$ if the eigenvalues of $\theta_{i,j}$ lie inside the unit circle in the complex plane, for all i, j . The space $\mathcal{F}_T^n$ is the subspace of all points satisfying the additional constraint that the ϵ_i 's and $\theta_{i,j}$'s be normal elements of the C^* -algebra $\text{End}_{\mathbb{C}}(V)$, where the Hermitian inner product on V is induced by the surjection $\mathbb{C}^n \twoheadrightarrow V$. A point of $\mathcal{F}_T^n$ belongs to $(\mathcal{F}_T^n)_C$ if and only if the eigenvalues of $\theta_{i,j}$ belong to the unit open disk in the complex plane.

Using these descriptions of the spaces $(\mathcal{F}_T^n)_C$ and $(\mathcal{G}_T^n)_C^{\text{an}}$, we construct a pair of compatible deformation retractions. Namely, letting $I = [0, 1]$, define maps

$$I \times (\mathcal{G}_T^n)_C^{\text{an}} \rightarrow (\mathcal{G}_T^n)_C^{\text{an}}, \quad \text{and} \quad I \times (\mathcal{F}_T^n)_C \rightarrow (\mathcal{F}_T^n)_C,$$

by sending $(\lambda, \mathbb{C}^n \twoheadrightarrow V, (\epsilon_i), (\theta_{i,j}))$ to $(\mathbb{C}^n \twoheadrightarrow V, (\epsilon_i), (\lambda \theta_{i,j}))$. One may readily verify that these definitions are independent of the representative chosen and define continuous maps. It follows that $(\mathcal{G}_T^n)_C^{\text{an}}$ (resp., $(\mathcal{F}_T^n)_C$) is homotopy equivalent to the subspace ${}^0(\mathcal{G}_T^n)_C^{\text{an}}$ (resp., ${}^0(\mathcal{F}_T^n)_C$) defined by the condition $\theta_{i,j} = 0$, for all i, j . The space ${}^0(\mathcal{G}_Y^n)_C^{\text{an}}$ is the space of points given by a quotient $[\mathbb{C}^n \twoheadrightarrow V]$ together with a vector space decomposition $V = V_1 \oplus \cdots \oplus V_\nu$ defined by idempotents. The space

${}^0(\mathcal{F}_Y^n)_C$ consists of a quotient $\mathbb{C}^n \rightarrow V$ together with an *orthogonal* decomposition $V = V_1 \perp \cdots \perp V_\nu$ defined by normal idempotents.

It remains to show that for any finite dimension Hermitian inner product space V , the inclusion of the space of all orthogonal decompositions $V = V_1 \perp \cdots \perp V_\nu$ into the space of all direct sum decompositions $V = V_1 \oplus \cdots \oplus V_\nu$ is a homotopy equivalence. Each of these spaces decomposes into a disjoint union indexed by the dimensions $d_1, \dots, d_\nu$ of the subspaces $V_1, \dots, V_\nu$ (where $d_1 + \cdots + d_\nu = k$). Write $\text{Grass}^{\text{orth}}(d_1, \dots, d_\nu) \subset \text{Grass}(d_1, \dots, d_\nu)$ for these connected components. We show this inclusion is a homotopy equivalence by induction on ν , with the case $\nu = 1$ being evident. The maps

$$\text{Grass}^{\text{orth}}(d_1, \dots, d_\nu) \rightarrow \text{Grass}^{\text{orth}}(d_1, \dots, d_{\nu-1} + d_\nu)$$

and

$$\text{Grass}(d_1, \dots, d_\nu) \rightarrow \text{Grass}(d_1, \dots, d_{\nu-1} + d_\nu),$$

given by taking the direct sum of the last two subspaces, are each fibrations with the fiber of the first map being $\text{Grass}^{\text{orth}}(d_{\nu-1}, d_\nu)$ and the fiber of the second map being $\text{Grass}(d_{\nu-1}, d_\nu)$. In general, the space $\text{Grass}^{\text{orth}}(a, b)$ is homeomorphic to $\text{Grass}_a(\mathbb{C}^{a+b})$ under the evident forgetful map while the analogous map $\text{Grass}(a, b) \rightarrow \text{Grass}_a(\mathbb{C}^{a+b})$ is a fibration with contractible fibers. $\square$

This completes the proof of Theorem 5.1, since the preceding lemma shows that $\mathfrak{ob}_n \mathcal{A}^{\text{top}}(S^0, Y^{\text{an}}) \rightarrow \mathfrak{ob}_n \mathcal{A}^{\text{semi}}(\mathbb{C}, Y)$ is a homology equivalence, for all n , and thus $\mathbf{bu}^c(S^0, Y^{\text{an}}) \rightarrow \mathcal{K}^{\text{semi}}(\mathbb{C}, Y)$ is a homotopy equivalence of spectra by Lemma 5.2. $\square$

We close this section with a corollary to Theorem 5.1 which will be needed to describe a map from $\mathcal{K}^{\text{semi}}$ to $\mathbf{bu}^c$. Observe that for a quasi-projective variety Y , the functor $\mathcal{A}^{\text{top}}(S^0, Y^{\text{an}}) \rightarrow \mathcal{A}^{\text{semi}}(\mathbb{C}, Y)$ extends in an evident manner to a functor $\mathcal{A}^{\text{top}}(W, Y^{\text{an}}) \rightarrow \mathcal{A}^{\text{qtop}}(W, Y)$, for any CW complex W .

Corollary 5.9. *Suppose W is an unpointed CW complex having the homotopy type of a finite CW complex and let Y be a quasi-projective, smooth complex variety. Then the natural continuous functor*

$$\mathcal{A}^{\text{top}}(W, Y^{\text{an}}) \rightarrow \mathcal{A}^{\text{qtop}}(W, Y)$$

induces a weak equivalence of spectra

$$\mathbf{bu}^c(W, Y^{\text{an}}) \rightarrow \mathcal{K}^{\text{qtop}}(W, Y).$$

Proof. Viewed as functors of W , the two spectra involved identify homotopy equivalent spaces, and so we may assume W is a finite CW complex. We need to show that the natural map of Γ -spaces

$$|\mathcal{A}^{\text{top}}(W, - \wedge Y_+^{\text{an}})| \rightarrow |\mathcal{A}^{\text{qtop}}(W, - \wedge Y_+)|$$

induces a weak equivalence on homotopy-theoretic group completions:

$$\Omega^1(S^1 \otimes |\mathcal{A}^{\text{top}}(W, - \wedge Y_+^{\text{an}})|) \rightarrow \Omega^1(S^1 \otimes |\mathcal{A}^{\text{qtop}}(W, - \wedge Y_+)|).$$

Using Lemmas 3.9, 3.10, 3.3, and 3.4, we have natural weak equivalences of Γ -spaces

$$|\mathcal{A}^{\text{top}}(W, - \wedge Y_+^{\text{an}})| \sim \underline{\text{Maps}}(W_+, |\mathcal{A}^{\text{top}}(S^0, - \wedge Y_+^{\text{an}})|)$$

and

$$|\mathcal{A}^{\text{qtop}}(W, - \wedge Y_+)| \sim \underline{\text{Maps}}(W_+, |\mathcal{A}^{\text{qtop}}(\text{Spec } \mathbb{C}, - \wedge Y_+)|).$$

Thus, it suffices to show that the left-hand vertical map in the commutative diagram

$$\begin{array}{ccc}
\Omega^1 (S^1 \otimes \underline{\text{Maps}}(W_+, |\mathcal{A}^{top}(S^0, - \wedge Y_+^{an})|)) & \longrightarrow & \Omega^1 \underline{\text{Maps}}(W_+, S^1 \otimes |\mathcal{A}^{top}(S^0, - \wedge Y_+^{an})|) \\
\downarrow & & \downarrow \\
\Omega^1 (S^1 \otimes \underline{\text{Maps}}(W_+, |\mathcal{A}^{qtop}(S^0, - \wedge Y_+)|)) & \longrightarrow & \Omega^1 \underline{\text{Maps}}(W_+, S^1 \otimes |\mathcal{A}^{qtop}(S^0, - \wedge Y_+)|)
\end{array}$$

is a weak equivalence. But the horizontal arrows of this diagram are each weak equivalences by Corollary 4.3 and the right-hand vertical map is a weak equivalence by the theorem. $\square$

Remark 5.10. *Corollary 5.9 will be applied in the case where $W = X^{an}$, where X is quasi-projective complex variety. Observe that the hypotheses of the corollary are met in this case. Indeed, embedding X as an open subvariety of a projective variety $\overline{X}$, there exists by [20] a triangulation of $\overline{X}^{an}$ such that $(\overline{X} - X)^{an}$ is a subcomplex. Thus X^{an} is the complement of a (closed) subcomplex in a finite simplicial complex. Define the core of X^{an} to be the union of all closed simplices in the first barycentric subdivision of the given triangulation of $\overline{X}^{an}$ which are entirely contained in X^{an} . Then X^{an} admits a deformation retraction onto its core, which is a finite CW complex.*

6. OPERATIONS AND COMPATIBILITY OF SUCH

Let X and Y be quasi-projective complex varieties. Observe that there are continuous functors of topological categories

$$\mathcal{A}^{semi}(X, Y) \longrightarrow \mathcal{A}^{qtop}(X^{an}, Y) \longleftarrow \mathcal{A}^{top}(X^{an}, Y^{an}),$$

which are covariantly natural in Y and contravariantly natural in X . These maps induce maps of Γ -spaces

$$|\mathcal{A}^{semi}(X, - \wedge Y_+)| \longrightarrow |\mathcal{A}^{qtop}(X^{an}, - \wedge Y_+)| \longleftarrow |\mathcal{A}^{top}(X^{an}, - \wedge Y_+^{an})|,$$

and thus define natural maps of spectra

$$\mathcal{K}^{semi}(X, Y) \longrightarrow \mathcal{K}^{qtop}(X^{an}, Y) \longleftarrow \mathbf{bu}^c(X^{an}, Y^{an}),$$

with the map on the right a weak equivalence of spectra when Y is smooth by Corollary 5.9 and Remark 5.10. Moreover, the map (11) extends this diagram to include the non-connective spectrum $\mathbf{bu}(-, -)$:

$$(15) \quad \mathcal{K}^{semi}(X, Y) \longrightarrow \mathcal{K}^{qtop}(X^{an}, Y) \longleftarrow \mathbf{bu}^c(X^{an}, Y^{an}) \longrightarrow \mathbf{bu}(X^{an}, Y^{an}).$$

The goal of this section is to define certain operations for each of the theories appearing in diagram (15) which are compatible with the displayed maps. For the theory $\mathbf{bu}$, such operations could be defined using classical machinery, since this theory is represented by a commutative ring spectrum. Notice that such an easy construction is not possible for the other three theories. In fact, we define operations for all four theories by using just two basic operations which represent an external product and a sort of composition operation. The operations obtained in this manner for $\mathbf{bu}$ will be shown to agree with the classical operations.

The first of these two basic operations is *external product* and is written using the symbol $\boxtimes$. If X, W, Y and Z are quasi-projective complex varieties, we define a continuous functor

$$\boxtimes : \mathcal{A}^{semi}(X, Y) \times \mathcal{A}^{semi}(W, Z) \rightarrow \mathcal{A}^{semi}(X \times W, Y \times Z)$$

by sending a pair of objects

$$[p : \mathcal{O}_{X \times Y}^m \twoheadrightarrow M], [q : \mathcal{O}_{W \times Z}^n \twoheadrightarrow N]$$

to the object given by composition of

$$\mathcal{O}_{X \times W \times Y \times Z}^{nm} \cong \mathcal{O}_{X \times Y}^m \otimes \mathcal{O}_{W \times Z}^n \twoheadrightarrow \pi_{X \times Y}^* M \otimes \pi_{W \times Z}^* N,$$

where the isomorphism is given using lexicographic ordering – that is, if $e_1, e_2, \dots$ denote the standard basis elements, the isomorphism sends $e_i \otimes e_j \in \mathcal{O}^m \otimes \mathcal{O}^n$ to $e_{(i-1)n+j} \in \mathcal{O}^{mn}$. Here, the maps $\pi_{X \times Y}$ and $\pi_{W \times Z}$ are the evident projections. We define the pairing $\boxtimes$ on morphisms by specifying an external product operation on the collection of injective functions $\underline{m} \hookrightarrow \underline{n}$. Namely, for functions $\alpha : \underline{m} \hookrightarrow \underline{n}$, $\beta : \underline{p} \hookrightarrow \underline{q}$, define $\alpha \boxtimes \beta : \underline{mp} \hookrightarrow \underline{nq}$ by sending $(i-1)p + j$ to $(\alpha(i)-1)q + \beta(j)$, where $i \in \underline{m}, j \in \underline{p}$. (This definition corresponds to taking the tensor product of the matrices $\tilde{\alpha}, \tilde{\beta}$ associated to α, β , following the lexicographic ordering convention above.)

The proof of the following lemma is straightforward.

Lemma 6.1. *The bi-functor $\boxtimes$ is associative, in the sense that if U, V are also varieties, then the two evident functors*

$$\mathcal{A}^{semi}(X, Y) \times \mathcal{A}^{semi}(W, Z) \times \mathcal{A}^{semi}(U, V) \rightarrow \mathcal{A}^{semi}(X \times W \times U, Y \times Z \times V)$$

coincide.

One may readily verify that the definition of $\boxtimes$ extends to give a continuous bi-functor

$$\boxtimes : \mathcal{A}^{qtop}(X^{an}, Y) \times \mathcal{A}^{qtop}(W^{an}, Z) \rightarrow \mathcal{A}^{qtop}((X \times W)^{an}, (Y \times Z)).$$

These pairings of categories induce external pairings of Γ -spaces (see the Appendix)

$$\begin{aligned} (m_+ \mapsto \mathcal{A}^{semi}(X, m_+ \wedge Y_+)) \times (n_+ \mapsto \mathcal{A}^{semi}(W, n_+ \wedge Z_+)) \\ \downarrow \boxtimes \\ (m_+, n_+ \mapsto \mathcal{A}^{semi}(X \times W, m_+ \wedge Z_+ \wedge n_+ \wedge Y_+)), \end{aligned}$$

and similarly for $\mathcal{A}^{qtop}$. As with any pairing of Γ -spaces (see Lemma A.3), there are associated pairings of spectra

$$\begin{aligned} \boxtimes : \mathcal{K}^{semi}(X, Y) \wedge \mathcal{K}^{semi}(W, Z) &\rightarrow \mathcal{K}^{semi}(X \times W, Y \times Z) \\ \boxtimes : \mathcal{K}^{qtop}(X^{an}, Y) \wedge \mathcal{K}^{qtop}(W^{an}, Z) &\rightarrow \mathcal{K}^{qtop}(X^{an} \times W^{an}, Y \times Z), \end{aligned}$$

and these pairings are natural and are compatible with the map between the two theories.

For CW complexes W, T, V , and U , with V and U pointed, there is an analogous topological version of the bifunctor $\boxtimes$,

$$\boxtimes : \mathcal{A}^{top}(W, V) \times \mathcal{A}^{top}(T, U) \rightarrow \mathcal{A}^{top}(W \times T, V \wedge U),$$

defined as follows. Recall that we can identify $\mathbf{ob}\mathcal{A}^{top}(W, V)$ with $\underline{\text{Maps}}(W_+, \mathbf{ob}\mathcal{A}^{top}(S^0, V))$, and thus the definition of $\boxtimes$ on objects can be given as the composition of

$$\begin{array}{c} \underline{\text{Maps}}(W_+, \mathbf{ob}\mathcal{A}^{top}(S^0, V)) \times \underline{\text{Maps}}(T_+, \mathbf{ob}\mathcal{A}^{top}(S^0, U)) \\ \downarrow \\ \underline{\text{Maps}}((W \times T)_+, \mathbf{ob}\mathcal{A}^{top}(S^0, V) \wedge \mathbf{ob}\mathcal{A}^{top}(S^0, U)) \\ \downarrow \\ \underline{\text{Maps}}((W \times T)_+, \mathbf{ob}\mathcal{A}^{top}(S^0, V \wedge U)), \end{array}$$

once the pairing

$$\mathbf{ob}\mathcal{A}^{top}(S^0, V) \wedge \mathbf{ob}\mathcal{A}^{top}(S^0, U) \rightarrow \mathbf{ob}\mathcal{A}^{top}(S^0, V \wedge U),$$

is specified. This latter pairing is defined in (7) of Section 3. The definition of $\boxtimes$ on morphisms is defined just as for $\mathcal{A}^{semi}$. As before, we obtain pairings of Γ -spaces

$$\boxtimes : |\mathcal{A}^{top}(W, - \wedge V)| \wedge |\mathcal{A}^{top}(T, - \wedge U)| \rightarrow |\mathcal{A}^{top}(W \times T, - \wedge V \wedge - \wedge U)|,$$

and consequently a pairing of spectra

$$\boxtimes : \mathbf{bu}^c(W, V) \wedge \mathbf{bu}^c(T, U) \rightarrow \mathbf{bu}^c(W \times T, V \wedge U).$$

The above pairing extends to a pairing of $\mathcal{W}$ -spaces (see the Appendix), and we therefore obtain a pairing of spectra

$$\boxtimes : \mathbf{bu}(W, V) \wedge \mathbf{bu}(T, U) \rightarrow \mathbf{bu}(W \times T, V \wedge U)$$

defined by the collection of pairings of spaces

$$|\mathcal{A}^{top}(W, S^j \wedge V)| \wedge |\mathcal{A}^{top}(T, S^k \wedge U)| \rightarrow |\mathcal{A}^{top}(W \times T, S^{j+k} \wedge V \wedge U)|, \quad j, k \geq 0.$$

The next result follows immediately from the definition of $\boxtimes$.

Proposition 6.2. *The external pairings $\boxtimes$ defined above are compatible with the maps between the theories involved. That is, for quasi-projective varieties X , W , Y , and Z , the diagram of spectra*

$$\begin{array}{ccc} \mathcal{K}^{semi}(X, Y) \wedge \mathcal{K}^{semi}(W, Z) & \xrightarrow{\boxtimes} & \mathcal{K}^{semi}(X \times W, Y \times Z) \\ \downarrow & & \downarrow \\ \mathcal{K}^{qtop}(X^{an}, Y) \wedge \mathcal{K}^{qtop}(W^{an}, Z) & \xrightarrow{\boxtimes} & \mathcal{K}^{qtop}(X^{an} \times W^{an}, Y \times Z) \\ \uparrow & & \uparrow \\ \mathbf{bu}^c(X^{an}, Y^{an}) \wedge \mathbf{bu}^c(W^{an}, Z^{an}) & \xrightarrow{\boxtimes} & \mathbf{bu}^c(X^{an} \times W^{an}, Y^{an} \times Z^{an}) \\ \downarrow & & \downarrow \\ \mathbf{bu}(X^{an}, Y^{an}) \wedge \mathbf{bu}(W^{an}, Z^{an}) & \xrightarrow{\boxtimes} & \mathbf{bu}(X^{an} \times W^{an}, Y^{an} \times Z^{an}). \end{array}$$

commutes.

Taking $Y = Z = \text{Spec } \mathbb{C}$ for the pairing $\boxtimes$, we obtain a natural pairing we call *external product for cohomology*. We write this pairing as

$$\overline{\wedge} : \mathcal{K}^{semi}(X, \mathbb{C}) \wedge \mathcal{K}^{semi}(W, \mathbb{C}) \rightarrow \mathcal{K}^{semi}(X \times W, \mathbb{C}),$$

and similarly for the other three theories. Taking $X = W = \text{Spec } \mathbb{C}$ for the pairing $\boxtimes$, we get the *external product for homology*, which we write as

$$\Delta : \mathcal{K}^{\text{semi}}(\mathbb{C}, Y) \wedge \mathcal{K}^{\text{semi}}(\mathbb{C}, Z) \rightarrow \mathcal{K}^{\text{semi}}(\mathbb{C}, Y \times Z),$$

and similarly for the other theories. Taking $X = W$ in the external product for cohomology, we can further compose with pullback along the diagonal map $\Delta : X \rightarrow X \times X$ to define *cup product* for cohomology, which we write as

$$\cup : \mathcal{K}^{\text{semi}}(X, \mathbb{C}) \wedge \mathcal{K}^{\text{semi}}(X, \mathbb{C}) \rightarrow \mathcal{K}^{\text{semi}}(X, \mathbb{C}),$$

and similarly for the other theories. Observe that the cup product pairings for $\mathcal{K}^{\text{semi}}$, $\mathcal{K}^{\text{qtop}}$, and $\mathbf{bu}^c$ are induced by the product pairing for the Γ -rings $|\mathcal{A}^{\text{semi}}(X, -)|$, $|\mathcal{A}^{\text{qtop}}(X^{\text{an}}, -)|$, and $|\mathcal{A}^{\text{top}}(X^{\text{an}}, -)|$, and thus by Lemma A.3, the cup product pairings endow the spectra $\mathcal{K}^{\text{semi}}(X, \mathbb{C})$, $\mathcal{K}^{\text{qtop}}(X^{\text{an}}, \mathbb{C})$, and $\mathbf{bu}^c(X^{\text{an}}, S^0)$ with the structures of commutative ring spectra. Similarly, the cup product pairing for $\mathbf{bu}$ is given by the product pairing for the FSP $|\mathcal{A}^{\text{top}}(X^{\text{an}}, -)|$, and so by Lemma A.5, cup product endows $\mathbf{bu}(X^{\text{an}}, S^0)$ with the structure of a commutative ring spectrum.

We use the same symbols $\overline{}$, Δ , and $\cap$ for the associated operations on homotopy groups. The following result follows immediately from the construction of cup product.

Proposition 6.3. *For any quasi-projective complex variety X , the natural maps*

$$\mathcal{K}_*^{\text{semi}}(X, \mathbb{C}) \longrightarrow \mathcal{K}_*^{\text{qtop}}(X^{\text{an}}, \mathbb{C}) \xleftarrow{\cong} \mathbf{bu}_*^c(X^{\text{an}}, S^0) \longrightarrow \mathbf{bu}_*(X^{\text{an}}, S^0)$$

are graded ring homomorphisms.

We now turn to the definition of another type of operation for the four bivariate theories we are considering. We begin with semi-topological K -theory. Given varieties X , Y , and Z , there is a “composition” functor

$$\theta_{X,Y,Z} : \mathcal{A}^{\text{semi}}(X, Y) \times \mathcal{A}^{\text{semi}}(Y, Z) \rightarrow \mathcal{A}^{\text{semi}}(X, Z)$$

defined on objects by sending a pair

$$([p : \mathcal{O}_{X \times Y}^m \rightarrow M], [q : \mathcal{O}_{Y \times Z}^n \rightarrow N])$$

to the object

$$[p \otimes q : \mathcal{O}_{X \times Y \times Z}^{mn} \cong \mathcal{O}_{X \times Y \times Z}^n \otimes \mathcal{O}_{X \times Y \times Z}^m \rightarrow \pi_{X \times Y}^* M \otimes \pi_{Y \times Z}^* N].$$

Here, the isomorphism is given by using the lexicographic indexing convention used to define external products above. The definition of this functor on morphisms is given by external product of injective functions just as with $\boxtimes$. Clearly, this gives a well-defined continuous functor, which is strictly associative in the sense that given another variety W , the two evident functors

$$\mathcal{A}^{\text{semi}}(X, Y) \times \mathcal{A}^{\text{semi}}(Y, Z) \times \mathcal{A}^{\text{semi}}(Z, W) \rightarrow \mathcal{A}^{\text{semi}}(X, W)$$

coincide.

The pairing $\theta_{X,Y,Z}$ extends to a pairing

$$\mathcal{A}^{\text{semi}}(X, m_+ \wedge Y_+) \times \mathcal{A}^{\text{semi}}(Y, n_+ \wedge Z_+) \rightarrow \mathcal{A}^{\text{semi}}(X, mn_+ \wedge Z_+)$$

by identifying $m_+ \wedge n_+$ with mn_+ via $(i, j) \mapsto (i-1)n + j$. Upon taking nerves, we obtain an external pairing of Γ -spaces, and thus a pairing of spectra

$$\theta_{X,Y,Z} : \mathcal{K}^{\text{semi}}(X, Y) \wedge \mathcal{K}^{\text{semi}}(Y, Z) \rightarrow \mathcal{K}^{\text{semi}}(X, Z).$$

We also have the pairing

$$\mathcal{A}^{\text{qtop}}(X^{\text{an}}, Y) \times \mathcal{A}^{\text{qtop}}(Y^{\text{an}}, Z) \rightarrow \mathcal{A}^{\text{qtop}}(X^{\text{an}}, Z),$$

defined in precisely the same manner, which gives a pairing of spectra

$$\theta_{X,Y,Z} : \mathcal{K}^{\text{qtop}}(X^{\text{an}}, Y) \wedge \mathcal{K}^{\text{qtop}}(Y^{\text{an}}, Z) \rightarrow \mathcal{K}^{\text{qtop}}(X^{\text{an}}, Z).$$

Finally, for (unpointed) CW complexes W , V , U , and V , we have the pairing

$$(16) \quad \mathcal{A}^{\text{top}}(W, V) \times \mathcal{A}^{\text{top}}(V, U) \rightarrow \mathcal{A}^{\text{top}}(W, U)$$

defined on objects by sending the pair

$$(\rho : \mathcal{C}(V) \rightarrow \text{Mat}_m(\mathcal{C}(W)), \eta : \mathcal{C}(U) \rightarrow \text{Mat}_n(\mathcal{C}(V)))$$

to the composition

$$\mathcal{C}(U) \xrightarrow{\eta} \text{Mat}_n(\mathcal{C}(V)) \xrightarrow{\text{Mat}_n(\rho)} \text{Mat}_n(\text{Mat}_m(\mathcal{C}(W))) \xrightarrow{\cong} \text{Mat}_{nm}(\mathcal{C}(W)),$$

where the isomorphism is given using the lexicographic ordering as before. The pairing is defined on morphisms just as for $\mathcal{A}^{\text{semi}}$. As above, this pairing of topological categories induces an external pairing of Γ -spaces and hence a pairing of spectra

$$\theta_{X,Y,Z} : \mathbf{bu}^c(X, Y) \wedge \mathbf{bu}^c(Y, Z) \rightarrow \mathbf{bu}^c(X, Z).$$

The pairing (16) of topological categories extends to give a pairing of $\mathcal{W}$ -spaces and therefore induces the pairing of spectra

$$\theta_{X,Y,Z} : \mathbf{bu}(X, Y) \wedge \mathbf{bu}(Y, Z) \rightarrow \mathbf{bu}(X, Z).$$

We also use θ to denote the induced maps on homotopy groups in each of the four settings. It is convenient to think of these θ pairings as analogous to composition rules. Thus, given $\alpha \in K_p^{\text{semi}}(X, Y)$ and $\beta \in K_q^{\text{semi}}(Y, Z)$, we will also write $\beta \circ \alpha$ for $\theta(\alpha, \beta)$, and similarly for the other three theories.

The following results follow immediately from the definitions. In this proposition and subsequent ones, we use the shorthand $\mathcal{K}$ to denote any of the four bivariate spectrum-valued theories under consideration and we let K_q stand for its q -th homotopy group. The reader should interpret $\text{Spec } \mathbb{C}$ to mean S^0 and X to mean X^{an} when appropriate.

Proposition 6.4. *Let $\mathcal{K}$ denote any of the four bivariate theories $\mathcal{K}^{\text{semi}}$, $\mathcal{K}^{\text{qtop}}$, $\mathbf{bu}^c$, or $\mathbf{bu}$. The θ pairings for $\mathcal{K}$ are natural and associative. By associative, we mean that given quasi-projective varieties X , Y , Z , and W , the diagram*

$$\begin{array}{ccc} \mathcal{K}(X, Y) \wedge \mathcal{K}(Y, Z) \wedge \mathcal{K}(Z, W) & \xrightarrow{\theta_{X,Y,Z} \wedge \text{id}} & \mathcal{K}(X, Z) \wedge \mathcal{K}(Z, W) \\ \text{id} \wedge \theta_{Y,Z,W} \downarrow & & \downarrow \theta_{X,Z,W} \\ \mathcal{K}(X, Y) \wedge \mathcal{K}(Y, W) & \xrightarrow{\theta_{X,Y,W}} & \mathcal{K}(X, W) \end{array}$$

commutes. Thus, for elements $\alpha \in K_p(X, Y)$, $\beta \in K_q(Y, Z)$, and $\gamma \in K_r(Z, W)$, we have the equation

$$\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha,$$

where $x \circ y = \theta(y, x)$.

Proposition 6.5. *Let $\mathcal{K}$ denote any of the four bivariant theories $\mathcal{K}^{semi}$, $\mathcal{K}^{qtop}$, $\mathbf{bu}^c$, or $\mathbf{bu}$. The θ pairing and external product operation $\boxtimes$ for $\mathcal{K}$ are compatible in the sense that*

$$\begin{array}{ccc}
\mathcal{K}(X, Z) \wedge \mathcal{K}(Y, W) \wedge \mathcal{K}(Z, S) \wedge \mathcal{K}(W, T) & \xrightarrow{1 \wedge \tau \wedge 1} & \mathcal{K}(X, Z) \wedge \mathcal{K}(Z, S) \wedge \mathcal{K}(Y, W) \wedge \mathcal{K}(W, T) \\
\downarrow \boxtimes \wedge \boxtimes & & \downarrow \theta \wedge \theta \\
& & \mathcal{K}(X, S) \wedge \mathcal{K}(Y, T) \\
& & \downarrow \boxtimes \\
\mathcal{K}(X \times Y, Z \times W) \wedge \mathcal{K}(Z \times W, S \times T) & \xrightarrow{\theta} & \mathcal{K}(X \times Y, S \times T)
\end{array}$$

commutes for all X, Y, Z, W, S , and T , where τ is the evident involution. In terms of homotopy groups, for all $\alpha \in K_i(X, Z)$, $\beta \in K_j(Y, W)$, $\gamma \in K_k(Z, S)$, and $\delta \in K_l(W, T)$, we have the equation

$$(\gamma \boxtimes \delta) \circ (\alpha \boxtimes \beta) = (\gamma \circ \alpha) \boxtimes (\delta \circ \beta).$$

Proposition 6.6. *The θ pairings are compatible with the natural maps between the four theories. That is, given quasi-projective varieties X, Y and Z the diagram of spectra*

$$\begin{array}{ccc}
\mathcal{K}^{semi}(X, Y) \wedge \mathcal{K}^{semi}(Y, Z) & \xrightarrow{\theta} & \mathcal{K}^{semi}(X, Z) \\
\downarrow & & \downarrow \\
\mathcal{K}^{qtop}(X^{an}, Y) \wedge \mathcal{K}^{qtop}(Y^{an}, Z) & \xrightarrow{\theta} & \mathcal{K}^{qtop}(X^{an}, Z) \\
\uparrow & & \uparrow \\
\mathbf{bu}^c(X^{an}, Y^{an}) \wedge \mathbf{bu}^c(Y^{an}, Z^{an}) & \xrightarrow{\theta} & \mathbf{bu}^c(X^{an}, Z^{an}) \\
\downarrow & & \downarrow \\
\mathbf{bu}(X^{an}, Y^{an}) \wedge \mathbf{bu}(Y^{an}, Z^{an}) & \xrightarrow{\theta} & \mathbf{bu}(X^{an}, Z^{an})
\end{array}$$

commutes.

We proceed to define two “slant product” operations for each of the four theories under consideration. We prove that these slant product operations are compatible with the maps between these theories. The reader should consult [1, III.9], [34, §13] for a construction of such operation in the context of generalized cohomology theories of CW complexes.

In the pairings of Proposition 6.2, we can take $W = Z$ and let 1_W denote the object of $\mathcal{A}^{semi}(W, W)$ given by the canonical quotient $\mathcal{O}_{W \times W}^1 \twoheadrightarrow \mathcal{O}_W$ associated to the diagonal embedding $W \hookrightarrow W \times W$. The topological analogue is the element $1_W \in \mathcal{A}^{top}(W^{an}, W^{an})$ given by the identity map $\mathcal{C}(W) \rightarrow \text{Mat}_1(\mathcal{C}(W))$, and there is a similarly defined object, also written 1_W , which belongs to $\mathcal{A}^{qtop}(W^{an}, W)$. These distinguished objects are compatible under the relevant maps so that we

obtain maps of spectra fitting into the commutative diagram

$$\begin{array}{ccc}
\mathcal{K}^{\text{semi}}(X, Y) & \xrightarrow{(-) \boxtimes 1_W} & \mathcal{K}^{\text{semi}}(X \times W, Y \times W) \\
\downarrow & & \downarrow \\
\mathcal{K}^{\text{qtop}}(X^{\text{an}}, Y) & \xrightarrow{(-) \boxtimes 1_W} & \mathcal{K}^{\text{qtop}}(X^{\text{an}} \times W^{\text{an}}, Y \times W) \\
\uparrow & & \uparrow \\
\mathbf{bu}^c(X^{\text{an}}, Y^{\text{an}}) & \xrightarrow{(-) \boxtimes 1_W} & \mathbf{bu}^c(X^{\text{an}} \times W^{\text{an}}, Y^{\text{an}} \times W^{\text{an}}) \\
\downarrow & & \downarrow \\
\mathbf{bu}(X^{\text{an}}, Y^{\text{an}}) & \xrightarrow{(-) \boxtimes 1_W} & \mathbf{bu}(X^{\text{an}} \times W^{\text{an}}, Y^{\text{an}} \times W^{\text{an}}).
\end{array}$$

The maps defined by $1_W \boxtimes (-)$ are compatible in this sense as well. These maps lead to the definitions of the slant products:

Definition 6.7. Let $\mathcal{K}$ denote any of the four bivariant theories $\mathcal{K}^{\text{semi}}$, $\mathcal{K}^{\text{qtop}}$, $\mathbf{bu}^c$, or $\mathbf{bu}$. For X and Y quasi-projective varieties, the slant product pairing $/$ is defined as the map of spectra

$$/ : \mathcal{K}(X \times Y, \mathbb{C}) \wedge \mathcal{K}(\mathbb{C}, Y) \rightarrow \mathcal{K}(X, \mathbb{C})$$

given by $(-)/(-) = \theta_{X, X \times Y, \mathbb{C}}(\tau(- \wedge (1_X \boxtimes -)))$, where τ is the evident involution.

The slant product pairing $\backslash$ is defined as the map of spectra

$$\backslash : \mathcal{K}(X, \mathbb{C}) \wedge \mathcal{K}(\mathbb{C}, X \times Y) \rightarrow \mathcal{K}(\mathbb{C}, Y)$$

by $(-)\backslash(-) = \theta_{\mathbb{C}, X \times Y, Y}(\tau((- \boxtimes 1_Y) \wedge -))$.

The symbols $/$ and $\backslash$ are also used to refer to the induced pairings on homotopy groups:

$$\begin{aligned}
(-)/(-) &: K_p(X \times Y, \mathbb{C}) \otimes K_q(\mathbb{C}, Y) \rightarrow K_{p+q}(X, \mathbb{C}) \\
(-)\backslash(-) &: K_p(X, \mathbb{C}) \otimes K_q(\mathbb{C}, X \times Y) \rightarrow K_{p+q}(\mathbb{C}, Y).
\end{aligned}$$

On the level of homotopy groups, we therefore have the defining equations

$$\begin{aligned}
\alpha/\beta &= \alpha \tilde{\circ} (1_X \boxtimes \beta) \\
\gamma \backslash \delta &= (\gamma \boxtimes 1_Y) \tilde{\circ} \delta
\end{aligned}$$

for $\alpha \in K_p(X \times Y, \mathbb{C})$, $\beta \in K_q(\mathbb{C}, Y)$, $\gamma \in K_m(X, \mathbb{C})$, and $\delta \in K_n(\mathbb{C}, X \times Y)$.

Finally, we define internal cap product operations, written $\cap$, in the usual manner using the diagonal map:

Definition 6.8. Let $\mathcal{K}$ denote any of the four bivariant theories $\mathcal{K}^{\text{semi}}$, $\mathcal{K}^{\text{qtop}}$, $\mathbf{bu}^c$, or $\mathbf{bu}$. For a quasi-projective complex variety X , the cap product pairing is the map of spectra

$$\cap : \mathcal{K}(X, \mathbb{C}) \wedge \mathcal{K}(\mathbb{C}, X) \rightarrow \mathcal{K}(\mathbb{C}, X)$$

defined by $(-)\backslash \Delta_*(-)$, where $\Delta : X \rightarrow X \times X$ is the diagonal embedding.

The symbol $\cap$ is used also to denote the induced operations on homotopy groups

$$(-) \cap (-) : K_p(X, \mathbb{C}) \otimes K_q(\mathbb{C}, X) \rightarrow K_{p+q}(\mathbb{C}, X),$$

and so we have the defining equation

$$\alpha \cap \beta = (\alpha \boxtimes 1_X) \circ \Delta_*(\beta),$$

for $\alpha \in K_p(X, \mathbb{C})$ and $\beta \in K_q(\mathbb{C}, X)$.

The following result is an immediate consequence of the definitions and Propositions 6.2 and 6.4.

Proposition 6.9. *All of the operations defined here – external product for homology Δ , external product for cohomology $\overline{\Delta}$, cup product $\cup$, cap product $\cap$, and the two slant products $\backslash$ and $/$ – for the theories $\mathcal{K}^{semi}$, $\mathcal{K}^{qtop}$, $\mathbf{bu}$, and $\mathbf{bu}^c$ are compatible with the natural maps between these theories.*

Observe that the operations we have defined for $\mathbf{bu}^c$ and $\mathbf{bu}$ extend in the obvious manner to define operations for these theories on all CW complexes. Indeed, we may define these operations in the category of *pointed* CW complexes, so that we have natural pairings

$$\boxtimes : \mathbf{bu}(X, Y) \wedge \mathbf{bu}(Z, W) \rightarrow \mathbf{bu}(X \wedge Z, Y \wedge W)$$

and

$$\theta_{X,Y,Z} : \mathbf{bu}(X, Y) \wedge \mathbf{bu}(Y, Z) \rightarrow \mathbf{bu}(X, Z)$$

for any pointed CW complexes X , Y , Z , and W , and similarly for $\mathbf{bu}^c$. These operations, in turn, define the external, cup, cap, and slant products in the pointed category. Moreover, the cup product pairing endows the spectrum $\mathbf{bu}^c(S^0, S^0)$ with the structure of a commutative ring spectrum, namely the commutative ring spectrum associated with the FSP $|\mathcal{A}^{top}(S^0, -)|$ (cf. Lemma A.5).

Proposition 6.10. *The operations Δ , $\overline{\Delta}$, $\cup$, $\cap$, $\backslash$, and $/$ defined on the cohomology theory $bu_*(-, S^0)$ and the homology theory $bu_*(S^0, -)$ (viewed as functors from the category of pointed CW complexes to graded abelian groups) agree with the classical operations of the same name associated to the commutative ring spectrum $\mathbf{bu}(S^0, S^0)$.*

Proof. This follows by inspection by comparing the definition of these operations given here with the classical definition presented, for example, in [1, III.9]. $\square$

Remark 6.11. *By Proposition 2.7 the spectrum $\mathbf{bu}(S^0, S^0)$ is weakly equivalent to the spectrum $\mathbf{bu}$ representing connective topological K-theory. Note, however, that we have not proven that these two spectra are equivalent as ring spectra (with the ring structure on $\mathbf{bu}$ defined, for example, as in [27, VIII.2]). Indeed, we will not prove this result since it is not needed. However, it follows from Proposition 6.18 below that the ring-valued cohomology theories associated to $\mathbf{bu}$ and $\mathbf{bu}(S^0, S^0)$ do in fact coincide, at least in positive degrees.*

We introduced the notation $\circ$ for θ to suggest θ is analogous to a composition operation. To further this analogy, for X and Y quasi-projective complex varieties, define a function $\text{Hom}(X, Y) \rightarrow K_0^{semi}(X, Y)$ by sending $f : X \rightarrow Y$ to the class of the structure sheaf of its graph: $[\mathcal{O}_{\Gamma_f}] \in K_0^{semi}(X, Y)$. We will write $[f]$ for the class $[\mathcal{O}_{\Gamma_f}]$ in $K_0^{semi}(X, Y)$. In case Y is smooth, we also write $[f]$ for the image of this class under any of the maps

$$K_0^{semi}(X, Y) \longrightarrow K_0^{qtop}(X^{an}, Y) \xleftarrow{\cong} bu_0^c(X^{an}, Y^{an}) \longrightarrow bu_0(X^{an}, Y^{an}).$$

Lemma 6.12. *Let $K_*(-, -)$ denote any of the four bivariate theories under consideration. For quasi-projective complex varieties X and Y , with Y smooth, given a morphism $f : X \rightarrow Y$ and classes $\alpha \in K_q(Y, Z)$ and $\beta \in K_p(Z, X)$, we have*

$$f^*(\alpha) = \alpha \tilde{\circ} [f], \quad f_*(\beta) = [f] \tilde{\circ} \beta.$$

Proof. In the case $K = K^{\text{semi}}$, the class $[f]$ is represented by the canonical quotient $[\mathcal{O}_{X \times Y} \rightarrow \mathcal{O}_{\Gamma_f}]$, regarded as the object of the topological category $\mathcal{A}^{\text{semi}}(X, Y)$. If we restrict the pairing of topological categories

$$\mathcal{A}^{\text{semi}}(X, Y) \times \mathcal{A}^{\text{semi}}(Y, Z) \rightarrow \mathcal{A}^{\text{semi}}(X, Z)$$

which defines the θ pairing to this object, we clearly obtain the continuous functor $\mathcal{A}^{\text{semi}}(Y, Z) \rightarrow \mathcal{A}^{\text{semi}}(X, Z)$ given by f^* . This establishes the first equation for K^{semi} . The proof of the second equation for K^{semi} and the proof for the other theories are very similar. $\square$

As in the classical setting, there are a plethora of equations involving the operations introduced here which are valid for all four theories. In the following proposition, we give the subset of these equations actually needed in this paper.

Proposition 6.13. *Let $K_*(-, -)$ denote any of the four bivariate theories K_*^{semi} , K_*^{qtop} , bu_*^c or bu_* . The operations Δ , $\overline{\Delta}$, $\cup$, $\cap$, $\setminus$, and $/$ satisfy the following equations (cf. [34, §13]):*

1. *for all $\alpha \in K_m(X, \mathbb{C})$, $\beta \in K_n(X, \mathbb{C})$, we have*

$$\alpha \cup \beta = (-1)^{mn} \beta \cup \alpha;$$

2. *for all $a \in K_l(X \times Y, \mathbb{C})$, $y \in K_m(Y, \mathbb{C})$, and $z \in K_n(\mathbb{C}, Y)$, we have*

$$a/(y \cap z) = (a \cup \pi_2^* y)/z, \quad \text{in } K_{l+m+n}(X, \mathbb{C}),$$

where $\pi_2 : X \times Y \rightarrow Y$ is the canonical projection;

3. *for all $a \in K_l(X \times Y, \mathbb{C})$, $y \in K_m(X, \mathbb{C})$, and $z \in K_n(\mathbb{C}, Y)$, we have*

$$y \cup (a/z) = (\pi_1^* y \cup a)/z, \quad \text{in } K_{l+m+n}(X, \mathbb{C}),$$

where $\pi_1 : X \times Y \rightarrow X$ is the canonical projection.

Proof. The graded-commutativity of cup product is an immediate consequence of the fact that underlying spectrum is defined by a commutative Γ -ring (or, an FSP , in the case of bu_*) – see the Appendix for details.

For the second identity, using Propositions 6.4 and 6.5 and Lemma 6.12 repeatedly, we obtain the following string of equalities:

$$\begin{aligned} a/(y \cap z) &= a \tilde{\circ} (1_X \boxtimes [(y \boxtimes 1_Y) \tilde{\circ} \Delta_Y \tilde{\circ} z]) \\ &= a \tilde{\circ} (1_X \boxtimes [(1_Y \boxtimes y) \tilde{\circ} \Delta_Y \tilde{\circ} z]) \\ &= a \tilde{\circ} (1_X \boxtimes 1_Y \boxtimes y) \tilde{\circ} (1_X \boxtimes \delta_Y) \tilde{\circ} (1_X \boxtimes z) \\ &= a \tilde{\circ} (1_X \boxtimes 1_Y \boxtimes y) \tilde{\circ} (1_X \boxtimes \pi_2) \tilde{\circ} \Delta_{X \times Y} \tilde{\circ} (1_X \boxtimes z) \\ &= (a \boxtimes y) \tilde{\circ} (1_X \boxtimes \pi_2) \tilde{\circ} \Delta_{X \times Y} \tilde{\circ} (1_X \boxtimes z) \\ &= (a \boxtimes \pi_2^* y) \tilde{\circ} \Delta_{X \times Y} \tilde{\circ} (1_X \boxtimes z) \\ &= (a \cup \pi_2^* y)/z. \end{aligned}$$

Similarly, the third equality is given by:

$$\begin{aligned}
y \cup (a/z) &= (y \boxtimes [a \tilde{\circ} (1_X \boxtimes z)]) \tilde{\circ} \Delta_X \\
&= y \tilde{\circ} (1_X \boxtimes a) \tilde{\circ} (1_X \boxtimes 1_X \boxtimes z) \tilde{\circ} \Delta_X \\
&= y \tilde{\circ} (1_X \boxtimes a) \tilde{\circ} (\Delta_X \boxtimes 1_Y) \tilde{\circ} (1_X \boxtimes z) \\
&= y \tilde{\circ} (1_X \boxtimes a) \tilde{\circ} (\pi_1 \boxtimes 1_X \boxtimes 1_Y) \tilde{\circ} \Delta_{X \times Y} \tilde{\circ} (1_X \boxtimes z) \\
&= y \tilde{\circ} (\pi_1 \boxtimes a) \tilde{\circ} \Delta_{X \times Y} \tilde{\circ} (1_X \boxtimes z) \\
&= ((y \tilde{\circ} \pi_1) \boxtimes a) \tilde{\circ} \Delta_{X \times Y} \tilde{\circ} (1_X \boxtimes z) \\
&= (\pi_1^* y \cup a)/z
\end{aligned}$$

□

We now indicate how to define transfer maps for finite (but not necessarily dominant) morphisms of smooth varieties using the θ pairings. Recall that given complex varieties X and Y , the group $K_q^{\text{alg}}(X, Y)$ is the Quillen K -group of the exact category, which we will write as $\mathcal{P}(X, Y)$, consisting of coherent sheaves M on $X \times Y$ which are finite and flat over X . (Recall that this means the support of M maps to X via a finite morphism and the pushforward of M to X is locally free.) Let $K'_q(X, Y)$ denote the q -th Quillen K -group of the abelian category $\mathcal{M}(X, Y)$ consisting of all coherent sheaves on $X \times Y$ which are finite over X (i.e., which are supported on subschemes of $X \times Y$ which map to X via a finite morphism). The following technical result will prove useful.

Lemma 6.14. *Suppose X and Y are smooth quasi-projective complex varieties with Y projective. Then the natural map*

$$K_q^{\text{alg}}(X, Y) \rightarrow K'_q(X, Y)$$

induced by the evident inclusion of exact categories is an isomorphism for all $q \geq 0$.

Proof. As in the proof of [37, 2.2], Quillen's Resolution Theorem [28] gives the result, provided we can establish the following property for the pair (X, Y) : Given a closed subscheme $W \subset X \times Y$ such that the projection map $W \rightarrow X$ is finite, there exists a closed subscheme W' with $W \subset W' \subset X \times Y$ such that $W' \rightarrow X$ is finite and flat. The proof of [37, 2.3] establishes this property in the special case where $Y = \mathbb{P}^n$, for some n . More generally, observe that we may assume Y is connected, say of dimension n . Then there is a finite and hence flat morphism $p : Y \rightarrow \mathbb{P}^n$, and we note that $V = (1_X \times p)(W)$ (the scheme theoretic image) is a closed subscheme of $X \times \mathbb{P}^n$ which is finite over X . Consequently, there exists a closed subscheme V' with $V \subset V' \subset X \times \mathbb{P}^n$. Now take $W' = (1_X \times p)^{-1}(V') = V' \times_{\mathbb{P}^n} Y$. □

Using Lemma 6.14, a class in $K'_0(X, Y)$ (with X and Y satisfying the stated hypotheses) determines classes in each of the four bivariant theories under consideration by using the string of maps

$$(17) \quad K_0^{\text{semi}}(X, Y) \longrightarrow K_0^{\text{qtop}}(X^{\text{an}}, Y) \xleftarrow{\cong} bu_0^{\text{c}}(X^{\text{an}}, Y^{\text{an}}) \longrightarrow bu_0(X^{\text{an}}, Y^{\text{an}}).$$

In particular, if $f : Y \rightarrow X$ is a finite morphism of smooth projective varieties, then the structure sheaf of its graph is an object of $\mathcal{M}(X, Y)$, and thus defines a class in $K_0^{\text{alg}}(X, Y)$, $K_0^{\text{semi}}(X, Y)$, etc. We write each of these classes as $[f^t]$.

Definition 6.15. Let $f : Y \rightarrow X$ be a finite morphism of smooth, projective complex varieties. Letting $K_*(-, -)$ denote any of the four bivariant theories under consideration, we define the transfer maps

$$f_* : K_*(Y, Z) \rightarrow K_*(X, Z) \quad \text{and} \quad f^* : K_*(Z, X) \rightarrow K_*(Z, Y)$$

(where Z is any quasi-projective variety) by the formulas $f_*(\alpha) = \alpha \circ [f^t] = \theta([f^t], \alpha)$ and $f^*(\beta) = [f^t] \circ \beta = \theta(\beta, [f^t])$.

When f is a closed immersion of smooth projective varieties, the map f_* is sometimes called the “Gysin map”. Note that, in general, the transfer maps f_* and f^* are obviously compatible with the maps (15) relating the four theories. We can also define a transfer map f_* on algebraic K_0 -groups,

$$f_* : K_0^{\text{alg}}(Y, Z) \rightarrow K_0^{\text{alg}}(X, Z),$$

in the same situation as in Definition 6.15, by use of the pairings $\mathcal{P}(X, Y) \times \mathcal{P}(Y, Z) \rightarrow \mathcal{P}(X, Z)$ defined in the evident manner via external tensor product. (Of course, f_* can be defined on the higher algebraic K -groups as well.) Moreover, the transfer map f_* on K_0^{alg} is clearly compatible with the transfer map f_* on K_0^{semi} under the natural transformation $K_0^{\text{alg}}(-, -) \rightarrow K_0^{\text{semi}}(-, -)$. The following lemma will prove useful.

Lemma 6.16. Let $f : Y \rightarrow X$ be a finite morphism of smooth, projective varieties and Z be any quasi-projective variety. Then the diagram

$$\begin{array}{ccc} K_0^{\text{alg}}(Y, Z) & \xrightarrow{f_*} & K_0^{\text{alg}}(X, Z) \\ \downarrow & & \downarrow \\ K'_0(Y, Z) & \xrightarrow{f_*} & K'_0(X, Z) \end{array}$$

commutes, where the bottom arrow is the map induced by the exact functor $\mathcal{M}(Y, Z) \rightarrow \mathcal{M}(X, Z)$ given by push-forward of coherent sheaves along $f \times 1_Z$.

Proof. The map along the bottom of the displayed diagram can also be described as the map induced by the exact functor $\mathcal{M}(Y, Z) \rightarrow \mathcal{M}(X, Z)$ defined by tensoring over $\mathcal{O}_Y$ with $\mathcal{O}_{\Gamma_f}$, the structure sheaf of the graph Γ_f of f . (Note that $\mathcal{O}_{\Gamma_f}$ is flat as a $\mathcal{O}_Y$ -module, so that this is indeed an exact functor.) This interpretation makes the commutativity of the diagram obvious. $\square$

We will need the following “projection formula” in the next section.

Proposition 6.17. Suppose $f : Y \rightarrow X$ is a finite morphism of smooth, projective complex varieties. If $K_*(-, -)$ denotes any of the four bivariant theories under consideration, then given classes $\alpha \in K_p(Y, \mathbb{C})$ and $\beta \in K_q(X, \mathbb{C})$, the equation

$$f_*(f^*(\beta) \cup \alpha) = \beta \cup f_*(\alpha)$$

holds in $K_{p+q}(X, \mathbb{C})$.

Proof. The equation $\Delta_X^*(1_X \boxtimes [f^t]) = (f, 1_Y)_*([f^t])$ is easily seen to hold in $K_0^{\text{alg}}(X, X \times Y)$ using Lemma 6.16. Since $(f, 1_Y)_*[f^t] = (f \boxtimes 1_Y) \circ \Delta_Y \circ [f^t]$ and using Propositions 6.4 and 6.5, we obtain the string of equations

$$\begin{aligned} f_*(f^*(\beta) \cup \alpha) &= \Delta_Y^*(f^*(\beta) \boxtimes \alpha) \circ [f^t] \\ &= (\beta \boxtimes \alpha) \circ (f \boxtimes 1_Y) \circ \Delta_Y \circ [f^t] \\ &= (\beta \boxtimes \alpha) \circ (1_X \boxtimes [f^t]) \circ \Delta_X \\ &= \Delta_X^*(\beta \boxtimes f_*(\alpha)) = \beta \cup f_*(\alpha). \end{aligned}$$

□

We close this sections by a brief discussion of the connection of the operations introduced here with the classical operations on topological K -theory. It follows from Theorem 3.17 that the graded abelian group valued functor on pointed CW complexes

$$W \mapsto \bigoplus_q \mathbf{bu}_q(W, S^0)$$

is isomorphic to the functor

$$X \mapsto \bigoplus_q ku^{-q}(W),$$

where the latter functor is represented by the classical model for the K -theory spectrum $\mathbf{bu}$. We give below an elementary proof that the this isomorphism is compatible with the rings structures, at least in positive degrees.

We first recall that classical definition of *non-connective* topological K -theory, or KU -theory, for short. Let $KU^0(W) = ku^0(X)$ denote usual topological K^0 -group of W , with generators given by vector bundles on W which are trivialized by a *finite open cover* of W . Then $KU^0(W)$ is a ring under tensor product of bundles. More generally, given another pointed CW complex V , there is an external pairing

$$KU^0(W) \otimes KU^0(V) \rightarrow KU^0(W \wedge V)$$

given by external product of vector bundles in the evident manner. Define $KU^{-q}(W) = KU^0(S^q \wedge W)$ for $q \geq 0$, so that $\bigoplus_{q \geq 0} KU^{-q}(W)$ is a ring under external tensor product of bundles:

$$KU^0(S^p \wedge W) \otimes KU^0(S^q \wedge W) \rightarrow KU^0(S^p \wedge W \wedge S^q \wedge W) \rightarrow KU^0(S^p \wedge S^q \wedge W),$$

where the last map is given by pull-back along the map

$$S^p \wedge S^q \wedge W \rightarrow S^p \wedge W \wedge S^q \wedge W$$

sending (x, y, w) to (x, w, y, w) . Observe that there are also external pairings of the form

$$KU^p(W) \otimes KU^q(V) \rightarrow KU^{p+q}(W \wedge V)$$

defined by the composition of

$$KU^0(S^p \wedge W) \otimes KU^0(S^q \wedge V) \rightarrow KU^0(S^p \wedge W \wedge S^q \wedge V) \rightarrow KU^0(S^p \wedge S^q \wedge W \wedge V),$$

where the last map is induced by the evident involution. Letting $\beta \in KU^{-2}(S^0) = KU^0(S^2) \cong \mathbb{Z}$ be a chosen generator (the “Bott element”), multiplication by β (regarded as an element of $KU^{-2}(W)$ via pull-back) induces an isomorphism

$$- \cup \beta : KU^q(X) \xrightarrow{\cong} KU^{q+2}(X)$$

for all $q \geq 0$. Finally, the $\mathbb{Z}$ -graded ring $KU^*(W)$ is defined by formally inverting β in the $\mathbb{N}$ -graded ring $\bigoplus_{q \geq 0} KU^{-q}(W)$.

Proposition 6.18. *For any pointed CW complex W , there is a natural isomorphism of graded rings:*

$$(18) \quad \bigoplus_{q \geq 0} \mathbf{bu}_q(W, S^0) \rightarrow \bigoplus_{q \geq 0} KU^{-q}(W)$$

Proof. The isomorphism $\mathbf{bu}_0(W, S^0) \cong KU^0(W)$ is given by the fact that each group is naturally the quotient of $[W_+, \text{Grass}]^+$ by the subgroup $[S^0, \text{Grass}]^+$. (Here, Grass denotes the infinite disjoint union $\coprod_n \text{Grass}_n(\mathbb{C}^\infty)^{\text{an}}$ of Grassmann manifolds.) This map is clearly a ring map, since the cup product rule for $\mathbf{bu}_0(W, S^0)$ is readily seen to be given by tensor product of vector bundles. Observe there is a natural isomorphism

$$\mathbf{bu}_q(W, S^0) \cong \mathbf{bu}_0(S^q \wedge W, S^0), \quad q \geq 0,$$

arising from the natural identifications:

$$\begin{aligned} \mathbf{bu}_q(W, S^0) &\cong [S^q, \Omega^1 \mathbf{ob}_\infty \mathcal{A}(W, S^1)] \\ &\cong [S^q, \mathbf{ob}_\infty \mathcal{A}(S^1 \wedge W, S^1)] \\ &\cong [S^0, \mathbf{ob}_\infty \mathcal{A}(S^q \wedge S^1 \wedge W, S^1)] \\ &\cong [S^0, \mathbf{ob}_\infty \mathcal{A}(S^1 \wedge S^q \wedge W, S^1)] \\ &\cong [S^0, \Omega^1 \mathbf{ob}_\infty \mathcal{A}(S^q \wedge W, S^1)] \\ &\cong \mathbf{bu}_0(S^q \wedge W, S^0). \end{aligned}$$

The natural map $\mathbf{bu}_q(W, S^0) \cong KU^{-q}(W)$, for $q > 0$, referred to in the statement of this proposition is defined by using the natural isomorphism $\mathbf{bu}_q(W, S^0) \cong \mathbf{bu}_q(S^q \wedge W, S^0)$. To show (18) is a ring map, it therefore suffices to show that diagram

$$\begin{array}{ccc} \mathbf{bu}_p(X, S^0) \otimes \mathbf{bu}_q(X, S^0) & \longrightarrow & \mathbf{bu}_{p+q}(X, S^0) \\ \cong \downarrow & & \cong \downarrow \\ \mathbf{bu}_0(S^p \wedge X, S^0) \otimes \mathbf{bu}_0(S^q \wedge X, S^0) & \longrightarrow & \mathbf{bu}_{p+q}(S^{p+q} \wedge X, S^0) \end{array}$$

commutes. This is verified immediately from the definition of cup product. $\square$

Under the natural isomorphism $\mathbf{bu}_2(S^0, S^0) \cong KU^{-2}(S^0)$, the chosen generator $\beta \in KU^{-2}(S^0)$ corresponds to a generator β of $\mathbf{bu}_2(S^0, S^0)$. From Proposition 6.18, multiplication by β induces an isomorphism

$$-\cup \beta : \mathbf{bu}_q(W, S^0) \xrightarrow{\cong} \mathbf{bu}_{q+2}(W, S^0),$$

for all $q \geq 0$. Since multiplication by β is defined on the full $\mathbb{Z}$ -graded ring $\bigoplus_{q \in \mathbb{Z}} \mathbf{bu}_q(W, S^0)$, we obtain a natural ring map

$$(19) \quad \bigoplus_{q \in \mathbb{Z}} \mathbf{bu}_q(W, S^0) \rightarrow \bigoplus_{q \geq 0} \mathbf{bu}_q(W, S^0)[1/\beta],$$

which is defined for $q < 0$ as the composition of

$$\mathbf{bu}_q(W, S^0) \xrightarrow{-\cup \beta^n} \mathbf{bu}_{q+2n}(W, S^0) \xrightarrow{-\cup \beta^{-n}} \left(\bigoplus_{i \geq 0} \mathbf{bu}_i(W, S^0)[1/\beta] \right)_q,$$

for $n \gg 0$. (Here, the second map is multiplication by the n -th power of the formal inverse of β .) The map (19) is an isomorphism in non-negative degrees, since multiplication by β induces an isomorphism in non-negative degrees. The following result is thus immediate.

Corollary 6.19. *For any pointed CW complex W , there is a natural map of graded rings*

$$\bigoplus_{q \in \mathbb{Z}} \mathbf{bu}_q(W, S^0) \rightarrow \bigoplus_{q \in \mathbb{Z}} KU^{-q}(W)$$

which is an isomorphism in non-negative degrees. Moreover, inverting $\beta \in \mathbf{bu}_2(W, S^0)$, we have a natural isomorphism of $\mathbb{Z}$ -graded rings

$$\bigoplus_{q \geq 0} \mathbf{bu}_q(W, S^0)[1/\beta] \rightarrow \bigoplus_{q \in \mathbb{Z}} KU^{-q}(W).$$

7. MAJOR APPLICATIONS

In this section, we use the results established in the previous sections to prove two major theorems. The first is that the “Bott-inverted” semi-topological K -theory of a smooth, projective complex variety coincides with the topological K -theory of the associated analytic space. Using [12, 3.8], this gives in particular a new proof of Thomason’s seminal result [35] in the special case of smooth, projective complex varieties. The second major application is that the semi-topological K -theory of a smooth, projective variety Y surjects onto the topological K -theory of Y^{an} in degrees at least twice the dimension of Y (as a complex variety). This generalizes (in the case of smooth, projective complex varieties) a result in [14]. The proofs of both of these assertions follow rather easily from the Fundamental Comparison Theorem (Theorem 5.1) and the properties of the cohomological operations established in the previous section.

Recall that $bu_*(-, S^0)$ and $bu_*(S^0, -)$ give the (generalized) cohomology and homology groups of a compact CW complex associated to the ring spectrum $\mathbf{bu}(S^0, S^0)$. Moreover, the operations introduced in Section 6 for $bu_*(-, -)$ coincide with the “classically defined” operations of the same name arising from the commutative ring spectrum $\mathbf{bu}(S^0, S^0)$. By Proposition 2.7, the spectrum $\mathbf{bu}(S^0, S^0) = \mathbf{bu}^c(S^0, S^0)$ is weakly equivalent to the spectrum representing connective topological K -theory; however, we have not proven that these two spectra agree as *ring* spectra. Nevertheless, we have proven enough to establish a Poincaré duality isomorphism for $bu_*(-, -)$, which we will exploit in this section.

Throughout this section, we will write $ku_q(-)$ and $ku^{-q}(-)$ for $bu_q(S^0, -)$ and $bu_q(-, S^0)$, respectively. This allows us to introduce the homology and cohomology of a pair $A \subset X$ of spaces without causing a notational conflict. That is, we will write $ku^q(X, A)$ and $ku_q(X, A)$ for the relative cohomology and homology groups associated to the spectrum $\mathbf{bu}(S^0, S^0)$, where A is a subspace of a topological space X ; the reader is alerted to avoid confusing this notation with the notation for the bivariant theory $bu_*(-, -)$. In particular, if (W, w_0) is a pointed space, by $ku^*(W, w_0)$ we mean the graded abelian group previously written as $bu_{-*}(W, S^0)$ (where in the latter, the base point of W is not given explicitly in the notation). From the results in the previous sections of this paper, we have natural maps

$$K_*^{\text{semi}}(X, \text{Spec } \mathbb{C}) \rightarrow ku^{-*}(X^{\text{an}}), \quad K_*^{\text{semi}}(\text{Spec } \mathbb{C}, X) \rightarrow ku_*(X^{\text{an}})$$

for any quasi-projective complex variety X , which are compatible with the cup, cap, and slant products defined in Section 6. In particular, $K_*^{\text{semi}}(X, \text{Spec } \mathbb{C}) \rightarrow ku^{-*}(X^{\text{an}})$ is map of graded rings.

Suppose M is a smooth compact m -manifold (i.e., a smooth compact manifold with each connected component having dimension m) and E is a commutative ring spectrum (such as $\mathbf{bu}(S^0, S^0)$) with associated homology and cohomology theories written as E_* and E^* . Recall that a *Thom class* for M with respect to E is an element t_M in the relative cohomology group $E^m(M \times M, M \times M \setminus \Delta)$ (where Δ is the diagonal embedding of M in $M \times M$), having the property that for all $x \in M$, the image of t_M in

$$E^m(M \times \{x\}, M \times \{x\} \setminus \{x\} \times \{x\}) \cong E^m(M, M - \{x\}) \cong E^m(S^m)$$

is a generator of $E^*(S^m)$ as a $E^*(S^0)$ -module.

Lemma 7.1. *For all $q \geq 0$, each of the two generators of $ku^q(S^q) \cong \mathbb{Z}$ as an abelian group forms a free basis for the graded module $ku^*(S^q)$ over the graded ring $ku^*(S^0) \cong \mathbb{Z}[\beta]$, where β is a generator of the abelian group $ku^2(S^0) \cong \mathbb{Z}$.*

Proof. Clearly, $ku^*(S^q)$ is a graded module over the graded ring $ku^*(S^0)$, and the latter is isomorphic to $\mathbb{Z}[\beta]$ since it coincides as a ring with the classical connective K -theory ring of S^0 by Corollary 6.19. Let $1_{S^q} \in |\mathcal{A}^{\text{top}}(S^q, S^q)|$ be the point represented by the canonical object $\mathcal{C}_0(S^q) \rightarrow \text{Mat}_1(\mathcal{C}_0(S^q))$. Then 1_{S^q} defines a class y in $\pi_0|\mathcal{A}^{\text{top}}(S^q, S^q)| \cong ku^q(S^q) \cong \mathbb{Z}$ which generates $ku^q(S^q)$ as an abelian group. Using Theorem 4.4, we see that the map

$$- \boxtimes 1_{S^q} : |\mathcal{A}^{\text{top}}(S^0, S^m)| \rightarrow |\mathcal{A}^{\text{top}}(S^q, S^{q+m})| \sim \Omega^q(|\mathcal{A}^{\text{top}}(S^0, S^m \wedge -)| \otimes S^q)$$

is a weak equivalence, from which it follows that

$$- \cup y : ku^*(S^0) \rightarrow ku^{*+q}(S^q)$$

is an isomorphism. □

Recall that ku^* coincides with classical connective topological K -theory as a graded abelian group valued functor. Therefore, given a smooth, compact, orientable m -manifold M , there exists a class $t_M \in ku^m(M \times M, M \times M \setminus \Delta)$ which maps to ± 1 in $ku^m(M, M \setminus \{x\}) \cong \mathbb{Z}$ for all $x \in M$. Using Lemma 7.1, this suffices to show t_M is a Thom class for M relative to the ring spectrum $bu(S^0, S^0)$. (If the reader finds the previous few paragraphs needlessly convoluted, recall that we have not proven $bu(S^0, S^0)$ coincides as a *ring spectrum* with the classical connective ring spectrum representing topological K -theory. Thus, we cannot merely cite the literature for the existence of a Thom class.)

Let $j : (M \times M, \emptyset) \rightarrow (M \times M, M \times M \setminus \Delta)$ be the evident map of pairs, so that a Thom class $t_M \in ku^m(M \times M, M \times M \setminus \Delta)$ determines a class $j^*(t_M)$ in $ku^m(M \times M)$. The following result is the formal statement of Poincaré duality.

Proposition 7.2. *(See [34, 14.13].) If M is a compact, smooth orientable m -manifold and $t_M \in ku^m(M \times M, M \times M \setminus \Delta)$ is a Thom class, then the map*

$$(20) \quad j^*(t_M)/- : ku_*(M) \xrightarrow{\cong} ku^{m-*}(M)$$

is an isomorphism in all degrees.

The inverse of (20) is given in terms of cap product by a fundamental class. Namely, given M with a chosen Thom class t_M as in Proposition 7.2, there is a

unique element of $ku_m(M)$ mapping to $1 \in ku^0(M)$ under this isomorphism. We call such an element a *fundamental class* of the manifold M relative to the homology theory ku_* and it is written $[M]$. Note that the class $[M]$ depends on the choice of Thom class t_M .

Proposition 7.3. (See [34, 14.17].) *If M is a compact, smooth, orientable m -manifold, t_M is a Thom class (relative to ku^*), and $[M] \in ku_m(M)$ is the corresponding fundamental class, then the map*

$$(21) \quad - \cap [M] : ku^*(M) \rightarrow ku_{m-*}(M)$$

is an isomorphism in all degrees and is the inverse up to a sign of the map (20).

If X is a smooth, connected, projective complex variety of dimension d , then there exists a Thom class $t_X \in ku^{2d}(X^{\text{an}} \times X^{\text{an}}, X^{\text{an}} \times X^{\text{an}} \setminus \Delta)$ and thus a corresponding fundamental class $[X^{\text{an}}] \in ku_{2d}(X^{\text{an}})$. In light of the Theorem 5.1, fundamental classes (at least formally speaking) exists in semi-topological K -theory as well:

Definition 7.4. *For a smooth, projective complex variety X of dimension d , a fundamental class of X , written $[X]$, is an element of $K_{2d}^{\text{semi}}(\mathbb{C}, X)$ which maps to a fundamental class $[X^{\text{an}}] \in ku_{2d}(X^{\text{an}})$ under the isomorphism given by Theorem 5.1.*

Of course, there is no reason to conclude that the analogue of the Poincaré Duality isomorphism (21) holds for K^{semi} . Indeed, the map

$$- \cap [X] : K_q^{\text{semi}}(X, \mathbb{C}) \rightarrow K_{q+2d}^{\text{semi}}(\mathbb{C}, X)$$

cannot be an isomorphisms in all degrees (even if one restricts to $q \geq 0$), since if it were, compatibility of the operations would imply that $K_0^{\text{semi}}(X, \mathbb{C}) \rightarrow ku^0(X^{\text{an}})$ is an isomorphism. However, in analogy with the Quillen-Lichtenbaum conjecture, the following conjecture is plausible (and, presumably, difficult to prove):

Conjecture 7.5. *For X a smooth, projective complex variety of dimension d , the map*

$$- \cap [X] : K_q^{\text{semi}}(X, \mathbb{C}) \rightarrow K_{q+2d}^{\text{semi}}(\mathbb{C}, X)$$

is an isomorphism for $q \geq d - 1$.

There clearly does not exist a strict analogue of the Thom class in K^{semi} , since it would have to live in the trivial group $K_{-2d}^{\text{semi}}(X, \mathbb{C})$. However, given a Thom class $t = t_{X^{\text{an}}}$ in $ku^{2d}(X^{\text{an}} \times X^{\text{an}}, X^{\text{an}} \times X^{\text{an}} \setminus \Delta)$, if we multiply $j^*(t)$ by β^d , where $\beta \in ku^{-2}(pt)$ is the Bott element (defined below), then we obtain a class in $ku^0(X^{\text{an}} \times X^{\text{an}})$ which ought to lift to an element of $K_0^{\text{semi}}(X \times X, \mathbb{C})$.

Definition 7.6. *The Bott element β is a choice of generator for the group $K_2^{\text{semi}}(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$. We also write β for its image in $K_2^{\text{semi}}(X, \mathbb{C})$ under the canonical map. By β^d , we mean the d -th power of β in the ring $K_*^{\text{semi}}(\mathbb{C}, \mathbb{C})$ and its image in $K_*^{\text{semi}}(X, \mathbb{C})$. We use the same notation for the images of these elements under the natural map $K_*^{\text{semi}}(X, \mathbb{C}) \rightarrow ku^{-*}(X^{\text{an}})$.*

Definition 7.7. *For a smooth, quasi-projective complex variety X , the element $\delta_X \in K_0^{\text{semi}}(X \times X, \mathbb{C})$ is defined as the class of the sheaf $\mathcal{O}_\Delta$ (which determines a class in $K_0^{\text{alg}}(X \times X, \mathbb{C})$, and hence in $K_0^{\text{semi}}(X \times X, \mathbb{C})$, since X is smooth). Write $\delta_{X^{\text{an}}}$ for the image of δ_X under the map $K_0^{\text{semi}}(X \times X, \mathbb{C}) \rightarrow bu_0(X^{\text{an}} \times X^{\text{an}}, S^0) = ku^0(X^{\text{an}} \times X^{\text{an}})$.*

Note that we could equivalently define δ_X to be the image of $1 \in K_0^{\text{semi}}(X, \mathbb{C})$ under the transfer map

$$\Delta_* : K_0^{\text{semi}}(X, \mathbb{C}) \rightarrow K_0^{\text{semi}}(X \times X, \mathbb{C})$$

defined in Section 6. We need to relate the class $\delta_{X^{\text{an}}}$ with a Thom class.

Lemma 7.8. *For a smooth, projective complex variety X of dimension d , the element $\delta_{X^{\text{an}}} \in ku^0(X^{\text{an}} \times X^{\text{an}})$ is equal to $\beta^d \cup j^*(t)$, for some Thom class $t \in ku^{2d}(X^{\text{an}} \times X^{\text{an}}, X^{\text{an}} \times X^{\text{an}} \setminus \Delta)$.*

Proof. For brevity, let $T = X^{\text{an}} \times X^{\text{an}}$.

The class $\delta_X \in K_0(X \times X)$ is given by the alternating sum $\sum_i (-1)^i [E_i]$ for any choice of a finite resolution

$$0 \rightarrow E_l \rightarrow \cdots \rightarrow E_0 \rightarrow \mathcal{O}_X \rightarrow 0$$

by locally free coherent sheaves on X . The bounded complex given by the E_i 's defines a class in the relative K -group $K_{\Delta, 0}^{\text{alg}}(X \times X)$ and hence a class $\tilde{\delta}$ in $ku_{\Delta}^0(T) \cong ku(T, T \setminus \Delta)$ by taking the associated complex of topological vector bundles involving the E_i 's. (See [13, Appendix A] for details on the relevant models for topological K -theory with supports.) The class $\tilde{\delta} \in ku^0(T, T \setminus \Delta)$ has the property that its restriction to $ku^0(X^{\text{an}}, X^{\text{an}} \setminus \{x\})$ is a generator, for all $x \in X^{\text{an}}$.

Now let $t' \in ku^{2d}(T, T \setminus \Delta)$ be any choice of Thom class. Then in $ku^0(T, T \setminus \Delta) = KU^0(T, T \setminus \Delta)$, the elements $\tilde{\delta}$ and $\beta^d \cup t'$ each represent Thom classes for X^{an} for 2-periodic topological K -theory KU^* . By [34, 14.7], there is a unit $u \in KU^0(X^{\text{an}}) = ku^0(X^{\text{an}})$ such that $\tilde{\delta} = \pi_1^*(u) \cup \beta^d \cup t' = \beta^d \cup (\pi_1^*u \cup t')$ and $t = \pi_1^*u \cup t'$ is a Thom class of X^{an} for the generalized cohomology theory ku^* . $\square$

The following proposition describes the effect of transfer maps (as defined in Section 6) associated to finite, flat morphisms on fundamental classes.

Proposition 7.9. *Let $\pi : X \rightarrow Y$ be a finite, flat morphism of smooth projective varieties, and let K_* denote any of the four bivariant theories under consideration. Then the diagram*

$$(22) \quad \begin{array}{ccc} K_*(\mathbb{C}, Y) & \xrightarrow{\delta_Y / -} & K_*(Y, \mathbb{C}) \\ \pi^* \downarrow & & \pi^* \downarrow \\ K_*(\mathbb{C}, X) & \xrightarrow{\delta_X / -} & K_*(X, \mathbb{C}) \end{array}$$

commutes. In particular, the transfer maps

$$\pi^* : K_{2d}^{\text{semi}}(\mathbb{C}, Y) \rightarrow K_{2d}^{\text{semi}}(\mathbb{C}, X), \quad \pi^* : ku_{2d}(Y^{\text{an}}) \rightarrow ku_{2d}(X^{\text{an}})$$

send a fundamental class of Y to a fundamental class of X .

Proof. Letting (temporarily) δ_X and δ_Y denote the elements of $K_0^{\text{alg}}(X \times X, \mathbb{C})$ and $K_0^{\text{alg}}(Y \times Y, \mathbb{C})$, respectively, given by the diagonal embeddings, we claim we have the equation

$$(23) \quad (1_X \times \pi)_*(\delta_X) = (\pi \times 1_Y)^*(\delta_Y) \in K_0^{\text{alg}}(X \times Y, \mathbb{C}).$$

Indeed, using Lemma 6.16 and the fact that

$$\begin{array}{ccc} X & \xrightarrow{(1_X, \pi)} & X \times Y \\ \pi \downarrow & & \downarrow \pi \times 1_Y \\ Y & \xrightarrow{\Delta_Y} & Y \times Y \end{array}$$

is a pullback square, each side of (23) is seen to coincide with the class of the structure sheaf of the closed embedding $X \xrightarrow{(1_X, \pi)} X \times Y$. For any $\alpha \in K_*(\mathbb{C}, Y)$, equation (23), Proposition 6.5, and Lemma 6.12 give the chain of equalities

$$\begin{aligned} \delta_X / \pi^*(\alpha) &= \delta_X \tilde{\circ} (1_X \boxtimes ([\pi^t] \tilde{\circ} \alpha)) \\ &= \delta_X \tilde{\circ} (1_X \boxtimes [\pi^t]) \tilde{\circ} (1_X \boxtimes \alpha) \\ &= (1_X \times \pi)_*(\delta_X) \tilde{\circ} (1_X \boxtimes \alpha) \\ &= (\pi \times 1_Y)^*(\delta_Y) \tilde{\circ} (1_X \boxtimes \alpha) \\ &= \delta_Y \tilde{\circ} ([\pi] \boxtimes 1_Y) \tilde{\circ} (1_X \boxtimes \alpha) \\ &= \delta_Y \tilde{\circ} (1_Y \boxtimes \alpha) \tilde{\circ} [\pi] \\ &= \pi^*(\delta_Y / \alpha), \end{aligned}$$

which establishes the commutativity of (22)

For the final assertion, it suffices by the compatibility of transfer to establish the result for ku_* . Let $[Y^{\text{an}}]$ denote any fundamental class of Y . Using Lemma 7.8 and Proposition 6.13 we have $\delta_{Y^{\text{an}}} / - = (\beta^d \cup j^*(t_{Y^{\text{an}}})) / - = \beta^d \cup (j^*(t_{Y^{\text{an}}}) / -)$ and $\delta_{X^{\text{an}}} / - = (\beta^d \cup j^*(t_{X^{\text{an}}})) / - = \beta^d \cup (j^*(t_{X^{\text{an}}}) / -)$ and for some Thom classes $t_{Y^{\text{an}}}$ and $t_{X^{\text{an}}}$. Using these equations and the fact that any two Thom classes differ by a unit of ku^0 (cf. [34, 14.7]), we have

$$\begin{aligned} \beta^d \cup (j^*(t_{X^{\text{an}}}) / \pi^*[Y^{\text{an}}]) &= \delta_{X^{\text{an}}} / \pi^*[Y^{\text{an}}] \\ &= \pi^*(\delta_{Y^{\text{an}}} / [Y^{\text{an}}]) \\ &= \pi^*(\beta^d \cup (j^*(t_Y) / [Y^{\text{an}}])) \\ &= \pi^*(\beta^d \cup u) \\ &= \beta^d \cup \pi^*(u) \end{aligned}$$

for some unit $u \in ku^0(Y^{\text{an}})$. Since $\beta^d \cup - : ku^0(X^{\text{an}}) \rightarrow ku^{-2d}(X^{\text{an}})$ is an isomorphism, we conclude that $v = j^*(t_{X^{\text{an}}}) / \pi^*[Y^{\text{an}}]$ is a unit of $ku^0(X^{\text{an}})$. It follows that $v^{-1} \cup t_{X^{\text{an}}}$ is also a Thom class for X^{an} whose associated fundamental class is $\pi^*[Y^{\text{an}}]$. $\square$

The following Theorem provides the key technical result leading to the major theorems which follow it.

Theorem 7.10. *For any smooth, projective complex variety X of dimension d , the composition*

$$K_*^{\text{semi}}(X, \mathbb{C}) \xrightarrow{-\cap[X]} K_{*+2d}^{\text{semi}}(\mathbb{C}, X) \xrightarrow{\delta_X / -} K_{*+2d}^{\text{semi}}(X, \mathbb{C})$$

coincides with multiplication by $\beta^d \cup u$, for some unit $u \in K_0^{\text{semi}}(X, \mathbb{C})$. Likewise, the composition

$$ku^*(X^{\text{an}}) \xrightarrow{-\cap[X^{\text{an}}]} ku_{2d-*}(X^{\text{an}}) \xrightarrow{\delta_{X^{\text{an}}} / -} ku^{*-2d}(X^{\text{an}})$$

coincides with multiplication by $\beta^d \cup v$, for some unit $v \in ku^0(X^{an})$.

Proof. Note that any two fundamental classes differ by a unit of K_0^{semi} or ku^0 . Thus, it suffices to prove the theorem for any choice of fundamental class.

For all $y \in K_*^{\text{semi}}(X, \mathbb{C})$, we have the string of equations

$$\begin{aligned} \delta_X / (y \cap [X]) &= (\delta_X \cup (\pi_2^* y)) / [X] \\ &= (\delta_X \cup (\pi_1^* y)) / [X] \\ &= ((\pi_1^* y) \cup \delta_X) / [X] \\ &= y \cup (\delta_X / [X]). \end{aligned}$$

showing that $\delta_X / (- \cap [X])$ coincides with multiplication by $\delta / [X]$. All of the equations in this list follow from the standard equations (cf. Proposition 6.13) involving the operations, except for the second equation. To establish the second equation, let τ be the evident involution on $X \times X$. Then we have $\tau^*(\delta_X \cup \alpha) = \tau^*(\Delta_*(\Delta^*(\alpha))) = \Delta_*(\Delta^*(\alpha)) = \delta_X \cup \alpha$ for all $\alpha \in K_*^{\text{semi}}(X \times X)$ by the projection formula (cf. Proposition 6.17) for the map $\Delta : X \rightarrow X \times X$. (Observe that $\tau^* = \tau_*$, and so $\tau^* \Delta_* = \Delta_*$, since $\tau \circ \Delta = \Delta$.) Thus for any $y \in K_*^{\text{semi}}(X, \mathbb{C})$, we have

$$\delta_X \cup (\pi_2^* y) = \tau^*(\delta_X \cup (\pi_2^* y)) = \tau^*(\delta_X) \cup \tau^*(\pi_2^* y) = \delta_X \cup \pi_1^* y.$$

A similar argument shows that $\delta_{X^{an}}(- \cap [X^{an}])$ is multiplication by $\delta_{X^{an}} / [X^{an}]$. It therefore suffices to show $\delta_X / [X] = \beta^d \cup u$ and $\delta_{X^{an}} / [X^{an}] = \beta^d \cup v$, for some units u and v . The latter equation follows from Lemma 7.8 and Proposition 6.13, as shown in the proof of Proposition 7.9. To establish the former, let $\pi : X \rightarrow \mathbb{P}^d$ be any finite (and hence flat) morphism. Letting $[X] = \pi^*[\mathbb{P}^d]$ for some choice of fundamental class $[\mathbb{P}^d]$, we have from Proposition 7.9 that $\delta_X / [X] = \pi^*(\delta_{\mathbb{P}^d} / [\mathbb{P}^d])$, and so it suffices to show $\delta_{\mathbb{P}^d} / [\mathbb{P}^d] = \beta^d \cup u'$, for some unit $u' \in K_0^{\text{semi}}(\mathbb{P}^d, \mathbb{C})$. This follows from the topological version of the result, since $K_*^{\text{semi}}(\mathbb{P}^d, \mathbb{C}) \cong bu_*^c((\mathbb{P}^d)^{an}, S^0)$ by [11, 7.1]. $\square$

Theorem 7.11. *For any smooth, projective complex variety X , the natural map*

$$K_*^{\text{semi}}(X, \text{Spec } \mathbb{C})[1/\beta] \rightarrow ku^{-*}(X^{an})[1/\beta]$$

is an isomorphism of graded rings.

Proof. This follows immediately from Theorems 5.1 and 7.10 and the compatibility of operations established in Section 6. For these results give the existence of the commutative diagram

$$\begin{array}{ccccc} K_*^{\text{semi}}(X, \mathbb{C}) & \xrightarrow{-\cap[X]} & K_*^{\text{semi}}(\mathbb{C}, X) & \xrightarrow{\delta_X / -} & K_{*+2d}^{\text{semi}}(X, \mathbb{C}) \\ \downarrow & & \cong \downarrow & & \downarrow \\ ku^{-*}(X^{an}) & \xrightarrow{-\cap[X^{an}]} & ku_*(X^{an}) & \xrightarrow{\delta_{X^{an}} / -} & ku^{-*-2d}(X^{an}) \end{array}$$

such that the composition of either row is multiplication by β^d up to units of K_0^{semi} and ku^0 and such that the middle vertical map is an isomorphism. A simple diagram chase completes the proof. $\square$

Recall from the end of Section 6 that KU^* denotes 2-periodic topological K -theory.

Corollary 7.12. *For any smooth, projective complex variety X , the natural map*

$$K_*^{\text{semi}}(X, \text{Spec } \mathbb{C})[1/\beta] \rightarrow KU^{-*}(X^{\text{an}})$$

is an isomorphism of graded rings.

Proof. This follows immediately from the theorem and the fact that $ku^*(X^{\text{an}})[1/\beta]$ is naturally isomorphic to the ring $KU^*(X^{\text{an}})$ by Corollary 6.19. $\square$

Recall that $K_q^{\text{alg}}(X; \mathbb{Z}/n)$ denotes the algebraic K -groups with coefficients in $\mathbb{Z}/n$, defined by homotopy classes of maps from the Moore spectrum $M(q, \mathbb{Z}/n)$ to $\mathcal{K}^{\text{alg}}(X)$. The groups $K_q^{\text{semi}}(X; \mathbb{Z}/n)$, and $KU^{-q}(X^{\text{an}}; \mathbb{Z}/n)$ are defined similarly.

Corollary 7.13. *For any smooth, projective complex variety X and integer $n > 0$, the natural map*

$$K_*^{\text{alg}}(X; \mathbb{Z}/n)[1/\beta] \rightarrow KU^{-*}(X^{\text{an}}; \mathbb{Z}/n)$$

is an isomorphism of graded rings.

Proof. By [12, 3.8], we have a natural isomorphism $K_*^{\text{alg}}(X; \mathbb{Z}/n) \cong K_*^{\text{semi}}(X; \mathbb{Z}/n)$ of graded rings, and so it suffices to establish an isomorphism

$$K_*^{\text{semi}}(X; \mathbb{Z}/n)[1/\beta] \rightarrow ku^{-*}(X^{\text{an}}; \mathbb{Z}/n)[1/\beta].$$

Letting $\mathcal{K}$ stand for either $\mathcal{K}^{\text{semi}}$ or **bu**, recall that multiplication by β is induced by the pairing of spectra

$$\mathcal{K}(\mathbb{C}, \mathbb{C}) \wedge \mathcal{K}(X, \mathbb{C}) \rightarrow \mathcal{K}(X, \mathbb{C}).$$

In particular, $\beta \cup -$ acts naturally on homotopy groups with finite coefficients (including the case $n = 2$ – cf. [10, B.4]) and is compatible with the boundary maps in the long exact sequence

$$\cdots \rightarrow K_q(X; \mathbb{Z}) \xrightarrow{\cdot n} K_q(X; \mathbb{Z}) \rightarrow K_q(X; \mathbb{Z}/n) \rightarrow K_{q-1}(X; \mathbb{Z}) \rightarrow \cdots$$

Since

$$K_q(X; A)[1/\beta] = \varinjlim \left(K_q(X; A) \xrightarrow{\cdot \cup \beta} K_{q+2}(X; A) \xrightarrow{\cdot \cup \beta} \cdots \right),$$

where $A = \mathbb{Z}$ or $A = \mathbb{Z}/n$, the result follows immediately from the Five Lemma and Theorem 7.11. $\square$

Theorem 7.14. *For any smooth, projective complex variety X of dimension d , the natural map*

$$K_q^{\text{semi}}(X) \rightarrow KU^{-q}(X^{\text{an}})$$

is a split surjection for $q \geq 2d$.

Proof. This follows immediately from the commutative diagram

$$(24) \quad \begin{array}{ccc} K_q^{\text{semi}}(\mathbb{C}, X) & \xrightarrow{\delta_X / -} & K_q^{\text{semi}}(X, \mathbb{C}) \\ \cong \downarrow & & \downarrow \\ bu_q(S^0, X^{\text{an}}) & \xrightarrow{\delta_{X^{\text{an}}} / -} & bu_q(X^{\text{an}}, S^0), \end{array}$$

since $bu_q(X^{\text{an}}, S^0) \cong KU^{-q}(X^{\text{an}})$ for $q \geq 0$ and the map along the bottom is an isomorphism for $q \geq 2d$. $\square$

Corollary 7.15. *For any smooth, projective complex variety X of dimension d and integer $n > 0$, the natural map*

$$K_q^{alg}(X; \mathbb{Z}/n) \rightarrow KU^{-q}(X^{an}; \mathbb{Z}/n)$$

is a split surjection for $q \geq 2d$.

Proof. Using [12, 3.8], it suffices to show $K_q^{\text{semi}}(X; \mathbb{Z}/n) \rightarrow KU^{-q}(X^{an}; \mathbb{Z}/n)$ is a split surjection of $q \geq 2d$. The maps $\delta_X/-$ and $\delta_{X^{an}}/-$ in the proof of Theorem 7.14 are each induced by certain maps of spectra, and thus are also defined for homotopy groups with finite coefficients. Thus we have a commutative diagram like (24) but with coefficients in $\mathbb{Z}/n$. Indeed, all of the assertions in the proof of Theorem 7.14 remain valid when coefficients of $\mathbb{Z}/n$ are taken. $\square$

APPENDIX A. Γ -SPACES, Γ -RINGS, $\mathcal{W}$ -SPACES, AND FSP'S

In this appendix, we provide basic definitions and results concerning the key homotopical tools used in the body of this paper. For additional background concerning the material in this appendix, see [1], [6], [30], [29], [4], [34], [32], [5], [23] and, especially, [24].

We begin with the notion of a Γ -space, due to Segal [30], and its extension to the notion of a Γ -ring, due to Schwede [29] and Lydakis [23]. Let n_+ denote the pointed finite set $\{*, 1, \dots, n\}$, with $*$ serving as the base point, and let Γ^{op} denote the category consisting of objects n_+ , $n \geq 0$, and with morphisms given by pointed functions. Here, 0_+ denotes the one-point set $\{*\}$. A Γ -object in a category $\mathcal{C}$ is a functor from Γ^{op} to $\mathcal{C}$. A Γ -object G in the category of topological spaces is more commonly referred to a “ T -space”, and in this case we require additionally that $G(0_+)$ be the one point space. Observe that G can thus be viewed as taking values in the category of pointed spaces, with the base point of $G(n_+)$ defined by the map $G(0_+) \rightarrow G(n_+)$ induced by the unique map $0_+ \rightarrow n_+$. A Γ -space G is *special* if the map

$$(p_{1*}, \dots, p_{n*}) : G(n_+) \rightarrow G(1_+)^{\times n},$$

where $p_i : n_+ \rightarrow 1_+$ is the projection map defined by

$$p_i(j) = \begin{cases} j, & \text{if } i = j \text{ and} \\ *, & \text{if } i \neq j \end{cases}$$

is a homotopy equivalence, for all $n \geq 0$. (Note that in Segal’s original formulation [30], every Γ -space is by definition special.) Similarly, a Γ -object G in the category of simplicial sets, multi-simplicial sets, or multi-simplicial spaces is assumed to be such that $G(0_+)$ consists of the evident one-point object, and moreover such a G is *special* if the associated Γ -space $n_+ \mapsto |G(n_+)|$ is special, where $|-|$ denotes taking geometric realization. In each of these cases, the Γ -object G can be viewed as taking values in the corresponding pointed category.

Given a Γ -space G and a pointed CW complex T , Segal’s *tensor construction* (cf. [30]) produces a space $T \otimes G$, defined as follows: Let T define a co- Γ -space (that is, a contravariant functor from Γ^{op} to pointed spaces) by sending n_+ to $\text{Maps}(n_+, T) = T^{n_+}$. Then $T \otimes G$ is defined as the quotient space

$$\coprod_{n \geq 0} T^{n_+} \times G(n_+) / \sim,$$

where the equivalence relation is defined by

$$(f_*(g), t) \sim (g, f^*(t))$$

for all $f : m_+ \rightarrow n_+$, $g \in G(m_+)$, and $t : n_+ \rightarrow T$. In this manner, the functor $G : \Gamma^{op} \rightarrow Spaces$ is “promoted” to a functor defined on all pointed CW complexes by the formula $G(T) = T \otimes G$. Letting CW^* denote the category of pointed CW complexes (i.e., spaces admitting the structure of a CW complex), this promotion is given by topological left Kan extension along the inclusion of categories $\Gamma^{op} \hookrightarrow CW^*$.

Given a Γ -space G and a CW complex T , we define another Γ -space, written $T \underline{\otimes} G$, by the formula

$$m_+ \mapsto T \otimes G(m_+ \wedge -).$$

Here, and everywhere else in this paper, given m_+ and n_+ in Γ^{op} , their smash product $m_+ \wedge n_+$ is identified with $(mn)_+$ via lexicographic ordering: $(i, j) \mapsto n(i-1) + j$. If S is another pointed CW complex, there is a natural homeomorphism

$$S \otimes (T \underline{\otimes} G) \cong (S \wedge T) \otimes G = G(S \wedge T).$$

In particular, observe that $T \underline{\otimes} G$ can be identified with the Γ -space $n_+ \mapsto G(n_+ \wedge T)$, and the promotion of $T \underline{\otimes} G$ to a functor defined on CW^* is given by $S \mapsto G(S \wedge T)$.

Given a Γ -space G and two pointed CW complexes S and T , there is a natural map of spaces

$$T \wedge G(S) \rightarrow T \otimes G(- \wedge S) = G(T \wedge S),$$

induced by the evident inclusion $T \times G(S) \hookrightarrow \coprod_n T^{n+} \times G(n_+ \wedge S)$. This map extends to a natural map of Γ -spaces

$$T \wedge G(- \wedge S) \rightarrow G(- \wedge T \wedge S),$$

which in particular allows us to associate a spectrum to any Γ -space. For the purposes of this paper, by a *spectrum* we mean a sequence of pointed spaces $E_0, E_1, \dots$ together with “bonding” maps $S^1 \wedge E_i \rightarrow E_{i+1}$, for $i \geq 0$. Morphisms of spectra are defined as in [1, §2]. The homotopy groups of a spectrum E are given by

$$\pi_n E = \varinjlim_j \pi_{n+j} E^j.$$

A morphism of spectra is a *stable weak equivalence* if it induces an isomorphism on homotopy groups. The spectrum associated to a Γ -space G , written $\mathbb{S}(G)$, is given by the sequence of spaces

$$G(S^0), \quad G(S^1), \quad G(S^2), \quad \dots$$

where the bonding maps $S^1 \wedge G(S^k) \rightarrow G(S^{k+1})$ are defined by the canonical maps

$$S^1 \wedge G(S^k) \rightarrow G(S^1 \wedge S^k) \cong G(S^{k+1}).$$

The following result is proven by Bousfield and Friedlander.

Lemma A.1. (cf. [6, 4.1]) *Let G be a Γ -space and T a pointed CW complex. Then the natural map*

$$T \wedge \mathbb{S}(G) \rightarrow \mathbb{S}(T \underline{\otimes} G)$$

is a stable weak equivalence of spectra.

It is also proven by Bousfield-Friedlander [6, 5.8] that the stable homotopy category of Γ -spaces (with the appropriate notion of stable equivalence) coincides with that of connective spectra.

We now discuss the notions of a smash product of Γ -spaces, an external pairing of Γ -spaces, and a Γ -ring (cf. [29], [23]). Given a triple of Γ -spaces X , Y and Z , an *external* pairing of Γ -spaces, written

$$\mu : X \overline{\wedge} Y \rightarrow Z,$$

is a map of bi- Γ -spaces (i.e., a natural transformation of functors from $\Gamma^{\text{op}} \times \Gamma^{\text{op}}$ to Spaces) of the form

$$(n_+, m_+ \mapsto X(n_+) \wedge Y(m_+)) \rightarrow (n_+, m_+ \mapsto Z(n_+ \wedge m_+)).$$

External pairings of Γ -spaces induce in a functorial manner pairings of spectra (as will be stated formally below). This may be treated as a “black box” construction, but since it is rather elementary and illustrates the essential motivation for using Γ -rings and FSP’s, we will explain the workings of this black box in a bit of detail.

Given any bi- Γ -space, we may convert it into a Γ -space by topological left Kan extension along the map $-\wedge - : \Gamma^{\text{op}} \times \Gamma^{\text{op}} \rightarrow \Gamma^{\text{op}}$ given by $(m_+, n_+) \mapsto m_+ \wedge n_+$. (Recall that $m_+ \wedge n_+$ is identified with $(mn)_+$ via lexicographic ordering.) In particular, we define the *smash product* $X \wedge Y$ of two Γ -spaces X and Y to be the Γ -space obtain from the bi- Γ -space $(m_+, n_+) \mapsto X(m_+) \wedge Y(n_+)$ in this fashion. Explicitly, we have the formula:

$$(X \wedge Y)(n_+) = \varinjlim_{p_+ \wedge q_+ \rightarrow n_+} X(p_+) \wedge Y(q_+).$$

Given an external pairing $X \overline{\wedge} Y \rightarrow Z$ of Γ -spaces, the universal property of topological left Kan extension gives a map of Γ -spaces $X \wedge Y \rightarrow Z$ (cf. [24, 21.4]). Explicitly, the map $X \wedge Y \rightarrow Z$ is given by composition with the canonical map

$$\varinjlim_{m_+ \wedge n_+ \rightarrow p_+} Z(m_+ \wedge n_+) \rightarrow Z(p_+).$$

Applying Z to the evident involution $\tau : \Gamma^{\text{op}} \times \Gamma^{\text{op}} \rightarrow \Gamma^{\text{op}}$, we obtain a morphism $Z(\tau) : Z(- \wedge -) \rightarrow \tau^* Z(- \wedge -)$ of bi- Γ -spaces $Z(- \wedge -)$. Since $\tau^2 = id$, it is essentially obvious that the diagram

$$(25) \quad \begin{array}{ccc} \varinjlim_{a_+ \wedge b_+ \rightarrow p_+} Z(a_+ \wedge b_+) & \longrightarrow & Z(p_+) \\ Z(\tau) \downarrow & & = \downarrow \\ \varinjlim_{a_+ \wedge b_+ \rightarrow p_+} Z(b_+ \wedge a_+) & \longrightarrow & Z(p_+) \end{array}$$

commutes. From this, we easily obtain:

Lemma A.2. *Given an external pairing of Γ -spaces $X \overline{\wedge} X \rightarrow Z$ which is symmetric in the sense that*

$$\begin{array}{ccc} X(m_+) \wedge X(n_+) & \longrightarrow & Z(m_+ \wedge n_+) \\ \tau \downarrow & & Z(\tau) \downarrow \\ X(n_+) \wedge X(m_+) & \longrightarrow & Z(n_+ \wedge m_+) \end{array}$$

commutes for all $m, n \geq 0$, the induced map of Γ -spaces $X \wedge X \rightarrow Z$ is symmetric in the sense that the diagram

$$\begin{array}{ccc} X \wedge X & \longrightarrow & Z \\ \tau \downarrow & & \downarrow \\ X \wedge X & \longrightarrow & Z \end{array}$$

commutes.

A Γ -ring (cf. [29, 2.1]) is a Γ -space G together with an external pairing of the form $\mu : G \overline{\wedge} G \rightarrow G$, together with a “unit map”, which is a map of Γ -spaces of the form

$$\mathbf{1} : S \rightarrow G.$$

Here, $S : \Gamma^{op} \rightarrow CW^*$ denotes the evident inclusion of categories, and is so named because the associated spectrum $\mathbb{S}(S)$ is clearly the sphere spectrum. The pairing μ and natural transformation $\mathbf{1}$ are required to satisfy the appropriate unital and associative axioms. Rather than list them explicitly here, we refer ahead to the definition of FSP below and mention that a Γ -ring is required to satisfy the axioms obtained by replacing X, Y, Z , and S^0 with l_+, m_+, n_+ , and 1_+ , respectively, in the diagrams (26) through (29). A Γ -ring G is *commutative* if diagram (30) commutes for $X = m_+$ and $Y = n_+$. (The map $\tau : m_+ \wedge n_+ = (mn)_+ \rightarrow (mn)_+ = n_+ \wedge m_+$ appearing at various places in these diagrams is defined to be the unique map causing the diagram

$$\begin{array}{ccc} m_+ \wedge n_+ & \xrightarrow[\cong]{\text{lex}} & (mn)_+ \\ \downarrow \text{flip} & & \downarrow \tau \\ n_+ \wedge m_+ & \xrightarrow[\cong]{\text{lex}} & (nm)_+ \end{array}$$

to commute, where the horizontal maps are given by lexicographic ordering.)

Lemma A.3. *Given an external pairing of Γ -spaces $X \overline{\wedge} Y \rightarrow Z$, there is a functorial construction of a map of spectra*

$$\mathbb{S}(X) \wedge \mathbb{S}(Y) \rightarrow \mathbb{S}(Z),$$

where the smash product here is the handicraft smash product (cf. [1, 4]). Moreover, if X is a Γ -ring, then $\mathbb{S}(X)$ has the natural structure of a ring spectrum; if X is a commutative Γ -ring, then $\mathbb{S}(X)$ is a commutative ring spectrum.

Proof. This follows from [24, 0.3 and 11.8]. Note that it suffices to prove the result after promoting all the Γ -spaces to $\mathcal{W}$ -spaces, using the terminology in loc. cit. $\square$

We now introduce the notion of a $\mathcal{W}$ -space and the related notion of an FSP. These are generalizations of the notions of a Γ -space and Γ -ring, respectively. To motivate the definition of a $\mathcal{W}$ -space, recall that a Γ -space G may be promoted to a functor on pointed CW complex via the rule $G(T) = T \otimes G$, and this promoted functor serves as a primary example of a $\mathcal{W}$ -space. Intuitively, then, a $\mathcal{W}$ -space is the result of replacing the indexing category Γ^{op} with the category of pointed CW complex. More formally, let $\mathcal{W}$ denote the category of pointed spaces which are homeomorphic to a CW complex with morphisms given by continuous maps – for technical reasons, objects of $\mathcal{W}$ are required to be homeomorphic to a CW complex having some fixed upper bound on the cardinality of the collection cells (see [24,

4.6]). Then a $\mathcal{W}$ -space (cf. [24, 4.6]) is a covariant functor from $\mathcal{W}$ to $Spaces$. Thus, with this new language, we can say that a Γ -space determines a $\mathcal{W}$ -space upon promotion (cf. [24, 3.2]). Moreover, all of the constructions introduced above for Γ -spaces carry over to $\mathcal{W}$ -spaces. In particular, the *spectrum associated to a $\mathcal{W}$ -space X* , written $\mathbb{S}(X)$, is given by the sequence of spaces

$$X(S^0), \quad X(S^1), \quad X(S^2), \quad \dots,$$

where the bonding maps $S^1 \wedge X(S^q) \rightarrow X(S^{q+1})$ are induced by the functoriality of X (cf. [24, 4.9]). An *external pairing* of $\mathcal{W}$ -spaces $X \overline{\wedge} Y \rightarrow Z$ is a map of bi- $\mathcal{W}$ -spaces of the form $X(-) \wedge Y(-) \rightarrow Z(- \wedge -)$. The *smash product* of two $\mathcal{W}$ -spaces X and Y , written $X \wedge Y$, is the $\mathcal{W}$ -space obtained by topological left Kan extension of the bi- $\mathcal{W}$ -space $X(-) \wedge Y(-)$ along the continuous functor $\mathcal{W} \times \mathcal{W} \rightarrow \mathcal{W}$ defined by $(A, B) \mapsto A \wedge B$. As with Γ -spaces, an external pairing of $\mathcal{W}$ -spaces $X \overline{\wedge} Y \rightarrow Z$ defines a map of $\mathcal{W}$ -spaces $X \wedge Y \rightarrow Z$ (cf. [24, 21.4]).

To motivate the definition of an *FSP*, observe that given a bi- Γ -space $Z(-, -)$, we can apply the tensor construction to either variable so as to promote $Z(-, -)$ to a functor from $\mathcal{W} \times \mathcal{W}$ to $Spaces$. (The order in which we apply the tensor construction does not matter up to homeomorphism). In particular, if G is a Γ -ring, then the induced functor $G : \mathcal{W} \rightarrow Spaces$ comes equipped with a “unit map”

$$\mathbf{1} : (-) \rightarrow G(-),$$

which is a natural transformation of functors with $(-)$ representing the identity functor, and a “multiplication” map

$$\mu : G(-) \wedge G(-) \rightarrow G(- \wedge -).$$

More generally, an *FSP* (cf. [5], [4, 3.1]) is a covariant functor F from $\mathcal{W}$ to spaces equipped with two natural transformations: a “unit” map

$$\mathbf{1} : (-) \rightarrow F(-).$$

and a “multiplication” map

$$\mu : F(-) \wedge F(-) \rightarrow F(- \wedge -).$$

These natural transformations are required to cause the following diagrams to commute, for all X, Y , and Z in $\mathcal{W}$:

“ $\mathbf{1}$ is an idempotent”

$$(26) \quad \begin{array}{ccc} X \wedge Y & \xrightarrow{\mathbf{1}_X \wedge \mathbf{1}_Y} & F(X) \wedge F(Y) \\ & \searrow \mathbf{1}_{X \wedge Y} \quad \swarrow \mu_{X,Y} & \\ & F(X \wedge Y) & \end{array}$$

“Centrality of $\mathbf{1}$ ”

$$(27) \quad \begin{array}{ccccc} X \wedge F(Y) & \xrightarrow{\tau} & F(Y) \wedge X & \xrightarrow{id \wedge \mathbf{1}_X} & F(Y) \wedge F(X) \\ \downarrow \mathbf{1}_X \wedge id & & & & \downarrow \mu_{Y,X} \\ F(X) \wedge F(Y) & \xrightarrow{\mu_{X,Y}} & F(X \wedge Y) & \xrightarrow{F(\tau)} & F(Y \wedge X) \end{array}$$

“Associativity of multiplication”

$$(28) \quad \begin{array}{ccc} F(X) \wedge F(Y) \wedge F(Z) & \xrightarrow{id \wedge \mu_{Y,Z}} & F(X) \wedge F(Y \wedge Z) \\ \downarrow \mu_{X,Y} \wedge id & & \downarrow \mu_{X,Y \wedge Z} \\ F(X \wedge Y) \wedge F(Z) & \xrightarrow{\mu_{X \wedge Y, Z}} & F(X \wedge Y \wedge Z) \end{array}$$

“Agreement of $\mathbf{1}$ and S^0 ”

$$(29) \quad \begin{array}{ccc} S^0 \wedge F(X) & \xrightarrow{\mathbf{1}_{S^0} \wedge id} F(S^0) \wedge F(X) \xrightarrow{\mu_{S^0, X}} & F(S^0 \wedge X) \\ \downarrow \cong & & \downarrow \cong \\ F(X) & \xrightarrow{=} & F(X) \\ \uparrow \cong & & \uparrow \cong \\ F(X) \wedge S^0 & \xrightarrow{id \wedge \mathbf{1}_{S^0}} F(X) \wedge F(S^0) \xrightarrow{\mu_{X, S^0}} & F(X \wedge S^0) \end{array}$$

An FSP is *commutative* if, additionally, the following diagram commutes, for all X and Y in $\mathcal{W}$:

“Commutativity”

$$(30) \quad \begin{array}{ccc} F(X) \wedge F(Y) & \xrightarrow{\mu_{X,Y}} & F(X \wedge Y) \\ \downarrow \tau & & \downarrow F(\tau) \\ F(Y) \wedge F(X) & \xrightarrow{\mu_{Y,X}} & F(Y \wedge X) \end{array}$$

In the above axioms, each instance of τ denotes the appropriate symmetry involution and $\cong$ denotes the appropriate canonical homeomorphism. An FSP F is *convergent* if for any $X \in \mathcal{W}$, the sequence of maps

$$F(S^j) \wedge X \rightarrow F(S^j \wedge X), \quad j \geq 0$$

induces an isomorphism on π_* stably:

$$\varinjlim_j \pi_{n+j}(F(S^j) \wedge X) \xrightarrow{\cong} \varinjlim_j \pi_{n+j} F(S^j \wedge X)$$

for all n and X . An FSP F is *strictly connective* if $F(A)$ is a n -connected whenever $A \in \mathcal{W}$ is n -connected.

Lemma A.4. (cf. [24, 3.2]) *If G is a Γ -ring, then the promoted functor $G : \mathcal{W} \rightarrow \text{Spaces}$ defined by $G(V) = V \otimes G$ has the natural structure of a strictly connective FSP. If G is a commutative Γ -space, then the promoted functor is a commutative FSP.*

The spectrum $\mathbb{S}(F)$ associated to an FSP F is that associated to the underlying $\mathcal{W}$ -space. More generally, given $V \in \mathcal{W}$, we define a spectrum $\mathbb{S}(F(V))$ whose constituent spaces are

$$F(V), \quad F(S^1 \wedge V), \quad F(S^2 \wedge V), \quad \dots$$

The bonding maps of $\mathbb{S}(F(V))$ are given by the compositions of

$$S^1 \wedge F(S^k \wedge V) \xrightarrow{\mathbf{1}_{S^1} \wedge id} F(S^1) \wedge F(S^k \wedge V) \xrightarrow{\mu_{S^1, S^k \wedge V}} F(S^1 \wedge S^k \wedge V) \xrightarrow{\cong} F(S^{k+1} \wedge V),$$

for $k \geq 0$. When $V = S^0$, we have $\mathbb{S}(F(S^0)) = \mathbb{S}(F)$. If F is the FSP arising from a Γ -ring G , then $\mathbb{S}(F(V)) = \mathbb{S}(n_+ \mapsto V \otimes G(n_+ \wedge -))$. Observe that for any $V \in \mathcal{W}$, there is a natural map of spectra

$$\mathbb{S}(F) \wedge V \rightarrow \mathbb{S}(F(V)),$$

which is a stable weak equivalence provided F is convergent.

Lemma A.5. (cf. [24, 0.3 and 22.6]) *Given an external pairing of $\mathcal{W}$ -spaces $X \overline{\wedge} Y \rightarrow Z$, there is a functorial construction of a map of spectra*

$$\mathbb{S}(X) \wedge \mathbb{S}(Y) \rightarrow \mathbb{S}(Z),$$

where the smash product here is the handcraft smash product (cf. [1, 4]). Moreover, if F is an FSP, then $\mathbb{S}(F)$ has the natural structure of a ring spectrum; if F is a commutative FSP, then $\mathbb{S}(F)$ is a commutative ring spectrum.

A Γ -ring determines an FSP by “promotion”; conversely, an FSP F determines a Γ -ring by restricting the domain of F to Γ^{op} viewed as a subcategory of $\mathcal{W}$. Moreover, for any V in $\mathcal{W}$, there is a natural map of spaces

$$V \otimes F(-) \rightarrow F(V)$$

induced by the pairings $\underline{\text{Maps}}(n_+, V) \times F(n_+) \rightarrow F(V)$, $n \geq 0$, arising from the functoriality of F . More generally, an FSP F and a V in $\mathcal{W}$ determine a Γ -space by the formula $n_+ \mapsto F(n_+ \wedge V)$, and as with the case above (given by taking $V = S^0$), there is a natural map

$$W \otimes F(- \wedge V) \rightarrow F(W \wedge V)$$

for any W in $\mathcal{W}$. Consequently, in addition to the spectrum $\mathbb{S}(F(V))$ determined by F and V , we may form the spectrum

$$F(V), S^1 \otimes F(- \wedge V), S^2 \otimes F(- \wedge V), \dots$$

associated to the Γ -space $F(- \wedge V)$. Since we have natural maps

$$S^q \otimes F(- \wedge V) \rightarrow F(S^q \wedge V),$$

for all $q \geq 0$, the following Lemma is immediate.

Lemma A.6. (cf. [24, 3.2 and 4.8]) *For an FSP F and pointed space V , there is a natural map of spectra from the connective spectrum associated to the Γ -space $F(- \wedge V)$ (consisting of the spaces*

$$F(V), S^1 \otimes F(- \wedge V), S^2 \otimes F(- \wedge V), \dots)$$

to the spectrum $\mathbb{S}(F(V))$ (consisting of the spaces

$$F(V), F(S^1 \wedge V), F(S^2 \wedge V), \dots).$$

Finally, we indicate how to define a homology and cohomology theory directly in terms of an FSP F and also how to define all of the standard cohomological operations. The theories so given and the operations so defined are equivalent to those arising from the spectrum $\mathbb{S}(F)$. We include the following definitions for the sake of explicitness. Indeed, the theories and operations associated to an FSP are defined more simply than those for ring spectra. The only reason to define cohomology groups and operations on them in terms of spectra (and, indeed, the only reason to introduce spectra at all, as opposed to Γ -spaces, Γ -rings, $\mathcal{W}$ -spaces, and FSP’s) is to conform with the historical development of the subject and to have the correct inputs for the Poincare Duality Theorem.

For an FSP F , the *cohomology theory associated to F* is the graded abelian group valued functor $F^*(-)$ on $\mathcal{W}$ defined by

$$F^q(X) = \varinjlim_k \pi_{k-q} \underline{\text{Maps}}(X, F(S^k)).$$

Similarly, the *homology theory associated to F* is defined by

$$F_q(X) = \varinjlim_k \pi_{k+q}(F(S^k) \wedge X).$$

If F is convergent, this coincides with the the direct limit

$$\varinjlim_k \pi_{k+q} F(S^k \wedge X).$$

Note that the cohomology and homology theories associated with F coincide up to natural isomorphism with the cohomology and homology theories associated to the spectrum $\mathbb{S}(F)$.

We now define cup product, cap product, two external products, and two slant products for the cohomology/homology theory associated to F . Given $X \in \mathcal{W}$, the *cup product* is the graded pairing

$$\cup : F^p(X) \otimes F^q(X) \rightarrow F^{p+q}(X)$$

defined in the evident manner from the collection of pairings

$$\underline{\text{Maps}}(X, F(S^j)) \wedge \underline{\text{Maps}}(X, F(S^k)) \rightarrow \underline{\text{Maps}}(X, F(S^j) \wedge F(S^k)) \rightarrow \underline{\text{Maps}}(X, F(S^{j+k})),$$

for $j, k \geq 0$. By “evident manner”, we mean that a generator $\alpha \otimes \beta \in F^p(X) \otimes F^q(X)$, represented by the pair of maps

$$S^{p+j} \rightarrow \underline{\text{Maps}}(X, F(S^j)), \quad S^{q+k} \rightarrow \underline{\text{Maps}}(X, F(S^k)),$$

for some $j, k \geq 0$, is sent to the element represented by the composition

$$\begin{aligned} S^{p+q+j+k} &= S^{p+j} \wedge S^{q+k} \rightarrow \underline{\text{Maps}}(X, F(S^j)) \wedge \underline{\text{Maps}}(X, F(S^k)) \\ &\rightarrow \underline{\text{Maps}}(X, F(S^j) \wedge F(S^k)) \rightarrow \underline{\text{Maps}}(X, F(S^{j+k})). \end{aligned}$$

Given $X, Y \in \mathcal{W}$, cup product generalizes in the obvious manner to *external pairing for cohomology*

$$\overline{\cup} : F^p(X) \otimes F^q(Y) \rightarrow F^{p+q}(X \wedge Y)$$

by use of the family of pairings of spaces

$$\begin{aligned} \underline{\text{Maps}}(X, F(S^j)) \wedge \underline{\text{Maps}}(Y, F(S^k)) &\rightarrow \underline{\text{Maps}}(X \wedge Y, F(S^j) \wedge F(S^k)) \\ &\rightarrow \underline{\text{Maps}}(X \wedge Y, F(S^{j+k})), \quad j, k \geq 0. \end{aligned}$$

Indeed, the former is obtained from the latter by setting $X = Y$ and composing with the pullback along the diagonal map $X \rightarrow X \wedge X$.

Given $X, Y \in \mathcal{W}$, the *external product for homology* is the pairing

$$\Delta : F_p(X) \otimes F_q(Y) \rightarrow F_{p+q}(X \wedge Y)$$

defined in the evident manner from the collection of pairings of spaces

$$F(S^j) \wedge X \wedge F(S^k) \wedge Y \rightarrow F(S^j \wedge S^k) \wedge X \wedge Y = F(S^{j+k}) \wedge X \wedge Y.$$

Given $X, Y \in \mathcal{W}$, the *slant product pairing* / is the pairing

$$/ : F^p(X \wedge Y) \otimes F_q(Y) \rightarrow F^{p-q}(X)$$

defined in the evident manner from the family of pairings of spaces

$$\begin{aligned}
& \underline{\text{Maps}}(X \wedge Y, F(S^j)) \wedge F(S^k) \wedge Y \\
& \longrightarrow \underline{\text{Maps}}(F(S^k) \wedge X \wedge Y, F(S^k) \wedge F(S^j)) \wedge F(S^k) \wedge Y \\
& \longrightarrow \underline{\text{Maps}}(F(S^k) \wedge X \wedge Y, F(S^{k+j})) \wedge F(S^k) \wedge Y \\
& \longrightarrow \underline{\text{Maps}}(X, F(S^{k+j})),
\end{aligned}$$

where the final map is given by the standard evaluation pairing. Similarly, the *slant product pairing* $\backslash$ is the pairing

$$\backslash : F^p(X) \otimes F_q(X \wedge Y) \rightarrow F_{q-p}(Y)$$

is defined by the family of pairings of spaces

$$\begin{aligned}
& \underline{\text{Maps}}(X, F(S^j)) \wedge F(S^k) \wedge X \wedge Y \\
& \longrightarrow \underline{\text{Maps}}(F(S^k) \wedge X \wedge Y, F(S^k) \wedge F(S^j) \wedge Y) \wedge F(S^k) \wedge X \wedge Y \\
& \longrightarrow \underline{\text{Maps}}(F(S^k) \wedge X \wedge Y, F(S^{j+k}) \wedge Y) \wedge F(S^k) \wedge X \wedge Y \\
& \longrightarrow F(S^{j+k}) \wedge Y.
\end{aligned}$$

Finally, the *cap product pairing* $\cap$ is the pairing

$$\cap : F^p(X) \otimes F_q(X) \rightarrow F_{q-p}(X)$$

is defined by $u \cap v = u \backslash \Delta_*(v)$, where $\Delta : X \rightarrow X \wedge X$ is the diagonal map.

The following lemma is immediately seen to hold by comparing the above definitions with those found, for example, in [34, §13].

Lemma A.7. *For any FSP F , the operations $\overline{\wedge}$, $\cap$, $\triangle$, $\sqcap$, $/$, and $\backslash$ defined on the cohomology and homology theories associated to F coincide with the operations bearing the same names associated to the ring spectrum $\mathbb{S}(F)$, as defined in [34, 13.50]. In particular, all of the usual formula relating these operations are valid (cf. [1, 13.52–13.62]).*

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