ALGEBRAIC COBORDISM OF SIMPLY CONNECTED LIE GROUPS

NOBUAKI YAGITA

ABSTRACT. Let $G_{\mathbb{C}}$ be the algebraic group over $\mathbb{C}$ corresponding a simply connected Lie group G . The algebraic cobordism $\Omega(G_{\mathbb{C}})$ defined by Levine and Morel is showed isomorphic to MU^* -subalgebra of $MU^*(G)$ with some modulus and is computed explicitly.

1. INTRODUCTION

For a smooth algebraic variety X over a field k of $ch(k) = 0$, recently Levine and Morel [L-M 1,2] defined the algebraic cobordism $\Omega^*(X)$ such that $\Omega^*(X) \otimes_{\Omega^*} Z \cong CH^*(X)$; the Chow ring of X and that there are maps

$$\Omega^*(X) \xrightarrow{\rho_{MGL}} MGL^{2*,*}(X) \xrightarrow{t_{MU}} MU^*(X)$$

where $MGL^{2*,*}(-) = \oplus_i MGL^{2i,i}(-)$ is the motivic cobordism theory defined by Voevodsky [V 1] and $MU^*(-)$ is the usual complex cobordism theory.

Let G be a simply connected Lie group and $G_{\mathbb{C}}$ the corresponding reductive algebraic group over $\mathbb{C}$; the complex number field. Let T be the maximal torus of G and $\pi : G \rightarrow G/T$ is the projection. By Grothendieck and Kac [K], it is known

$$CH^*(G_{\mathbb{C}}) \cong \pi^*(H^*(G/T)).$$

We study the cobordism version of this result. Fix a prime p . Recall $\Omega^* \cong MU^* \cong \mathbb{Z}[x_1, \dots]$ with $|x_i| = -2i$ and identify $v_i = x_{p^{i-1}}$.

Theorem 1.1. *Let $I = (p, v_1, \dots)$ be the invariant prime ideal of MU^* . Then we have the isomorphism $\Omega^*(G_{\mathbb{C}})/I^2 \cong \pi^*(MU^*(G/T))/I^2$.*

The above isomorphism seems to hold without I^2 , however we can not prove it now.

By Borel theorem, we can write $H^*(G; \mathbb{Z}/p) \cong P(y_{even})/p \otimes \Lambda(x_{odd})$ where $P(y_{even})$ is a truncated polynomial algebra of even degree generators y_{even} and $\Lambda(x_{odd})$ is the exterior algebra of odd degree generators x_{odd} . When $p = 2$ we take y_{even} as a power of some x_{odd} . Then the result of Grothendieck and Kac [K] is stated as

$$CH^*(G_{\mathbb{C}})_{(p)} \cong P(y_{even})/p.$$

Let Q_i be the Milnor primitive operation inductively defined by $Q_i = [Q_{i-1}, P^{p^{i-1}}]$ and $Q_0 = \beta$; the Bockstein operation. It is known that $Q_i(x_{odd}) \in P(y_{even})/p$ for all $i \geq 0$.

Theorem 1.2. *There is an Ω^* -algebra isomorphism*

$$\Omega^*(G_{\mathbb{C}})/I^2 \cong \Omega^* \otimes P(y_{even})/(I^2, \sum_i v_i Q_i(x_{odd})).$$

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For all simply connected simple Lie groups, more explicit forms of this theorem are shown in Theorem 5.1-5.4.

As for motivic cobordism theory $MGL^{*,*}(G_{\mathbb{C}})$, we do not know well. Indeed, even $MU^*(G)_{(p)}$ are quite complicated and unknown, in general. Let $MGL/p^{*,*}$ be the mod p motivic cobordism theory defined by the cofiber

$$MGL \xrightarrow{p} MGL \rightarrow MGL/p$$

in the stable $\mathbb{A}^1$ -homotopy category. Let $MU/p^*(X) = MU^*(-; \mathbb{Z}/p)$ the mod p complex cobordism theory defined similarly.

Theorem 1.3. *Let G be a simple Lie group in Case I in §5, e.g., $G = F_4, E_6, E_7$ for $p = 3$. Then we can give bidegree to $MU/p^*(G)$ such that degree of its MU^* -algebra generators are $(2n, n)$ or $(2m - 1, m)$ for $n, m \geq 3$, and that*

$$MU/p^*(G) \subset MGL/p^{*,*}(G_{\mathbb{C}}).$$

In §2, we recall the definition of the algebraic cobordism $\Omega^*(X)$ and the relation to the complex cobordism $MU^*(X)$. We also remark the BP -version of $\Omega^*(X)$. In §3, we note that Theorem 1.1 holds if $\pi^*MU^*(G/T) \cong MU^*(G/T)/Ideal(i^*\tilde{MU}^*(BT))$. Here

$$G \xrightarrow{\pi} G/T \xrightarrow{i} BT.$$

is the usual fibering. In §4, we will prove the above isomorphism $mod(I^2)$ with some assumption of the Milnor operation Q_i on $H^*(G; \mathbb{Z}/p)$. Here we use the spectral sequence for connected Morava K -theory (or other cohomology theories)

$$E_2^{*,*} \cong H^*(BT; k(1)^*(G)) \implies k(1)^*(G/T).$$

Indeed, $E_{\infty}^{*,*}$ shows the image $i^*k(1)^*(BT)$. In §5, we check the condition of Q_i , for each simple Lie group which has p -torsion, and consequently we see Theorem 1.1. Explicit results of $\Omega^*(G_{\mathbb{C}})/I^2$ also given here. In §6, we give examples of computations of the above spectral sequence for the easy cases G . In §7, for this type group G , we also study the motivic cobordism $MGL/p^{*,*}(G_{\mathbb{C}})$. The last section is very short remark for classifying spaces BG .

2. ALGEBRAIC COBORDISM

By extending the arguments by Quillen [Q], Levine and Morel defined the algebraic cobordism theory $\Omega^*(-)$ as the universal theory in theories having transfers and Chern classes [L-M 1,2] (We say that $h^*(X)$ is a theory having transfers and Chern classes if this theory satisfies the axioms A1 to A4 in [L-M 1]). Here we note that $\Omega^*(-)$ is not cohomology theory. The ring $\Omega^*(X)$ is constructed as

$$\Omega^*(X) = \{[f : M \rightarrow X]\} / (relations).$$

Here f is a map from a smooth variety M to X of pure codimension, namely, $\dim_{f(y)}(X) - \dim_y(M)$ is constant for all y in the same connected component of M . Relations are given so that we can define Chern classes or formal group laws (for details, see [L-M 1]). Given theory $h^*(-)$ having transfers and Chern classes, the map

$$\rho_h : \Omega^*(-) \rightarrow h^*(-)$$

is defined by $\rho_h([f : M \rightarrow X]) = f_*(1_M)$ where $1_M \in h^0(M)$ represents the identity element.

Let $H^{*,*}(-)$ (resp. $MGL^{*,*}(-)$, $MU^*(-)$, $CH^*(-)$, $H^*(-)$) be the motivic cohomology theory defined by Suslin and Voevodsky (resp. the motivic cobordism theory, the complex

cobordism theory, the Chow ring, the usual cohomology theory). Then we have commutative diagram

$$\begin{array}{ccccc} \Omega^*(X) & \xrightarrow{\rho_{MGL}} & MGL^{2*,*}(X) & \xrightarrow{t_M} & MU^*(X) \\ \downarrow = & & \downarrow \rho^{2*,*} & & \downarrow \rho^* \\ \Omega^*(X) & \xrightarrow{\rho_{CH}} & CH^*(X) \cong H^{2*,*}(X) & \xrightarrow{t_H} & H^*(X) \end{array}$$

where $\rho^{2*,*}, \rho^*$ (resp. t_m, t_H) are Thom maps (resp. realization maps.) Levine and Morel proves that

$$t_M \rho_{MGL} : \Omega^*(pt) \cong MU^*(pt) \cong \mathbb{Z}[x_1, x_2, \dots] \quad \text{with} \quad |x_i| = -2i$$

$$\text{and} \quad \rho_{CH} \otimes_{\Omega^*} Z : \Omega^*(X) \otimes_{\Omega^*} Z \cong CH^*(X).$$

Moreover they conjecture that ρ_{MGL} are always isomorphisms.

For the localized theories $h^*(-)_{(p)}$, we can consider the BP -version by using the universal p -typical formal group laws. However we construct here the BP -version using the Novikov's technique (5.4 in [N]). Given a sequence $\alpha = (a_1, a_2, \dots), a_i \geq 0$, recall the Landweber-Novikov operation is defined by

$$S_\alpha(f_*(1_M)) = f_*(c_\alpha(N_f)) \quad \text{for } [f : M \rightarrow X] \in \Omega^*(X)$$

where $c_\alpha(N_f)$ is the Chern class of the normal bundle of $f(M)$ in X so that S_α in $MU^*(X)$ is the usual Landweber-Novikov operation. This operations satisfy the Cartan formula $S_\alpha(xy) = \sum_{\alpha=\beta+\gamma} S_\beta(x)S_\gamma(y)$ for $x, y \in \Omega^*(X)$. Define an operation

$$\Delta_{x_i} = \sum_{q \geq 1} (x_i / S_{\Delta_i}(x_i))^{q-1} S_{q\Delta_i}.$$

Note that $\Delta_{x_i}(x_i) = 1$ and $S_{\Delta_i}(x_i) \neq 0 \pmod p$ if $i \neq p^j - 1$. Then we can easily prove that $\pi_i = 1 - x_i \Delta_{x_i}$ is a multiplicative projection such that $\pi_i(x_j) = (1 - \delta_{ij})x_j$. Essentially composing (for details, see p587 in [N]) π_i for all $i \neq p^j - 1$, we get the multiplicative projection $\Phi : \Omega^*(-)_{(p)} \rightarrow \Omega^*(-)_{(p)}$ such that

$$\Phi(x_i) = \begin{cases} x_i & \text{(if } i = p^j - 1 \text{ for some } j) \\ 0 & \text{(otherwise)} \end{cases}$$

Define the algebraic Brown-Peterson theory

$$\Omega_{BP}^*(X) = \text{Im}(\Phi(\Omega^*(X)_{(p)})) \subset \Omega^*(X)_{(p)}.$$

Hence if $h^*(-)$ is a theory having transfers and Chern classes, then there is the natural map $\rho_{BP,h} : \Omega_{BP}^*(X) \rightarrow h^*(X)_{(p)}$ compatible with $\rho_{h(p)}$.

Lemma 2.1. *Identifying $\Omega_{BP}^* = MU_{(p)}^* / (x_i | i \neq p^j - 1)$, we have the isomorphism $\Omega_{BP}^*(X) \cong \Omega_{BP}^* \otimes_{\Omega_{(p)}^*} \Omega^*(X)_{(p)}$.*

Proof. Since $\pi_{x_i}(a) = (1 - x_i \Delta_{x_i})a = a \pmod{(x_i)}$, we see $\Phi(a) = a \pmod{(x_i | i \neq p^j - 1)}$ for all $a \in \Omega^*(X)_{(p)}$. $\square$

In this paper we mainly consider $\Omega^*(X)_{(p)}$ but not $\Omega_{BP}^*(X)$. However we use notations $v_i = x_{p^i-1} \in \Omega_{(p)}^*$ for ease of expressions.

3. LIE GROUP AND ITS MAXIMAL TORUS

Let G be a simply connected Lie group and $G_{\mathbb{C}}$ the corresponding reductive algebraic group over $\mathbb{C}$. Let T be the maximal torus of G . then we have the fibering

$$(3.1) \quad G \xrightarrow{\pi} G/T \xrightarrow{i} BT = ET/T.$$

Taking $BT = \text{colim}_{N \rightarrow \infty} ((\mathbb{A}^N - \{0\})^{\times n} / (\mathbb{G}_m)^n)$, we get the maps of algebraic groups

$$G_{\mathbb{C}} \xrightarrow{\pi_{\mathbb{C}}} G_{\mathbb{C}}/T_{\mathbb{C}} \xrightarrow{i_{\mathbb{C}}} BT.$$

If an algebraic variety X has a cellular decomposition, i.e., $X = X_n \supset \dots \supset X_0$ with $X_i - X_{i-1} = \cup \mathbb{A}^{n_{ij}}$ where $\mathbb{A}^{n_{ij}}$ is the affine space of $\dim = n_{ij}$, then the Chow ring is isomorphic to the ordinary cohomology i.e., $CH^*(X) \cong H^*(X)$ and moreover $MU^*(X) \cong MU^* \otimes H^*(X)$. Hence these facts hold for $X = G_{\mathbb{C}}$ and $BT_{\mathbb{C}}$.

From Grothendieck and Kac [K], it is known that

$$(3.2) \quad CH^*(G_{\mathbb{C}}) \cong \text{Im}(\pi^* : H^*(G/T) \rightarrow H^*(G)).$$

Lemma 3.1. *Let $\tilde{MU}^*(-)$ be the reduced $MU^*(-)$ theory. If there is the isomorphism*

$$\pi^*(MU^*(G/T)) \cong MU^*(G/T)/(Ideal(i^* \tilde{MU}^*(BT))),$$

then we have

$$MGL^{2*,*}(G_{\mathbb{C}}) \supset \Omega^*(G_{\mathbb{C}}) \cong \pi^*(MU^*(G/T)).$$

Proof. Consider the diagram

$$\begin{array}{ccccc} \Omega^*(G_{\mathbb{C}}) & \xleftarrow{\pi_{\mathbb{C}}^*} & \Omega^*(G_{\mathbb{C}}/T_{b\mathbb{C}}) & \xleftarrow{i_{\mathbb{C}}^*} & \Omega^*(BT) \\ (1) \downarrow & & (2) \downarrow & & (3) \downarrow \\ MU^*(G) & \xleftarrow{\pi_{MU}^*} & MU^*(G/T) & \xleftarrow{i_{MU}^*} & MU^*(BT). \end{array}$$

where (1),(2),(3) are maps induced from $t_{MU}\rho_{MGL}$. For $X = G/T, BT$, we already know that $MU^*(X) \cong MU^* \otimes H^*(X) \cong MU^* \otimes CH^*(X)$. Hence (2) and (3) are isomorphisms. While the rows are not exact, we get the map

$$\Omega^*(G_{\mathbb{C}}) \xleftarrow{\pi^*} MU^*(G/T)/(Ideal(i_{MU}^* \tilde{MU}^*(BT)))$$

where π^* is induced from $\pi_{\mathbb{C}}^*(2)^{-1}$. Tensoring $\otimes_{\Omega^*} \mathbb{Z}$, we have

$$CH^*(G) \cong \Omega^*(G) \otimes_{\Omega^*} \mathbb{Z} \xleftarrow{\pi^* \otimes_{MU^*} \mathbb{Z}}$$

$$(MU^*(G/T)/(Ideal(i_{MU}^*))) \otimes_{MU^*} \mathbb{Z} \cong H^*(G/T)/(Ideal(i_H^*)).$$

By Grothendieck and Kac theorem the above map is epic. Hence π^* itself also epic. From the assumption of this lemma, we get $\text{Im}(\pi_{MU}^*) \cong MU^*(G/T)/(Ideal(i_{MU}^*))$. Hence the map π^* is an isomorphism.

Since (1) = $t_{MU}\rho_{MGL}$ is injective, the map ρ_{MGL} is a split injection. $\square$

4. $BP^*(G/T)/Ideal(i^*(\tilde{BP}^*(BT)))$

In this section, we consider $\Omega_{BP}^*(G_{\mathbb{C}})$ or $\Omega^*(G_{\mathbb{C}})_{(p)}$. Moreover we study $BP^*(G)$ instead of $MU^*(G)_{(p)}$. Let G be a simply connected Lie group. By the Borel's theorem, we have the ring isomorphism for p odd

$$(4.1) \quad H^*(G; \mathbb{Z}/p) \cong P(y)/p \otimes \Lambda(x_1, \dots, x_l) \quad \text{with } P(y) = \mathbb{Z}[y_1, \dots, y_k]/(y_1^{p^{r_1}}, \dots, y_k^{p^{r_k}})$$

where $|y_i| = \text{even}$ and $|x_j| = \text{odd}$. When $p=2$, the righthand side is isomorphic to $gr H^*(G; \mathbb{Z}/2)$ and for each y_i , there is x_j with $x_j^2 = y_i$. By Grothendieck and Kac [K], we know that

$$(4.2) \quad CH^*(G_{\mathbb{C}})/p \cong P(y)/p.$$

Consider the spectral sequence induced from the cofiber (3.1)

$$(4.3) \quad E_2^{*,*} = H^*(BT; H^*(G; \mathbb{Z}/p)) \implies H^*(G/T; \mathbb{Z}/p).$$

The cohomology of the classifying space of the torus is

$$H^*(BT) \cong \mathbb{Z}[t_1, \dots, t_l] \quad \text{with } |t_i| = 2$$

where l is also the number of the odd degree generators x_k in $H^*(G; \mathbb{Z}/p)$. It is known that there is a regular sequence $(b_1, \dots, b_l)$ in $H^*(BT)/p$ such that $d_{|x_i|+1}(x_i) = b_i$. Thus we get

$$(4.4) \quad H^*(G/T; \mathbb{Z}/p) \cong P(y) \otimes \mathbb{Z}/p[t_1, \dots, t_l]/(b_1, \dots, b_l).$$

Recall the connected Morava K-theory $k(n)^*(-)$ with the coefficient $k(n)^* = \mathbb{Z}/p[v_n]$. Then the usual Morava K-theory is $K(n)^*(X) = [v_n^{-1}]k(n)^*(X)$.

Since $H^*(G/T)$ has no torsion, we get the BP^* -modules isomorphism

$$(4.5) \quad BP^*(G/T) \cong BP^* \otimes P(y) \otimes \mathbb{Z}[t_1, \dots, t_l]/(\tilde{b}_1, \dots, \tilde{b}_l)$$

where $\tilde{b}_i = b_i \bmod(p)$. However note that this is not a BP^* -algebras isomorphism. (The righthand side above is isomorphic to $gr BP^*(G/T)$.) We also note $k(n)^*(G/T) \cong k(n)^* \otimes_{BP^*} BP^*(G/T)$ and

$$(4.6) \quad k(n)^*(G/T) \cong k(n)^* \otimes P(y) \otimes \mathbb{Z}/p[t_1, \dots, t_l]/(b_1, \dots, b_l).$$

We want to study the spectral sequence

$$(4.7) \quad E_2^{*,*} = H^*(BT; k(1)^*(G)) \implies k(1)^*(G).$$

Let $\rho : k(n)^*(X) \rightarrow H^*(X; \mathbb{Z}/p)$ be the natural (Thom) map. Let us write by X^s the s -skeleton of X . Recall also Q_i is the Milnor primitive operation inductively defined by $Q_i = P^{p^{i-1}}Q_{i-1} - Q_{i-1}P^{p^{i-1}}$ where P^k are reduced power operations and $Q_0 = \beta$; Bockstein operation. Here we recall lemmas related about relations between BP^* -modules structure of $BP^*(X)$ and Q_i -actions on $H^*(X; \mathbb{Z}/p)$.

Lemma 4.1. ([Y 1]) *Let $\sum_n v_n y_n = 0 \in BP^*(X)$. Then there is $x \in H^*(X; \mathbb{Z}/p)$ such that $Q_n(x) = \rho(y_n)$.*

Lemma 4.2. *For $x \in H^*(X; \mathbb{Z}/p)$, there is $y' \in k(n)^*(X)$ with $Q_n(x) = \rho(y')$ and $v_n y' = 0$. Conversely if $v_n y' = 0$, then there is x with $Q_n(x) = \rho(y')$.*

Proof. There is the Sullivan-Bockstein exact sequence

$$k(n)^*(X) \xrightarrow{v_n} k(n)^*(X) \xrightarrow{\rho} H^*(X; \mathbb{Z}/p) \xrightarrow{\delta} k(n)^*(X) \xrightarrow{v_n}$$

Since $\rho\delta = Q_n$, if $Q_n x = y$, then $\delta(x) = y'$ is v_n -torsion. Conversely if $v_n y' = 0$, then there is $x \in H^*(X; \mathbb{Z}/p)$ with $\delta(x) = y'$ and $Q_n(x) = \rho(y')$. $\square$

Lemma 4.3. *Let $F \rightarrow E \xrightarrow{p} B$ be a fiber bundle such that $H^*(B)$ is a p -torion free simply connected space. In the Serre spectral sequence converging to $H^*(E; \mathbb{Z}/p)$, let $0 \neq y \in H^*(F; \mathbb{Z}/p)$ and $0 \neq b \in H^*(B; \mathbb{Z}/p)$ such that if $Q_n x = y$, then $d_r(x) = b$ and that there exists such $x \in H^*(F; \mathbb{Z}/p)$. Then there are $y' \in k(n)^*(F)$ and $b' \in k(n)^*(B)$ such that $\rho(y') = y, \rho(b') = b$ and*

$$v_n y' = \lambda b', \quad \lambda \neq 0 \in \mathbb{Z}/p \quad \text{in } k(n)^*(p^{-1}B^{|b|})$$

Proof. Let $B' = B^{|b|-1}$ and $E' = p^{-1}(B')$. Consider the Serre spectral sequence

$$E_2^{*,*} = H^*(B'; H^*(F; \mathbb{Z}/p)) \implies H^*(E'; \mathbb{Z}/p).$$

Since $d_r x = b = 0 \in H^*(B')$, there is $x \in H^*(E'; \mathbb{Z}/p)$ with $Q_n x = y$. This means there exists $v_n y' = 0$ in $k(n)^*(E')$.

On the otherhand, let $B'' = B^{|b|-1} \cup e_b$ and $E'' = p^{-1}B''$ where e_b is the normal cell representing b . Then $d_r x = b \neq 0$ and there is no element $x \in H^*(E''; \mathbb{Z}/p)$ with $Q_n x = \rho(y)$. From Lemma 4.2, we get $v_n y' \neq 0 \in k(n)^*(E'')$ and have this lemma. $\square$

Remark (4.8). Lemma 4.1 and Lemma 4.2 are also hold for $n = 0$ replacing $k(0)^*(-) = H^*(-)$: the integral cohomology and $v_0 = p, \lambda \neq 0 \pmod{p}$.

The Q_i actions on $H^*(G; \mathbb{Z}/p)$ are quite restrictive. We consider the following assumption.

Assumption (4.9)

(A.1) if $i \neq j$, then $|x_i| \neq |x_j|$.

(A.2) $Q_k x_i \in P(y)$ for all k, i and if $Q_k x_i \neq 0$, then $Q_0 x_i = y_i$ or, $Q_0 x_i = 0$ and $Q_1 x_i = y_{i'}$ for some i'' .

The above assumptions are satisfied for all simply connected simple Lie groups except for $p = 2, Spin(n), E_8$. However quite similar conditions also hold for these cases. (We will explain these in the next section.)

Lemma 4.4. *Assume (A.1), (A.2). Then $BP^*(G/T)/Ideal(i^*(\tilde{BP}^*(BT)), I^2)$ is a quotient ring of $BP^* \otimes P(y)/(I^2, (1), (2))$ where*

$$(1) \quad py_i = 0 \pmod{v_1, \dots} \quad \text{if } Q_0 x_i = y_i$$

$$(2) \quad v_1 y_j = 0 \pmod{v_2, \dots} \quad \text{if } Q_1 x_{j'} = y_j \text{ but } Q_0 x_{j'} = 0.$$

Proof. First note that from (4.5), the BP^* -algebra is a quotient of $BP^* \otimes P(y)$. Recall in the spectral sequence (4.3), $d_r(x_i) = b_i$ and $Q_0(x_i) = y_i$. Letting $F = G, G/T = E, B = BT$, from Remark (4.8), we know

$$py_i = i^*(b_i) \quad \text{in } H^*(G/T) = BP^*(G/T)/(v_1, v_2, \dots).$$

From (4.5), we have $py_i + \sum v_n a_n \in Ideal(i^*(\tilde{BP}^*(BT)))$ for $a_n \in BP^* \otimes P(y)$. Hence the relation (1) contained in $Ideal(i^*(\tilde{BP}^*(BT)))$.

Let $Q_1 x_{i'} = y_i$. Recall $b_{i'} = d_{|x_{i'}|+1}(x_{i'})$ in the spectral sequence (4.3). From Lemma 4.3, we already know in $k(1)^*(G/T)$

$$(*) \quad v_1 y_i = \lambda b_{i'} + b' \quad \lambda \neq 0 \in \mathbb{Z}/p, \quad b' \in E_\infty^{|b_{i'}|+1,*}.$$

Since $b_{i'} = 0$ in $H^*(G/T; \mathbb{Z}/p)$, we know $b' = 0 \pmod{v_1}$ in $k(1)^*(G/T)$. Here we recall all elements are represented as in (4.7). Hence we can write

$$b' = v_1 d(y) + v_1 g \quad \text{with } d(y) \in P(y), \quad g \in I(t) = Ideal(t_1, \dots, t_l) \subset k(1)^*(G/T).$$

Let $E' = p^{-1}G/T^{|b_{i'}|}$. Since $b' \in E_\infty^{|b_{i'}|+1,*}$, we see $b' = 0 \in k(1)^*(E')$. Hence there is $x \in H^*(E'; \mathbb{Z}/p)$ such that $Q_1(x) = b'/v_1 = d(y) + g$. But if $b' \neq 0 \in k(1)^*(G/T)$, then x does not exist in $H^*(G/T; \mathbb{Z}/p)$, namely, the corresponding element x in $E_2^{*,*}$ of (4.7) is not

a permanent cycle. In particular if $d(y) \neq 0$, then $x \neq 0 \in H^*(G; \mathbb{Z}/p)$ but $|x| = |x_{i'}|$ and there is no differential $d_r(x) \neq 0$ for $r > |x_{i'}| + 1 = |b_{i'}|$; this is a contradiction. Hence we get $b' \in I(t)$ and so $v_1 y_i \in I(t)$.

Let $Q_0 x_{i'} = 0$ and $Q_1 x_{i'} = y_i$. From (*), we get in $BP^*(G/T)$,

$$v_1 y_i = \lambda \tilde{b}_{i'} + \tilde{b}' + pf(y) + pb'' \mod(p^2, v_2, v_3, \dots),$$

where $\tilde{b}_{i'} = b_{i'} \mod(p)$, $\tilde{b}' = b' \mod(p)$, $f(y) \in P(y)$ and $b'' \in \text{Ideal}(t_1, \dots, t_l) \subset BP^*(G/T)$. But $f(y) = 0 \mod(p^2)$ otherwise $Q_0 x_{i'} = -f(y) \neq 0$ from Remark 4.8. . Thus we get (2) is contained in $\text{Ideal}(i^*(\tilde{B}P^*(BT)))$. $\square$

Lemma 4.5. *Assume (A,2). The image $\pi^* BP^*(G/T)/I^2$ is isomorphic to*

$$BP^* \otimes P(y)/(I^2, (1), (2) \text{ in Lemma 4.3}).$$

Proof. Suppose that there is a relation in $\pi^* BP^*(G/T)/I^2$

$$(*) \quad v_n y = 0 \mod(I^2, v_{n+1}, \dots) \text{ with } y \in P(y).$$

From Lemma 4.1, there is $x \in H^*(G; \mathbb{Z}/p)$ such that $Q_n x = y$ and $Q_j(x) = 0 \mod(I^2)$ for $j < n$. Let $H^*(G; \mathbb{Z}/p) = \bigoplus_s X_s$ where X_s is the free $P(y)$ -module generated by monomials $x_{i_1} \dots x_{i_s}$. Since all $Q_i(x_j) \in P(y)$ from (A.2), we know that $Q_i : X_s \rightarrow X_{s-1}$. In particular, we can write $x = \sum_j u_j x_j$, $u_j \in P(y)$. Then we have the relation $(\mod(I^2, v_{n+1}, \dots))$

$$\begin{aligned} \sum_j u_j \sum_k (v_k Q_k(x_j)) &= \sum_k v_k \sum_j u_j Q_k(x_j) \\ &= \sum_k v_k Q_k(\sum_j u_j x_j) = \sum_k v_k Q_k(x) = v_n y. \end{aligned}$$

Thus we can express the relation type (*) by taking (1),(2) in Lemma 4.3. By the induction on n , we can prove the lemma. $\square$

From Lemma 3.1. the following corollary is immediate.

Corollary 4.6. *If (A.1),(A,2) are satisfied for simply connected Lie group G_i , $1 \leq i \leq n$, then the product $G = G_1 \times \dots \times G_n$ satisfies Theorem 1.1 in the introduction.*

Proof. Let T_i be the maximal torus of G_i . Since G_i/T_i and BT_i are torsion free, we see

$$BP^*(B/T)/\text{Ideal}(i^*) \cong \bigotimes_{BP^*} BP^*(G_i/T_i)/(\text{Ideal}(\sum i_i^*)).$$

Hence the result of Lemma 4.4 holds for this G .

On the otherhand (A.2) is satisfied also in $H^*(G; \mathbb{Z}/p) \cong \bigotimes H^*(G_i; \mathbb{Z}/p)$. Hence Lemma 4.5 also holds. Thus we get Theorem 1.1 from the $\mod(I^2)$ version of Lemma 3.1. $\square$

5. SIMPLE GROUPS

We study simple groups. The simple Lie groups which have p -torsion in $H^*(G)$ are divided to the following cases.

Case I. The (G, p) are the exceptional Lie groups $(G_2, 2), (F_4, 2), (E_6, 2), (F_4, 3), (E_6, 3), (E_7, 3)$ and $(E_8, 5)$.

Case II. $(E_8, 3)$.

Case III. The cases $(E_7, 2), (E_8, 2)$.

Cases IV. The classical cases $(Spin(n), 2)$.

Case I. We at first study $k(1)^*(G)$ for the groups (G, p) in Case I. The ordinary mod p -cohomology is written

$$H^*(G; \mathbb{Z}/p) \cong \mathbb{Z}/p[y]/(y^p) \otimes \Lambda(x_1, x_2, \dots, x_l)$$

where $|x_1| = 3, |x_2| = 2p+1, |y| = 2p+2, Q_1x_1 = Q_0x_2 = y$. In this case $k(1)^*(G)$ is known. Consider the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*} = H^*(G; k(1)^*) \implies k(1)^*(G).$$

The only nonzero differential is $d_{2p-1}(x) = v_1 \otimes Q_1(x)$. Thus we can prove

$$k(1)^*(G) \cong (k(1)^*[y]/(y^p, v_1y) \oplus k(1)^*\{xy^{p-1}\}) \otimes \Lambda(x_2, \dots, x_l)$$

Moreover, $BP^*(G)$ is computed in [Y2],

$$BP^*(G) \cong (BP^* \otimes P(y)/(py, v_1y) \oplus BP^*\{w_1, w_2\}/(pw_1 + v_1w_2) \oplus BP^*\{x'_1, x''\}) \otimes \Lambda(x_3, \dots, x_l).$$

Here $w_1 = x_1y^{p-1}, w_2 = x_2y^{p-1}, x'_1 = px_1$ and $x'' = x_1x_2y^{p-1}$ in the Atiyah-Hirzebruch spectral sequence converging to $BP^*(G)$.

Of course this case satisfies (A,1),(A,2) in §4. Moreover it is easy proved that Corollary 4.6 holds without *modulo*(I^2) for all products of Lie groups of Case I.

Theorem 5.1. *For the group (G, p) in Case I, we have the isomorphism*

$$\Omega^*(G)_{(p)} \cong \Omega^*[y]/(py, v_1y, y^p).$$

Case II. Let $(G, p) = (E_8, 3)$. The mod 3 cohomology is

$$H^*(E_8; \mathbb{Z}/3) \cong \mathbb{Z}/3[y_8, y_{20}]/(y_8^3, y_{20}^3) \otimes \Lambda(x_3, x_7, x_{15}, x_{19}, \dots)$$

where the suffix means its degree, i.e., $|y_8| = 8$. The Minor operations are

$$Q_1x_3 = Q_0x_7 = y_8, \quad Q_2x_3 = Q_1x_{15} = Q_0x_{19} = y_{20}$$

and $Q_ix_j = 0$ for other i, j . From this we have relations, e.g.,

$$v_1y_8 + v_2y_{20} = 0, \quad v_1y_{20} = 0.$$

We easily see the conditions (A,1),(A,2). The $BP^*(E_8)$ is computed ([M],[Y 3]). We also get the result without *mod*(I^2), however for $BP^*(E_8 \times E_8)$ I can not show the result without *mod*(I^2).

Theorem 5.2. *Then Theorem 1.1 holds for $(E_8, 3)$ without *mod*(I^2), i.e.,*

$$\Omega(E_8)_{(3)} \cong \Omega^*[y_8, y_{20}]/(3y_8, 3y_{20}, y_8^3, y_{20}^3, v_1y_8 + v_2y_{20}, v_1y_{20}).$$

Case III. $(E_7, 2), (E_8, 2)$.

The mod 2-cohomology of E_8 is

$$H^*(E_8; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_3, x_5, x_9, x_{15}]/(x_3^{16}, x_5^8, x_9^4, x_{15}^2) \otimes \Lambda(x_{17}, x_{23}, x_{27}, x_{29}),$$

$$\text{and} \quad H^*(E_7; \mathbb{Z}/2) \cong H^*(E_8; \mathbb{Z}/2)/(x_3^4, x_5^4, x_{15}^2, x_{29}).$$

Let us write $y_{2i} = x_i^2$ if it is not zero. The Q_i -actions are completely determined by Hunton-Mimura-Nishimoto-Schuster. We have for example, [H-M-N-S], [M-N]

$$Q_1x_3 = Q_0x_5 = y_6, \quad Q_0x_9 = y_{10} \quad Q_1x_{15} = Q_0x_{17} = y_{18},$$

$$Q_2x_{23} = Q_1x_{27} = Q_0x_{29} = y_{30}$$

$$\text{with} \quad 0 = Q_0x_3 = Q_0x_{15} = Q_0x_{27}, \quad 0 = Q_0x_{23} = Q_1x_{23}.$$

(Of course there many other nonzero operations, e.g., $Q_2x_3 = y_{10}, Q_3x_3 = y_{18}$.)

Assumption (A.2) holds except for $Q_0x_{23} = Q_1x_{23} = 0$ but $Q_2x_{23} = y_{30}$.

If there is a relation in $BP^*(X)/I^2$

$$v_n a_n + v_{n+1} a_{n+1} + \dots = 0 \quad a_n \neq 0,$$

then we say it *the relation starting with $v_n a_n$* . For example, the relations in $BP^*(E_8)/I^2$ starting with $2y_6, v_1 y_6$ are

$$2y_6 + v_2 y_6^2 + v_3 y_{10}^2 = 0, \quad v_1 y_6 + v_2 y_{10} + v_3 y_{18} = 0.$$

Theorem 5.3. *The cases $p = 2$ and $G = E_8, E_7$ satisfy Theorem 1.1, e.g.,*

$$(1) \quad \Omega^*(E_8)/I^2 \cong \Omega^*[y_6, y_{10}, y_{18}, y_{30}]/(y_6^8, y_{10}^4, y_{18}^2, y_{30}^2, R, I^2)$$

where R is generated by relations starting with

$$2y_6, v_1 y_6, 2y_{10}, 2y_{18}, v_1 y_{18}, 2y_{30}, v_1 y_{30}, v_2 y_{30},$$

$$(2) \quad \Omega^*(E_7)/I^2 \cong \Omega^*(E_8)/(I^2, y_6^2, y_{10}^2, y_{30}).$$

Proof. We will prove only for $G = E_8$ but E_7 is much easier. Note that the product $\mu : G \times G \rightarrow G$ induces the coproduct $\mu^* : BP^*(G) \rightarrow BP^*(G \times G)$, which is not isomorphic to $BP^*(G) \otimes_{BP^*} BP^*(G)$. Since y_i is primitive in $H^*(G; \mathbb{Z}/2)$, we can write in $BP^*(G \times G)$

$$\mu^*(y_i) = y_i \otimes 1 + 1 \otimes y_i + a \quad \text{with } a \in I = (2, v_1, \dots).$$

Hence $y_i^{2^k}, k \geq 1$ is always primitive $\text{mod}(I^2)$. Conversely $\text{mod}(I^2)$ primitive elements are of the form $y_i^{2^{k'}}, k' \geq 0$ and x_i . By the dimensionl reason, we see $y_6^8 = 0, y_{10}^4 = 0, \dots$

The part which does not satisfy (A,2) is come from

$$Q_0 x_{23} = Q_1 x_{23} = 0, \quad Q_2 x_{23} = y_{30}.$$

Here we do $k(2)$ version of the proof of Lemma 4.4. Letting $y_i = y_{30}$ and $x_{i'} = x_{23}$, we can take in $k(2)^*(G/T), v_2 y_i = b_{i'} + b'$ same as (*) in the proof of Lemma 4.4. All arguments work similarly and we get the theorem. $\square$

Case IV ; classical group cases.

We first consider the orthogonal groups $G = SO(m)$, while these are not simply connected. The mod 2-cohomology is written as (see for example [N])

$$grH^*(SO(m); \mathbb{Z}/2) \cong \Lambda(x_1, x_2, \dots, x_{m-1})$$

where the multiplications are given by $x_s^2 = x_{2s}$. We write $y_{2(\text{odd})} = x_{\text{odd}}^2$. Hence we can write

$$H^*(SO(m); \mathbb{Z}/2) \cong \mathbb{Z}/2[y_{4i+2} | 2 \leq 4i+2 \leq m-1] / (y_{2i+1}^{s(i)}) \otimes \Lambda(x_1, x_3, \dots)$$

where $s(i)$ is the smallest number such that $2^{s(i)}(4i+2) \geq m$.

The Q_i -operations are given by Nishimoto [N]

$$Q_n x_{\text{odd}} = x_{\text{odd}+|Q_n|}, \quad Q_n x_{\text{even}} = Q_n y_{\text{even}} = 0.$$

We note the operations

$$Q_0 x_1 = y_2, \quad Q_1 x_{2(\text{odd})-3} = Q_0 x_{2(\text{odd})-1} = y_{2(\text{odd})} \text{ if } (\text{odd}) > 1.$$

Take new generators $x'_{4m-1} = x_{4m-1} + x_{2m-1} y_{2m}, y'_{4m+2} = y_{4m+2} + y_{2m+2} y_{2m}, x'_{4m+1} = x_{4m+1} + x_{2m+1} y_{2m}$ so that

$$Q_0(x'_{4m-1}) = 0, \quad Q_1(x'_{4m-1}) = Q_0(x'_{4m+1}) = y'_{4m+2}.$$

Thus we see that the assumption (A.1),(A.2) are satisfied and Theorem 1.1 holds for $G = SO(m)$. Relations are given by

$$\sum_n v_n Q_n(x_{odd}) = \sum_n v_n x_{odd+|Q_n|} = 0 \quad \text{mod}(I^2).$$

For example, the relations in $BP^*(SO(m))/I^2$ starting with $2y_6$ and $v_1(y_6 + y_2^3)$

$$\begin{aligned} 2y_6 + v_1 y_2^3 + v_2 y_6^2 + v_3 y_{10}^2 + \dots &= 0, \\ v_1(y_6 + y_2^3) + v_2(y_{10} + y_2^5) + v_3(y_{18} + y_2^9) + v_4(y_{34} + y_2^{17}) + \dots &= 0 \end{aligned}$$

Next consider the case $G = Spin(m)$. Let $2^{t-1} < m \leq 2^t$. Then the mod 2-cohomology is

$$H^*(BSpin(m); \mathbb{Z}/2) \cong H^*(BSO(m); \mathbb{Z}/2)/(y_2) \otimes \Lambda(z)$$

with $|z| = 2^t - 1$. Here this element z is defined as following. Consider the spectral sequence

$$E_2^{*,*} \cong H^*(B\mathbb{Z}/2; H^*(Spin(m); \mathbb{Z}/2)) \implies H^*(BSO(m); \mathbb{Z}/2)$$

induced from the cofiber $Spin(m) \rightarrow SO(m) \rightarrow B\mathbb{Z}/2$. Let $x \in H^1(B\mathbb{Z}/2; \mathbb{Z}/2)$ be the generator. Then z is defined as an element with $d_{2^t}(z) = x^{2^t}$ since this element is zero in $H^*(SO(m); \mathbb{Z}/2)$.

The Q_i -actions of z are also given by Nishimoto [N]

$$\begin{aligned} Q_0 z &= \sum_{i+j=2^{t-1}, i < j} x_{2i} x_{2j} \\ Q_n z &= \sum_{i=2^n+1}^{2^{t-2}+2^{n-1}-1} x_{2i} x_{2^t+2^{n+1}-2-2i} \quad \text{for } n \geq 1 \end{aligned}$$

It is immediate $Q_n z \in P(y)$ and decomposable as a product of generators x_{even} .

Theorem 5.4. *The groups $G = SO(m)$, $Spin(m)$ satisfy Theorem 1.1. There are $\Omega_{(2)}^*$ -algebra isomorphisms*

$$(1) \quad \Omega^*(SO(m))/I^2 \cong \Omega^*[y_{4i+2} | 2 \leq 4i+2 \leq m-1] / (R, y_{4i+2}^{2^{s(i)}}, I^2)$$

where $R = \{\text{relations starting with } 2y_{4i+2}, v_1(y_{2i}y_{2i+2} + y_{4i+2}) \text{ for } y_{4j} = y_{2j}^2\}$.

$$(2) \quad \Omega^*(Spin(m))/I^2 \cong \Omega^*(Spin(m))/(y_2, \sum v_n Q_n(z), I^2)$$

Proof. Let $G = Spin(m)$. To see the results of Lemma 4.4, Lemma 4.5, we need to see that the relation

$$\sum v_n Q_n(z) = 0 \quad \text{in } BP^*(G)/I^2$$

comes from an element in $Ideal \tilde{BP}^*(G/T)$.

Let $b_z \in H^*(BT; \mathbb{Z}/2)$ be the element with $d_{|z|+1}(z) = b_z$ in the spectral sequence converging $H^*(G/T; \mathbb{Z}/2)$. There are many elements $x \in H^*(G; \mathbb{Z}/2)$ such that $Q_n x = Q_n z$. Let us write $x = \sum f_{odd}(y) x_{odd}$ with $f_{odd}(y) \in P(y)/2$. Recall that $Q_n z$ is the decomposable element for generators x_{even} from the result of Nishimoto. Hence we need only consider the cases $|f_{odd}(y)| > 0$, namely, $|x_{odd}| < |z|$. Since $d_{|x_{odd}|+1}(x_{odd}) \neq 0$, this element disappears in $H^*(p^{-1}(G/T^{|b_z|-1}); \mathbb{Z}/2)$ from $|x_{odd}| + 1 < |z| + 1 = |b_z|$.

For each n , we consider the spectral sequence

$$H^*(BT; k(n)^*(G)) \implies k(n)^*(G/T).$$

By the arguments similar to the proof of Lemma 4.3, we get

$$(*) \quad v_n Q_n(z) = b_z + b'_n \quad b'_n \in E_\infty^{|b_z|+1,*}.$$

Moreover the arguments similar to the proof of Lemma 4.5, we can see that $b'_n = v_n b''_n$ and $b''_n \in I(t)$.

Let us write

$$\sum v_n Q_n(z) - b_z = \sum v_n a_n \quad \text{in } BP^*(G/T)/I^2.$$

Considering above equation in $k(n)^*(G/T)$, we have $a_n = b''_n \bmod(2, v_1, \dots, \hat{v}_n, \dots)$. Hence we have $\sum v_n Q_n(z) + b_z \in I(t) \bmod(I^2)$ i.e., the sum $\sum v_n Q_n(z)$ itself in $I(t)$ modulo I^2 . $\square$

From Theorem 5.1- 5.4 and Corollary 4.6, we get Theorem 1.1, and moreover we explicitly know $\Omega^*(G)/I^2$ for all simply connected Lie groups (and orthogonal groups).

6. EXAMPLE FOR THE SPECTRAL SEQUENCES

In this section, we give examples for spectral sequences for groups G in the most easy case Case I. We consider the spectral sequence

$$(6.1) \quad E(h)_2^{*,*} = H^*(BT; h^*(G/T)) \implies h^*(G/T)$$

for homology theories, $h = H\mathbb{Z}/p, H\mathbb{Z}, H\mathbb{Q}, k(1), K(1), BP\langle 1 \rangle$. Since G/T has no torsion, in any h we know that

$$(6.2) \quad h^*(G/T) \cong h^* \otimes P(y) \otimes P'(t)/(b_1, \dots, b_l) \quad \text{where } P'(t) = \mathbb{Z}[t_1, \dots, t_l].$$

First recall the case $h = H\mathbb{Z}/p$, the cohomology is $h^*(G; \mathbb{Z}/p) \cong P(y)/p \otimes \Lambda(x_1, \dots, x_l)$. The differentials are $d_{|x_i|+1}(x_i) = b_i$ and we still know

$$(6.3) \quad E(H\mathbb{Z}/p)_\infty^{*,*} \cong P(y)/p \otimes P(t)'/(b_1, \dots, b_l).$$

Hereafter we restrict ourself for Case I. This case $P(y) \cong \mathbb{Z}[y]/(y^p)$ and $Q_1 x_1 = Q_0 x_2 = y$.

Next we consider the spectral sequences for Morava K -theories. First recall again

$$k(1)^*(G) \cong (P(y)^+/p \oplus k(1)^* \Lambda(x_1 y^{p-1})) \otimes \Lambda(x_2, \dots, x_l).$$

We also note that $BP^*(G; \mathbb{Z}/p) \cong BP^* \otimes_{k(1)^*} k(1)^*(G)$ identifying $BP^*/p \supset \mathbb{Z}/p[v_1] \cong k(1)^*$.

Proposition 6.1. *Let G be a simple Lie group in Case I. Then we get isomorphisms*

$$E(k(1))_\infty^{*,*} \cong k(1)^* \otimes P(y)/(v_1 y) \otimes P(t)'/(b_2, \dots, b_l, b_1^s y^{p-s} | 1 \leq s \leq p).$$

$$E(K(1))_\infty^{*,*} \cong K(1)^* \otimes P(t)'/(b_1^p, b_2, \dots, b_l).$$

Proof. First note some facts of the differentials in $K(1)^*$ -theory. Since $K(1)^*(-)$ holds Kunneth formula, we can defined the coalgebra map μ^* on also $E(K(1))_r^{*,*}$ considering the fibering

$$G \times G \rightarrow G \times G/T \rightarrow G \times BT.$$

Each element $x_1 y^{p-1}, x_i$ are primitive in $K(1)^*(G)$, but

$$\mu^* d_r(x_i) = d_r(\mu^* x_i) = d_r(x_i \otimes 1 + 1 \otimes x_i) = 1 \otimes d_r(x_i).$$

The differentials images are written as $d_r(x_i) = \sum f(x)g(t)$ for $g(t) \in K(1)^* P(t)'$ and $f(x) \in \Lambda(x_1 y^{p-1}, x_2, \dots, x_l)$. However if $|f(x)| > 0$, then $\mu^* d_r(x_i) \neq 1 \otimes d_r(x_i)$. Thus we get

$$(*, *) \quad d_r(x_i), d_r(x_1 y^{p-1}) \in K(1)^* P(t)'.$$

Now we study the $k(1)^*$ -theory. Recall $E(k(1))_2^{*,*} \cong k(1)^*(G \otimes P(t)')$. Suppose $d_2(x) \neq 0$.

(1) The case $d_2(x)$ is v_1 -torsion.

This case $d_2(x)$ is a $k(1)^*$ -module generator, since $E(k(1))_2^{*,*}$ does not has higher v_1 -torsion. Hence $d_2(x) \neq 0$ also in $E(H\mathbb{Z}/p)_2^{*,*}$ because the Thom map $E(k(1))_2^{*,*}/(v_1) \rightarrow E(H\mathbb{Z}/p)_2^{*,*}$ is injective. This is a contradiction to $d_2(x) = 0 \in E(H\mathbb{Z}/p)_2^{*,*}$.

(2) The case $d_2(x)$ is in v_1 -image.

Since also $E(k(1))_2^{*,*}$ has no higher v_1 -torsion, we see $d_2(x) \neq 0$ also in $E(K(1))_2^{*,*}$. Let us write $d_2(x) = \sum f(x)g(t)$. Then by dimensional reason, it is necessary $|f(x)| > 0$. This is a contradiction to $(*,*)$. Thus we get $d_2(x) = 0$.

Next consider $d_4(x)$. By the same reasons as (1),(2), we see $d_4(x_i) = 0$ for $i \neq 1$. By the reason as (1), we see $d_4(x_1 y^{p-1}) = b_1 y^{p-1}$. Since $b_1 y^{p-1}$ is v_1 -torsion, the element $v_1 x_1 y^{p-1}$ is a $k(1)^*$ -module generator in $E(k(1))_5^{*,*}$. The differentials are defined as some boundary maps, we get

$$\partial(x_1 y^{p-1}) = b_1 y^{p-1} \mod(\text{some filtration}),$$

which implies that

$$\partial(v_1 x_1 y^{p-1}) = b_1(v_1 y) y^{p-2} = b_1^2 y^{p-2} \mod(\text{some filtration})$$

where we take y such that $v_1 y = b_1 \in k(1)^*(G/T)$. This follows the fact

$$d_{2|b_1|}(v_1 x_1 y^{p-1}) = b_1^2 y^{p-2} \mod(\text{some filtration}).$$

By induction on r for $d_r(x_i)$, we can prove $d_{|b_i|}(x_i) = b_i$ for $i \neq 1$, and $d_{s|b_1|}(v_1^{s-1} x_1 y^{p-1}) = b_1^s y^{p-s}$ for $1 \leq s \leq p$ by the arguments similar to (1),(2). For $K(1)^*$ -theory we can prove that $d_{p|b_1|}(x_2 y^{p-1}) = v_1^{-(p-1)} b_1^p$ and the result of the proposition. $\square$

We consider the integral cohomology $h = H\mathbb{Z}$. Since these groups G has no higher p -torsion, we have

$$H^*(G; \mathbb{Z}) \cong (P(y)^+/(py) \oplus \Lambda(x_2 y^{p-1})) \otimes \Lambda(x_1, x_3, \dots, x_l),$$

$$H^*(G; \mathbb{Q}) \cong \mathbb{Q} \otimes \Lambda(x_2 y^{p-1}, x_1, x_3, \dots, x_l).$$

By the arguments similar to the case Morava K -theories, but more easily, we get the following differentials.

The differentials in $E(\mathbb{Q})_r^{*,*}$ are $d_{b_i|}(x_i) = b_i$ for $i \neq 2$, and $d_{p|b_2|}(x_2 y^{p-1}) = b_2^p$.

The differentials in $E(H\mathbb{Z})_r^{*,*}$ are $d_{s|b_2|}(p^{s-1} x_2 y^{p-1}) = b_2^s y^{p-s}$ and $d_{|x_i|+1}(x_i) = b_i$ for $i \neq 2$.

Proposition 6.2. *Let G be a simple Lie group in Case I. Then we get isomorphism*

$$E(H\mathbb{Z})_\infty^{*,*} \cong P(y)/(py) \otimes P(t)'/(b_1, b_3, \dots, b_l, b_2^s y^{p-s} | 1 \leq s \leq p),$$

$$E(H\mathbb{Q})_\infty^{*,*} \cong P(t)'/(b_1, b_2^p, b_3, \dots, b_l)$$

Now we recall the $BP\langle 1 \rangle^*$ theory with the coefficient $BP\langle 1 \rangle^* = \mathbb{Z}_{(p)}[v_1]$ so that $BP\langle 1 \rangle^*/p = k(1)^*$. For G in Case I, it is known that $BP\langle 1 \rangle^*(G) \cong BP\langle 1 \rangle^* \otimes_{BP^*} BP^*(X)$. (See Case I in §5). The rational BP -theory is immediate

$$BP^*(G; \mathbb{Q}) \cong BP^* \otimes \mathbb{Q} \otimes \Lambda(x'_1, w_2) \otimes \Lambda(x_3, \dots, x_l).$$

The differentials in $E(BP \otimes \mathbb{Q})_r^{*,*}$ are given by $d_4 x'_1 = p b_1$, $d w_2 = b_2^p$. Here we use notations that x'_1, w_1, w_2 correspond elements $p x_1, x_1 y^{p-1}, x_2 y^{p-1}$ respectively, in the spectral sequence. (Recall also Case I in §5.)

Theorem 6.3. *Let G be a simple Lie group in Case I and $E(BP\langle 1 \rangle)_r^{*,*}$ be the spectral sequence (6.1) for $h = BP\langle 1 \rangle$. Then we have the isomorphism*

$$E(BP\langle 1 \rangle)_\infty^{*,*} \cong BP\langle 1 \rangle^* \otimes P(y)/(py, v_1 y) \otimes P(t)'/(p b_1, b_3, \dots, b_l, b_1^s b_2^r y^{p-s-r} | 1 \leq s+r \leq p)$$

Proof. Compare this spectral sequence to those of the other theories $k(1), H\mathbb{Z}, BP\langle 1 \rangle \otimes \mathbb{Q}$. By induction on r of $d_r(x_i)$, we show in $E(BP\langle 1 \rangle)_{r,*}^*$, $d_{|b_i|}(x_i) = b_i$ for $i \neq 1, 2$ and $d_4(x'_1) = pb_1$, and moreover

$$\begin{aligned} d_{s|b_1|+r|b_2|}(p^r v_1^{s-1} w_1) &= b_1^s b_2^r y^{p-s-r} \text{ for } 1 \leq s+r \leq p \\ d_{r|b_2|}(p^{r-1} w_2) &= b_2^r y^{p-r} \text{ for } 1 \leq r \leq p. \end{aligned}$$

Thus we can prove this theorem. $\square$

The cohomology $BP^*(G/T)$ is still given in (6.2), while it does not give the information of $i^*BP^*(BT)$. However the above theorem says more strong facts. For example

$$p^s v_1^r y^{p-s-r-1} \notin \text{Im}(i^*(BP^*(BT))) \text{ mod}(v_2, \dots), \quad \text{if } s+r < p-1$$

while of course it is in $\text{Ideal}(i^*(\tilde{BP}^*(BT)))$. Hence we can know that these facts also hold for $\Omega^*(-)$ and $MGL^{2*,*}(-)$ although they have not the Serre spectral sequences.

7. MOTIVIC THEORIES

In this section, we study motivic generalized theories defined by Voevodsky [Vo1],[Vo2]. When X has a cellular decomposition, we can show

$$h^{*,*}(X) \cong h^{*,*}(pt) \otimes H^{*,*}(X),$$

for $h^{*,*}(-)$ the generalized motivic cohomology, e.g., $h = H\mathbb{Z}/p, MGL$. The spaces $X = G/T, BT$ have this property.

We first study the mod p motivic cohomology theory. It is known that there is an element $\tau \in H^{0,1}(X; \mathbb{Z}/p)$ such that $t_{\mathbb{Z}/p}(\tau) = 1$ where $t_{\mathbb{Z}/p} : H^{*,*} \rightarrow H^*(X; \mathbb{Z}/p)$ is the realization map. Recall also $H^*(G; \mathbb{Z}/p) \cong P(y)/p \otimes \Lambda(x_1, \dots, x_l)$. (when $p = 2$, we let $y_i = x_i^2$.)

Theorem 7.1. *Let G be a simply connected Lie group. Giving bidegree to $H^*(G; \mathbb{Z}/p)$ by $\deg(y_i) = (|y_i|, |y_i|/2)$ and $\deg(x_i) = (|x_i|, (|x_i|+1)/2)$, we have the injection*

$$P(y)/p \otimes \Lambda(x_1, \dots, x_l) \otimes \mathbb{Z}/p[\tau] \subset H^{*,*}(G_{\mathbb{C}}; \mathbb{Z}/p)$$

such that for $p = \text{odd}$, it is a ring monomorphism, and for $p = 2$, it is a ring monomorphism to $gr H^{,*}(G_{\mathbb{C}}; \mathbb{Z}/2)$ and $x_i^2 - y_i \tau \in \text{Ker}(t_{\mathbb{Z}/p})$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} H^{2*,*}(G_{\mathbb{C}}/T_{\mathbb{C}}; \mathbb{Z}/p) & \longleftarrow & H^{2*,*}(G_{\mathbb{C}}/T_{\mathbb{C}}, G_{\mathbb{C}}; \mathbb{Z}/p) & \xleftarrow{\delta} & H^{2*,*}(G_{\mathbb{C}}; \mathbb{Z}/p) \\ t_{\mathbb{Z}/p} \downarrow & & t_{\mathbb{Z}/p} \downarrow & & t_{\mathbb{Z}/p} \downarrow \\ H^*(G/T; \mathbb{Z}/p) & \longleftarrow & H^*(G/T, G; \mathbb{Z}/p) & \xleftarrow{\delta} & H^*(G; \mathbb{Z}/p) \end{array}$$

where rows are exact. The cohomology $H^*(G/T, G; \mathbb{Z}/p)$ is computed by using the spectral sequence

$$E_2^{*,*}(G/T, G) = (E_2^{*,*} - E_2^{0,*}) \implies H^*(G/T, G; \mathbb{Z}/p)$$

where $E_2^{*,*}$ is the spectral sequence converging to $H^*(G/T; \mathbb{Z}/p)$.

Since the differential in $E_r^{*,*}$ are given $d_{|b_i|}(x_i) = b_i$, we easily seen from the definition of differential, $\delta(x_j) = b_j \text{ mod}(E_{\infty}^{|b_i|+1,*})$ in $H^*(G/T; \mathbb{Z}/p)$. Here note that $t_1, \dots, t_l$ are in $H^*(G/T, G; \mathbb{Z}/p)$, while y_i is not in $H^*(G/T, G; \mathbb{Z}/p)$ because $t_i = 0 \in H^*(G; \mathbb{Z}/p)$. Moreover $\delta(x_i) = b_i$ implies $b_i \neq 0 \in H^*(G/T, G; \mathbb{Z}/p)$.

Similarly $t_1, \dots, t_l$ are in $H^{2*,*}(G/T, G_{\mathbb{C}}; \mathbb{Z}/p)$, since the corresponding elements in $H^{2*,*}(G/T; \mathbb{Z}/p)$ goes to zero in $H^{2*,*}(G_{\mathbb{C}}; \mathbb{Z}/p)$. Moreover $b_i \neq 0 \in H^{2*,*}(G/T, G_{\mathbb{C}}; \mathbb{Z}/p)$ since it is nonzero in $H^*(G/T, G; \mathbb{Z}/p)$. Hence there is the element $x_i \in H^{2*-1,*}(G_{\mathbb{C}}; \mathbb{Z}/p)$ such that $\delta(x_i) = b_i$.

Since $t_{\mathbb{Z}/p}(\tau) = 1$, we also see $t_{\mathbb{Z}/p}(y_i\tau) = t_{\mathbb{Z}/p}(x_i^2)$ for $p = 2$. $\square$

The injection in the above theorem seems an isomorphism but I can not prove it now.

Other types of motivic cohomology seem quite complicated and we consider only group G in Case I. By using arguments similar to the mod p case, we have ;

Proposition 7.2. *Let G be a simple Lie group in Case I. Giving the bidegree by $\deg(y) = (|y|, |y|/2)$, $\deg(x) = (|x|, (|x| + 1)/2)$ for $x = x_i, x_2y^{p-1}$, we have the ring monomorphism*

$$H^*(G) \cong (\mathbb{Z}[y]/(py, y^p) \oplus \mathbb{Z}\{x_2y^{p-1}\}) \otimes \Lambda(x_1, x_3, \dots, x_l) \subset H^{*,*}(G_{\mathbb{C}}).$$

Recall the motivic cobordism $MGL^{*,*}(G_{\mathbb{C}})$ defined by Voevodsky. Let $MGL/p^{*,*}(-)$ the motivic cohomology theory defined by the cofiber sequence of stable $\mathbb{A}^1$ -homotopy category

$$MGL \xrightarrow{p} MGL \longrightarrow MGL/p$$

so that we have the long exact sequence

$$\longrightarrow MGL^{*,*}(X) \xrightarrow{p} MGL^{*,*}(X) \longrightarrow MGL/p^{*,*}(X) \longrightarrow \dots$$

Theorem 7.3. *Let G be a simple Lie group in Case I. Giving the bidegree by $\deg(y) = (|y|, |y|/2)$, $\deg(x) = (|x|, (|x| + 1)/2)$ for $x = x_i, x_1y^{p-1}$, we have the MU^*/p -algebra injection*

$$MU/p^*(G) \cong (MU^*/p \otimes (\mathbb{Z}[y]/(y^p, v_1y) \oplus \mathbb{Z}\{x_1y^{p-1}\}) \otimes \Lambda(x_2, \dots, x_l) \subset MGL/p^{*,*}(G_{\mathbb{C}}).$$

Proof. The proof is quite similar to the case of ordinary mod p motivic cohomology. Here we use the realization map

$$MGL/p^{*,*}(G_{\mathbb{C}}) \xrightarrow{t_{MGL}} MU^*(G; \mathbb{Z}/p) \rightarrow BP^*(G; \mathbb{Z}/p) \rightarrow k(1)^*(G)$$

and the isomorphism $BP^*(G; \mathbb{Z}/p) \cong BP^* \otimes_{k(1)^*} k(1)^*(G)$. $\square$

8. CLASSIFYING SPACES

Here we consider the other types of algebraic spaces. For an algebraic group G (not assumed here the connectness) over $\mathbb{C}$, we can construct the classifying space BG as a limit of smooth algebraic varieties. (Indeed we still considered BT .) B.Totaro ([To1],[To2]) first studied the Chow ring of BG . He computed most important cases and conjectured $CH^*(BG) \cong MU^*(BG) \otimes_{MU^*} \mathbb{Z}$. He first found the modified cycle map ([To1])

$$(7.1) \quad \bar{c}l : CH^*(X) \rightarrow MU^*(X) \otimes_{MU^*} \mathbb{Z}$$

such that its composition with the Thom map is the usual cycle map. Now this map is extended by Levine and Morel as the map

$$(7.2) \quad t_{MU} \rho_{MGL} : \Omega^*(X) \rightarrow MU^*(X).$$

By results of Totaro, the map (7.2) are isomorphism for products of $X = BGL_n, BO(n), BZ/p^n$, and hence groups whose p -Sylow subgroups are abelian p -groups ([To1],[To2]). Moreover it is known that (7.1) is epic, for example, $G = PGL_3, SO(4), Spin(m), m \leq 9, G_2$ and the extraspecial p groups of order p^3 ([Pa],[Ve],[S-Y],[Y4]). Hence all these cases (7. 2) are epic. Thus (7.2) seems to be isomorphism for each $X = BG$.

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DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION, IBARAKI UNIVERSITY, MITO, IBARAKI, JAPAN
E-mail address: yagita@mito.ipc.ibaraki.ac.jp