ALGEBRAIC COBORDISM OF SIMPLY CONNECTED LIE GROUPS

NOBUAKI YAGITA

Abstract. Let $G_{\mathbb{C}}$ be the algebraic group over $\mathbb{C}$ corresponding a simply connected Lie group G. The algebraic cobordism $\Omega(G_{\mathbb{C}})$ defined by Levine and Morel is showed isomorphic to MU^*-subalgebra of $MU^*(G)$ with some modulous and is computed explicitely.

1. Introduction

For a smooth algebraic variety X over a field k of $ch(k) = 0$, recently Levine and Morel [L-M 1,2] defined the algebraic cobordism $\Omega^*(X)$ such that $\Omega^*(X) \otimes_{\Omega^*} Z \cong CH^*(X)$; the Chow ring of X and that there are maps

$$\Omega^*(X) \xrightarrow{\rho_{MGL}} MGL^{2*,*}(X) \xrightarrow{t_{MU}} MU^*(X)$$

where $MGL^{2*,*}(-) = \oplus_i MGL^{2i,i}(-)$ is the motivic cobordism theory defined by Voevodsky [V 1] and $MU^*(-)$ is the usual complex cobordism theory.

Let G be a simply connected Lie group and $G_{\mathbb{C}}$ the corresponding reductive algebraic group over $\mathbb{C}$; the complex number field. Let T be the maximal torus of G and $\pi : G \to G/T$ is the projection. By Grothendieck and Kac [K], it is known

$$CH^*(G_{\mathbb{C}}) \cong \pi^*(H^*(G/T)).$$

We study the cobordism version of this result. Fix a prime p. Recall $\Omega^* \cong MU^* \cong \mathbb{Z}[x_1, ...]$ with $|x_i| = -2i$ and identify $v_i = x_{p^i-1}$.

Theorem 1.1. *Let $I = (p, v_1, ...)$ be the invariant prime ideal of MU^*. Then we have the isomorphism $\Omega^*(G_{\mathbb{C}})/I^2 \cong \pi^*(MU^*(G/T))/I^2$.*

The above isomorphism seems to hold without I^2, however we can not prove it now.

By Borel theorem, we can write $H^*(G; \mathbb{Z}/p) \cong P(y_{even})/p \otimes \Lambda(x_{odd})$ where $P(y_{even})$ is a truncated polynomial algebra of even degree generators y_{even} and $\Lambda(x_{odd})$ is the exterior algebra of odd degree generators x_{odd}. When $p = 2$ we take y_{even} as a power of some x_{odd}. Then the result of Grothendieck and Kac [K] is stated as

$$CH^*(G_{\mathbb{C}})_{(p)} \cong P(y_{even})/p.$$

Let Q_i be the Milnor primitive operation inductively defined by $Q_i = [Q_{i-1}, P^{p^{i-1}}]$ and $Q_0 = \beta$; the Bockstein operation. It is known that $Q_i(x_{odd}) \in P(y_{even})/p$ for all $i \geq 0$.

Theorem 1.2. *There is an Ω^*-algebra isomorphism*

$$\Omega^*(G_{\mathbb{C}})/I^2 \cong \Omega^* \otimes P(y_{even})/(I^2, \sum_i v_i Q_i(x_{odd})).$$

1991 *Mathematics Subject Classification.* Primary 55P35, 57T25; Secondary 55R35, 57T05.

Key words and phrases. algebraic cobordism, Chow ring, motivic cohomology, BP-theory, simply connected Lie groups.

For all simply connected simple Lie groups, more explicit forms of this theorem are shown in Theorem 5.1-5.4.

As for motivic cobordism theory $MGL^{*,*}(G_{\mathbb{C}})$, we do not know well. Indeed, even $MU^*(G)_{(p)}$ are quite complicated and unknown, in general. Let $MGL/p^{*,*}$ be the mod p motivic cobordism theory defined by the cofibering

$$MGL \xrightarrow{p} MGL \to MGL/p$$

in the stable $\mathbb{A}^1$-homotopy category. Let $MU/p^*(X) = MU^*(-;\mathbb{Z}/p)$ the mod p complex cobordism theory defined similarly.

Theorem 1.3. *Let G be a simle Lie group in Case I in §5, e.g., $G = F_4, E_6, E_7$ for $p = 3$. Then we can give bidegree to $MU/p^*(G)$ such that degree of its MU^*-algebra generators are $(2n, n)$ or $(2m-1, m)$ for $n, m \geq 3$, and that*

$$MU/p^*(G) \subset MGL/p^{*,*}(G_{\mathbb{C}}).$$

In §2, we recall the definition of the algebraic cobordism $\Omega^*(X)$ and the relation to the complex cobordism $MU^*(X)$. We also remark the BP-version of $\Omega^*(X)$. In §3, we note that Theorem 1.1 holds if $\pi^*MU^*(G/T) \cong MU^*(G/T)/Ideal(i^*\tilde{MU}^*(BT))$. Here

$$G \xrightarrow{\pi} G/T \xrightarrow{i} BT.$$

is the usual fibering. In §4, we will prove the above isomorphism $mod(I^2)$ with some assumption of the Milnor operation Q_i on $H^*(G;\mathbb{Z}/p)$. Here we use the spectral sequence for connected Morava K-theory (or other cohomology theories)

$$E_2^{*,*} \cong H^*(BT; k(1)^*(G)) \Longrightarrow k(1)^*(G/T).$$

Indeed, $E_\infty^{*,0}$ shows the image $i^*k(1)^*(BT)$. In §5, we check the condition of Q_i, for each simple Lie group which has p-torsion, and consequently we see Theorem 1.1. Explicit results of $\Omega^*(G_{\mathbb{C}})/I^2$ also given here. In §6, we give examples of computations of the above spectral sequence for the easy cases G. In §7, for this type group G, we also study the motivic cobordism $MGL/p^{*,*}(G_{\mathbb{C}})$. The last section is very short remark for classifying spaces BG.

2. ALGEBRAIC COBORDISM

By extending the arguments by Quillen [Q], Levine and Morel defined the algebraic cobordism theory $\Omega^*(-)$ as the universal theory in theories having transfers and Chern classes [L-M 1,2] (We say that $h^*(X)$ is a theory having transfers and Chern classes if this theory satifies the actioms A1 to A4 in [L-M 1]). Here we note that $\Omega^*(-)$ is not cohomology theory. The ring $\Omega^*(X)$ is constructed as

$$\Omega^*(X) = \{[f : M \to X]\}/(relations).$$

Here f is a map from a smooth variety M to X of pure codimension, namely, $dim_{f(y)}(X) - dim_y(M)$ is constant for all y in the same connected component of M. Relations are given so that we can define Chern classes or formal group laws (for details, see [L-M 1]). Given theory $h^*(-)$ having transfers and Chern classes, the map

$$\rho_h : \Omega^*(-) \to h^*(-)$$

is defined by $\rho_h([f : M \to X]) = f_*(1_M)$ where $1_M \in h^0(M)$ represents the identity element.

Let $H^{*,*}(-)$ (resp. $MGL^{*,*}(-)$, $MU^*(-)$, $CH^*(-)$, $H^*(-)$) be the motivic cohomology theory defined by Suslin and Voevodsky (resp. the motivic cobordism theory, the complex cobordism theory, the Chow ring, the usual cohomology theory). Then we have commutaive diagram

$$\begin{array}{ccccc}
\Omega^*(X) & \xrightarrow{\rho_{MGL}} & MGL^{2*,*}(X) & \xrightarrow{t_M} & MU^*(X) \\
{=}\downarrow & & \rho^{2*,*}\downarrow & & \rho^*\downarrow \\
\Omega^*(X) & \xrightarrow{\rho_{CH}} & CH^*(X)\cong H^{2*,*}(X) & \xrightarrow{t_H} & H^*(X)
\end{array}$$

where $\rho^{2*,*},\rho^*$ (resp. t_m, t_H) are Thom maps (resp. realization maps.) Levine and Morel proves that

$$t_M\rho_{MGL}:\Omega^*(pt)\cong MU^*(pt)\cong \mathbb{Z}[x_1,x_2,...] \quad with \quad |x_i|=-2i$$

$$and \quad \rho_{CH}\otimes_{\Omega^*}Z:\Omega^*(X)\otimes_{\Omega^*}Z\cong CH^*(X).$$

Moreover they conjecture that ρ_{MGL} are always isomorphisms.

For the loaclized theories $h^*(-)_{(p)}$, we can consider the BP-version by using the universal p-typical formal group laws. However we construct here the BP-version using the Novikov's technique (5.4 in [N]). Given a sequence $\alpha=(a_1,a_2,..)$, $a_i\ge 0$, recall the Landweber-Novikov operation is defined by

$$S_\alpha(f_*(1_M))=f_*(c_\alpha(N_f)) \quad for\ [f:M\to X]\in\Omega^*(X)$$

where $c_\alpha(N_f)$ is the Chern class of the normal bundle of $f(M)$ in X so that S_α in $MU^*(X)$ is the usual Landweber-Novikov operation. This operations satisfy the Cartan formula $S_\alpha(xy)=\Sigma_{\alpha=\beta+\gamma}S_\beta(x)S_\gamma(y)$ for $x,y\in\Omega^*(X)$. Define an operation

$$\Delta_{x_i}=\Sigma_{q\ge1}(x_i/S_{\Delta_i}(x_i))^{q-1}S_{q\Delta_i}.$$

Note that $\Delta_{x_i}(x_i)=1$ and $S_{\Delta_i}(x_i)\ne 0 \bmod p$ if $i\ne p^j-1$. Then we can easily prove that $\pi_i=1-x_i\Delta_{x_i}$ is a multiplicative projection such that $\pi_i(x_j)=(1-\delta_{ij})x_j$. Essentially composing (for details, see p587 in [N]) π_i for all $i\ne p^j-1$, we get the multiplicative projection $\Phi:\Omega^*(-)_{(p)}\to\Omega^*(-)_{(p)}$ such that

$$\Phi(x_i)=\begin{cases} x_i & (if\ i=p^j-1\ for\ some\ j)\\ 0 & (otherwise)\end{cases}$$

Define the algebraic Brown-Peterson theory

$$\Omega^*_{BP}(X)=Im(\Phi(\Omega^*(X)_{(p)})\subset\Omega^*(X)_{(p)}.$$

Hence if $h^*(-)$ is a theory having transfers and Chern classes, then there is the natural map $\rho_{BP,h}:\Omega^*_{BP}(X)\to h^*(X)_{(p)}$ compatible with $\rho_{h(p)}$.

Lemma 2.1. *Identifying* $\Omega^*_{BP}=MU^*_{(p)}/(x_i|i\ne p^j-1)$, *we have the isomorphism* $\Omega^*_{BP}(X)\cong\Omega^*_{BP}\otimes_{\Omega^*_{(p)}}\Omega^*(X)_{(p)}$.

Proof. Since $\pi_{x_i}(a)=(1-x_i\Delta_{x_i})a=a\ mod(x_i)$, we see $\Phi(a)=a\ mod(x_i|i\ne p^j-1)$ for all $a\in\Omega^*(X)_{(p)}$. □

In this paper we mainly consider $\Omega^*(X)_{(p)}$ but not $\Omega^*_{BP}(X)$. However we use notations $v_i=x_{p^i-1}\in\Omega^*_{(p)}$ for ease of expressions.

3. Lie group and its maximal torus

Let G be a simply connected Lie group and $G_{\mathbb{C}}$ the corresponding reductive algebraic group over $\mathbb{C}$. Let T be the maximal torus of G. then we have the fibering

$$(3.1)\quad G \xrightarrow{\pi} G/T \xrightarrow{i} BT = ET/T.$$

Taking $BT = colim_{N\to\infty}((\mathbb{A}^N - \{0\})^{\times n}/(\mathbb{G}_m)^n)$, we get the maps of algebraic groups

$$G_{\mathbb{C}} \xrightarrow{\pi_{\mathbb{C}}} G_{\mathbb{C}}/T_{\mathbb{C}} \xrightarrow{i_{\mathbb{C}}} BT.$$

If an algebraic variety X has a cellular decomposition,i.e., $X = X_n \supset ... \supset X_0$ with $X_i - X_{i-1} = \cup\mathbb{A}^{n_{ij}}$ where $\mathbb{A}^{n_{ij}}$ is the affine space of $dim = n_{ij}$, then the Chow ring is isomorphic to the ordinary cohomology i.e.,$CH^*(X) \cong H^*(X)$ and moreover $MU^*(X) \cong MU^* \otimes H^*(X)$. Hence these facts hold for $X = G_{\mathbb{C}}$ and $BT_{\mathbb{C}}$.

From Grothendieck and Kac [K], it is known that

$$(3.2)\quad CH^*(G_{\mathbb{C}}) \cong Im(\pi^* : H^*(G/T) \to H^*(G)).$$

Lemma 3.1. *Let $\tilde{MU}^*(-)$ be the reduced $MU^*(-)$ theory. If there is the isomorphism*

$$\pi^*(MU^*(G/T)) \cong MU^*(G/T)/(Ideal(i^*\tilde{MU}^*(BT)),$$

then we have

$$MGL^{2*,*}(G_{\mathbb{C}}) \supset \Omega^*(G_{\mathbb{C}}) \cong \pi^*(MU^*(G/T)).$$

Proof. Consider the diagram

$$\begin{array}{ccccc}
\Omega^*(G_{\mathbb{C}}) & \xleftarrow{\pi_{\mathbb{C}}^*} & \Omega^*(G_{\mathbb{C}}/T_{bC}) & \xleftarrow{i_{\mathbb{C}}^*} & \Omega^*(BT) \\
{\scriptstyle(1)}\downarrow & & {\scriptstyle(2)}\downarrow & & {\scriptstyle(3)}\downarrow \\
MU^*(G) & \xleftarrow{\pi_{MU}^*} & MU^*(G/T) & \xleftarrow{i_{MU}^*} & MU^*(BT).
\end{array}$$

where (1),(2),(3) are maps induced from $t_{MU}\rho_{MGL}$. For $X = G/T, BT$, we already know that $MU^*(X) \cong MU^* \otimes H^*(X) \cong MU^* \otimes CH^*(X)$. Hence (2) and (3) are isomorphisms. While the rows are not exact, we get the map

$$\Omega^*(G_{\mathbb{C}}) \xleftarrow{\pi^*} MU^*(G/T)/(Ideal(i_{MU}^*\tilde{MU}^*(BT)))$$

where π^* is induced from $\pi_{\mathbb{C}}^*(2)^{-1}$. Tensoring $\otimes_{\Omega^*}\mathbb{Z}$, we have

$$CH^*(G) \cong \Omega^*(G) \otimes_{\Omega^*} \mathbb{Z} \quad \xleftarrow{\pi^*\otimes_{MU^*}\mathbb{Z}}$$

$$(MU^*(G/T)/(Ideal(i_{MU}^*)) \otimes_{MU^*} \mathbb{Z} \cong H^*(G/T)/(Ideal(i_H^*)).$$

By Grothendieck and Kac theorem the above map is epic. Hence π^* itself also epic. From the assumption of this lemma, we get $Im(\pi_{MU}^*) \cong MU^*(G/T)/(Ideal(i_{MU}^*))$. Hence the map π^* is an isomorphism.

Since $(1) = t_{MU}\rho_{MGL}$ is injective, the map ρ_{MGL} is a split injection. $\square$

4. $BP^*(G/T)/Ideal(i^*(\tilde{BP}^*(BT))$

In this section, we consider $\Omega^*_{BP}(G_{\mathbb{C}})$ or $\Omega^*(G_{\mathbb{C}})_{(p)}$. Moreover we study $BP^*(G)$ instead of $MU^*(G)_{(p)}$. Let G be a simply connected Lie group. By the Borel's therorem, we have the ring isomorphism for p odd

$$(4.1)\quad H^*(G;\mathbb{Z}/p)\cong P(y)/p\otimes\Lambda(x_1,...,x_l)\qquad with\ P(y)=\mathbb{Z}[y_1,...,y_k]/(y_1^{p^{r_1}},...y_k^{p^{r_k}})$$

where $|y_i|=even$ and $|x_j|=odd$. When p=2, the righthand side is isomorphic to $grH^*(G;Z/2)$ and for each y_i, there is x_j with $x_j^2=y_i$. By Grothendieck and Kac [K], we know that

$$(4.2)\qquad CH^*(G_{\mathbb{C}})/p\cong P(y)/p.$$

Consider the spectral sequence induced from the cofibering (3.1)

$$(4.3)\quad E_2^{*,*}=H^*(BT;H^*(G;\mathbb{Z}/p))\Longrightarrow H^*(G/T;\mathbb{Z}/p).$$

The cohomology of the classifying space of the torus is

$$H^*(BT)\cong\mathbb{Z}[t_1,...,t_l]\qquad with\quad |t_i|=2$$

where l is also the number of the odd degree generatos x_k in $H^*(G;\mathbb{Z}/p)$. It is known that there is a regular sequence $(b_1,...,b_l)$ in $H^*(BT)/p$ such that $d_{|x_i|+1}(x_i)=b_i$. Thus we get

$$(4.4)\quad H^*(G/T;\mathbb{Z}/p)\cong P(y)\otimes\mathbb{Z}/p[t_1,...,t_l]/(b_1,...,b_l).$$

Recall the connected Morava K-theory $k(n)^*(-)$ with the coefficient $k(n)^*=Z/p[v_n]$. Then the usual Morava K-theory is $K(n)^*(X)=[v_n^{-1}]k(n)^*(X)$.

Since $H^*(G/T)$ has no torsion, we get the BP^*-modules isomorphism

$$(4.5)\quad BP^*(G/T)\cong BP^*\otimes P(y)\otimes\mathbb{Z}[t_1,...,t_l]/(\tilde{b}_1,...,\tilde{b}_l)$$

where $\tilde{b}_i=b_i\ mod(p)$. However note that this is not a BP^*-algebras isomorphism. (The righthand side above is isomorphic to $grBP^*(G/T)$.) We also note $k(n)^*(G/T)\cong k(n)^*\otimes_{BP^*}BP^*(G/T)$ and

$$(4.6)\quad k(n)^*(G/T)\cong k(n)^*\otimes P(y)\otimes\mathbb{Z}/p[t_1,...,t_l]/(b_1,...,b_l).$$

We want to study the spectral sequence

$$(4.7)\quad E_2^{*,*}=H^*(BT;k(1)^*(G))\Longrightarrow k(1)^*(G).$$

Let $\rho: k(n)^*(X)\to H^*(X;\mathbb{Z}/p)$ be the natural (Thom) map. Let us write by X^s the s-skeleton of X. Recall also Q_i is the Milnor primitive operation inductively defined by $Q_i=P^{p^{i-1}}Q_{i-1}-Q_{i-1}P^{p^{i-1}}$ where P^k are reduced power operations and $Q_0=\beta$; Bockstein operation. Here we recall lemmas related about relations between BP^*-modules structure of $BP^*(X)$ and Q_i-acotions on $H^*(X;\mathbb{Z}/p)$.

Lemma 4.1. *([Y 1]) Let $\sum_n v_ny_n=0\in BP^*(X)$. Then there is $x\in H^*(X;\mathbb{Z}/p)$ such that $Q_n(x)=\rho(y_n)$.*

Lemma 4.2. *For $x\in H^*(X;\mathbb{Z}/p)$, there is $y'\in k(n)^*(X)$ with $Q_n(x)=\rho(y')$ and $v_ny'=0$. Conversely if $v_ny'=0$, then there is x with $Q_n(x)=\rho(y')$.*

Proof. There is the Sullivan-Bockstein exact sequence

$$k(n)^*(X)\xrightarrow{v_n}k(n)^*(X)\xrightarrow{\rho}H^*(X;\mathbb{Z}/p)\xrightarrow{\delta}k(n)^*(X)\xrightarrow{v_n}$$

Since $\rho\delta=Q_n$, if $Q_nx=y$, then $\delta(x)=y'$ is v_n-torsion. Conversely if $v_ny'=0$, then there is $x\in H^*(X;\mathbb{Z}/p)$ with $\delta(x)=y'$ and $Q_n(x)=\rho(y')$. □

Lemma 4.3. *Let $F \to E \xrightarrow{p} B$ be a fiber bundle such that $H^*(B)$ is a p-torion free simply connected space. In the Serre spectral sequence converging to $H^*(E;\mathbb{Z}/p)$, let $0 \neq y \in H^*(F;\mathbb{Z}/p)$ and $0 \neq b \in H^*(B;\mathbb{Z}/p)$ such that if $Q_n x = y$, then $d_r(x) = b$ and that there exists such $x \in H^*(F;\mathbb{Z}/p)$. Then there are $y' \in k(n)^*(F)$ and $b' \in k(n)^*(B)$ such that $\rho(y') = y, \rho(b') = b$ and*

$$v_n y' = \lambda b', \quad \lambda \neq 0 \in \mathbb{Z}/p \quad in\ k(n)^*(p^{-1}B^{|b|})$$

Proof. Let $B' = B^{|b|-1}$ and $E' = p^{-1}(B')$. Consider the Serre spectral sequence

$$E_2^{*,*} = H^*(B'; H^*(F;\mathbb{Z}/p)) \Longrightarrow H^*(E';\mathbb{Z}/p).$$

Since $d_r x = b = 0 \in H^*(B')$, there is $x \in H^*(E';Z/p)$ with $Q_n x = y$. This means there exists $v_n y' = 0$ *in* $k(n)^*(E')$.

On the otherhand, let $B'' = B^{|b|-1} \cup e_b$ and $E'' = p^{-1}B''$ where e_b is the normal cell representing b. Then $d_r x = b \neq 0$ and there is no element $x \in H^*(E'';\mathbb{Z}/p)$ with $Q_n x = \rho(y)$. From Lemma 4.2, we get $v_n y' \neq 0 \in k(n)^*(E'')$ and have this lemma. □

Remark (4.8). Lemma 4.1 and Lemma 4.2 are also hold for $n = 0$ replacing $k(0)^*(-) = H^*(-):$ the integral cohomology and $v_0 = p$, $\lambda \neq 0\ mod(p)$.

The Q_i actions on $H^*(G;\mathbb{Z}/p)$ are quite restrictive. We consider the following assumption.

Assumption (4.9)

(A.1) if $i \neq j$, then $|x_i| \neq |x_j|$.

(A.2) $Q_k x_i \in P(y)$ for all k, i and if $Q_k x_i \neq 0$, then $Q_0 x_i = y_i$ or, $Q_0 x_i = 0$ and $Q_1 x_i = y_{i''}$ for some i''.

The above assumptions are satisfied for all simply connected simple Lie groups except for $p = 2$, $Spin(n)$, E_8. However quite similar conditions also hold for these cases. (We will expain these in the next section.)

Lemma 4.4. *Assume (A,1),(A.2). Then $BP^*(G/T)/Ideal(i^*(\tilde{BP}^*(BT)), I^2))$ is a quotient ring of $BP^* \otimes P(y)/(I^2,(1),(2))$ where*

$$(1) \quad py_i = 0 \ \ mod(v_1,...) \quad if\ Q_0 x_i = y_i$$

$$(2) \quad v_1 y_j = 0 \ \ mod(v_2,...) \quad if \quad Q_1 x_{j'} = y_j \ \ but \ \ Q_0 x_{j'} = 0.$$

Proof. First note that from (4.5) , the BP^*-algebra is a quotient of $BP^* \otimes P(y)$. Recall in the spectral sequence (4.3), $d_r(x_i) = b_i$ and $Q_0(x_i) = y_i$. Letting $F = G, G/T = E, B = BT$, from Remark (4.8), we know

$$py_i = i^*(b_i) \qquad in \ \ H^*(G/T) = BP^*(G/T)/(v_1, v_2, ...).$$

From (4.5), we have $py_i + \sum v_n a_n \in Ideal(i^*(\tilde{BP}^*(BT))$ for $a_n \in BP^* \otimes P(y)$. Hence the relation (1) containd in $Ideal(i^*(\tilde{BP}^*(BT))$.

Let $Q_1 x_{i'} = y_i$. Recall $b_{i'} = d_{|x_{i'}|+1}(x_{i'})$ in the spectral sequence (4.3). From Lemma 4.3, we already know in $k(1)^*(G/T)$

$$(*) \qquad v_1 y_i = \lambda b_{i'} + b' \quad \lambda \neq 0 \in \mathbb{Z}/p,\ b' \in E_\infty^{|b_{i'}|+1,*}.$$

Since $b_{i'} = 0$ in $H^*(G/T;\mathbb{Z}/p)$, we know $b' = 0\ mod(v_1)$ in $k(1)^*(G/T)$. Here we recall all elements are represented as in (4.7). Hence we can write

$$b' = v_1 d(y) + v_1 g \qquad with \quad d(y) \in P(y),\ g \in I(t) = Ideal(t_1, ..., t_l) \subset k(1)^*(G/T).$$

Let $E' = p^{-1}G/T^{|b_{i'}|}$. Since $b' \in E_\infty^{|b_{i'}|+1,*}$,we see $b' = 0 \in k(1)^*(E')$. Hence there is $x \in H^*(E';\mathbb{Z}/p)$ such that $Q_1(x) = b'/v_1 = d(y) + g$. But if $b' \neq 0 \in k(1)^*(G/T)$, then x does not exist in $H^*(G/T;\mathbb{Z}/p)$, namely, the corresponding element x in $E_2^{*,*}$ of (4.7) is not a permanent cycle. In particular if $d(y)\neq 0$, then $x\neq 0\in H^*(G;\mathbb{Z}/p)$ but $|x|=|x_{i'}|$ and there is no differential $d_r(x)\neq 0$ for $r>|x_{i'}|+1=|b_{i'}|$; this is a contradiction. Hence we get $b'\in I(t)$ and so $v_1y_i\in I(t)$.

Let $Q_0x_{i'}=0$ and $Q_1x_{i'}=y_i$. From (*), we get in $BP^*(G/T)$,

$$v_1y_i=\lambda\tilde{b}_{i'}+\tilde{b}'+pf(y)+pb'' \quad mod(p^2,v_2,v_3,...),$$

where $\tilde{b}_{i'}=b_{i'}\ mod(p)$, $\tilde{b}'=b'\ mod(p)$, $f(y)\in P(y)$ and $b''\in Ideal(t_1,...,t_l)\subset BP^*(G/T)$. But $f(y)=0\ mod(p^2)$ otherwise $Q_0x_{i'}=-f(y)\neq 0$ from Remark 4.8. . Thus we get (2) is contained in $Ideal(i^*(\tilde{BP}^*(BT))$. □

Lemma 4.5. *Assume (A,2). The image $\pi^*BP^*(G/T)/I^2$ is isomorphic to*

$$BP^*\otimes P(y)/(I^2,(1),(2)\ in\ Lemma\ 4.3).$$

Proof. Suppose that there is a relation in $\pi^*BP^*(G/T)/I^2$

$$(*)\qquad v_ny=0\quad mod(I^2,v_{n+1},...)\quad with\quad y\in P(y).$$

From Lemma 4.1, there is $x\in H^*(G;\mathbb{Z}/p)$ such that $Q_nx=y$ and $Q_j(x)=0\ mod(I^2)$ for $j<n$. Let $H^*(G;\mathbb{Z}/p)=\oplus_sX_s$ where X_s is the free $P(y)$-module generated by monomials $x_{i_1}...x_{i_s}$. Since all $Q_i(x_j)\in P(y)$ from (A.2), we know that $Q_i:X_s\to X_{s-1}$, In particular, we can write $x=\sum_j u_jx_j$, $u_j\in P(y)$. Then we have the relation $(mod(I^2),v_{n+1},...))$

$$\sum_j u_j\sum_k(v_kQ_k(x_j))=\sum_k v_k\sum_j u_jQ_k(x_j)$$

$$=\sum_k v_kQ_k(\sum_j u_jx_j)=\sum_k v_kQ_k(x)=v_ny.$$

Thus we can express the relation type (*) by taking (1),(2) in Lemma 4.3. By the induction on n, we can prove the lemma. □

From Lemma 3.1. the following corollary is immediate.

Corollary 4.6. *If (A.1),(A,2) are satisfied for simply connected Lie group G_i, $1\leq i\leq n$, then the product $G=G_1\times...\times G_n$ satisfies Theorem 1.1 in the introduction.*

Proof. Let T_i be the maximal torus of G_i. Since G_i/T_i and BT_i are torsion free, we see

$$BP^*(B/T)/Ideal(i^*)\cong\otimes_{BP^*}BP^*(G_i/T_i)/(Ideal(\sum i_i^*)).$$

Hence the result of Lemma 4.4 holds for this G.

On the otherhand (A.2) is satisfied also in $H^*(G;Z/p)\cong\otimes H^*(G_i;\mathbb{Z}/p)$. Hence Lemma 4.5 also holds. Thus we get Theorem 1.1 from the $mod(I^2)$ version of Lemma 3.1. □

5. SIMPLE GROUPS

We study simple groups. The simple Lie groups which have p-torsion in $H^*(G)$ are divided to the following cases.

Case I. The (G,p) are the exceptional Lie groups $(G_2,2)$, $(F_4,2)$,$(E_6,2)$, $(F_4,3)$,$(E_6,3)$,$(E_7,3)$ and $(E_8,5)$.

Case II. $(E_8,3)$.

Case III. The cases $(E_7,2),(E_8,2)$.

Cases IV. The classical cases $(Spin(n),2)$.

Case I. We at first study $k(1)^*(G)$ for the groups (G,p) in Case I. The ordinary mod p-cohomology is written

$$H^*(G;\mathbb{Z}/p) \cong \mathbb{Z}/p[y]/(y^p) \otimes \Lambda(x_1,x_2,...,x_l)$$

where $|x_1|=3, |x_2|=2p+1, |y|=2p+2, Q_1x_1=Q_0x_2=y$. In this case $k(1)^*(G)$ is known. Consider the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*}=H^*(G;k(1)^*) \Longrightarrow k(1)^*(G).$$

Th only nonzero differential is $d_{2p-1}(x)=v_1\otimes Q_1(x)$. Thus we can prove

$$k(1)^*(G) \cong (k(1)^*[y]/(y^p,v_1y) \oplus k(1)^*\{xy^{p-1}\}) \otimes \Lambda(x_2,...,x_l)$$

Moreover, $BP^*(G)$ is computed in [Y2],

$$BP^*(G) \cong (BP^*\otimes P(y)/(py,v_1y)\oplus BP^*\{w_1,w_2\}/(pw_1+v_1w_2)\oplus BP^*\{x_1',x''\})\otimes\Lambda(x_3,...x_l).$$

Here $w_1=x_1y^{p-1}$, $w_2=x_2y^{p-1}$, $x_1'=px_1$ and $x''=x_1x_2y^{p-1}$ in the Atiyah-Hirzebruch spectral sequence converging to $BP^*(G)$.

Of course this case satisfies (A,1),(A,2) in §4. Moreover it is easy proved that Corollary 4.6 holds without *modulo*(I^2) for all products of Lie groups of Case I.

Theorem 5.1. *For the group (G,p) in Case I, we have the isomorphism*

$$\Omega^*(G)_{(p)} \cong \Omega^*[y]/(py,v_1y,y^p).$$

Case II. Let $(G,p)=(E_8,3)$. The mod 3 cohomology is

$$H^*(E_8;\mathbb{Z}/3) \cong \mathbb{Z}/3[y_8,y_{20}]/(y_8^3,y_{20}^3)\otimes\Lambda(x_3,x_7,x_{15},x_{19},...)$$

where the suffix means its degree,i.e., $|y_8|=8$. The Minor operations are

$$Q_1x_3=Q_0x_7=y_8, \quad Q_2x_3=Q_1x_{15}=Q_0x_{19}=y_{20}$$

and $Q_ix_j=0$ for other i,j. From this we have relations,e.g.,

$$v_1y_8+v_2y_{20}=0, \qquad v_1y_{20}=0.$$

We easily see the conditions (A,1),(A,2). The $BP^*(E_8)$ is computed ([M],[Y 3]). We also get the result without *mod*(I^2), however for $BP^*(E_8\times E_8)$ I can not show the result without *mod*(I^2).

Theorem 5.2. *Then Theorem 1.1 holds for $(E_8,3)$ without mod(I^2) ,i.e.,*

$$\Omega(E_8)_{(3)} \cong \Omega^*[y_8,y_{20}]/(3y_8,3y_{20},y_3^3,y_{20}^3,v_1y_8+v_2y_{20},v_1y_{20}).$$

Case III. $(E_7,2)$, $(E_8,2)$.

The mod 2-cohomology of E_8 is

$$H^*(E_8;\mathbb{Z}/2) \cong \mathbb{Z}/2[x_3,x_5,x_9,x_{15}]/(x_3^{16},x_5^8,x_9^4,x_{15}^2)\otimes\Lambda(x_{17},x_{23},x_{27},x_{29}),$$

$$\textit{and} \qquad H^*(E_7;\mathbb{Z}/2) \cong H^*(E_8;\mathbb{Z}/2)/(x_3^4,x_5^4,x_{15}^2,x_{29}).$$

Let us write $y_{2i}=x_i^2$ if it is not zero. The Q_i-actions are completely determined by Hunton-Mimura-Nishimoto-Schuster. We have for example, [H-M-N-S], [M-N]

$$Q_1x_3=Q_0x_5=y_6, \qquad Q_0x_9=y_{10} \qquad Q_1x_{15}=Q_0x_{17}=y_{18},$$

$$Q_2x_{23}=Q_1x_{27}=Q_0x_{29}=y_{30}$$

$$\textit{with} \qquad 0=Q_0x_3=Q_0x_{15}=Q_0x_{27}, \quad 0=Q_0x_{23}=Q_1x_{23}.$$

(Of course there many other nonzero operations , e.g., $Q_2x_3=y_{10}$, $Q_3x_3=y_{18}$.)

Assumption (A.2) holds except for $Q_0x_{23}=Q_1x_{23}=0$ but $Q_2x_{23}=y_{30}$.

If there is a relation in $BP^*(X)/I^2$

$$v_na_n + v_{n+1}a_{n+1} + ... = 0 \qquad a_n \neq 0,$$

then we say it *the relation starting with* v_na_n. For example, the relations in $BP^*(E_8)/I^2$ starting with $2y_6$, v_1y_6 are

$$2y_6 + v_2y_6^2 + v_3y_{10}^2 = 0, \quad v_1y_6 + v_2y_{10} + v_3y_{18} = 0.$$

Theorem 5.3. *The cases $p = 2$ and $G = E_8$, E_7 satify Theorem 1.1 ,e.g.,*

$$(1) \qquad \Omega^*(E_8)/I^2 \cong \Omega^*[y_6, y_{10}, y_{18}, y_{30}]/(y_6^8, y_{10}^4, y_{18}^2, y_{30}^2, R, I^2)$$

where R is generated by relations starting with

$$2y_6,\ v_1y_6,\ 2y_{10},\ 2y_{18},\ v_1y_{18},\ 2y_{30},\ v_1y_{30},\ v_2y_{30},$$

$$(2) \qquad \Omega^*(E_7)/I^2 \cong \Omega^*(E_8)/(I^2, y_6^2, y_{10}^2, y_{30}).$$

Proof. We will prove only for $G = E_8$ but E_7 is much easier. Note that the product $\mu : G \times G \to G$ induces the coproduct $\mu^* : BP^*(G) \to BP^*(G \times G)$, which is not isomorphic to $BP^*(G) \otimes_{BP^*} BP^*(G)$. Since y_i is primitive in $H^*(G; \mathbb{Z}/2)$, we can write in $BP^*(G \times G)$

$$\mu^*(y_i) = y_i \otimes 1 + 1 \otimes y_i + a \qquad \textit{with } a \in I = (2, v_1, ..).$$

Hence $y_i^{2^k}$, $k \geq 1$ is always primitive $mod(I^2)$. Conversely $mod(I^2)$ primitive elements are of the form $y_i^{2k'}$, $k' \geq 0$ and x_i. By the dimensionl reason, we see $y_6^8 = 0$, $y_{10}^4 = 0$,...

The part which does not satify (A,2) is come from

$$Q_0x_{23} = Q_1x_{23} = 0, \quad Q_2x_{23} = y_{30}.$$

Here we do $k(2)$ version of the proof of Lemma 4.4. Letting $y_i = y_{30}$ and $x_{i'} = x_{23}$, we can take in $k(2)^*(G/T)$, $v_2y_i = b_{i'} + b'$ same as (*) in the proof of Lemma 4.4. All arguments work similarly and we get the theorem. □

Case IV ; classical group cases.

We first consider the orthogonal groups $G = SO(m)$, while these are not simply connected. The mod 2-cohomology is written as (see for example [N])

$$grH^*(SO(m); \mathbb{Z}/2) \cong \Lambda(x_1, x_2, ..., x_{m-1})$$

where the multiplications are given by $x_s^2 = x_{2s}$. We write $y_{2(odd)} = x_{odd}^2$. Hence we can write

$$H^*(SO(m); \mathbb{Z}/2) \cong \mathbb{Z}/2[y_{4i+2} | 2 \leq 4i + 2 \leq m - 1]/(y_{2i+1}^{s(i)}) \otimes \Lambda(x_1, x_3, ...)$$

where $s(i)$ is the smallest number such that $2^{s(i)}(4i + 2) \geq m$.

The Q_i-operations are given by Nishimoto [N]

$$Q_nx_{odd} = x_{odd+|Q_n|}. \qquad Q_nx_{even} = Q_ny_{even} = 0.$$

We note the operations

$$Q_0x_1 = y_2, \qquad Q_1x_{2(odd)-3} = Q_0x_{2(odd)-1} = y_{2(odd)} \ \ if\ (odd) > 1.$$

Take new generators $x'_{4m-1} = x_{4m-1} + x_{2m-1}y_{2m}$, $y'_{4m+2} = y_{4m+2} + y_{2m+2}y_{2m}$, $x'_{4m+1} = x_{4m+1} + x_{2m+1}y_{2m}$ so that

$$Q_0(x'_{4m-1}) = 0, \quad Q_1(x'_{4m-1}) = Q_0(x'_{4m+1}) = y'_{4m+2}.$$

Thus we see that the assumption (A.1),(A,2) are satisfied and Theorem 1.1 holds for $G = SO(m)$. Relations are given by

$$\sum_n v_n Q_n(x_{odd}) = \sum_n v_n x_{odd+|Q_n|} = 0 \qquad mod(I^2).$$

For example, the relations in $BP^*(SO(m))/I^2$ starting with $2y_6$ and $v_1(y_6+y_2^3)$

$$2y_6 + v_1y_2^3 + v_2y_6^2 + v_3y_{10}^2 + ... = 0,$$
$$v_1(y_6+y_2^3) + v_2(y_{10}+y_2^5) + v_3(y_{18}+y_2^9) + v_4(y_{34}+y_2^{17}) + ... = 0$$

Next consider the case $G = Spin(m)$. Let $2^{t-1} < m \leq 2^t$. Then the mod 2-cohomology is

$$H^*(BSpin(m);\mathbb{Z}/2) \cong H^*(BSO(m);\mathbb{Z}/2)/(y_2) \otimes \Lambda(z)$$

with $|z| = 2^t - 1$. Here this element z is defined as following. Consider the spectral sequence

$$E_2^{*,*} \cong H^*(B\mathbb{Z}/2; H^*(Spin(m);\mathbb{Z}/2)) \Longrightarrow H^*(BSO(m);\mathbb{Z}/2)$$

induced from the cofibering $Spin(m) \to SO(m) \to B\mathbb{Z}/2$. Let $x \in H^1(B\mathbb{Z}/2;\mathbb{Z}/2)$ be the genrator. Then z is defined as an element with $d_{2^t}(z) = x^{2^t}$ since this element is zero in $H^*(SO(m);\mathbb{Z}/2)$.

The Q_i-actions of z are also given by Nishimoto [N]

$$Q_0 z = \sum_{i+j=2^{t-1}, i<j} x_{2i}x_{2j}$$

$$Q_n z = \sum_{i=2^n+1}^{2^{t-2}+2^{n-1}-1} x_{2i}x_{2^t+2^{n+1}-2-2i} \qquad for \quad n \geq 1$$

It is immediate $Q_n z \in P(y)$ and decomposable as a product of generators x_{even}.

Theorem 5.4. *The groups $G = SO(m),\ Spin(m)$ satisfy Theorem 1.1. There are $\Omega^*_{(2)}$-algebra isomorphisms*

$$(1) \quad \Omega^*(SO(m))/I^2 \cong \Omega^*[y_{4i+2} | 2 \leq 4i+2 \leq m-1]/(R, y_{4i+2}^{2^{s(i)}}, I^2)$$

$$where \quad R = \{relations\ starting\ with \quad 2y_{4i+2}, \quad v_1(y_{2i}y_{2i+2} + y_{4i+2}) \quad for\ y_{4j} = y_{2j}^2\}.$$

$$(2) \qquad \Omega^*(Spin(m)/I^2 \cong \Omega^*(Spin(m))/(y_2, \sum v_nQ_n(z),\ I^2)$$

Proof. Let $G = Spin(m)$. To see the results of Lemma 4.4, Lemma 4.5, we need to see that the relation

$$\sum v_n Q_n(z) = 0 \qquad in\ BP^*(G)/I^2$$

comes from an element in $Ideal\tilde{B}P^*(G/T)$.

Let $b_z \in H^*(BT;\mathbb{Z}/2)$ be the element with $d_{|z|+1}(z) = b_z$ in the spectral sequence converging $H^*(G/T;\mathbb{Z}/2)$. There are many elements $x \in H^*(G;\mathbb{Z}/2)$ such that $Q_n x = Q_n z$. Let us write $x = \sum f_{odd}(y)x_{odd}$ with $f_{odd}(y) \in P(y)/2$. Recall that $Q_n z$ is the decomposable element for generators x_{even} from the result of Nishimoto. Hence we need only consider the cases $|f_{odd}(y)| > 0$, namely, $|x_{odd}| < |z|$. Since $d_{|x_{odd}|+1}(x_{odd}) \neq 0$, this element disappears in $H^*(p^{-1}(G/T^{|b_z|-1});\mathbb{Z}/2)$ from $|x_{odd}|+1 < |z|+1 = |b_z|$.

For each n, we consider the spectral sequence

$$H^*(BT; k(n)^*(G)) \Longrightarrow k(n)^*(G/T).$$

By the arguments similar to the proof of Lemma 4.3, we get

$$(*) \qquad v_nQ_n(z) = b_z + b_n' \qquad b_n' \in E_\infty^{|b_z|+1,*}.$$

Moreover the arguments similar to the proof of Lemma 4.5, we can see that $b_n' = v_n b_n''$ and $b_n'' \in I(t)$.

Let us write

$$\sum v_n Q_n(z) - b_z = \sum v_n a_n \qquad in \;\; BP^*(G/T)/I^2.$$

Considering above equation in $k(n)^*(G/T)$, we have $a_n = b_n''\ mod(2, v_1, ..., \hat{v}_n, ...)$. Hence we have $\sum v_n Q_n(z) + b_z \in I(t)\ mod(I^2)$ i.e., the sum $\sum v_n Q_n(z)$ itself in $I(t)$ modulo I^2. □

From Theorem 5.1- 5.4 and Corollary 4.6, we get Theorem 1.1, and moreover we explicitely know $\Omega^*(G)/I^2$ for all simply connected Lie groups (and orthogonal groups).

6. Example for the spectral sequences

In this section, we give examples for spectral sequences for groups G in the most easy case Case I. We consider the spectral sequence

$$(6.1) \quad E(h)_2^{*,*} = H^*(BT; h^*(G/T)) \Longrightarrow h^*(G/T)$$

for homology theories, $h = H\mathbb{Z}/p, H\mathbb{Z}, H\mathbb{Q}, k(1), K(1), BP\langle 1\rangle$. Since G/T has no torsion, in any h we know that

$$(6.2) \quad h^*(G/T) \cong h^* \otimes P(y) \otimes P'(t)/(b_1, ..., b_l) \qquad where \;\; P'(t) = \mathbb{Z}[t_1, ..., t_l].$$

First recall the case $h = HZ/p$, the cohomology is $h^*(G; \mathbb{Z}/p) \cong P(y)/p \otimes \Lambda(x_1, ...x_l)$. The diffirentials are $d_{|x_i|+1}(x_i) = b_i$ and we still know

$$(6.3) \quad E(H\mathbb{Z}/p)_\infty^{*,*} \cong P(y)/p \otimes P(t)'/(b_1, ..., b_l).$$

Hereafter we restrict ourself for Case I. This case $P(y) \cong \mathbb{Z}[y]/(y^p)$ and $Q_1 x_1 = Q_0 x_2 = y$.

Next we consider the spectral sequences for Morava K-theories. First recall again

$$k(1)^*(G) \cong (P(y)^+/p \oplus k(1)^*\Lambda(x_1 y^{p-1})) \otimes \Lambda(x_2, ..., x_l).$$

We also note that $BP^*(G; \mathbb{Z}/p) \cong BP^* \otimes_{k(1)^*} k(1)^*(G)$ identifying $BP^*/p \supset \mathbb{Z}/p[v_1] \cong k(1)^*$.

Proposition 6.1. *Let G be a simple Lie group in Case I. Then we get isomorphisms*

$$E(k(1))_\infty^{*,*} \cong k(1)^* \otimes P(y)/(v_1 y) \otimes P(t)'/(b_2, ..., b_l, b_1^s y^{p-s} | 1 \le s \le p).$$

$$E(K(1))_\infty^{*,*} \cong K(1)^* \otimes P(t)'/(b_1^p, b_2, ..., b_l).$$

Proof. First note some facts of the differentials in $K(1)^*$-theory. Since $K(1)^*(-)$ holds Kunneth formula, we can defined the coalgebra map μ^* on also $E(K(1))_r^{*,*}$ considering the fibering

$$G \times G \to G \times G/T \to G \times BT.$$

Each element $x_1 y^{p-1}, x_i$ are primitive in $K(1)^*(G)$, but

$$\mu^* d_r(x_i) = d_r(\mu^* x_i) = d_r(x_i \otimes 1 + 1 \otimes x_i) = 1 \otimes d_r(x_i).$$

The differntials images are written as $d_r(x_i) = \sum f(x)g(t)$ for $g(t) \in K(1)^*P(t)'$ and $f(x) \in \Lambda(x_1 y^{p-1}, x_2, ..., x_l)$. However if $|f(x)| > 0$, then $\mu^* d_r(x_i) \ne 1 \otimes d_r(x_i)$. Thus we get

$$(*,*) \quad d_r(x_i),\ d_r(x_1 y^{p-1}) \in K(1)^* P(t)'.$$

Now we study the $k(1)^*$-theory. Recall $E(k(1))_2^{*,*} \cong k(1)^*(G) \otimes P(t)'$. Suppose $d_2(x) \ne 0$.

(1) The case $d_2(x)$ is v_1-torsion.

This case $d_2(x)$ is a $k(1)^*$-module generator, since $E(k(1))_2^{*,*}$ does not has higher v_1-torsion. Hence $d_2(x) \ne 0$ also in $E(H\mathbb{Z}/p)_2^{*,*}$ because the Thom map $E(k(1))_2^{*,*}/(v_1) \to E(H\mathbb{Z}/p)_2^{*,*}$ is injective. This is a contradiction to $d_2(x) = 0 \in E(H\mathbb{Z}/p)_2^{*,*}$.

(2) The case $d_2(x)$ is in v_1-image.
Since also $E(k(1))_2^{*,*}$ has no higher v_1-torsion, we see $d_2(x)\neq 0$ also in $E(K(1))_2^{*,*}$. Let us write $d_2(x)=\sum f(x)g(t)$. , Then by dimensional reason , it is neccesary $|f(x)|>0$. This is a contradiction to (*,*). Thus we get $d_2(x)=0$.

Next consider $d_4(x)$. By the same reasons as (1),(2), we see $d_4(x_i)=0$ for $i\neq 1$. By the reason as (1), we see $d_4(x_1y^{p-1})=b_1y^{p-1}$. Since b_1y^{p-1} is v_1-torsion, the elemnt $v_1x_1y^{p-1}$ is a $k(1)^*$-module generator in $E(k(1))_5^{*,*}$. The differentials are defined as some boundary maps, we get

$$\partial(x_1y^{p-1})=b_1y^{p-1}\quad mod(some\ filtration),$$

which implies that

$$\partial(v_1x_1y^{p-1})=b_1(v_1y)y^{p-2}=b_1^2y^{p-2}\quad mod(some\ filtration)$$

where we take y such that $v_1y=b_1\in k(1)^*(G/T)$. This follows the fact

$$d_{2|b_1|}(v_1x_1y^{p-1})=b_2^2y^{p-2}\qquad mod(some\ filtrarion).$$

By induction on r for $d_r(x_i)$, we can prove $d_{|b_i|}(x_i)=b_i$ for $i\neq 1$, and $d_{s|b_1|}(v_1^{s-1}x_1y^{p-1})=b_1^sy^{p-s}$ for $1\leq s\leq p$ by the arguments similar to (1),(2). For $K(1)^*$-theory we can prove that $d_{p|b_1|}(x_2y^{p-1})=v_1^{-(p-1)}b_1^p$ and the result of the proposition. □

We consider the integral cohomology $h=H\mathbb{Z}$. Since these groups G has no higher p-torsion, we have

$$H^*(G;\mathbb{Z})\cong(P(y)^+/(py)\oplus\Lambda(x_2y^{p-1}))\otimes\Lambda(x_1,x_3,...,x_l),$$

$$H^*(G;\mathbb{Q})\cong\mathbb{Q}\otimes\Lambda(x_2y^{p-1},x_1,x_3,...,x_l).$$

By the arguments similar to the case Morava K-theories, but more easily, we get the following differentials.

The differentials in $E(\mathbb{Q})_r^{*,*}$ are $d_{b_i|}(x_i)=b_i$ for $i\neq 2$, and $d_{p|b_2|}(x_2y^{p-1})=b_2^p$.

The differentials in $E(H\mathbb{Z})_r^{*,*}$ are $d_{s|b_2|}(p^{s-1}x_2y^{p-1})=b_2^sy^{p-s}$ and $d_{|x_i|+1}(x_i)=b_i$ for $i\neq 2$.

Proposition 6.2. *Let G be a simple Lie group in Case I. Then we get isomorphism*

$$E(H\mathbb{Z})_\infty^{*,*}\cong P(y)/(py)\otimes P(t)'/(b_1,b_3,...,b_l,b_2^sy^{p-s}|1\leq s\leq p),$$

$$E(H\mathbb{Q})_\infty^{*,*}\cong P(t)'/(b_1,b_2^p,b_3,...,b_l)$$

Now we recall the $BP\langle 1\rangle^*$ theory with the coefficient $BP\langle 1\rangle^*=\mathbb{Z}_{(p)}[v_1]$ so that $BP\langle 1\rangle^*/p-k(1)^*$. For G in Case I, it is known that $BP\langle 1\rangle^*(G)\cong BP\langle 1\rangle^*\otimes_{BP^*}BP^*(X)$. (See Case I in §5). The rational BP-theory is immediate

$$BP^*(G;\mathbb{Q})\cong BP^*\otimes Q\otimes\Lambda(x_1',w_2)\otimes\Lambda(x_3,...,x_l).$$

The differentials in $E(BP\otimes\mathbb{Q})_r^{*,*}$ are given by $d_4x_1'=pb_1$, $dw_2=b_2^p$. Here we use notations that x_1',w_1,w_2 correspond elemnts $px_1,x_1y^{p-1},x_2y^{p-1}$ respectively, in the spectral sequence. (Recall also Case I in §5.)

Theorem 6.3. *Let G be a simple Lie group in Case I and $E(BP\langle 1\rangle)_r^{*,*}$ be the spectral sequence (6.1) for $h=BP\langle 1\rangle$. Then we have the isomorphism*

$$E(BP\langle 1\rangle)_\infty^{*,*}\cong BP\langle 1\rangle^*\otimes P(y)/(py,v_1y)\otimes P(t)'/(pb_1,b_3,...,b_l,b_1^sb_2^ry^{p-s-r}|1\leq s+r\leq p)$$

Proof. Compare this spectral sequence to those of the other theories $k(1), H\mathbb{Z}, BP\langle 1\rangle\otimes\mathbb{Q}$. By induction on r of $d_r(x_i)$, we show in $E(BP\langle 1\rangle)_r^{*,*}$, $d_{|b_i|}(x_i)=b_i$ for $i\neq 1,2$ and $d_4(x_1')=pb_1$, and moreover

$$d_{s|b_1|+r|b_2|}(p^rv_1^{s-1}w_1)=b_1^sb_2^ry^{p-s-r}\ for\ 1\le s+r\le p$$
$$d_{r|b_2|}(p^{r-1}w_2)=b_2^ry^{p-r}\ for\ 1\le r\le p.$$

Thus we can prove this theorem. □

The cohomology $BP^*(G/T)$ is still given in (6.2), while it does not give the imformation of $i^*BP^*(BT)$. However the above theorem says more strong facts. For example

$$p^sv_1^ry^{p-s-r-1}\notin Im(i^*(BP^*(BT))\quad mod(v_2,...),\qquad if\ s+r<p-1$$

while of course it is in $Ideal(i^*(\tilde{BP}^*(BT)))$. Hence we can know that these facts also hold for $\Omega^*(-)$ and $MGL^{2*,*}(-)$ although they have not the Serre spectral sequences.

7. motivic theories

In this section, we study motivic generalized theories defined by Voevodsky [Vo1],[Vo2]. When X has a cellular decomposition, we can show

$$h^{*,*}(X)\cong h^{*,*}(pt)\otimes H^{*,*}(X),$$

for $h^{*,*}(-)$ the generalalized motivic cohomology, e.g., $h=H\mathbb{Z}/p, MGL$. The spaces $X=G/T, BT$ have this property.

We first study the mod p motivic cohomology theory. It is known that there is an element $\tau\in H^{0,1}(X;\mathbb{Z}/p)$ such that $t_{\mathbb{Z}/p}(\tau)=1$ where $t_{\mathbb{Z}/p}: H^{*,*'}\to H^*(X;\mathbb{Z}/p)$ is the realization map. Recall also $H^*(G;\mathbb{Z}/p)\cong P(y)/p\otimes\Lambda(x_1,...,x_l)$. (when $p=2$, we let $y_i=x_i^2$.)

Theorem 7.1. *Let G be a simply connected Lie group. Giving bidegree to $H^*(G;\mathbb{Z}/p)$ by $deg(y_i)=(|y_i|,|y_i|/2)$ and $deg(x_i)=(|x_i|,(|x_i|+1)/2)$, we have the injection*

$$P(y)/p\otimes\Lambda(x_1,...,x_l)\otimes\mathbb{Z}/p[\tau]\subset H^{*,*}(G_{\mathbb{C}};\mathbb{Z}/p)$$

such that for $p=odd$, it is a ring monomorphism, and for $p=2$, it is a ring monomorphism to $grH^{*,*}(G_{\mathbb{C}};\mathbb{Z}/2)$ and $x_i^2-y_i\tau\in Ker(t_{\mathbb{Z}/p})$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} H^{2*,*}(G_{\mathbb{C}}/T_{\mathbb{C}};\mathbb{Z}/p) & \longleftarrow & H^{2*,*}(G_{\mathbb{C}}/T_{\mathbb{C}},G_{\mathbb{C}};\mathbb{Z}/p) & \xleftarrow{\delta} & H^{2*,*}(G_{\mathbb{C}};\mathbb{Z}/p)\\ \downarrow t_{\mathbb{Z}/p} & & \downarrow t_{\mathbb{Z}/p} & & \downarrow t_{\mathbb{Z}/p}\\ H^*(G/T;\mathbb{Z}/p) & \longleftarrow & H^*(G/T,G;\mathbb{Z}/p) & \xleftarrow{\delta} & H^*(G;\mathbb{Z}/p)\end{array}$$

where rows are exact. The cohomology $H^*(G/T,G;\mathbb{Z}/p)$ is computed by using the spectral sequence

$$E_2^{*,*}(G/T,G)=(E_2^{*,*}-E_2^{0,*})\Longrightarrow H^*(G/T,G;\mathbb{Z}/p)$$

where $E_2^{*,*}$ is the spectral sequence converging to $H^*(G/T;\mathbb{Z}/p)$.

Since the differential in $E_r^{*,*}$ are given $d_{|b_i|}(x_i)=b_i$, we easily seen from the definition of differential, $\delta(x_j)=b_j\ mod(E_\infty^{|b_i|+1,*})$ in $H^*(G/T;\mathbb{Z}/p)$. Here note that $t_1,...,t_l$ are in $H^*(G/T,G;\mathbb{Z}/p)$, while y_i is not in $H^*(G/T,G;\mathbb{Z}/p)$because $t_i=0\in H^*(G;\mathbb{Z}/p)$. Moreover $\delta(x_i)=b_i$ implies $b_i\neq 0\in H^*(G/T.G;\mathbb{Z}/p)$.

Similarly $t_1,...,t_l$ are in $H^{2*,*}(G/T,G_{\mathbb{C}};\mathbb{Z}/p)$, since the corresponding elements in $H^{2*,*}(G/T;\mathbb{Z}/p)$ goes to zero in $H^{2*,*}(G_{\mathbb{C}};\mathbb{Z}/p)$. Moreover $b_i\neq 0\in H^{2*,*}(G/T,G_{\mathbb{C}};\mathbb{Z}/p)$ since it is nonzero in $H^*(G/T,G;\mathbb{Z}/p)$. Hence there is the element $x_i\in H^{2*-1,*}(G_{\mathbb{C}};\mathbb{Z}/p)$ such that $\delta(x_i)=b_i$.

Since $t_{\mathbb{Z}/p}(\tau)=1$, we also see $t_{\mathbb{Z}/p}(y_i\tau)=t_{\mathbb{Z}/p}(x_i^2)$ for $p=2$. □

The injection in the above theorem seems an isomorphism but I can not prove it now.

Other types of motivic cohomology seem quite complicated and we consider only group G in Case I. By using arguments similar to the mod p case, we have ;

Proposition 7.2. *Let G be a simple Lie group in Case I. Giving the bidegree by $deg(y)=(|y|,|y|/2), deg(x)=(|x|,(|x|+1)/2)$ for $x=x_i, x_2y^{p-1}$, we have the ring monomorphism*

$$H^*(G)\cong(\mathbb{Z}[y]/(py,y^p)\oplus\mathbb{Z}\{x_2y^{p-1}\})\otimes\Lambda(x_1,x_3,...,x_l)\subset H^{*,*}(G_{\mathbb{C}}).$$

Recall the motivic cobortdism $MGL^{*,*}(G_{\mathbb{C}})$ defined by Voevodsky. Let $MGL/p^{*,*}(-)$ the motivic cohohomology theory defined by the cofiber sequence of stable $\mathbb{A}^1$-homotopy category

$$MGL\xrightarrow{p}MGL\longrightarrow MGL/p$$

so that we have the long exact sequence

$$\longrightarrow MGL^{*,*}(X)\xrightarrow{p}MGL^{*,*}(X)\longrightarrow MGL/p^{*,*}(X)\longrightarrow ...$$

Theorem 7.3. *Let G be a simple Lie group in Case I. Giving the bidegree by $deg(y)=(|y|,|y|/2), deg(x)=(|x|,(|x|+1)/2)$ for $x=x_i, x_1y^{p-1}$, we have the MU^*/p-algebra injection*

$$MU/p^*(G)\cong(MU^*/p\otimes(\mathbb{Z}[y]/(y^p,v_1y)\oplus\mathbb{Z}\{x_1y^{p-1}\})\otimes\Lambda(x_2,...,x_l)\subset MGL/p^{*,*}(G_{\mathbb{C}}).$$

Proof. The proof is quite similar to the case of ordinary mod p motivic cohomology. Here we use the realization map

$$MGL/p^{*,*}(G_{\mathbb{C}})\stackrel{t_{MGL}}{\longrightarrow}MU^*(G;\mathbb{Z}/p)\to BP^*(G;\mathbb{Z}/p)\to k(1)^*(G)$$

and the isomorphism $BP^*(G;\mathbb{Z}/p)\cong BP^*\otimes_{k(1)^*}k(1)^*(G)$. □

8. Classifying spaces

Here we consider the other types of algebraic spaces. For an algebraic group G (not assumed here the connectness) over $\mathbb{C}$, we can construct the classifying space BG as a limit of smooth algebraic varieties. (Indeed we still considered BT.) B.Totaro ([To1],[To2]) first studied the Chow ring of BG. He computed most important cases and conjectured $CH^*(BG)\cong MU^*(BG)\otimes_{MU^*}\mathbb{Z}$. He first found the modified cycle map ([To1])

$$(7.1)\qquad \bar{cl}:CH^*(X)\to MU^*(X)\otimes_{MU^*}\mathbb{Z}$$

such that its compostion with the Thom map is the usual cycle map. Now this map is extended by Levine and Morel as the map

$$(7.2)\qquad t_{MU}\rho_{MGL}:\Omega^*(X)\to MU^*(X).$$

By results of Totaro, the map (7.2) are isomorphism for products of $X=BGL_n, BO(n), BZ/p^n$, and hence groups whose p-Sylow subgroups are abelian p-groups ([To1],[To2]). Moreover it is known that (7.1) is epic ,for example, $G=PGL_3, SO(4), Spin(m), m\le 9, G_2$ and the extraspecial p groups of order p^3 ([Pa],[Ve].[S-Y],[Y4]). Hence all these cases (7. 2) are epic. Thus (7.2) seems to be isomorphism for each $X=BG$.

References

[G] A. Grothendieck. Torsion homologique et sections rationnelles. *Sem. C.Chevalley, ENS 1958,expose 5, Secreatariat Math., IHP, Paris* (1958).

[H-M-N-S] J. Hunton, M. Mimura, T. Nishimoto and B. Schuster. Higher v_n torsion in Lie groups. *J.Mth.Soc.Japan* **50** (1998), 801-818.

[I-K-T] K. Ishitoya, A. Kono and H. Toda Hopf algebra structure of mod 2 cohomology of simple Lie groups. *Puble. RIMS Kyoto Univ* **12** (1976-77), 141-167.

[K] V. G. Kac. Torsion in cohomology of compact Lie groups and Chow rings of reductive algebraic groups. *Invent. Math.* **80** (1985), 69-79.

[L-M 1] M. Levine and F. Morel. Coborsime algébrique I. *C. R. Acad. Sci. Paris* **332** (2001), 1-6.

[L-M 2] M. Levine and F. Morel. Coborsime algébrique II. *C. R. Acad. Sci. Paris* **332** (2001), 815-820.

[M] M. Mimura. Homotopy theory of Lie groups. *Hand book of algebraic topology, Edited by I.James, Elsevier Science B¡V.* (1995), 951-991.

[M-N] M. Mimura and T. Nishimoto. Hopf algebra structure of Morava K-theory of exceptional Lie groups. *To appear.* (2001).

[M-V] F. Morel and V. Voevodsky. $\mathbb{A}^1$-homotopy theory of schemes. *IHES Publ. Math.* **90** (2001), 45-143.

[Ni] T. Nishimoto Higher torsion in Morava K-thoeory of $SO(m)$ and $Sin(m)$. *J.Math.Soc. Japan.* **52** (2001), 383-394.

[No] P. Novikov The methods of algebraic topology from the view point of cobordism theory. *Math. USSR. Izv.* **1** (1967), 827-913.

[P] R. Pandharipande. Equivariant Chow rings of $O(k), SO(2k+1)$, and $SO(4)$. *J. Reine Angew. Math.* **496** (1998) 131-148.

[Q] D. Quillen. Elementary proofs of some results of cobordism theory using Steenrod operations. *Adv. Math.* **7** (1971), 29-56.

[Sc-Y] B. Schuster and N. Yagita. Transfer of Chern classes in BP-cohomology and Cow rings. *Trans. AMS.* **353** (2001), 1039-1054.

[To1] B. Totaro. Torsion algebraic cycles and complex cobordism. *J. Amer. Math. Soc.* **10** (1997), 467–493.

[To2] B. Totaro. The Chow ring of classifying spaces. *Proc.of Symposia in Pure Math. "Algebraic K-theory" (1997:University of Washington,Seattle)* **67** (1999), 248-281.

[Ve] G.Vezzosi. On the Chow ring of the classifying stack of $PGL_{3,C}$. *J.Reine Angew. Math.* **523** (2000), 1-54.

[Vo1] V. Voevodsky. The Milnor conjecture. *Preprint* (1996).

[Vo2] V. Voevodsky (Noted by Weibel). Voevodosky's Seattle lectures : K-theory and motivic cohomology *Proc.of Symposia in Pure Math. "Algebraic K-theory" (1997:University of Washington,Seattle)* **67** (1999), 283-303.

[Y1] N. Yagita. On relations between Brown-Peterson cohomology and the ordinary mod p cohomology theory. *Kodai Math.J* **7** (1984), 273-285.

[Y2] N. Yagita. Brown-Peterson cohomology groups of exceptional Lie groups. J. Pure and Apllied algebra. **17**, (1980) 223-226.

[Y3] N. Yagita. On mod odd prime Brown-Peterson cohomology groups of exceptional Lie groups. J. Math. Soc. Japan. **34**, (1982), 293-305.

[Y4] N. Yagita. Chow ring of classifying spaces of extraspecial p-groups. To appear. (2000).

Department of Mathematics, Faculty of Education, Ibaraki University, Mito, Ibaraki, Japan
E-mail address: yagita@mito.ipc.ibaraki.ac.jp