ALGEBRAIC ORIENTED COHOMOLOGY THEORIES

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ABSTRACT. For every smooth projective variety over an infinite field F we define its fundamental polynomial in $\mathbb{Z}[\mathbf{b}] = \mathbb{Z}[b_1, b_2, \dots]$ and prove that the fundamental polynomials generate the Lazard ring $\text{Laz} \subset \mathbb{Z}[\mathbf{b}]$. Using description of invariant prime ideals in Laz , due to Landweber, we assign to every smooth projective variety X the numbers $n_p(X)$ for every prime integer p . Inequality $n_p(Y) > n_p(X)$ for some prime p is an obstruction for existence of a morphism $Y \rightarrow X$ over F .

1. INTRODUCTION

Let $\mathbf{Sm}(F)$ be the category of smooth quasi-projective varieties over a field F . M. Levine and F. Morel have defined in [7] an oriented cohomology theory over F as a contravariant functor A^* from the category $\mathbf{Sm}(F)$ to the category of graded commutative rings satisfying certain properties (see section 2). Examples of the oriented cohomology theories are K^* given by the Grothendieck rings of varieties in $\mathbf{Sm}(F)$ (Example 2.2) and H^* given by the Chow rings (Example 2.1).

It is proved in [7] that if $\text{char } F = 0$ (resolution of singularities is used) then there exists a universal oriented *algebraic cobordism* cohomology theory Ω^* . For every oriented cohomology theory A^* there is unique morphism of cohomology theories $\Omega^* \rightarrow A^*$ commuting with the push-forward homomorphisms. For a variety $X \in \mathbf{Sm}(F)$ the group $\Omega^*(X)$ is generated by the classes $[f]$ corresponding to projective morphisms $f : Y \rightarrow X$ in $\mathbf{Sm}(F)$. The homomorphism $\Omega^*(X) \rightarrow A^*(X)$ takes the class $[f]$ to $f_A(1_Y)$, where f_A is the push-forward homomorphism in A^* . Thus, the image of the morphism $\Omega^*(X) \rightarrow A^*(X)$, which we denote by $A_c^*(X)$, can be defined just in terms of the theory A : the group $A_c^*(X)$ is generated by the elements $f_A(1_Y)$ for all projective morphisms $f : Y \rightarrow X$.

To every oriented cohomology theory A one has associated a commutative formal group law Φ^A over the *coefficient ring* $A^*(\text{pt})$. The formal group law Φ^Ω is the universal one and the coefficient ring $\Omega^*(\text{pt})$ is the Lazard ring.

In the present paper we consider oriented cohomology theories on $\mathbf{Sm}(F)$ for infinite fields F of arbitrary characteristic and don't refer to the problem of resolution of singularities and existence of the cobordism theory. The idea is to consider "large" oriented cohomology theories A^* such that the natural

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homomorphism $\Omega^*(X) \rightarrow A^*(X)$ is injective at least for $X = \text{pt}$ and work inside A^* instead of Ω^* .

How to construct “large” cohomology theories? In section 4 we consider an operation (we call it *tilde operation*), which assigns to every oriented cohomology theory A^* another theory $\tilde{A}^*$ defined by

$$\tilde{A}^*(X) = A^*(X) \otimes \mathbb{Z}[\mathbf{b}] = A^*(X)[\mathbf{b}],$$

where $\mathbb{Z}[\mathbf{b}] = \mathbb{Z}[b_1, b_2, \dots]$ is the polynomial ring in infinitely many variables. We define the push-forward homomorphisms in $\tilde{A}^*$ in such a way that the inverse Todd genus of the natural embedding $A^* \hookrightarrow \tilde{A}^*$ is the universal one (with the coefficient being the indeterminates b_i). We prove that the two theories $\tilde{H}^*$ and $\tilde{K}^*$ are large enough so that the coefficient rings $\tilde{H}_c^*(\text{pt})$ and $\tilde{K}_c^*(\text{pt})$ are both isomorphic to the Lazard ring. In sections 6 and 7 we follow closely the method of [10].

For every projective variety $X \in \mathbf{Sm}(F)$ we define the *fundamental polynomial* $\mathbf{F}_X^H \in \mathbb{Z}[\mathbf{b}]$ and prove that for any infinite field F the fundamental polynomials of all projective $X \in \mathbf{Sm}(F)$ generate the same ring - the Lazard ring Laz considered as a subring of $\mathbb{Z}[\mathbf{b}]$. The fundamental polynomials $\mathbf{F}_X^H$ do not change under field extensions (and therefore can be computed over an algebraic closure of F); nevertheless, they keep track of an arithmetic information on X . Namely, all the coefficients of $\mathbf{F}_X^H$ are divisible by the greatest common divisor of the degrees $[F(x) : F]$ of all closed points of X . For example, existence of division algebras of a given dimension over an extension of F explains the fact that the fundamental polynomial of the projective space $\mathbb{P}_F^n$ is divisible by $n + 1$ in $\mathbb{Z}[\mathbf{b}]$ (Example 3.7), the well known fact in topology (see [9, Ch. VII]).

In section 9 we prove that characteristic classes of vector bundles over $X \in \mathbf{Sm}(F)$ take values in the subgroup $A_c^*(X) \subset A^*(X)$. We use this result in section 10 where we study the Landweber-Novikov operations on Laz . In section 11 we introduce ideals $I(X) \subset \text{Laz}$ for every projective variety $X \in \mathbf{Sm}(F)$, consisting of the fundamental polynomials of all projective varieties $Y \in \mathbf{Sm}(F)$ such that there is a morphism $Y \rightarrow X$ (defined over F). We prove that the ideal $I(X)$ is invariant under the Landweber-Novikov operations and so are all the associated prime ideals. Invariant prime ideals were described by Landweber in [5]. Based on this description one can associate to every projective variety $X \in \mathbf{Sm}(F)$ and every prime integer p an integer $n_p(X) \in \{0, 1, \dots, \infty\}$. Inequality $n_p(Y) > n_p(X)$ for some prime p is an obstruction for existence of a morphism $Y \rightarrow X$ defined over F .

Although the paper is purely algebraic, the most of the constructions are borrowed from topology. The class $[-T_X] \in K_0(X)$ of the tangent bundle T_X over X is a replacement for the stable normal bundle of X . The tilde operation is analogous to the wedge product with the Thom spectrum MU . The embedding of the Lazard ring into $\mathbb{Z}[\mathbf{b}]$ is the Hurewicz homomorphism

$\pi_*(MU) \rightarrow H_*(MU)$. The Landweber-Novikov operations are induced by those on the spectrum MU .

2. DEFINITION OF AN ORIENTED COHOMOLOGY THEORY

Let F be a field, $\mathbf{Sm}(F)$ the category of smooth quasi-projective varieties over F . Let A^* be a functor from $\mathbf{Sm}(F)^{op}$ to the category $\mathbf{GrRings}$ of $\mathbb{Z}$ -graded commutative rings. For a morphism $f : Y \rightarrow X$ in $\mathbf{Sm}(F)$ the (pull-back) ring homomorphism $A^*(f)$ is denoted by f^A .

An *oriented cohomology theory over F* (see [7]) is a functor

$$A^* : \mathbf{Sm}(F)^{op} \rightarrow \mathbf{GrRings}$$

together with a graded (push-forward) group homomorphism

$$f_A : A^*(Y) \longrightarrow A^{*+d}(X)$$

for every projective morphism $f : Y \rightarrow X$ in $\mathbf{Sm}(F)$ of pure codimension d , satisfying the following:

- (i) For a pair of projective morphisms $f : Y \rightarrow X$ and $g : Z \rightarrow Y$, one has $(f \circ g)_A = f_A \circ g_A$.
- (ii) Let $E \rightarrow X$ be a vector bundle over $X \in \mathbf{Sm}(F)$ of rank r , $\mathbb{P}(E) \rightarrow X$ associated projective bundle. Then $A^*(\mathbb{P}(E))$ is a free $A^*(X)$ -module with basis $1, \xi, \xi^2, \dots, \xi^{r-1}$, where $\xi = s^A s_A(1_{\mathbb{P}(E)})$, and s is the zero section of the tautological line bundle over $\mathbb{P}(E)$.
- (iii) Let

$$(1) \quad \begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ h \downarrow & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array}$$

be a transverse Cartesian square in $\mathbf{Sm}(F)$ with f a projective morphism, i.e. the sequence of tangent bundles over Y'

$$0 \longrightarrow T_{Y'} \xrightarrow{df' + dh} f'^* T_{X'} \oplus h^* T_Y \xrightarrow{dg - df} (fh)^* T_X \longrightarrow 0$$

is exact. Then $f'_A \circ h^A = g^A \circ f_A$.

- (iv) Let $p : V \rightarrow X$ be an affine bundle (a torsor for a vector bundle over X). Then $p^A : A^*(X) \rightarrow A^*(V)$ is an isomorphism.

- (v) (Projection formula) Let $f : Y \rightarrow X$ be a projective morphism in $\mathbf{Sm}(F)$. Then for every $a \in A^*(X)$ and $b \in A^*(Y)$, $f_A(b \cdot f^A(a)) = f_A(b) \cdot a$.

The ring $A^*(\text{pt})$, where $\text{pt} = \text{Spec}(F)$, is called the *coefficient ring of A^** . For every $X \in \mathbf{Sm}(F)$, $A^*(X)$ is an algebra over $A^*(\text{pt})$.

Example 2.1. ([7, Ex. 1.2]) The *Chow cohomology theory* H^* assigns to every variety $X \in \mathbf{Sm}(F)$ the Chow ring $H^*(X) = \mathrm{CH}^*(X)$. The push-forward and pull-back homomorphisms are defined in [2]. The coefficient ring $\mathrm{CH}^*(\mathrm{pt})$ is equal to $\mathbb{Z}$.

Example 2.2. ([7, Ex. 1.3]) The *K-theory* assigns to every variety $X \in \mathbf{Sm}(F)$ the Laurent polynomial ring $K^*(X) = K_0(X)[t, t^{-1}]$ graded by $\deg(t) = -1$, i.e. $K^i(X) = K_0(X)t^{-i}$. If $f : Y \rightarrow X$ is a projective morphism of pure codimension d , then for every $a \in K^*(Y)$, $f_*(at^{-i}) = f_*(a)t^{-i-d}$, where f_* is the push-forward homomorphism in algebraic K-theory. We have also $K^*(\mathrm{pt}) = \mathbb{Z}[t, t^{-1}]$.

Let $p : X \rightarrow \mathrm{pt}$ be the structure morphism. If X is projective of dimension d , we define the *fundamental class* $[X]^A$ of X in the theory A as the element

$$[X]^A = p_A(1_X) \in A^{-d}(\mathrm{pt}).$$

For example, $[\mathrm{pt}]^A = 1$, $[X]^H = 0$ if $d > 0$ and $[X]^K = td(X)t^d$, where $td(X) = p_*([\mathcal{O}_X]) \in \mathbb{Z}$ is the Todd number of X [2, Example 15.2.13].

Proposition 2.3. *Let X and Y be projective varieties in $\mathbf{Sm}(F)$. Then $[X \times Y]^A = [X]^A \cdot [Y]^A$.*

Proof. Consider Cartesian transverse square

$$\begin{array}{ccc} X \times Y & \xrightarrow{p} & X \\ q \downarrow & & \downarrow s \\ Y & \xrightarrow{r} & \mathrm{pt}. \end{array}$$

We have

$$\begin{aligned} [X \times Y]^A &= (sp)_A(1_{X \times Y}) \\ &= (s_A p_A q^A)(1_Y) \quad (\text{property (iii)}) \\ &= (s_A s^A r_A)(1_Y) \quad (\text{projection formula}) \\ &= s_A(1_X) \cdot r_A(1_Y) \\ &= [X]^A \cdot [Y]^A. \end{aligned}$$

□

For every smooth variety X consider the graded subgroup $A_c^*(X)$ in $A^*(X)$ generated by the elements $f_A(1_Y)$ for all projective morphisms $f : Y \rightarrow X$ in $\mathbf{Sm}(F)$. Clearly, $A_c^i(X) = 0$ if $i > \dim(X)$. For a projective morphism $g : X \rightarrow X'$ the push-forward map g_A takes $A_c^*(X)$ to $A_c^*(X')$.

The subgroup $A_c^*(\mathrm{pt}) \subset A^*(\mathrm{pt})$ is generated by the fundamental classes $[X]^A$ for all smooth projective varieties X . Proposition 2.3 shows that $A_c^*(\mathrm{pt})$ is a subring in $A^*(\mathrm{pt})$.

Example 2.4. $H_c^*(\text{pt}) = H^*(\text{pt}) = \mathbb{Z}$, $K_c^*(\text{pt}) = \mathbb{Z}[t]$.

3. CHERN CLASSES

Let $p : L \rightarrow X$ be a line bundle over $X \in \mathbf{Sm}(F)$. We define the first Chern class of L in an oriented cohomology theory A^* over F by

$$c_1^A(L) = s^A s_A(1_X) \in A^1(X),$$

where $s : X \rightarrow L$ is the zero section of p (see [7]). Since $p \circ t = \text{id}_X$ for every section t of p , we have $t^A = (p^A)^{-1}$ (property (iv)). Hence,

$$c_1^A(L) = (p^A)^{-1} s_A(1_X).$$

Example 3.1. The first Chern class of a vector bundle $E \rightarrow X$ in K -theory is defined by

$$c_1^K(E) = (\text{rank}(E) - [E^\vee])t^{-1} \in K_0(X)t^{-1} = K^1(X).$$

Proposition 3.2. *Let $p : L \rightarrow X$ be a line bundle over $X \in \mathbf{Sm}(F)$ and let $i : Y \hookrightarrow X$ be the zero subscheme of a section t of p . If Y is a smooth divisor in X , then $i_A(1_Y) = c_1^A(L) \in A^1(X)$.*

Proof. The diagram

$$\begin{array}{ccc} Y & \xrightarrow{i} & X \\ \downarrow i & & \downarrow t \\ X & \xrightarrow{s} & L \end{array}$$

where s is the zero section of p , is transverse. Hence,

$$i_A(1_Y) = i_A i^A(1_X) = t^A s_A(1_X) = (p^A)^{-1} s_A(1_X) = c_1^A(L).$$

□

The standard method by Grothendieck (see [7]) gives Chern classes $c_i^A(E) \in A^i(X)$ for every vector bundle $p : E \rightarrow X$ of rank r . They satisfy the equation

$$\sum_{i=0}^r (-1)^i p^A(c_i^A(E)) \xi^{r-i} = 0 \in A^r(\mathbb{P}(E)),$$

where ξ is the first Chern class of the tautological line bundle over $\mathbb{P}(E)$.

A *partition* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ is a sequence of integers (possibly empty) $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_k > 0$. The *degree* of α is the integer

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_k.$$

The integer k is called the *length* $l(\alpha)$ of the partition α . Denote by $p(d)$ the number of all partitions of degree d .

Consider the polynomial ring $\mathbb{Z}[b_1, b_2, \dots] = \mathbb{Z}[\mathbf{b}]$ in infinitely many variables $b_1, b_2, \dots$ as a graded ring with $\deg b_i = i$. For every partition α set

$$b_\alpha = b_{\alpha_1} b_{\alpha_2} \dots b_{\alpha_k}.$$

The monomials b_α form a basis of the polynomial ring over $\mathbb{Z}$, and more precisely, the b_α with $|\alpha| = d$ form a basis of the d -graded component $\mathbb{Z}[\mathbf{b}]_d$. Thus, $\mathbb{Z}[\mathbf{b}]_d$ is a free abelian group of rank $p(d)$.

Consider another polynomial ring $\mathbb{Z}[c_1, c_2, \dots] = \mathbb{Z}[\mathbf{c}]$ with the similar grading $\deg c_i = i$. The elements of $\mathbb{Z}[\mathbf{c}]$ we call the *characteristic classes* and the c_n - the *Chern classes*.

For any partition α we define the “smallest” symmetric polynomial

$$P_\alpha(x_1, x_2, \dots) = \sum_{(i_1, i_2, \dots, i_k)} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_k}^{\alpha_k} = Q_\alpha(\sigma_1, \sigma_2, \dots),$$

containing the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_k^{\alpha_k}$, where the σ_i are the standard symmetric functions, and set

$$c_\alpha = Q_\alpha(c_1, c_2, \dots).$$

For example, $c_n = c_{(1,1,\dots,1)}$ (n units). The characteristic classes c_α with $|\alpha| = d$ form a basis of $\mathbb{Z}[\mathbf{c}]_d$.

Let A^* be an oriented cohomology theory over a field F . For every element (*characteristic class*) $c \in \mathbb{Z}[\mathbf{c}]$ and every vector bundle E over a variety $X \in \mathbf{Sm}(F)$ there is a well defined class $c^A(E) \in A^*(X)$. In particular, for every partition α there are *generalized Chern classes*

$$c_\alpha^A(E) \in A^{|\alpha|}(X).$$

We define the *characteristic polynomial of E in the theory A^** by the formula

$$\mathbf{P}^A(E) = \sum_{\alpha} c_\alpha^A(E) b_\alpha \in A^*(X)[\mathbf{b}].$$

Example 3.3. If L is a line bundle, then $\mathbf{P}^A(L) = \sum_{i \geq 0} c_1^A(L)^i b_i$.

Assume that a vector bundle $E \rightarrow X$ has a filtration with line factors $L_1, L_2, \dots, L_r$. Then it follows from definition of the generalized Chern classes that

$$\mathbf{P}^A(E) = \mathbf{P}^A(L_1) \cdot \mathbf{P}^A(L_2) \cdot \dots \cdot \mathbf{P}^A(L_r).$$

Hence, by the splitting principle, for an exact sequence of vector bundles $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$ over X ,

$$(2) \quad \mathbf{P}^A(E) = \mathbf{P}^A(E') \cdot \mathbf{P}^A(E'').$$

The value of the i -th Chern class $c_i(E)$ is nilpotent for $i > 0$ (see [7]), hence for every $\alpha \neq \emptyset$, the class $c_\alpha^A(E)$ is also nilpotent. The constant term of the polynomial $\mathbf{P}^A(E)$ is equal to 1, so that the polynomial $\mathbf{P}^A(E)$ is invertible in the polynomial ring $A^*(X)[\mathbf{b}]$. Thus, there is well defined group homomorphism

$$\mathbf{P}^A : K_0(X) \longrightarrow A^*(X)[\mathbf{b}]^\times, \quad [E] \mapsto \mathbf{P}^A(E).$$

For a variety $X \in \mathbf{Sm}(F)$ we define the *characteristic polynomial of X in the theory A^** :

$$\mathbf{P}_X^A = \mathbf{P}^A(T_X)^{-1} = \mathbf{P}^A(-T_X) \in A^*(X)[\mathbf{b}],$$

where T_X is the tangent bundle of X .

Assume that X is projective. Let $p : X \rightarrow \text{pt}$ be the structure morphism. The polynomial

$$\mathbf{F}_X^A = p_A \mathbf{P}_X^A = \sum_{\alpha} p_A c_{\alpha}(-T_X) b_{\alpha} \in A^*(\text{pt})[\mathbf{b}]$$

is called the *fundamental polynomial of X in the theory A^** . The coefficients of the polynomial $\mathbf{F}_X^A$, the elements $p_A c_{\alpha}(-T_X) \in A^*(\text{pt})$, are called the *characteristic numbers of X in the theory A^** . Clearly, the fundamental class $[X]^A$ is the constant term of the fundamental polynomial $\mathbf{F}_X^A$.

Example 3.4. Let $X \in \mathbf{Sm}(F)$ be a variety of dimension d . Then the polynomial $\mathbf{F}_X^H \in \mathbb{Z}[\mathbf{b}]$ is either zero or homogeneous of degree d . The class of the tangent bundle of the projective space $\mathbb{P}_F^n$ is equal to $[L_{can}]^{n+1} - 1$, where L_{can} is the canonical bundle over $\mathbb{P}_F^n$. Since $c_1^H(L_{can})^n$ is the class of a rational point, the polynomial $\mathbf{F}_{\mathbb{P}^n}^H$ is equal to the degree n part of the power series $(1 + b_1 + b_2 + \dots)^{-n-1}$. In particular, the b_n -coefficient of $\mathbf{F}_{\mathbb{P}^n}^H$ equals $-(n+1)$. For example,

$$\begin{aligned} \mathbf{F}_{\mathbb{P}^1}^H &= -2b_1, \\ \mathbf{F}_{\mathbb{P}^2}^H &= -3b_2 + 6b_1^2. \end{aligned}$$

Example 3.5. Note that for every vector bundle E over a variety $X \in \mathbf{Sm}(F)$,

$$c_{\alpha}^K(E) \in K_0(X)^{(|\alpha|)} t^{-|\alpha|},$$

where $K_0(X)^{(i)}$ is the i -th term of the topological filtration on $K_0(X)$. Therefore, $c_{\alpha}^K(E) = 0$ if $|\alpha| > d = \dim X$ (cf. Corollary 9.6). Hence, $\mathbf{F}_X^K \in \mathbb{Z}[t, \mathbf{b}]$ is a homogeneous polynomial (with t of degree 1). The t^d -coefficient of $\mathbf{F}_X^K$ is the Todd number of X . For example,

$$\begin{aligned} \mathbf{F}_{\mathbb{P}^1}^K &= t - 2b_1, \\ \mathbf{F}_{\mathbb{P}^2}^K &= t^2 - 3tb_1 - 3b_2 + 6b_1^2. \end{aligned}$$

We will prove (Proposition 6.1) that $\mathbf{F}_X^K|_{t=0} = \mathbf{F}_X^H$ for every $X \in \mathbf{Sm}(F)$.

Thus, every projective variety $X \in \mathbf{Sm}(F)$ has the class $\mathbf{F}_X^H$ in $\mathbb{Z}[\mathbf{b}]$. Clearly, the class does not change under field extensions: for every field extension E/F the varieties X and $X_E = X \times_{\text{Spec } F} \text{Spec } E$ have the same class in Laz. Hence, if X and Y are twisted forms of each other (if they are isomorphic over a separable closure of F), then $\mathbf{F}_X^H = \mathbf{F}_Y^H$.

For a variety $X \in \mathbf{Sm}(F)$ denote by n_X the gcd of $\deg(x) = [F(x) : F]$ over all closed points $x \in X$. By the very definition, for a projective variety X , all the coefficients of $\mathbf{F}_X^H$ (the characteristic numbers) are divisible by n_X . We have proved

Proposition 3.6. 1. *For every projective variety $X \in \mathbf{Sm}(F)$, the polynomial $\mathbf{F}_X^H$ is divisible by n_Y for every twisted form Y of X_E over a field extension E/F .*

2. *Let n be the gcd of all the coefficients of $\mathbf{F}_X^H$ for a projective variety $X \in \mathbf{Sm}(F)$. Then X has a zero-cycle of degree n .*

Example 3.7. For every $d \in \mathbb{N}$ and a field F there is a field extension E/F and a division algebra A over E of dimension $(d+1)^2$. Let Y be the Severi-Brauer variety over E corresponding to A (see [4]). The variety Y is a twisted form of the projective space $\mathbb{P}_F^d$. Since $n_Y = d+1$, the Proposition explains why the characteristic polynomial of the projective space $\mathbb{P}_F^d$ in $\mathbb{Z}[\mathbf{b}]$ is divisible by $n+1$.

4. TILDE OPERATION

Let A^* be an oriented cohomology theory over F . We associate to A^* a new cohomology theory $\tilde{A}^*$ defined by

$$\tilde{A}^*(X) = A^*(X) \otimes \mathbb{Z}[\mathbf{b}] = A^*(X)[\mathbf{b}].$$

The structure of a graded ring on $\tilde{A}^*(X)$ is given by the one of the graded ring $A^*(X)$ and by assigning degree $-|\alpha|$ to every b_α . In particular, for every $X \in \mathbf{Sm}(F)$, $\mathbf{P}_X^A \in \tilde{A}^0(X)$ and, if X is projective, $\mathbf{F}_X^A \in \tilde{A}^{-d}(\text{pt})$, where $d = \dim(X)$.

The pull-back homomorphism $f^{\tilde{A}} : \tilde{A}^*(X) \rightarrow \tilde{A}^*(Y)$ associated to a morphism $f : Y \rightarrow X$ is equal to $f^A \otimes \text{id}_{\mathbb{Z}[\mathbf{b}]}$. The push-forward map $f_{\tilde{A}}$ associated to a projective morphism $f : Y \rightarrow X$ is defined by

$$(3) \quad f_{\tilde{A}}(a) = f_A(a \cdot \mathbf{P}_Y^A) \cdot (\mathbf{P}_X^A)^{-1} = f_A(a \cdot \mathbf{P}_Y^A \cdot f^A(\mathbf{P}_X^A)^{-1}).$$

If f is a closed embedding, then $[f^*T_X] - [T_Y]$ is equal to the class of the normal bundle $N_Y X$ of Y in X . Hence $\mathbf{P}_Y^A \cdot f^A(\mathbf{P}_X^A)^{-1} = \mathbf{P}^A(N_Y X)$ and

$$f_{\tilde{A}}(a) = f_A(a \cdot \mathbf{P}^A(N_Y X)).$$

Lemma 4.1. *Let $p : L \rightarrow X$ be a line bundle. Then*

$$c_1^{\tilde{A}}(L) = c_1^A(L) \cdot \mathbf{P}^A(L) = \sum_{i \geq 0} c_1^A(L)^{i+1} b_i \in A^*(X)[\mathbf{b}].$$

Proof. Let $s : X \rightarrow L$ be the zero section. The normal bundle of s is equal to L . Hence,

$$\begin{aligned} c_1^{\tilde{A}}(L) &= s^{\tilde{A}} s_{\tilde{A}}(1_X) \\ &= s^A s_A(\mathbf{P}^A(L)) \quad (\text{projection formula}) \\ &= s^A(s_A(1) \cdot p^A \mathbf{P}^A(L)) \\ &= c_1^A(L) \cdot \mathbf{P}^A(L). \end{aligned}$$

□

Proposition 4.2. *The functor $\tilde{A}^*$ is an oriented cohomology theory.*

Proof. We need to check properties (i)-(v) in the definition of an oriented cohomology theory.

(i) Let $f : Y \rightarrow X$ and $g : Z \rightarrow Y$ be two projective morphisms. Then for any $a \in A^*(Z)$,

$$\begin{aligned} (f_{\tilde{A}} \circ g_{\tilde{A}})(a) &= f_{\tilde{A}}(g_A(a \mathbf{P}_Z^A) \cdot (\mathbf{P}_Y^A)^{-1}) \\ &= f_A(g_A(a \mathbf{P}_Z^A) \cdot (\mathbf{P}_Y^A)^{-1} \cdot \mathbf{P}_Y^A) \cdot (\mathbf{P}_X^A)^{-1} \\ &= (f_A \circ g_A)(a \mathbf{P}_Z^A) \cdot (\mathbf{P}_X^A)^{-1} \\ &= (f \circ g)_{\tilde{A}}(a). \end{aligned}$$

(ii) Follows from Lemma 4.1 and invertibility of $\mathbf{P}^A(L)$.

(iii) Consider a transverse Cartesian square (1). We have

$$\mathbf{P}_{Y'}^A \cdot f'^A(\mathbf{P}_{Y'}^A)^{-1} = h^A \mathbf{P}_Y^A \cdot h^A f^A(\mathbf{P}_X^A)^{-1}$$

and therefore, for every $a \in A^*(Y)$,

$$\begin{aligned} (f'_{\tilde{A}} \circ h^{\tilde{A}})(a) &= f'_A(h^A(a) \cdot \mathbf{P}_{Y'}^A \cdot f'^A(\mathbf{P}_{Y'}^A)^{-1}) \\ &= f'_A(h^A(a) \cdot h^A \mathbf{P}_Y^A \cdot h^A f^A(\mathbf{P}_X^A)^{-1}) \\ &= f'_A h^A(a \cdot \mathbf{P}_Y^A \cdot f^A(\mathbf{P}_X^A)^{-1}) \\ &= g^A f_A(a \cdot \mathbf{P}_Y^A \cdot f^A(\mathbf{P}_X^A)^{-1}) \\ &= g^A(f_A(a \cdot \mathbf{P}_Y^A) \cdot (\mathbf{P}_X^A)^{-1}) \\ &= (g^{\tilde{A}} \circ f_{\tilde{A}})(a). \end{aligned}$$

(iv) Obvious.

(v) Let $f : Y \rightarrow X$ be a projective morphism in $\mathbf{Sm}(F)$, $a \in A^*(X)$ and $b \in A^*(Y)$. We have

$$\begin{aligned} f_{\tilde{A}}(b \cdot f^{\tilde{A}}(a)) &= f_A(b \cdot f^A(a) \cdot \mathbf{P}_Y^A) \cdot (\mathbf{P}_X^A)^{-1} \quad (\text{projection formula}) \\ &= f_A(b \cdot \mathbf{P}_Y^A) \cdot a \cdot (\mathbf{P}_X^A)^{-1} \\ &= f_{\tilde{A}}(b) \cdot a \end{aligned}$$

□

Remark 4.3. The correspondence $E \mapsto \mathbf{P}^A(E)$ is given by the characteristic class $\mathbf{P}^A = \sum_{\alpha} c_{\alpha} b_{\alpha}$ over $\mathbb{Z}[\mathbf{b}]$. In view of [8], $\mathbf{P}^A$ is the inverse Todd genus of the natural embedding of A^* into $\tilde{A}^*$ and the formula (3) is the Riemann-Roch theorem for this embedding.

Note that if $X \in \mathbf{Sm}(F)$ is projective, the fundamental class $[X]^{\tilde{A}} \in \tilde{A}(\text{pt}) = A(\text{pt})[\mathbf{b}]$ coincides with the fundamental polynomial $\mathbf{F}_X^A$. In particular, by Proposition 2.3, $\mathbf{F}_{X \times Y}^A = \mathbf{F}_X^A \cdot \mathbf{F}_Y^A$ for all $X, Y \in \mathbf{Sm}(F)$.

5. FORMAL GROUP LAW OF A THEORY

Let A^* be an oriented cohomology theory over F . By [7], there is unique commutative formal group law

$$\Phi^A = \sum_{i,j \geq 0} a_{ij}^A x^i y^j = x + y + \sum_{i,j \geq 1} a_{ij}^A x^i y^j,$$

over the coefficient ring $A^*(\text{pt})$ with $a_{ij}^A \in A^{1-i-j}(\text{pt})$, such that for every two line bundles L and L' over a variety $X \in \mathbf{Sm}(F)$,

$$c_1(L \otimes L') = c_1(L) + c_1(L') + \sum_{i,j \geq 1} a_{ij}^A c_1(L)^i c_1(L')^j \in A^1(X).$$

Example 5.1. Since $c_1^H(L \otimes L') = c_1^H(L) + c_1^H(L')$ for two line bundles L and L' [2, Prop. 2.5(e)], $\Phi^H(x, y) = x + y$ is the *additive group law*. It follows from the description of the first Chern class in K -theory (Example 3.1) that $\Phi^K(x, y) = x + y - xyt$ (called the *multiplicative group law*).

In the rest of the section we prove that the coefficients of the group law Φ^A belong to the subring $A_c^*(\text{pt}) \subset A^*(\text{pt})$.

Lemma 5.2. *Let L_{can} be the canonical line bundle over the projective space $\mathbb{P}_F^n$ over F . Then $p_A(c_1^A(L_{\text{can}})^i) = [\mathbb{P}_F^{n-i}]^A$ for every $i \geq 0$.*

Proof. Induction on n . Let $p : \mathbb{P}_F^n \rightarrow \text{pt}$ the structure morphism, $j : \mathbb{P}_F^{n-1} \hookrightarrow \mathbb{P}_F^n$ an embedding, $q = p \circ j$, $\xi = c_1^A(L_{\text{can}})$. Then $j^A(\xi) = c_1^A(L'_{\text{can}})$, where $L'_{\text{can}} = j^*(L_{\text{can}})$ is the canonical vector bundle over $\mathbb{P}_F^{n-1}$. Proposition 3.2 gives $\xi = j_A(1_{\mathbb{P}_F^{n-1}})$, hence, by the induction hypothesis,

$$p_A(\xi^i) = p_A(j_A(1) \cdot \xi^{i-1}) = p_A j_A(j^A(\xi)^{i-1}) = q_A(c_1^A(L'_{\text{can}})^{i-1}) = [\mathbb{P}_F^{n-i}]^A.$$

□

Lemma 5.3. (cf. [1, Prop. II.10.6]) *Let V be a smooth hypersurface in $\mathbb{P}_F^n \times \mathbb{P}_F^m$ of type $(1, 1)$ for some n and m . Then*

$$[V]^A = \sum_{i=0}^n \sum_{j=0}^m a_{ij}^A [\mathbb{P}_F^{n-i}]^A \cdot [\mathbb{P}_F^{m-j}]^A \in A^{1-n-m}(\text{pt}).$$

Proof. Let $i : V \hookrightarrow \mathbb{P}_F^n \times \mathbb{P}_F^m$ be the embedding of V as a divisor. The corresponding line bundle is the tensor product $L_1 \otimes L_2$ of two canonical line bundles on $\mathbb{P}_F^n$ and $\mathbb{P}_F^m$ respectively. Hence, by Proposition 3.2,

$$i_A(1_V) = c_1^A(L_1 \otimes L_2) = \Phi^A(\xi, \eta) = \sum_{i,j \geq 0} a_{ij}^A \xi^i \eta^j,$$

where $\xi = c_1^A(L_1)$, $\eta = c_1^A(L_2)$.

Let $p : \mathbb{P}_F^n \times \mathbb{P}_F^m \rightarrow \text{pt}$ be the structure morphism. By Lemma 5.2,

$$p_A(\xi^i) = [\mathbb{P}_F^{n-i}]^A, \quad p_A(\eta^j) = [\mathbb{P}_F^{m-j}]^A,$$

hence, by projection formula, $p_A(\xi^i \eta^j) = [\mathbb{P}_F^{n-i}]^A \cdot [\mathbb{P}_F^{m-j}]^A$ and therefore,

$$[V]^A = p_A(i_A(1_V)) = p_A\left(\sum_{i,j \geq 0} a_{ij}^A \xi^i \eta^j\right) = \sum_{i=0}^n \sum_{j=0}^m a_{ij}^A [\mathbb{P}_F^{n-i}]^A \cdot [\mathbb{P}_F^{m-j}]^A.$$

□

Corollary 5.4. $a_{nm}^A \in A_c^{1-n-m}(\text{pt})$ for every n and m .

Proof. We prove the statement by induction on $n + m$. By Lemma 5.3 and induction hypothesis, $a_{nm} - [V]^A \in A_c^{1-n-m}(\text{pt})$, whence the result. □

Corollary 5.5. Let $i(t) = \sum_{k \geq 1} b_k^A t^k$ be the additive inverse power series of Φ^A , that is $\Phi_A(t, i(t)) = 0$. Then $b_k^A \in A_c^{1-k}(\text{pt})$ for every $k \geq 1$.

6. K -THEORY VERSUS CHOW THEORY

A relation between K -theory and (rational) Chow-theory is given by the Chern character

$$\text{ch}_X : K^*(X) \longrightarrow H^*(X) \otimes \mathbb{Q}$$

for every $X \in \mathbf{Sm}(F)$. It is the ring homomorphism defined by

$$\text{ch}([E]t^{-k}) = \text{ch}([E]) = \text{rank}(E) + \sum_{i=1}^{\infty} \frac{1}{i!} c_{(i)}^H(E)$$

for a vector bundle $E \rightarrow X$ [2, Ch. 15]. In particular, the homomorphism

$$\text{ch}_{\text{pt}} : K^*(\text{pt}) = \mathbb{Z}[t, t^{-1}] \longrightarrow \mathbb{Q} = H^*(\text{pt}) \otimes \mathbb{Q}$$

is the evaluation at $t = 1$.

For a projective morphism $f : X \rightarrow Y$, by the classical Riemann-Roch formula [2, Th. 15.2], for every $a \in K^*(X)$:

$$f_H(\text{ch}_X(a) \cdot td(T_X)) = \text{ch}_Y(f_K(a)) \cdot td(T_Y),$$

where $td \in \mathbb{Q}[\mathbf{c}]$ is the (rational) Todd characteristic class.

Let $X \in \mathbf{Sm}(F)$ be a projective variety of dimension d and $p : X \rightarrow \text{pt}$ the structure morphism. Applying the Riemann-Roch formula for p and $a = c_\alpha^K(-T_X)$, we get

$$(4) \quad \deg(\text{ch}_X(c_\alpha^K(-T_X)) \cdot td(T_X)) = \text{ch}_{\text{pt}}(p_K(c_\alpha^K(-T_X))) = p_K c_\alpha^K(-T_X)|_{t=1},$$

where $\deg = p_H$ is the degree homomorphism.

Proposition 6.1. 1. For every projective $X \in \mathbf{Sm}(F)$, $\mathbf{F}_X^K|_{t=0} = \mathbf{F}_X^H$.

2. The evaluation homomorphism $\mathbb{Z}[t, \mathbf{b}] \rightarrow \mathbb{Z}[\mathbf{b}]$ at $t = 0$ induces ring isomorphism between $\tilde{K}_c(\text{pt})$ and $\tilde{H}_c(\text{pt})$. In particular, for every d , the degree d components $\tilde{K}_c(\text{pt})_d$ and $\tilde{H}_c(\text{pt})_d$ are free abelian groups of rank at most $p(d)$.

Proof. 1. Let $d = \dim X$. For every partition α ,

$$p_K c_\alpha^K(-T_X) \in K^{|\alpha|-d}(X) = K_0(X)t^{d-|\alpha|}.$$

- a) If $|\alpha| > d$, then $c_\alpha^K(-T_X)$ and $c_\alpha^H(-T_X)$ are both zero (Examples 3.4 and 3.5).
- b) If $|\alpha| < d$, then $p_K c_\alpha^K(-T_X)|_{t=0} = 0 = \deg c_\alpha^H(-T_X)$.
- c) Assume that $|\alpha| = d$. We have

$$\text{ch}(c_\alpha^K) \cdot td^{-1} = c_\alpha^H + \text{characteristic class of degree } > d.$$

Hence, by (4),

$$p_K c_\alpha^K(-T_X)|_{t=0} = p_K c_\alpha^K(-T_X) = p_K c_\alpha^K(-T_X)|_{t=1} = \deg c_\alpha^H(-T_X).$$

2. By the first statement, the evaluation at $t = 0$ takes $\tilde{K}_c(\text{pt})$ onto $\tilde{H}_c(\text{pt})$. We need to prove injectivity of the evaluation. Let $X_1, \dots, X_s \in \mathbf{Sm}(F)$ be projective varieties of the same dimension and $m_1, \dots, m_s \in \mathbb{Z}$ such that

$$\sum_{i=1}^s m_i \mathbf{F}_{X_i}^K|_{t=0} = \sum_{i=1}^s m_i \mathbf{F}_{X_i}^H = 0.$$

Equivalently,

$$(5) \quad \sum_{i=1}^s m_i \deg c^H(-T_{X_i}) = 0$$

for every generalized Chern class $c = c_\alpha$. Since c_α generate $\mathbb{Q}[\mathbf{c}]$, the formula (5) holds for every characteristic class $c \in \mathbb{Q}[\mathbf{c}]$. Taking $c = \text{ch}(c_\alpha) \cdot td^{-1}$ and applying formula (4) for every X_i , we get

$$\sum_{i=1}^s m_i p_K^{(i)} c_\alpha^K(-T_{X_i})|_{t=1} = 0$$

for every α , where $p^{(i)} : X_i \rightarrow \text{pt}$ is the structure morphism. But the sum $\sum_{i=1}^s m_i p_K^{(i)} c_\alpha^K(-T_{X_i})$ is a monomial in t and hence it is zero for every α . It follows that $\sum_{i=1}^s m_i \mathbf{F}_{X_i}^K = 0$.

The group $\tilde{H}_c(\text{pt})_d$ is a subgroup of the free group $\mathbb{Z}[\mathbf{b}]_d$ of rank $p(d)$, whence the last statement of the Proposition. $\square$

7. HYPERSURFACES $V(n_1, n_2, \dots, n_k)$

From now on we assume that the base field is infinite. We will use the following variant of Bertini theorem [3, Th.II.8.18]:

Theorem 7.1. *Let X be a smooth variety over an infinite field, L a very ample line bundle over X . Then there is a section of L with smooth zero divisor.*

Let $\mathbb{P}$ be the product of projective spaces $\mathbb{P}_F^{n_1} \times \mathbb{P}_F^{n_2} \times \dots \times \mathbb{P}_F^{n_k}$. Denote by L_i the pull-back on $\mathbb{P}$ of the canonical vector bundle over $\mathbb{P}_F^{n_i}$ and by L the tensor product of the L_i . By Theorem 7.1, there is a smooth hypersurface

$$V = V(n_1, n_2, \dots, n_k) \subset \mathbb{P},$$

which is the zero divisor of a section of L .

Let $i : V \hookrightarrow \mathbb{P}$ be the embedding. For an oriented cohomology theory A^* over F , by Proposition 3.2,

$$i_{\tilde{A}}(1_V) = c_1^{\tilde{A}}(L) = c_1^A(L) \cdot \mathbf{P}^A(L).$$

Denote by $q : \mathbb{P} \rightarrow \text{pt}$ the structure morphism. Hence,

$$\mathbf{F}_V^A = [V]^{\tilde{A}} = q_{\tilde{A}} i_{\tilde{A}}(1_V) = q_{\tilde{A}}(c_1^A(L) \cdot \mathbf{P}^A(L)) = q_A(c_1^A(L) \cdot \mathbf{P}^A(L) \cdot \mathbf{P}_{\mathbb{P}}^A).$$

The class in $K_0(\mathbb{P})$ of the tangent bundle of $\mathbb{P}$ equals $\sum [L_i]^{n_i+1} - k1$. We have then

$$\mathbf{P}_{\mathbb{P}}^A = \prod_{i=1}^k \mathbf{P}^A(L_i)^{-n_i-1}.$$

Thus,

$$\mathbf{F}_V^A = q_A(c_1^A(L) \cdot \mathbf{P}^A(L) \cdot \prod_{i=1}^k \mathbf{P}^A(L_i)^{-n_i-1}).$$

Set $\xi_i = c_1^A(L_i)$, $\xi = c_1^A(L)$. Therefore,

$$(6) \quad \mathbf{F}_V^A = q_A \left(\left(\sum_{j \geq 0} \xi^{j+1} b_j \right) \cdot \prod_{i=1}^k \left(\sum_{j \geq 0} \xi_i^j b_j \right)^{-n_i-1} \right).$$

Note that

$$\xi = \Psi^A(\xi_1, \xi_2, \dots, \xi_k),$$

where Ψ^A is the iterated group law of A .

Assume that $A^* = H^*$, so that $\xi = \sum \xi_i$. We would like to compute the α -characteristic number of V for $\alpha = (n-1)$, where $n = \sum n_i$, that is the coefficient of b_{n-1} in $\mathbf{F}_V^H$. Assume that $n_i > 1$ for at least two values of i , so that $n-1 \geq n_i+1$ for all i . Since $\xi_i^{n_i+1} = 0$, we can ignore the second multiple in (6). Hence

$$\deg c_{(n-1)}^H(-T_V) = q_A(\xi^n) = \frac{n!}{n_1! n_2! \dots n_k!} q_A(\xi_1^{n_1} \dots \xi_k^{n_k}) = \frac{n!}{n_1! n_2! \dots n_k!}.$$

We have proved

Proposition 7.2. (cf. [9, Lemma VII.6.8], [10]) *Let $V = V(n_1, n_2, \dots, n_k)$, $n = \sum n_i$. If $n_i > 1$ for at least two values of i , then*

$$\deg c_{(n-1)}^H(V) = \frac{n!}{n_1! n_2! \dots n_k!}.$$

Now consider K -theory $A^* = K^*$. Let p be a prime integer. Assume that for some s , $n_i = p^{s-1}$ for every i and $k = p$, so that $n = p^s$. We have

$$\xi = \Phi(\xi_1, \xi_2, \dots, \xi_p) = v_1 - v_2 t + \dots + (-1)^p v_p t^{p-1},$$

where v_j are the standard symmetric functions on the ξ_i . Note that the r.h.s. of (6) is a polynomial in the b_α with the coefficients of the form $q_K(P(v))$, where P is a polynomial over $\mathbb{Z}$.

Lemma 7.3. *Let v be a monomial $v_1^{\alpha_1} \dots v_p^{\alpha_p}$. If $\alpha_i > 0$ for some $i = 1, 2, \dots, p-1$, then $q_K(v)$ is divisible by p .*

Proof. Assume $\alpha_i > 0$, so that $v = v_i u$, where $u = v_i^{-1} v$ is a monomial. For every permutation $\sigma \in S_p$ let $\xi_\sigma = \xi_{\sigma(1)} \xi_{\sigma(2)} \dots \xi_{\sigma(i)}$. Then $v = \sum_{\sigma \in S_p} \xi_\sigma u$ and hence $q_K(v)$ is divisible by p since $q_K(\xi_\sigma u) = q_K(\xi_\tau u)$ for every $\sigma, \tau \in S_p$ and the number $\binom{p}{i}$ of terms in the sum is divisible by p . $\square$

Thus, we can delete all monomials in the v_i 's containing v_i for $i = 1, 2, \dots, p-1$ and rewrite (6) modulo p :

$$(7) \quad \mathbf{F}_V^K \equiv q_K \left(\left(\sum_{j \geq 0} [(-1)^p v_p t^{p-1}]^{j+1} b_j \right) \cdot \prod_{i=1}^p \left(\sum_{j \geq 0} \xi_i^j b_j \right)^{-p^{s-1}-1} \right) \pmod{p}.$$

Recall that $\mathbf{F}_V^K$ is a homogeneous polynomial in $\mathbb{Z}[t, \mathbf{b}]$ of degree $\dim(V) = p^s - 1$.

Proposition 7.4. (cf. [10, Lemma, p.121]) *Let $V = V(p^{s-1}, p^{s-1}, \dots, p^{s-1})$ (p terms), $\alpha = (p^{s-1} - 1, p^{s-1} - 1, \dots, p^{s-1} - 1)$. Then the b_α -coefficient of $\mathbf{F}_V^K$ is not divisible by p . If $\deg \beta \geq p^s - p$ and b_β -coefficient of $\mathbf{F}_V^K$ is not divisible by p , then $\deg \beta = p^s - p$ and β is a refinement of α .*

Proof. A typical monomial of the r.h.s. of (7) is of the form

$$t^{(p-1)(j+1)} b_j b_{\alpha^1} \dots b_{\alpha^p} = t^{(p-1)(j+1)} b_\beta$$

for partitions $\alpha^1, \dots, \alpha^p$. Note that since $v_p \xi_i^{p^s} = 0$ we may assume that $|\alpha^i| \leq p^{s-1} - 1$ for all i . We have $|\beta| = p^s - 1 - (p-1)(j+1) \leq p^s - p$ and equality holds iff $j = 0$. Hence, if $\deg \beta \geq p^s - p$ and the b_β -coefficient of $\mathbf{F}_V^K$ is not divisible by p , then $\deg \beta = p^s - p$ and $j = 0$. Therefore, $|\alpha^i| = p^{s-1} - 1$ for all i and β is a refinement of α .

It follows from (7) that modulo p , the b_α -coefficient of $\mathbf{F}_V^K$ is equal to $(-1)^p t^{p-1} q_K(v_p^{p^{s-1}}) = (-1)^p t^{p-1}$ and hence it is not trivial. $\square$

Define the following partial ordering on the set of all partitions. We write $\alpha \leq \beta$ if $|\alpha| < |\beta|$ or $|\alpha| = |\beta|$ and $l(\alpha) \geq l(\beta)$. We consider largest monomials of polynomials in the b_i 's with respect to this ordering.

Lemma 7.5. (cf. [10, Proposition, p.125]) *For every prime integer p and every integer $d \geq 1$ there exists a projective variety $M_d^p \in \mathbf{Sm}(F)$ of dimension d such that the polynomial $\mathbf{F}_{M_d^p}^K$ has largest monomial b_d modulo p if $d \neq p^s - 1$ for any s or $t^{p-1}(b_{p^{s-1}-1})^p$ if $d = p^s - 1$ for some $s > 0$.*

Proof. Assume first that $d+1$ is not divisible by p and set $M_d^p = \mathbb{P}_F^d$. By Proposition 6.1, the b_d -coefficients of $\mathbf{F}_{M_d^p}^K$ and $\mathbf{F}_{M_d^p}^H$ coincide. By Example 3.4, this coefficient is equal to $-(d+1)$ and it is not divisible by p .

Assume now that $d+1$ is divisible by p but $d+1 \neq p^s$ for any s . We write $d+1 = p^r(pu+v)$ with $r > 0$ and $0 < v < p$. If $u = 0$, $v > 1$, we set

$M_d^p = V(p^r, p^r(v-1))$. By Proposition 7.2, the b_d -coefficient of $\mathbf{F}_{M_d^p}^H$ is equal to $\binom{p^r v}{p^r}$ and hence it is not divisible by p .

If $u > 0$, let $M_d^p = V(p^r v, p^{r+1}u)$ and again by Proposition 7.2, the b_d -coefficient of $\mathbf{F}_{M_d^p}^H$ is equal to $\binom{p^r(pu+v)}{p^r v}$ and it is not divisible by p .

If $d+1 = p^s$ for some s , let $M_d^p = V(p^{s-1}, p^{s-1}, \dots, p^{s-1})$ (p terms). Then by Proposition 7.4, the b_d -coefficient of $\mathbf{F}_{M_d^p}^K$ is zero modulo p if $|\alpha| \geq p^s - p$ unless $|\alpha| = p^s - p$ and α refines $(p^{s-1} - 1, \dots, (p^{s-1} - 1))$. $\square$

Corollary 7.6. (cf. [10, Corollary, p.126]) *For a partition α and let $M_\alpha^p = M_{\alpha_1}^p \times \dots \times M_{\alpha_r}^p$. Then for every integer $d \geq 0$, the polynomials $\mathbf{F}_{M_\alpha^p}^K \pmod{p}$ in $(\mathbb{Z}/p\mathbb{Z})[t, \mathbf{b}]$ with $|\alpha| = d$ are linearly independent.*

Proposition 7.7. 1. *The ring $\tilde{K}_c(\text{pt}) \otimes \mathbb{Z}/p\mathbb{Z}$ (resp. $\tilde{H}_c(\text{pt}) \otimes \mathbb{Z}/p\mathbb{Z}$) is a polynomial ring over $\mathbb{Z}/p\mathbb{Z}$ in the variables $\mathbf{F}_{M_d^p}^K$ (resp. $\mathbf{F}_{M_d^p}^H \pmod{p}$), $d \geq 1$.*
 2. *The degree d component $\tilde{K}_c(\text{pt})_d$ is a direct summand of $\mathbb{Z}[t, \mathbf{b}]_d$ of rank $p(d)$.*
 3. *The degree d component $\tilde{H}_c(\text{pt})_d \subset \mathbb{Z}[\mathbf{b}]_d$ is a free subgroup of (maximal) rank $p(d)$.*

Proof. By Corollary 7.6, the rank of the image of $\tilde{K}_c(\text{pt})_d$ in $(\mathbb{Z}/p\mathbb{Z})[t, \mathbf{b}]$ for every prime integer p is at least $p(d)$. On the other hand, the rank of $\tilde{K}_c(\text{pt})_d$ is at most $p(d)$ by Proposition 6.1. Hence the classes $\mathbf{F}_{M_\alpha^p}^K \pmod{p}$ generate $\tilde{K}_c(\text{pt}) \otimes \mathbb{Z}/p\mathbb{Z}$. The map $\tilde{K}_c(\text{pt})_d \otimes \mathbb{Z}/p\mathbb{Z} \rightarrow (\mathbb{Z}/p\mathbb{Z})[t, \mathbf{b}]$ is injective for every p , therefore $\tilde{K}_c(\text{pt})_d$ is a direct summand of $\mathbb{Z}[t, \mathbf{b}]_d$. The statements about $\tilde{H}_c(\text{pt})$ follow from Proposition 6.1. $\square$

Remark 7.8. The second statement of the Proposition is an algebraic analog of the Hattori-Stong Theorem [10, Theorem, p.129].

Let J be the ideal in $\tilde{K}_c(\text{pt}) \otimes \mathbb{Z}/p\mathbb{Z}$ generated by $\mathbf{F}_X^K$ for all projective $X \in \mathbf{Sm}(F)$ of positive dimension. For every projective $X \in \mathbf{Sm}(F)$ of dimension d ,

$$(8) \quad \mathbf{F}_X^K \equiv \lambda \mathbf{F}_{M_d^p}^K \pmod{J^2}$$

for a uniquely determined $\lambda \in \mathbb{Z}/p\mathbb{Z}$. Recall that the b_d -coefficients of $\mathbf{F}_X^K$ and $\mathbf{F}_X^H$ coincide and are equal to $\deg c_{(d)}^H(-T_X)$. Note that the b_d -coefficient of every element of J^2 is trivial for every d .

Proposition 7.9. *For a projective variety $X \in \mathbf{Sm}(F)$ of dimension $d = p^s - 1$, the characteristic number $\deg c_{(d)}^H(-T_X)$ is divisible by p .*

Proof. The statement follows from (8) since by Lemma 7.5, $\deg c_{(d)}^H(-T_{M_d^p})$ is divisible by p . $\square$

Lemma 7.10. *Let S a set of smooth projective varieties over F . Assume that for every prime integer p and every $d \geq 1$ there is $X \in S$ such that*

$\deg c_{(d)}^H(-T_X)$ is not divisible by p if $d \neq p^s$ for any s and $\deg c_{(d)}^H(-T_X)$ is not divisible by p^2 if $d = p^s - 1$ for some $s > 0$. Then the ring $\tilde{K}_c(\text{pt})$ is generated by the $\mathbf{F}_X^K$, $X \in S$.

Proof. Let p be a prime integer. For every $d \geq 1$, there is $X \in S$ such that λ in (8) is not zero modulo p . Hence the polynomials $\mathbf{F}_X^K$ generate $\tilde{K}_c(\text{pt})$ modulo p for every p , whence the statement. $\square$

Proposition 7.11. *The subring $\tilde{K}_c(\text{pt}) \subset \mathbb{Z}[t, \mathbf{b}]$ is generated by the classes of projective spaces $\mathbb{P}_F^n$ and smooth hypersurfaces $V(n, m)$.*

Proof. Let S be the set of all projective spaces $\mathbb{P}_F^n$ and smooth hypersurfaces $V(n, m)$. Let p be a prime integer and $d \geq 1$. If $d \neq p^s$ for any s , the proof of Lemma 7.5 shows there is $X \in S$ such that $\deg c_{(d)}^H(-T_X)$ is not divisible by p .

Assume that $d = p^s - 1$ for some s . If $s > 1$, then by Proposition 7.2,

$$\deg c_{(p^s-1)}^H(V_{p^{s-1}, p^s-p^{s-1}}) = \binom{p^s}{p^{s-1}}$$

is not divisible by p^2 . If $s = 1$, by Example 3.4,

$$\deg c_{(p-1)}^H(\mathbb{P}^{p-1}) = -p.$$

By Lemma 7.10, the set S generates $\tilde{K}_c(\text{pt})$. $\square$

Propositions 6.1 and 7.11 imply

Corollary 7.12. *The subring $\tilde{H}_c(\text{pt}) \subset \mathbb{Z}[\mathbf{b}]$ is generated by the fundamental polynomials of projective spaces $\mathbb{P}_F^n$ and smooth hypersurfaces $V(n, m)$.*

8. LAZARD RING

Let Laz be the Lazard ring, the coefficient ring of the universal (one-dimensional, commutative) group law [9, Prop. VII.5.3]. For a commutative ring R , the set of R -points

$$\text{Spec}(\text{Laz})(R) = \text{Mor}(\text{Spec}(R), \text{Spec}(\text{Laz})) = \text{Hom}_{\text{rings}}(\text{Laz}, R)$$

is identified with the set of all formal group laws over R .

Denote the scheme $\text{Spec } \mathbb{Z}[\mathbf{b}]$ by G . For a commutative ring R the set of R -points $G(R) = \text{Hom}_{\text{rings}}(\mathbb{Z}[\mathbf{b}], R)$ can be identified with the set of sequences $(r_1, r_2, \dots)$ of elements of R (r_i being the image of the b_i) and therefore with the set of power series

$$t + r_1 t^2 + r_2 t^3 + \dots \in R[[t]].$$

The composition of power series makes G a group scheme over $\mathbb{Z}$.

The group $\text{Spec}(\mathbb{Z}[\mathbf{b}](R))$ acts on $\text{Spec}(\text{Laz})(R)$ by conjugation

$$(f \cdot \Phi)(x, y) = f(\Phi(f^{-1}(x), f^{-1}(y))).$$

Thus, the group scheme G acts on the scheme $\text{Spec}(\text{Laz})$. Denote by

$$\log t = t + m_1 t^2 + m_2 t^3 + \dots \in \mathbb{Z}[\mathbf{b}][[t]]$$

the formal inverse of

$$\exp t = t + b_1 t^2 + b_2 t^3 + \dots \in \mathbb{Z}[\mathbf{b}][[t]].$$

It is known that $m_d = \mathbf{F}_{\mathbb{P}^d}^H / (d+1)$ [9, VII, Cor. 6.12].

Lemma 8.1. *For every oriented cohomology theory A^* ,*

$$\Phi^{\tilde{A}}(x, y) = \exp \Phi^A(\log x, \log y).$$

Proof. For a line bundle L ,

$$c_1^{\tilde{A}}(L) = \exp c_1^A(L), \quad c_1^A(L) = \log c_1^{\tilde{A}}(L).$$

Hence for a pair of line bundles L and L' ,

$$\begin{aligned} c_1^{\tilde{A}}(L \otimes L') &= \exp c_1^A(L \otimes L') \\ &= \exp \Phi^A(c_1^A(L), c_1^A(L')) \\ &= \exp \Phi^A(\log c_1^{\tilde{A}}(L), \log c_1^{\tilde{A}}(L')). \end{aligned}$$

□

By Lemma 8.1, the group law

$$\Phi = \exp(\log x + \log y) = x + y + \sum_{i,j \geq 1} a_{ij} x^i y^j$$

over $\mathbb{Z}[\mathbf{b}]$ coincides with $\Phi^{\tilde{H}}$. It defines a ring homomorphism $\text{Laz} \rightarrow \mathbb{Z}[\mathbf{b}]$ which is, in fact, injective [9, VII, §5]. We will identify Laz with its image in $\mathbb{Z}[\mathbf{b}]$. The ring Laz is generated by the coefficients a_{ij} and Φ is the universal group law over Laz .

Theorem 8.2. *For every infinite field F , the subgroup in $\mathbb{Z}[\mathbf{b}]$, generated by the fundamental polynomials $\mathbf{F}_X^H$ for all $X \in \mathbf{Sm}(F)$, coincides with $\text{Laz} \subset \mathbb{Z}[\mathbf{b}]$.*

Proof. The differential form

$$d \log(x) = (1 + 2m_1 x + 3m_2 x^2 + \dots) dx = (1 + \mathbf{F}_{\mathbb{P}^1}^H x + \mathbf{F}_{\mathbb{P}^2}^H x^2 + \dots) dx$$

can be computed out of the formal group law by the formula [9, Prop. VII.5.7]

$$d \log(x) = \frac{dx}{\Phi_y(x, 0)}.$$

Hence, the classes of the projective spaces $\mathbb{P}_F^n$ can be expressed in terms of the a_{ij} , so that $\mathbf{F}_{\mathbb{P}^n}^H \in \text{Laz}$. By Lemma 5.3, $\mathbf{F}_{V(n,m)}^H \in \text{Laz}$ for every n and m . It follows from Corollary 7.12 that $\tilde{H}_c(\text{pt}) \subset \text{Laz}$.

Conversely, the inclusion $\text{Laz} \subset \tilde{H}_c(\text{pt})$ follows from Corollary 5.4 since Laz is generated by the coefficients a_{ij} . □

Thus, every projective variety $X \in \mathbf{Sm}(F)$ has the class $\mathbf{F}_X^H$ in the Lazard ring Laz .

9. VALUES OF CHARACTERISTIC CLASSES

In this section we prove that characteristic classes in an oriented cohomology theory A^* over F take values in $A_c^* \subset A^*$. For $X \in \mathbf{Sm}(F)$ denote by $A_{cl}^*(X)$ the subgroup in $A_c^*(X)$ generated by the elements $i_A(1_Z)$, where $i : Z \hookrightarrow X$ is a smooth closed subvariety.

Lemma 9.1. *Let L be a very ample line bundle over $X \in \mathbf{Sm}(F)$. Then $c_1^A(L) \cdot A_{cl}^*(X) \subset A_{cl}^*(X)$.*

Proof. Let $i : Z \hookrightarrow X$ be a smooth closed subvariety. The restriction $L' = L|_Z$ is very ample over Z . By Bertini theorem 7.1, there exists a section of L' with the smooth zero divisor Z' . Denote by $j : Z' \hookrightarrow Z$ the embedding. Then by Proposition 3.2, $j_A(1_{Z'}) = c_1^A(L')$ and hence

$$c_1^A(L) \cdot i_A(1_Z) = i_A(i_A^A c_1(L)) = i_A(c_1^A(L')) = i_A j_A(1_{Z'}) = (ij)_A(1_{Z'}) \in A_{cl}^*(X).$$

□

Proposition 9.2. *Let L be a line bundle over $X \in \mathbf{Sm}(F)$. Then $c_1^A(L) \cdot A_c^*(X) \subset A_c^*(X)$.*

Proof. Let $f : Y \rightarrow X$ be a projective morphism with $Y \in \mathbf{Sm}(F)$ and let $L' = f^*(L)$. Choose very ample line bundles L_1 and L_2 over Y such that $L' = L_1 \otimes L_2^{-1}$. By Lemma 9.1,

$$c_1^A(L_1)^i \cdot c_1^A(L_2)^j \in A_{cl}^*(Y) \subset A_c^*(Y)$$

for all i and j . Then by Proposition 5.4 and Corollary 5.5,

$$c_1^A(L') = \Phi^A(c_1^A(L_1), ic_1^A(L_2)) \in A_c^*(Y).$$

Finally,

$$c_1^A(L) \cdot f_A(1_Y) = f_A(f^A c_1^A(L)) = f_A(c_1^A(L')) \in f_A(A_c^*(Y)) \subset A_c^*(X).$$

□

Let E be a vector bundle of rank $r > 0$ over X . Consider the projection $p : \mathbb{P}(E) \rightarrow X$ and set

$$\xi = c_1^A(L_{can}) \in A^1(\mathbb{P}(E)),$$

where L_{can} is the canonical line bundle over $\mathbb{P}(E)$.

Lemma 9.3. *For every $i \geq 0$,*

$$p_A(\xi^{r-1+i}) = c_i^A(-E) + \sum_{j>i} a_j d_j^A(E) \in A^*(X),$$

for some $a_j \in A_c^*(\text{pt})$ and characteristic classes d_j of degree j .

Proof. By the splitting principle, we may assume that E is a subbundle of a trivial bundle E' of rank n and the factor bundle E'/E has a filtration by line bundles $L_1, L_2, \dots$. Let $l : \mathbb{P}(E) \rightarrow \mathbb{P}(E')$ be the closed embedding, $q : \mathbb{P}(E') \rightarrow X$ the projection, L'_{can} the canonical line bundle over

$\mathbb{P}(E'), \zeta = c_1^A(L'_{can}) \in A^1(\mathbb{P}(E'))$. We can consider l as a composition of closed embeddings of codimension 1 corresponding to the line bundles $q^*L_k \otimes L'_{can}$. Hence, by Proposition 3.2,

$$(9) \quad l_A(\xi^{r-1+i}) = l_A(1 \cdot l^A \zeta^{r-1+i}) = \zeta^{r-1+i} \cdot \prod_k c_1^A(q^*L_k \otimes L'_{can}).$$

We can compute $c_1^A(q^*L_k \otimes L'_{can})$ using the formal group law Φ^A :

$$(10) \quad c_1^A(q^*L_k \otimes L'_{can}) = q^A c_1^A(L_k) + \zeta + \sum_{l,m \geq 1} a_{lm} q^A c_1^A(L_k)^l \zeta^m.$$

Applying q_A to (9) and using (10), we get the formula we need, since by Lemma 5.2, $q_A(\zeta^s) = [\mathbb{P}_F^{n-1-s}]^A \in A_c^*(\text{pt})$, $a_{lm} \in A_c^*(\text{pt})$ (Lemma 5.4) and $\sigma_i(L_j) = c_i^A(E'/E) = c_i^A(-E)$. $\square$

Lemma 9.4. *For every $s \geq 0$, $p_A(\xi^s) \cdot A_c^*(X) \subset A_c^*(X)$.*

Proof. Let $f : Y \rightarrow X$ be a projective morphism in $\mathbf{Sm}(F)$, $E' = f^*(E)$. Consider the Cartesian transverse square

$$\begin{array}{ccc} \mathbb{P}(E') & \xrightarrow{g} & \mathbb{P}(E) \\ p' \downarrow & & \downarrow p \\ Y & \xrightarrow{f} & X. \end{array}$$

We have

$$p_A(\xi^s) \cdot f_A(1_Y) = p_A(\xi^s \cdot p^A f_A(1_Y)) = p_A(\xi^s \cdot g_A p'^A(1_Y)) = p_A(\xi^s \cdot g_A(1)) \in A_c^*(X)$$

since by Proposition 9.2, $\xi^s \cdot g_A(1) \in A_c^*(\mathbb{P}(E))$. $\square$

Theorem 9.5. *For every vector bundle E over X and every characteristic class c , $c^A(E) \cdot A_c^*(X) \subset A_c^*(X)$.*

Proof. Since $c(E) = c'(-E)$ for some characteristic class c' , it is sufficient to prove that $c^A(-E) \cdot A_c^*(X) \subset A_c^*(X)$ for every c . We may assume that

$$c = c_{\alpha_1} c_{\alpha_2} \dots c_{\alpha_k}$$

for some partition $\alpha = (\alpha_1, \dots, \alpha_k)$. Let $a \in A_c^*(X)$. By Lemma 9.3,

$$(11) \quad p_A(\xi^{r-1-\alpha_1}) \dots p_A(\xi^{r-1-\alpha_k}) a = c^A(-E) a + \sum_{j>i} a_j d_j^A(E) a \in A^*(X),$$

for some $a_j \in A_c^*(\text{pt})$ and characteristic classes d_j of degree bigger than the degree of c . By the reverse induction on the degree of c , we have $d_j^A(E) a \in A_c^*(X)$. By Lemma 9.4, the left hand side of (11) also belongs to $A_c^*(X)$. Hence, $c^A(-E) a \in A_c^*(X)$. $\square$

Corollary 9.6. *For every smooth X and every vector bundle E over X , the classes $c_\alpha^A(E)$ belong to $A_c^*(X)$ for every partition α . Moreover, $c_\alpha^A(E) = 0$ if $|\alpha| > \dim(X)$. In particular, if X is projective, the fundamental polynomial $\mathbf{F}_X^A$ is of degree at most $\dim(X)$.*

Proof. The group $A_c^i(X)$ is trivial if $i > \dim(X)$. $\square$

10. LANDWEBER-NOVIKOV OPERATIONS

Let R be a commutative ring. Assume that the group scheme $G = \operatorname{Spec} \mathbb{Z}[\mathbf{b}]$ acts on $\operatorname{Spec} R$. The co-morphism of the action we denote by

$$\theta_R : R \longrightarrow R \otimes \mathbb{Z}[\mathbf{b}] = R[\mathbf{b}].$$

For every $r \in R$,

$$\theta(r) = \sum_{\alpha} s_{\alpha}^R(r) \otimes b_{\alpha}$$

for uniquely determined elements $s_{\alpha}^R(r) \in R$. We call the group endomorphisms

$$s_{\alpha}^R : R \longrightarrow R$$

for all partitions α the *Landweber-Novikov operations on R* .

Now consider the natural action of G on $\operatorname{Spec}(\operatorname{Laz})$ (section 8). The corresponding operations $s_{\alpha}^{\operatorname{Laz}}$ we simply denote by s_{α} .

Let $\varepsilon : \operatorname{Laz} \rightarrow \mathbb{Z}$ be the restriction of the augmentation map $\mathbb{Z}[\mathbf{b}] \rightarrow \mathbb{Z}$.

Lemma 10.1. *The composition*

$$\operatorname{Laz} \xrightarrow{\theta_{\operatorname{Laz}}} \operatorname{Laz} \otimes \mathbb{Z}[\mathbf{b}] \xrightarrow{\varepsilon \otimes \operatorname{id}} \mathbb{Z}[\mathbf{b}]$$

coincides with the embedding $\operatorname{Laz} \hookrightarrow \mathbb{Z}[\mathbf{b}]$.

Proof. The homomorphism $\theta_{\operatorname{Laz}}$ corresponds to the group law

$$\exp(\Phi(\log x, \log y))$$

on $\operatorname{Laz} \otimes \mathbb{Z}[\mathbf{b}]$, where Φ is the universal group law on Laz . The augmentation of Φ is the additive group law over $\mathbb{Z}$, whence the result. $\square$

Denote by

$$\mu : \mathbb{Z}[\mathbf{b}] \longrightarrow \mathbb{Z}[\mathbf{b}] \otimes \mathbb{Z}[\mathbf{b}]$$

the co-multiplication ring homomorphism for the group scheme G . Let A^* be an oriented ring cohomology theory over F . Consider the ring homomorphism of cohomology theories

$$\tilde{\mu} = \operatorname{id}_A \otimes \mu : \tilde{A}^* \longrightarrow \widetilde{\tilde{A}^*}.$$

Lemma 10.2. *For every $X \in \mathbf{Sm}(F)$ and $a \in K_0(X)$,*

$$\tilde{\mu}(\mathbf{P}^A(a)) = \mathbf{P}^A(a) \cdot \mathbf{P}^{\tilde{A}}(a).$$

(In the r.h.s. the first term is a polynomial in the b'_{α} and the second - in the b''_{α} .)

Proof. The co-multiplication $\mu : \mathbb{Z}[\mathbf{b}] \rightarrow \mathbb{Z}[\mathbf{b}] \otimes \mathbb{Z}[\mathbf{b}] = \mathbb{Z}[\mathbf{b}', \mathbf{b}'']$ satisfies

$$\sum_{i \geq 0} t^{i+1} \mu(b_i) = \sum_{j \geq 0} \left(\sum_{k \geq 0} t^{k+1} b'_k \right)^{j+1} b''_j.$$

By the splitting principle and multiplicativity property (2), we may assume that $a = [L]$, where L is a line bundle. Hence (with $\xi = c_1(L)$),

$$\begin{aligned} \tilde{\mu}(\mathbf{P}^A(L)) &= \sum_{i \geq 0} \xi^i \mu(b_i) \\ &= \sum_{j \geq 0} \left(\sum_{k \geq 0} \xi^k b'_k \right)^{j+1} \xi^j b''_j \\ &= \sum_{j \geq 0} \mathbf{P}^A(L)^{j+1} \xi^j b''_j \\ &= \mathbf{P}^A(L) \cdot \sum_{j \geq 0} c_1^{\tilde{A}}(L)^j b''_j \\ &= \mathbf{P}^A(L) \cdot \mathbf{P}^{\tilde{A}}(L). \end{aligned}$$

□

Corollary 10.3. *For every projective variety $X \in \mathbf{Sm}(F)$,*

$$\mu(\mathbf{F}_X^A) = \mathbf{F}_X^{\tilde{A}}.$$

Proof. We apply Lemma 10.2 for $a = [-T_X]$:

$$\mu(\mathbf{F}_X^A) = \mu(p_A \mathbf{P}_X^A) = p_A \tilde{\mu}(\mathbf{P}_X^A) = p_A(\mathbf{P}_X^A \cdot \mathbf{P}_X^{\tilde{A}}) = p_{\tilde{A}}(\mathbf{P}_X^{\tilde{A}}) = \mathbf{F}_X^{\tilde{A}}.$$

□

We can express the Landweber-Novikov operations in terms of characteristic numbers in $\tilde{H}$. This is an analog of Novikov's formula [1, Th. I.8.3] with the cobordism theory replaced by its approximation $\tilde{H}$.

Proposition 10.4. *For every projective variety $X \in \mathbf{Sm}(F)$,*

$$s_\alpha(\mathbf{F}_X^H) = p_{\tilde{H}} c_\alpha^{\tilde{H}}(-T_X) \in \mathbb{Z}[\mathbf{b}],$$

where $p : X \rightarrow \text{pt}$ is the structure morphism.

Proof. Consider the following commutative diagram

$$\begin{array}{ccccc} \text{Laz} & \xrightarrow{\theta_{\text{Laz}}} & \text{Laz} \otimes \mathbb{Z}[\mathbf{b}] & \xrightarrow{\varepsilon \otimes \text{id}} & \mathbb{Z}[\mathbf{b}] \\ \theta_{\text{Laz}} \downarrow & & \text{id} \otimes \mu \downarrow & & \mu \downarrow \\ \text{Laz} \otimes \mathbb{Z}[\mathbf{b}] & \xrightarrow{\theta_{\text{Laz}} \otimes \text{id}} & \text{Laz} \otimes \mathbb{Z}[\mathbf{b}', \mathbf{b}''] & \xrightarrow{\varepsilon \otimes \text{id}} & \mathbb{Z}[\mathbf{b}', \mathbf{b}'']. \end{array}$$

By Lemma 10.1 and Corollary 10.3, the composition $\mu \circ (\varepsilon \otimes \text{id}) \circ \theta_{\text{Laz}}$ takes the class $\mathbf{F}_X^H$ to

$$\mathbf{F}_X^{\tilde{H}} = \sum_{\alpha} p_{\tilde{H}} c_{\alpha}^{\tilde{H}}(-T_X) b''_{\alpha}.$$

By Lemma 10.1, another composition $(\varepsilon \otimes \text{id}) \circ (\theta_{\text{Laz}} \otimes \text{id}) \circ \theta_{\text{Laz}}$ takes $\mathbf{F}_X^H$ to

$$\sum_{\alpha} s_{\alpha}(\mathbf{F}_X^H) b''_{\alpha}.$$

□

11. INVARIANT IDEALS

Let R be a commutative ring. Assume that the group scheme $G = \text{Spec } \mathbb{Z}[\mathbf{b}]$ acts on $\text{Spec } R$. An ideal $I \subset R$ is called *invariant* if $s_{\alpha}^R(I) \subset I$ for every α .

Let p be a prime integer. The ideal $p\mathbb{Z}[\mathbf{b}]$ in $\mathbb{Z}[\mathbf{b}]$ is obviously prime and invariant with respect to the action of G on itself by left translations. Therefore, the intersection $I(p) = \text{Laz} \cap p\mathbb{Z}[\mathbf{b}] \subset \text{Laz}$ is a prime invariant ideal in Laz .

Let $n = 0, 1, 2, \dots, \infty$. Denote by $I(p, n)$ the ideal in $I(p)$ generated by all $a \in I(p)$ of degree $\leq p^n - 1$. For example, $I(p, 0) = p\text{Laz}$ and $I(p, \infty) = I(p)$.

Thus, for every prime p we have a chain of prime invariant ideals in Laz :

$$p\text{Laz} = I(p, 0) \subset I(p, 1) \subset \dots \subset I(p, n) \subset \dots \subset I(p, \infty) = I(p).$$

It is known (see [9, Prop. VII.4.21] and [5, Th. 2.7]) that every ideal $I(p, n)$ is prime and invariant and the only nonzero prime invariant ideals in Laz are $I(p, n)$ for all prime p and $n \geq 0$.

Let X be a projective smooth variety over a field F . The set

$$I(X) = \{\mathbf{F}_Y^H \in \text{Laz} \text{ for all } Y \text{ such that } \text{Mor}_F(Y, X) \neq \emptyset\}$$

is a graded ideal in Laz . Let $q : X \rightarrow \text{pt}$ be a structure morphism. For every projective morphism $f : Y \rightarrow X$,

$$q_{\tilde{H}} f_{\tilde{H}}(1_Y) = (qf)_{\tilde{H}}(1_Y) = \mathbf{F}_Y^H.$$

Hence

$$I(X) = q_{\tilde{H}} \tilde{H}_c(X).$$

Recall that n_X is the gcd of $\deg(x)$ over all closed points x of a variety X .

Example 11.1. $I(X)_0 = n_X \mathbb{Z}$. If $X(F) \neq \emptyset$, $I(X) = \text{Laz}$.

Theorem 11.2. *For a projective variety $X \in \mathbf{Sm}(F)$ over an infinite field F , the ideal $I(X) \subset \text{Laz}$ is invariant.*

Proof. Let $f : Y \rightarrow X$ be a morphism, $q : X \rightarrow \text{pt}$ the structure morphism. By Proposition 10.4 and Corollary 9.6,

$$s_{\alpha}(\mathbf{F}_Y^H) = q_{\tilde{H}} f_{\tilde{H}}(c^{\tilde{H}}(-T_Y)) \in q_{\tilde{H}}(\tilde{H}_c(X)) = I(X).$$

□

Let P be a minimal prime ideal in Laz containing $I(X)$. By [6, Th. 3.1], P is invariant and hence $P = I(p, n)$ for some prime integer p and $n = 0, 1, \dots, \infty$. Clearly, P is the only minimal prime ideal containing $I(X)$ and p . We set $n_p(X) = n$. If for a prime integer p there is no invariant prime ideal containing $I(X)$ and p , we set $n_p(X) = \infty$. Thus, for every projective variety X we have the numbers $n_p(X)$ assigned for each prime integer p .

Proposition 11.3. *Let $X \in \mathbf{Sm}(F)$ be a projective variety, p a prime integer. Then the following conditions are equivalent:*

- (1) $p \mid n_X$;
- (2) *There exists an invariant prime ideal of Laz containing $I(X)$ and p .*

Proof. If $I(p, n_p(X))$ is the minimal prime ideal, then $I(X)_0 \subset p\mathbb{Z}$, i.e. $p \mid n_X$. Conversely, let $p \mid n_X$. Since the radical of $I(X)$ is the intersection of all minimal ideals, one of the minimal ideals contains p . $\square$

Proposition 11.4. *Let X and Y be projective smooth varieties such that $\text{Mor}(Y, X) \neq \emptyset$. Then $n_p(Y) \leq n_p(X)$ for every prime p .*

Proof. We have $I(Y) \subset I(X) \subset I(p, n_p(X))$. The minimal prime ideal between $I(Y)$ and $I(p, n_p(X))$ is equal $I(p, n_p(Y))$, hence $n_p(Y) \leq n_p(X)$. $\square$

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