ALGEBRAIC COBORDISM I

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Contents

1991 *Mathematics Subject Classification.* Primary 19E15; Secondary 14C99, 14C25.

Key words and phrases. Cobordism.

First author partially supported by the NSF.

0. Introduction : Chern classes and cobordism

Fix a base field k. We denote by $\mathbf{Sch}_k$ the category of separated schemes of finite type over k and by $\mathbf{Sm}_k$ its full subcategory consisting of smooth quasi-projective k-schemes.

Let $d \in \mathbb{Z}$ be an integer. A morphism $f : Y \to X$ in $\mathbf{Sm}_k$ has *relative dimension* d if, for each $y \in Y$, we have $\dim_k(Y, y) - \dim_k(X, f(y)) = d$, where for $z \in Z \in \mathbf{Sm}_k$ we denote by $\dim_k(Z, z)$ the Krull dimension of the component of Z containing z. We shall also say in that case that f has relative codimension $-d$.

We let $\mathbf{R}^*$ denote the category of *commutative graded rings with unit*. Observe that a commutative graded ring is not necessarily *graded commutative*. We say that a functor $A^* : (\mathbf{Sm}_k)^{\mathrm{op}} \to \mathbf{R}^*$ is *additive* if $A^*(\emptyset) = 0$ and for any pair $(X, Y) \in (\mathbf{Sm}_k)^2$ the canonical ring map $A^*(X \amalg Y) \to A^*(X) \times A^*(Y)$ is an isomorphism.

For a vector bundle $E \to X$ on a scheme X, we have the projective bundle $q : \mathbb{P}(E) \to X$ representing the functor of rank one quotients of E. If $\mathcal{E}$ is the sheaf of sections of E, then $\mathbb{P}(E) = \mathrm{Proj}(\mathrm{Sym}^*(\mathcal{E}))$. We let $O(1) \to \mathbb{P}(E)$ denote the canonical quotient line bundle of q^*E.

The following notion is directly taken from Quillen's paper [20]:

Definition 1. An *oriented cohomology theory* on $\mathbf{Sm}_k$ is given by

(D1). An additive functor $A^* : (\mathbf{Sm}_k)^{\mathrm{op}} \to \mathbf{R}^*$.

(D2). For each projective morphism $f : Y \to X$ in $\mathbf{Sm}_k$ of relative codimension d, a homomorphism of graded $A^*(X)$-modules:

$$f_* : A^*(Y) \to A^{*+d}(X)$$

(Observe that the ring homomorphism $f^* : A^*(X) \to A^*(Y)$ gives $A^*(Y)$ the structure of an $A^*(X)$-module).

These satisfy

(A1). One has $(\mathrm{Id}_X)_* = \mathrm{Id}_{A^*(X)}$ for any $X \in \mathbf{Sm}_k$ and moreover given projective morphisms $f : Y \to X$ and $g : Z \to Y$ in $\mathbf{Sm}_k$, with f of relative codimension d and g of relative codimension e, one has

$$(f \circ g)_* = f_* \circ g_* : A^*(Z) \to A^{*+d+e}(X).$$

(A2). Let $f : X \to Z$, $g : Y \to Z$ be transverse morphisms in $\mathbf{Sm}_k$, giving the cartesian square (in $\mathbf{Sm}_k$)

$$\begin{tikzcd} W \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

Suppose that f is projective of relative dimension d (thus so is f'). Then $g^*f_* = f'_*g'^*$.

(PB). Let $E \to X$ be a rank n vector bundle over some X in $\mathbf{Sm}_k$, $O(1) \to \mathbb{P}(E)$ the canonical quotient line bundle with zero section $s : \mathbb{P}(E) \to O(1)$. Let $1 \in A^0(\mathbb{P}(E))$ denote the multiplicative unit element. Define $\xi \in A^1(\mathbb{P}(E))$ by

$$\xi := s^*(s_*(1)).$$

Then $A^*(\mathbb{P}(E))$ is a free $A^*(X)$-module, with basis

$$(1, \xi, \dots, \xi^{n-1}).$$

(EH). Let $E \to X$ be a vector bundle over some X in $\mathbf{Sm}_k$, and let $p : V \to X$ be an E-torsor. Then $p^* : A^*(X) \to A^*(V)$ is an isomorphism.

A morphism of oriented cohomology theories is a natural transformation of functors $(\mathbf{Sm}_k)^{\mathrm{op}} \to \mathbf{R}^*$ which commutes with the the maps f_*.

The morphism of the form f^* are called *pull-backs* and the morphism of the form f_* are called *push-forwards*. Axiom (PB) will be refered to as the *projective bundle formula* and axiom (EH) as the *extended homotopy property*.

Given an oriented cohomology theory A^*, one may use Grothendieck's method [6] to define Chern classes $c_i(E) \in A^i(X)$ of a vector bundle $E \to X$ of rank n over X as follows. Using the notations of the previous definition, axiom (PB) implies that there exists unique elements $c_i(E) \in A^i(X)$, $i \in \{0, \dots, n\}$, such that $c_0(E) = 1$ and

$$\sum_{i=0}^{n} (-1)^i \, c_i(E) \, \xi^{n-i} = 0.$$

One can check all the standard properties of Chern classes as in [6] using the axioms listed above. Moreover, these Chern classes are characterized by the following properties:

(1) For any line bundle L over $X \in \mathbf{Sm}_k$, $c_1(L)$ equals $s^*s_*(1) \in A^1(X)$, where $s : X \to \mathcal{L}$ denotes the zero section.

(2) For any morphism $Y \to X \in \mathbf{Sm}_k$, and any vector bundle E over X, one has for each $i \geq 0$

$$c_i(f^*\mathcal{E}) = f^*(c_i(\mathcal{E})).$$

(3) Additivity formula: if

$$0 \to E' \to E \to E'' \to 0$$

is an exact sequence of vector bundles, then one has for each integer $n \geq 0$:

$$c_n(E) = \sum_{i=0}^{n} c_i(E')\, c_{n-i}(E'').$$

Sometime, to avoid confusion, we will write $c_i^A(E)$ for the Chern classes of E computed in the oriented cohomology theory A^*.

The fundamental insight of Quillen in [20], and the main difference with Grothendieck's axioms in [6], is that it is not true in general that one has the formula

$$c_1(L \otimes M) = c_1(L) + c_1(M)$$

for line bundles L and M over the same base. In other words the map

$$c_1 : \mathrm{Pic}(X) \to A^1(X),\ L \mapsto c_1(L),$$

is not assumed to be a group homomorphism, but only a natural transformation of *pointed sets*. In fact, a classical remark due to Quillen [20, Proposition 2.7] (see also Corollary 10.9 below) describes the way c_1 is not additive as follows:

Lemma 1. *Let A^* be an oriented cohomology theory on $\mathbf{Sm}_k$. Then for any line bundle $\mathcal{L}$ on $X \in \mathbf{Sm}_k$ the class $c_1(\mathcal{L})^n$ vanishes for n large enough*[1]*. Moreover, there is a unique power series*

$$F_A(u,v) = \sum_{i,j} a_{i,j}\, u^i\, v^j \in A^*(k)[[u,v]]$$

with $a_{i,j} \in A^{1-i-j}(k)$, such that, for any $X \in \mathbf{Sm}_k$ and any pair of line bundles L, M on X, we have

$$F_A(c_1(L), c_1(M)) = c_1(L \otimes M).$$

In addition, the pair $(A^*(k), F_A)$ is a commutative formal group.

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[1] In fact we will prove later on that $n > \dim_k(X)$ suffices.

Recall from [10] that a commutative formal group law of rank one with coefficients in A is a pair (A, F) consisting of a commutative ring A and a formal power series

$$F(u,v) = \sum_{i,j} a_{i,j}\, u^i\, v^j \in A[[u,v]]$$

such that the following holds:

1. $F(u,0) = F(0,u) = u \in A[[u]]$.
2. $F(u,v) = F(v,u) \in A[[u,v]]$.
3. $F(u, F(v,w)) = F(F(u,v),w) \in A[[u,v,w]]$

These properties of F_A reflects the fact that, for line bundles L, M, N on $X \in \mathbf{Sm}_k$, one has (denoting by O_X the trivial line bundle of rank one over X):

1' $L \otimes O_X = O_X \otimes L = L \in \mathrm{Pic}(X)$.
2' $L \otimes M = M \otimes L \in \mathrm{Pic}(X)$.
3' $L \otimes (M \otimes N) = (L \otimes M) \otimes N \in \mathrm{Pic}(X)$.

Lazard pointed out in [10] that there exists a universal commutative formal group law of rank one $(\mathbb{L}, F_{\mathbb{L}})$ and proved that the ring $\mathbb{L}$ (also called the *Lazard ring*) is a polynomial ring with integers coefficients on a countable set of variables x_i, $i \geq 1$. The construction of $(\mathbb{L}, F_{\mathbb{L}})$ is rather easy. Set $\tilde{\mathbb{L}} := \mathbb{Z}[\{A_{i,j} |\ (i,j) \in \mathbb{N}^2\}]$, and $\tilde{F}(u,v) = \sum_{i,j} A_{i,j}\, u^i\, v^j \in \tilde{\mathbb{L}}[[u,v]]$. Then define $\mathbb{L}$ to be the quotient ring of $\tilde{\mathbb{L}}$ by the relations obtained by imposing the relations (1), (2) and (3) above to $\tilde{F}$, and let

$$F_{\mathbb{L}} = \sum_{i,j} a_{i,j}\, u^i\, v^j \in \mathbb{L}[[u,v]]$$

denote the image of F by the homomorphism $\tilde{\mathbb{L}} \to \mathbb{L}$. It is clear that the pair $(\mathbb{L}, F_{\mathbb{L}})$ is the universal commutative formal group law of rank one, which means that to define a commutative formal group law of rank one (F, A) on A is equivalent to define a ring homomorphism $\Phi_F : \mathbb{L} \to A$.

The Lazard ring can be graded by assigning the degree $i + j - 1$ to the coefficient $a_{i,j}$. We denote by $\mathbb{L}_*$ this commutative graded ring. We could as well have graded it by assigning the degree $1 - i - j$ to the coefficient $a_{i,j}$, in which case we denote by $\mathbb{L}^*$ the corresponding commutative graded ring. For instance $\mathbb{L}^0 = \mathbb{L}_0 = \mathbb{Z}$ and $\mathbb{L}^{-n} = \mathbb{L}_n = 0$ if $n < 0$.

One can check then that for any oriented cohomology theory A^* the homomorphism of rings induced by the formal group law given by lemma 1 is indeed a homomorphism of graded rings

$$\Phi_A : \mathbb{L}^* \to A^*(k)$$

Example 1. The Chow ring $X \mapsto \mathrm{CH}^*(X)$ is a basic example of oriented cohomology theory on $\mathbf{Sm}_k$; this follows from [5]. In that case, the formal group law obtained on $\mathbb{Z} = \mathrm{CH}^*(k)$ by lemma 1 is the *additive* formal group law $F_a(u, v) = u + v$.

Example 2. Another fundamental example of oriented cohomology theory is given by the Grothendieck K^0 functor $X \mapsto K^0(X)$, where for X a smooth k-scheme, $K^0(X)$ denotes the Grothendieck group of locally free coherent sheaves on X. For $\mathcal{E}$ a locally free sheaf on X we denote by $[\mathcal{E}] \in K^0(X)$ its class. The tensor product of sheaves induces a unitary, commutative ring structure on $K^0(X)$. In fact we rather consider the graded ring $K^0(X)[\beta, \beta^{-1}] := K^0(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\beta, \beta^{-1}]$, where $\mathbb{Z}[\beta, \beta^{-1}]$ is the ring of Laurent polynomial in a variable β of degree -1.

It is endowed with pull-backs for any morphism $f : Y \to X$ by the formula:

$$f^*([\mathcal{E}] \cdot \beta^n) := f^*([\mathcal{E}]) \cdot \beta^n$$

for $\mathcal{E}$ a locally free coherent sheaf on X and $n \in \mathbb{Z}$. We identify $K^0(X)$ with the Grothendieck group $G_0(X)$ of all coherent sheaves on X by taking a finite locally free resolution of a coherent sheaf (X is assumed to be regular). This allows one to define push-forwards for a projective morphism $f : Y \to X$ of pure codimension d by the formula

$$f_*([\mathcal{E}] \cdot \beta^n) := \Sigma_{i=0}^{\infty} (-1)^i [R^i f_*(\mathcal{E})] \cdot \beta^{n-d} \in K_0(X)[\beta, \beta^{-1}]$$

for $\mathcal{E}$ a locally free sheaf on Y and $n \in \mathbb{Z}$. One can easily check using standard results that this is an oriented cohomology theory.

Moreover, for a line bundle L over X with with projection $\pi : L \to X$, zero section $s : X \to L$ and sheaf of sections $\mathcal{L}$, one has

$$s^*(s_*(1_X)) = s^*([\mathcal{O}_{s(X)}]\beta^{-1}) = s^*(1 - [\pi^*(\mathcal{L})^\vee])\beta^{-1} = (1 - [\mathcal{L}^\vee])\beta^{-1}$$

so that $c_1^K(L) := (1 - [\mathcal{L}^\vee])\beta^{-1}$. We thus find that the associated power series F_K is the *multiplicative formal group law*

$$F_m(u, v) := u + v - \beta uv$$

as this follows easily from the relation

$$(1 - [(\mathcal{L} \otimes \mathcal{M})^\vee]) = (1 - [\mathcal{L}^\vee]) + (1 - [\mathcal{M}^\vee]) - (1 - [\mathcal{L}^\vee])(1 - [\mathcal{M}^\vee])$$

in $K^0(X)$, where $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on X.

Definition 2. Let A^* be an oriented cohomology theory on $\mathbf{Sm}_k$ with associated formal group law F_A.

1) We shall say that A^* is *ordinary* if $F_A(u,v)$ is the additive formal group law.

2) We shall say that it is *multiplicative* if $F_A(u,v) = u + v - buv$ for some (uniquely determined) $b \in A^{-1}(k)$; we shall say moreover that A^* is *periodic* if b is a unit in $A^*(k)$.

Our main results on oriented cohomology theories are the following three theorems. In each of these statements, A^* denoted a fixed oriented cohomology theory on $\mathcal{M}_k$:

Theorem 1. *Assume k has characteristic zero. If A^* is ordinary then there exists one and only one morphism of oriented cohomology theories*

$$\vartheta_A^{\mathrm{CH}} : \mathrm{CH}^* \to A^*.$$

Theorem 2. *If A^* is multiplicative and periodic then there exists one and only one morphism of oriented cohomology theories*

$$\vartheta_A^K : K^0[\beta, \beta^{-1}] \to A^*.$$

Theorem 1 says that, in characteristic zero, the Chow ring functor is the universal ordinary oriented cohomology theory on $\mathbf{Sm}_k$. It seems reasonable to conjecture that this statement still holds over any field. Theorem 2 says that $K^0[\beta, \beta^{-1}]$ is the universal multiplicative and periodic oriented cohomology theory on $\mathbf{Sm}_k$.

Remark 1. The classical Grothendieck-Riemann-Roch theorem can be easily deduced from theorem 2, see remark 8.

Remark 2. Using theorem 2 and the fact that for any smooth k-scheme the Chern character induces an isomorphism

$$ch : K^0(X) \otimes \mathbb{Q} \cong \mathrm{CH}(X) \otimes \mathbb{Q}$$

(where CH denotes the ungraded Chow ring), it is possible to prove $\mathbb{Q}$-versions of theorem 2 and theorem 3 below over *any* field.

Well-known examples of ordinary cohomology theories are given by the "classical" ones: the even part of étale ℓ-adic cohomology theory (with $\ell \neq \mathrm{char}(k)$ a prime number), the de Rham cohomology theory over a field of characteristic zero, the even part of Betti cohomology associated to a complex embedding of the base field. In some sense theorem 1 and its rational analogue over any field explains, a priori, the existence of the cycle map in all these classical cohomology theories.

The following result introduces our main object of study:

Theorem 3. *Assume k has characteristic zero. Then there exists a universal oriented cohomology theory, denoted by*

$$X \mapsto \Omega^*(X),$$

which we call algebraic cobordism. *Thus, given an oriented cohomology theory A^*, there is a unique morphism*

$$\vartheta : \Omega^* \to A^*$$

of oriented cohomology theories.

In addition, we have two main results describing properties of the universal theory Ω^* which do not obviously follow from universality. The first may be viewed as an algebraic version of Quillen's identification of $MU^*(pt.)$ with $\mathbb{L}$:

Theorem 4. *For any field k of characteristic zero, the canonical homomorphism classifying F_Ω*

$$\Phi : \mathbb{L}^* \to \Omega^*(k)$$

is an isomorphism.

The second reflects the strongly algebraic nature of Ω_*:

Theorem 5. *Let $i : Z \to X$ be a closed immersion between smooth varieties over k, d the codimension of Z in X and $j : U \to X$ the open immersion of the complement of Z. Then the sequence*

$$\Omega^{*-d}(Z) \xrightarrow{i_*} \Omega^*(X) \xrightarrow{j^*} \Omega^*(U) \to 0$$

The construction of Ω^* is directly inspired by Quillen's description of complex cobordism [20]: For $f : Y \to X$ a projective morphism of codimension d from a smooth k-scheme Y to X denote by $[f : Y \to X]_A \in A^d(X)$ the element $f_*(1_Y)$. Then $\Omega^d(X)$ is generated as a group by all the isomorphisms classes of projective morphisms $Y \to X$ of codimension d with Y smooth. The morphism ϑ necessarily maps $Y \to X$ to $[f : Y \to X]_A$, which proves uniqueness of ϑ. Observe that $\Omega^n(X) = 0$ for $n > \dim(X)$. When $X = \operatorname{Spec} k$ we simply denote by $[Y] \in \Omega^{-d}(k)$ and $[Y]_A \in A^{-d}(k)$ the class of the projective smooth variety $Y \to \operatorname{Spec} k$ of dimension d.

Remark 3. One should note that the relations defining Ω_* are not just the obvious "algebraization" of the complex cobordism relations. Indeed, one can consider projective morphisms of the form $f : Y \to$

$X \times \mathbb{A}^1$ with Y smooth and f transverse to the inclusion $X \times \{0,1\} \to X \times \mathbb{A}^1$. Letting $f_0 : Y_0 \to X$, $f_1 : Y_1 \to X$ be the pull-backs of f via $X \times 0 \to X \times \mathbb{A}^1$ and $X \times 1 \to X \times \mathbb{A}^1$, respectively, we do have the relation

$$[f_0 : Y_0 \to X] = [f_1 : Y_1 \to X]$$

in $\Omega^*(X)$. However, imposing only relations of this form on the free abelian group of isomorphism classes of projective morphisms $f : Y \to X$ (with Y irreducible and smooth over k) does not give $\Omega^*(X)$, even for $X = \operatorname{Spec} k$, and even for algebraically closed k. To see this, consider $\Omega^{-1}(k)$, i.e., the part of $\Omega^*(k)$ generated by the classes of smooth projective curves C over k. Clearly, the genus is invariant under the "naive" cobordisms given by maps $Y \to \mathbb{A}^1$, but we know that $\mathbb{L}^{-1} \cong \mathbb{Z}$, generated by the class of $\mathbb{P}^1$. Thus, if one uses only the naive notion of algebraic cobordism, it would not be possible to make a curve of genus $g > 0$ equivalent to $(1-g)\mathbb{P}^1$, as it should be.

Example 3. In [20], Quillen defines a notion of *complex oriented cohomology theory* on the category of differentiable manifolds and pointed out that complex cobordism theory $X \mapsto MU^*(X)$ can be interpreted as the universal such theory. Our definition 1 is so inspired by Quillen's axioms that given a complex imbedding $\sigma : k \to \mathbb{C}$, it is clear that the functor $X \mapsto MU^{2*}(X_\sigma(\mathbb{C}))$ admits a canonical structure of oriented cohomology theory ($X_\sigma(\mathbb{C})$ denoting the differentiable manifold of complex points of $X \times_k \mathbb{C}$). From the universality of algebraic cobordism we get for any $X \in \mathbf{Sm}_k$ a canonical morphism of graded rings

$$\Omega^*(X) \to MU^{2*}(X_\sigma(\mathbb{C})).$$

Given a complex embedding $\sigma : k \to \mathbb{C}$ the previous considerations define a ring homomorphism

$$\Phi^{top} : \Omega^* \to MU^{2*}.$$

In very much the same way, given an extension of fields $k \subset K$ and a k-scheme X denote by X_K the scheme $X \times_{\operatorname{Spec} k} \operatorname{Spec} K$. For any oriented cohomology theory A^* on $\mathbf{Sm}_K$, the functor

$$(\mathbf{Sm}_k)^{op} \to R^*, X \mapsto A^*(X_K)$$

is an oriented cohomology theory on $\mathbf{Sm}_k$. In particular, we get natural morphisms $\Omega^*(X) \to \Omega^*(X_K)$, giving in the case $X = \operatorname{Spec} k$ a canonical ring homomorphism

$$\Omega^*(k) \to \Omega^*(K).$$

Theorem 4 easily implies:

Corollary 6. *Let k be a field of characteristic zero.*

1) Given a complex embedding $\sigma : k \to \mathbb{C}$ the canonical homomorphism

$$\Phi^{top} : \Omega^*(k) \to MU^{2*}$$

is an isomorphism.

2) Given a field extension $k \subset F$, the canonical homomorphism

$$\Omega^*(k) \to \Omega^*(F)$$

is an isomorphism.

Remark 4. Suppose $\mathrm{char}(k) = 0$. Let X be a smooth irreducible quasi-projective k-scheme, with field of functions K. One then has a canonical homomorphism of rings $\Omega^*(X) \to \Omega^*(K)$ defined as the composition of the canonical morphism $\Omega^*(X) \to \Omega^*(X_K)$ (extension of scalars) with the restriction $\Omega^*(X_K) \to \Omega^*(K)$ to the tautological K-point of X_K. It corresponds to "taking the generic fiber" in the sense that given a projective morphism $f : Y \to X$ of relative codimension d and generic fiber $Y_K \to \mathrm{Spec}\, K$, a smooth projective K-scheme, its image by the previous homomorphism is the class $[Y_K] \in \Omega^d(K)$.

The composition $\Omega^*(k) \to \Omega^*(X) \to \Omega^*(K)$ is an isomorphism by 2) of corollary 6. We denote by

$$\delta : \Omega^*(X) \to \Omega^*(k)$$

the composition of $\Omega^*(X) \to \Omega^*(K)$ and the inverse isomorphism $\Omega^*(K) \to \Omega^*(k)$. Now, for a morphism $f : Y \to X$ of relative codimension 0, we have the *degree* of f, denoted $\deg(f)$, which is zero if f is not dominant and equal to the degree of the field extension $k(X) \to k(Y)$ if f is dominant. We observe that $\Omega^0(k)$ is canonically isomorphic to $\mathbb{Z}$ and that through this identification, $\delta([f : Y \to X]) = \deg(f)$ in case f has relative codimension zero.

From theorem 5 and corollary 6 we get the following result, which is a very close analogue of the fundamental results in [20] concerning complex cobordism.

Corollary 7. *Let k be a field of characteristic zero and let X be in $\in \mathbf{Sm}_k$. Then $\Omega^*(X)$ is generated as an $\mathbb{L}^*$-module by the classes of non-negative degrees (Recall that $\mathbb{L}^*$ is concentrated in degrees ≤ 0).*

Indeed corollary 6, with $F = k(X)$, implies that a given element $\eta \in \Omega^*(X)$ is "constant" over some open subscheme $j : U \to X$ of X:

$$j^*\eta = \delta(\eta) \cdot 1_U.$$

By theorem 5, the difference $\eta - \delta(\eta)\cdot 1_X$ comes from Ω^* of some proper closed subscheme Z (after removing the singular locus of Z), and noetherian induction completes the proof. In fact, since each reduced closed subscheme Z of X has a smooth birational model $\tilde{Z} \to Z$, we get the following more precise version, which we call the *generalized degree formula*:

Theorem 8. *Let k be a field of characteristic zero. Let X be in $\mathbf{Sm}_k$. For each closed integral subscheme $Z \subset X$ let $\tilde{Z} \to Z$ be a projective birational morphism with $\tilde{Z}$ smooth quasi-projective over k and $[\tilde{Z} \to X] \in \Omega^*(X)$ denote the class of the projective morphism $\tilde{Z} \to X$. Then $\Omega^*(X)$ is generated as an $\mathbb{L}^*$-module by the classes $[\tilde{Z} \to X]$.*

In particular, for any irreducible $X \in \mathbf{Sm}_k$, $\Omega^*(X)$ is generated as an $\mathbb{L}^*$-module by the unit $1 \in \Omega^0(X)$ and by the elements $[\tilde{Z} \to X]$ with $\dim(\tilde{Z}) < \dim(X)$, that is to say of degrees > 0 in $\Omega^*(X)$. More precisely, for $\eta \in \Omega^*(X)$, there are integral proper closed subschemes Z_i of X, and elements $\alpha_i \in \Omega^*(k)$, $i = 1, \ldots, r$, such that

$$\eta = \delta(\eta)\cdot[\mathrm{Id}_X] + \sum_{i=1}^{r} \alpha_i \cdot [\tilde{Z}_i \to X]. \tag{0.1}$$

Given a smooth projective irreducible k-scheme X of dimension $d > 0$, Rost introduces (see [15]) the ideal $M(X) \subset \mathbb{L}^* = \Omega^*(\operatorname{Spec} k)$ generated by classes $[Y] \in \mathbb{L}^*$ of smooth projective k-schemes Y of dimension $< d$ for which there exists a morphism $Y \to X$ over k. The following result establishes Rost's degree formula as conjectured in [15]. It is an obvious corollary to theorem 8 and remark 4.

Theorem 9. *Let k be a field of characteristic zero. For any morphism $f : Y \to X$ between smooth projective irreducible k-schemes then the class $[Y] - \delta(f)[X]$ of $\mathbb{L}^*$ lies in the ideal $M(X)$. In other words, one has the following equality in the quotient ring $\mathbb{L}^*/M(X)$:*

$$[Y] = \delta(f)\cdot[X] \in \mathbb{L}^*/M(X).$$

We shall also deduce the following

Theorem 10. *Let k be a field of characteristic zero. Let X be a smooth projective k-variety.*

1) The ideal $M(X)$ is a birational invariant of X.

2) Moreover, the class of X modulo $M(X)$:

$$[X] \in \mathbb{L}^*/M(X)$$

is a birational invariant of X as well.

For instance, let $d \geq 1$ be an integer and N_d the d-th Newton polynomial in the variables $c_1, \ldots, c_d$. Recall that if we consider the c_i's as the d elementary symmetric functions in the $x_1, \ldots, x_d$, then

$$N_d(c_1, \ldots, c_d) = \Sigma_i x_i^d.$$

If X is smooth projective of dimension d, we set

$$s_d(X) := -\deg N_d(\tau_X) \in \mathbb{Z},$$

τ_X denoting the tangent bundle of X and $\deg : \mathrm{CH}^d(X) \to \mathbb{Z}$ the usual degree homomorphism. One checks that if X and Y are smooth projective k-scheme of dimension d and d', one has $s_{d+d'}(X \times Y) = 0$ if both $d > 0$ and $d' > 0$. We also know (see [2]) that if d is of the form $p^n - 1$ where p is a prime number and $n > 0$, then $\frac{s_d(X)}{p}$ is always an integer. In that case, using theorem 8 and observing that if $\dim(Z) < d$ then $s_d([W] \cdot [Z]) \neq 0$ implies $\dim(Z) = 0$ one obtains the following result:

Corollary 11. *Let $f : Y \to X$ be a morphism between smooth projective varieties of dimensions $d > 0$. Assume that $d = p^n - 1$ where p is a prime number and $n > 0$. Then there exists a 0-cycle on X with integral coefficients whose degree is the integer*

$$\frac{s_d(Y)}{p} - \deg(f) \cdot \frac{s_d(X)}{p}.$$

This formula was first proven by Rost[2], and then generalized further by Borghesi [3].

Consider now the graded rings homomorphisms

$$\Phi_a : \mathbb{L}^* \to \mathbb{Z}$$

and

$$\Phi_m : \mathbb{L}^* \to \mathbb{Z}[\beta, \beta^{-1}]$$

classifying the respectively the additive and multiplicative formal group laws.

Theorem 3 obviously implies that, over a field of characteristic zero, the ordinary oriented cohomology theory

$$X \mapsto \Omega^*(X) \otimes_{\mathbb{L}^*} \mathbb{Z}$$

obtained by extension of scalars from Ω^* via Φ_a, is the universal ordinary oriented cohomology theory. In the same way 3 implies that, over

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[2] V. Voevodsky had considered weaker forms before.

a field of characteristic zero, the multiplicative oriented cohomology theory

$$X \mapsto \Omega^*(X) \otimes_{\mathbb{L}^*} \mathbb{Z}[\beta, \beta^{-1}]$$

obtained by extending the scalars from Ω^* via Φ_m is the universal multiplicative periodic oriented cohomology theory. Over a field of characteristic zero, we get from theorem 3 canonical morphisms of oriented cohomology theories

$$\Omega^* \to \mathrm{CH}^*$$

and

$$\Omega^* \to K^0[\beta, \beta^{-1}].$$

We immediately deduce from theorems 2 and 3 the following result:

Theorem 12. *Over a field of characteristic zero, the canonical morphism*

$$\Omega^* \to K^0[\beta, \beta^{-1}]$$

induces an isomorphism

$$\Omega^* \otimes_{\mathbb{L}^*} \mathbb{Z}[\beta, \beta^{-1}] \cong K^0[\beta, \beta^{-1}].$$

Theorem 12 is the analogue of a well-known theorem of Conner and Floyd [4]. Theorems 1 and 3 similarly imply the analogous relation between Ω^* and CH^*:

Theorem 13. *Let k be a field of characteristic zero. Then the canonical morphism*

$$\Omega^* \to \mathrm{CH}^*$$

induces an isomorphism

$$\Omega^* \otimes_{\mathbb{L}^*} \mathbb{Z} \to \mathrm{CH}^*.$$

In fact, we prove theorem 13 first, using theorem 3, theorem 4 and some explicit computations of the class of a blow-up of a smooth variety along a smooth subvariety. We then deduce theorem 1 from theorems 3 and 13.

Remark 5. Theorem 13, together with the natural transformation described in example 3, immediately implies a result of B. Totaro [22] constructing for any smooth $\mathbb{C}$-variety X, a map

$$\mathrm{CH}^*(X) \to MU^{2*}(X) \otimes_{\mathbb{L}^*} \mathbb{Z}$$

factoring the topological cycle class map $\mathrm{CH}^*(X) \to H^{2*}(X, \mathbb{Z})$ through the natural map $MU^{2*}(X) \otimes_{\mathbb{L}^*} \mathbb{Z} \to H^{2*}(X, \mathbb{Z})$.

Remark 6. *Unoriented cobordism.* Let $X \mapsto MO^*(X)$ denote unoriented cobordism theory and $MO^* := MO^*(point)$ the unoriented cobordism of a point, as studied by Thom [21]. Given a real embedding $\sigma : k \to \mathbb{R}$, then for any smooth k-scheme X of dimension d denote by $X_\sigma(\mathbb{R})$ the differentiable manifold (of dimension d) of real points of X. Then clearly, the assignment

$$X \mapsto [X_\sigma(\mathbb{R})] \in MO^*(X)$$

has a structure of oriented cohomology theory on $\mathbf{Sm}_k$ (one can use [20] ; observe in that case that the associated theory of Chern classes is nothing but the theory of Stiefel-Whitney classes in $MO^*(X)$). Thus we get from universality of Ω^* a natural transformation

$$\Omega^*(X) \to MO^*(X_\sigma(\mathbb{C}))$$

From theorem 4 we thus get for any real embedding $k \to \mathbb{R}$ a natural homomorphism:

$$\Psi_{k\to\mathbb{R}} : \mathbb{L}^* \cong \Omega^*(k) \to MO^*$$

which (using corollary 6) doesn't depend on k, so that we assume $k = \mathbb{R}$. Then $\Psi_\mathbb{R} : \mathbb{L}^* = \Omega^*(\operatorname{Spec}\mathbb{R}) \to MO^*$, is the map which sends the class $[X]$ of a smooth projective variety X over $\mathbb{R}$ to the unoriented class of the differentiable manifold $X(\mathbb{R})$ of real points.

From [20], the theory of Stiefel-Whitney classes in MO^* defines an isomorphism of rings

$$\mathbb{L}^*/[2] \to MO^*$$

where $[2]$ denotes the (coefficients of the) power series $[2](u) := F_\mathbb{L}(u,u)$. One easily checks that the induced epimorphism $\mathbb{L}^* \to MO^*$ is the homomorphism $\Psi_\mathbb{R}$ above.

From all this follows a geometric interpretation of the map $\Psi : \mathbb{L}^* \to \mathbb{L}^*/[2]$ using the identifications $\mathbb{L}^* = \Omega^*(\mathbb{R}) = MU^{2*}$ and $\mathbb{L}^*/[2] = MO^*$: let $x \in MU^{2n}$ be an element represented by a smooth projective variety X over $\mathbb{R}$. Then $\Psi(x)$ is equal to the unoriented cobordism class $[X(\mathbb{R})]$ (which thus only depends on x).

Remark 7. The hypothesis of characteristic zero in theorems 3, and the related theorem 12 is needed only to allow the use of resolution of singularities, and so these results are valid over any field admitting resolution of singularities in the sense of Appendix A. Theorem 4 uses resolution of singularities as well as the weak factorization theorem of [1]. Thus theorems 1 and 13 rely on both resolution of singularities and the weak factorization theorem.

Our definition of the homomorphism δ, on the other hand, relies at present on the generic smoothness of a morphism $Y \to X$ of smooth

k-schemes, hence is restricted to characteristic zero, regardless of any assumptions on resolution of singularities. Thus, the explicit formula (0.1) in theorem 8 relies on characteristic zero for its very definition. However, corollary 7 and the other statements of theorem 8 can be proved directly from theorem 13, hence are valid over a field k admitting resolution of singularities and the weak factorization theorem.

At this point, let's give some heuristic explanation of the whole picture.

For X a finite CW-complex, one can define its singular cohomology groups with integral coefficients $H^*(X;\mathbb{Z})$, its complex K-theory $K^*(X)$, and its complex cobordim $MU^*(X)$ see [2] for instance. These are complex oriented cohomology theories, they admit a theory of Chern classes and the analogue of lemma 1 implies the existence of a canonical ring homomorphism from $\mathbb{L}^*$ to the coefficient ring of the theory (which double the degrees with our conventions).

Quillen in [19] refined Milnor's computation [14] that that the complex cobordism MU^* of a point is a polynomial algebra with integral coefficient by showing that the map

$$\Phi^{top} : \mathbb{L}^* \to MU^{2*}$$

is an isomorphism (here we mean that Φ^{top} double the degrees and that the odd part of MU^* vanishes). Then in [20], Quillen produced a geometric proof of that fact emphasizing that MU^* is the universal complex oriented cohomology theory on the category of differentiable manifolds.

The theorem of Conner-Floyd [4] now asserts that for each CW-complex X the map

$$MU^*(X) \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}] \to K^*(X)$$

is an isomorphism (beware that in topology β has degree -2).

However, in general for a CW-complex X the homomorphism

$$MU^*(X) \otimes_{\mathbb{L}} \mathbb{Z} \to H^*(X;\mathbb{Z})$$

is not an isomorphism (not even surjective), even when restricted to the even part. Thus contrary to theorem 12, theorem 13 has no obvious counterpart in topology.

To give a heuristic explanation of our results we should mention that for smooth varieties over a field singular cohomology is replaced by motivic cohomology $H^{*,*}(X;\mathbb{Z})$, complex K-theory by Quillen's algebraic K-theory $K^{*,*}(X)$ and complex cobordism by the theory $MGL^{*,*}$ represented by the algebraic Thom complex MGL see [25]. Observe here that the theories takes values in the category of bigraded rings, the first degree corresponding to the cohomological degree and the second to the weight. Then in that case one should still have the Conner-Floyd isomorphism[3]

$$MGL^{*,*}(X) \otimes_{\mathbb{L}^*} \mathbb{Z}[\beta, \beta^{-1}] \cong K^{*,*}(X)\mathbb{Z}[\beta, \beta^{-1}]$$

for any simplicial smooth k-variety X (beware here that β has bidegree $(-2, -1)$). However the map $MGL^{*,*}(X) \otimes_{\mathbb{L}^*} \mathbb{Z} \to H^{*,*}(X)$ would almost never be an isomorphism. Instead one expects a spectral sequence[4] from motivic cohomology to $MGL^{*,*}(X)$; the filtration considered in section 14.8 should by the way be the one induced by that spectral sequence. Then theorem 13 is explained by its degeneration in the area computing the bidegrees of the form $(2n, n)$.

In fact, the geometric approach taken in the present work only deal with bidegrees of the form $(2n, n)$. Indeed, one can check that for any oriented bigraded cohomology theory $A^{*,*}$ in that setting the associated functor $X \mapsto \oplus_n A^{2n,n}(X)$ (graded by n) has a structure of oriented cohomology theory on $\mathbf{Sm}_k$ in our sens. In particular one gets morphisms $\Omega^*(X) \to \oplus_n MGL^{2n,n}(X)$ which we conjecture to be an isomorphism.

We are hopeful that our geometric approach can be extended to describe the whole bigraded algebraic cobordism, and that our results are only the first part of a general description of the functor $MGL^{*,*}$.

Most of the main results in this paper where announced in [12, 13]. The reader should notice that we have made a change of convention on degrees. In [12, 13] our cohomology theories were assumed to be take values in the category of graded commutative rings, and the push-forward maps were assumed to increase the degree by 2 times the codimension, and of course the Chern classes c_i were of degree $2i$. This had the advantage of fitting well with the notation used in topology. But as is clear from our constructions, we only deal with the even part, and for notational simplicity we have divided the degrees by 2.

The paper is organized as follows. In order to work in greater generality as in [5], instead of dealing only with cohomology theories on smooth varieties, we will construct Ω^* as an oriented Borel-Moore homology theory $X \mapsto \Omega_*(X)$ for X a finite type k-scheme.

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[3] This has been proven over any field by the second author jointly with M. Hopkins, unpublished.

[4] This spectral sequence has been announced in characteristic zero by the second author jointly with M. Hopkins, unpublished.

In Part I, we construct algebraic cobordism over any field as the universal “oriented Borel-Moore $\mathbb{L}_*$-functor of geometric type” on the category of finite type k-schemes. These functors are endowed with projective push-forwards, smooth pull-backs and action of the first Chern classes of line bundle. We also define algebraic cobordism by explicit generators and relations.

In Part II of our work, we establish our fundamental technical result: the localization theorem 5, when k is of characteristic zero. The rest of the Part II deduces from this theorem the standard properties of algebraic cobordism such as: the projective bundle formula and the extended homotopy invariance.

Part III introduces the dual notions of weak oriented cohomology theories and of weak oriented Borel-Moore homology theories. We then develop the theory of Chern classes for these theories, give some applications, and then prove all the theorems announced in the introduction. One should notice however that theorems 1, and 3 are only proven here in the weaker form where one replaces the notion of oriented cohomology theory by the notion of weak oriented cohomology theory. One should observe however that the proofs of the other theorems such as 2, the various degree formulas and 4) use only those weak forms.

Part IV of this work will appear in [11]. The difference between the notion of oriented cohomology theories (considered above) and its dual notion of oriented Borel-Moore homology theory introduced in Part IV, and the notion of weak oriented cohomology theories and its dual notion of weak Borel-Moore homology theory is that the latter have only pull-backs for smooth morphisms while the former have pull-backs for any local complete intersection morphisms (such as any morphism between smooth k-schemes). The construction of pull-backs in algebraic cobordism with respect to any local complete intersection morphism is given in Part IV (assuming resolution of singularities), which finishes the proof of theorems 1 and 3.

Notations and conventions. Throughout this paper, we let k be any field, unless otherwise stated. $\mathcal{V}$ will denote a category which is either the category $\mathbf{Sch}_k$ or the category $\mathbf{Sm}_k$. For any $X \in \mathbf{Sch}_k$, we shall always denote by $\pi_X : X \to \operatorname{Spec} k$ the structural morphism. By a smooth morphism, we will always mean a smooth and quasi-projective morphism. In particular, a smooth k-scheme will always be assumed to be quasi-projective over k.

For a scheme X, we have the equivalence of the categories of vector bundles over X with the category of locally free coherent sheaves on X, defined by sending a vector bundle to its sheaf of sections. We will use this equivalence to pass freely between vector bundles and locally free coherent sheaves.

We denote by $\mathcal{O}_X$ the structure sheaf over any scheme X and by O_X, or simply O when no confusion can arise, the trivial line bundle over X. Given a Cartier divisor $D \subset X$ we let $\mathcal{O}_X(D)$ denote the invertible determined by D and $O_X(D)$ the line bundle whose $\mathcal{O}_X$-module of section is $\mathcal{O}_X(D)$. For a vector bundle $E \to X$, we write $\mathcal{O}_X(E)$ for the sheaf of (germs of) sections of E.

For a locally free coherent sheaf $\mathcal{E}$ on a scheme X, we let $q : \mathbb{P}(\mathcal{E}) \to X$ denote the projective bundle $\mathrm{Proj}_{\mathcal{O}_X}(\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{E}))$, and $q^*\mathcal{E} \to \mathcal{O}(1)_{\mathcal{E}}$ the canonical quotient. For a vector bundle $E \to X$, we write $\mathbb{P}(E)$ for $\mathbb{P}(\mathcal{O}(E))$, and $q^*E \to O(1)_E$ for the canonical quotient. For $n > 0$, O^n_X will denote the trivial vector bundle of rank n over X, and we write γ_n for the line bundle $O(1)_{O^n_X}$ on $\mathbb{P}^n_X$.

We let $\mathbf{Ab}_*$ denote the category of graded abelian groups where we put the index down and we let $\mathbf{Ab}^*$ denote the category of graded abelian groups where we put the index up. We will identify these two categories using the functor $M_* \mapsto M^{-*}$. For a functor F defined on a sub-category of $\mathbf{Sch}_k$ we will usually write $F(k)$ instead of $F(\mathrm{Spec}\, k)$.

Part 1. Definition of algebraic cobordism

The basic structures that we emphasize in Part 1 consists of three types of operations: push-forwards f_* for projective morphisms, pull-backs f^* for smooth morphisms, and a first Chern class endomorphism $\tilde{c}_1(L)$ for each line bundle L.

In section 1 we develop the required formalism of oriented Borel-Moore functors which encodes these three structures. In section 2 we give the definition of algebraic cobordism as the universal oriented Borel-Moore functors satisfying some explicit geometric axioms. In the rest of Part 1 we give some basic computations and some elementary properties.

In this part k denotes any field and $\mathcal{V}$ denote either the category $\mathbf{Sch}_k$ or the category $\mathbf{Sm}_k$. We let $\mathcal{V}'$ denote the subcategory of $\mathcal{V}$ whose morphisms are the projective morphisms.

1. *Oriented Borel-Moore functors*

1.1. Push-forwards, pull-backs and first Chern classes.

Definition 1.2. A functor $H_* : \mathcal{V}' \to \mathbf{Ab}_*$ is called *additive* if for any finite family $(X_1, \ldots, X_r)$ of finite type k-schemes, the homomorphism

$$\oplus_{i=1}^r H_*(X_i) \to H_*(\amalg_{i=1}^r X_i)$$

induced by the (projective) morphisms $X_i \subset \amalg_{i=1}^r X_i$ is an isomorphism. Observe that in particular we must have

$$H_*(\emptyset) = 0.$$

Definition 1.3. An *oriented Borel-More functor* on $\mathcal{V}$ is given by:

(D1). An additive functor $H_* : \mathcal{V}' \to \mathbf{Ab}_*$.

(D2). For each smooth equidimensional morphism $f : Y \to X$ in $\mathcal{V}$ of relative dimension d a homomorphism of graded groups

$$f^* : H_*(X) \to H_{*+d}(Y).$$

(D3). For each line bundle L on X a homomorphism of graded abelian groups:

$$\tilde{c}_1(\mathcal{L}) : H_*(X) \to H_{*-1}(X).$$

These data satisfy the following axioms:

(A1). For any pair of composable smooth equidimensional morphisms $(f : Y \to X, g : Z \to Y)$ respectively of dimension d and e, one has

$$(f \circ g)^* = g^* \circ f^* : H_*(X) \to H_{*+d+e}(Z),$$

and $\mathrm{Id}_X^* = \mathrm{Id}_{H_*(X)}$ for any $X \in \mathcal{V}_k$.

(A2). Let $f : X \to Z$, $g : Y \to Z$ be morphisms in $\mathcal{V}$, giving the cartesian square

$$\begin{tikzcd} W \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z. \end{tikzcd}$$

Suppose that f is projective and g is smooth equidimensional. Then

$$g^* f_* = f'_* g'^*.$$

(A3). Given a projective morphism $f : Y \to X$ and a line bundle L over X, one has

$$f_* \circ \tilde{c}_1(f^* L) = \tilde{c}_1(L) \circ f_*$$

(A4). Given a smooth equidimensional morphism $f : Y \to X$ and a line bundle L over X, one has

$$\tilde{c}_1(f^* L) \circ f^* = f^* \circ \tilde{c}_1(L).$$

(A5). Given line bundles L and M on $X \in \mathcal{V}$ one has:

$$\tilde{c}_1(L) \circ \tilde{c}_1(M) = \tilde{c}_1(M) \circ \tilde{c}_1(L).$$

Moreover, if L and M are isomorphic, then $\tilde{c}_1(L) = \tilde{c}_1(M)$.

Remark 1.4. Let $L \to X$ be a line bundle on some $X \in \mathcal{V}$, and let $\mathcal{L}$ be the invertible sheaf of sections of L. As a matter of notation, we define $\tilde{c}_1(\mathcal{L}) : H_*(X) \to H_{*-1}(X)$ to be $\tilde{c}_1(L)$.

Given an oriented Borel-Moore functor H_* and a projective morphism $f : Y \to X$ the homomorphism $f_* : H_*(Y) \to H_*(X)$ is called the *push-forward* along f. For a smooth equidimensional morphism $f : Y \to X$ of relative dimension d , the homomorphism $f^* : H_*(X) \to H_{*+d}(Y)$ is called the *pull-back* along g. And for a line bundle L on X, the homomorphism $\tilde{c}_1(L)$ is called the first Chern class (operator) of L.

A morphism $\vartheta : G_* \to H_*$ of oriented Borel-Moore functors is a natural transformation $\vartheta : G_* \to H_*$ of functors $\mathcal{V}' \to \mathbf{Ab}_*$ which moreover commutes with the smooth pull-backs and the operators $\tilde{c}_1$.

Remark 1.5. Any oriented Borel-Moore functor $A_*(-)$ on $\mathbf{Sch}_k$ defines by restriction an oriented Borel-Moore functor on $\mathbf{Sm}_k$.

Conversaly, an oriented Borel-Moore functor A_* on $\mathbf{Sm}_k$ determines an oriented Borel-Moore functor A_*^{BM} on $\mathbf{Sch}_k$ as follows.

One first takes $A_*^{BM}(Y)$ for Y smooth to be $A_*(Y)$. Then for any finite type k-scheme X, we consider the category $\mathcal{C}/X$ whose objects are projective morphisms $Y \to X$ with Y smooth and morphisms from $Z \to X$ to $Y \to X$ are (projective) morphisms $Z \to Y$ over X. Then one sets

$$A_*^{BM}(X) := \mathrm{colim}_{Y \to X \in \mathcal{C}/X} A_*(Y).$$

Push-forward morphisms $f_* : A_*^{BM}(Y) \to A_*^{BM}(X)$ for a projective morphism $f : Y \to X$ in $\mathbf{Sch}_k$, pull-backs along smooth equidimensional morphisms, action of $\tilde{c}_1(L)$ and external products are induced in the obvious way by the same operation on the subcategory of smooth k-schemes. All the axioms are easy consequences of those on the category $\mathbf{Sm}_k$.

Of course, the restriction of the oriented Borel-Moore functor A_*^{BM} to $\mathbf{Sm}_k$ equals A_*. But the converse of course is not true in general. Given an oriented Borel-Moore functor A_* on $\mathbf{Sch}_k$ and given $X \in \mathbf{Sch}_k$ there is a canonical morphism

$$A_*^{BM}(X) = \mathrm{colim}_{Y \to X \in \mathcal{C}/X} A_*(Y) \to A_*(X),$$

which define a morphism of oriented Borel-Moore functors. In general this is not an isomorphism. In case it is we shall say that A_* is *generated by smooth schemes.*

For X a scheme of finite type over k, let's denote by $\mathcal{M}(X)$ the set of isomorphism classes (over X) of projective morphisms $Y \to X$ with Y in $\mathbf{Sm}_k$; $\mathcal{M}(X)$ becomes a monoid for the disjoint union. We let $\mathcal{M}_*^+(X)$ denote its group completion graded by the dimension over k of the Y's. Given a projective morphism $f : Y \to X$ with Y smooth, we let either $[f : Y \to X]$, or $[Y \to X]$ or $[f]$, depending on the context, denote the image of $f : Y \to X$ in $\mathcal{M}^+(X)$. We observe that the class of the empty scheme $\emptyset \to X$ is $[\emptyset \to X] = 0$ and that $\mathcal{M}_*^+(X)$ is the free abelian group on classes $[Y \to X]$ with $Y \to X$ projective and Y smooth and irreducible.

Given a projective morphism (in $\mathcal{V}'$) $f : Y \to X$, composition with f defines a graded group homomorphism $f_* : \mathcal{M}_*^+(Y) \to \mathcal{M}_*^+(X)$. Given a smooth equidimensional morphism $f : X' \to X$, of relative dimension d, one has the homomorphism $g^* : \mathcal{M}_*^+(X) \to \mathcal{M}_{*+d}^+(Y), [Z \to X] \mapsto [Z \times_X Y \to Y]$. It is easy to check that the operations f_* and f^* satisfies Axioms (A1) and (A2) of the above definition.

Let H_* be an oriented Borel-Moore functor and choose an element $a \in H_0(k)$. Following Quillen, we construct a canonical natural transformation

$$\vartheta_{H,a} : \mathcal{M}_*^+ \to H_*$$

of functors $\mathcal{V}' \to \mathbf{Ab}_*$ as follows: for each projective morphism $f : Y \to X$ with Y smooth and irreducible, set

$$\vartheta_{H,a}([f : Y \to X]) := f_* \circ \pi_Y^*(a),$$

where $\pi_Y : Y \to \operatorname{Spec} k$ denotes the structural morphism. Clearly the above natural transformation commutes with smooth pull-backs in $\mathcal{M}_*^+$.

Remark 1.6. It would have been possible to define algebraic cobordism only working with $\mathcal{M}_*^+(X)$ (see 4.17). Instead we will consider a slightly more sophisticated theory, mainly the universal oriented Borel-Moore functor $X \mapsto \mathcal{Z}_*(X)$ which we are going to construct ; $\mathcal{Z}_*$ is obtained from $\mathcal{M}_*^+$ by formally adding the first Chern classes operator. This approach simplifies the definition of algebraic cobordism.

Definition 1.7. Let X be a k-scheme of finite type.

1) A *cobordism cycle* over X is a family $(f : Y \to X, L_1, \ldots, L_r)$ consisting of:

1. a projective morphism $f : Y \to X$ with Y integral, smooth over k.
2. a finite sequence $(L_1, \ldots, L_r)$ of r line bundles over Y (this sequence has to be interpreted as empty if $r = 0$).

The dimension of $(f : Y \to X, L_1, \ldots, L_r)$ is $\dim_k(Y) - r \in \mathbb{Z}$.

2) An *isomorphism* Φ of cobordism cycles $(Y \to X, L_1, \ldots, L_r) \cong (Y' \to X, L'_1, \ldots, L'_{r'})$ is a triple $\Phi = (\phi : Y \to Y', \sigma, (\psi_1, \ldots, \psi_r))$ consisting of:

1. an isomorphism $\phi : Y \to Y'$ of X-schemes.
2. a bijection $\sigma : \{1, \ldots, r\} \cong \{1, \ldots, r'\}$ (so that r must equal r').
3. for each $i \in \{1, \ldots, r\}$ an isomorphism of line bundles over Y: $\psi_i : L_i \cong \phi^*(L'_{\sigma(i)})$.

3) We let $\mathcal{C}(X)$ be the set of isomorphism classes of cobordism cycles over X and $\mathcal{Z}(X)$ be the free abelian group on $\mathcal{C}(X)$. We observe that

the dimension of cobordism cycles makes $\mathcal{Z}_*(X)$ into a graded abelian group called the group of cobordism cycles on X. The image of a cobordism cycle $(f : Y \to X, L_1, \ldots, L_r)$ in this group is denoted $[f : Y \to X, L_1, \ldots, L_r]$, or simply $[f, L_1, \ldots, L_r]$, or $[Y \to X, L_1, \ldots, L_r]$, depending on the context.

If $Y \to X$ is a projective morphism with Y smooth over k, we denote by $[Y \to X] \in \mathcal{Z}_*(X)$ the sum of the classes $[Y_\alpha \to X]$ corresponding to the irreducible components Y_α of Y. We thus gets a natural graded homomorphism

$$\mathcal{M}_*^+(X) \to \mathcal{Z}_*(X),$$

which is easily seen to be a monomorphism. When X is smooth and equidimensional of dimension d, the class $[\mathrm{Id}_X : X = X] \in \mathcal{Z}_d(X)$ is simply denoted 1_X or even 1.

Remark 1.8. Clearly given finite type k-schemes X and X', the natural homomorphism

$$\mathcal{Z}_*(X) \oplus \mathcal{Z}_*(X') \to \mathcal{Z}_*(X \coprod X')$$

is an isomorphism of graded abelian groups, so that $\mathcal{Z}_*$ is additive.

Moreover if X is a finite type k-scheme and X_α are the irreducible components of X then clearly the homomorphism

$$\oplus_\alpha \mathcal{Z}_*(X_\alpha) \to \mathcal{Z}_*(X)$$

is an epimorphism.

Let $g : X \to X'$ be a projective morphism, with X (and X') in $\mathcal{V}$. Composition with g defines the map of graded groups

$$\begin{aligned} &g_* : \mathcal{Z}_*(X) \to \mathcal{Z}_*(X') \\ &[f : Y \to X, L_1, \ldots, L_r] \mapsto [g \circ f : Y \to X', L_1, \ldots, L_r], \end{aligned}$$

called the push-forward along g.

If $g : X \to X'$ is a smooth equidimensional morphism of relative dimension d, sending $[f : Y \to X, (L_1, \ldots, L_r)]$ to $[p_2 : (Y \times_X X') \to X', p_1^*(L_1), \ldots, p_1^*(L_r)]$ defines the homomorphism

$$g^* : \mathcal{Z}_*(X) \to \mathcal{Z}_{*+d}(X')$$

called pull-back along g.

Let X be k-scheme of finite type and let L be a line bundle on X. We call the homomorphism

$$\begin{aligned} &\tilde{c}_1(L) : \mathcal{Z}_*(X) \to \mathcal{Z}_{*-1}(X) \\ &[f : Y \to X, L_1, \ldots, L_r] \mapsto [Y \to X, L_1, \ldots, L_r, f^*(L)] \end{aligned}$$

the *first Chern class homomorphism* of L.

Remark 1.9. Let $(f : Y \to X, L_1, \dots, L_r)$ be a standard cobordism cycle on X. Then one obviously has the formulas (in $\mathcal{Z}_*(X)$):

$$\begin{aligned}[f : Y \to X, L_1, \dots, L_r] &= f_* \circ [\mathrm{Id}_Y, L_1, \dots, L_r] \\ &= f_* \circ \tilde{c}_1(L_r)([\mathrm{Id}_Y, L_1, \dots, L_{r-1}]) \\ &\vdots \\ &= f_* \circ \tilde{c}_1(L_r) \circ \cdots \circ \tilde{c}_1(L_1)(1_Y) \\ &= f_* \circ \tilde{c}_1(L_r) \circ \cdots \circ \tilde{c}_1(L_1) \circ \pi_Y^*(1),\end{aligned}$$

where $\pi_Y : Y \to \operatorname{Spec} k$ is the structural morphism and $1 \in \mathcal{Z}_0(k)$ is the class of the identity of $\operatorname{Spec} k$.

Lemma 1.10. *The functor*

$$\begin{aligned}\mathcal{Z}_* : \mathcal{V}' &\to \mathbf{Ab}_* \\ X &\mapsto \mathcal{Z}_*(X),\end{aligned}$$

endowed with the above operations of smooth pull-backs, and first Chern classes is an oriented Borel-Moore functor on $\mathcal{V}$. Moreover it is the universal one in the sense that given any oriented Borel-Moore functor H_* on $\mathcal{V}$ and an element $a \in H_0(k)$ there is one and only one morphism of oriented Borel-Moore functors

$$\vartheta_{H,a} : \mathcal{Z}_* \to H_*$$

such that $\vartheta_{H,a}(1) = a \in H_*(k)$.

The proof is rather easy. To prove the universality one uses remark 1.9 to show that one must have

$$\vartheta_{H,a}([f : Y \to X, L_1, \dots, L_r]) = f_* \circ \tilde{c}_1(L_r) \circ \cdots \circ \tilde{c}_1(L_1)(a)$$

Remark 1.11. *External products.* We define an external product on the functor $\mathcal{Z}_*$ as follows:

$$\begin{aligned}&\mathcal{Z}_*(X) \times \mathcal{Z}_*(Y) \to \mathcal{Z}_*(X \times_k Y) \\ &([f : X' \to X, L_1, \dots, L_r], [g : Y' \to Y, M_1, \dots, M_s]) \mapsto \\ &[f \times g : X' \times Y' \to X \times Y, p_1^*(L_1), \dots, p_1^*(L_r), p_2^*(M_1), \dots, p_2^*(M_s)]\end{aligned}$$

It is associative and commutative. Moreover $1 := [\mathrm{Id}_k] \in \mathcal{Z}_0(k)$ is a unit element for that product. In particular, $\mathcal{Z}_*(k)$ becomes a unitary, associative, commutative graded ring and for any $X \in \mathcal{V}$ the group $\mathcal{Z}_*(X)$ then becomes equiped with a natural structure of graded $\mathcal{Z}_*(k)$-module. We are led to the following definition:

Definition 1.12. An *oriented Borel-Moore functor with product* consists of an oriented Borel-Moore functor H_* together with:

(D4). An element $1 \in H_0(k)$ and, for each pair (X, Y) of finite type k-schemes, a bilinear graded pairing (called the external product)

$$\begin{aligned} \times : H_*(X) \times H_*(Y) &\to H_*(X \times Y) \\ (\alpha, \beta) &\mapsto \alpha \times \beta \end{aligned}$$

which is (strictly) commutative, associative, and admits 1 as unit.

These satisfy

(A6). Given projective morphisms f and g one has

$$f_* \times g_* = (f \times g)_*.$$

(A7). Given smooth equidimensional morphisms f and g, one has

$$f^* \times g^* = (f \times g)^*.$$

(A8). Given finite type k-schemes X and Y and a line bundle L on X one has for any classes $\alpha \in H_*(X)$ and $\beta \in H_*(Y)$

$$(\tilde{c}_1(L)(\alpha)) \times \beta = \tilde{c}_1(p_1^*(L))\big(\alpha \times \beta\big).$$

Given an oriented Borel-Moore functor with product A_* we observe that the axioms give $A_*(k)$ a commutative, graded ring structure, give to each $A_*(X)$ a structure of $A_*(k)$-module, and imply that all the operations f_*, f^* and $\tilde{c}_1(L)$ preserve the $A_*(k)$-module structure.

We also observe that, endowed with its external product, $\mathcal{Z}_*$ is an oriented Borel-Moore functor functor with product. Moreover, one easily checks that it is in fact the universal one: given an oriented Borel-Moore functor with product A_*, there exists one and only one morphism of oriented Borel-Moore functors with product

$$\vartheta_A : \mathcal{Z}_* \to A_*.$$

In fact, one simply checks that the transformation $\vartheta_{A,1}$ given by lemma 1.10 commutes with the external products.

Definition 1.13. Let R_* be a commutative graded ring with unit. An *oriented Borel-Moore R_*-functor* A_* is an oriented Borel-Moore functor with product together with a graded ring homomorphism

$$\Phi : R_* \to A_*(k).$$

For such a functor, one gets the structure of an R_*-module on $H_*(X)$ for each $X \in \mathcal{V}$, by using Φ and the external product. All the operations of projective push-forward, smooth pull-back, and $\tilde{c}_1$ of line bundles are R_*-linear.

For instance, given a oriented Borel-Moore R_*-functor A_* and a homomorphism of commutative graded ring $R_* \to S_*$, one can construct a multiplicative oriented S_* functor, denoted by $A_* \otimes_{R_*} S_*$, by the assignment $X \mapsto A_*(X) \otimes_{R_*} S_*$. The push-forward, smooth pull-back, and $\tilde{c}_1$ of line bundles are obtained by extension of scalars $(-) \otimes_{R_*} S_*$.

1.14. **Imposing relations.** Let H_* be an oriented Borel-Moore functor and, for each $X \in \mathcal{V}$, let $\mathcal{R}_*(X) \subset H_*(X)$ be a set of homogeneous elements. We will construct a new oriented Borel-Moore functor denoted by $H_*/\mathcal{R}_*$ together with a morphism of oriented Borel-Moore functors $\pi : H_* \to H_*/\mathcal{R}_*$ with the following universal property: given any oriented Borel-Moore functor G_* and any morphism of oriented functors $\vartheta : H_* \to G_*$ such that, for each X, the homomorphism $\vartheta(X) : H_*(X) \to G_*(X)$ vanishes on $\mathcal{R}_*(X)$, then there is one and only one morphism of oriented Borel-Moore functors $\varphi : H_*/\mathcal{R}_* \to G_*$ such that $\varphi \circ \pi = \vartheta$. This oriented Borel-Moore functor will then be said to be obtained from H_* by *killing the elements in the* $R_*(X)$, or that $H_*/\mathcal{R}_*$ is the *quotient of* H_* *by the relations* $\mathcal{R}_*$.

To construct $H_*/\mathcal{R}_*$, we proceed as follows: For $X \in \mathcal{V}$ denote by $<\mathcal{R}_*>(X) \subset H_*(X)$ the subgroup generated by elements of the following form:

$$f_* \circ \tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_r) \circ g^*(\rho) \tag{1.1}$$

with $f : Y \to X$ a projective morphism, $(L_1, \ldots, L_r)$, $r \geq 0$, a family of line bundles over Y, $g : Y \to Z$ a smooth equidimensional morphism and $\rho \in \mathcal{R}_*(Z)$. Then $<\mathcal{R}_*>$ is a oriented Borel-Moore sub-functor of H_*, and is the smallest one which contains each of the $\mathcal{R}_*(X)$.

The assignment $X \mapsto H_*(X)/<\mathcal{R}_*>(X)$ has thus a unique structure of oriented Borel-Moore functor which makes the canonical projection $\pi : H_*(X) \to H_*(X)/<\mathcal{R}_*>(X)$ a morphism of oriented Borel-Moore functors. We denote by $H_*/\mathcal{R}_*$ this oriented Borel-Moore functor. It is clear that the morphism $\pi : H_* \to H_*/\mathcal{R}_*$ is a solution to our problem.

Remark 1.15. Let A_* be an oriented Borel-Moore functor with product. Assume we are given for each X a set $\mathcal{R}_*(X)$ of homogeneous elements in $A_*(X)$ such that for $\rho \in A_*(X)$ and $\sigma \in A_*(Y)$ one has

$$\rho \times \sigma \in \mathcal{R}_*(X \times Y)$$

if either $\rho \in \mathcal{R}_*(X)$ or $\sigma \in \mathcal{R}_*(Y)$, then obviously, there is one and only one external product on the oriented Borel-Moore functor $H_*/\mathcal{R}_*$

compatible with the projection $H_* \to H_*/\mathcal{R}_*$. This statement easily follows from 1.1.

1.16. **Cohomological notations.** Le A_* be a oriented Borel-Moore functor on $\mathbf{Sm}_k$. For X in $\mathbf{Sm}_k$, let the X_α denote the irreducible components of X and set $d_\alpha := \dim_k(X_\alpha)$. We introduce the following notation

$$A^n(X) = \oplus_\alpha A_{d_\alpha - n}(X_\alpha)$$

Given a smooth morphism $f : Y \to X$ the pull-back morphism in A_* associated to f now defines a homomorphism of degree 0 $A^*(X) \to A^*(Y)$. Given a projective morphism $f : Y \to X$ of relative codimension d, the push-forward morphism in A_* now define push-forward $f_* : A^*(Y) \to A^{*+d}(X)$. The endomorphism $\tilde{c}_1(L)$ of $A_*(X)$ associated to a line bundle L on X induces an endomorphism, still denoted by $\tilde{c}_1(L)$, $A^*(X) \to A^{*+1}(X)$. Finally, an external product on A_* induces an external product $A^*(X) \otimes A^*(Y) \to A^*(X \times Y)$.

The assignment $X \mapsto A^*(X)$ will be called the *oriented cohomological functor* on $\mathbf{Sm}_k$ associated to A_*. One can rewrite all the axioms for an oriented Borel-Moore functor A_* on $\mathbf{Sm}_k$ and in terms of A^*. Clearly A^* and A_* are thus determined by each other and the category of oriented Borel-Moore functors on $\mathbf{Sm}_k$ is equivalent to that of oriented cohomological functors on $\mathbf{Sm}_k$.

2. *Algebraic cobordism*

2.1. **Oriented Borel-Moore functors of geometric type.** Recall from the introduction that $\mathbb{L}_*$ denotes the Lazard ring homologically graded (which means that $\mathbb{L}_n = 0$ if $n < 0$) and that $F_{\mathbb{L}}(u, v) \in \mathbb{L}_*[[u, v]]$ denotes the universal formal group law.

Definition 2.2. An oriented Borel-Moore $\mathbb{L}_*$-functor A_* on $\mathcal{V}$ is said to be of *geometric type* if the following three axioms holds:

(Dim). For any smooth k-scheme Y and any family $(L_1, \ldots, L_n)$ of line bundles on Y with $n > \dim_k(Y)$, one has

$$\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_n)(1_Y) = 0 \in A_*(Y).$$

(Sect). For any smooth k-scheme Y, any line bundle L on Y, any section s of L which is transverse to the zero section of L, one has

$$\tilde{c}_1(L)(1_Y) = [i : Z \to Y],$$

where $i : Z \to Y$ is the closed immersion of the zeros of s.

(FGL). Let $F_A \in A_*(k)[[u,v]]$ be the image by the homomorphism $\mathbb{L}_* \to A_*(k)$ (giving the $\mathbb{L}_*$-structure) of the power series $F_{\mathbb{L}}$. Then for any smooth k-scheme Y and any pair (L, M) of line bundles on Y, one has

$$F_A(\tilde{c}_1(L), \tilde{c}_1(M))(1_Y) = \tilde{c}_1(L \otimes M)(1_Y) \in A_*(Y).$$

Remark 2.3. For the left-hand side of Axiom (FGL) to make sense, we need the vanishing stated in (Dim) and the fact that $\tilde{c}_1(L)$ and $\tilde{c}_1(M)$ commute. In fact, only the following weak form of Axiom (Dim) is needed for the left-hand side of (FGL) to make sense:

(Nilp). For each smooth k-scheme Y there exists an integer N_Y such that, for each family $(L_1, \ldots, L_n)$ of line bundles on Y with $n > N_Y$, one has

$$\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_n)(1_Y) = 0 \in H_*(Y).$$

For instance, this property holds in H_* if, for each given X, $H_*(X)$ vanishes below some degree.

We will prove below (in remark 4.12) that for an oriented Borel-Moore $\mathbb{L}_*$-functor H_* which satisfies (Nilp), (Sect) and (FGL) then property (Dim) holds.

Example 2.4. The Chow group functor

$$X \mapsto \mathrm{CH}_*(X)$$

endowed with projective push-forward, smooth pull-backs, the action of the $\tilde{c}_1$ of line bundles and the external product of cycles (see [5, Chapters 1 & 2]) is an oriented Borel-Moore functor. Moreover, given line bundles L and M (over the same base) one has the formula

$$\tilde{c}_1(L \otimes M) = \tilde{c}_1(L) + \tilde{c}_1(M).$$

This gives CH_* an $\mathbb{L}_*$-structure: the formal group law is given by

$$F_a(u, v) = u + v$$

One easily checks (using the results of [5]) that CH_* is of geometric type. Moreover, the Chow group functor $X \mapsto \mathrm{CH}_*(X)$ can be seen to be detected by smooth k-schemes (in the sense of Section 1.5) when k has characteristic zero.

Example 2.5. Another fundamental example of oriented Borel-Moore functor is given by the G_0-theory $X \mapsto G_0(X)$, where $G_0(X)$ denotes Grothendieck K-group of the category of coherent $\mathcal{O}(X)$-modules (see Fulton [5, Chapter 15] for instance). If we denote by $K^0(X)$ the Grothendieck K-group of vector bundles then for X an arbitrary finite type k-scheme, the tensor product of locally free sheaves induces a unitary, commutative ring structure on $K^0(X)$ and $G_0(X)$ has a natural structure of $K^0(X)$-module: the action of a locally free sheaf $\mathcal{E}$ on $G_0(X)$ is just given by $\mathcal{E}. : G_0(X) \to G_0(X), [\mathcal{M}] \mapsto [\mathcal{M} \otimes_{\mathcal{O}(X)} \mathcal{E}]$.

Let $G_0(X)[\beta,\beta^{-1}]$ be $G_0(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\beta,\beta^{-1}]$, where $\mathbb{Z}[\beta,\beta^{-1}]$ is the ring of Laurent polynomial in a variable β of degree $+1$. Define push-forwards for a projective morphism $f : Y \to X$ by

$$f_*([\mathcal{M}] \cdot \beta^n) := \Sigma_{i=0}^{\infty}(-1)^i [R^i f_*(\mathcal{M})] \cdot \beta^n \in G_0(X)[\beta,\beta^{-1}]$$

for $\mathcal{M}$ a coherent $\mathcal{O}(Y)$-module and $n \in \mathbb{Z}$, thus defining a functor

$$\mathcal{V}' \to \mathbf{Ab}_*, X \mapsto G_0(X)[\beta,\beta^{-1}].$$

It is endowed with pull-backs along smooth equidimensional morphism $f : Y \to X$ of relative dimension d by the formula:

$$f^*([\mathcal{M}] \cdot \beta^n) := [f^*(\mathcal{M})] \cdot \beta^{n+d}.$$

The first Chern class endomorphism associated to the line bundle L on $X \in \mathbf{Sch}_k$ is defined by the multiplication by $(1 - [\mathcal{L}^\vee]) \cdot \beta$:

$$\tilde{c}_1(L) := (1 - [\mathcal{L}^\vee]) \cdot \beta : G_0(X)[\beta,\beta^{-1}] \to G_0(X)[\beta,\beta^{-1}],$$

where $\mathcal{L}$ is the invertible sheaf of sections of L. One can easily check that together with the external product

$$G_0(X)[\beta,\beta^{-1}] \times G_0(Y)[\beta,\beta^{-1}] \to G_0(X \times Y)[\beta,\beta^{-1}]$$

$$([\mathcal{M}],[\mathcal{N}]) \mapsto [\pi_X^*(\mathcal{M}) \otimes_{X\times Y} \pi_Y^*(\mathcal{N})]$$

our functor $X \mapsto G_0(X)[\beta,\beta^{-1}]$ is an oriented Borel-Moore functor with product on $\mathbf{Sch}_k$. Moreover, if L and M are line bundles over X the formula $\tilde{c}_1(L \otimes M) = \tilde{c}_1(L) + \tilde{c}_1(M) - \beta \circ \tilde{c}_1(L) \circ \tilde{c}_1(M)$ mentioned in the introduction, gives $G_0[\beta,\beta^{-1}]$ an $\mathbb{L}_*$-structure with associated formal group law

$$F_m(u,v) = u + v - \beta uv$$

One can then checks that $G_0[\beta,\beta^{-1}]$ is of geometric type. However, we do not know whether or not the functor $X \mapsto G_0(X)[\beta,\beta^{-1}]$ is detected by smooth k-schemes, even in characteristic zero.

Power series. Let A_* be an oriented Borel-Moore functor on $\mathcal{V}$. By lemma 1.10, we have the morphism of oriented Borel-Moore functors $\vartheta_{A_*,1} : \mathcal{Z}_* \to A_*$. For X in $\mathcal{V}$, let $\bar{A}_*(X)$ be the sub-$A_*(k)$-module of $A_*(X)$ generated by $\vartheta_{A_*,1}(\mathcal{Z}_*(X))$. It is easy to see that this defines an oriented Borel-Moore functor $\bar{A}_*$.

Now suppose that A_* satisfies axiom (Dim). Let $F(u_1, \ldots, u_r) \in A_*(k)[[u_1, \ldots, u_r]]$ be a formal power series in $(u_1, \ldots, u_r)$ with coefficients in the graded ring $A_*(k)$. Expanding

$$F(u_1, \ldots, u_n) = \sum_I a_I u^I$$

where I runs over the set of r-tuples $I = (n_1, \ldots, n_r)$ of integers and $a_I \in A_*(k)$, we will say that F is *absolutely homogeneous of degree* n if for each I, a_I is in $A_{|I|-n}(k)$, where $|I| = n_1 + \cdots + n_r$.

Given line bundles $(L_1, \ldots, L_r)$ on $X \in \mathbf{Sch}_k$ the operations $\tilde{c}_1(L_1), \ldots, \tilde{c}_1(L_r)$ are locally nilpotent on $\bar{A}_*(X)$ (by axiom (Dim)) and commute with each other. In the endomorphism ring $End(\bar{A}_*(X))$ we may thus substitute $\tilde{c}_1(L_i)$ for u_i in F and get a well-defined homogeneous element of degree $-n$:

$$F(\tilde{c}_1(L_1), \ldots, \tilde{c}_1(L_r)) : \bar{A}_*(X) \to \bar{A}_{*-n}(X) \subset A_{*-n}(X). \tag{2.1}$$

If X is a smooth equidimensional k-scheme of dimension d, we have the class $1_X \in \bar{A}_d(X)$ and we set

$$[F(L_1, \ldots, L_r)] := F(\tilde{c}_1(L_1), \ldots, \tilde{c}_1(L_r))(1_X) \in A_{d-n}(X)$$

Using these notations, the equation in Axiom (Sect) can be written as

$$[\mathcal{L}] = [Z \to Y]$$

and that in Axiom (FGL) as

$$[F_A(L, M)] = [L \otimes M].$$

Definition 2.6. Let X be a smooth and irreducible k-scheme. We let $\mathcal{R}_*^{Dim}(X) \subset \mathcal{Z}_*(X)$ denote the subset generated by all elements of the form

$$[Y \to X, L_1, \ldots, L_r], \text{ where } \dim_k(Y) < r.$$

We denote by $\underline{\mathcal{Z}}_*$ the oriented Borel-Moore functor $\mathcal{Z}_*/\mathcal{R}_*^{Dim}$.

Of course, $\underline{\mathcal{Z}}_*$ is by construction the universal oriented Borel-Moore functor on $\mathcal{V}$ satisfying (Dim).

The following lemma is easy to prove using 1.1:

Lemma 2.7. *Let X be a finite type k-scheme. The subgroup $<\mathcal{R}_*^{Dim}>(X)$ of $\mathcal{Z}_*(X)$ is exactly the one generated by standard cobordism cycles of the form:*

$$[Y \to X, \pi^*(L_1), \ldots, \pi^*(L_r), M_1, \ldots, M_s]$$

where Z is a smooth quasi-projective irreducible k-scheme, $(L_1, \ldots, L_r)$ are line bundles on Z, $\pi : Y \to Z$ is a smooth quasi-projective equidimensional morphism and $r > \dim_k(Z)$.

One easily checks using this lemma and remark 1.15 that the external product on $\mathcal{Z}_*$ descends to give $\underline{\mathcal{Z}}_*$ an external product which makes $\underline{\mathcal{Z}}_*$ an oriented Borel-Moore functor with product.

Remark 2.8. Of course, for any $X \in \mathbf{Sch}_k$ one has $\underline{\mathcal{Z}}_n(X) = 0$ if $n < 0$ by construction.

The following lemma is immediate:

Lemma 2.9. *Let X be a k-scheme of finite type and L a line bundle on X. Then the endomorphism $\tilde{c}_1(L)$ of $\underline{\mathcal{Z}}_n(X)$ is locally nilpotent, i.e., for each $a \in \underline{\mathcal{Z}}_n(X)$ there is an $n \in \mathbb{N}$ such that $(\tilde{c}_1(L))^n(a) = 0$.*

2.10. Algebraic pre-cobordism.

Definition 2.11. Let Y be a smooth and irreducible k-scheme. We let $\mathcal{R}_*^{Sect}(Y) \subset \underline{\mathcal{Z}}_*(Y)$ denote the subset composed by all elements of the form

$$[L] - [Z \to Y],$$

where L is a line bundle over Y, $s : Y \to L$ is a transverse section and $Z \to Y$ is the closed subscheme of zeroes of s (which is smooth over k by assumption on s). We denote by $\underline{\Omega}_*$ the oriented Borel-Moore functor $\underline{\mathcal{Z}}_*/\mathcal{R}^{Sect}$. It is called *algebraic pre-cobordism.*

Remark 2.12. In other words, the elements in $\mathcal{R}^{Sect}(Y)$ are exactly those of the form $[O_Y(Z)] - [Z \to Y]$ for $Z \to Y$ a smooth divisor.

The following lemma is easy to prove using 1.1:

Lemma 2.13. *Let X be a finite type k-scheme. The subgroup $<R_*^{Sect}>(X)$ of $\underline{\mathcal{Z}}_*(X)$ is exactly the one generated by elements of the form:*

$$[Y \to X, L_1, \ldots, L_r] - [Z \to X, i^*(L_1), \ldots, i^*(L_{r-1})]$$

with $r > 0$, $[Y \to X, L_1, \ldots, L_r]$ a standard cobordism cycle on X, and $i : Z \to Y$ the closed immersion of the subscheme defined by the vanishing of a transverse section $s : Y \to L_r$. Thus one has

$$[Y \to X, L_1, \ldots, L_r] = [Z \to X, i^*(L_1), \ldots, i^*(L_{r-1})]$$

in $\underline{\Omega}_*(X)$.

The elements of the above form are called *elementary cobordisms.* One easily checks using the previous lemma and remark 1.15 that the external product on $\underline{\mathcal{Z}}_*$ descends to give $\underline{\Omega}_*$ an external product which makes $\underline{\Omega}_*$ an oriented Borel-Moore functor with product.

Remark 2.14. Let $f : W \to X \times \mathbb{A}^1$ be a projective morphism with W smooth and irreducible. Suppose that the composite $p_2 \circ f : W \to \mathbb{A}^1$ is transverse to both $\{0\}$ and $\{1\}$, so that the closed subschemes $W_0 := f^{-1}(\{0\})$ and $W_1 := f^{-1}(\{1\})$ are smooth over k. We call the difference $[W_0 \to X] - [W_1 \to X] \in \mathcal{M}(X)^+$ a *naive cobordism*, we let $\mathcal{N}_*(X) \subset \mathcal{M}_*^+(X)$ denote the subgroup generated by those naive cobordisms and we denote by $\Omega_*^{naive}(X)$ the quotient $\mathcal{M}_*^+(X)/\mathcal{N}_*(X)$. It is clear that the image of this group vanishes in $\underline{\Omega}_*(X)$. In fact the difference $[W_0 \to W] - [W_1 \to W]$ even vanishes in $\underline{\Omega}_*(W)$ because

$$[W_0 \to W] = [W, O_W] = [W_1 \to W],$$

where O_W denotes the trivial line bundle. We thus get a homomorphism $\Omega_*^{naive}(X) \to \underline{\Omega}_*(X)$. This homomorphism won't be an isomorphism in general. In fact it won't be a surjection because there are line bundles which have no sections transverse to the zero section.

2.15. **Definition of algebraic cobordism.** Recall from the introduction that we denote the Lazard ring by $\mathbb{L}_*$, and let $F_{\mathbb{L}}(u,v) = \sum_{i,j} a_{i,j} u^i v^j$ denote the power series defining the universal formal group law. $\mathbb{L}_*$ is graded and the degree of $a_{i,j}$ is $i+j-1$. Thus $F_{\mathbb{L}}(u,v)$ is absolutely homogeneous of degree 1. We also observe that $a_{i,j} = 0$ when $ij = 0$ unless $(i,j) = (1,0)$ or $(i,j) = (0,1)$, in which case $a_{1,0} = a_{0,1} = 1$. In the sequel we will consider the oriented Borel-Moore functor $X \mapsto \mathbb{L}_* \otimes \underline{\Omega}_*$ obtained by extension of scalars. This functors satisfies the Axiom (Dim) (and of course (Sect)).

Definition 2.16. Let Y be a smooth irreducible k-scheme. We let $\mathcal{R}_*^{FGL}(Y) \subset \mathbb{L}_* \otimes \underline{\Omega}_*(X)$ be the subset of elements of the form

$$[F_{\mathbb{L}}(L, M)] - [L \otimes M]$$

for any pair (L, M) of line bundles over Y.

If $S_* \subset \mathbb{L}_* \otimes \underline{\Omega}_*(X)$ is a graded subset, we denote by $\mathbb{L}_* \otimes S \subset \mathbb{L}_* \otimes \underline{\Omega}_*(X)$ the subset of elements of the form $a \otimes \rho$ with $a \in \mathbb{L}_*$ and $\rho \in S_*$.

Definition 2.17. We define *Algebraic cobordism*:

$$\Omega_* : X \mapsto \Omega_*(X),$$

to be the oriented Borel-Moore $\mathbb{L}_*$-functor on $\mathbf{Sch}_k$ which is the quotient of $\mathbb{L}_* \otimes \underline{\Omega}_*$ by the relations $\mathbb{L}_* \otimes \mathcal{R}_*^{FGL}$,

$$\Omega := \mathbb{L}_* \otimes \underline{\Omega}_* / {<}\mathbb{L}_* \otimes \mathcal{R}_*^{FGL}{>}.$$

Remark 2.18. We could have define algebraic cobordism directly as the quotient of $\mathbb{L}_*\otimes\underline{\mathcal{Z}}_*$ by $\mathbb{L}_*\otimes\big(\mathcal{R}_*^{Sect}\bigcup\mathcal{R}_*^{FGL}\big)$. One should observe that for the elements in $\mathcal{R}_*^{FGL}$ to be defined, however, we need to have some vanishing of products of the $\tilde{c}_1$, which is guaranted by axiom (Dim). This forces us to start by killing $\mathcal{R}_*^{Dim}$ first.

One easily observes that the sets $<\mathbb{L}_*\otimes\mathcal{R}_*^{FGL}>(X)$ satisfy the conditions of remark 1.15. The external product on $\mathbb{L}_*\otimes\underline{\Omega}_*$ thus descends to Ω_*, making algebraic cobordism an oriented Borel-Moore functor with product.

Let's denote the composite homomorphism $\mathbb{L}_*\to\mathbb{L}_*\otimes\underline{\Omega}_*(k)\to\Omega_*(k)$ by

$$\Phi(k):\mathbb{L}_*\to\Omega_*(k), a\mapsto[a].$$

$\Phi(k)$ turns Ω_* into an oriented Borel-Moore $\mathbb{L}_*$-functor of geometric type. The following theorem is clear by construction.

Theorem 2.19. *Algebraic cobordism is the universal oriented Borel-Moore $\mathbb{L}_*$-functor of geometric type. More precisely, given any oriented Borel-Moore $\mathbb{L}_*$-functor of geometric type A_*, there is a unique morphism of oriented Borel-Moore $\mathbb{L}_*$-functors*

$$\vartheta_A:\Omega_*\to A_*.$$

Remark 2.20. In fact, given any oriented Borel-Moore $\mathbb{L}_*$-functor of geometric type A_*, the morphism ϑ_A clearly induces a morphism

$$\tilde{\vartheta}_A:\Omega_*\otimes_{\mathbb{L}_*}A_*(k)\to A_*$$

from the oriented Borel-Moore $\mathbb{L}_*$-functor obtained from Ω_* by the extension of scalars $\mathbb{L}_*\to A_*$. Observe that the left hand side $X\mapsto\Omega_*(X)\otimes_{\mathbb{L}_*}A_*(k)$ is still an oriented Borel-Moore $\mathbb{L}_*$-functor of geometric type.

We shall say that A_* is *free* if the morphism $\tilde{\vartheta}_A:\Omega_*\otimes_{\mathbb{L}_*}A_*(k)\to A_*$ is an isomorphism. This means that to give a morphism of oriented Borel-Moore $\mathbb{L}_*$-functors from A_* to an oriented Borel-Moore $\mathbb{L}_*$-functor of geometric type B_* is the same as to give a factorization $\mathbb{L}_*\to A_*(k)\to B_*(k)$.

Many of our main results can be rephrased by saying that a given theory A_* is free.

For instance, we conjecture that the Chow groups functors as well as the K-theory functor are free, over any field. We will prove this conjecture in characteristic zero in Part III. For the K-theory functor we "only" need the resolution of singularities, for the Chow groups we need the resolution of singularities and the weak factorization theorem of [1].

Remark 2.21. Algebraic cobordism is detected by smooth k-schemes :

Lemma 2.22. *For any $X \in \mathbf{Sch}_k$ the homomorphism*

$$\Omega_*^{BM}(Y) = \mathrm{colim}_{Y \to X \in \mathcal{C}/X} \Omega_*(Y) \to \Omega_*(X)$$

is an isomorphism.

Proof. Indeed generators of $\Omega_*(X)$ clearly come from the left hand side, and this is still true for the relations: they all come from explicit relations on smooth k-schemes. □

3. *Some computations in algebraic cobordism*

In this section, A_* denotes any oriented Borel-Moore functor of geometric type and $F_A(u,v) = \sum_{i,j} a_{i,j} u^i v^j \in A_*(k)[[u,v]]$ its associated formal group law. For a projective morphism $Y \to X$ with Y smooth equidimensional, we simply denote by $[Y \to X] \in A_*(X)$, the class $[Y \to X]_A$.

Recall the existence of a unique power series $\chi_A(u) = \sum_{i>0} \alpha_i u^i \in \mathbb{L}[[u]]$ which satisfies the equality

$$F_A(u, \chi_A(u)) = 0$$

Thus, for each a line bundle L on $X \in \mathbf{Sm}_k$, we have in $A_*(X)$

$$\chi_A([L]) = [L^\vee]$$

An easy computation gives:

$$\chi_A(u) = -u + a_{1,1}u^2 - (a_{1,1})^2 u^3 + (\,(a_{1,1})^3 \;+\; a_{1,1} \cdot a_{2,1}\,)\,u^4 \\ + \text{ terms of degree } \geq 5 \tag{3.1}$$

3.1. **A universal formula for the blow-up.** Let $i : Z \to X$ be a closed embedding between smooth k-varieties. Let $X_Z \to X$ the blow-up of X at Z, η_i the conormal sheaf $\mathcal{I}_Z/\mathcal{I}_Z^2$ of i. Recall[5] that the *deformation to the normal cone* of i is the blow-up of $X \times \mathbb{P}^1$ along the closed subscheme $Z \times \{0\}$, which we denote by

$$\pi : Y \to X \times \mathbb{P}^1$$

We have the identification of the exceptional divisor $\mathbb{P}$ of π with the projective bundle of the conormal sheaf of $Z \times \{0\} \subset X \times \mathbb{P}^1$, i.e.

$$\mathbb{P} = \mathbb{P}(\eta_i \oplus \mathcal{O}_Z).$$

---

[5] see [5].

Also, X_Z is a closed subscheme in Y (the proper transform of $X \times \{0\}$ in fact).

Similarly, since π is an isomorphism over a neighborhood of $X \times \{1\}$ in $X \times \mathbb{P}^1$, we have the closed embedding $i_1 : X \to Y$.

Proposition 3.2. *Let $i : Z \to X$ be a closed embedding of smooth k-schemes. Let $X_Z \to X$ the blow-up of X along Z, η_i the conormal sheaf of i, $\mathbb{P} := \mathbb{P}(\eta_i \oplus \mathcal{O}_Z)$ and $q : \mathbb{P} \to Z$ the obvious projection. Then we has the equality (in $\mathbb{A}_d(Y)$):*

$$[X_Z \to Y] = [X \to Y] + \chi_A([O_Y(\mathbb{P})])$$

Proof. We have the identiy of divisors $\pi^*(X \times \{0\} = \mathbb{P} + X_Z$. Thus we have the isomorphisms

$$O_Y(\mathbb{P}) \otimes O_Y(X_Z) \cong (pr_{\mathbb{P}^1} \circ \pi)^*(O(1)),$$

where $O_Y(\mathbb{P})$ and $O_Y(X_Z)$ are the line bundles on Y of the divisors $\mathbb{P}$ and X_Z, respectively, and $O(1)$ is the tautological quotient line bundle on $\mathbb{P}^1$ (with sheaf of sections $\mathcal{O}(1)$).

Similarly, we have $O_Y(X) \cong (pr_{\mathbb{P}^1} \circ \pi)^*(O(1))$, and hence

$$O_Y(X_Z) \cong O_Y(X) \otimes O_Y(\mathbb{P})^\vee. \tag{3.2}$$

From lemma 4.2 and the isomorphism (3.2)

(3.3)

(3.4)

$$\begin{aligned}
[X_Z \to Y] &= \tilde{c}_1(O_Y(X_Z))(1_Y) \\
&= \sum_{i,j} a_{i,j}\tilde{c}_1(O_Y(X))^i \circ (\chi_A(\tilde{c}_1(O_Y(\mathbb{P})))^j\,(1_Y) \\
&= [X \to Y] + [\chi_A O_Y(\mathbb{P})] \\
&\qquad + \sum_{i\geq 1, j\geq 1} a_{i,j}\tilde{c}_1(O_Y(X))^i \circ (\chi_A(\tilde{c}_1(O_Y(\mathbb{P}))))^j\,(1_Y).
\end{aligned}$$

Now the restriction of $O_Y(\mathbb{P})$ to X, is clearly trivial since $\mathbb{P} \cap X = \emptyset$. In particular, in $A_*(Y)$ one has

$$\begin{aligned}
\tilde{c}_1(O_Y(\mathbb{P})) \circ \tilde{c}_1(O_Y(X))(1_Y) &= \tilde{c}_1(O_Y(\mathbb{P}))[X \to Y] \\
&= \tilde{c}_1(O_Y(\mathbb{P}))i_{1*}(1_X) \\
&= i_{1*}\big(\,\tilde{c}_1(i_1^* O_Y(\mathbb{P}))(1_X)\,\big) \\
&= i_{1*}\big(\,\tilde{c}_1(O_X)(1_X)\,\big) = 0.
\end{aligned}$$

Thus, in our formula (3.3), the terms with $i \geq 1$ and $j \geq 1$, all vanish, completing the proof. □

We can further simplify this formula. Let $g(u) \in A_*(k)[[u]]$ be the power series uniquely determined by the equation

$$\chi_A(u) = u \cdot g(u)$$

We thus have by 3.1

$$g(u) = -1 + a_{1,1}u - (a_{1,1})^2u^2 + (\,(a_{1,1})^3 \,+\, a_{1,1}.a_{2,1}\,)\,u^3 \\ +\ \text{terms of degree } \geq 4 \tag{3.5}$$

$g(u) = -1+$

Proposition 3.3. *Let $i : Z \to X$ be a closed immersion of smooth k-schemes. Let $X_Z \to X$ the blow-up of X at Z, η_i the conormal sheaf of i, $\mathbb{P} := \mathbb{P}(\eta_i \oplus \mathcal{O}_Z) \subset Y$ the exceptional divisor of π, and $q : \mathbb{P} \to Z$ the projection. Let $O_{\mathbb{P}}(-1)$ denote the dual of the canonical quotient line bundle $O_{\mathbb{P}}(1)$. Then one has the equality (in $A_d(X)$):*

$$[X_Z \to X] = [\mathrm{Id}_X] + i_* \circ q_*\big(\,[g(O_{\mathbb{P}}(-1))]\,\big)$$

Proof. Let $\phi : \mathbb{P} \to Y$ denote the inclusion, and let $E = \mathbb{P} \cap X_Z$; E is the exceptional divisor of the blow-up $X_Z \to X$. It is easy to see that E is defined by the vanishing of the composition

$$\mathcal{O}_{\mathbb{P}} \cong q^*\mathcal{O}_Z \to q^*(\eta_i \oplus \mathcal{O}_Z) \to \mathcal{O}(1),$$

so $\mathcal{O}_{\mathbb{P}}(E) \cong \mathcal{O}(1)$.

Since $\phi_*(1_{\mathbb{P}}) = \tilde{c}_1(O_Y(\mathbb{P}))(1_Y)$, one has the equality (in $A_*(Y)$):

$$\chi_A([O_Y(\mathbb{P})]) = \phi_*\big(\,g([\phi^*O_Y(\mathbb{P})])\,\big).$$

Since $X_Z \cap \mathbb{P} = \emptyset$, we have $\phi^*O_Y(X) \cong O_{\mathbb{P}}$. Thus, from (3.2) we see that

$$\phi^*O_Y(\mathbb{P}) \cong \phi^*O_Y(-X_Z) \cong O_{\mathbb{P}}(-E).$$

Thus $\phi^*O_Y(\mathbb{P}) \cong O_{\mathbb{P}}(-1)$, giving us the identity in in $A_*(Y)$:

$$\chi_A([O_Y(\mathbb{P})]) = \phi_*\big(\,g([O_{\mathbb{P}}(-1)])\,\big).$$

Substituting this identity in the formula of proposition 3.2, and pushing forward to $A_*(X)$ by the projective morphism $Y \to X \times \mathbb{P}^1 \to X$ yields the desired formula. □

3.4. **Projective spaces and Milnor's hypersurfaces.** Let $n > 0$ and $m > 0$ be integers. Recall that γ_n denotes the line bundle on $\mathbb{P}^n$ whose sheaf of sections is $\mathcal{O}(1)$. Write $\gamma_{n,m}$ for the line bundle $p_1^*(\gamma_n) \otimes p_2^*(\gamma_m)$ on $\mathbb{P}^n \times \mathbb{P}^m$.

We let $i : H_{n,m} \to \mathbb{P}^n \times \mathbb{P}^m$ denote the smooth closed subscheme defined by the vanishing of a transverse section of $\gamma_{n,m}$.

Remark 3.5. The smooth projective k-schemes $H_{n,m}$ are known as the Milnor hypersurfaces [2]. Taking $m \leq n$, it is easy to see that, choosing suitable homogeneous coordinates $X_0, \dots, X_n$ for $\mathbb{P}^n$ and $Y_0, \dots, Y_m$ for $\mathbb{P}^m$, $H_{n,m}$ is defined by the vanishing of $\sum_{i=0}^m X_i Y_i$. In particular, for $m = 1 < n$, $H_{n,m} \subset \mathbb{P}^n \times \mathbb{P}^1$ is the standard embedding of the blow-up of a linear $\mathbb{P}^{n-2}$ in $\mathbb{P}^n$. In general, the projection $H_{n,m} \to \mathbb{P}^m$ makes $H_{n,m}$ a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^m$. In the special case $n = m = 1$, we see that $H_{1,1}$ is (up to a change of coordinates) the diagonal $\mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 3.6. *Write the formal group law of A_* as $F_A(u,v) = \sum_{i,j} a_{ij} \cdot u^i \cdot v^j$. Then we have the equation*

$$[H_{n,m} \to \mathbb{P}^n \times \mathbb{P}^m] = \sum_{i\geq 0}^{n} \sum_{j \geq 0}^{m} a_{ij} [\mathbb{P}^{n-i} \times \mathbb{P}^{m-j} \to \mathbb{P}^n \times \mathbb{P}^m] \tag{3.6}$$

in $A_*(\mathbb{P}^n \times \mathbb{P}^m)$, where $\mathbb{P}^{n-i} \times \mathbb{P}^{m-j} \to \mathbb{P}^n \times \mathbb{P}^m$ is the product of linear embeddings $\mathbb{P}^{n-i} \to \mathbb{P}^n$, $\mathbb{P}^{m-j} \to \mathbb{P}^m$. We also have the equation

$$[H_{n,m}] = [\mathbb{P}^n] \cdot [\mathbb{P}^{m-1}] + [\mathbb{P}^{n-1}] \cdot [\mathbb{P}^m] + \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} [\mathbb{P}^{n-i}] \cdot [\mathbb{P}^{m-j}] \tag{3.7}$$

in $A_*(k)$.

Proof. The formula (3.7) follows from (3.6) by pushing forward from $\mathbb{P}^n \times \mathbb{P}^m$ to $\operatorname{Spec} k$, and noting that $[\mathbb{P}^{n-i}] \cdot [\mathbb{P}^{m-j}] = [\mathbb{P}^{n-i} \times \mathbb{P}^{m-j}]$, and that $a_{10} = a_{01} = 1$, $a_{n0} = a_{0m} = 0$ for $n > 1$ or $m > 1$.

To prove (3.6), one has the following computation in $A_*(\mathbb{P}^n \times \mathbb{P}^m)$:

$$\begin{aligned}
[H_{n,m} \to \mathbb{P}^n \times \mathbb{P}^m] &= \tilde{c}_1(\gamma_{n,m})(1_{\mathbb{P}^n \times \mathbb{P}^m}) = \tilde{c}_1(p_1^*(\gamma_n) \otimes p_2^*(\gamma_m)) \\
&= F_A(\tilde{c}_1(p_1^*(\gamma_n)), \tilde{c}_1(p_2^*(\gamma_m))) \\
&= \Big(\sum_{i \geq 1, j \geq 1} [a_{ij}].\tilde{c}_1(p_1^*(\gamma_n))^i \circ \tilde{c}_1(p_2^*(\gamma_m))^j \Big)(1_{\mathbb{P}^n \times \mathbb{P}^m}).
\end{aligned}$$

The last expression can be computed easily: since the sections of γ_n define hyperplanes in $\mathbb{P}^n$, applying the axioms (A3) and (Sect) repeatedly yields

$$\tilde{c}_1(p_1^*(\gamma_n))^i \circ \tilde{c}_1(p_2^*(\gamma_m))^j (1_{\mathbb{P}^n \times \mathbb{P}^m}) = [\mathbb{P}^{n-i} \times \mathbb{P}^{m-j} \to \mathbb{P}^n \times \mathbb{P}^m].$$

One thus gets the right hand side of formula (**??**), proving the lemma. □

Remark 3.7. We observe that the previous formula for $n = m = 1$ gives:

$$[H_{1,1}] = [\mathbb{P}^1] + [\mathbb{P}^1] + [a_{1,1}] \cdot 1.$$

Following remark 3.5, $H_{1,1}$ is isomorphic to the diagonal $\mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$, and the formula (3.7) yields

$$[a_{1,1}] = -[\mathbb{P}^1] \in A_1(k).$$

Remark 3.8. Following remark 3.7, we find that $H_{2,1} \subset \mathbb{P}^2 \times \mathbb{P}^1$ is isomorphic to the blow-up of a k-rational point $p = \operatorname{Spec} k$ in $\mathbb{P}^2$. Using the notation of proposition 3.3, we have $\eta_i \cong \mathcal{O}_p^2$, and hence $\mathbb{P} = \mathbb{P}(k \oplus k \oplus k) = \mathbb{P}^2$ and $O_{\mathbb{P}}(-1) = \gamma_2^\vee$. Using this with our formula for g and remark 3.3, we find

$$[H_{2,1} \to \mathbb{P}^2] = [\mathrm{Id}_{\mathbb{P}^2}] + i_* q_* (- [\mathrm{Id}_{\mathbb{P}^2}] - [\mathbb{P}^1] \cdot c_1(\gamma_2^\vee) - [\mathbb{P}^1]^2 c_1(\gamma_2^\vee)^2.$$

Let $\mathbb{P}^1 \to \mathbb{P}^2$ be the linear embedding. Since $c_1(\gamma_2^\vee) = \chi(c_1(\gamma_2) = \chi([\mathbb{P}^1 \to \mathbb{P}^2])$, we find that

$$[H_{2,1} \to \mathbb{P}^2] = [\mathrm{Id}_{\mathbb{P}^2}] + i_*(-[\mathbb{P}^2 \to p] + [\mathbb{P}^1] \cdot [\mathbb{P}^1 \to p]).$$

Pushing forward to $\operatorname{Spec} k$ yields $[H_{2,1}] = [\mathbb{P}^1]^2$.

The formula of lemma 3.6 therefore gives $[H_{2,1}] = [\mathbb{P}^2] + [\mathbb{P}^1] \cdot [\mathbb{P}^1] + a_{1,1} \cdot [\mathbb{P}^1] + a_{2,1}$ which finally implies (in $A_*(k)$)

$$a_{2,1} = [\mathbb{P}^1]^2 - [\mathbb{P}^2].$$

Remark 3.9. When $n > 1$ and $m > 1$, the lemma implies that in $A_*(k)$, $[H_{n,m}]$ equals $a_{n,m}$ modulo decomposable elements (sum of products of element of degree > 0).

4. *Some elementary properties*

Throughout this section A_* denotes an oriented Borel-Moore $\mathbb{L}_*$-functor of geometric type on $\mathcal{V}$,

4.1. **Properties of the action of first Chern classes.** Recall the existence of a (unique) power series $\chi(u) \in \mathbb{L}[[u]]$, with leading term $-u$ and which satisfies the equality

$$F_{\mathbb{L}}(u, \chi(u)) = 0.$$

We observe that χ is absolutely homogeneous of degree 1. In the sequel, we will freely use the following notations:

$$\begin{aligned} u +_F v &:= F_{\mathbb{L}}(u, v) \in \mathbb{L}[[u, v]], \\ [-1]_F u &:= \chi(u) \in \mathbb{L}[[u]], \\ u -_F v &:= F_{\mathbb{L}}(u, \chi(v)) \in \mathbb{L}[[u, v]]. \end{aligned}$$

We will also use the same notation for the image of these power series by the homomorphism $\mathbb{L}_* \to A_*(k)$. If f, g are locally nilpotent endomorphisms of an $\Omega_*(k)$-module, we also denote by $f +_F g$ the endomorphism obtained by subtituting f for u and g for v in $u +_F v$,

$f -_F g$ and $-_F$ for those obtained in the same way from $u -_F v$ and $-_F f$.

The following lemma is an immediate consequence of the definition of algebraic cobordism (remember that the endomorphisms $\tilde{c}_1(L)$, $\tilde{c}_1(M)$ are locally nilpotent and commute with each other):

Lemma 4.2. *For any finite type k-scheme X and any pair (L, M) of line bundles on X one has the following relations in $End(\Omega_*(X))$:*

1. $\tilde{c}_1(L) +_F \tilde{c}_1(M) = \tilde{c}_1(L \otimes M)$
2. $\tilde{c}_1(L) -_F \tilde{c}_1(M) = \tilde{c}_1(L \otimes M^\vee)$
3. $[-1]_F \tilde{c}_1(L) = \tilde{c}_1(L^\vee)$

where $^\vee$ denotes the operation of dualization.

Remark 4.3. Let X be a finite type k-scheme. Denote by $End^{\tilde{c}_1}(\Omega_*(X))$ the sub-$\Omega_*(k)$-algebra of the $\Omega_*(k)$-algebra $End_{\Omega_*(k)}(\Omega_*(X))$ generated by the $\tilde{c}_1(L)$. Observe that this algebra is commutative graded. Then each element in $End^{\tilde{c}_1}_{-1}(\Omega_*(X))$ (the subgroup of elements of degree -1 of that algebra) is locally nilpotent, and the map $(f, g) \mapsto f +_F g$ defines an abelian group structure on this set, with $f \mapsto [-1]_F f$ as inverse. Moreover, the map

$$\tilde{c}_1 : \mathrm{Pic}(X) \to \mathrm{End}^{\tilde{c}_1}_{-1}(\Omega_*(X))$$

is a group homomorphism. This is a reformulation of the previous lemma.

Lemma 4.4. *Let O_X be the trivial line bundle on $X \in$ $\mathbf{Sch}_k$. Then the homomorphism*

$$\tilde{c}_1(O_X) : \Omega_*(X) \to \Omega_{*-1}(X)$$

is the zero homomorphism.

Proof. For any standard cobordism cycle $[Y \to X, L_1, \ldots, L_r]$ on X, one has

$$\begin{aligned}\tilde{c}_1(O_X)([Y \to X, L_1, \ldots, L_r]) &= [Y \to X, L_1, \ldots, L_r, O_Y] \\ &= [\emptyset \to X, L_1, \ldots, L_r] = 0,\end{aligned}$$

because the constant unit section of O_X never vanishes. □

Remark 4.5. Suppose that k have characteristic zero. Let $f : Y \to X$ be a projective morphism with X finite type over k and Y smooth quasi-projective over k. Choose a line bundle L on X generated by sections $s_1, \ldots, s_n$. Then the set of points $\underline{x} = (x_1, \ldots, x_n) \in \mathbb{A}^n$ such that $\sum_{i=1}^n x_i f^*(s_i)$ is a section transverse to the zero section of $f^*(L)_{|Y_{\kappa(\underline{x})}}$ on $Y_{\kappa(\underline{x})}$ is an open dense subset; see [17, Bertini's theorem 2.3] for instance.

Suppose that k has positive characteristic, $Y = X$ and L is very ample. If $s_1, \dots, s_n$ are sections of L which give a closed immersion of Y into $\mathbb{P}^{n-1}$, then as above, the subscheme of sections $\sum_{i=1}^n x_i f^*(s_i)$ of L which are transverse to the sero section a dense open subset of $\mathbb{A}^n$.

Thus:

1) If k has characteristic zero, or has positive characteristic, $Y = X$ and L is very ample, such a open set has always a rational point over k, and thus there is always a section of L transverse to the zero section.

2) When k is finite, it may not have a rational point but will always have two closed point x and y with $[\kappa(x) : k]$ and $[\kappa(y) : k]$ prime to each other. Thus there are always two finite extensions $k \subset F_1$ and $k \subset F_2$ such that $[F_1 : k]$ and $[F_2 : k]$ are prime to each other and such that the pull-back of L to X_{F_1} and X_{F_2} have sections transverse to the zero section, assuming $Y = X$ and L is very ample.

Lemma 4.6. *(1) Let Y be in $\mathbf{Sm}_k$, and let $(L_1, \dots, L_n)$ be a family of line bundles over Y. Suppose that each of the L_i is very ample. Then for all line bundles $M_1, \dots, M_r$ on Y,*

$$\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_n)([\mathrm{Id}_Y, M_1, \dots, M_r]) = 0 \in \underline{\Omega}_*(Y)$$

(2) Suppose k has characteristic zero. Let X be a finite type k-scheme, n be an integer such that $n > \dim_k(X)$ and $(L_1, \dots, L_n)$ be a family of line bundles over X. Assume that each of the L_i is generated by its global sections. Then

$$\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_n) = 0 \in \ End(\underline{\Omega}_*(X))$$

In particular, for a globally generated line bundle L, one has

$$\tilde{c}_1(L)^n = 0$$

Proof. For (2), we may obviously assume that X is irreducible using remark 1.8. We proceed by induction on $\dim_k(X)$. If $\dim_k(X) = 0$, the line bundles are trivial and the result is corollary 4.4. We assume now that $\dim_k(X) > 0$. Let's choose a standard cobordism cycle $[f : Y \to X, M_1, \dots, M_r]$ on X.

We know by remark 4.5 that there exists a section s of L_n such that the pull-back of s to Y is transverse to the zero section of $f^*(L_n)$. Let $i : S \subset X$ denote the closed immersion of the zeros of s, $j : Z \to Y$ denote the closed immersion of the zeros of $f^*(s)$ and by $g : Z \to S$ the obvious morphism. One observe then that either $S = X$ which means L_n is trivial, or $\dim_k(S) < \dim_k(X)$ because X is irreducible. In the first case $\tilde{c}_1(L_n) = 0$ and there is nothing to prove. So we assume the second case. One then has (using lemma **??**):

$$\begin{aligned}
&\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_n)([f : Y \to X, M_1, \ldots, M_r]) \\
&\qquad = \tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_{n-1})([f : Y \to X, M_1, \ldots, M_r, f^*L_n]) \\
&\qquad = \tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_{n-1})([f \circ j : Z \to X, j^*M_1, \ldots, j^*M_r]) \\
&\qquad = \tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_{n-1}) i_*\big([g : Z \to S, j^*M_1, \ldots, j^*M_r]\big) \\
&\qquad = i_*\left(\tilde{c}_1(i^*L_1) \circ \cdots \circ \tilde{c}_1(i^*L_{n-1})\big([g : Z \to S, j^*M_1, \ldots, j^*M_r]\big) \right)
\end{aligned}$$

Since $\dim_k(S) < \dim_k(X)$, the last term vanishes by the inductive hypothesis.

For (1), if k has characteristic zero, we may use (2). If k has positive characteristic and is infinite, the same argument as above, using remark rem:useful2(2) proves (1). If k is finite, then one chooses (using 4.5(2) again) two finite extensions $k \subset F_1$ and $k \subset F_2$ such that $[F_1 : k]$ and $[F_2 : k]$ are prime together and such that the pull-back of L to X_{F_1} and X_{F_2} have sections transverse to the zero section. The same reasoning shows that the element

$$\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_n)([\mathrm{Id}_Y, M_1, \ldots, M_r])$$

maps to zero into both $\Omega_*(X_{F_1})$ and $\Omega_*(X_{F_1})$. The result easily follows then from lemma 4.7 below. □

Lemma 4.7. *1) Let $k \subset L$ be a finite separable extension of fields. For a scheme X of finite type over k, denote by $\pi(L/k) : X_L \to X$ the natural morphism. Then the composition:*

$$A_*(X) \xrightarrow{\pi(L/k)^*} A_*(X_L) \xrightarrow{\pi(L/k)_*} A_*(X)$$

is multiplication by $[L : k]$.

2) Let $k \subset F_1$ and $k \subset F_2$ be finite separable fields extensions of k, of degrees prime together, and let u_1 and u_2 be integers such that $u_1.[F_1 : k] + u_2.[F_2 : k] = 1$. Then for any scheme X of finite type over k the homomorphism:

$$A_*(X) \xrightarrow{\pi(F_1/k)^* + \pi(F_2/k)^*} A_*(X_{F_1}) \oplus A_*(X_{F_2})$$

is a split monomorphism with left inverse

$$u_1\pi(F_1/k)_* + u_2\pi(F_2/k)_* : A_*(X_{F_1}) \oplus A_*(X_{F_2}) \to A_*(X).$$

Proof. 1). By definition the composition

$$A_*(X) \xrightarrow{\pi(L/k)^*} A_*(X_L) \xrightarrow{\pi(L/k)_*} A_*(X)$$

is the external multiplication by the class of $(\operatorname{Spec} L \to \operatorname{Spec} k)$ in $A_0(k)$. The lemma then follows from lemma 4.8 below.

2) Let u and v be integers such that $u.[F_1 : k] + v.[F_2] = 1$ Then the homomorphism

$$A_*(X_{F_1}) \oplus A_*(X_{F_2}) \xrightarrow{u.\pi(F_1/k)_* + v.\pi(F_2/k)_*} A_*(X)$$

is a left inverse to the given homomorphism. □

Lemma 4.8. *Let L be a finite separable k-algebra. Then*

$$[\operatorname{Spec} L] = [L : k] \cdot 1$$

in $\Omega_0(k)$.

Proof. We may of course reduce to the case $\operatorname{Spec} L$ is connected which means that L is a finite separable field extension of k. Thus there exists $x \in L$ such that $L = k[x]$. Let $f \in k[X]$ be the irreducible polynomial of x, of degree $d = [L : k]$.

Assume first that k is infinite. Choose distinct elements $a_1, \dots, a_d \in k$, and let $g = \prod_{i=1}^{d}(X - a_i)$. Since f is irreducible, no a_i is a root of f. Let $F = Y \cdot f + (1 - Y) \cdot g \in k[X, Y]$, and let $W \subset \mathbb{A}^2_k$ be the closed subscheme defined by F. Using the Jacobian criterion, one checks that W is smooth over k. Via the projection on $\operatorname{Spec} k[Y]$, W is finite over $\mathbb{A}^1_k$. It clearly gives a naive cobordism between $\operatorname{Spec} L$ and $d \cdot \operatorname{Spec} k$.

If k is finite, we proceed by induction on $d = [L : k]$. If $d = 2$ the same argument as above applies. So we may assume $d > 2$. We choose an irreducible polynomial $h \in k[U]$ of degree $d - 1$ (such an h always exists) and an $a \in k$. We set $g = (X - a) \times h$. Note that h is automatically separable since k is perfect. Then the above reasoning applies to show that $[\operatorname{Spec} L] = 1 + [\operatorname{Spec} k[X]/h]$, and the inductive hypothesis gives the result. □

Remark 4.9. Let X be a quasi-projective k-scheme. For each line bundle L on X, there is a very ample line bundle M such that $M \otimes L$ is very ample. From lemma 4.2, the homomorphism $\tilde{c}_1(L)$ can be computed as

$$\tilde{c}_1(L) = \tilde{c}_1(M \otimes L) -_F \tilde{c}_1(M)$$

Theorem 4.10. *(1) Let k be a field of characteristic zero. For any $X \in \mathbf{Sch}_k$ and any family $(L_1, \dots, L_n)$ of line bundles on X with $n > \dim_k(X)$, such that one of the following two conditions is satisfied:*

(a) *The line bundles are all globally generated.*
(b) *X is a quasi-projective k-scheme.*

Then one has

$$\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_n) = 0$$

in the endomorphism ring of $\Omega_*(X)$.
(2) Let k *be an arbitrary field. Let* Y *be in* $\mathbf{Sm}_k$. *Then for any family* $(L_1, \ldots, L_n)$ *of line bundles on* Y *with* $n > \dim_k(Y)$, *one has*

$$\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_n)(1_Y) = 0.$$

In particular, Ω_* *satisfies the axiom (Dim).*

Proof. Case (1a) follows from lemma 4.6. Assume now that X is a quasi-projective scheme. Using remark 4.9, we see that for each $i \in \{1, \ldots, \}$ there exists two very ample line bundles M_i and N_i such that $L_i \cong M_i \otimes (N_i)^\vee$. But then from lemma 4.2 we have

$$\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_n) = \Pi_{i=1}^n \big(\tilde{c}_1(M_i) -_F \tilde{c}_1(N_i)\big).$$

Thus the endomorphism $\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_n)$ is a sum of terms of the form $a \cdot \tilde{c}_1(L'_1) \circ \cdots \circ \tilde{c}_1(L'_m)$, with $a \in \Omega_*(k)$, each L'_j very ample, and $m \geq n$. Using lemma 4.6 again completes the proof of (1b)

The proof of (2) is essentially the same as the proof of (1b), where we use the portion of lemma 4.6 dealing with k of positive characteristic. □

Remark 4.11. We do not know in general whether or not the previous theorem holds for any finite type k-scheme and any family of line bundles.

Remark 4.12. The method of proof of theorem 4.10 obviously applies to prove that an oriented Borel-Moore $\mathbb{L}_*$-functor H_* for which Axioms (Nilp), (Sect) and (FGL) hold satisfies axiom (Dim) as well (see remark 2.3).

Remark 4.13. Beware that the theorem 4.10 doesn't hold for A_* in general. However, lemmas 4.2, 4.4 and 4.6 as well as theorem 4.10 hold for A_* if the canonical morphism

$$\vartheta : \Omega_* \to A_*$$

is surjective, that is to say that for any $X \in \mathcal{V}$, $A_*(X)$ is generated over $A_*(k)$ by the classes $[Y \to X]_A$ of projective morphisms with smooth irreducible source.

4.14. **Generators for algebraic cobordism.**

Lemma 4.15. *Let* X *be a finite type* k*-scheme. Then* $\Omega_*(X)$ *is generated as a group by standard cobordism cycles*

$$[Y \to X, (L_1, \ldots, L_r)]$$

In other words, the obvious homomorphism $\underline{\Omega}_*(X) \to \Omega_*(X)$ *is surjective.*

Proof. The $\mathbb{L}_*$-module $\Omega_*(X)$ is clearly generated by the standard cobordism cycles. Using functoriality, it is easy to see that we only have to show that the homomorphism of rings

$$\mathcal{Z}_*(k) \to \Omega_*(k)$$

is surjective. As the homomorphism of rings $\mathbb{L}_* \otimes \mathcal{Z}_*(k) \to \Omega_*(k)$ is surjective by definition, it is thus sufficient to prove that the image of $\mathbb{L}_* \to \Omega_*(k), x \mapsto [x]$ is in the image of $\mathcal{Z}_*(k) \to \Omega_*(k)$.

We will show that for all n, m, $a_{n,m}$ is in the image by induction on $n+m$. This implies the lemma because the Lazard ring is generated by the $a_{n,m}$. Assume then that a_{ij} is in the image of $\Omega_*(k)$ for $i+j \leq n+m-1$. Then the formula **??** immediately implies, using the inductive hypothesis, that $a_{n,m}$ is also in the image of $\mathcal{Z}_*(k) \to \Omega_*(k)$ (see remark 3.9). $\square$

Lemma 4.16. *For any* $X \in \mathbf{Sch}_k$ *then* $\Omega_*(X)$ *is generated as a group by classes of the form*

$$[Y \to X, L_1, \ldots, L_r]$$

where each of the line bundle L_i *on* Y *is very ample.*

Proof. Given any standard cobordism cycle $(f : Y \to X, L_1, \ldots, L_r)$ on X, on $X \in \mathbf{Sch}_k$, we have the formula from remark 1.9

$$[f : Y \to X, L_1, \ldots, L_r] = f_* \circ \tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_r)(1_Y).$$

The lemma follows easily from this identity, remark 4.9 and lemma 4.15. $\square$

Lemma 4.17. *Let* X *be a finite type* k*-scheme. Then the canonical homomorphism*

$$\mathcal{M}_*^+(X) \to \Omega_*(X)$$

is an epimorphism. In other words, the graded abelian group $\Omega_*(X)$ *is generated as a group by the classes* $[Y \to X]$ *of projective morphisms* $Y \to X$ *with* Y *smooth quasi-projective and irreducible.*

Proof. From lemma 4.16 we know that $\Omega_*(X)$ is generated as a group by classes of the form

$$[Y \to X, L_1, \ldots, L_r],$$

where each of the line bundle L_i on Y is very ample. If k is infinite, using remark 4.5 we may find sections s_i of L_i which are transverse to the zero section. But then using lemma 2.13, we see that

$$\begin{aligned}[][Y \to X, L_1, \dots, L_r] = [Y_1 \to X, L_2, \dots, L_r]\\ = \dots = [Y_r \to X]\end{aligned}$$

where $Y_i = \bigcap_{j=1}^{i}\{s_j = 0\}$, thus proving the statement.

If k is finite, the same reasoning works using remark 4.5 and lemma 4.7. ∎

□

Theorem 4.18. *Let k be a field. Then the homomorphism $\Phi_0(k) : \mathbb{L}_0 \to \Omega_0(k)$ is an isomorphism and $\Omega_0(k)$ is the free abelian group on $1 = [\operatorname{Spec} k]$.*

Moreover, given any smooth variety $X = \operatorname{Spec} A$ of dimension zero over k, then $[X] = \dim_k(A) \cdot 1$ in $\Omega_0(k)$.

Proof. The last formula has been established in lemma 4.8. The surjectivity follows from that formula and from lemma 4.17 which proves that $\Omega_0(k)$ is generated by classes $[\operatorname{Spec} L]$ where L is a (separable) finite field extension of k. Thus $\mathbb{Z} = \mathbb{L}_0 \to \Omega_0(k)$ is surjective. But the map $\Omega_0(k) \to \mathrm{CH}_0(k) = \mathbb{Z}$ is a left inverse to $\mathbb{L}_0 \to \Omega_0(k)$, which is thus injective. □

4.19. **Relations defining algebraic cobordism.** It will be useful to give explicit generators for the kernel of the natural surjection (see 4.17):

$$\underline{\Omega}_* \to \Omega_*$$

For this, first choose, for each (i, j) with $i \le j$, an element $a'_{ij} \in \mathcal{Z}_{i+j-1}(k)$ lifting $[a_{ij}] \in \Omega_{i+j-1}(k)$ (to do that one can use 3.4) ; for $j < i$ we set $a'_{ji} = a'_{ij}$. Let $F(u, v) \in \underline{\Omega}_*(k)[[u, v]]$ be the power series

$$F(u, v) = u + v + \sum_{i,j\ge1} a'_{ij} u^i v^j$$

Definition 4.20. Let $X \in \mathbf{Sch}_k$. Let $\bar{\mathcal{R}}_*(X)$ denote the subgroup of $\underline{\Omega}_*(X)$ generated by elements of the form

$$f_* \circ \tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_r)\big([F(L, M)] - [L \otimes M]\big),$$

where $f : Y \to X$ is in $\mathcal{M}(X)$, and $(L_1, \dots, L_r, L, M)$ are line bundles on Y. We denote by

$$\bar{\Omega}_*(X)$$

the quotient group $\underline{\Omega}_*(X)/\bar{\mathcal{R}}_*(X)$.

Lemma 4.21. *Let* $X \in \mathbf{Sch}_k$. *Then* $\bar{\mathcal{R}}_*(X)$ *is also the subgroup of* $\underline{\Omega}_*(X)$ *generated by elements of the form*

$$f_* \circ \tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_r)\big(F(\tilde{c}_1(L), \tilde{c}_1(M))(\eta) - \tilde{c}_1(L \otimes M)(\eta) \big),$$

where $(f : Y \to X, L_1, \ldots, L_r,)$ *is a standard cobordism cycle on* X, L, M *are line bundles on* Y, *and* η *is in* $\underline{\Omega}_*(Y)$.

Proof. Indeed, $\underline{\Omega}_*(Y)$ is generated by the standard cobordism cycles $(g : Z \to Y, M_1, \ldots, M_s)$ on Y. But then using the notations of the lemma with $\eta = [g : Z \to Y, M_1, \ldots, M_s]$ we have

$$\begin{aligned} f_* \circ \tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_r) \circ F(\tilde{c}_1(L), \tilde{c}_1(M))(\eta) = (f \circ g)_* \circ \tilde{c}_1(g^*L_1) \circ .. \\ \cdots \circ \tilde{c}_1(g^*L_r) \circ \tilde{c}_1(M_1) \circ \cdots \circ \tilde{c}_1(M_s)([F(g^*L, g^*M)]) \end{aligned}$$

and

$$\begin{aligned} f_* \circ \tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_r) \circ \tilde{c}_1(L \otimes M)(\eta) = (f \circ g)_* \circ \tilde{c}_1(g^*L_1) \circ .. \\ \cdots \circ \tilde{c}_1(g^*L_r) \circ \tilde{c}_1(M_1) \circ \cdots \circ \tilde{c}_1(M_s)([g^*L \otimes g^*M]) \end{aligned}$$

which verifies our assertion.

□

As a consequence of the lemma, together with lemma **??** and remark 1.15, one sees that $X \mapsto \bar{\Omega}_*(X)$ is a multiplicative oriented Borel-Moore functor.

It is easy to see that the elements in $\bar{\mathcal{R}}_*(X)$ become zero in $\Omega_*(X)$ through the projection $\underline{\Omega}_*(X) \to \Omega_*(X)$, giving a natural epimorphism $\tau_X : \bar{\Omega}_*(X) \to \Omega_*(X)$.

Proposition 4.22. *Let* X *be a finite type* k*-scheme. Then the homomorphism*

$$\tau_X : \bar{\Omega}_*(X) \to \Omega_*(X)$$

is an isomorphism.

Proof. We know that $X \mapsto \bar{\Omega}_*(X)$ is a multiplicative oriented Borel-Moore functor. It suffices to show that $(F, \bar{\Omega}_*(k))$ is a commutative formal group.

Indeed, if this is so, we have the canonical homomorphism

$$\phi : \mathbb{L} \to \bar{\Omega}_*(k),$$

with $\phi(a_{ij}) = a'_{ij}$. Using the $\bar{\Omega}_*(k)$-module structure on $\bar{\Omega}_*$, we get the surjective morphism of multiplicative oriented Borel-Moore functors

$$\vartheta : \mathbb{L} \otimes \underline{\Omega}_* \to \bar{\Omega}_*,$$

$$\vartheta(a \otimes b) = \phi(a)b,$$

extending the natural transformation $\underline{\Omega}_* \to \bar{\Omega}_*$. If L and M are line bundles on X, we have

$$\vartheta(F_L(\tilde{c}_1(L), \tilde{c}_1(M))) = F(\tilde{c}_1(L), \tilde{c}_1(M))$$

as endomorphisms of $\bar{\Omega}_*(X)$, hence ϑ descends to a surjective natural transformation $\Omega_* \to \bar{\Omega}_*$, which is easily seen to be inverse to τ.

Now, to show that $(F, \bar{\Omega}_*(k))$ is a commutative formal group, we need only verify the associativity $F(F(u,v),w) = F(u,F(v,w)$ in $\bar{\Omega}_*(k)[[u,v,w]]$. Suppose the associativity relation $F(F(u,v),w) = F(u,F(v,w)$ is satisfied modulo (u^n, v^{m+1}, w^{p+1}). Write

$$F(F(u,v),w) = \sum_{ijl} a_{ijl}u^iv^jw^l; \quad F(u,F(v,w)) = \sum_{ijl} a'_{ijl}u^iv^jw^l.$$

Let (a,b,c) be integers, and let $O_X(a,b,c)$ denote the line bundle whose sheaf of section is $\mathcal{O}(a,b,c) := p_1^*\mathcal{O}(a) \otimes p_2^*\mathcal{O}(b) \otimes p_3^*\mathcal{O}(c)$ on $X := \mathbb{P}^n \times \mathbb{P}^m \times \mathbb{P}^p$. Then, as endomorphisms of $\bar{\Omega}_*(X)$, we have

$$\begin{aligned}
F(F(\tilde{c}_1(O_X(1,0,0)),&\tilde{c}_1(O_X(0,1,0)), \tilde{c}_1(O_X(0,0,1)) \\
&= F(\tilde{c}_1(O_X(1,1,0)), \tilde{c}_1(O_X(0,0,1)) \\
&= \tilde{c}_1(O_X(1,1,1)) \\
&= F(\tilde{c}_1(O_X(1,0,0)), \tilde{c}_1(O_X(0,1,1)) \\
&= F(\tilde{c}_1(O_X(1,0,0)), F(\tilde{c}_1(O_X(0,1,0), \tilde{c}_1(O_X(0,0,1))).
\end{aligned}$$

Evaluating both sides on $\mathrm{Id}_{\mathbb{P}^n \times \mathbb{P}^m \times \mathbb{P}^p}$ gives

$$\sum_{ijl} a_{ijl}[H^{(i)} \times H^{(j)} \times H^{(l)}] = \sum_{ijl} a'_{ijl}[H^{(i)} \times H^{(j)} \times H^{(l)}]$$

in $\bar{\Omega}_*(\mathbb{P}^n \times \mathbb{P}^m \times \mathbb{P}^p)$, where $H^{(i)}$ stands for the intersection of i distinct hyperplanes. Pushing forward to $\operatorname{Spec} k$, using our induction hypothesis, and the fact that $H^{(n)} \times H^{(m)} \times H^{(p)}$ pushes forward to the identity in $\bar{\Omega}_*(k)$, we find that the associativity relation $F(F(u,v),w) = F(u,F(v,w)$ is satisfied modulo $(u^{n+1}, v^{m+1}, w^{p+1})$. The same argument allows us to increase the degree in v and in w, which completes the proof. $\square$

Part 2. Fundamental properties of algebraic cobordism

5. *Divisor classes*

Definition 5.1. Let W be in $\mathbf{Sm}_k$. Recall that an *strict normal crossing divisor* E on W is a Weil divisor $E = \sum_{i=1}^m n_i \cdot E_i$ where each n_i is ≥ 1, each E_i is an integral closed subscheme of W, of pure codimension 1, and smooth over k, and moreover for each $I \subset \{1, \dots, m\}$ the intersection

$$E_I := \bigcap_{i \in I} E_i$$

is transverse[6] for any $I \subset \{1, \dots, m\}$.

We denote by $i : |E| \subset W$ the support of E, i.e. the reduced closed subscheme whose underlying space is the union of the E_i. We write $O_W(E)$ for the line bundle on W corresponding to E, which means that its $\mathcal{O}_W$-module of sections is $\mathcal{O}_W(E)$.

Recall that we denote by $[O_W(E)] \in \Omega_*(W)$ the class $\tilde{c}_1(O_W(E))(1_W) \in \Omega_*(W)$. The main object of this section is to define a class $[E \to |E|] \in \Omega_*(|E|)$ lifting $[O_W(E)] \in \Omega_*(W)$, i.e. such that $i_*([E \to |E|]) = [O_W(E)]$.

5.2. Intersection of smooth divisors.

Proposition 5.3. *Let W be in $\mathbf{Sm}_k$ and equidimensional, and let $E = \sum_{i=1}^m E_i$ be a strict normal crossing divosr on W. Let $i : E_{\{1,\dots,m\}} \to W$ be the inclusion. Let $L_1, \dots, L_n$ be line bundles on W. Let $H(u_1, \dots, u_m)$ be any power series with $\Omega_*(k)$ coefficients. Then*

$$i_*[H(i^*L_1, \dots, i^*L_n)] = \tilde{c}_1(O_W(E_1)) \circ \ldots \circ \tilde{c}_1(O_W(E_m))[H(L_1, \dots, L_n)].$$

in $\Omega_*(W)$.

Proof. By induction on m it suffices to prove the case $r = 1$; write E for E_1. We have

$$\begin{aligned} i_*[H(i^*L_1, \dots, i^*L_n)] &= i_*\big(H(\tilde{c}_1(i^*L_1), \dots, \tilde{c}_1(i^*L_n))(\mathrm{Id}_E)\big) \\ &= H(\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_n))([E \to W]) \\ &= H(\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_n)) \circ \tilde{c}_1(O_W(E))(\mathrm{Id}_W) \\ &= \tilde{c}_1(O_W(E))\big(H(\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_n))(\mathrm{Id}_W)\big) \\ &= \tilde{c}_1(O_W(E))[H(L_1, \dots, L_n)] \end{aligned}$$

---

[6] This the same as requiring that these intersections are smooth over k of codimension $\sharp I$.

□

5.4. **Some power series.** For a formal group (F, R), recall that we simply write $u +_F v$ for $F(u, v)$, and extend this notation in the evident way for the other formal group operations such as *formal opposite* denoted $[-1] \cdot_F v$ (which satisfies $u +_F [-1] \cdot_F u = 0$), *formal difference* denoted $u -_F v$ (equal to $u +_F [-1] \cdot_F (v)$), n-fold formal sum $u_1 +_F + \cdots +_F u_n$, and formal multiplication by $n \in \mathbb{Z}$ denoted $[n] \cdot_F u$.

Let $n_1, \ldots, n_m$ be integers. We will also use the notation

$$G^{n_1,\cdots,n_m}(u_1, \ldots, u_m) := [n_1] \cdot_F u_1 +_F \ldots +_F [n_m] \cdot_F u_m$$

Example 5.5. By the relations encoded in the definition of Ω_*, we have for line bundles $L_1, \ldots, L_r$ on a finite type k-scheme X:

$$G_\Omega^{n_1,\cdots,n_m}(\tilde{c}_1(L_1), \ldots, \tilde{c}_1(L_r)) = \tilde{c}_1(L_1^{\otimes n_1} \otimes \ldots \otimes L_r^{\otimes n_r})$$

For a sequence $J = (j_1, \ldots, j_m) \in (\mathbb{N})^m$ of non-negative integers j_i, let $u^J = u_1^{j_1} \cdot \ldots \cdot u_m^{j_m}$. We set $||J|| := Sup_i(j_i)$.

Lemma 5.6. *Any power series $F(u_1, \ldots, u_m) \in R[[(u_1, \ldots, u_m)]]$ can be uniquely written as*

$$F(u_1, \ldots, u_m) = \sum_{J,\ ||J|| \leq 1} u^J \cdot F_J(u_1, \ldots, u_m),$$

where the sum runs over each J for which $||J|| \leq 1$ and each monomial $h_{J,J'} u^{J'}$, $J' = (j'_1, \ldots, j'_m)$, occuring in $F_J^{n_1,\cdots,n_m}$ has $j'_s = 0$ if $j_s = 0$.

Proof. Write $F(u_1, \ldots, u_m) := \sum_{K \in (\mathbb{N})^m} f_K \cdot u^K$. For any K define $J(K) = (j_1, \ldots, j_m)$ such that $j_i = 0$ if $k_i = 0$ and $j_i = 1$ if $k_i \geq 1$ and $J' := K - J(K)$ (so that $J' = (j'_1, \ldots, j'_m)$ with $j'_i = 0$ if $k_i = 0$ and $j'_i = k_i - 1$ else). and $j'_i = j_i - 1 if i \in I(J)$. Then clearly for each $I \subset \{1, \ldots, m\}$, the power series

$$F_I(u_1, \ldots, u_m) := \sum_{J | I(J) = I} f_J . u^{J'}$$

satisfies the equation of the lemma. Uniqueness is easy and left to the reader. □

For any sequence $n_1, \ldots, n_m$ of integers and any $J \in (\mathbb{N})^m$ with $||J|| \leq 1$ we let

$$H_J^{n_1,\cdots,n_m}(u_1, \ldots, u_m) \in \Omega_*(k)[[u_1, \ldots, u_m]]$$

denote the power series given by applying the lemma to $G^{n_1,\cdots,n_m}$. So that we have in particular

$$G^{n_1,\cdots,n_m}(u_1, \ldots, u_m) = \sum_{J,\ ||J|| \leq 1} u^J \cdot H_J^{n_1,\cdots,n_m}(u_1, \ldots, u_m)$$

Example 5.7. Assume $m = 2$ and $n_1 = n_2 = 1$. Then

$$\begin{aligned} G^{1,1}(u,v) = F(u,v) &= \sum_{i,j} a_{i,j} u^i v^j \\ &= u + v + \sum_{i\geq 1, j\geq 1} a_{i,j} u^i v^j \\ &= u + v + u \cdot v \cdot \big(\sum_{i\geq 1, j\geq 1} a_{i,j} u^{i-1} v^{j-1} \big) \end{aligned}$$

so that $H^{1,1}_{(1,0)}(u,v) = 1$, $H^{1,1}_{(0,1)}(u,v) = 1$ and

$$H^{1,1}_{(1,2)}(u,v) = \sum_{i\geq 1, j\geq 1} a_{i,j} u^{i-1} v^{j-1}$$

5.8. **Normal crossing divisors.** Let W be in $\mathbf{Sm}_k$ and let E be an strict normal crossing divisor on W. Write $E = \sum_{j=1}^m n_j E_j$, with the E_j irreducible. For each index $J = (j_1, \ldots, j_m)$ with $||J|| \leq 1$, we have the *face* $E^J := \cap_{i,\ j_i=1} E_i$ of E. Alternatively, for each $I \subset \{1, \ldots, m\}$ we set by $E_I := \cap_{i\in I} E_i$. Of course, $E_I = E^{J(I)}$ where $J(I) = (j_1, \ldots, j_m)$ with $j_i = 0$ if $i \notin I$ and $j_i = 1$ if $i \in I$.

Let $\iota^J : E^J \to |E|$ be the inclusion. Let $L_i := O_W(E_i)$, and let $L_i^J = (\iota^J)^* L_i$.

Definition 5.9. Keeping the previous notations and assumptions, we define the class $[E \to |E|] \in \Omega_*(|E|)$ by the formula

$$[E \to |E|] := \sum_{J,\ ||J||\leq 1} \iota_*^J \big([H_J^{n_1,\ldots,n_m}(L_1^J, \ldots, L_m^J)] \big). \tag{5.1}$$

If $f : |E| \to X$ is a projective morphism, we write $[E \to X] \in \Omega_*(X)$ for $f_*([E \to |E|])$ if there is no ambiguity.

Example 5.10. Assume $m = 2$ and $n_1 = n_2 = 1$, so that $E = E_1 + E_2$ and $|E| = E_1 \bigcup E_2$. From example 5.7, we see that

$$[E \to |E|] := \iota_*^{(1,0)}(1_{E_1}) + \iota_*^{(0,1)}(1_{E_2}) + \iota^{(1,1)}[H^{1,1}_{(1,1)}(L_1^{(1,1)}, L_2^{(1,2)})]$$

$$[E_1 \to |E|] + [E_2 \to |E|] + \iota^{(1,1)}[H^{1,1}_{(1,1)}(L_1^{(1,1)}, L_2^{(1,1)})]$$

In particular, assume that L_1 and L_2 are trivial. Then equation 5.1 becomes

$$[E \to |E|] = [E_1 \to |E|] + [E_2 \to |E|] + [a_{1,1}] \cdot [E_{\{1,2\}} \to |E|]$$

which equals (using remark 3.7):

$$[E \to |E|] = [E_1 \to |E|] + [E_2 \to |E|] - [\mathbb{P}^1 \to \operatorname{Spec} k] \cdot [E_{\{1,2\}} \to |E|]$$

Proposition 5.11. *Let W be in $\mathbf{Sm}_k$, let E be a strict normal crossing divisor on W. Then*

$$[E \to W] = [O_W(E)].$$

In particular, let X be a finite type k-scheme, let $f : W \to X$ be a projective morphism, and let E, E' be strict normal crossing divisors on W with $O_W(E) \cong O_W(E')$. Then

$$[E \to X] = [E' \to X]$$

in $\Omega_*(X)$.

Proof. Write $E = \sum_{i=1}^m n_i E_i$ with the E_i smooth and irreducible. Write G for $G^{n_1,\dots,n_m}$, H_J for $H_J^{n_1,\dots,n_m}$. Let $G_J = u^J H_J$, so that

$$G = \sum_J G_J$$

Let $i^J : E^J \to W$ be the inclusion. By proposition 5.3 we have

$$i^J_*\big([H_J(L_1^J, \dots, L_m^J)]\big)\big) = [G_J(L_1, \dots, L_m)].$$

Thus

$$\begin{aligned}
[E \to W] &= \sum_J i^J_*\big(H_J[\tilde{c}_1(L_1^J), \dots, \tilde{c}_1(L_m^J)]\big)\\
&= \sum_J [G_J(L_1, \dots, L_m)]\\
&= [G(L_1, \dots, L_m)]\\
&= G\big(\tilde{c}_1(L_1^{\otimes n_1}), \dots, \tilde{c}_1(L_m^{\otimes n_m})\big)(\mathrm{Id}_W)\\
&= \tilde{c}_1\big(L_1^{\otimes n_1} \otimes \dots \otimes L_m^{\otimes n_m}\big)(\mathrm{Id}_W) \text{ (cf. ex. 5.5)}\\
&= [O_W(E)].
\end{aligned}$$

□

6. *Localization*

Let X be a finite type k-scheme $i : Z \to X$ a closed subscheme, and $j : U \to X$ the open complement. It is obvious that the composite $\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X) \xrightarrow{j^*} \Omega_*(U)$ is zero so that we can consider the following sequence

$$\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X) \xrightarrow{j^*} \Omega_*(U) \to 0, \tag{6.1}$$

as a complex. Our main task in this section will be to show that (6.1) is exact (theorem 6.10), at least under the assumption that the base field k admits resolution of singularities.

6.1. **Blow-ups.** Let $f : W' \to W$ be a projective birational map, with W' and W smooth over k. In this section, we consider the problem of writing the class $[W' \to W]$ in $\Omega_*(W)$.

By [7], there is a closed subscheme T of W such that f is the blow-up of W along T. Since W is smooth, we may assume[7] that the support of T is the fundamental locus of f, i.e. $W \setminus |T|$ is exactly the set of $x \in W$ over which f is an isomorphism. Let $E \subset W'$ be the exceptional divisor, and $T = f(E)$ the fundamental locus of f.

We form the "deformation to the normal cone" as follows: let $\mu : Y \to W \times \mathbb{A}^1$ be the blow-up of $W \times \mathbb{A}^1$ along $T \times 0$. Let $<W \times 0>$ denote the proper transform of $W \times 0$. Let $<T \times \mathbb{A}^1>$ denote the proper transform of $T \times \mathbb{A}^1$. Let $\tilde{E}$ be the exceptional divisor of μ.

Lemma 6.2. *The restriction of μ, $\mu : <W \times 0> \to W$ is isomorphic over W to $f : W' \to W$. In addition, $Y \setminus (\tilde{E} \cap <T \times \mathbb{A}^1>)$ is smooth, and contains $<W \times 0>$. Finally, if E is a strict normal crossing divisor, then $(\tilde{E} + <W \times 0>) \setminus <T \times \mathbb{A}^1>$ is a strict normal crossing divisor on $Y \setminus (\tilde{E} \cap <T \times \mathbb{A}^1>)$*

Proof. The assertions being local over W, we may assume that $W = \operatorname{Spec} A$ for some k-algebra A. Let $I \subset A$ be the ideal defining T, $J \subset A[t]$ the ideal defining $T \times 0$, so $J = (I, t)$.

Suppose $I = (f_0, \ldots, f_m)$. Then $W' = \operatorname{Proj}_A(\oplus_n I^n)$ is the subscheme of $\mathbb{P}^m_A$ defined by the kernel $N_{W'}$ of the surjection of graded rings

$$\begin{aligned} A[X_0, \ldots, X_n] &\to \oplus_n I^n \\ g(X_0, \ldots, X_n) &\mapsto g(f_0, \ldots, f_n). \end{aligned}$$

Similarly, Y is the subscheme of $\mathbb{P}^{m+1}_A$ defined by the kernel N_Y of the surjection of graded rings

$$\begin{aligned} A[X_0, \ldots, X_n, X_{n+1}] &\to \oplus_n J^n \\ g(X_0, \ldots, X_n, X_{n+1}) &\mapsto g(f_0, \ldots, f_n, t). \end{aligned}$$

We claim that N_Y is the ideal generated by $N_{W'}$ and the elements $f_j X_{n+1} - tX_j$, $j = 0, \ldots, n$. To see this, take $g \in N_Y$ of degree d, and expand g as a sum of monomials $X^I X^j_{n+1}$, ($I = (i_0, \ldots, i_n)$, $|I| = \sum_j i_j$):

$$g = \sum_{|I|+i=d} g_I X^I X^i_{n+1},$$

---

[7] See [7, Exercise 7.11 (c)].

with

$$g_I = \sum_{j=0}^{N_I} g_I^j t^j; \quad g_I^j \in A,\ g_I^{N_I} \neq 0.$$

Modulo the elements $f_j X_{n+1} - tX_j$, we may rewrite g as

$$g' := \sum_{i=0}^{d} g'_i X_{n+1}^i + (\sum_{j=1}^{M} a_j t^j) X_{n+1}^d,$$

with $g'_i \in A[X_0, \ldots, X_n]$ homogeneous of degree $d - i$, and $a_j \in A$. Evaluating g', we have

$$0 = g'(f_0, \ldots, f_n, t) = \sum_{i=0}^{d} g'_i(f_0, \ldots, f_n) t^i + \sum_{i=1}^{M} a_i t^{d+i},$$

hence each $a_i = 0$, and each g'_i is in $N_{W'}$, proving our claim.

Next, we note that multiplication by X_{n+1}, $\times X_{n+1} : \mathcal{O}_Y \to \mathcal{O}_Y(1)$, is injective. Indeed, multiplication by X_{n+1} on the homogeneous coordinate ring of Y is just multiplication by t on $\oplus_n J^n$, which is evidently injective. This implies the injectivity of $\times X_{n+1}$ on the sheaf level. Additionally, it is clear from our description of N_Y that

$$A[X_0, \ldots, X_n]/N_{W'} \cong A[X_0, \ldots, X_{n+1}]/(N_Y, X_{n+1}),$$

so W' is isomorphic to the subscheme of Y defined by $X_{n+1} = 0$. Since W' is smooth, and X_{n+1} is a non-zero divisor on $\mathcal{O}_Y$, this implies that Y is smooth in a neighborhood of $X_{n+1} = 0$. Finally, since the proper transform $[W \times 0]$ is irreducible and dense in $(X_{n+1} = 0)$, we see that the equation $X_{n+1} = 0$ defines the subscheme $<W \times 0>$.

We now look at the proper transform $<T \times \mathbb{A}^1>$. Let $\mathcal{N}$ denote the sheaf of ideals on Y defining $<T \times \mathbb{A}^1>$, and let $\mathcal{I}$ be the sheaf of ideals defined by the homogeneous ideal $(X_0, \ldots, X_n)$. We claim that $\mathcal{N} = \mathcal{I}$. Indeed, on the subscheme of Y defined by $(X_0, \ldots, X_n)$, X_{n+1} is invertible, hence the relations $f_j X_{n+1} - tX_j$ in N_Y imply that $\mathcal{I} \supset (f_0, \ldots, f_n)\mathcal{O}_Y$, and we have equality of ideal sheaves after inverting t. Thus $\mathcal{N} = \mathcal{I}$ after inverting t, hence $\mathcal{N} \supset \mathcal{I}$. Since the subscheme of Y defined by $(X_0, \ldots, X_n)$ is evidently isomorphic to $T \times \mathbb{A}^1$ via the projection to $X \times \mathbb{A}^1$, we have $\mathcal{N} = \mathcal{I}$, as claimed.

On the other hand, consider the affine open subscheme U_i of Y defined by $X_i \neq 0$, and the similarly defined subscheme V_i of W'. Our description of N_Y in terms of $N_{W'}$ implies that $U_i \cong V_i \times \mathbb{A}^1$, using X_{n+1}/X_i as the map to $\mathbb{A}^1$. Thus U_i is smooth. Since $Y \to X \times \mathbb{A}^1$ is an isomorphism over $X \times \mathbb{A}^1 \setminus T \times 0$, this proves that $Y \setminus (\tilde{E} \cap <T \times \mathbb{A}^1>)$ is smooth.

Finally, both $\tilde{E}\cap U_i \subset U_i$ and $E \subset V_i$ are the subschemes of U_i and V_i, respectively, defined by f_i. Again referring to the explicit equations defining Y, we see that $\tilde{E}\cap U_i \subset U_i$ is isomorphic to $E\times\mathbb{A}^1 \subset V_i\times\mathbb{A}^1$, using as above the coordinate X_{n+1}/X_i. Thus $\tilde{E}\setminus{<}T\times\mathbb{A}^1{>}$ is a strict normal crossing divisor on $Y\setminus(\tilde{E}\cap{<}T\times\mathbb{A}^1{>})$. We note that ${<}W\times 0{>}$ is smooth, and ${<}W\times 0{>}\cap\tilde{E}$ is the strict normal crossing divisor E on W'. Write $E=\sum_i n_iD_i$. Since $\tilde{E}=E\times\mathbb{A}^1$ in a neighborhood of ${<}W\times 0{>}$, this implies that $\tilde{E}=\sum_i n_iD_i\times\mathbb{A}^1$ in a neighborhood of ${<}W\times 0{>}$. Thus ${<}W\times 0{>}$ intersects each irreducible component of $\tilde{E}$ transversely, hence $(\tilde{E}+{<}W\times\mathbb{A}^1{>})\setminus{<}T\times\mathbb{A}^1{>}$ is a strict normal crossing divisor on $Y\setminus(\tilde{E}\cap{<}T\times\mathbb{A}^1{>})$. ☐

6.3. **Preliminaries on classes of divisors.** Let E be an effective strict normal crossing divisor on some $W\in\mathbf{Sm}_k$. We have defined in section 5 the class $[E\to|E|]$ in $\Omega_*(|E|)$; for later use we will need a somewhat more detailed description of this class. Write $E=\sum_{i=1}^m n_iD_i$ with the D_i distinct and irreducible, so $D:=\sum_{i=1}^m D_i$ is a reduced strict normal crossing divisor. Write $E_{\text{mult}}:=E-D$, and $E_{\text{red}}=\sum_{n_i=1}D_i$. Define

$$|E|_{\text{sing}}:=\big(\cup_{i<j}|D_i|\cap|D_j|\big)\cup|E_{\text{mult}}|,$$

so $|E|_{\text{sing}}$ is the singular locus of $|D|$, together with the union of the $|D_j|$ with $n_j>1$.

Lemma 6.4. *Let W be in $\mathbf{Sm}_k$, and let $E=\sum_{i=1}^m n_iD_i$ be a strict normal crossing divisor on W. Let $\iota_{\text{sing}}:|E|_{\text{sing}}\to|E|$ be the inclusion. Then there is an element η of $\Omega_*(|E|_{\text{sing}})$ such that*

$$[E\to|E|]=[E_{\text{red}}\to|E|]+\iota_{\text{sing}*}\eta.$$

Proof. We may suppose that $n_j=1$ for $j=1,\dots,s$, and $n_j>1$ for $j=s+1,\dots,m$. Let $n=\sum_i n_i$, and let F_n denote the n-fold sum

$$F_n(u_1,\dots,u_n)=u_1+_{F_\Omega}\dots+_{F_\Omega}u_n.$$

We have $F_\Omega(u,v)=u+v+\sum_{i\geq1,\ j\geq1}a_{ij}u^iv^j$, from which it follows that

$$F_n(u_1,\dots,u_n)=\sum_{i=1}^n u_i+\sum_{i_1\geq1,\dots,i_n\geq1}a_{i_1\dots i_n}u_1^{i_1}\cdots u_n^{i_n}.$$

Thus, if we expand

$$G^{1,\dots,1,n_{s+1},\dots,n_m}=F(u_1,\dots,u_s,u_{s+1},\dots,u_{s+1},\dots,u_m,\dots,u_m)$$

as required for the definition of $[E\to|E|]$,

$$G^{1,\dots,1,n_{s+1},\dots,n_m}=\sum_J u^JH_J(u_1,\dots,u_m);\ \ J=(j_1,\dots,j_m),0\leq j_l\leq 1,$$

we find

$$G^{1,\dots,1,n_{s+1},\dots,n_m} = \sum_{i=1}^{s} u_i + \sum_{J'} u^{J'} H_{J'}(u_1,\dots,u_m),$$

where each J' has either two nonzero entries, or an entry $j'_l = 1$ with $l > s$. We thus have

$$\begin{aligned}[E \to |E|] &= \sum_J \iota^J_*[H_J(\iota^{J*}L(D_1),\dots,\iota^{J*}L(D_m))] \\ &= \sum_{j=1}^{s} \iota_{j*}(\mathrm{Id}_{D_j}) \\ &\qquad\qquad + \sum_{J'} \iota^{J'}_*[H_{J'}(\iota^{J*}L(D_1),\dots,\iota^{J*}L(D_m))],\end{aligned}$$

where $\iota_j : D_j \to |E|$ is the inclusion. This gives us the desired decomposition, because each $\iota^{J'}_*$ factors through $\iota_{\mathrm{sing}*}$. □

Lemma 6.5. *Let $j : V \to Y$ be an open subscheme of some $Y \in \mathbf{Sm}_k$, and let $\tilde{D}$ be an effective strict normal crossing divisor on Y such that $D := j^*\tilde{D}$ is smooth (and reduced). Let $j_D : D \to |\tilde{D}|$ be the inclusion. Then there is a class $[\tilde{D}]* \in \mathcal{M}^+(|\tilde{D}|)$ such that*

1. $j_D^*[\tilde{D}]* = \mathrm{Id}_D \in \mathcal{C}(D)$.
2. *The image of $[\tilde{D}]*$ in $\Omega_*(|\tilde{D}|)$ is $[\tilde{D} \to |\tilde{D}|]$.*

Proof. Suppose $\tilde{D} = \tilde{D}_1 + \sum_{j=2}^m n_j \tilde{D}_j$, with the $\tilde{D}_j$ smooth, and with $\tilde{D}_1$ the closure of D in Y. Let $\iota^j : \tilde{D}_j \to |\tilde{D}|$, and $i : |\tilde{D}|_{\mathrm{sing}} \to |\tilde{D}|$ be the inclusions. By lemma 6.4, there is a class $\eta \in \Omega_*(|\tilde{D}|_{\mathrm{sing}})$ such that

$$[\tilde{D} \to |\tilde{D}|] = \sum_j \iota^j_*[\tilde{D}_j] + i_*\eta.$$

Let $\eta*$ be a lifting of η to an element of $\mathcal{M}^+(|\tilde{D}|_{\mathrm{sing}})$ (use 4.17). Since $|\tilde{D}|_{\mathrm{sing}}$ and $\tilde{D}_j$, $j > 1$ are contained in $Y \setminus V$, and since $j_D^*[\tilde{D}_1 \to |\tilde{D}|] = \mathrm{Id}_D$, taking

$$[\tilde{D}]* = \sum_j (\tilde{D}_j \to |\tilde{D}|) + i_*\eta*$$

gives the desired element of $\mathcal{M}^+(|\tilde{D}|)$. □

6.6. **Main result.** In this section we assume that k admits resolution of singularities.

Proposition 6.7. *Let $\mu : W' \to W$ be a birational projective morphism, with W and W' in $\mathbf{Sm}_k$. Let $F \subset W$ be a closed subset containing the fundamental locus of μ, and let E be the exceptional divisor of μ. Suppose that E is a strict normal crossing divisor. Then there is an element $\eta \in \Omega_*(F)$ such that*

$$[\mu : W' \to W] = \mathrm{Id}_W + i_{F*}(\eta)$$

Proof. We may suppose that F is the fundamental locus of μ. Let T be a closed subscheme of W supported in F such that μ is the blow-up of W along T. Let $q : Y \to W \times \mathbb{P}^1$ be the blow-up of $W \times \mathbb{P}^1$ along $T \times 0$, and let $\tilde{E}$ be the exceptional divisor. By lemma 6.2, we have the identification of $W' \to W$ with the restriction of q to $q_0 : {<}W \times 0{>} \to W \times 0$. Furthermore, the singular locus of Y is contained in $|\tilde{E}| \cap {<}F \times \mathbb{P}^1{>}$, which is disjoint from ${<}W \times 0{>}$. Finally, ${<}W \times 0{>} + \tilde{E}$ is a reduced strict normal crossing divisor away from Y_{sing}.

Thus, by the resolution of singularities, we may find a projective birational map $p : \tilde{Y} \to Y$ in $\mathbf{Sm}_k$, which is an isomorphism over $Y \setminus Y_{\mathrm{sing}}$, such that $(qp)^{-1}(W \times 0)$ is a strict normal crossing divisor. Thus, $(qp)^{-1}(W \times 0) = 1 \cdot {<}W \times 0{>} + \sum_i n_i \tilde{D}_i$ with ${<}W \times 0{>} + \sum_{i=1}^m \tilde{D}_i$ a reduced normal crossing divisor, and with $(qp)(\tilde{D}_i) \subset F$ for all i.

Let $\tilde{D} = \sum_{i=1}^m \tilde{D}_i$, and let

$$\begin{aligned} &i : |{<}W \times 0{>} + \tilde{D}| \to \tilde{Y}, \\ &i_{<W\times 0>} : {<}W \times 0{>} \to |{<}W \times 0{>} + \tilde{D}|, \\ &i_{|\tilde{D}|} : |\tilde{D}| \to |{<}W \times 0{>} + \tilde{D}| \end{aligned}$$

be the inclusions. Let $f : \tilde{Y} \to W$ be the morphism $p_1 qp$ and let $f^F : |\tilde{D}| \to F$ be the restriction of f.

Since the divisors $(qp)^{-1}(W \times \infty)$ and $(qp)^{-1}(W \times 0)$ are linearly equivalent strict normal crossing divisors on $\tilde{Y}$, it follows from proposition 5.11 that

$$f_*([{<}W \times 0{>} + \sum_i n_i \tilde{D}_i \to \tilde{Y}]) = f_*[(qp)^{-1}(W \times \infty) \to \tilde{Y}]$$

in $\Omega_*(W)$. Since qp is an isomorphism over $W \times (\mathbb{P}^1 \setminus \{0\})$, we have $f_*([(qp)^{-1}(W \times \infty) \to \tilde{Y}]) = \mathrm{Id}_W$. By lemma 6.4, there is an element $\tau \in \Omega_*(|\tilde{D}|)$ such that

$$\begin{aligned} &[{<}W \times 0{>} + \sum_i n_i \tilde{D}_i \to |{<}W \times 0{>} + \tilde{D}|] \\ &= [{<}W \times 0{>} \to |{<}W \times 0{>} + \tilde{D}|] + i_{|\tilde{D}|*}(\tau) \end{aligned}$$

in $\Omega_*(|{<}W \times 0{>} + \tilde{D}|)$. Let $\eta = f_*^D(\tau) \in \Omega_*(F)$. We thus have

$$f_*([{<}W \times 0{>} \to \tilde{Y}]) + i_{F*}(\eta) = \mathrm{Id}_W.$$

Since $f : {<}W \times 0{>} \to W$ is isomorphic to $\mu : W' \to W$, this proves the proposition. □

Lemma 6.8. *Let Y be in $\mathbf{Sm}_k$, $j : U \to Y$ an open subscheme, $L'_1, L_1, \ldots, L_m$ line bundles on Y. Suppose that*

1. *The complement $i : Z \to Y$ of U is a strict normal crossing divisor.*
2. *$j^*L_1 \cong j^*L'_1$.*

Then $(\mathrm{Id}_Y, L'_1, \ldots, L_m) - (\mathrm{Id}_Y, L_1, \ldots, L_m)$ is in $i_*(\Omega_*(Z))$.

Proof. It suffices to show that $[L'_1] - [L_1] = i_*x$ for some $x \in \Omega_*(Z)$. Indeed, if this is the case, then

$$\begin{aligned}
(\mathrm{Id}_Y, L'_1, \ldots, L_m) &- (\mathrm{Id}_Y, L_1, \ldots, L_m)\\
&= \tilde{c}_1(L_m) \circ \ldots \circ \tilde{c}_1(L_2)([L'_1] - [L_1])\\
&= \tilde{c}_1(L_m) \circ \ldots \circ \tilde{c}_1(L_2)(i_*x)\\
&= i_*(\tilde{c}_1(i^*L_m) \circ \ldots \circ \tilde{c}_1(i^*L_2)(x)).
\end{aligned}$$

The kernel of $j^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(U)$ is the set of line bundles of the form $O_Y(A)$, where A is a divisor supported on the normal crossing divisor Z. Thus, there are effective divisors A and B, supported on Z, such that $L_1 \otimes O_Y(A) \cong L'_1 \otimes O_Y(B)$. It clearly suffices to handle the case $L'_1 = L_1 \otimes O_Y(A)$.

In this case, $[L'_1] = [L_1] +_F [O_Y(A)]$. Since

$$F(u, v) = u + v \bmod (uv)\Omega_*(k),$$

there is a polynomial $g(u, v)$ in $\Omega_*(k)[u, v]$ with

$$[L'_1] = [L_1] + g(\tilde{c}_1(L_1), \tilde{c}_1(O_Y(A)))([O_Y(A)]).$$

Arguing as above, it suffices to show that $[O_Y(A)]$ is in $i_*(\Omega_*(Z))$. But from proposition 5.11 we have

$$[O_Y(A)] = i_*[A \to Z],$$

which completes the proof of the lemma. □

Lemma 6.9. *Let X be a finite type k-scheme, $[Y \xrightarrow{f} X, L_1, \ldots, L_m]$ a standard cobordism cycle on X. Let $i : Z \to X$ be a closed subscheme with complement $j : U \to X$ and let $j_Y : Y_U \to Y$ denote the inclusion of the open subscheme $Y_U := Y \times_X U$. Suppose there is a smooth, quasi-projective k-scheme T, a smooth equi-dimensional morphism $\pi : Y_U \to T$ and line bundles $M_1, \ldots, M_r$ on T with $j_Y^*L_i \cong \pi^*M_i$, $i = 1, \ldots, r$*

and with $r > \dim_k T$. *Then the class of* $[Y \xrightarrow{f} X, L_1, \ldots, L_m]$ *in* $\Omega_*(X)$ *is in* $i_*(\Omega_*(Z))$.

Proof. Using lemma 4.7, we may assume that k is infinite. We proceed by induction on $\dim_k T$. We may assume that $Y = X$ and $f = \mathrm{Id}_Y$, so that π is a morphism $\pi : U \to T$. It suffices to prove the case $r = m = \dim_k T + 1$.

We note that

$$[\mathrm{Id}_Y, L_1, \ldots, L_r] = \tilde{c}_1(L_1) \circ \ldots \circ \tilde{c}_1(L_r)([\mathrm{Id}_Y]).$$

If L is a line bundle on Y, then $\tilde{c}_1(L)(i_*\eta) = i_*\tilde{c}_1(i^*L)(\eta)$, for $\eta \in \Omega_*(Z)$. Thus $\tilde{c}_1(L)$ sends $i_*(\Omega_*(Z))$ into itself. Since $\tilde{c}_1(L \otimes M^{\pm 1}) = \tilde{c}_1(L) \pm_F \tilde{c}_1(M)$, the result for $L_1 = L$ and $L_1 = M$ implies the result for $L_1 = L \otimes M^{\pm 1}$.

Next, let $g : \tilde{Y} \to Y$ be a projective birational morphism which is an isomorphism over U, and with exceptional divisor a strict normal crossing divisor. We identify U with $g^{-1}(U)$ and let $\tilde{Z} = \tilde{Y} \setminus U$. Since $[\tilde{Y} \to Y] - [\mathrm{Id}_Y]$ is in $i_*(\Omega_*(Z))$ (proposition 6.7) and since

$$\begin{aligned} g_*\tilde{c}_1(g^*L_1) \circ \ldots \circ \tilde{c}_1(g^*L_r)([\mathrm{Id}_{\tilde{Y}}]) - \tilde{c}_1(L_1) \circ \ldots \circ \tilde{c}_1(L_r)([\mathrm{Id}_Y]) \\ = \tilde{c}_1(L_1) \circ \ldots \circ \tilde{c}_1(L_r)([\tilde{Y} \to Y] - [\mathrm{Id}_Y]), \end{aligned}$$

it suffices to prove the result with $\tilde{Y}$ replacing Y and g^*L_j replacing L_j. Thus, using resolution of singularities, we may assume that Z is a strict normal crossing divisor on Y.

By resolution of singularities, there is a smooth projective k-scheme $\tilde{T}$ containing T as a dense open subscheme. Also by resolution of singularities, there is a projective birational morphism $g : \tilde{Y} \to Y$, which is an isomorphism over U, and with the exceptional divisor of π a strict normal crossing divisor, such that π extends to a morphism $\tilde{\pi} : \tilde{Y} \to \tilde{T}$. Since $\mathrm{Pic}(\tilde{T}) \to \mathrm{Pic}(T)$ is surjective, we may replace T with $\tilde{T}$. As above, we may replace Y with $\tilde{Y}$. Thus, changing notation, we may assume that $\pi : U \to T$ extends to $\tilde{\pi} : Y \to T$.

Since the kernel of $\mathrm{Pic}(Y) \to \mathrm{Pic}(U)$ is the set of line bundles of the form $O_Y(A)$, with A a divisor supported in Z, there are divisors A_j supported in Z such that $L_j = \tilde{\pi}^*(M_j) \otimes O_Y(A_j)$, $j = 1, \ldots, r$. By lemma 6.8 (take $T = \operatorname{Spec} k$, $\pi : Y \setminus Z \to \operatorname{Spec} k$ the structure morphism), $\tilde{c}_1(O(A_j))([\mathrm{Id}_Y])$ is in $i_*\Omega_*(Z)$. Following our comment above, we may assume that $L_j = \tilde{\pi}^*(M_j)$ for $j = 1, \ldots, r$.

Suppose that $\dim Z = 0$, then $M_1 = O_Z$, so $L_1 = O_Y$. Since $\tilde{c}_1(O_Y)$ is the zero endomorphism, the case $\dim Z = 0$ is settled.

Suppose that $\dim_k Z = r > 0$. We may write $M_1 = N \otimes M^{-1}$, with both N and M very ample line bundles on T. Using our comments above, we may assume that M_1 is very ample on T.

Since k is infinite, the Bertini theorem tells us that there is a section s of M_1 with smooth divisor T_1. Let $\bar{Y}$ be the subscheme of Y defined by $\tilde{\pi}^*s = 0$. Since $\bar{Y}$ is a divisor on Y, we may write $\bar{Y} = Y_1 + A$, with A and Y_1 effective, having no common components, and with A supported in the strict normal crossing divisor Z. Since $\pi : U \to T$ is smooth, it follows that $U_1 := Y_1 \cap U$ is a smooth dense open subscheme of Y_1. Let $r : T_1 \to T$, $j_1 : U_1 \to Y_1$ be the inclusions.

By resolution of singularities, there is a projective birational morphism $g : \tilde{Y} \to Y$ which is an isomorphism over U, such that $g^*(\bar{Y})$ and the exceptional divisor of g are strict normal crossing divisors on $\tilde{Y}$. As above, we may replace Y with $\tilde{Y}$ and L_j with g^*L_j. Changing notation, we may assume that Y_1 is smooth.

Since $L_1 = O_Y(Y_1 + A)$, and A is supported on Z, it suffices to show that the class of $[Y, O_Y(Y_1), L_2, \ldots, L_r]$ is in $i_*\Omega_*(Z)$. Letting $i_1 : Y_1 \to Y$ be the inclusion, we have

$$[Y, O_Y(Y_1), \ldots, L_r] = i_{1*}[Y_1, i_1^*L_2, \ldots, i_1^*L_r].$$

Since $\pi_{|Y_1} : Y_1 \to T_1$ is smooth and equi-dimensional on the dense open subscheme U_1, and since the restriction of $i_1^*L_j$ to Y_1 is $\pi_{|Y_1}^*(r^*M_j)$, we may use induction to conclude that $(Y_1, i_1^*L_2, \ldots, i_1^*L_r)$ is in the image of $\Omega_*(Y_1 \cap Z)$, completing the proof. ☐

Theorem 6.10. *Let X be a finite type k-scheme $i : Z \to X$ a closed subscheme, and $j : U \to X$ be the open complement. Then the sequence*

$$\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X) \xrightarrow{j^*} \Omega_*(U) \to 0,$$

is exact.

Proof. Step I. We first prove that the map

$$j^* : \mathcal{Z}_*(X) \to \mathcal{Z}_*(U)$$

is surjective. Let $[f : Y \to U, L_1, \ldots, L_r]$ be a cobordism cycle on U. As $f : Y \to U$ is a projective morphism and Y is smooth and quasi-projective over k, there exists a closed immersion of $Y \to U \times \mathbb{P}^N$ for some N, with f being the projection on U. Let $\bar{Y}$ be the closure of Y in $X \times \mathbb{P}^N$. Applying resolution of singularities, there is a projective birational morphism $\mu : \tilde{Y} \to \bar{Y}$ which is an isomorphism over Y, such that $\tilde{Y}$ is smooth (and quasi-projective as well). Thus $\tilde{f} := p_1 \circ \mu : \tilde{Y} \to X$ lifts f. Moreover, as the restriction map $\mathrm{Pic}(\tilde{Y}) \to \mathrm{Pic}(Y)$ is onto, one can extend the line bundles $L_1, \ldots, L_r$ on Y to line bundles $\tilde{L}_1, \ldots, \tilde{L}_r$

on $\tilde{Y}$. The cobordism cycle $[\tilde{f} : \tilde{Y} \to X, \tilde{L}_1, \ldots, \tilde{L}_r]$ on X clearly lifts $[f : Y \to U, L_1, \ldots, L_r]$, thus proving the surjectivity. In particular this implies the surjectivity of the maps $j^* : \underline{\mathcal{Z}}_*(X) \to \underline{\mathcal{Z}}_*(U)$, $j^* : \underline{\Omega}_*(X) \to \underline{\Omega}_*(U)$ and $j^* : \Omega_*(X) \to \Omega_*(U)$.

Step II. We now prove that the map

$$\ker(\mathcal{Z}_*(X) \to \Omega_*(X)) \xrightarrow{j^*} \ker(\mathcal{Z}_*(U) \to \Omega_*(U))$$

is surjective. By the construction of algebraic cobordism, an easy diagram chasing argument shows it is sufficient to prove that the maps

$$\ker(\underline{\mathcal{Z}}_*(X) \to \underline{\Omega}_*(X)) \xrightarrow{j^*} \ker(\underline{\mathcal{Z}}_*(U) \to \underline{\Omega}_*(U))$$

and

$$\ker(\underline{\Omega}_*(X) \to \Omega_*(X)) \xrightarrow{j^*} \ker(\underline{\Omega}_*(U) \to \Omega_*(U))$$

are surjective, and that

$$j^*(\ker(\mathcal{Z}_*(X) \to \Omega_*(X))) \supset \ker(\mathcal{Z}_*(U) \to \underline{\mathcal{Z}}_*(U)).$$

By lemma 2.13, the subgroup $<R_*^{Sect}>(U)$ of $\underline{\mathcal{Z}}_*(U)$ is exactly the one generated by elements of the form:

$$[Y \to U, L_1, \ldots, L_r] - [Z \to U, i^*(L_1), \ldots, i^*(L_{r-1})]$$

with $r > 0$, $[Y \to U, L_1, \ldots, L_r]$ a standard cobordism cycle on U, and $i : Z \to Y$ the closed immersion of a smooth divisor in Y such that $L_r \cong O_Y(Z)$. By step I, one may find a standard cobordism cycle $[\tilde{f} : \tilde{Y} \to X, \tilde{L}_1, \ldots, \tilde{L}_r]$ on X lifting $[Y \to X, L_1, \ldots, L_r]$. Let $\tilde{i} : \tilde{Z} \to \tilde{Y}$ be the closure of Z in $\tilde{Y}$.

Applying resolution of singularities (to $\tilde{Z} \subset \tilde{Y}$), there is a projective birational morphism $\mu : \tilde{Y}' \to \tilde{Y}$, such that μ is an isomorphism outside of $\tilde{Z} \setminus Z$, and such that the proper transform $\mu^{-1}[\tilde{Z}]$ is smooth. Replacing $\tilde{Y}$ with $\tilde{Y}'$, and $\tilde{L}_i$ with $\mu^*\tilde{L}_j$ and changing notation, we may assume that the closure $\tilde{Z}$ of Z is smooth. Since $L_r \cong O_Y(Z)$, we may take $\tilde{L}_r = O_Y(\tilde{Z})$. Thus the element

$$[\tilde{Y} \to X, \tilde{L}_1, \ldots, \tilde{L}_r] - [\tilde{Z} \to X, \tilde{i}^*(\tilde{L}_1), \ldots, \tilde{i}^*(\tilde{L}_{r-1})]$$

is an element of $<R_*^{Sect}>(X)$ lifting the given element of $<R_*^{Sect}>(U)$.

We now show that

$$\ker(\underline{\Omega}_*(X) \to \Omega_*(X)) \xrightarrow{j^*} \ker(\underline{\Omega}_*(U) \to \Omega_*(U))$$

is surjective. By proposition 4.22, we know that $\ker(\underline{\Omega}_*(U) \to \Omega_*(U))$ is generated as a group by the elements of the form (see 4.20)

$$f_* \circ \tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_r)\big([F(L, M)] - [L \otimes M]\big),$$

where $f : Y \to U$ is a projective morphism with Y irreducible and smooth, and $(L_1, \ldots, L_r, L, M)$ are line bundles on Y. Again by Step I, we may lift this element to the element

$$\tilde{f}_* \circ \tilde{c}_1(\tilde{L}_1) \circ \cdots \circ \tilde{c}_1(\tilde{L}_r)\big([F(\tilde{L}, \tilde{M})] - [\tilde{L} \otimes \tilde{M}]\big),$$

which obviously lies in $\ker(\underline{\Omega}_*(X) \to \Omega_*(X))$.

Finally, we show that

$$j^*(\ker(\mathcal{Z}_*(X) \to \Omega_*(X))) \supset \ker(\mathcal{Z}_*(U) \to \underline{\mathcal{Z}}_*(U)).$$

Indeed, lemma 2.7 shows that $\ker(\mathcal{Z}_*(U) \to \underline{\mathcal{Z}}_*(U))$ is generated by elements of the form

$$x := [Y \to U, \pi^*M_1, \ldots , \pi^*M_r, L_{r+1}, \ldots , L_m],$$

where $\pi : Y \to T$ is a smooth equi-dimensional morphism to a smooth quasi-projective k-scheme T of dimension $< r$, and $M_1, \ldots , M_r$ are line bundles on T. By Step I, we can lift x to an element $\tau := [\tilde{Y} \to X, \tilde{L}_1, \ldots , \tilde{L}_m]$ of $\mathcal{Z}_*(X)$. By lemma 6.9, there is an element η in $\mathcal{Z}_(Z)$ such that $i_*(\eta) - \tau$ is in $\ker(\mathcal{Z}_*(X) \to \Omega_*(X))$. Thus $\tau - i_*(\eta)$ is a lifting of x to $\ker(\mathcal{Z}_*(X) \to \Omega_*(X))$. This completes Step II.

Step III. The kernel of $j^* : \Omega_*(X) \to \Omega_*(U)$ is generated by differences

$$[f : Y \to X, L_1, \ldots, L_r] - [f' : Y' \to X, L'_1, \ldots, L'_r]$$

of standard cobordism cycles which agree on U. Indeed, take $x \in \mathcal{Z}_*(X)$ whose class in $\Omega_*(X)$ lies in the kernel of $j^* : \Omega_*(X) \to \Omega_*(U)$. By step II, we may modify x by an element in $\ker(\mathcal{Z}_*(X) \to \Omega_*(X)))$, so that $j^*x = 0$ in $\mathcal{Z}_*(U)$. Since $\mathcal{Z}_*(U)$ and $\mathcal{Z}_*(X)$ are the free abelian groups on the standard corbordism cycles, it follows that x can be expressed in $\mathcal{Z}_*(X)$ as a sum of differences of standard cobordism cycles on X which agree on U, as required.

Step IV. We finally prove that the differences $[f : Y \to X, L_1, \ldots, L_r]$ $-[f' : Y' \to X, L'_1, \ldots, L'_r]$ of standard cobordism cycles which agree on U lie in the image of $\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X)$. Let α denote $[f : Y \to X, L_1, \ldots, L_r]$ and α' denote $[f' : Y' \to X, L'_1, \ldots, L'_r]$.

If $Y \times_X U$ and $Y' \times_X U$ are both empty, then clearly $[f : Y \to X, L_1, \ldots, L_r]$ and $[f' : Y' \to X, L'_1, \ldots, L'_r]$ both are in the image of i_*.

If $Y \times_X U$ and $Y' \times_X U$ are (both) non-empty, then choose a U-isomorphism $\phi : Y \times_X U \to Y' \times_X U$. Let $\Gamma \subset Y \times_X Y'$ be the closure of the graph of ϕ. Resolving the singularities of Γ, we have a Y'' in $\mathbf{Sm}_k$, with projective morphisms $\mu : Y'' \to Y$, $\mu' : Y'' \to Y'$

which are isomorphisms over U, and with $f \circ \mu = f' \circ \mu'$. We may also assume that $(f\mu)^{-1}(X \setminus U)$ is a normal crossing divisor on Y''. Write $\alpha - \alpha' = (\alpha - \beta) - (\alpha' - \beta') + (\beta' - \beta)$, with $\beta = [Y'' \to X, \mu^* L_1, \ldots, \mu^* L_r]$ and $\beta' = [Y'' \to X, \mu'^* L_1', \ldots, \mu'^* L_r']$. We have to prove that $\alpha - \beta$, $\alpha' - \beta'$ and $\beta' - \beta$ lie in the image of $\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X)$.

As $(f\mu)^{-1}(X \setminus U)$ is a normal crossing divisor on Y'', we may apply proposition 6.7 to $Y'' \to Y$: if $F = Y \setminus Y_U$, there is an $\eta \in \Omega_*(F)$ such that $[\mu] - [Y = Y] = i_{F*}(\eta)$ in $\Omega_*(Y)$. Applying the $\tilde{c}_1(L_i)$ and pushing-forward by f gives that $\alpha - \beta$ lies in the image of $\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X)$. This reasoning shows the same holds for $\alpha' - \beta'$.

It remains to show that $\beta' - \beta$ lies in the image of $\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X)$ as well.

Since f_* maps $\Omega_*((f\mu)^{-1}(Z))$ to $\Omega_*(Z)$, we may replace X with Y''. Changing notation, we may assume that X is smooth and quasi-projective over k and that Z is a strict normal crossing divisor on X. Since $j^* L_1 \cong j^* L_1'$, it follows from lemma 6.8 that $\beta' - \beta$ is in $i_*(\Omega_*(Z))$.

This finishes the proof of theorem 6.10. □

Here is an immediate corollary :

Corollary 6.11. *Let k be a field admitting resolution of singularities and let $f : Y \to X$ be a projective morphism between finite type k-schemes. Suppose that there is an increasing filtration $\emptyset = X_0 \subset X_1 \subset \ldots X_{n-1} \subset X_n = X$ of X by closed subschemes with the property that, for each $i \in \{1, \ldots, n\}$, the restriction $f^{-1}(X_i) - f^{-1}(X_{i-1}) \to X_i - X_{i-1}$ of f admits a section. Then the homomorphism*

$$f_* : \Omega_*(Y) \to \Omega_*(X)$$

is onto.

For instance one can apply the lemma in case f is the projective bundle associated to some vector bundle over X.

Proof. We prove the result by induction on n. Denoting by U the open complement $X - X_{n-1}$, one has the diagram of localization sequences

$$\begin{tikzcd}
\Omega_*(f^{-1}(X_{n-1})) \arrow[r] \arrow[d] & \Omega_*(Y) \arrow[r] \arrow[d] & \Omega_*(f^{-1}(U)) \arrow[r] \arrow[d] & 0 \\
\Omega_*(X_{n-1}) \arrow[r] & \Omega_*(X) \arrow[r] & \Omega_*(U) \arrow[r] & 0.
\end{tikzcd}$$

The left vertical map is onto by the inductive assumption and the right vertical map is onto because there exists a section of $f^{-1}(U) \to U$. An easy diagram chase gives the lemma. □

7. *Homotopy invariance*

In this section, we show that, for a finite type k-scheme X, the smooth pull-back $p^* : \Omega_*(X) \to \Omega_{*+1}(X \times \mathbb{A}^1)$ is an isomorphism. We assume that k is a field admitting resolution of singularities.

Lemma 7.1. *Suppose that k is an infinite field. Let W be in $\mathbf{Sm}_k$ and let $i : Z \to W$ be a smooth closed subscheme. Then $\Omega_*(W)$ is generated by standard cobordism cycles of the form $[f : Y \to W]$ with f transverse to i.*

Proof. By lemma 4.17, $\Omega_*(W)$ is generated by the cobordism cycles of the form $[f : Y \to W]$.

Given one such cobordism cycle, there is a closed immersion $Y \to \mathbb{P}^n \times W$ such that f is the restriction of the projection. Let $Y_0 \subset Y$ be the singular locus of $f^{-1}(Z)$, that is, the minimal closed subset C of Y such that the $f : Y \setminus C \to W$ is transverse to $Z \to W$. We proceed by induction on $\dim Y_0$ and $\dim Y$ to show that $[f : Y \to W]$ is equivalent to a sum of cobordism cycles $[g : Y' \to W]$ with g transverse to $Z \to W$. In particular, if $\eta \in \Omega_{\dim Y}(W)$ is *decomposable*, i.e., is in $\Omega_s(k)\Omega_{\dim Y - s}(W)$ for some $s > 0$, then we can write η in the desired form.

W is quasi-projective, so there is a locally closed immersion $\mathbb{P}^n \times W \to \mathbb{P}^M$; let $\mathcal{O}(1)$ be the restriction to $\mathbb{P}^n \times W$ of $\mathcal{O}_{\mathbb{P}^M}(1)$. Let $\mathcal{I}$ be the ideal sheaf of Y in $\mathbb{P}^n \times W$. For N sufficiently large the linear system $H^0(\mathbb{P}^n \times W, \mathcal{I}(N))$ defines a locally closed immersion $f_N : \mathbb{P}^n \times W \setminus Y \to \mathbb{P}^r$ (r depending on N). Taking N larger if necessary, we may assume that $\mathcal{I}(N)$ is generated by global sections. Noting that Y is smooth, it follows from the Bertini theorem that, for s in a Zariski open subset of $H^0(\mathbb{P}^n \times W, \mathcal{I}(N))$, the divisor D defined by $s = 0$ is smooth, and the scheme-theoretic intersection $D \cap \mathbb{P}^n \times Z \setminus Y_0$ is also smooth. Proceeding inductively, we find a smooth irreducible closed subscheme T of $\mathbb{P}^n \times W$ such that

1. $Y \subset T$
2. $p_2 : T \setminus Y_0 \to W$ is transverse to i.
3. $\dim T = \dim Y + 1$.

We can find very ample line bundles L and M on T such that $O_T(Y) = L \otimes M^{-1}$. Using the formal group law, and working modulo decomposable elements, we see that we may replace Y with the divisor of general sections of L and M. Now, since the singular locus of $T \cap \mathbb{P}^n \times Z$ is contained in Y_0, if Y' is the divisor of a sufficiently general section of L, then Y' is smooth, $\dim(Y' \cap Y_0) = \dim Y_0 - 1$, and the singular locus of $Y' \cap \mathbb{P}^n \times Z$ is contained in $Y' \cap Y_0$. By induction, we may write $[p_2 : Y' \to W]$ as a sum of cobordism cycles of the desired form. The same works for a divisor of a general section of M, completing the proof. $\square$

Proposition 7.2. *Let X be a finite type k-scheme, and let $p : X \times \mathbb{A}^1 \to X$ be the projection. Then $p^* : \Omega_*(X) \to \Omega_{*+1}(X \times \mathbb{A}^1)$ is surjective.*

Proof. If X is a finite type k-scheme, then $\Omega_*(X) = \Omega_*(X_{\mathrm{red}})$, so we may assume that X is reduced. Since k admits resolution of singularities, k is necessarily perfect, hence X has a filtration by closed subschemes

$$\emptyset = X_0 \subset X_1 \subset \ldots \subset X_N = X$$

such that $X_i \setminus X_{i-1}$ is in $\mathbf{Sm}_k$. Using noetherian induction, we may assume the result is true for X_{N-1}. Letting $U = X_N \setminus X_{N-1}$, the commutative diagram of localization sequences

$$\begin{tikzcd}
\Omega_*(X_{N-1}) \arrow[r] \arrow[d] & \Omega_*(X) \arrow[r] \arrow[d] & \Omega_*(U) \arrow[r] \arrow[d] & 0 \\
\Omega_{*+1}(X_{N-1} \times \mathbb{A}^1) \arrow[r] & \Omega_{*+1}(X \times \mathbb{A}^1) \arrow[r] & \Omega_{*+1}(U \times \mathbb{A}^1) \arrow[r] & 0
\end{tikzcd}$$

and a diagram chase reduces us to the case of $X \in \mathbf{Sm}_k$.

Take X in $\mathbf{Sm}_k$. Using lemma 4.7 and the standard trick of taking extensions of k of relatively prime degrees, we reduce to the case of an infinite field k.

By resolution of singularities, there is a smooth projective k-scheme $\bar{X}$ containing X as an open subscheme. Since $\Omega_*(\bar{X} \times \mathbb{A}^1) \to \Omega_*(X \times \mathbb{A}^1)$ is surjective, it suffices to prove the result for $\bar{X}$. Changing notation, we may assume that X is projective.

By lemma 7.1, it suffices to show that cobordism cycles of the form $[f : Y \to X \times \mathbb{A}^1]$ such that $f^{-1}(X \times 0)$ is smooth and codimension one on Y are in the image of p^*. If $f : Y \to X \times \mathbb{A}^1$ is such a projective morphism, then, as X is projective, $p_2 f : Y \to \mathbb{A}^1$ is smooth over a neighborhood of 0 in $\mathbb{A}^1$. Let $m : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ be the multiplication map $m(x, y) = xy$. The map m is flat, and is smooth over $\mathbb{A}^1 \setminus \{0\}$. Since Y is smooth over a neighborhood of 0, it follows that $Y_m := Y \times_{\mathbb{A}^1} (\mathbb{A}^1 \times \mathbb{A}^1)$ is in $\mathbf{Sm}_k$. Let $g : Y_m \to X \times \mathbb{A}^1 \times \mathbb{A}^1$ be the projection. Then $g^{-1}(X \times \mathbb{A}^1 \times 0) = f^{-1}(X \times 0) \times \mathbb{A}^1$ and $g^{-1}(X \times \mathbb{A}^1 \times 1) = Y$. It is easy to check that $g : g^{-1}(X \times \mathbb{A}^1 \times 0) \to X \times \mathbb{A}^1$ is isomorphic to $p^*(f : f^{-1}(X \times 0) \to X)$, so g gives a cobordism between f and something in the image of p^*. $\square$

Theorem 7.3. *Let X be a finite type k-scheme. Then $p^*:\Omega_*(X)\to\Omega_{*+N}(X\times\mathbb{A}^N)$ is an isomorphism for all N.*

Proof. As for proposition 7.2, we may assume that k is infinite. It suffices to prove the case $N=1$. Having proved surjectivity in proposition 7.2, it suffices to prove the injectivity of p^*. Let $i_\infty: X\to X\times\mathbb{P}^1$ be the inclusion $i_\infty(x)=(x,(1:0))$. Consider the localization sequence

$$\Omega_*(X)\xrightarrow{i_{\infty *}}\Omega_*(X\times\mathbb{P}^1)\xrightarrow{j^*}\Omega^*(X\times\mathbb{A}^1)\to 0.$$

Let $q: X\times\mathbb{P}^1\to X$ be the projection, and let $\psi:\Omega_*(X\times\mathbb{P}^1)\to\Omega_{*-1}(X)$ be the map

$$\psi(\eta)=q_*(\tilde{c}_1(\mathcal{O}(1))(\eta)).$$

If $\eta=i_{\infty *}(\tau)$ for some τ in $\Omega_*(X)$, then

$$\begin{aligned}\tilde{c}_1(\mathcal{O}(1))(\eta)&=i_{\infty *}(\tilde{c}_1(i_\infty^*(\mathcal{O}(1)))(\tau))\\&=i_{\infty *}(\tilde{c}_1(\mathcal{O}_X)(\tau))\\&=0.\end{aligned}$$

Thus, $\psi\circ i_{\infty *}=0$, and ψ descends to a well-defined homomorphism

$$\bar{\psi}:\Omega_*(X\times\mathbb{A}^1)\to\Omega_{*-1}(X).$$

On the other hand, for τ in $\Omega_*(X)$ of the form $[f:Y\to X]$,

$$\begin{aligned}\psi\circ q^*(\tau)&=q_*(\tilde{c}_1(\mathcal{O}(1))([f\times\mathrm{Id}:Y\times\mathbb{P}^1\to X\times\mathbb{P}^1])\\&=q_*(i_\infty\circ f:Y\to X\times\mathbb{P}^1]\\&=[f:Y\to X]=\tau,\end{aligned}$$

since $(f\times\mathrm{Id})^*(\mathcal{O}(1))$ has the section X_0 with smooth divisor $Y\times\infty$. Since classes of the form $[f:Y\to X]$ generate $\Omega_*(X)$ by lemma 4.17, it follows that $\psi\circ q^*=\mathrm{Id}$, hence $\bar{\psi}\circ p^*=\mathrm{Id}$, and p^* is injective. □

8. *The projective bundle formula*

To simplify the notation, we freely pass between the category of line bundles and the category of invertible sheaves, writing for instance $\tilde{c}_1(\mathcal{L})$ for the endomorphism $\tilde{c}_1(L)$, if L is a line bundle with sheaf of sections $\mathcal{L}$.

8.1. **Support conditions.** Let X be a finite type k-scheme, and $i: F\to X$ a closed subset; give F the reduced scheme structure. Let $\Omega_*^F(X)\subset\Omega_*(X)$ denote the image of $i_*:\Omega_*(F)\to\Omega_*(X)$.

Lemma 8.2. *Let F be a closed subset of X.*

1. *Let* $f(u_1, \ldots, u_n)$ *be a power series with* $\Omega_*(k)$ *coefficients. Choose invertible sheaves* $\mathcal{L}_1, \ldots, \mathcal{L}_m$ *on* X. *Then* $f(\tilde{c}_1(\mathcal{L}_1), \ldots, \tilde{c}_1(\mathcal{L}_m))$ *preserves the subgroup* $\Omega^F_*(X)$.
2. *Let* $p : Y \to X$ *be a smooth quasi-projective morphism,* $q : X \to Z$ *a projective morphism. Then* p^* *maps* $\Omega^F_*(X)$ *to* $\Omega^{p^{-1}F}_*(Y)$, *and* q_* *maps* $\Omega^F_*(X)$ *to* $\Omega^{q(F)}_*(X)$.

Proof. (2) follows directly from the definition of smooth pull-back and proper pushforward. Similarly, (1) follows from the projection formula

$$\tilde{c}_1(\mathcal{L})(i_{F*}(\eta)) = i_{F*}(\tilde{c}_1(i_F^*\mathcal{L})(\eta)),$$

for $\mathcal{L}$ an invertible sheaf on X, η an element of $\Omega_*(F)$. ☐

8.3. **Projective bundles.** Let X be a k-scheme of finite type, let $p : \mathcal{E} \to X$ be a vector bundle of rank $n+1$, giving the $\mathbb{P}^n$-bundle $q : \mathbb{P}(\mathcal{E}) \to X$, with canonical quotient invertible sheaf $\mathcal{O}(1)$. Write ξ for the operator $\tilde{c}_1(\mathcal{O}(1))$. We have the group homomorphisms

$$\xi^j \circ q^* : \Omega_{*-n+j}(X) \to \Omega_*(\mathbb{P}(\mathcal{E}));$$

let

$$\Phi_{X,\mathcal{E}} : \oplus_{j=0}^n \Omega_{*-n+j}(X) \to \Omega_*(\mathbb{P}(\mathcal{E}))$$

be the sum of the $\xi^j \circ q^*$.

In the case of a trivial bundle $\mathcal{E} = \mathcal{O}_X^{n+1}$, we have

$$\mathbb{P}(\mathcal{E}) = \mathrm{Proj}_{\mathcal{O}_X}\mathcal{O}_X[X_0, \ldots, X_n] = \mathbb{P}^n_X,$$

and $\mathcal{O}(1)$ is the invertible sheaf with $q_*\mathcal{O}(1)$ the $\mathcal{O}_X$-module generated by $X_0, \ldots, X_n$. Let $i_m : \mathbb{P}^m_X \to \mathbb{P}^n_X$ be the subscheme defined by $X_{m+1} = \ldots = X_n = 0$, and let $q_m : \mathbb{P}^m_X \to X$ be the projection.

The following is an elementary computation:

Lemma 8.4. *For* $\mathcal{E} = \mathcal{O}_X^{n+1}$, *we have* $\xi^{n-m} \circ q^*(\eta) = i_{m*}(q_m^*(\eta))$, *and* $q_*(\xi^{n-m} \circ q^*(\eta)) = [\mathbb{P}^m_k]\eta$. *Also* $\xi^{n+1} = 0$.

8.5. **Some operators.** We proceed to define $\mathbb{Z}$-linear combinations of composable expressions in ξ, q_* and q^*, which we write as $\psi_0, \ldots, \psi_n$. Evaluating the expression ψ_j as an operator will define a graded map $\psi_j : \Omega_{*-n+j}(X) \to \Omega_*(\mathbb{P}(\mathcal{E}))$, and, in case $\mathcal{E} = \mathcal{O}_X^{n+1}$, we will have

$$\psi_i \circ (\xi^j \circ q^*) = \begin{cases} \mathrm{Id}_{\Omega_*(X)} & \text{for } i = j \\ 0 & \text{for } i \neq j. \end{cases} \tag{8.1}$$

We define the the ψ_i inductively, starting with $\psi_0(\eta) := q_*(\xi^n(\eta))$. Suppose we have defined ψ_i for $i = 0, \dots, m-1$, $1 \le m \le n$. Let

$$\psi_m = q_* \circ \big(\xi^m \circ \big(\mathrm{Id} - \sum_{j=0}^{m-1} \xi^j \circ q^* \circ \psi_j\big)\big).$$

It follows directly from lemma 8.4 that ψ_m satisfies the conditions (8.1), and the induction continues. We let Ψ be the formal expression $\oplus_{j=0}^n \psi_j$; having chosen a finite type k-scheme X and a rank $n+1$ bundle $\mathcal{E}$ on X, the expression Ψ determines the homomorphism

$$\Psi_{X,\mathcal{E}} : \Omega_*(\mathbb{P}(\mathcal{E})) \to \oplus_{j=0}^n \Omega_{*-n+j}(X),$$

natural in the pair $(X, \mathcal{E})$.

Lemma 8.6. *Suppose* $\mathcal{E} = \mathcal{O}_X^{n+1}$. *Then* $\Psi_{X,\mathcal{E}} \circ \Phi_{X,\mathcal{E}} = \mathrm{Id}$.

Proof. This follows directly from the identities (8.1). ☐

Theorem 8.7. *Let k be a field admitting resolution of singularities. Let X be a k-scheme of finite type, $\mathcal{E}$ a rank $n+1$ vector bundle on X. Then*

$$\Phi_{X,\mathcal{E}} : \oplus_{j=0}^n \Omega_{*-n+j}(X) \to \Omega_*(\mathbb{P}(\mathcal{E}))$$

is an isomorphism.

Proof. We first consider the case of the trivial bundle $\mathcal{E} = \mathcal{O}_X^{n+1}$. We have shown the injectivity of Φ in lemma 8.6. We show surjectivity by induction on n, the case $n = 0$ being trivial.

From lemma 8.4, we have the commutative diagram, where the first row is the evident inclusion,

$$\begin{tikzcd}
\oplus_{j=0}^{n-1} \Omega_{*-n+1+j}(X) \arrow[r] \arrow[d, "\Phi_{X,\mathcal{O}_X^n}"'] & \oplus_{j=0}^n \Omega_{*-n+j}(X) \arrow[d, "\Phi_{X,\mathcal{O}_X^{n+1}}"] \\
\Omega_*(\mathbb{P}_X^{n-1}) \arrow[r, "i_{n-1*}"'] & \Omega_*(\mathbb{P}_X^n).
\end{tikzcd}$$

The image of $\oplus_{j=1}^n \Omega_{*-n+j}(X)$ under $\Phi_{X,\mathcal{O}_X^{n+1}}$ therefore contains the image of i_{n-1*}. On the other hand, we have

$$j^* \circ \psi_i = j^* \circ \xi^i \circ q^* = (j^* \tilde{c}_1(\mathcal{O}(1))^j \circ p^*$$

for all i. Since $j^*\mathcal{O}(1) \cong \mathcal{O}_{\mathbb{A}_X^n}$, and $\tilde{c}_1(\mathcal{O}_{\mathbb{A}_X^n}) = 0$ by lemma **??**(6), we have $j^* \circ \psi_i = 0$ for $i > 0$. Thus, using the localization sequence

$$\Omega_*(\mathbb{P}_X^{n-1}) \xrightarrow{i_{n-1*}} \Omega_*(\mathbb{P}_X^n) \xrightarrow{j^*} \Omega_*(\mathbb{A}_X^n) \to 0,$$

the surjectivity of $\Phi_{X,\mathcal{O}_X^{n+1}}$ follows from the surjectivity of

$$p^* = j^* \circ \psi_0 : \Omega_{*-n}(X) \to \Omega_*(\mathbb{A}^n_X).$$

The surjectivity of p^* follows from the homotopy theorem 7.3.

We now pass to the general case. Choose a filtration of X by closed subschemes

$$\emptyset = X^{N+1} \subset X^N \subset \ldots \subset X^1 \subset X^0 = X$$

such that the restriction of $\mathcal{E}$ to $X^m \setminus X^{m+1}$ is trivial. To simplify the text, we omit the mention of the appropriate restriction of $\mathcal{E}$ in the notation for Φ and Ψ; similarly, we write simply $\mathcal{E}$ for the restriction of $\mathcal{E}$ to the various locally closed subsets $X_m \setminus X_{m+1}$ or $X \setminus X_m$, etc.

Assume by induction that $\Phi_{X\setminus X_m}$ is an isomorphism. By the case of the trivial bundle, $\Phi_{X_m\setminus X_{m+1}}$ is an isomorphism.

It follows from lemma 8.2 that the maps $\Phi_{X_m\setminus X_{m+1}}$ and $\Psi_{X_m\setminus X_{m+1}}$ descend to maps on the images

$$\Phi^{X_m\setminus X_{m+1}} : \oplus_{j=0}^n \Omega_{*-n+j}^{X_m\setminus X_{m+1}}(X \setminus X_{m+1}) \to \Omega_*^{\mathbb{P}^n_{X_m\setminus X_{m+1}}(\mathcal{E})}(\mathbb{P}^n_{X\setminus X_{m+1}}(\mathcal{E}))$$

$$\Psi^{X_m\setminus X_{m+1}} : \Omega_*^{\mathbb{P}^n_{X_m\setminus X_{m+1}}}(\mathbb{P}^n_{X\setminus X_{m+1}}(\mathcal{E})) \to \oplus_{j=0}^n \Omega_{*-n+j}^{X_m\setminus X_{m+1}}(X \setminus X_{m+1})$$

with $\Psi^{X_m\setminus X_{m+1}} \circ \Phi^{X_m\setminus X_{m+1}} = \mathrm{Id}$. Thus $\Phi^{X_m\setminus X_{m+1}}$ is an isomorphism.

The localization sequences for the inclusions $X_m \setminus X_{m+1} \to X \setminus X_{m+1}$ and $\mathbb{P}^n_{X_m\setminus X_{m+1}}(\mathcal{E}) \to \mathbb{P}^n_{X\setminus X_{m+1}}(\mathcal{E})$ give us the commutative diagram with exact columns

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\oplus_{j=0}^n \Omega_{*-n+j}^{X_m\setminus X_{m+1}}(X \setminus X_{m+1}) \arrow[r, "\Phi^{X_m\setminus X_{m+1}}"] \arrow[d, "i_*"'] & \Omega_*^{\mathbb{P}^n_{X_m\setminus X_{m+1}}(\mathcal{E})}(\mathbb{P}^n_{X\setminus X_{m+1}}(\mathcal{E})) \arrow[d, "i_*"] \\
\oplus_{j=0}^n \Omega_{*-n+j}(X \setminus X_{m+1}) \arrow[r, "\Phi_{X\setminus X_{m+1}}"] \arrow[d, "j^*"'] & \Omega_*(\mathbb{P}^n_{X\setminus X_{m+1}}(\mathcal{E})) \arrow[d, "j^*"] \\
\oplus_{j=0}^n \Omega_{*-n+j}(X \setminus X_m) \arrow[r, "\Phi_{X_m\setminus X_{m+1}}"] \arrow[d] & \Omega_*(\mathbb{P}^n_{X\setminus X_m}(\mathcal{E})) \arrow[d] \\
0 & 0.
\end{tikzcd}$$

As the top and bottom horizontal maps are isomorphisms, $\Phi_{X\setminus X_{m+1}}$ is an isomorphism, and the induction continues. □

9. *The extended homotopy property*

Lemma 9.1. *Let* $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$ *be an exact sequence of locally free coherent sheaves on a finite type* k*-scheme* X*, with* $\mathcal{L}$ *an invertible sheaf, let* $i : \mathbb{P}(\mathcal{F}) \to \mathbb{P}(\mathcal{E})$ *be the associated closed imbedding of projective bundles over* X*. Let* $q : \mathbb{P}(\mathcal{E}) \to X$*,* $\bar{q} : \mathbb{P}(\mathcal{F}) \to X$ *be the projections. Then*

$$i_* \circ \bar{q}^* = \tilde{c}_1(q^*\mathcal{L}^{-1} \otimes \mathcal{O}_{\mathcal{E}}(1)) \circ q^*.$$

Proof. The subscheme $\mathbb{P}(\mathcal{F})$ of $\mathbb{P}(\mathcal{E})$ is defined by the vanishing of the map $\rho : q^*\mathcal{L} \to \mathcal{O}_{\mathcal{E}}(1)$ induced by the inclusion $\mathcal{L} \to \mathcal{E}$; letting s be the section of $q^*\mathcal{L}^{-1} \otimes \mathcal{O}_{\mathcal{E}}(1)$ induced by ρ, $\mathbb{P}(\mathcal{F})$ is defined by $s = 0$, so

$$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(\mathbb{P}(\mathcal{F})) \cong q^*\mathcal{L}^{-1} \otimes \mathcal{O}_{\mathcal{E}}(1).$$

Let $f : Y \to X$ be a projective morphism with $Y \in \mathbf{Sm}_k$. The map f induces the map $\tilde{f} : \mathbb{P}(f^*\mathcal{E}) \to \mathbb{P}(\mathcal{E})$, and the inclusion $i_Y : \mathbb{P}(f^*\mathcal{F}) \to \mathbb{P}(f^*\mathcal{E})$. Then $q^*([f])$ is represented by $\tilde{f}$, and $(i_*\bar{q}^*)([f])$ is represented by $\tilde{f} \circ i_Y$. By the above computation and the relations defining Ω_*, we have

$$[\mathbb{P}(f^*\mathcal{F}) \to \mathbb{P}(f^*\mathcal{E})] = [\tilde{f}^*(q^*\mathcal{L}^{-1} \otimes \mathcal{O}_{\mathcal{E}}(1))].$$

Applying $\tilde{f}_*$, and using the definition of $\tilde{c}_1$, we find

$$\begin{aligned}[\tilde{f} \circ i_Y] &= \tilde{f}_*([\mathbb{P}(f^*\mathcal{F}) \to \mathbb{P}(f^*\mathcal{E})]) \\ &= \tilde{c}_1(q^*\mathcal{L}^{-1} \otimes \mathcal{O}_{\mathcal{E}}(1))([\tilde{f}]) \\ &= \tilde{c}_1(q^*\mathcal{L}^{-1} \otimes \mathcal{O}_{\mathcal{E}}(1))(q^*([f])).\end{aligned}$$

□

Lemma 9.2. $0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{F} \to 0$ *be an exact sequence of vector bundles on a finite type* k*-scheme* X*, let* $i : \mathbb{P}(\mathcal{F}) \to \mathbb{P}(\mathcal{E})$ *be the associated closed imbedding of projective bundles over* X*. Let* $q : \mathbb{P}(\mathcal{E}) \to X$*,* $\bar{q} : \mathbb{P}(\mathcal{F}) \to X$ *be the projections, and let* $i : \mathbb{P}(\mathcal{F}) \to \mathbb{P}(\mathcal{E})$ *be the inclusion corresponding to the projection* $\mathcal{E} \to \mathcal{F}$*. Let* $\xi = \tilde{c}_1(\mathcal{O}_{\mathcal{E}}(1))$*, and let* $\bar{\xi} = \tilde{c}_1(\mathcal{O}_{\mathcal{F}}(1))$*. Then*

$$i_* \circ (\bar{\xi}^j \circ \bar{q}^*) = \xi^{j+1} \circ q^*$$

for all $j \geq 0$*.*

Proof. For $j = 0$, this is just a special case of lemma 9.1. In general, we apply the projection formula:

$$i_* \circ (\bar{\xi}^j \circ \bar{q}^*) = i_* \circ (i^*\xi^j \circ \bar{q}^*) = \xi^j \circ i_* \circ \bar{q}^*,$$

which reduces us to the case $j = 0$. □

Corollary 9.3. *Let k be a field admitting resolution of singularities. Let $p : F \to X$ be a rank n vector bundle over a k-scheme of finite type X, and let $\tilde{p} : V \to X$ be a principal homogeneous space for F (i.e., an affine space bundle over X). Then $\tilde{p}^* : \Omega_*(X) \to \Omega_{*+n}(V)$ is an isomorphism.*

Proof. Let $\mathcal{F}$ be the sheaf of sections of F, and $\mathcal{F}^D$ the dual. V is classified by an element $v \in H^1(X, \mathcal{F}) = \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F})$. Let

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{F}^D \to 0$$

be the extension dual to v, giving the projective bundles $q : \mathbb{P}(\mathcal{E}) \to X$, $\bar{q} : \mathbb{P}(\mathcal{F}^D) \to X$, and the inclusion $i : \mathbb{P}(\mathcal{F}^D) \to \mathbb{P}(\mathcal{E})$.

The complement of $\mathbb{P}(\mathcal{F}^D)$ in $\mathbb{P}(\mathcal{E})$ is isomorphic to V, as an X-scheme. Thus, we have the localization sequence

$$\Omega_*(\mathbb{P}(\mathcal{F}^D)) \xrightarrow{i_*} \Omega_*(\mathbb{P}(\mathcal{E})) \xrightarrow{j^*} \Omega_*(V) \to 0. \tag{9.1}$$

Let

$$\kappa : \oplus_{j=0}^{n-1} \Omega_{*-n+1-j}(X) \to \oplus_{j=0}^{n} \Omega_{*-n-j}(X)$$
$$\pi : \oplus_{j=0}^{n} \Omega_{*-n-j}(X) \to \Omega_*(X)$$

be the evident inclusion and projection, giving the exact sequence

$$0 \to \oplus_{j=0}^{n-1} \Omega_{*-n+1-j}(X) \xrightarrow{\kappa} \oplus_{j=0}^{n} \Omega_{*-n-j}(X) \xrightarrow{\pi} \Omega_*(X) \to 0 \tag{9.2}$$

It follows from lemma 9.2 that $i_* \circ \Phi_{X,\mathcal{F}^D} = \Phi_{X,\mathcal{E}} \circ \kappa$. Since $\mathbb{P}(\mathcal{F}^D)$ is the subscheme of $\mathbb{P}(\mathcal{E})$ defined by the vanishing of the map $\mathcal{O} \to \mathcal{O}(1)$ corresponding to the inclusion $\mathcal{O}_X \to \mathcal{E}$, it follows that $j^*\mathcal{O}(1) = \mathcal{O}$. Thus, $j^* \circ \Phi_{X,\mathcal{E}} = p^* \circ \pi$. Therefore, $(\Phi_{X,\mathcal{F}^D}, \Phi_{X,\mathcal{E}}, p^*)$ defines a map of sequences (9.2) → (9.1). Since $\Phi_{X,\mathcal{E}^D}$ and $\Phi_{X,\mathcal{E}^D \oplus \mathcal{O}_X}$ are isomorphisms by theorem 8.7, it follows that p^* is an isomorphism. □

Remark 9.4. The method used above can also be applied to prove the analogue of Corollary 9.3 for Chow groups as well as for the K_0 functor.

Part 3. Algebraic cobordism and the Lazard ring

10. *Chern classes*

In this section, we introduce the notion of oriented Borel-Moore weak homology theories. This notion is, as the terminolgy suggests, stronger than the notion of oriented Borel-Moore functors and slightly weaker than the notion of oriented Borel-Moore homology theory which will be considered in [11]. What we have proven in Part II immediately implies that algebraic cobordism is such a theory assuming the base field admits resolution of singularities.

We then develop the theory of Chern classes for these oriented Borel-Moore weak homology theories. We then deduce several results: comparison of algebraic cobordism with K-theory, definition of the Landweber-Novikov operations in algebraic cobordism, comparison of rational algebraic cobordism with rational Chow groups.

10.1. **Weak Borel-Moore homology theories.**

We introduce two axioms concerning an oriented functor A_*:

(PB). Given a rank n vector bundle $E \to X$ on $X \in \mathbf{Sch}_k$ with sheaf of sections $\mathcal{E}$, let $q : \mathbb{P}(\mathcal{E}) \to X$ denote the projective bundle, and let $O(1)_E \to \mathbb{P}(\mathcal{E})$ be the canonical quotient line bundle of q^*E. For any $i \in \{0, \ldots, n\}$, let

$$\xi^{(i)} : A_{*+i-n}(X) \to A_*(\mathbb{P}(\mathcal{E}))$$

be the composition of $q^* : A_{*+i-n}(X) \to A_{*+i}(\mathbb{P}(\mathcal{E}))$ followed by $\tilde{c}_1(O(1)_E)^i : A_{*+i}(\mathbb{P}(\mathcal{E})) \to A_*(\mathbb{P}(E))$. Then the homomorphism

$$\Sigma_{i=0}^{n-1}\xi^{(i)} : \oplus_{i=0}^{n-1} A_{*+i-n}(X) \to A_*(\mathbb{P}(\mathcal{E}))$$

is an isomorphism.

(H). Let $E \to X$ be a vector bundle of rank r over $X \in \mathbf{Sch}_k$, and let $p : V \to X$ be an E-torsor. Then $p^* : A_*(X) \to A_{*+r}(V)$ is an isomorphism.

If (PB) holds, we shall say that A_* satisfies the projective bundle formula.

In what follows A_* denotes an oriented Borel-Moore functor on $\mathcal{V}$ which moreover satisfies axioms (Sect), (PB) and (H). We will freely use the associated weak cohomological functor $Y \to A^*(Y)$ for smooth k-schemes Y.

10.2. **Functorialities.** Let $i : \mathbb{P}^{n-r} \to \mathbb{P}^n$ be a linear embedding and let X be in $\mathbf{Sch}_k$. We define a pull-back map

$$(\mathrm{Id}_X \times i)^* : A_*(X \times_k \mathbb{P}^n) \to A_{*-r}(X \times_k \mathbb{P}^{n-r}),$$

which enjoys the usual functorialities. For this, let $\mathbb{P}^{r-1} \to \mathbb{P}^n$ be a linearly embedding projective subspace, with $\mathbb{P}^{r-1} \cap \mathbb{P}^{n-r} = \emptyset$, let $j : U \to \mathbb{P}^n$ be the inclusion of the complement $\mathbb{P}^n \setminus \mathbb{P}^{r-1}$, and let $\pi : U \to \mathbb{P}^{n-r}$ be the projection. The projection π makes U into a rank r vector bundle over $\mathbb{P}^{n-r}$; in particular, the pull-back map

$$(\mathrm{Id}_X \times \pi)^* : A_{*-r}(X \times \mathbb{P}^{n-r}) \to A_*(X \times U)$$

is defined and is an isomorphism. We then set $(\mathrm{Id}_X \times i)^* := (\pi^*)^{-1} \circ j^*$.

The evident extension of this procedure allows one to define the pull-back

$$(\mathrm{Id}_X \times i_1 \times \ldots \times i_s)^* : A_*(X \times \mathbb{P}^{n_1} \times \mathbb{P}^{n_s}) \to A_{n-r}(X \times \mathbb{P}^{n_1 - r_1} \times \mathbb{P}^{n_s - r_s});$$
$$r = \sum r_j,$$

given linear embeddings $i_j : \mathbb{P}^{n_j - r_j} \to \mathbb{P}^{n_j}$, $j = 1, \ldots, s$.

The following list of functorialities is easily checked; we leave the details to the reader:

1. For a composite of linear embeddings $i_0 : \mathbb{P}^{n_0} \to \mathbb{P}^{n_1}$, $i_1 : \mathbb{P}^{n_1} \to \mathbb{P}^{n_2}$, we have
$$(\mathrm{Id}_X \times i_0)^* \circ (\mathrm{Id}_X \times i_1)^* = (\mathrm{Id}_X \times (i_1 \circ i_0))^*.$$
2. Let $i : \mathbb{P}^{n-r} \to \mathbb{P}^n$ be a linear embedding, $L \to X \times \mathbb{P}^n$ a line bundle, and $\eta \in A_*(X \times \mathbb{P}^n)$. Then
$$i^*(\tilde{c}_1(L)(\eta)) = \tilde{c}_1(i^*L)(i^*\eta).$$
3. If A_* has products, then $(\mathrm{Id}_X \times i)^*(a \cdot b) = a \cdot i^*b)$, for $a \in A_*(X)$, $b \in A_*(\mathbb{P}^n)$.
4. Given linear embeddings $i_j : \mathbb{P}^{n_j - r_j} \to \mathbb{P}^{n_j}$, $j = 1, \ldots, s$, the pull-back $(\mathrm{Id}_X \times i_1 \times \ldots \times i_s)^*$ is the composition of the pull-backs corresponding to the s individual linear embeddings, taken in any order.

The following lemma establishes axiom (Nilp) of 2.3 for A_*.

Lemma 10.3. *Let Y be in $\mathbf{Sm}_k$. Then there is an integer N_Y such that for any family $(L_1, \ldots, L_n)$ of line bundles on Y with $n > N_Y$, one has*

$$\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_n)(1_Y) = 0 \in A_*(Y)$$

Proof. Using Jouanolou's trick [9] we see there is a vector bundle $E \to X$ and a rank r a torsor $\pi : T \to X$ under $E \to X$ such that T is an affine (and smooth) scheme. Set $N_Y := \dim(T) = \dim(Y) + r$. We claim this integer satisfies the property. By the homotopy Axiom (H) of A_* we see that $\pi^* : A_*(Y) \to A_{*+d}(T)$ is an isomorphism. Thus it suffices to show that $\tilde{c}_1(\pi^*L_1) \circ \cdots \circ \tilde{c}_1(\pi^*L_n)(1_T) = 0 \in A_*(T)$. But because T is affine each $\pi^*(L_i)$ is very ample and the method of proof of lemma 4.6(2) (which only uses Axiom (Sect)) implies the result. □

10.4. **The formal group law.**

Lemma 10.5. *Let $n \geq 0$ be an integer.*

1) For $X \in \mathbf{Sch}_k$, each element $\eta \in A_*(X \times \mathbb{P}^n)$ is a sum

$$\eta = \sum_{i=0}^{n} \eta_i \cdot [\mathbb{P}^{n-i} \to \mathbb{P}^n],$$

with the η_i uniquely determined elements of $A_*(X)$.

2) $A_*(\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_r})$ is the free $A_*(k)$-module on the classes

$$[\mathbb{P}^{n_1-i_1} \to \mathbb{P}^{n_1}] \cdot \ldots \cdot [\mathbb{P}^{n_r-i_r} \to \mathbb{P}^{n_r}]; \quad 0 \leq i_j \leq n_j, j = 1, \ldots, r,$$

where for $i \in \{0, \ldots, m\}$, $\mathbb{P}^{m-i} \to \mathbb{P}^m$ denotes (any choice of) a linearly embedded projective space of dimension $m - i$.

Proof. (2) follows from (1) by induction on r. To prove (1), take $\eta \in A_*(X \times \mathbb{P}^n)$. Writing $X \times \mathbb{P}^n$ as $\mathbb{P}(\mathcal{O}_X^{n+1})$, we have the canonical quotient line bundle $\gamma_{X,n} := O(1)$ on $X \times \mathbb{P}^n$. From axiom (PB) applied to the trivial vector bundle of rank $n + 1$ over X , there are uniquely determined elements $\eta_i \in A_*(X)$ with

$$\eta = \sum_{i=0}^{n} \tilde{c}_1(\gamma_{X,n})^i (p_1^* \eta_i).$$

Let $\pi_n : \mathbb{P}^n \to \operatorname{Spec} k$ be the structure morphism. Since $\gamma_{X,n} = p_2^* \gamma_n$, and $p_1^* \eta_i = \eta_i \cdot \pi^*(1)$, we have

$$\tilde{c}_1(\gamma_{X,n})^i (p_1^* \eta_i) = \eta_i \cdot \tilde{c}_1(\gamma_n)^i (\pi_n^*(1)).$$

Thus, it suffice to show that

$$\tilde{c}_1(\gamma_n)^i (\pi_n^* 1) = [\mathbb{P}^{n-i} \to \mathbb{P}^n].$$

For $i = 1$, this follows from the axiom (Sect), together with the fact that each hyperplane $\mathbb{P}^{n-1} \to \mathbb{P}^n$ is defined by a transverse section of

γ_n. For $i > 1$, let $\iota : \mathbb{P}^{n-i+1} \to \mathbb{P}^n$ be a linear embedding. By induction on i we have

$$\begin{aligned}\tilde{c}_1(\gamma_n)^i(\pi_n^*1) &= \tilde{c}_1(\gamma_n)\big(\tilde{c}_1(\gamma_n)^{i-1}(\pi^*1)\big)\\ &= \tilde{c}_1(\gamma_n)\big([\mathbb{P}^{n-i+1} \to \mathbb{P}^n]\big)\\ &= \tilde{c}_1(\gamma_n)\big(\iota_*(\pi_{n-i+1}^*(1))\big)\\ &= \iota_*(\tilde{c}_1(\gamma_{n-i+1})(\pi_{n-i+1}^*(1)))\\ &= \iota_*([\mathbb{P}^{n-i} \to \mathbb{P}^{n-i+1}]\\ &= [\mathbb{P}^{n-i} \to \mathbb{P}^n],\end{aligned}$$

and the induction goes through. □

Remark 10.6. From this lemma we easily deduce that $\tilde{c}_1(\gamma_n)^{n+1} = 0$. Indeed, using the axiom (Sect) of oriented Borel-Moore weak homology theory one has $\tilde{c}_1(\gamma_n)([\mathbb{P}^i \to \mathbb{P}^n]) = [\mathbb{P}^{i-1} \to \mathbb{P}^n]$ unless $i = 0$ in which case $\tilde{c}_1(\gamma_n)([\mathbb{P}^0 \to \mathbb{P}^n]) = 0$.

Remark 10.7. The proof of the lemma yields the formula:

$$[\mathbb{P}^{n_1-i_1} \subset \mathbb{P}^{n_1}] \cdot \ldots \cdot [\mathbb{P}^{n_r-i_r} \subset \mathbb{P}^{n_r}] = \tilde{c}_1(p_1^*\gamma_{n_1})^{i_1} \circ \ldots \circ \tilde{c}_1(p_1^*\gamma_{n_r})^{i_r}.$$

It is also easy to check that, given a linear embedding $i : \mathbb{P}^{n-r} \to \mathbb{P}^n$, we have

$$i^*([\mathbb{P}^s \to \mathbb{P}^n]) = \begin{cases}[\mathbb{P}^{s-r} \to \mathbb{P}^{n-r}] & \text{for } r \leq s\\ 0 & \text{for } r > s.\end{cases}$$

Indeed, if $r > s$, we can take the $\mathbb{P}^s$ inside the complementary linear space $\mathbb{P}^r$, and if $r \leq s$, we can take the $\mathbb{P}^s$ to be the closure of $\pi^*(\mathbb{P}^{s-r})$, for some linearly embedded $\mathbb{P}^{s-r} \to \mathbb{P}^r$.

Remark 10.8. Let X be in $\mathbf{Sch}_k$, and let $i : X \to X \times \mathbb{P}^n$ be the closed embedding $i(x) = x \times \mathbb{P}^0$, where $\mathbb{P}^0$ is a chosen k-rational point of $\mathbb{P}^n$. Let $j : \mathbb{P}^{n-1} \to \mathbb{P}^n$ be a linear embedding. It follows directly from lemma 10.5 and remark 10.7 that the sequence

$$0 \to A_*(X) \xrightarrow{i_*} A_*(X \times \mathbb{P}^n) \xrightarrow{j^*} A_*(X \times \mathbb{P}^{n-1}) \to 0$$

is exact. Additionally, one has $i_*(\eta) = \eta \cdot [\mathbb{P}^0 \to \mathbb{P}^n]$.

Corollary 10.9. *There is a unique power series*

$$F_A(u, v) = \sum_{i,j} a_{i,j}\, u^i\, v^j \in A_*(k)[[u, v]]$$

with $a_{i,j} \in A_{i+j-1}(k)$, such that, for any integers $n > 0$ and $m > 0$ we have in the endomorphism ring of $A_*(\mathbb{P}^n \times \mathbb{P}^m)$:

$$F_A(\tilde{c}_1(pr_1^*(\gamma_n)), \tilde{c}_1(pr_2^*(\gamma_m))) = \tilde{c}_1(pr_1^*(\gamma_n) \otimes pr_2^*(\gamma_m)). \tag{10.1}$$

Moreover, $(A_*(k), F_A(u,v))$ *is a commutative formal group law.*

Proof. Let $n > 0$ and $m > 0$ be integers and consider the line bundle $O_{n,m}(a,b) := pr_1^*(\gamma_n^{\otimes a}) \otimes pr_2^*(\gamma_m^{\otimes b})$ over $\mathbb{P}^n \times \mathbb{P}^m$. Let $\pi_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \operatorname{Spec} k$ be the structure morphism. We write $1_{n,m}$ for $\pi_{n,m}^*(1_{\operatorname{Spec} k})$.

By lemma 10.5, we can write

$$\tilde{c}_1(O_{n,m}(1,1))(1_{n,m}) = \sum_{\substack{0\le i\le n\\ 0\le j\le m}} a_{i,j}^{n,m}[\mathbb{P}^{n-i} \to \mathbb{P}^n] \cdot [\mathbb{P}^{m-j} \to \mathbb{P}^m],$$

for unique elements $a_{i,j}^{n,m} \in A_{i+j-1}(k)$. We first check that the $a_{i,j}^{n,m}$ are independent of (n,m). Indeed, if $n \le N$, $m \le M$, we have linear embeddings $i_1 : \mathbb{P}^n \to \mathbb{P}^N$, $i_2 : \mathbb{P}^m \to \mathbb{P}^M$. By checking on basis elements and using the properties of $(i_1 \times i_2)^*$ described above, we

$$\begin{aligned}
\tilde{c}_1(O_{n,m}(1,1))(1_{n,m}) &= \tilde{c}_1((i_1 \times i_2)^* O_{N,M}(1,1))((i_1 \times i_2)^* 1_{N,M}) \\
&= (i_1 \times i_2)^*(O_{N,M}(1,1))(1_{N,M}) \\
&= \sum_{\substack{0\le i\le N\\ 0\le j\le M}} a_{i,j}^{N,M} \cdot i_1^*[\mathbb{P}^{N-i} \to \mathbb{P}^N] \cdot i_2^*[\mathbb{P}^{M-j} \to \mathbb{P}^M] \\
&= \sum_{\substack{0\le i\le n\\ 0\le j\le m}} a_{i,j}^{N,M} \cdot [\mathbb{P}^{n-i} \to \mathbb{P}^n] \cdot [\mathbb{P}^{m-j} \to \mathbb{P}^m].
\end{aligned}$$

Equating coefficients with respect to our basis for $A_*(\mathbb{P}^n \times \mathbb{P}^m)$ over $A_*(k)$ yields $a_{i,j}^{n,m} = a_{i,j}^{N,M}$ for $0 \le i \le n$, $0 \le j \le m$, as desired. We may therefore write $a_{i,j}$ for $a_{i,j}^{n,m}$.

Next, we claim that

$$\tilde{c}_1(O_{n,m}(1,1)) = \sum_{\substack{0\le i\le n\\ 0\le j\le m}} a_{i,j}^{n,m} \tilde{c}_1(O_{n,m}(1,0))^i \circ \tilde{c}_1(O_{n,m}(0,1))^j$$

as endomorphisms of $A_*(\mathbb{P}^n \times \mathbb{P}^m)$. As both sides are $A_*(k)$-linear, we need only check on the basis elements $[\mathbb{P}^r \to \mathbb{P}^n][\mathbb{P}^s \to \mathbb{P}^m]$. Letting $i_1 : \mathbb{P}^r \to \mathbb{P}^n$ and $i_2 : \mathbb{P}^s \to \mathbb{P}^m$ be linear embeddings, we have

$$\begin{aligned}
&\tilde{c}_1(O_{n,m}(1,1))([\mathbb{P}^r \to \mathbb{P}^n][\mathbb{P}^s \to \mathbb{P}^m]) \\
&\qquad = \tilde{c}_1(O_{n,m}(1,1))((i_1 \times i_2)_*(1_{r,s}) \\
&\qquad = (i_1 \times i_2)_*(\tilde{c}_1(O_{r,s}(1,1)(1_{r,s}) \\
&\qquad = \sum_{\substack{0\le i\le r\\ 0\le j\le s}} a_{i,j} \cdot [\mathbb{P}^{r-i} \to \mathbb{P}^r] \cdot [\mathbb{P}^{s-j} \to \mathbb{P}^s].
\end{aligned}$$

Similarly

$$\begin{aligned}\tilde{c}_1(O_{n,m}(1,0))^i \circ \tilde{c}_1(O_{n,m}(0,1))^j([\mathbb{P}^r \to \mathbb{P}^n][\mathbb{P}^s \to \mathbb{P}^m])\\ = (i_1 \times i_2)_*(O_{n,m}(1,0))^i \circ \tilde{c}_1(O_{n,m}(0,1))^j(1_{r,s})\\ = \begin{cases} [\mathbb{P}^{r-i} \to \mathbb{P}^n][\mathbb{P}^{s-j} \to \mathbb{P}^m] & \text{if } i \le r, j \le s\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

This verifies our claim.

Setting $F_A(u,v) = \sum_{ij} a_{ij}u^iv^j$ and noting that $\tilde{c}_1(\gamma_n)^{n+1} = 0$ (remark 10.6) completes the proof of formula (10.1).

Taking $n = 0$ or $m = 0$, and using (10.1), we see that $a_{i0} = 0 = a_{0j}$ for $i, j \ge 2$, and that $a_{10} = a_{01} = 1$. Similarly, using lemma 10.5, the isomorphism $\tau^*O_{n,m}(1,1) \cong O_{m,n}(1,1)$, where $\tau : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^m$ is the exchange of factors, yields the commutativity $F_A(u,v) = F_A(v,u)$.

For associativity, let $O_{n,m,\ell}(a,b,c)$ be the line bundle $p_1^*\gamma_n^{\otimes a} \otimes p_2^*\gamma_m^{\otimes b} \otimes p_3^*\gamma_\ell^{\otimes c}$ on $\mathbb{P}^n \otimes \mathbb{P}^m \otimes \mathbb{P}^\ell$. The same argument as above gives us a unique power series $G_A(u,v,w) = \sum a_{ijk}u^iv^jw^k$ with

$$\begin{aligned}G_A(\tilde{c}_1(O_{n,m,\ell}(1,0,0), \tilde{c}_1(O_{n,m,\ell}(0,1,0), \tilde{c}_1(O_{n,m,\ell}(0,0,1))\\ = \tilde{c}_1(O_{n,m,\ell}(1,1,1)).\end{aligned}$$

We claim that

$$F_A(u, F(v,w)) = G_A(u,v,w) = F_A(F_A(u,v),w).$$

It suffices to prove the first equality.

Arguing as in the proof of formula (10.1) reduces us to showing

$$\begin{aligned}F_A(\tilde{c}_1(O_{n,m,\ell}(1,0,0)), F_A(\tilde{c}_1(O_{n,m,\ell}(0,1,0)), \tilde{c}_1(O_{n,m,\ell}(0,0,1))))(1_{n,m,\ell})\\ = G_A(\tilde{c}_1(O_{n,m,\ell}(1,0,0)), \tilde{c}_1(O_{n,m,\ell}(0,1,0)), \tilde{c}_1(O_{n,m,\ell}(0,0,1)))(1_{n,m,\ell})\end{aligned}$$

for all n, m, ℓ, where $1_{n,m,\ell}$ is the pull-back of $1 \in A_*(k)$ via the structure morphism. For $\ell = 0$, this identity follows directly from formula (10.1); we proceed by induction on ℓ. If we write

$$F_A(u, F_A(v,w)) = \sum_{ijk} a'_{ijk}u^iv^jw^k,$$

our induction hypothesis together with the exact sequence of remark 10.8 implies that

$$\begin{aligned}F_A(\tilde{c}_1(O_{n,m,\ell}(1,0,0)), F_A(\tilde{c}_1(O_{n,m,\ell}(0,1,0)), \tilde{c}_1(O_{n,m,\ell}(0,0,1))))(1_{n,m,\ell})\\ - G_A(\tilde{c}_1(O_{n,m,\ell}(1,0,0)), \tilde{c}_1(O_{n,m,\ell}(0,1,0)), \tilde{c}_1(O_{n,m,\ell}(0,0,1)))(1_{n,m,\ell})\\ = \sum_{ij}(a'_{ij0} - a_{ij0})[\mathbb{P}^i \to \mathbb{P}^n][\mathbb{P}^j \times \mathbb{P}^0 \to \mathbb{P}^m \times \mathbb{P}^\ell].\end{aligned}$$

Let $\iota : \mathbb{P}^m \times \mathbb{P}^\ell \to \mathbb{P}^N$ be the Segre embedding ($N = (m+1)(\ell+1)-1$). Since $\iota_*([\mathbb{P}^j \times \mathbb{P}^0 \to \mathbb{P}^m \times \mathbb{P}^\ell]) = [\mathbb{P}^j \to \mathbb{P}^N]$, $(\mathrm{Id} \times \iota)_* : A_*(\mathbb{P}^n \times \mathbb{P}^m \times \mathbb{P}^\ell) \to A_*(\mathbb{P}^n \times \mathbb{P}^N)$ is injective on the $A_*(k)$ submodule generated by the classes $[\mathbb{P}^i \to \mathbb{P}^n][\mathbb{P}^j \times \mathbb{P}^0 \to \mathbb{P}^m \times \mathbb{P}^\ell]$. Thus we need only check our identity after pushing forward to $A_*(\mathbb{P}^n \times \mathbb{P}^N)$.

By smooth functoriality, we have

$$F_A(\tilde{c}_1(O_{n,m,\ell}(0,1,0)), \tilde{c}_1(O_{n,m,\ell}(0,0,1)) = \tilde{c}_1(O_{n,m,\ell}(0,1,1)),$$

so

$$\begin{aligned} &F_A(\tilde{c}_1(O_{n,m,\ell}(1,0,0)), F_A(\tilde{c}_1(O_{n,m,\ell}(0,1,0)), \tilde{c}_1(O_{n,m,\ell}(0,0,1))))(1_{n,m,\ell}) \\ &\qquad = F_A(\tilde{c}_1(O_{n,m,\ell}(1,0,0)), \tilde{c}_1(O_{n,m,\ell}(0,1,1))(1_{n,m,\ell}). \end{aligned}$$

Similarly,

$$\begin{aligned} &G_A(G_A(\tilde{c}_1(O_{n,m,\ell}(1,0,0)), \tilde{c}_1(O_{n,m,\ell}(0,1,0)), \tilde{c}_1(O_{n,m,\ell}(0,0,1)))(1_{n,m,\ell}) \\ &\qquad = \tilde{c}_1(O_{n,m,\ell}(1,1,1))(1_{n,m,\ell}). \end{aligned}$$

Since $O_{n,m,\ell}(a,1,1) = (\mathrm{Id} \times \iota)^*(O_{n,N}(a,1))$, we have

$$\begin{aligned} &(\mathrm{Id} \times \iota)_*(F_A(\tilde{c}_1(O_{n,m,\ell}(1,0,0)), \tilde{c}_1(O_{n,m,\ell}(0,1,1))(1_{n,m,\ell})) \\ &\qquad = F_A(\tilde{c}_1(O_{n,N}(1,0), \tilde{c}_1(O_{n,N}(0,1))((\mathrm{Id} \times \iota)_*(1_{n,m,\ell}), \\ &(\mathrm{Id} \times \iota)_*(\tilde{c}_1(O_{n,m,\ell}(1,1,1))(1_{n,m,\ell})) \\ &\qquad = \tilde{c}_1(O_{n,N}(1,1))((\mathrm{Id} \times \iota)_*(1_{n,m,\ell}). \end{aligned}$$

Thus, we need only check that

$$\begin{aligned} &F_A(\tilde{c}_1(O_{n,N}(1,0), \tilde{c}_1(O_{n,N}(0,1))((\mathrm{Id} \times \iota)_*(1_{n,m,\ell})) \\ &\qquad = \tilde{c}_1(O_{n,N}(1,1))((\mathrm{Id} \times \iota)_*(1_{n,m,\ell})). \end{aligned}$$

This this follows from our formula (10.1). □

Definition 10.10. An *oriented Borel-Moore weak homology theory* on $\mathcal{V}$ is an oriented Borel-Moore functor with product A_* which moreover satisfies the axioms (PB), (H), as well as (Sect) and (FGL) for the formal group law $F_A(u, v)$ given by Corollary 10.9.

One observes that, because of lemma 10.3, A_* satisfies the axiom (Nilp), so that one can make sense of the axiom (FGL). By remark 4.12) we see that A_* satisfies axiom (Dim), so A_* is of geometric type.

Theorem 10.11. *Let k be a field admitting resolution of singularities. Then algebraic cobordism*

$$X \mapsto \Omega_*(X)$$

is an oriented Borel-Moore weak homology theory on $\mathbf{Sch}_k$. *Moreover,* Ω_* *is the universal oriented Borel-Moore weak homology theory on* $\mathbf{Sm}_k$ *and on* $\mathbf{Sch}_k$.

Proof. This follows directly from the results proven in Part II. ☐

Remark 10.12. We call a formal group law F on a ring R *multiplicative* if $F(u, v) = u + v - buv$ for some element $b \in R$, *additive* if $F(u, v) = u + v$. We say a multiplicative formal group on R is *periodic* if b is invertible in R. Thus, the universal multiplicative formal group is classified by the homomorphism $\mathbb{L} \to \mathbb{Z}[\beta]$ sending a_{11} to $-\beta$, and a_{ij} to 0 for $(i, j) \neq (1, 1)$ where β is an indeterminant. Similarly, the universal periodic multiplicative formal group is given by $\mathbb{L} \to \mathbb{Z}[\beta, \beta^{-1}]$, and the universal additive formal group is $\mathbb{L} \to \mathbb{Z}$, sending all a_{ij} to zero. From theorem 10.11, it follows that $\Omega_* \otimes_{\mathbb{L}} \mathbb{Z}[\beta]$, $\Omega_* \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}]$ and $\Omega_* \otimes_{\mathbb{L}} \mathbb{Z}$ are respectively the universal multiplicative, periodic multiplicative and additive oriented Borel-Moore weak homology theories on $\mathbf{Sm}_k$ and on $\mathbf{Sch}_k$.

The Chow groups

$$X \mapsto \mathrm{CH}_*(X)$$

and the K-theory functor

$$X \mapsto K^0(X)[\beta, \beta^{-1}]$$

can be seen to be oriented Borel-Moore weak homology theories on $\mathbf{Sch}_k$ as well. CH_* is additive and $K^0[\beta, \beta^{-1}]$ is multiplicative and perodic.

Definition 10.13. An *oriented weak cohomology theory* on $\mathbf{Sm}_k$ is an oriented cohomological functor with product A^* (see 1.16) such that the associated oriented Borel-Moore functor A_* is an oriented Borel-Moore weak homology theory on $\mathbf{Sm}_k$.

For instance, any oriented cohomology theory on $\mathbf{Sm}_k$ in the sense of the introduction defines an oriented weak cohomology theory on $\mathbf{Sm}_k$.

Remark 10.14. The only difference between the notion of oriented weak cohomology theory on $\mathbf{Sm}_k$ and that of oriented cohomology theory A^* on $\mathbf{Sm}_k$ in the sense of the introduction is the existence of pull-backs $f^* : A^*(X) \to A^*(Y)$ for any morphism $f : Y \to X$ between smooth k-schemes (see [11]). Indeed, one can recover the ring structure on $A^*(X)$ as the external product $A^*(X) \otimes A^*(X) \to A^*(X \times X)$ followed by pull back along the diagonal: $\Delta^* : A^*(X \times X) \to A^*(X)$.

10.15. **Chern classes and Conner-Floyd Chern classes.** In this paragraph A_* denote an oriented Borel-Moore weak homology theory on $\mathcal{V}$. Let $E \to X$ be a vector bundle of rank n over $X \in \mathbf{Sch}_k$ with sheaf of section $\mathcal{E}$. One observes that the isomorphism in (PB) gives $A_*(\mathbb{P}(\mathcal{E}))$ the structure of a graded left $End(A_*(X))$-module and implies the existence of unique homomorphisms

$$\tilde{c}_i(E) : A_*(X) \to A_{*-i}(X)$$

for $i \in \{0, \dots, n\}$, with $\tilde{c}_0(E) = 1$, and satisfying the equation (in the endomorphism ring of $A_*(\mathbb{P}(\mathcal{E}))$):

$$\sum_{i=0}^{n} (-1)^i \tilde{c}_i(E) \tilde{c}_1(O(1)_E)^{n-i} = 0.$$

The homomorphism $\tilde{c}_i(E)$ is called the i-th Chern class of E.

Lemma 10.16. *Let A_* be an oriented Borel-Moore weak homology theory. Then the Chern classes satisfy the following properties:*

(0) Given vector bundles $E \to X$ and $F \to X$ on $X \in \mathbf{Sch}_k$ one has

$$\tilde{c}_i(E) \circ \tilde{c}_j(F) = \tilde{c}_j(F) \circ \tilde{c}_i(E)$$

for any (i, j).

(1) For any line bundle L, $\tilde{c}_1(L)$ agrees with the one given in the structure of oriented Borel-Moore weak homology theory on A_*.

(2) For any smooth equidimensional morphism $Y \to X \in \mathbf{Sch}_k$, and any vector bundle $E \to X$ over X one has

$$\tilde{c}_i(f^*E) \circ f^* = f^* \circ \tilde{c}_i(E)$$

(3) If $0 \to E' \to E \to E'' \to 0$ is an exact sequence of vector bundles over X, then for each integer $n \geq 0$ one has the following equation in $End(A_*(X)$:

$$\tilde{c}_n(E) = \sum_{i=0}^{n} \tilde{c}_i(E')\, \tilde{c}_{n-i}(E'')$$

(4) For any projective morphism $Y \to X \in \mathbf{Sch}_k$, and any vector bundle $E \to X$ over X one has

$$f_* \circ \tilde{c}_i(f^*E) = \tilde{c}_i(E) \circ f_*.$$

Moreover these Chern classes are characterized by (0-3).

The proof uses the "usual" procedure. We skip it. See [6, 5] for instance.

Remark 10.17. Given an oriented cohomology theory A^* on $\mathbf{Sm}_k$ in the sense of the introduction, one can associate Chern classes of vector bundles in the classical way. The relationship between the two approaches is that for a vector bundle E on a smooth scheme X one has

$$c_i(E) = \tilde{c}_i(E)(1_X),$$

and more generally

$$\tilde{c}_i(E) : A^*(X) \to A^{*+1}(X)$$

is just the cup product by $c_i(E)$. For an oriented weak cohomology theory however we don't have the (internal) cup product in the structure, so that this formula has no meaning.

As an example, the formula $\tilde{c}_1(O_X) = 0$ would follow from the previous formula and the obvious fact that $c_1(O_X) = 0$. But in general it doesn't hold.

The following lemma is proven using exactly the same method.

Lemma 10.18. *Let A_* be an oriented Borel-Moore weak homology theory on $\mathcal{V}$ and $\tau = (\tau_i)_{i\in\mathbb{N}} \in \Pi_{i=0}^{\infty} A^i(k)$, with $\tau_0 = 1$. Then one can define in a unique way for each $X \in \mathcal{V}$ and each vector bundle E on X an endomorphism (of degree zero)*

$$\tilde{c}_\tau(E) : A_*(X) \to A_*(X)$$

such that the following holds:

(0) Given vector bundles $E \to X$ and $F \to X$ one has

$$\tilde{c}_\tau(E) \circ \tilde{c}_\tau(F) = \tilde{c}_\tau(F) \circ \tilde{c}_\tau(E).$$

(1) For a line bundle L one has:

$$\tilde{c}_\tau(L) = \sum_{i=0}^{\infty} \tilde{c}_1(L)^i \, \tau_i.$$

(2) For any smooth equidimensional morphism $Y \to X \in \mathbf{Sch}_k$, and any vector bundle $E \to X$ over X one has

$$\tilde{c}_\tau(f^*E) \circ f^* = f^* \circ \tilde{c}_\tau(E).$$

(3) If $0 \to E' \to E \to E'' \to 0$ is an exact sequence of vector bundles over X, then one has:

$$\tilde{c}_\tau(E) = \tilde{c}_\tau(E') \circ \tilde{c}_\tau(E'').$$

Remark 10.19. Observe that $\tilde{c}_\tau(L)$ is an automorphism because the power series

$$\sum_{i=0}^{\infty} \tau_i\, u^i \,\in A_*(k)[[u]]$$

has leading term 1. Using axiom (H) and Jouanolou's trick (to reduce to the case X is affine) we see that given any vector bundle E, $\tilde{c}_{\mathbf{t}}(E)$ is always an automorphism as well. Clearly one may then use (3) to extend the definition of $\tilde{c}_\tau(\xi)$ to any element $\xi \in K^0(X)$ so that $\tilde{c}_\tau(\xi)$ is always an automorphism and the map

$$K^0(X) \to Aut(A_*(X))$$

a group homomorphism, making $A_*(X)$ a $K^0(X)$-module.

Example 10.20. *Universal example.* Let $\mathbb{Z}[\mathbf{t}] := \mathbb{Z}[t_1, \ldots, t_n, \ldots]$ be the graded ring of polynomials with integral coefficients on variables t_i, $i > 0$, of degree i. We apply the above construction to the oriented Borel-Moore weak homology theory $X \mapsto A_*(X)[\mathbf{t}] := A_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbf{t}]$ obtained from A_* by extension of scalars. We take for family $\tau = (\tau_i)_{i\in\mathbb{N}}$ the "universal" one given by $\tau_i = t_i$. For any line bundle L on X one then write $\tilde{c}_{\mathbf{t}}(L)$ for the automorphism

$$\tilde{c}_\tau(L) = \sum_{i=0}^{\infty} \tilde{c}_1(L)^i\, t_i \;: A_*(X)[\mathbf{t}] \to A_*(X)[\mathbf{t}]$$

and, for each vector bundle E over X, we denote by $\tilde{c}_{\mathbf{t}}(E)$ the automorphism $\tilde{c}_\tau(E)$. We may expand $\tilde{c}_{\mathbf{t}}(E)$ as:

$$\tilde{c}_{\mathbf{t}}(E) = \sum_{I=(n_1,\ldots,n_r,\ldots)} \tilde{c}_{(n_1,\ldots,n_r,\ldots)}(E) t_1^{n_1} \ldots t_r^{n_r} \ldots$$

The $\tilde{c}_I := \tilde{c}_{(n_1,\ldots,n_r,\ldots)}(E)$ are the *Conner-Floyd Chern classes endomorphisms* $A_*(X) \to A_{*-\sum_i n_i}(X)$. We recover for instance the ith Chern class $\tilde{c}_i(E)$ as the coefficient of t_i.

Now, to give $\tau = (\tau_i)_{i\in\mathbb{N}} \in \Pi_{i=0}^{\infty} A^i(k)$, with $\tau_0 = 1$, is the same as to give a morphism

$$\vartheta_\tau : A_*[\mathbf{t}] \to A_*, t_i \mapsto \tau_i.$$

If we consider $A_*(k)$ as an $A_*(k)[\mathbf{t}]$-algebra via ϑ_τ, this induces an isomorphsim

$$A_*(X)[\mathbf{t}] \otimes_{A_*(k)[\mathbf{t}]} A_*(k) \cong A_*(X)$$

which maps $\tilde{c}_{\mathbf{t}}(E)$ to $\tilde{c}_\tau(E)$.

10.21. **Twisting a weak Borel-Moore homology theory.** The ideas in this paragraph come from Quillen's paper [20].

Let A_* be a Borel-Moore weak homology theory on $\mathcal{V}$ and $\tau = (\tau_i)_{i\in\mathbb{N}} \in \Pi_{i=0}^{\infty} A_i(k)$, with $\tau_0 = 1$. We construct an new Borel-Moore weak homology theory on $\mathbf{Sch}_k$ denoted by $A_*^{(\tau)}$ as follows. The push-forward are unchanged so that $f_*^{(\tau)} = f_*$. For any smooth equidimensional morphism $f : Y \to X$ we have the bundle of *vertical tangent vectors* T_f, defined as the kernel of the surjection $df : T_Y \to f^*T_X$. The *virtual normal bundle of* f, N_f, is the element of $K^0(Y)$ defined as

$$N_f := -T_f.$$

We define

$$f^*_{(\tau)} := \tilde{c}_\tau(N_f) \circ f^*,$$

and for any line bundle L over X we set

$$\tilde{c}_1^{(\tau)}(L) := \tilde{c}_\tau(L) \circ \tilde{c}_1(L)$$

This clearly defines a Borel-Moore weak homology theory on $\mathcal{V}$, denoted by $A_*^{(\tau)}$. The definition of $\tilde{c}_1^{(\tau)}(L)$ can be rewritten as

$$\tilde{c}_1^{(\tau)}(L) = \lambda_{(\tau)}(\tilde{c}_1(L))$$

where $\lambda_{(\tau)}(u) = \sum_{i\geq 0} \tau_i \,.\, u^{i+1} \in A_*(k)[[u]]$. Thus there is a unique power series $\lambda^{-1}_{(\tau)}(u)$ such that $\lambda^{-1}_{(\tau)}(\lambda_{(\tau)}(u)) = u$. This easily implies that the formal group law $F_A^{(\tau)}$ associated to $A_*^{(\tau)}$ is given by the following equation:

$$F_A^\tau(u,v) = \lambda_{(\tau)}\,(\,F_A(\,\lambda^{-1}_{(\tau)}(u), \lambda^{-1}_{(\tau)}(v)\,)\,) \tag{10.2}$$

If we restrict attention to $\mathbf{Sm}_k$ we can do something slightly different. For $X \in \mathbf{Sm}_k$, let T_X denote the tangent bundle of X. For a morphism $f : Y \to X$ in $\mathbf{Sm}_k$, we have the *virtual tangent bundle* $T_f \in K^0(Y)$:

$$T_f = T_Y - f^*T_X \in K^0(Y).$$

(In case f is smooth, the exact sequence $0 \to T_f \to T_Y \to f^*T_X \to 0$ shows that our two definitions of T_f agree in this case). We define the virtual normal bundle of f, N_f, by $N_f := -T_f$. In case $f : Y \to \operatorname{Spec} k$ is the structure morphism, we write N_Y for N_f, and call N_Y the *virtual normal bundle of* Y.

Define a weak Borel-Moore homology theory on $\mathbf{Sm}_k$, denoted by A_*^τ, by requiring that the pull-backs are unchanged ($f_\tau^* = f^*$), that for a projective morphism $f : Y \to X$, we have

$$f_*^\tau := f_* \circ \tilde{c}_\tau(T_f),$$

and that for any line bundle L over X we have

$$\tilde{c}_1^{\tau}(L) := \tilde{c}_1(L) \circ \tilde{c}_{\mathbf{t}}(-L) \tag{10.3}$$

One easily checks that this defines a Borel-Moore weak homology theory on $\mathbf{Sm}_k$, denoted by A_*^{τ}.

Let $\lambda_\tau(u)$ be the power series such that $\tilde{c}_1^{(\tau)}(L) = \lambda_{(\tau)}(\tilde{c}_1(L))$. If we set $\vartheta(u) := \sum_{i\geq 0} \tau_i\, u^i$, then one has $\lambda_\tau(u) = u \cdot (\vartheta(u))^{-1}$ by 10.3, so that[8]:

$$\vartheta(u) = \frac{u}{\lambda_\tau(u)}.$$

Lemma 10.22. *Let X be in $\mathbf{Sm}_k$, with virtual normal bundle N_X. Then the automorphism*

$$\tilde{c}_{\mathbf{t}}(N_X) : A_*^{(\tau)}(X) \cong A_*^{\tau}(X)$$

determines an isomorphism of Borel-Moore weak homology theories on $\mathbf{Sm}_k$. In particular the two theories have the same formal group law.

Of course to give the family τ is the same as to give either $\lambda_{(\tau)}$ or λ_τ.

Example 10.23. Let us consider the weak Borel-Moore homology theory on $\mathbf{Sch}_k$

$$X \mapsto \mathrm{CH}_*(X) \otimes \mathbb{Q}[\beta, \beta^{-1}]$$

obtained from CH^* by the extension of scalars $\mathbb{Z} \subset \mathbb{Q}[\beta, \beta^{-1}]$ (where β has degree one). We apply our construction for the family τ given by

$$\lambda_\tau(u) = \frac{1 - e^{-\beta\, u}}{\beta}.$$

This gives

$$\vartheta(u) = \frac{\beta\, u}{1 - e^{-\beta\, u}},$$

and thus $\tilde{c}_1^{\tau}(L) = Td(L)$ is the Todd class for any line bundle L. More generally

$$f_*^{td}(x) = f_*(x \cup td(T_f)).$$

This new theory is denoted $X \mapsto \mathrm{CH}_*(X) \otimes \mathbb{Q}[\beta, \beta^{-1}]^{Todd}$. The formula

$$\begin{aligned} 1 - e^{-(u+v)} &= 1 - e^{-u} \cdot e^{-v} \\ &= (1 - e^{-u}) + (1 - e^{-v}) - (1 - e^{-u})(1 - e^{-v}) \end{aligned}$$

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[8] Compare with [18, §5].

implies that the formal group law for this theory is the multiplicative one:

$$F_m(u,v) = u + v - \beta\, u\, v.$$

Example 10.24. Assume k admits resolution of singularities. Following Quillen [20], the Landweber-Novikov operations are obtained as follows. We consider the weak Borel-Moore homology theory on $\mathbf{Sm}_k$.

$$X \mapsto \Omega_*(X)[\mathbf{t}].$$

Via the process described above, we modify this into $\Omega_*(X)[\mathbf{t}]^{\mathbf{t}}$, using the family $\tau = \mathbf{t} = (t_i)$. By the universality of algebraic cobordism, we have a canonical morphism

$$\vartheta^{LN} : \Omega_* \to \Omega_*(X)[\mathbf{t}]^{\mathbf{t}},$$

which we then expand as

$$\vartheta^{LN} = \sum_I s_I\, t^I,$$

where I runs over all finite sequences $(n_1, \ldots, n_r)$ (of arbitrary length), and $t^I = t_1^{n_1} \ldots t_r^{n_r}$. The natural transformations

$$s_{(n_1,\ldots,n_r,\ldots)} : \Omega_* \longrightarrow \Omega_{*-(\sum n_i \cdot i)}$$

are called the *Landweber-Novikov operations*.

Example 10.25. Still assuming that k admits resolution of singularities, we can also consider the other twisting

$$X \mapsto \Omega_*(X)[\mathbf{t}]^{(\mathbf{t})},$$

which is now a weak Borel-Moore theory on $\mathbf{Sch}_k$.

By universality of Ω_* we get a canonical morphism

$$\Omega_* \to \Omega_*[\mathbf{t}]^{(\mathbf{t})}.$$

Using the morphism $\Omega_* \to \mathrm{CH}_*$ we get by composition

$$\vartheta : \Omega_* \to \mathrm{CH}_*[\mathbf{t}]^{(\mathbf{t})}.$$

We shall prove later on that this morphism is an isomorphism after $\otimes\mathbb{Q}$.

At this point we can prove something weaker as follows.

Definition 10.26. Assume k admits resolution of singularities. We denote by Ω_*^{ad} the weak Borel-Moore homology theory

$$X \mapsto \Omega_*^{ad}(X) := \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}.$$

We have an obvious morphism $\Omega_* \to \Omega_*^{ad}$ and as above we consider the composition

$$\vartheta : \Omega_* \to \Omega_*^{ad}[\mathbf{t}]^{(\mathbf{t})} \tag{10.4}$$

From Equation 10.2 we know that the formal group law on $\Omega_*^{ad}[\mathbf{t}]^{(\mathbf{t})}$ is given by

$$\lambda_{(\mathbf{t})}\,(\,F_a(\,\lambda^{-1}_{(\mathbf{t})}(u), \lambda^{-1}_{(\mathbf{t})}(v)\,)\,) = \lambda_{(\mathbf{t})}\,(\,\lambda^{-1}_{(\mathbf{t})}(u) + \lambda^{-1}_{(\mathbf{t})}(v)\,) \tag{10.5}$$

Theorem 10.27. *Let k be a field admitting resolution of singularities. Then the morphism* (10.4)*induces an isomorphism:*

$$\vartheta \otimes \mathbb{Q} : \Omega_* \otimes \mathbb{Q} \to \Omega_*^{ad}[\mathbf{t}]^{(\mathbf{t})} \otimes \mathbb{Q}.$$

The idea behind this theorem is the following classical lemma :

Lemma 10.28. *Let R be a commutative $\mathbb{Q}$-algebra and let $F(u,v) \in R[[u,v]]$ be a commutative formal group law of rank one over R. Then there exists a unique power series $\ell_F(u) = \sum_i \tau_i\; u^{i+1} \in R[[u]]$ such that $\tau_0 = 1$ and satisfying*

$$\ell_F(F(u,v)) = \ell_F(u) + \ell_F(v).$$

This power series is called the *logarithm of F*. Thus to give a commutative formal group law of rank one over a $\mathbb{Q}$-algebra is exactly the same as to give its logarithm. If the logarithm of the universal formal group law on $\mathbb{L}_* \otimes \mathbb{Q}$ is denoted by $\ell_{\mathbb{L}_*}(u) = \sum_i b_i\; u^{i+1} \in \mathbb{L}_* \otimes \mathbb{Q}[[u]]$ the previous lemma and universality clearly imply that the canonical morphism

$$\mathbb{Q}[b_0, \ldots, b_n, \ldots] \to \mathbb{L}_* \otimes \mathbb{Q}$$

is an isomorphism. In fact it is equivalent to give ℓ_F or to give ℓ_F^{-1}. And, as well, we get an isomorphism $\mathbb{L}_* \otimes \mathbb{Q} = \mathbb{Q}[t_0, \ldots, t_n, \ldots]$ whith $\ell^{-1}_{\mathbb{L}_*}(u) = \sum_i t_i\; u^{i+1} \in \mathbb{L}_* \otimes \mathbb{Q}[\mathbf{t}][[u]]$. Setting $\lambda_{(\mathbf{t})}(u) := \ell^{-1}_{\mathbb{L}_*}(u)$ we see that the formal group law on $\mathbb{Q}[t_0, \ldots, t_n, \ldots]$ determined by this isomorphism is

$$\lambda_{(\mathbf{t})}\,(\,\lambda^{-1}_{(\mathbf{t})}(u) + \lambda^{-1}_{(\mathbf{t})}(v)\,)$$

(Compare 10.5.)

Our proof below of theorem 10.27 is directly inspired by the construction of the previous isomorphism.

Proof. Let $\ell(u) \in \Omega_*(k) \otimes \mathbb{Q}[[u]]$ denote the the logarithm of $F_\Omega(u,v)$. Setting $\ell(u) = \sum_{i\ge 0} b_i \,.\, u^{i+1}$. We can twist rational algebraic cobordism using this family $\mathbf{b} = \{b_i\}_i$ to get the weak Borel-Moore homology theory on $\mathbf{Sch}_k$

$$X \mapsto \Omega_*(X) \otimes \mathbb{Q}^{(\mathbf{b})}.$$

By formula 10.2, the formal group law on $\Omega_* \otimes \mathbb{Q}^{(\mathbf{b})}$ is thus the additive one. Thus the canonical morphism of weak Borel-Moore homology theories on $\mathbf{Sch}_k$, $\Omega_* \to \Omega_* \otimes \mathbb{Q}^{(\mathbf{b})}$ given by universality of Ω_* factors through the morphism $\Omega_* \to \Omega_*^{ad}$ inducing a canonical morphism

$$\phi : \Omega_*^{ad} \to \Omega_* \otimes \mathbb{Q}^{(\mathbf{b})}.$$

Let $\lambda(u)$ denote $\ell^{-1}(u)$; define the τ_i's by the equation : $\lambda(u) = \sum_{i\geq 0} \tau_i \cdot u^{i+1}$. Using the universal construction of Example 10.20, we may extend ϕ to

$$\Phi : \Omega_*^{ad}[\mathbf{t}] \to \Omega_* \otimes \mathbb{Q}^{(\mathbf{b})},$$

so that t_i is mapped to τ_i. By twisting the morphism Φ both at the source and the target we get a morphism

$$\Omega_*^{ad}[\mathbf{t}]^{(\mathbf{t})} \to (\Omega_* \otimes \mathbb{Q}^{(\mathbf{b})})^{(\tau)} = \Omega_* \otimes \mathbb{Q}$$

because as λ is the inverse to ℓ, twisting by the τ_i's is the inverse operation of twisting by the b_i's. One then easily checks that this morphism is an inverse to $\vartheta \otimes \mathbb{Q}$. Indeed, the composition

$$\Omega_* \otimes \mathbb{Q}\Omega_*^{ad}[\mathbf{t}]^{(\mathbf{t})} \to \Omega_* \otimes \mathbb{Q}$$

is the identity by the universality of Ω_*. Thus $\vartheta \otimes \mathbb{Q}$ is a monomorphism. To prove it is an epimorphism, it clearly suffices to prove that the composition $\Omega_* \otimes \mathbb{Q} \to \Omega_*^{ad}[\mathbf{t}]^{(\mathbf{t})} \to \Omega_*^{ad}$ is surjective, which is obvious by construction, and universality of Ω_*. □

Remark 10.29. As a corollary we see that

$$\vartheta : \Omega_*(k) \to \Omega_*^{ad}(k)[\mathbf{t}]$$

induces an isomorphism after $\otimes\mathbb{Q}$, proving that $\Omega_*(k) \otimes \mathbb{Q}$ is a polynomial algebra on $\Omega_*^{ad}(k) \otimes \mathbb{Q}$.

11. *Algebraic cobordism and K-theory*

In this section, we give a proof of theorem 2 and Corollary 12.

11.1. **Projective bundles.** In this section A_* denotes an oriented Borel-Moore functor of geometric type (in the sense of **??**) on $\mathbf{Sm}_k$ for which the formal group law is multiplicative. Thus if $b = [\mathbb{P}^1] \in A^{-1}$ one has $F_A(u, v) = u + v - buv$. However, we never assume that A_* is periodic so that the results below still hold if $b = 0$ (so that A_* additive is permitted)! Throughout this section, we will drop the index $_A$ in the notation $[-]_A$.

Clearly the power series $\chi_m(u) = [-1]_m(u)$ is given by $[-1]_m(u) = \frac{-u}{1-b\cdot u}$ Write $u \cdot g(u) = [-1]_m(u)$ so that $g(u) = \frac{-1}{1-b\cdot u}$. We obviously have:

$$[-1]_m(u) \cdot g([-1]_m(u)) = u$$

(because $[-1]_m([-1]_m(u)) = u$) proving that $g([-1]_m(u)) = b \cdot u - 1$.

Proposition 11.2. *Let $i : Z \to X$ be a closed immersion of codimension c between smooth k-schemes, X being of dimension d. Let $X_Z \to X$ be the blow-up of X at Z, η_i the conormal sheaf of i. Then one has the following equality in $A_d(X)$:*

$$[X_Z \to X] = [\mathrm{Id}_X] + b \cdot i_*[\mathbb{P}(\eta_i) \to Z] - i_*[\mathbb{P}(\eta_i \oplus \mathcal{O}_Z) \to Z]$$

Proof. Let $q : \mathbb{P} := \mathbb{P}(\eta_i \oplus \mathcal{O}_Z) \to Z$ be the structure morphism. We apply proposition 3.3. The computation of $g(u)$ given yield the identities in $A_*(\mathbb{P}(\nu_i \oplus \mathcal{O}))$:

$$g([O(-1)]) = g([-1]_m([O(1)])) = b \cdot [O(1)] - 1$$

As the closed subscheme $\mathbb{P}(\eta_i)$ of $\mathbb{P}(\eta_i \oplus \mathcal{O}_Z)$ is defined by the vanishing of the section

$$\mathcal{O}_{\mathbb{P}} \to q^*(\eta_i \oplus \mathcal{O}_Z) \to \mathcal{O}(1),$$

we have $[O(1)] = [\mathbb{P}(\nu_i) \to \mathbb{P}(\eta_i \oplus \mathcal{O}_Z)]$ and the formula follows. □

Take $X \in \mathbf{Sm}_k$. If $D \subset X$ is a divisor on X and E is a vector bundle on X we denote by $E(D)$ the tensor product $O_X(D) \otimes E$.

Lemma 11.3. *Let $\mathcal{L}$ be an invertible sheaf on $X \in \mathbf{Sm}_k$, $\mathcal{E} \to X$ a locally free sheaf of rank $r \geq 0$ and let $i : D \to X$ be a smooth closed subscheme of X of codimension one. Then one has the following formula in $A_*(X)$:*

$$\begin{aligned}[\mathbb{P}(\mathcal{L} \oplus \mathcal{E}) \to X] &= [\mathbb{P}(\mathcal{L}(D) \oplus \mathcal{E})) \to X] \\ &\quad + i_*[\mathbb{P}(i^*(\mathcal{L}(D) \oplus \mathcal{L} \oplus \mathcal{E})) \to D] - b \cdot i_*[\mathbb{P}(i^*(\mathcal{L}(D) \oplus \mathcal{E})) \to D]\end{aligned}$$

Proof. Consider the X-scheme $q : Y := \mathbb{P}(\mathcal{L}(D) \oplus \mathcal{L} \oplus \mathcal{E}) \to X$. The surjections

$$\mathcal{L}(D) \oplus \mathcal{L} \oplus \mathcal{E} \to \mathcal{L}(D) \oplus \mathcal{E};$$
$$\mathcal{L}(D) \oplus \mathcal{L} \oplus \mathcal{E} \to \mathcal{L} \oplus \mathcal{E},$$

determine the closed embeddings

$$i_0 : \tilde{D}_0 := \mathbb{P}(\mathcal{L}(D) \oplus \mathcal{E}) \to Y$$
$$i_1 : \tilde{D}_1 := \mathbb{P}(\mathcal{L} \oplus \mathcal{E}) \to Y.$$

Note that $q^*D = \mathbb{P}(i^*(\mathcal{L}(D) \oplus \mathcal{L} \oplus \mathcal{E}))$ as an X-scheme.

We have $\mathcal{O}_Y(\tilde{D}_0) \cong (q^*\mathcal{L})(1)$, $\mathcal{O}_Y(\tilde{D}_1) \cong q^*(\mathcal{L}(D))(1)$ and $\mathcal{O}_Y(q^*D) = q^*\mathcal{O}_X(D)$. This gives an isomorphism

$$\mathcal{O}_Y(q^*D + \tilde{D}_0) \cong q^*\mathcal{O}_X(D) \otimes \mathcal{O}_Y(\tilde{D}_0) \cong \mathcal{O}_Y(\tilde{D}_1).$$

This yields the identity of endomorphisms of $A_*(Y)$:

$$\begin{aligned}\tilde{c}_1(\mathcal{O}_Y(\tilde{D}_1)) &= \tilde{c}_1(q^*\mathcal{O}_X(D) \otimes \mathcal{O}_Y(\tilde{D}_0)) \\ &= \tilde{c}_1(q^*\mathcal{O}_X(D)) + \tilde{c}_1(\mathcal{O}_Y(\tilde{D}_0)) \\ &\qquad - b \cdot \tilde{c}_1(q^*\mathcal{O}_X(D)) \circ \tilde{c}_1(\mathcal{O}_Y(\tilde{D}_0)).\end{aligned}$$

Let $\tilde{i} : q^{-1}D \to Y$ be the inclusion. Applying this to 1_Y and using the axiom (Sect) gives

$$\begin{aligned}[\tilde{D}_1 \to Y] &= [q^*D \to Y] + [\tilde{D}_0 \to Y] - b \cdot [q^*D \cdot \tilde{D}_0 \to Y](1_X) \\ &= \tilde{i}_*[\mathbb{P}(i^*(\mathcal{L}(D) \oplus \mathcal{L} \oplus \mathcal{E}) \to q^{-1}(D)] + [\mathbb{P}(\mathcal{L}(D) \oplus \mathcal{E}) \to Y] \\ &\qquad - b \cdot \tilde{i}_*[\mathbb{P}(i^*(\mathcal{L}(D) \oplus \mathcal{E})) \to q^{-1}(D)].\end{aligned}$$

Pushing forward to X and D gives the desired formula. □

Lemma 11.4. *Let $\mathcal{E}$ be a direct sum of $n+1$ invertible sheaves on some $X \in \mathbf{Sm}_k$. Then in $A_*(X)$,*

$$[\mathbb{P}(\mathcal{E}) \to X] = b^n \cdot 1_X.$$

In particular, one has:

$$[\mathbb{P}^n] = b^n$$

in $A_*(k)$.

Proof. We proceed by induction on X. We first consider the case $\dim X = 0$, $X = \operatorname{Spec} F$ for F a finite extension field of k. Since $\mathbb{P}^n_F = \mathbb{P}^n_k \times_k F$, we need only consider the case $F = k$, i.e., we must show that $[\mathbb{P}^N] = b^N$. We proceed by induction on N.

For $N = 0$ there is nothing to prove, and for $N = 1$ the result follows from the equation $[\mathbb{P}^1] = b$ in remark 3.7).

Take $N \geq 2$. We apply proposition 11.2 with $X = \mathbb{P}^N$, D a linearly embedded $\mathbb{P}^{N-1}$, $\mathcal{L} = \mathcal{O}$, and $\mathcal{E} = 0$. Pushing forward the identity of classes in $A_*(X)$ to $A_*(k)$ yields

$$[\mathbb{P}_{\mathbb{P}^N}(\mathcal{O})] = [\mathbb{P}_{\mathbb{P}^N}(\mathcal{O}(1))] + [\mathbb{P}_{\mathbb{P}^{N-1}}(\mathcal{O}(1) \oplus \mathcal{O})] - b \cdot [\mathbb{P}_{\mathbb{P}^{N-1}}(\mathcal{O}(1))].$$

Since $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{E} \otimes \mathcal{L})$ for $\mathcal{E}$ a locally free sheaf and $\mathcal{L}$ an invertible sheaf, this gives

$$[\mathbb{P}_{\mathbb{P}^{N-1}}(\mathcal{O}(1) \oplus \mathcal{O})] = b \cdot [\mathbb{P}_{\mathbb{P}^{N-1}}(\mathcal{O}(1))] = b \cdot [\mathbb{P}^{N-1}] = b^n,$$

the last identity using our induction hypothesis. Now suppose we have shown that $[\mathbb{P}_{\mathbb{P}^{N-r}}(\mathcal{O} \oplus \mathcal{O}(1)^r)] = b^N$ for some r with $1 \leq r < N$.

Apply proposition 11.2 with $X = \mathbb{P}^{N-r}$, D a linearly embedded $\mathbb{P}^{N-r-1}$, $\mathcal{L} = \mathcal{O}$, and $\mathcal{E} = \mathcal{O}(1)^r$. Pushing forward to $A_*(k)$ gives

$$\begin{aligned} b^N &= [\mathbb{P}_{\mathbb{P}^{N-r}}(\mathcal{O} \oplus \mathcal{O}(1)^r)] \\ &= [\mathbb{P}_{\mathbb{P}^{N-r}}(\mathcal{O}(1) \oplus \mathcal{O}(1)^r)] + [\mathbb{P}_{\mathbb{P}^{N-r-1}}(\mathcal{O} \oplus \mathcal{O}(1)^{r+1})] \\ &\qquad - b \cdot [\mathbb{P}_{\mathbb{P}^{N-r-1}}(\mathcal{O}(1) \oplus \mathcal{O}(1)^r)] \\ &= [\mathbb{P}^r \times \mathbb{P}^{N-r}] + [\mathbb{P}_{\mathbb{P}^{N-r-1}}(\mathcal{O} \oplus \mathcal{O}(1)^{r+1})] - b \cdot [\mathbb{P}^r \times \mathbb{P}^{N-r-1}] \\ &= [\mathbb{P}^r][\mathbb{P}^{N-r}] - b \cdot [\mathbb{P}^r][\mathbb{P}^{N-r-1}] + [\mathbb{P}_{\mathbb{P}^{N-r-1}}(\mathcal{O} \oplus \mathcal{O}(1)^{r+1})]. \end{aligned}$$

By induction, $[\mathbb{P}^r] = b^r$, $[\mathbb{P}^{N-r-1}] = b^{N-r-1}$ and $[\mathbb{P}^{N-r}] = b^{N-r}$, giving $[\mathbb{P}_{\mathbb{P}^{N-r-1}}(\mathcal{O} \oplus \mathcal{O}(1)^{r+1})] = b^N$. Taking $r = N-1$ gives us $[\mathbb{P}^N] = b^N$, as desired.

Now let $\mathcal{L}_0, \ldots, \mathcal{L}_r$ be invertible sheaves on $X \in \mathbf{Sm}_k$. We prove that

$$[\mathbb{P}(L_0 \oplus \cdots \oplus L_r) \to X] = b^r.[X]$$

by induction on $n = \dim_k(X)$, the case $n = 0$ having been settled above. Assume $n > 0$ and that we have proven the above formula when the dimension of the base is $< n$.

Let $\mathcal{E}$ be the sum $\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$. Given a smooth divisor $i : D \to X$ in X, lemma 11.3 gives

$$\begin{aligned} &[\mathbb{P}(\mathcal{L}_0 \oplus \mathcal{E}) \to X] = [\mathbb{P}(\mathcal{L}_0(D) \oplus \mathcal{E}) \to X] \\ &\quad + i_*[\mathbb{P}(i^*(\mathcal{L}_0(D) \oplus \mathcal{L}_0 \oplus \mathcal{E})) \to D] - b \cdot i_*[\mathbb{P}(i^*(\mathcal{L}_0(D) \oplus \mathcal{E})) \to D]. \end{aligned}$$

Since $\dim D < n$, we may use our inductive hypothesis, giving

$$\begin{aligned} [\mathbb{P}(\mathcal{L}_0 \oplus \mathcal{E}) \to X] &= [\mathbb{P}(\mathcal{L}_0(D) \oplus \mathcal{E}) \to X] + i_* b^r[\mathrm{Id}_D] - b \cdot i_* b^{r-1}[\mathrm{Id}_D] \\ &= [\mathbb{P}(\mathcal{L}_0(D) \oplus \mathcal{E}) \to X]. \end{aligned}$$

Doing the same for each $\mathcal{L}_i$, it follows that, given smooth divisors $D_0, \ldots, D_r$ on X, we have

$$[\mathbb{P}(\oplus_{i=0}^r \mathcal{L}_i) \to X] = [\mathbb{P}(\oplus_{i=0}^r \mathcal{L}_i(D_i)) \to X].$$

Since X is quasi-projective, there is an invertible sheaf $\mathcal{L}$ on X such that $\mathcal{L} \otimes \mathcal{L}_i^\vee$ is generated by global sections for each i. Let D_i be the divisor of a section s_i of $\mathcal{L} \otimes \mathcal{L}_i^\vee$, where s_i is chosen sufficiently general so that D_i is smooth. Then $\mathcal{L}_i(D_i) \cong \mathcal{L}$ for each i, so

$$\begin{aligned} [\mathbb{P}(\oplus_{i=0}^r \mathcal{L}_i) \to X] &= [\mathbb{P}(\oplus_{i=0}^r \mathcal{L}_i(D_i)) \to X] \\ &= [\mathbb{P}(\oplus_{i=0}^r \mathcal{L}) \to X] \\ &= [p_2 : \mathbb{P}^r \times X \to X] \\ &= b^r[\mathrm{Id}X]. \end{aligned}$$

The lemma is now proven.

□

Remark 11.5. If we assume that A_* satisfies the extended homotopy property on $\mathbf{Sm}_k$, the previous lemma implies that for *any* locally free sheaf $\mathcal{E}$ on X of rank $r+1$ one has

$$[\mathbb{P}(\mathcal{E}) \to X] = b^r \cdot 1_X$$

Indeed, using the full flag variety $q : \mathcal{F}l_X(\mathcal{E}) \to X$ of $\mathcal{E}$ one can reduce to the case where E admits a complete flag of sub-vector bundles with quotients of rank 1; the projective bundle formula implies that $q^* : A_*(X) \to A_{*+M}(\mathcal{F}l_X(\mathcal{E}))$ is injective. Jouanolou's trick and homotopy invariance for A_* reduce further to the case of an affine base X, which reduces us to the case of a direct sum of invertible sheaves, since on an affine scheme, every exact sequence of locally free sheaves splits.

We also get the following simplification of proposition 11.2:

Corollary 11.6. *Let $i : Z \to X$ be a closed immersion of codimension c between smooth k-schemes. Let $X_Z \to X$ be the blow-up of X at Z. Then one has in $A_*(X)$:*

$$[X_Z \to X] = [\mathrm{Id}_X]$$

In particular, when X is smooth projective over k, one has in $A_*(k)$:

$$[X_Z] = [X]$$

Remark 11.7. The universal oriented Borel-Moore functor of geometric type having multiplicative formal group law is of course given by:

$$X \mapsto \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta]$$

It thus satisfies Corollary 11.6 above. It is tempting to make the following:

Conjecture 1. Let k be any field. Then the oriented Borel-Moore functor of geometric type

$$X \mapsto \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta]$$

is the universal oriented Borel-Moore functor of geometric type which has "birational invariance" in the following sense: given a birational projective morphism $f : Y \to X$ then $f_* 1_Y = 1_X$.

In fact we shall establish Conjecture 1 over a field of characteristic zero in **??** below.

11.8. **Universal property of K-theory.** We can now prove the following version of theorem 2 involving oriented weak cohomology theories on $\mathbf{Sm}_k$ (rather than genuine oriented cohomology theories):

Theorem 11.9. *Let A^* be an oriented cohomological functor on $\mathbf{Sm}_k$. Assume that A^* is multiplicative and periodic (in the sense of the introduction). Then there exists one and only one morphism of oriented weak cohomology theories*

$$\vartheta_A : K^0[\beta, \beta^{-1}] \to A^*$$

Proof. We first observe that ϑ_A has to map $\beta = [\mathbb{P}^1]$ to $b = [\mathbb{P}^1]_A$. Moreover, as for $X \in \mathbf{Sm}_k$ and any line bundle L over X with sheaf of section $\mathcal{L}$, one has in $K^0(X)$

$$[\mathcal{L}] = 1 - (1 - [(\mathcal{L}^\vee)^\vee] = 1 - c_1^K(L^\vee) \cdot \beta,$$

one must have

$$\vartheta_A([\mathcal{L}]) = \vartheta_A(1 - c_1^K(L^\vee) \cdot \beta) = 1 - c_1^A(L^\vee) \cdot b.$$

Using the splitting principle, this establishes uniqueness.

Now, for $X \in \mathbf{Sm}_k$, define

$$\vartheta_A : K^0(X)[\beta, \beta^{-1}] \to A^*(X)$$

by

$$[\mathcal{E}] \cdot \beta^n \mapsto (r - c_1^A(E^\vee) \cdot b) \cdot b^n$$

(where E is a vector bundle of rank r over X with sheaf of sections $\mathcal{E}$). One easily checks (using the splitting principle again) that this defines a homomorphism of graded rings. It obviously commutes with smooth pull-backs.

Now we prove that ϑ_A commutes with projective push-forwards. It sufficient to check this for the projection $\mathbb{P}^n \times X \to X$ to a smooth k-scheme X and for a closed immersion $Y \to X$ between smooth k-schemes.

We first consider the case of the projection $\mathbb{P}^n \times X \to X$ for some $n > 0$. We observe (using the external product) that it is sufficient to check the compatibility of ϑ_A with respect to the push forward along $\pi : \mathbb{P}^n \to \operatorname{Spec} k$. We note that $[\mathbb{P}^n]_K = \beta^n$, since $\chi(\mathcal{O}_{\mathbb{P}^N}) = 1$. By lemma **??**, $[\mathbb{P}^n] = b^n \in A^*(k)$, which verifies the compatibility of ϑ_A with π.

We now proceed to the case of a closed immersion $f : X \to Y$. We use the argument used of the proof of Grothendieck-Riemann-Roch for closed embedding as in [5, §15.2]. We keep the notation from there, so that $M \to Y \times \mathbb{P}^1$ denotes the deformation to the normal bundle of f

(in [5], M is the blow-up of $X \times \{\infty\}$ in $Y \times \mathbb{P}^1$). The computation is essentially the same, with just a slight modification: one has

$$j_{0*}[Y] = [O_M(\mathbb{P}(\eta_f \oplus \mathcal{O})] + [O_M(\tilde{Y})] - b \cdot \tilde{c}_1(O_M(\mathbb{P}(\eta_f \oplus \mathcal{O})))[O_M(\tilde{Y})],$$

with η_f the conormal sheaf of X in Y. The rest of the argument goes through. This finishes the proof of the theorem. □

Remark 8. The classical Grothendieck-Riemann-Roch theorem can be easily deduced from theorem 11.9. Indeed, one consider the oriented weak cohomology theory

$$X \mapsto \mathrm{CH}^*(X) \otimes \mathbb{Q}[\beta, \beta^{-1}]^{td}$$

constructed in remark 10.23. Now by theorem 11.9 there exists one (and only one) morphism

$$\vartheta : K^0[\beta, \beta^{-1}] \to \mathrm{CH}^* \otimes \mathbb{Q}[\beta, \beta^{-1}]$$

of oriented cohomology theories. One then checks by the splitting principle that ϑ is equal in degree 0 to the Chern character

$$ch : K^0(X) \to \mathrm{CH}(X) \otimes \mathbb{Q}$$

(where $\mathrm{CH}(X)$ denotes the ungraded Chow ring) giving Grothendieck-Riemann-Roch's theorem.

Remark 11.10. Theorem 2 follows from theorem 11.9. Indeed, we only have to check that given an oriented multiplicative periodic cohomology theory A^* the morphism of oriented weak cohomology theories

$$K^0[\beta, \beta^{-1}] \to A^*$$

constructed by theorem 11.9 commutes with all the pull-backs (not only with smooth one ; see 10.14). But pull-backs in K-theory come from pulling-back vector bundles, and the result follows from naturality of Chern classes.

Corollary 11.11. *Let k be a field admitting resolution of singularities. Then for any smooth k-scheme X the natural homomorphism:*

$$\Omega^*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta, \beta^{-1}] \to K^0(X)[\beta, \beta^{-1}]$$

is an isomorphism.

Proof. By remark 10.12, $X \mapsto \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta, \beta^{-1}]$ is the universal oriented multiplicative periodic Borel-Moore weak homology theory (either on $\mathbf{Sch}_k$ or on $\mathbf{Sm}_k$.) Theorem 11.9 implies on the other hand that $X \mapsto K_0(X)[\beta, \beta^{-1}]$ is also the universal oriented multiplicative periodic Borel-Moore weak homology theory, whence the result. □

Remark 11.12. We do not know whether or not the canonical homomorphism

$$\Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta, \beta^{-1}] \to G_0(X)[\beta, \beta^{-1}]$$

is an isomorphism for all finite type k-schemes X. What follows from the previous theorem is that

$$\Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta, \beta^{-1}] \to K_0^{BM}(X)[\beta, \beta^{-1}]$$

is an isomorphism for any finite type k-scheme X. The remaining problem is to decide whether or not

$$K_0^{BM}(X)[\beta, \beta^{-1}] \to G_0(X)[\beta, \beta^{-1}]$$

is an isomorphism for any finite type k-scheme X, with k admitting the resolution of singularities.

12. *The cobordism ring of a point*

In this section, we compute $\Omega_*(k)$ by showing that the canonical homomorphism

$$\Phi : \mathbb{L}_* \to \Omega_*(k)$$

is an isomorphism over any field of characteristic zero.

Our stategy to prove that $\mathbb{L}_* = \Omega_*(k)$ is to show first that Φ is always injective, for any field. Then to prove surjectivity it is sufficient to prove that the augmentation $\Omega_*(k) \to \mathbb{Z}$ induces an isomorphism $\Omega_*(k) \otimes_{\mathbb{L}*} \mathbb{Z} \cong \mathbb{Z}$. To do this we proceed inductively in the dimension.

12.1. **The canonical splitting.** In this section we prove that Φ is always a monomorphism, over any field. The first method is only valid when the field admits a complex imbedding.

Lemma 12.2. *Let $\sigma : k \to \mathbb{C}$ be a complex embedding. Then the induced homomorphism*

$$\Omega_*(k) \to MU_{2*}$$

doesn't depend on σ and its compostion with Φ induces Quillen's isomorphism

$$\Phi : \mathbb{L}_* \cong MU_{2*}$$

In particular, Φ is injective.

Proof. An embedding σ gives a functor from the category of smooth projective k-schemes to the category of compact complex manifolds, which we denote by $X \mapsto X_\sigma(\mathbb{C})$. The functor $X \mapsto MU^{2*}(X_\sigma(\mathbb{C}))$ defines by [20] an oriented cohomology theory on $\mathbf{Sm}_k$ in the sense of the introduction and thus also an oriented weak cohomological theory on $\mathbf{Sm}_k$ in the sense of section **??**. By universality of algebraic cobordism we get a canonical map of oriented weak cohomological theories $\vartheta_{MU,\sigma}(X) : \Omega^*(X) \to MU^{2*}(X_\sigma(\mathbb{C}))$. In particular, $\vartheta_{MU,\sigma}(k)$ yields the ring homomorphism

$$\Omega_*(k) \to MU_{2*}$$

By construction, the composition $\mathbb{L}_* \to \Omega_*(k) \to MU_{2*}$ is the canonical homomorphism Φ_{MU}, which, by Quillen's result [20], is an isomorphism.

We conclude by showing that $\Omega_*(k) \to MU_{2*}$ doesn't depends on σ. Indeed, $\Omega_*(k)$ is generated by classes $[X]$ of smooth projective varieties over k. In addition, by Milnor [14], the class $[X_\sigma(\mathbb{C})] \in MU_{2*}$ only depends on the Chern numbers of X. But the latter can be computed algebraically and are thus independent of the choice of complex embedding. In other words, given two complex embedding σ and τ, the varieties $X_\sigma(\mathbb{C})$ and $X_\tau(\mathbb{C})$ are automatically cobordant. □

An other way to prove the injectivity, over any field, is as follows. We use the morphism

$$\vartheta^{LN} : \Omega_* \to \mathrm{CH}_*[\mathbf{t}]^{(\mathbf{t})}$$

defined in **??**. Its associated formal group law is by construction the power series

$$F_{\mathbf{t}}(u,v) = \lambda^{-1}(\lambda(u) + \lambda(v)),$$

where

$$\lambda(u) = \sum_{i\geq 0} t_i.u^i \in \mathbb{Z}[t_1, \ldots, t_n, \ldots][[u]],$$

with the convention that $t_0 = 1$, and where $\lambda^{-1}(u)$ is the inverse power series, satisfying $\lambda(\lambda^{-1}(u)) = u$. The proof of the following lemma is obvious, by construction.

Lemma 12.3. *Let k be a field. Then the composition of the homomorphism*

$$\Omega_*(k) \to \mathbb{Z}[t_1, \ldots, t_n, \ldots]$$

induced by ϑ^{LN} and of

$$\Phi : \mathbb{L}_* \to \Omega_*(k)$$

is the canonical monomorphism

$$\mathbb{L}_* \to \mathbb{Z}[t_1, \ldots, t_n, \ldots]$$

classifying the formal group law $F_{\mathbf{t}}$.

We can thus deduce:

Corollary 12.4. *Let k be any field. Then*

$$\Phi : \mathbb{L}_* \to \Omega_*(k)$$

is a monomorphism.

Indeed, we know (see [2, 20]) that the homomorphism

$$\mathbb{L}_* \to \mathbb{Z}[t_1, \ldots, t_n, \ldots]$$

classifying the formal group law $F_{\mathbf{t}}$ is a monomorphism.

Remark 12.5. The homomorphism

$$\Omega_*(k) \to \mathbb{Z}[t_1, \ldots, t_n, \ldots]$$

can be checked to send the class $[X]$ of a smooth projective k-scheme X of dimension d to the sum

$$chern(X) := \sum_{\alpha_1, \ldots, \alpha_d} <c_1(\nu(X))^{\alpha_1} \ldots c_d(\nu(X))^{\alpha_d}, [X]>$$

where the α_i's are integers, $\nu(X) = -T_X \in K^0(X)$ is the virtual normal bundle[9] of X and $<x, [X]> \in \mathbb{Z}$ denotes the degree of a class $x \in \mathrm{CH}^*(X)$ (which is zero unless x has degree d).

Thus this homomorphism is just "computing all the Chern numbers" of X. A. Merkurjev has proven in [16] that all these $chern(X) \in \mathbb{Z}[t_1, \ldots, t_n, \ldots] =: \mathbb{Z}[\mathbf{t}]$ indeed lie in the (image of the) Lazard ring in $\mathbb{Z}[\mathbf{t}]$. This provides in fact a ring homomorphism

$$\Psi : \Omega_*(k) \to \mathbb{L}_*$$

which is left inverse to Φ, over any field. In the sequel, however, we will not use this fact, only corollary 12.4.

12.6. **The main theorem.** In this section we assume everywhere that $\mathrm{char}(k) = 0$.

We are now prepared to begin the proof of the surjectivity of $\Phi : \mathbb{L}_* \to \Omega_*(k)$ which will thus finish the proof of theorem 4, restated as theorem 12.8 below. In the sequel, we denote by $\Omega_*^{ad}(k)$ the ring $\Omega_*(k) \otimes_{\mathbb{L}_*} \mathbb{Z}$.

We first show that the class of W in $\Omega_*^{ad}(k)$ is a birational invariant.

Proposition 12.7. *Let W and W' be smooth projective varieties over k. Suppose that W and W' are birationally isomorphic. Then $[W'] = [W]$ in $\Omega_*^{ad}(k)$.*

---

[9] Voevodsky has proven that ν_X can always be represented as a difference $V - O_X^n$ for some vector bundle V over X [23].

Proof. The proof uses the factorization results of [1] in an essential way. By *loc. cit.* there is a sequence of birational morphisms

$$W = W_0 \leftarrow Y_0 \rightarrow W_1 \leftarrow \ldots \leftarrow Y_n \rightarrow W_n = W'$$

each of which is the blow-up along a smooth center. This reduces us to the case $W' = W_F \rightarrow W$, the blow-up of W along a smooth center F. We conclude by using Corollary 11.6. □

Theorem 12.8. *Let k be a field of characteristic zero. Then the natural map $\mathbb{L}_* \rightarrow \Omega_*(k)$ is an isomorphism.*

Proof. By lemma 12.4, we need only to show that $\Omega_*^{ad}(k) \cong \mathbb{Z}$. By theorem 4.18, the degree map $\Omega_0(k) \rightarrow \mathbb{Z}$ is an isomorphism. We now show that $\Omega_n^{ad}(k) = 0$ for all $n > 0$.

Let Y be a smooth irreducible projective variety of dimension n over k. Embed Y in a $\mathbb{P}^N$, and take a general linear projection of Y to a $\bar{Y} \subset \mathbb{P}^{n+1}$; with $Y \rightarrow \bar{Y}$ finite and birational. Let $\mu : S \rightarrow \mathbb{P}^{n+1}$ be a sequence of blow-ups with smooth centers such that $\mu^*(\bar{Y})$ is a strict normal crossing divisor. Write

$$\mu^*(\bar{Y}) = \tilde{Y} + \sum_i n_i E_i,$$

where $\tilde{Y}$ is the proper transform of $\bar{Y}$, and the E_i are components of the exceptional divisor of μ.

Since $\tilde{Y} \rightarrow \bar{Y}$ and $Y \rightarrow \bar{Y}$ are birational isomorphisms, $\tilde{Y}$ is birationally isomorphic to Y. Thus, by proposition 12.7, we have $[\tilde{Y}] = [Y]$ in $\Omega_n^{ad}(k)$. Write μ as a composition of blow-ups of S_i along the smooth center F_i:

$$S = S_0 \xrightarrow{\mu_1} S_1 \xrightarrow{\mu_2} \ldots \xrightarrow{\mu_r} S_r = \mathbb{P}^{n+1}.$$

Let $\bar{E}_i \subset S_{i-1}$ be the exceptional divisor $\mu_i^{-1}(F_i)$; $\bar{E}_i$ is the projective bundle $\mathbb{P}(\mathcal{N}_i) \rightarrow F_i$, where $\mathcal{N}_i$ is the conormal sheaf of F_i in S_i. Reordering the E_i, the map $S \rightarrow S_{i-1}$ restricts to a birational morphism $E_i \rightarrow \bar{E}_i$. By proposition **??** and proposition 12.7, it follows that $[E_i] = 0$ in $\Omega_n^{ad}(k)$.

Suppose $\bar{Y}$ has degree d in $\mathbb{P}^{n+1}$. Let $D \subset \mathbb{P}^{n+1}$ be the divisor of a general section of $\mathcal{O}(d)$. Then both D and μ^*D are smooth and irreducible, and $\mu : \mu^*D \rightarrow D$ is birational. Thus $[\mu^*D] = [D]$ in $\Omega_n^{ad}(k)$ by proposition 12.7. On the other hand, μ^*D is linearly equivalent to $\mu^*(\bar{Y})$, hence $[\mu^*D \rightarrow S] = [\mu^*\bar{Y} \rightarrow S]$ in $\Omega_n(S)$. Pushing forward to $\operatorname{Spec} k$, we find

$$[D] = [\mu^*D] = [\mu^*\bar{Y}] = [\tilde{Y}] = [Y] \in \Omega_n^{ad}(k)$$

Furthermore, D is linearly equivalent to d hyperplanes in $\mathbb{P}^{n+1}$, so by remark **??** and lemma **??**

$$[D] = d[\mathbb{P}^n] = 0 \in \Omega_n^{ad}(k)$$

completing the proof. □

Remark 12.9. It is reasonable to conjecture:

Conjecture 2. Let k be a field. Then

$$\Phi(k) : \mathbb{L}_* \to \Omega_*(k)$$

is an isomorphism.

We have proven that conjecture in characteristic zero and we know that the map is always injective. Moreover, theorem 4.18 shows that this conjecture is always true in dimension 0 over any field. Furthermore, one can check that our previous proof can be carried for smooth curves over a field, proving the conjecture in dimension 1 as well.

In fact the group $\Omega_1(k)$ is the free abelian group generated by $[\mathbb{P}^1]$. Moreover, let C be a smooth projective curve over k, g its genus (the genus of its extension to an algebraic closure of k). Then

$$[C] = (1 - g) \cdot [\mathbb{P}^1]$$

in $\Omega_1(k)$.

Finally, the technique used to prove theorem 12.8 requires characteristic zero in two places: first so that k admits resolution of singularities, and second so that the weak factorization results of [**?**] are valid. It seems reasonable to suppose that once one assumes that k admits resolution of singularities, the weak factorization theorem can be proved using the methods of [1], in which case theorem 12.8 would be valid over a field admitting resolution of singularities.

12.10. **Birational invariant theories.** In this section we prove the following theorem establishing Conjecture 1 for fields of characteristic zero.

Theorem 12.11. *Let k be a field of characteristic zero. Then the oriented Borel-Moore functor of geometric type*

$$X \mapsto \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta]$$

is the universal oriented Borel-Moore functor of geometric type which has "birational invariance" in the following sense: given a birational projective morphism $f : Y \to X$ between smooth irreducible varieties, then $f_* 1_Y = 1_X$.

We start by proving the following :

Proposition 12.12. *Let $I \in \mathbb{L}_*$ denote the ideal generated by elements of the form $[W] - [W']$, with W and W' be smooth projective varieties over k which are birationally isomorphic. Then I is the kernel of the map*

$$\Omega_*(k) = \mathbb{L}_* \to \mathbb{Z}[\beta]$$

classifying the multiplicative formal group law.

Proof. Let W be a smooth projective irreducible k-variety. The image of $[W]$ in $\mathbb{Z}[\beta]$ will be denoted by $[W]_\beta$ in the sequel.

First, if W and W' are smooth projective varieties over k which are birationally isomorphic then by [1] there is a sequence of birational morphisms

$$W = W_0 \leftarrow Y_0 \to W_1 \leftarrow \ldots \leftarrow Y_n \to W_n = W'$$

each of which is the blow-up along a smooth center. By Corollary 11.6, we conclude that $[W]_\beta = [W']_\beta$ proving that I is contained in the kernel Ker of $\Omega_*(k) = \mathbb{L}_* \to \mathbb{Z}[\beta]$.

To prove the converse inclusion $Ker \subset I$ we proceed as follows. Let $\mathbb{Z}[\beta] \to \mathbb{L}_*$ be the ring homomorphism which maps β to $[\mathbb{P}^1]$. It is a section of $\mathbb{L}_* \to \mathbb{Z}[\beta]$. To finish the proof it thus clearly sufficies to show that $\mathbb{Z}[\beta] + I = \Omega_*(k)$. It is thus sufficient to prove that each of the coefficients $a_{i,j}$ of the formal group law in $\mathbb{L}_*$. But using the formula in 3.6 and induction on the degree, this follows from the following lemma : □

Lemma 12.13. *For each pair (n, m) of positive integers, the Milnor hypersurface $H_{n,m}$ is birational to $\mathbb{P}^{n+m-1}$.*

Proof. Assume $n \leq m$. Then the projection $H_{n,m} \to \mathbb{P}^n$ (composition of the closed immersion $H_{n,m} \to \mathbb{P}^n \times \mathbb{P}^m$ and the first projection, realizes $H_{n,m}$ as a $\mathbb{P}^{m-1}$-bundle over $\mathbb{P}^n$. More precisely, if $X_0, \ldots, X_n$ and $Y_0, \ldots, Y_m$ are standard coordinates on $\mathbb{P}^n$ and $\mathbb{P}^m$, one can use the section $\sum_{i=0}^n X_i Y_i$ of $O(1,1)$ to define $H_{n,m}$, and then $H_{n,m} = \mathrm{Proj}_{\mathbb{P}^n}(E)$, where E is the kernel of the surjection $O_{\mathbb{P}^n}^{m+1} \to O_{\mathbb{P}^n}(1)$ with matrix $(X_0, ..., X_n, 0, \ldots, 0)$. Thus $H_{n,m}$ is birational to $\mathbb{P}^{n+m-1}$. □

Proof. (of theorem 12.11) Let A_* be an oriented Borel-Moore functor of geometric type which has birational invariance. By birational invariance of A_*, we see that the map $\mathbb{L}_* = \Omega_*(k) \to A_*(k)$ vanishes on the ideal I considered in proposition 12.12. From that proposition then follows that the map $\mathbb{L}_* = \Omega_*(k) \to A_*(k)$ factors through $\mathbb{L}_* \to \mathbb{Z}[\beta]$. Thus the map

$$\Omega_*(X) \to A_*(X)$$

given by the universality of Ω_* induces a canonical morphism

$$\Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta] \to A_*(X)$$

Thus it remains only to prove that the theory $X \mapsto \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta]$ itself has birational invariance. Given a birational projective morphism $f : Y \to X$ between smooth irreducible varieties, then by theorem 0.3.1 of [1] there exists a finite sequence of blow-up and blow-down with smooth centers in the category of X-schemes starting with Y and ending with X. So we reduce to proving $f_* 1_Y = 1_X$ when f is a blow-up with smooth center, and this clearly follows from proposition 11.6. ☐

13. *Degree formulas*

13.1. **The degree homomorphism.** Let A_* be an oriented Borel-Moore weak homology theory on $\mathbf{Sch}_k$. For any field extension $k \subset F$ which is of finite type, one denotes by

$$A_*(F/k)$$

the colimit over the category of all models [10] X of F over k of the groups $A_{*+\deg .tr(F/k)}(X)$. We observe that the assignment $F \mapsto A_*(F/k)$ is covariantly functorial with respect to field extensions: given an extension $\phi : F \subset L$ of fields of finite type over k, one has a homomorphism $\phi^* : A_*(F/k) \to A_*(L/k)$.

For instance, given an integral k-scheme X with function field F, then $A_*(F/k)$ can be identified with the colimit

$$\mathrm{colim}_{U \subset X} A_*(U)$$

where U ranges over the set of non-empty open subsets of X. Let $i : \eta \to X$ denotes the generic point of X. Then we denote by

$$i^* : A_*(X) \to A_*(F/k)$$

the canonical homomorphism.

Definition 13.2. The oriented Borel-Moore weak homology theory A_* is said to be *generically constant* if for each field extension $k \subset F$ of finite type over k the canonical morphism $A_*(k) \to A_*(F/k)$ is an isomorphism.

For instance, the Chow group functor has this property ; recall that $\mathrm{CH}_*(F/k) = \mathbb{Z}$ placed in degree zero, for any finite type field extension $k \subset F$. The K-theory functor as well. We now proceed to prove that algebraic cobordism also satisfies this property in characteristic zero; the proof will crucially rely on theorem **??**.

---

[10] A model of F over k is an integral finite type k-scheme X together with an isomorphism between the field F and the field of functions of X.

Let $k \subset F$ be a finite type field extension of characteristic zero. We define a ring homomorphism

$$\Omega_*(F/k) \to \Omega_*(F)$$

as follows: The group $\Omega_*(F/k)$ is generated by classes of the form $[f : Y \to X]$, with $f : Y \to X$ a projective morphism, Y smooth and irreducible, and X integral with field of function F. Let η be the generic point of X. Then, since the characteristic is zero, the generic fiber Y_η of f, which is a projective F-scheme, is also a smooth F-scheme. The assignment $[Y \to X] \mapsto [Y_\eta]$ then induces the desired homomorphism. Indeed, it is easy to check that the kernel of $\mathcal{M}(X)^+ \to \Omega_*(X)$ map to zero, and that the resulting homomorphism $\Omega_*(X) \to \Omega_*(F)$ is natural on the category of models of F over k, hence descends to the direct limit $\Omega_*(F/k)$.

Lemma 13.3. *Let k be a field of characteristic zero, then for a finite type field extension $k \subset F$ the homomorphism*

$$\Omega_*(F/k) \to \Omega_*(F)$$

is an isomorphism

Proof. Let X denote an integral model for F. Since each $f : Y \to \eta$ in $\mathcal{M}(\eta)$ is projective over η, it is clear that i^* induces an isomorphism

$$i^* : \varinjlim_U \mathcal{M}(U) \to \mathcal{M}(F).$$

If $Y \to \eta$ is in $\mathcal{M}(\eta)$, then each invertible sheaf $\mathcal{L}$ on Y is the restriction of an invertible sheaf $\tilde{\mathcal{L}}$ on $\tilde{Y}$ for some open U and some $\tilde{f} : \tilde{Y} \to U$ in $\mathcal{M}(U)$ inducing $Y \to \eta$. Thus

$$i^* : \varinjlim_U \underline{\Omega}_*(U) \to \underline{\Omega}_*(F)$$

is an isomorphism. As similar argument easily shows that

$$i^* : \varinjlim_U \Omega_*(U) \to \Omega_*(F)$$

is an isomorphism. ☐

Corollary 13.4. *In characteristic zero, algebraic cobordism is generically constant.*

Proof. One easily checks that the diagram

$$\begin{tikzcd}
\Omega_*(k) \arrow[rr] & & \Omega_*(F) \\
& \mathbb{L}_* \arrow[ul, "\Phi_k"] \arrow[ur, "\Phi_F"'] &
\end{tikzcd}$$

commutes. By theorem **??** the map $\Omega_*(k) \to \Omega_*(F)$ is an isomorphism. Using lemma 13.3 completes the proof. $\square$

Let A_* be a generically constant oriented Borel-Moore weak homology theory. Let X be an irreducible finite type k-scheme. Let η be the generic point of X, with inclusion $i_\eta : \eta \to X$, and let $p_\eta : \eta \to \operatorname{Spec} k$ be the structure morphism. The map $p_\eta^* : A_*(k) \to A_*(k(\eta)/k)$ is an isomorphism by assumption, hence we have the homomorphism $\deg : A_*(X) \to A_*(k)$ defined by $\deg = (p_\eta^*)^{-1} \circ i_\eta^*$.

More generally, if X is a finite type k-scheme with irreducible components $X_1, \ldots, X_r$, we have the homomorphisms

$\deg_i : A_*(X) \to A_*(k)$, $i = 1, \ldots, r$, defined by

$$\deg_i := (p_{\eta_i}^*)^{-1} \circ i_{\eta_i}^*$$

where η_i is the generic point of X_i.

13.5. **The generalized degree formula.**

Definition 13.6. Let A_* be a generically constant oriented Borel-Moore weak homology theory. We say that A_* has the localization property if for any closed immersion $i : Z \to X$ with $j : U \subset X$ the complementary open immersion, the sequence:

$$A_*(Z) \xrightarrow{i_*} A_*(X) \xrightarrow{j^*} A_*(U) \to 0$$

is exact.

For instance, the Chow group functor and the K-theory functor have this property over any field. Algebraic cobordism has the localization property as well, at least in characteristic zero, by theorem **??**.

Theorem 13.7 (Generalized degree formula)**.** *Let k be a field. Let A_* denote an oriented Borel-Moore weak homology theory. Assume A_* is generically constant and that it has the localization property.*

Let X be a finite type k-scheme. Assume that, for each closed integral subscheme $Z \subset X$, we are given a projective birational morphism $\tilde{Z} \to Z$ with $\tilde{Z}$ in $\mathbf{Sm}_k$. Then the $A_*(k)$-module $A_*(X)$ is generated by the classes $[\tilde{Z} \to X]$.

More precisely, let $X_1, \ldots, X_r$ be the irreducible components of X. Let α be an element of $A_*(X)$. Then, for each $Z \subset X$ of codimension > 0, there are elements $\omega_Z \in A_*(k)$, all but a finite number being zero, such that

$$\alpha - \sum_{i=1}^{r} \deg_i(\alpha) \cdot [\tilde{X}_i \to X] = \sum_{Z, \operatorname{codim}(Z) > 0} \omega_Z \cdot [\tilde{Z} \to X].$$

One should observe that we don't use the resolution of singularities in the proof. The problem of course is to find some desingularisations $\tilde{Z} \to Z$ of each closed integral subscheme of X. See **??** below for a variant of theorem 13.7 which uses De Jong's theorem [**?**].

proof of theorem 13.7. We proceed by induction on the dimension of X. For U_i an open subscheme of X_i, we let $\tilde{U}_i$ denote the the inverse image of U_i by $\tilde{X}_i \to X$.

Let $\alpha \in A_*(X)$. Then the element

$$\alpha - \sum_{i=1}^{r} \deg_i(\alpha) \cdot [\tilde{X}_i \to X]$$

vanishes upon applying each of the homomorphisms $\deg_i$. Thus there is for each i an open subscheme $j_i : U_i \to X$, containing the generic point of X_i and disjoint from $\cup_{j \neq i} X_j$, such that $j_i^* \alpha = \deg_i(\alpha) \cdot [\tilde{U}_i \to U_i]$ in $\Omega_*(U_i)$.

Thus, letting $U = \cup_{i=1}^{r} U_i$, with inclusion $j : U \to X$, we have

$$j^*\big(\alpha - \sum_{i=1}^{r} \deg_i(\alpha) \cdot [\tilde{X}_i \to X]\big) = 0$$

in $\Omega_*(U)$. Let $W = X \setminus U$ with closed immersion $i : W \to X$. By the localization property of A_*, there is an element α^1 of $A_*(W)$ such that

$$\alpha = \sum_{i=1}^{r} \deg_i(\alpha) \cdot [\tilde{X}_i \to X] + i_*(\alpha^1)$$

Each closed integral subscheme $Z \subset W$ is also a closed integral subscheme in X thus we have our projective birational morphisms $\tilde{Z} \to Z$ with $\tilde{Z} \in \mathbf{Sm}_k$. We then apply the inductive hypothesis to W together with the given family of projective morphisms $\tilde{Z} \to W$, where Z ranges over all the closed integral subscheme $Z \subset W$. We get an expression of our class $\alpha^1 \in A_*(W)$ as

$$\alpha^1 = \sum_{Z \subset W} \omega_Z \cdot [\tilde{Z} \to W]$$

Together with our previous expression we thus get

$$\begin{aligned} \alpha &= \sum_{i=1}^{r} \deg_i(\alpha) \cdot [\tilde{X}_i \to X] + i_*(\alpha^1) \\ &= \sum_{i=1}^{r} \deg_i(\alpha) \cdot [\tilde{X}_i \to X] + \sum_{Z \subset W} \omega_Z \cdot [\tilde{Z} \to X] \end{aligned}$$

proving the theorem. □

Corollary 13.8. *With the assumptions as in theorem 13.7, suppose in addition that X is in $\mathbf{Sm}_k$ and is irreducible.*
1) Let $f : Y \to X$ be a projective morphism with Y in $\mathbf{Sm}_k$. Then there are elements $\omega_Z \in A_*(k)$, all but a finite number being zero, such that

$$[f : Y \to X]_A - \deg(f) \cdot [\mathrm{Id}_X]_A = \sum_{Z,\mathrm{codim}(Z)>0} \omega_Z \cdot [\tilde{Z} \to X]$$

2) Let $f : Y \to X$ be a projective birational morphism with Y in $\mathbf{Sm}_k$. Then there are elements $\omega_Z \in A_*(k)$, all but a finite number being zero, such that

$$[f : Y \to X]_A = [\mathrm{Id}_X]_A + \sum_{Z,\mathrm{codim}(Z)>0} \omega_Z \cdot [\tilde{Z} \to X]$$

Proof. Since X is in $\mathbf{Sm}_k$, we may take $\tilde{X} \to X$ to be $\mathrm{Id} : X \to X$. The first assertion then follows from theorem 13.7. The second follows from the first, noting that $\deg f = 1 \in A_*(k)$ if f is birational, by theorem 4.18. ☐

Theorem 13.9 (Rational generalized degree formula)**.** *Let k be perfect field. Let A_* denote an oriented Borel-Moore weak homology theory such that $A_*(k)$ is a $\mathbb{Q}$-algebra. Assume A_* is generically constant and that it satisfies the localization property.*

Let X be a finite type k-scheme. For each closed integral subscheme $Z \subset X$ choose a projective morphism $\tilde{Z} \to Z$ with $\tilde{Z}$ smooth over k and generically étale[11]*.*

Then the $A_*(k)$-module $A_*(X)$ is generated by the classes $[\tilde{Z} \to X]$. More precisely, let $X_1, \ldots, X_r$ be the irreducible components of X, and let α be an element of $A_*(X)$. Then, for each $Z \subset X$ of codimension > 0, there exists elements $\omega_Z \in A_*(k)$, all but a finite number being zero, such that

$$\alpha - \sum_{i=1}^{r} \deg_i(\alpha) \cdot [\tilde{X}_i \to X] = \sum_{Z,\mathrm{codim}(Z)>0} \omega_Z \cdot [\tilde{Z} \to X].$$

Remark 13.10. Under the assumptions of theorem 13.9, the analog of corollary 13.8 is also valid. In particular, if X is in $\mathbf{Sm}_k$ and is irreducible, then, given $\alpha \in A_*(X)$, there exists, for each $Z \subset X$ of codimension > 0, elements $\omega_Z \in A_*(k)$, all but a finite number being zero, such that

$$\alpha - \deg(\alpha) \cdot [\mathrm{Id}_X] = \sum_{Z,\mathrm{codim}(Z)>0} \omega_Z \cdot [\tilde{Z} \to X]$$

---

[11] This is possible by De Jong's theorem.

The proof of theorem 13.9 is exactly the same as for theorem 13.7 using the following:

Lemma 13.11. *Let A_* be any oriented Borel-Moore weak homology theory and let $f : Y \to X$ be a finite étale morphism between smooth irreducible k-schemes. Then one has the equality*

$$\deg([Y \to X]) = [k(Y) : k(X)] \cdot 1_X$$

in $A_*(k(X)/k$.

Proof. The proof is basically the same as that of lemma 13.11, taking into account that one may replace X by any of its non-empty open subset, so one may assume that $X = \operatorname{Spec} R$ is affine and that f corresponds to an elementary étale algebra $(R[T]/P)_r$. ☐

Remark 13.12. Let A_* denote an oriented Borel-Moore weak homology theory which is generically constant and satisfies the localization property. Then theorem 13.7 implies that the natural map

$$A_*(k) \otimes_{\mathbb{Z}} \mathcal{M}(X)^+_* \to A_*(X), (\omega \otimes [Y \to X]) \mapsto \omega \cdot [Y \to X]_A$$

is surjective. Thus the morphism $\Omega_*(X) \otimes_Z A_*(k) \to A_*(X)$ must be always surjective as well.

In particular, if we assume further that the ring $A_*(k)$ is generated as a group by classes $[X]$ of smooth projective varieties over k, then it follows that $\Omega_*(X) \to A_*(X)$ is surjective.

We also get:

Corollary 13.13. *Let k be a field of characteristic zero. Let A_* denote an oriented Borel-Moore weak homology theory. Assume A_* is generically constant and moreover that it satisfies the localization property. Then for any finite type k-scheme X the $A_*(k)$-module $A_*(X)$ is generated over $A_*(k)$ by the classes of degree $\in \{0, \ldots, \dim(X)\}$.*

If in addition X is irreducible, let $\tilde{X} \to X$ be a projectve birational morphism with $\tilde{X}$ smooth. Then $A_*(X)$ is generated over $A_*(k)$ by $\tilde{1}_X := [\tilde{X} \to X] \in A_{\dim(X)}$ and classes of degree $\in \{0, \ldots, \dim(X) - 1\}$.

Let A_* be an oriented Borel-Moore weak homology theory and let X be a smooth projective irreducible k-variety. We let $\tilde{A}_*(X)$ denote the kernel of the degree homomorphism $\deg : A_*(X) \to A_*(F/k)$, F denoting the function field of X. Assume that A_* is generically constant. Then the composition $A_*(k) \to A_*(X) \to A_*(F/k)$ is an isomorphism, so that one gets a direct sum decomposition as $A_*(k)$-module

$$A_*(X) = A_*(k) \oplus \tilde{A}_*(X)$$

even if X has no rational k-point!

Corollary 13.14. *Let k be a field of characteristic zero. Let A_* be an oriented Borel-Moore weak homology theory. Assume A_* is generically constant and that it satisfies the localization property. Then for any irreducible k-variety X, the $A_*(k)$-module $\tilde{A}_*(X)$ is generated by the classes $[\tilde{Z} \to X]_A$.*

This is clear from theorem 13.7.

Given an oriented Borel-Moore weak homology theory A_* and a projective irreducible k-scheme X of dimension $d > 0$, denote by $M(X) \subset A_*(k)$ the ideal generated by classes $[Y]_A \in A_*(k)$ of smooth projective k-schemes Y of dimension $\dim_k(Y) < d$ for which there exists a (projective) morphism $Y \to X$ over k.

Theorem 13.15. *With the previous notations, for any irreducible projective k-scheme X, the ideal $M(X)$ is the image of the push-forward associated to $\pi : X \to \operatorname{Spec} k$:*

$$\pi_* : A_*(X) \to A_*(k)$$

of $\tilde{A}_*(X)$.

This easily follows from corollary 13.14.

13.16. **Rost's degree formulas.** Rost has described a number of what are called "degree formulas" which relate the degree of a map $f : Y \to X$ of smooth projective varieties, the Segre numbers of X and Y, and the degrees of zero-cycles on X. As pointed out in [15], these all follow from a formula in the cobordism of X, called the generalized degree formula.

Given a smooth projective irreducible k-scheme X of dimension $d > 0$, Rost introduces the ideal $M(X) \subset \mathbb{L}_* = \Omega_*(\operatorname{Spec} k)$ generated by classes $[Y] \in \mathbb{L}_*$ of smooth projective k-schemes Y of dimension $\dim_k(Y) < d$ for which there exists a morphism $Y \to X$ over k.

We now recall the statement of theorem 9:

Theorem 13.17. *Let k be a field of characteristic zero. For a morphism $f : Y \to X$ between smooth projective irreducible k-schemes one has*

$$[Y] - \delta(f) \cdot [X] \in M(X)$$

This is an immediate consequence of the theorem 13.7 applied to algebraic cobordism.

Using again the weak factorization theorem, we also deduce :

Theorem 13.18. *Let k be a field of characteristic zero. Let X be a smooth projective k-variety.*

1) The ideal $M(X)$ is a birational invariant of X.

2) Moreover, the class of X modulo $M(X)$:

$$[X] \in \mathbb{L}^*/M(X)$$

is a birational invariant of X as well.

Proof. 1) If $Y \to X$ is a birational morphism, then clearly $M(Y) \subset M(X)$. The converse inclusion $M(X) \subset M(Y)$ is proven by induction on $d = \dim(X)$ as follows. For $d = 0$ it is a triviality ($Y = X$). Assume $d > 0$. By the weak factorization theorem, we may assume that $Y \to X$ is a blow-up with smooth center C in X. By the generalized degree formula a set of generators of $M(X)$ can be obtained by choosing a smooth projective birational resolutions $\tilde{(Z)} \to Z$ for each irreducible closed subscheme Z of X. We know proceed by induction on the dimension of the generators. If $\dim(Z) = 0$ then then Z is a closed point in X with residue field K. As the center C of the blow-up is smooth, the map $Y(K) \to X(K)$ is onto and thus $[Z] \in M(Y)$. Assume now that $\dim(Z) > 0$. Then either Z is contained in C either the fiber product $Z' := \tilde{Z} \times_X Y$ maps birationaly to Z. In the latter case, we resolve Z' as $Z'' \to Z'$. By the generalized degree formula, we see that $[Z'']$ equals $[Z]$ modulo a linear combination on generators of $M(X)$ of smaller dimension with coefficients in the Lazard ring. In the former case, the projection $Z' \to \tilde{Z}$ is the pull back of the projective bundle $\mathbb{P}(\nu) \to C$ (of the normal bundle of the immersion $C \to X$) and is a projective bundle on $\tilde{Z}$. But we know by Corollary 6.11 that $\Omega_*(Z') \to \Omega_*(\tilde{Z})$ is onto so that the class $[\tilde{Z} \to \tilde{Z}]$ lifts to some class $z' \in \Omega_*(Z')$. But now clearly, because the push-forward of z' to $\Omega_*(k)$ lies in the image of the push-forward of $\Omega_*(\mathbb{P}(\nu))$ which is contained in $M(Y)$ because $\mathbb{P}(\nu)$ is a smooth divisor in Y. This achieves the proof of 1).

2) is an immediate consequence of theorem 13.17 with $\delta(f) = 1$. □

We also prove :

Theorem 13.19. *The ideal $M(X) \subset \mathbb{L}_*$ is the image of the push-forward associated to $\pi : X \to \operatorname{Spec} k$:*

$$\pi_* : \Omega_*(X) \to \Omega_*(k)$$

of $\tilde{\Omega}_*(X)$.

This an immediate consequence of corollary 13.14.

Corollary 13.20. *For any irreducible* k*-variety* X *the ideal* $M(X)$ *is stable under the action of Landweber-Novikov operations.*

Proof. It is easy to see that $\tilde{\Omega}_*(X)$ is stable under the Landweber-Novikov operations. Also, by definition 10.24, the Landweber-Novikov operations define a morphism of weak oriented Borel-Moore homology theories:

$$\vartheta^{LN} = \sum_I s_I\, t^I : \Omega_* \to \Omega_*(X)[\mathbf{t}]^{\mathbf{t}}$$

which in particular commutes with push-forward. Noting these facts, the result follows directly from theorem 13.19. □

A. Merkurjev has given a proof of corollary 13.20 over any field in [16].

Remark 13.21. Let k be any field. Let $f : Y \to X$ be a morphism between smooth projective varieties of dimension $d > 0$. Then there always exists a 0-cycle on X with integral coefficients $\Sigma_\alpha n_\alpha \cdot z_\alpha$ (where the z_α are closed points in X) and satisfying

$$s_d(Y) - \deg(f) \cdot s_d(X) = \Sigma_\alpha n_\alpha [\kappa(z_\alpha) : k]. \tag{13.1}$$

In characteristic zero this easily follows from the above considerations but this can proven over any field as follows. One consider the oriented Borel-Moore weak homology theory given by $X \mapsto \mathrm{CH}^\nu_*(X)[\mathbf{t}]$ and constructed in **??**. It is then obvious that the class $[f : Y \to X]$ can be written

$$[f : Y \to X] = \deg(f) \cdot [\mathrm{Id}_X] + \Sigma_\alpha \omega_\alpha \cdot [Z_\alpha \subset X]$$

with $\omega_\alpha \in \mathbb{Z}[\mathbf{t}]$ and $\mathrm{codim}_X(Z) > 0$. Pushing forward this to $\mathrm{CH}^\nu_*(k)[\mathbf{t}] = \mathbb{Z}[\mathbf{t}]$ gives

$$[Y] - \deg(f) \cdot [X] = \Sigma_\alpha \omega_\alpha \cdot (\pi_X)^\nu_*[Z_\alpha]$$

from which the formula (13.1) follows by taking s_d. However one cannot deduce the more subtle corollary 11, because it is not true in general that, if $d = p^n - 1$ for some prime number p and $n > 0$, that $s_d(\omega)$ is divisible by p for $\omega \in \mathbb{Z}[\mathbf{t}]$, even though this holds for the elements in $\mathbb{L}_* \subset \mathbb{Z}[\mathbf{t}]$. Thus the difficulty is that, if one uses only the theory CH^*, one doesn't know that the ω_Z lie in $\mathbb{L}_*$.

14. *Comparison to the Chow groups*

In this section we prove theorem 13, which we restate:

Theorem 14.1. *Let k be a field of characteristic zero. Then the canonical morphism*

$$\Omega_* \otimes_{\mathbb{L}_*} \mathbb{Z} \to \mathrm{CH}_*$$

is an isomorphism.

As the theory $\Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}$ is evidently the universal ordinary weak oriented Borel-Moore homology theory on $\mathcal{V}$, we can reformulate this theorem by saying that the Chow groups functor

$$X \mapsto \mathrm{CH}_*(X)$$

is the universal ordinary weak oriented Borel-Moore homology theory. It is reasonable to conjecture that this statement still holds over any field.

To prove the theorem, we construct an explicit inverse morphism

$$\mathrm{CH}_* \to \Omega_* \otimes_{\mathbb{L}_*} \mathbb{Z}$$

14.2. **The map $Z_*(X) \to \Omega_* \otimes_{\mathbb{L}_*} \mathbb{Z}$.** Given any finite type scheme X over k, we denote by $Z_*(X)$ the free abelian group on the set of closed integral subschemes $Z \subset X$, graded by the dimension of Z.

Lemma 14.3. *Let $\pi : \tilde{Z} \to Z$ be a projective birational morphism with Z and $\tilde{Z}$ smooth over k. Then the class of the projective morphism $\tilde{Z} \to Z$:*

$$[\tilde{Z} \to Z] = \pi_* 1_{\tilde{Z}} \in \Omega_*(Z) \otimes_{\mathbb{L}_*} \mathbb{Z}$$

equals 1_Z.

Proof. This clearly follows from Corollary **??** because the classes ω involved, being of non-negative degree, vanish in $\Omega_*(Z) \otimes_{\mathbb{L}_*} \mathbb{Z}$. □

Let X denote a finite type k-scheme, and let $Z \subset X$ be a closed integral subscheme of X.

Lemma 14.4. *Let $\tilde{Z} \to Z$ be a projective birational morphism with $\tilde{Z}$ smooth over k. Then the class of the projective morphism $\tilde{Z} \to X$:*

$$[\tilde{Z} \to X] \in \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}$$

depends only on Z. We denote this class by $[Z \subset X]$.

Proof. Indeed, let $f_1 : \tilde{Z}_1 \to Z$ and $f_2 : \tilde{Z}_2 \to Z$ be projective birational morphisms with $\tilde{Z}_1$ and $\tilde{Z}_2$ smooth. Choosing a smooth resolution $\tilde{Z}_3$ of the fiber product $\tilde{Z}_1 \times_Z \tilde{Z}_2$ we have that the projective morphisms $\pi_1 : \tilde{Z}_3 \to \tilde{Z}_1$ and $\pi_2 : \tilde{Z}_3 \to \tilde{Z}_2$ are projective birational and we deduce from lemma 14.3 that, in $\Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}$, one has:

$$[\tilde{Z}_1 \to X] = (f_1)_* 1_{\tilde{Z}_1} = (f_1)_*(\pi_1)_* 1_{\tilde{Z}_3} = (f_1 \circ \pi_1)_* 1_{\tilde{Z}_3} = (f_2 \circ \pi_2)_* 1_{\tilde{Z}_3}$$

$$= (f_2)_*(\pi_2)_* 1_{\tilde{Z}_3} = (f_2)_* 1_{\tilde{Z}_2} = [\tilde{Z}_2 \to X]$$

thus establishing the lemma. ☐

Definition 14.5. For a finite type k-scheme X, we denote by

$$\begin{gathered} \phi : Z_*(X) \to \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z} \\ Z \subset X \mapsto [Z \subset X] \end{gathered}$$

the induced group homomorphism.

It is clear that the composition

$$Z_*(X) \to \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z} \to \mathrm{CH}_*(X)$$

is the canonical morphism.

Moreover we observe that $\phi : Z_*(X) \to \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}$ is an epimorphism: this follows easily from theorem 13.7.

Thus to finish the proof of theorem 14.1, it suffices to prove:

Lemma 14.6. *Let X be a finite type k-scheme, let $W \subset X$ be an integral closed subscheme, and let $f \in k(W)^*$ be a rational function on W with divisor*

$$div(f) \in Z_*(X).$$

Then one has

$$\phi(div(f)) = 0 \in \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}$$

Indeed, by definition, $\mathrm{CH}_*(X)$ is the quotient of $Z_*(X)$ by the subgroup generated by the cycles of the form $div(f)$. Thus lemma 14.6 implies that ϕ induces a homomorphism

$$\mathrm{CH}_*(X) \to \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}$$

which is surjective and right inverse to $\Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z} \to \mathrm{CH}_*(X)$; ϕ is thus an isomorphism.

proof of lemma 14.6. Let $W \subset X$ an integral closed subscheme $W \subset X$ and f a non-zero rational function on W. By resolution of singularities, there is a projective birational morphism $\pi : \tilde{W} \to W$ such that $\tilde{W}$ is in $\mathbf{Sm}_k$ and such that the induced rational function $\tilde{f}$ on $\tilde{W}$ defines a morphism $\tilde{f} : \tilde{W} \to \mathbb{P}^1$. We may also assume that $\mathrm{div}\tilde{f}$ is a strict normal crossing divisor on $\tilde{W}$; write $\mathrm{div}\tilde{f} = D_0 - D_\infty$, with D_0 and D_∞ effective and having no common components. In particular, the effective strict normal crossing divisors D_0 and D_∞ have classes $[D_0 \to \tilde{W}]$, $[D_\infty \to \tilde{W}]$ in $\Omega_*(\tilde{W})$. Additionally, from our explicit formula for the class of a normal crossing divisor, and the isomorphism $\Omega_*^{ad}(k) \cong \mathbb{Z}$ (theorem 12.8), it follows that

$$\phi(D_0) = [D_0 \to \tilde{W}]_{\Omega^{ad}}; \quad \phi(D_\infty) = [D_\infty \to \tilde{W}]_{\Omega^{ad}}.$$

Since

$$\mathrm{div} f = \pi_*(D_0 - D_\infty)$$

in $Z_*(W)$, it suffices to show that

$$[D_0 \to \tilde{W}] = [D_0 \to \tilde{W}]$$

in $\Omega_*(\tilde{W})$. Since $\mathcal{O}_{\tilde{W}}(D_0) \cong \tilde{f}^*\mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_{\tilde{W}}(D_\infty)$, we have

$$[D_0 \to \tilde{W}] = \tilde{c}_1(\tilde{f}^*\mathcal{O}_{\mathbb{P}^1}(1))(1_{\tilde{W}}) = [D_\infty \to \tilde{W}]$$

by proposition 5.11. This completes the proof of the lemma. □

Remark 14.7. Let $X \in \mathbf{Sch}_k$. We denote by $I_*(X) \subset \Omega_*(X)$ the kernel of $\Omega_*(X) \to \mathrm{CH}_*(X)$. Then theorem 10 implies that

$$\mathbb{L}_{\geq 1} \cdot \Omega_*(X) = I_*(X)$$

14.8. **A filtration of algebraic cobordism.** Let X be a finite type k-scheme and let $n \geq 0$ be an integer. We define the sub-graded group

$$F^{(n)}\Omega_*(X)$$

to be the one generated by classes $[f : Y \to X]$ with Y smooth, irreducible and $\dim(Y) - \dim f(Y) \geq n$. We observe that this is a sub-$\Omega_*(k)$-module of $\Omega_*(X)$. For $X = \operatorname{Spec} k$ one has $F^{(n)}\Omega_*(k) = \Omega_{*\geq n}(k)$, the subgroup of elements of degree $\geq n$. In characteristic zero, we moreover know that $\mathbb{L}_* \cong \Omega_*(k)$. Observe that using the result of the previous section we get $F^{(1)}\Omega_*(X) = I_*(X)$.

Theorem 14.9. *Assume* $\mathrm{char}(k) = 0$. *Let X be a finite type k-scheme and let $n \geq 0$ be an integer. Then one has*

$$F^{(n)}(\Omega_*(X)) = L_{\geq n}.\Omega_*(X)$$

Proof. This follows easily from theorem 13.7. □

The associated bigraded abelian group $(F^{(n)}\Omega_*(k)/F^{(n+1)}\Omega_*(k))_n$ is denoted by $Gr_*\Omega_*(X)$. For $X = \operatorname{Spec} k$ it is canonically isomorphic to $\Omega_*(k)$ via the obvious isomorphism: $F^{(n)}\Omega_*(k)/F^{(n+1)}\Omega_*(k) = \Omega_n(k)$.

Corollary 14.10. *For any finite type k-scheme X the canonical homomorphism (of bigraded abelian groups)*

$$\Phi_X : \mathbb{L}_* \otimes \mathrm{CH}_*(X) \to Gr_*(\Omega_*(X))$$

is an epimorphism of $\mathbb{L}_*$-modules.

Remark 14.11. Using theorem 10.27 one can show in fact that $\Phi_X \otimes \mathbb{Q}$ is an isomorphism.

14.12. **Some computations.** In this paragraph we assume that $\mathrm{char}(k) = 0$.

The following lemma is an immediate consequence of Corollary 14.10.

Lemma 14.13. *Let X be a finite type k-scheme. The group $F^{(1)}\Omega_0(X)$ vanishes so that the homomorphism*

$$\Phi_X : \mathbb{L}_* \otimes \mathrm{CH}_*(X) \to Gr_*(\Omega_*(X))$$

induces in degree 0 an epimorphism :

$$\Phi_X : \mathbb{L}_0 \otimes \mathrm{CH}_0(X) \to Gr_0(\Omega_0(X)) = \Omega_0(X)$$

which is left inverse to the canonical morphism

$$\Omega_0(X) \to \mathrm{CH}_0(X)$$

which is thus always an isomorphism.

Now we are going to study $\Omega_1(X)$. By the theorem 14.9 we have an exact sequence of abelian groups

$$L_1 \otimes \Omega_0(X) \to \Omega_1(X) \to \mathrm{CH}_1(X) \to 0$$

We recall that L_1 is a free abelian group on the class $[\mathbb{P}^1]$ so that the left hand side is isomorphic to $\mathbb{Z} \otimes \Omega_0(X) = \Omega_0(X)$.

Lemma 14.14. *Let X be a smooth k-scheme. Then the composition*

$$\Omega_0(X) = \mathbb{Z} \otimes \Omega_0(X) \to \Omega_1(X) \to K_0(X)$$

is the canonical homomorphism $\mathrm{CH}_0(X) = \Omega_0(X) \to K_0(X)$ which maps a 0-cycle to the class of its associated $\mathcal{O}_X$-module..

This is easy to prove and left to the reader.

For a finite type k-scheme X, we have the reduced K_0 of X, $\tilde{K}_0(X)$, defined as the kernel of the the rank map $K_0(X) \to H^0(X_{\mathrm{Zar}}, \mathbb{Z})$.

Corollary 14.15. *1) Let X be a smooth k-scheme of dimension 1. Then the homomorphism*

$$\Omega_1(X) \to K_0(X)$$

is an isomorphism.

2) Let X be a smooth k-scheme of dimension 2. Then the homomorphism

$$\Omega_1(X) \to K_0(X)$$

is an monomorphism which identifies $\Omega_1(X)$ with $\tilde{K}_0(X)$.

Proof. 1) By lemma 14.14, we know that the homomorphism $\Omega_1(X) \to K_0(X)$ induces a map from the short exact sequence

$$0 \to \mathrm{CH}_0(X) \to \Omega_1(X) \to \mathrm{CH}_1(X) \to 0$$

to the short exact sequence

$$0 \to \mathrm{CH}_0(X) \to K_0(X) \to \mathrm{CH}_1(X) \to 0$$

thus is an isomorphism.

2) For a smooth surface, it is well known that the group $\tilde{K}_0(X)$ is inside a short exact sequence

$$0 \to \mathrm{CH}_0(X) \to \tilde{K}_0(X) \to \mathrm{CH}_1(X) \to 0$$

(where the morphism $\tilde{K}_0(X) \to \mathrm{CH}_1(X)$ is induced by assigning to a vector bundle E of rank n the isomorphism class of its maximal exterior power $\Lambda^n(E) \in \mathrm{CH}_1(X) = \mathrm{Pic}(X)$. One concludes that $\Omega_1(X) \to \tilde{K}_0(X)$ is an isomorphism using the morphism of the short exact sequence $0 \to \mathrm{CH}_0(X) \to \Omega_1(X) \to \mathrm{CH}_1(X) \to 0$ to the previous one (using lemma 14.14). □

Appendix A. **Resolution of singularities**

In this appendix, we collect the various results we need from the theory of resolution of singularities, for lack of a good reference in the literature. We are indebted to D. Cutkosky for supplying the arguments given below.

Theorem A.1. *Let X be a smooth quasi-projective variety over a field k of characteristic zero, let D be an effective strict normal crossing divisor on X, and let S be a reduced and irreducible codimension one subscheme of X. Let V be an open subset of X containing each generic point of $|D+S|$ such that $V\cap(D+S)$ is a strict normal crossing divisor on V. Then there is a sequence of blow-ups of smooth centers lying over $X\setminus V$,*

$$X'=X_r\to\ldots\to X_0=X$$

such that, letting E_j be the exceptional divisor of $X_j\to X$, and D_j, S_j the proper transforms of D, S to X_j, E_j+D_j is a strict normal crossing divisor for all j, and $E_r+D_r+S_r$ is a strict normal crossing divisor on X'.

Proof. This is a special case of [8, Theorem $\mathrm{I}_2^{N,n}$, pg. 170], where, in the notation of that result, we take $N=\dim_k X$, $n=N-1$, and $(\mathfrak{R}_I^{N,N-1},U)$ is the resolution datum $((|D|;X;S),V)$. □

Corollary A.2. *Let X be a quasi-projective variety over a field k of characteristic zero, and let D and D' be effective divisors on X, with D' a strict normal crossing divisor on X. Let U be a smooth open subscheme of X, containing each generic point of $|D+D'|$, such that $(D+D')\cap U$ is a strict normal crossing divisor on U. Then there is a sequence of blow-ups of smooth centers lying over $X\setminus U$*

$$X_r\to\ldots\to X_0=X$$

such that, letting E be the exceptional divisor of $\mu:X_r\to X$ and D_r, D'_r the proper transform of D to X_r, $E+D_r+D'_r$, is a strict normal crossing divisor on X_r. If X is smooth, we may take U such that $X\setminus U\subset|D+D'|$. In this case, E is supported in $\mu^*(D+D')$, and we may take the sequence of blow-ups so that, letting E_j be the exceptional divisor of $X_j\to X$ and D'_j the proper transform of D' to X_j, $E_j+D'_j$ is a strict normal crossing divisor on X_j for all j.

Proof. We note that $|D'|$ is contained in the smooth locus of X. By resolving the singularities of X via a sequence of blow-ups of smooth centers lying over X_{sing} [8, Main Theorem I*, pg.132], and taking the proper transforms of D and D', we reduce to the case of a smooth X.

We may assume that D is reduced; write $D = \sum_{i=1}^m D^i$ with the D^i irreducible. We proceed by induction on m.

Let $D^* = \sum_{i=1}^{m-1} D^i$. Assuming the result for $m-1$, we have a sequence of blow-ups as above such that $E_r + D_r^* + D_r'$ is a strict normal crossing divisor, and $E_j + D_j'$ is a strict normal crossing divisor for all j. Replacing X with X_r, D with the proper transform of D^m, and D' with $E_r + D_r^* + D_r'$, we reduce to the case $m = 1$, which is theorem A.1. $\square$

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