ALGEBRAIC COBORDISM II

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Abstract. We complete and extend the construction of algebraic cobordism from [4]. Let k be a field admitting resolution of singularities, let $\mathbf{Sch}_k$ denote the category of finite type schemes over a field k, and let $\mathbf{Sm}_k$ be the full subcategory of smooth quasi-projective k-schemes. For an l.c.i. morphism $f : X \to Y$ of finite type k-schemes, we define a functorial pull- back morphism f^*. With these pull-back maps, Ω_* becomes what we call a *oriented Borel-Moore homology theory* on $\mathbf{Sch}_k$. Restricting Ω_* to smooth quasi-projective k-schemes, this defines Ω^* as an oriented cohomology theory on $\mathbf{Sm}_k$. Relying on the results of [4], we show that Ω_* is the universal oriented Borel-Moore homology theory on $\mathbf{Sch}_k$ and the universal oriented cohomology theory on $\mathbf{Sm}_k$. This completes the proofs of some of the main results of [4]. In addition, we extend the results of [4] concerning Rost's degree formulas from smooth k-schemes to local-complete-intersection k-schemes (for k of characteristic zero).

Contents

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0. *Introduction*

In the joint work [4] with F. Morel, we have described the theory of *algebraic cobordism*, $X \mapsto \Omega_*(X)$. This paper completes the constructions and proofs of some of the main results of [4].

What remains to be done? Let k be a field, and let $\mathbf{Sch}_k$ be the category of separated k-schemes of finite type. In [4], the theory Ω_* is constructed as an *oriented Borel-Moore weak homology theory on* $\mathbf{Sch}_k$ [4, Definition 10.7]. Roughly speaking, such a theory A_* is the assignment of a graded group $A_*(X)$ for each finite type k-scheme, with functorial push-forward maps $f_* : A_*(X) \to A_*(Y)$ for each projective morphism $f : X \to Y$, functorial pull-back maps $f^* : A_*(X) \to A_{*+d}(Y)$ for each smooth quasi-projective morphism $f : Y \to X$ of relative dimension d, external products $\times : A_*(X) \otimes A_*(Y) \to A_*(X \times Y)$, and, for each line bundle $L \to X$ on $X \in \mathbf{Sch}_k$, a degree -1 endomorphism $\tilde{c}_1(L) : A_*(X) \to A_{*-1}(X)$, all satisfying a number of properties and compatibilities.

There is a related notion, that of an *oriented cohomology theory on* $\mathbf{Sm}_k$. Here, we have a contravariant functor A^* from $\mathbf{Sm}_k$ to the category of graded rings, together with push-forward maps $f_* : A^*(X) \to A^{*-d}(Y)$ for each projective morphism $f : X \to Y$ in $\mathbf{Sm}_k$, of relative dimension d, again with certain properties and compatibilities. As we shall see below, the essential difference between these two notions is that in a cohomology theory, one has pull-back maps for an *arbitrary* morphism in $\mathbf{Sm}_k$, whereas in a weak homology theory, one only has pull-backs for smooth quasi-projective morphisms.

In any case, some of the main results of [4] are stated in the context of oriented cohomology theories, but the proofs in [4] are given only for the related oriented Borel-Moore weak homology theories. The main purpose of this paper is to finish the task of showing that the weak homology theories considered in [4] have the structure of oriented cohomology theories, and that the results proved in [4] for the weak homology theories extend to the oriented cohomology theories. For the convenience of the reader, we recall the statements here:

Recall from [4, Introduction, Lemma 1] that for each oriented cohomology theory A^* on $\mathbf{Sm}_k$, there is a formal group law $F_A(u, v)$ over $A^*(k)$. We call A^* *ordinary* if $F_A(u, v) = u + v$. Call A^* *multiplicative* if $F_A(u, v) = u + v - b \cdot uv$ for some $b \in A^*(k)$; if in addition b is a unit in $A^*(k)$, call A^* *periodic.*

Theorem 1 ([4, Introduction, Theorem 1])**.** *Let k be a field of characteristic zero, and let A^* be an oriented cohomology theory on $\mathbf{Sm}_k$. If A^* is ordinary then there exists one and only one morphism of oriented* *cohomology theories*

$$\vartheta_A^{\mathrm{CH}} : \mathrm{CH}^* \to A^*.$$

Theorem 2 ([4, Introduction, Theorem 3])**.** *Let k be a field of characteristic zero. Then there exists a universal oriented cohomology theory on $\mathbf{Sm}_k$, denoted by*

$$X \mapsto \Omega^*(X),$$

which we call algebraic cobordism*. Thus, given an oriented cohomology theory A^* on $\mathbf{Sm}_k$, there is a unique morphism*

$$\vartheta : \Omega^* \to A^*$$

of oriented cohomology theories.

As mentioned above, our main task will be the extension of the pull-back morphisms $f^* : \Omega^*(X) \to \Omega^*(Y)$ from smooth quasi-projective morphisms to arbitary morphisms $f : Y \to X$ in $\mathbf{Sm}_k$ (of relative dimension d). In fact, we will work in a more general context, giving pull-back morphisms $f^* : \Omega^*(X) \to \Omega^*(Y)$ for each *local complete intersection morphism* $f : Y \to X$ in $\mathbf{Sch}_k$. The proper context for this construction is that of an *oriented Borel-Moore homology theory* on $\mathbf{Sch}_k$. As we shall see in §1, this notion simultaneously extends both that of an oriented Borel-Moore weak homology theory on $\mathbf{Sch}_k$ and that of an oriented cohomology theory on $\mathbf{Sm}_k$.

The paper is organized as follows: We begin with the definition of an oriented Borel-Moore homology theory in §1. We also relate this notion to that of an oriented Borel-Moore weak homology theory on $\mathbf{Sch}_k$, and an oriented cohomology theory on $\mathbf{Sm}_k$. We devote most of the rest of the paper to the construction of pull-back maps in Ω_* for l.c.i. morphisms. We closely follow the strategy laid out by Fulton in [1], where similar pull-back maps are defined for the Chow groups. In §2, we recall from [1] the notion of a *pseudo-divisor*, and we define a modified version of algebraic cobordism for a finite type k-scheme X with a pseudo-divisor D, $\Omega_*(X)_D$. This group comes equipped with a homomorphism $\Omega_*(X)_D \to \Omega_*(X)$.

In §3, we define the operation of "intersection with the pseudo-divisor D",

$$D(-) : \Omega_*(X)_D \to \Omega_*(|D|),$$

where $|D|$ is the support of D. Again, this parallels the construction given in [1]. We actually need to consider a version of this construction involving two pseudo-divisors D, D', in order to prove later the crucial commutativity of intersection. In §4, we establish some of the basic properties of the operation $D(-)$. In §5, we prove the crucial "moving lemma", showing that the map $\Omega_*(X)_D \to \Omega_*(X)$ is an isomorphism. This allows us to use the operation $D(-)$ to define the pull-back

$$i_D^* : \Omega^*(X) \to \Omega_*(|D|)$$

if D is a pseudo-divisor on X with support $|D|$. We complete the discussion of pull-back maps for l.c.i. morphisms in §6, where we use the method of deformation to the normal bundle from [1] to extend the pull-back map i_D^* first to the case of a regular imbedding $i : Z \to X$, and then finally to an arbitrary l.c.i. morphism. We prove as well all the important properties of the l.c.i. pull-back in this section.

We prove the results outlined at the beginning of this introduction in §7, where we show that the l.c.i. pull-backs make Ω_* into the universal Borel-Moore homology theory on $\mathbf{Sch}_k$. We also fill in the missing pieces in the proofs of Theorems 1 and 3 of [4]. In §8, we extend some of the degree formulas of [4] from smooth varieties to l.c.i. schemes.

We let $(F_{\mathbb{L}}, \mathbb{L})$ denote the universal rank one commutative formal group. Here $F_{\mathbb{L}}(u, v) = u + v + \sum_{i,j \geq 1} a_{ij} u^i v^j$, and the *Lazard ring* $\mathbb{L}$ is generated by the coefficients a_{ij} as an algebra over $\mathbb{Z}$. We grade $\mathbb{L}$ by giving a_{ij} degree $i + j - 1$, which defines the graded ring $\mathbb{L}_*$; giving a_{ij} degree $1 - i - j$ defines the graded ring $\mathbb{L}^*$ with $\mathbb{L}^n = \mathbb{L}_{-n}$.

Although Fabien Morel is not listed as an author of this work, I would like to gratefully acknowledge the important influence he has had on this paper. In particular, the notion of an oriented Borel-Moore homology theory on $\mathbf{Sch}_k$, and the suggestion that Ω_* should have this structure, are entirely due to him.

1. *Oriented Borel-Moore homology theories*

1.1. **Admissible subcategories and l.c.i. morphisms.** Let S be a noetherian separated scheme. We let $\mathbf{Sch}_S$ denote the category of finite type separated S-schemes and $\mathbf{Sm}_S$ the full subcategory of smooth quasi-projective S-schemes.

Recall that a closed imbedding $i : Z \to X$ is called a *regular imbedding* if the ideal sheaf of Z in X is locally generated by a regular sequence. Also, a local complete intersection morphism in $\mathbf{Sch}_S$, an *l.c.i. morphism* for short, is a morphism $f : X \to Y$ of flat finite type S-schemes which admits a factorization as $f = q \circ i$, where $i : X \to P$ is a regular imbedding and $q : P \to Y$ is a smooth, quasi-projective morphism.

Let $\mathcal{V} \subset \mathbf{Sch}_S$ be a full subcategory containing S and $\emptyset$ which satisfies:

(1.1)

1. If $Y \to X$ is a smooth quasi-projective morphism with $X \in \mathcal{V}$, then $Y \in \mathcal{V}$.
2. If $X \to S$ and $Y \to S$ are in $\mathcal{V}$, then so is the fiber product $X \times_S Y \to S$.
3. If X and Y are in $\mathcal{V}$, so is $X \coprod Y$.

Following the usage in [4], we call such a $\mathcal{V}$ an *admissible* subcategory of $\mathbf{Sch}_S$. We note that each admissible subcategory of $\mathbf{Sch}_S$ contains $\mathbf{Sm}_S$. We let $\mathcal{V}'$ denote the subcategory of $\mathcal{V}$ consisting of only projective morphisms.

We sometimes require additional objects in $\mathcal{V}$, as described by the modified version of (1):

4. If $Y \to X$ is an l.c.i. morphism with $X \in \mathcal{V}$, then $Y \in \mathcal{V}$.

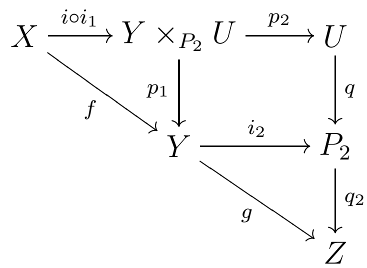
We will refer to a full subcategory $\mathcal{V}$ of $\mathbf{Sch}_S$ satisfying (1)-(4) as an *l.c.i.-closed* admissible subcategory of $\mathbf{Sch}_S$.

Remarks 1.1. 1) Our notion of an l.c.i. morphism $f : X \to Y$ may differ somewhat from other texts, as we require that the smooth morphism in the factorization be quasi-projective, and that X and Y be flat over S.

2) For the basic properties of regular imbeddings and l.c.i. morphisms, we refer the reader to [1, Appendix B.7]. For example, if $f : X \to Y$ is an l.c.i. morphism, and if we have any factorization of f as $q \circ i$, where $i : X \to P$ is a closed imbedding and $q : P \to Y$ is smooth, then i is automatically a regular imbedding. In particular, if $f : X \to Y$ is a quasi-projective morphism of flat finite type S-schemes, then the condition that f be an l.c.i. morphism is local on X.

3) If $f : X \to Y$ and $g : Y \to Z$ are l.c.i. morphisms, then so is $g \circ f : X \to Z$. Indeed, factor f and g as $f = q_1 i_1$, $g = q_2 i_2$, with $i_1 : X \to P_1$, $i_2 : Y \to P_2$ regular imbeddings, and $q_1 : P_1 \to Y$, $q_2 : P_2 \to Z$ smooth and quasi-projective. Since q_1 is quasi-projective, we can factor q_1 as a closed immersion $i' : P_1 \to Y \times_S \mathbb{P}^N$ followed by the smooth projection $p_1 : Y \times_S \mathbb{P}^N \to Y$. Since $Y \to P_2$ is quasi-projective, there is an open subscheme $U \subset P_2 \times_S \mathbb{P}^N$ containing $(i_2 \times \mathrm{id})(i(P_1))$ such that $P_1 \to Y \times_{P_2} U$ is a closed imbedding. By (2), we may replace

P_1 with $Y \times_{P_2} U$, giving the commutative diagram



with $i \circ i_1$ a regular imbedding, q smooth and quasi-projective, and the square cartesian. This gives the desired factorization $g \circ f = (q_2 q) \circ (p_2 i i_1)$.

Similarly, if $f_i : X_i \to Y_i$, $i = 1, 2$ are l.c.i. morphisms, then the product $f_1 \times f_2 : X_1 \times_S X_2 \to Y_1 \times_S Y_2$ is also an l.c.i. morphism. This follows from the fact that a flat pull-back of a regular imbedding is a regular imbedding.

4) We call a finite type S-scheme $p : X \to S$ an *l.c.i. S-scheme* if p is an l.c.i. morphism, and let $\mathbf{Lci}_S$ denote the full subcategory of $\mathbf{Sch}_S$ with objects the l.c.i. S-schemes. From our above remarks, $\mathbf{Lci}_S$ satisfies the conditions (1.1)(1)-(4), i.e., $\mathbf{Lci}_S$ is an l.c.i.-closed admissible subcategory of $\mathbf{Sch}_S$. Clearly every l.c.i.-closed admissible subcategory of $\mathbf{Sch}_S$ contains $\mathbf{Lci}_S$.

If $f : X \to Z$ and $g : Y \to Z$ are morphisms in $\mathcal{V}$, we say that f and g are *transverse in* $\mathcal{V}$ if

1. f and g are Tor-independent, that is
$$\mathrm{Tor}_i^{\mathcal{O}_Z}(\mathcal{O}_X, \mathcal{O}_Y) = 0; \quad i > 0.$$
2. The fiber product $X \times_Z Y$ is in $\mathcal{V}$.

For example, if $\mathcal{V} = \mathbf{Sm}_k$, k a field, we recover the usual notion of transverse morphisms. For another example, take $\mathcal{V} = \mathbf{Sch}_k$, k a field. Let $f : X \to Z$ be an l.c.i. morphism and let $g : Y \to Z$ be an arbitrary morphism. If f and g are transverse in $\mathbf{Sch}_S$, then the projection $X \times_Z Y \to Y$ is an l.c.i. morphism. As a third example, let f be an l.c.i. morphism in $\mathbf{Lci}_S$, and g an arbitrary morphism in $\mathbf{Lci}_S$. If f and g are Tor-independent, then $X \times_Z Y$ is in $\mathbf{Lci}_S$, hence f and g are transverse in $\mathbf{Lci}_S$.

1.2. **Oriented Borel-Moore homology.** We introduce the notion of an oriented Borel-Moore homology theory.

Given a rank n locally free sheaf $\mathcal{E}$ on $X \in \mathcal{V}$, let $q : \mathbb{P}(\mathcal{E}) \to X$ denote the projective bundle of rank one quotients of $\mathcal{E}$, with tautological quotient invertible sheaf $q^*\mathcal{E} \to \mathcal{O}(1)_{\mathcal{E}}$. We let $O(1)_{\mathcal{E}}$ denote the line bundle on $\mathbb{P}(\mathcal{E})$ with sheaf of sections $\mathcal{O}(1)_{\mathcal{E}}$.

We call a functor $F : \mathcal{V}' \to \mathbf{Ab}_*$ *additive* if $F(\emptyset) = 0$ and the canonical map $F(X) \oplus F(Y) \to F(X \coprod Y)$ is an isomorphism for all X, Y in $\mathcal{V}$.

Definition 1.2. Let $\mathcal{V}$ be an admissible subcategory of $\mathbf{Sch}_S$. An *oriented Borel-Moore homology theory* A on $\mathcal{V}$ is given by

(D1). An additive functor

$$A_* : \mathcal{V}' \to \mathbf{Ab}_*\,,\ X \mapsto A_*(X).$$

(D2). For each l.c.i. morphism $f : Y \to X$ in $\mathcal{V}$ of relative dimension d, a homomorphism of graded groups

$$f^* : A_*(X) \to A_{*+d}(Y).$$

(D3). An element $1 \in A_0(S)$ and, for each pair (X, Y) of objects in $\mathcal{V}$, a bilinear graded pairing:

$$\begin{aligned} A_*(X) \otimes A_*(Y) &\to A_*(X \times_S Y) \\ u \otimes v &\mapsto u \times v, \end{aligned}$$

called the external product, which is associative, commutative and admits 1 as unit element.

These satisfy

(BM1). One has $Id_X^* = Id_{A_*(X)}$ for any $X \in \mathcal{V}$. Moreover, given composable l.c.i. morphisms $f : Y \to X$ and $g : Z \to Y$ in $\mathcal{V}$ of pure relative dimension, one has $(f \circ g)^* = g^* \circ f^*$.

(BM2). Let $f : X \to Z$ and $g : Y \to Z$ be morphisms in $\mathcal{V}$. Suppose that f and g are transverse in $\mathcal{V}$, that f is projective and that g is an l.c.i. morphism, giving the cartesian square

$$\begin{array}{ccc} W & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z. \end{array}$$

Note that f' is projective and g' is an l.c.i. morphism. Then $g^* f_* = f'_* g'^*$.

(BM3). Let $f : X' \to X$ in $\mathcal{V}$ and $g : Y' \to Y$ be morphisms in $\mathcal{V}$. If f and g are projective, then for $u' \in A_*(X')$ and $v' \in A_*(Y')$ one has

$$(f \times g)_*(u' \times v') = f_*(u') \times g_*(v').$$

If f and g are l.c.i. morphisms, then for $u \in A_*(X)$ and $v \in A_*(Y)$ one has

$$(f \times g)^*(u \times v) = f^*(u) \times g^*(u')$$

(PB). For $L \to Y$ a line bundle on $Y \in \mathcal{V}$ with zero-section $s : Y \to L$, define the operator

$$\tilde{c}_1(L) : A_*(Y) \to A_{*-1}(Y)$$

by $\tilde{c}_1(L)(\eta) = s^*(s_*(\eta))$. Let $\mathcal{E}$ be a rank $n+1$ locally free coherent sheaf on $X \in \mathcal{V}$, with projective bundle $q : \mathbb{P}(\mathcal{E}) \to X$. For $i = 0, \ldots, n$, let

$$\xi^{(i)} : A_{*+i-n}(X) \to A_*(\mathbb{P}(\mathcal{E}))$$

be the composition of $q^* : A_{*+i-n}(X) \to A_{*+i}(\mathbb{P}(\mathcal{E}))$ followed by $\tilde{c}_1(O(1)_{\mathcal{E}})^i : A_{*+i}(\mathbb{P}(\mathcal{E})) \to A_*(\mathbb{P}(\mathcal{E}))$. Then the homomorphism

$$\Sigma_{i=0}^{n-1}\xi^{(i)} : \oplus_{i=0}^{n} A_{*+i-n}(X) \to A_*(\mathbb{P}(\mathcal{E}))$$

is an isomorphism.

(H). Let $E \to X$ be a vector bundle of rank r over $X \in \mathbf{Sch}_k$, and let $p : V \to X$ be an E-torsor. Then $p^* : A_*(X) \to A_{*+r}(V)$ is an isomorphism.

(CD). For integers $r, N > 0$, let $W = \mathbb{P}^N \times_S \ldots \times_S \mathbb{P}^N$ (r factors), and let $p_i : W \to \mathbb{P}^N$ be the ith projection. Let $X_0, \ldots, X_N$ be the standard homogeneous coordinations on $\mathbb{P}^N$, let $n_1, \ldots, n_r$ be non-negative integers, and let $i : E \to W$ be the subscheme defined by $\prod_{i=1}^{r} p_i^*(X_N)^{n_i} = 0$. Suppose that E is in $\mathcal{V}$. Then $i_* : A_*(E) \to A_*(W)$ is injective.

Remark 1.3. The axiom (CD) may appear at first glance to be somewhat unnatural, but it is implied by a cellular decomposition property enjoyed by many examples of oriented Borel-Moore homology theories, namely:

(CD′). Let E be a scheme in $\mathcal{V}$. Suppose that the reduced subscheme E_{red} has a filtration by reduced closed subschemes

$$\emptyset = E_0 \subset E_1 \subset \ldots \subset E_N = E_{\mathrm{red}}$$

such that
(a) $E_i \setminus E_{i-1}$ is a disjoint union of its irreducible components.

(b) Each irreducible component E^0_{ij} of $E_i \setminus E_{i-1}$ is an affine space $\mathbb{A}^{N_{ij}}_S$.

(c) Let E_{ij} be the closure of E^0_{ij} in E_i. Then E_{ij} is smooth over S.

Then the evident map $\coprod_{i,j} E_{ij} \to E$ induces a surjection

$$\oplus_{i,j} A_*(E_{ij}) \to A_*(E).$$

We will verify in §1.7 that (CD′) implies (CD). Since we will only need this special consequence of the cellular decomposition property, we list only the property (CD) as an axiom, even though it may seem less natural than the axiom (CD′).

We will be mostly interested in the sequel in the case S is the spectrum of a field and $\mathcal{V}$ is the category of finite type k-schemes. However, all the problems we have considered in [4] have interesting generalizations for other choices of $\mathcal{V}$. One can easily develop a general theory of Chern classes of vector bundles as in [4]. Taking for instance $\mathcal{V}$ to be the category of finite type S-schemes which are regular, one has the oriented Borel-Moore homology theory

$$X \mapsto K^0(X)[\beta, \beta^{-1}].$$

Then, for a given multiplicative and periodic theory A^*, the assignment $E \mapsto \mathrm{rank}(E) - c^A_1(E^\vee)$ gives a natural transformation $ch_A : K^0[\beta, \beta^{-1}] \to A^*$, which makes $K^0[\beta, \beta^{-1}]$ the universal multiplicative and periodic oriented Borel-Moore homology theory. This raises the question of whether the analogues of Theorems 1 and 3 of [4, Introduction] are still valid in the general situation. When $S = \mathrm{Spec}\,(k)$ for a field k which is not perfect, for example, then one can take for $\mathcal{V}$ one of the three following categories: that of all finite type k-schemes, that of all regular finite type k-schemes, that of all smooth finite type k-schemes, and we do not know if the analogues of Theorems 1 and 3 remain true in these cases.

Another interesting example is the category $\mathbf{Lci}_S$. This may be viewed as the largest full subcategory of $\mathbf{Sch}_S$ for which each object $\pi_X : X \to S$ has an obvious *fundamental class* $1_X := \pi^*_X(1)$[1]. As we shall see in §8, from the point of view of algebraic cobordism an l.c.i. S-scheme is essentially the same as a smooth S-scheme. For instance, one has a virtual normal bundle for an l.c.i. S-scheme, and a reasonable theory of Chern "numbers" for those which are projective over S.

[1]It is not clear if $\mathbf{Lci}_S$ is in fact the largest full subcategory of $\mathbf{Sch}_S$ for which one can define natural fundamental classes, but one can show by examples that it is impossible to define natural fundamental classes on all of $\mathbf{Sch}_S$, even for $S = \mathrm{Spec}\,k$, k a field.

1.3. **Oriented cohomology.** Let $\mathcal{V}$ be an admissible subcategory of $\mathbf{Sch}_S$. The definition of an *oriented cohomology theory* on $\mathbf{Sm}_k$, k a field, is given in [4, Introduction]. Changing $\mathbf{Sm}_k$ to $\mathcal{V}$ throughout, one extends this to the notion of oriented cohomology theory on $\mathcal{V}$. In general, an oriented cohomology theory on $\mathcal{V}$ is not equivalent to an oriented Borel-Moore homology theory on $\mathcal{V}$, but for $\mathcal{V} = \mathbf{Sm}_S$, the two notions coincide.

Indeed, let A_* be an oriented Borel-Moore homology theory on $\mathbf{Sm}_S$, and let A^* be the theory A with cohomological grading: $A^n(X) = A_{n-\dim_S X}(X)$ for $X \to S$ smooth and of pure dimension over S; in general, one takes direct sums over the connected components of X. The cup product on $A^*(X)$ is defined by

$$\alpha \cup \beta := \delta_X^*(\alpha \times \beta),$$

where $\delta_X : X \to X \times X$ is the diagonal and $\times : A^*(X) \otimes A^*(X) \to A^*(X \times X)$ is the external product. Conversely, if A^* is a cohomology theory on $\mathbf{Sm}_S$, let A_* be A with homological grading, $A_n(X) = A^{\dim_S X - n}(X)$. Define the external product of $\alpha \in A_*(X)$, $\beta \in A_*(Y)$ by $\alpha \times \beta := p_1^*(\alpha) \cup p_2^*(\beta) \in A^*(X \times_S Y)$.

Proposition 1.4. *The operations $A_* \mapsto A^*$, $A^* \mapsto A_*$ give equivalences of the category of oriented Borel-Moore homology theories on $\mathbf{Sm}_S$ with the category of oriented cohomology theories on $\mathbf{Sm}_S$.*

Proof. Suppose we are given an oriented Borel-Moore homology theory A_* on $\mathbf{Sm}_S$. Since the external product is unital, commutative and associative, the same is true for the cup product, where the unit is $1_X := p_X^*(1)$, $p_X : X \to \operatorname{Spec} k$ being the structure morphism. Noting that each morphism $X \to Y$ in $\mathbf{Sm}_S$ is an l.c.i. morphism, axiom (BM1) defines A^* as a functor from $\mathbf{Sm}_k$ to graded groups, and axiom (BM3) shows that the cup product is functorial. Thus, A^* is a commutative ring valued functor on $\mathbf{Sm}_S$.

To show that A^* is an oriented cohomology theory on $\mathbf{Sm}_S$, we need to show that

1. If $f : Y \to X$ is a projective morphism in $\mathbf{Sm}_S$ of relative dimension d, then the push-forward $f_* : A^*(Y) \to A^{*-d}(X)$ is $A^*(X)$-linear (the projection formula).
2. For a line bundle $p : L \to X$ on $X \in \mathbf{Sm}_S$, the Chern class endomorphism $\tilde{c}_1(L) : A^*(X) \to A^{*+1}(X)$ is given by cup product with $c_1(L)$.

For (1), we have the commutative diagram, in which the square is cartesian:

$$
\begin{array}{ccc}
 & Y \times_S Y & \\
 {}^{\delta_Y}\nearrow & & \searrow^{\mathrm{id}\times f} \\
Y & \xrightarrow[(\mathrm{id}, f)]{} & Y \times_S X \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\times \mathrm{id}} \\
X & \xrightarrow[\delta_X]{} & X \times_S X.
\end{array}
$$

In addition, the maps $f \times \mathrm{id}$ and δ_X are transverse in $\mathbf{Sm}_S$. Using axioms (BM1), (BM2) and (BM3), we have

$$
\begin{aligned}
f_*(\alpha \cup f^*(\beta)) &= f_*(\delta_Y^*(\alpha \times f^*(\beta))) \\
&= f_*(\delta_Y^* \circ (\mathrm{id} \times f)^*(\alpha \times \beta)) \\
&= f_* \circ (\mathrm{id}, f)^*(\alpha \times \beta) \\
&= \delta_X^* \circ (f \times \mathrm{id})_*(\alpha \times \beta) \\
&= \delta_X^*(f_*(\alpha) \times \beta) \\
&= f_*(\alpha) \cup \beta.
\end{aligned}
$$

For (2), let $s : X \to L$ be the zero section. Then $c_1(L) = s^*(s_*(1_X))$ by definition, while for $\eta \in A^*(X)$,

$$
\begin{aligned}
\tilde{c}_1(L)(\eta) &:= s^*(s_*(\eta)) \\
&= s^*(s_*(1_X \cup s^*(p^*\eta))) \\
&= s^*(s_*(1_X) \cup p^*\eta) \\
&= s^*(s_*(1_X)) \cup \eta \\
&= c_1(L) \cup \eta.
\end{aligned}
$$

Similarly, given an oriented cohomology theory A^* on $\mathbf{Sm}_S$, the functor A_* on the projective morphisms of $\mathbf{Sm}_S$ evidently satisfies all the axioms of an oriented Borel-Moore homology theory, with the possible exception of (BM2) for push-forward and the axiom (CD). For the axiom (BM2), it suffices to show that $(f \times \mathrm{id})_*(\alpha \times \beta) = f_*(\alpha) \times \beta$ for a projective morphism $f : X' \to X$ in $\mathbf{Sm}_S$; this follows easily from the projection formula.

For the axiom (CD), since $\mathcal{V} = \mathbf{Sm}_S$, the only choice for E (up to permuting the factors in $W := (\mathbb{P}^N)^r$) is $E = \mathbb{P}^{N-1} \times \mathbb{P}^N \ldots \times \mathbb{P}^N$. By repeated applications of the projective bundle formula, $A^*(W)$ is the free $A^*(S)$-module on the classes $\xi_1^{n_1} \cdot \ldots \cdot \xi_r^{n_r}$, $0 \leq n_i \leq N$, where $\xi_i = c_1(p_i^*(O(1)))$, and $p_i : W \to \mathbb{P}^N$ is the ith projection. Similarly, $A^*(E)$ is the free $A^*(S)$-module on the classes $\bar{\xi}_1^{n_1} \cdot \ldots \cdot \bar{\xi}_r^{n_r}$, $0 \leq n_1 \leq N-1$,

$0 \le n_i \le N$, $i = 2, \dots, r$, where $\bar{\xi}_i$ is the restriction of ξ_i to E. Let $i : E \to W$ be the inclusion, and let 1_E be the unit in $A^*(E)$. By the projection formula and lemma 1.5 below, we have

$$\begin{aligned} i_*(\bar{\xi}_1^{n_1} \cdot \ldots \cdot \bar{\xi}_r^{n_r}) &= i_*(i^*(\xi_1^{n_1} \cdot \ldots \cdot \xi_r^{n_r}) \cup 1_E) \\ &= \xi_1^{n_1} \cdot \ldots \cdot \xi_r^{n_r} \cup i_*(1_E) \\ &= \xi_1^{n_1+1} \cdot \ldots \cdot \xi_r^{n_r} \end{aligned}$$

Thus $i_* : A^*(E) \to A^{*+1}(W)$ is injective, verifying (CD). □

Lemma 1.5. *Let A^* be an oriented cohomology theory on $\mathbf{Sm}_S$. Consider the following property (taken from* [4, Definition 2.1]*):*

(Sect*). *Let Y be in $\mathbf{Sm}_S$. Let $L \to Y$ be a line bundle on some $Y \in \mathbf{Sm}_S$, $s : Y \to L$ a section transverse to the zero-section (in $\mathbf{Sm}_S$) and $i : Z \to Y$ the closed imbedding of the zero subscheme of s. Then*

$$c_1(L) = i_*(1_Z).$$

The property (Sect) holds for A^*.*

Proof. We actually prove a more general result in the context of an oriented Borel-Moore homology theory in proposition 7.2, relying on lemma 7.1. In both these proofs, we only use the axioms (BM2) and (H). For lemma 7.1, the same proof yields the analogous result for A^*, where E_1 and E_2 are now required to be smooth divisors, intersecting transversely on some $Y \in \mathbf{Sm}_k$. Similarly, the proof of proposition 7.2 verifies the axiom (Sect*) for an arbitrary oriented cohomology theory A^* on $\mathbf{Sm}_S$. □

Remark 1.6. We have actually proved a bit more in proposition 1.4, in that we never used the axiom (CD) in showing that an oriented Borel-Moore homology theory gives rise to a cohomology theory. In particular, this shows that, for $\mathcal{V} = \mathbf{Sm}_S$, the axiom (CD) is a consequence of the other axioms.

1.4. **Weak homology theories.** Fix a field k. In this section $\mathcal{V}$ will be an admissible subcategory of $\mathbf{Sch}_k$. The notion of an oriented Borel-Moore weak homology theory A_* on $\mathbf{Sch}_k$ has been defined in [4, Definition 10.7]; changing $\mathbf{Sch}_k$ to $\mathcal{V}$, one has the notion of an oriented Borel-Moore weak homology theory A_* on $\mathcal{V}$.

Proposition 1.7. *Let A_* be an oriented Borel-Moore homology theory on $\mathcal{V}$. By restricting the pull-back maps f^* to smooth quasi-projective morphisms $f : Y \to X$ in $\mathcal{V}$ having pure relative dimension, A_* defines an oriented Borel-Moore weak homology theory on $\mathcal{V}$, also denoted A_*.*

Proof. We need to show:

1. Let L be a line bundle on some $X \in \mathcal{V}$. If $f : Y \to X$ is a smooth morphism in $\mathcal{V}$, then $\tilde{c}_1(f^*L) \circ f^* = f^* \circ \tilde{c}_1(L)$. If $f : Y \to X$ is a projective morphism in $\mathcal{V}$, then $f_* \circ \tilde{c}_1(f^*L) = \tilde{c}_1(L) \circ f_*$.
2. If L and M are line bundles on $X \in \mathcal{V}$, then $\tilde{c}_1(L) \circ \tilde{c}_1(M) = \tilde{c}_1(M) \circ \tilde{c}_1(L)$.
3. Let X and Y be in $\mathcal{V}$, and $L \to X$ be a line bundle on X. For $\alpha \in A_*(X)$, $\beta \in A_*(Y)$, we have
$$\tilde{c}_1(L)(\alpha) \times \beta = \tilde{c}_1(p_1^*L)(\alpha \times \beta),$$
where $p_1 : X \times_k Y \to X$ is the projection.
4. The axioms (Sect) and (FGL) [4, Definition 2.1] are valid for A_*.

The property (1) follows easily from the functoriality of smooth pull-back (BM1) and projective push-forward, plus axiom (BM2) applied to the transverse cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{f^*s} & f^*L \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_L} \\ X & \xrightarrow[s]{} & L. \end{array}$$

For (2), we have the transverse cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{s_L} & L \\ {\scriptstyle s_M}\downarrow & & \downarrow{\scriptstyle \tilde{s}_M} \\ M & \xrightarrow[\tilde{s}_L]{} & L \oplus M. \end{array}$$

Applying (BM2) and the functoriality of smooth pull-back and projective push-forward, we have

$$\begin{aligned} \tilde{c}_1(M) \circ \tilde{c}_1(L) &= s_M^* s_{M*} s_L^* s_{L*} \\ &= s_M^* \tilde{s}_L^* \tilde{s}_{M*} s_{L*} \\ &= s_L^* \tilde{s}_M^* \tilde{s}_{L*} s_{M*} \\ &= s_L^* s_{L*} s_M^* s_{M*} \\ &= \tilde{c}_1(L) \circ \tilde{c}_1(M). \end{aligned}$$

(3) is an easy consequence of (BM3).

For (4), we note that A_* satisfies the axiom (Sect): Let $p : Y \to \operatorname{Spec} k$ be in $\mathbf{Sm}_k$ and let $1_Y = p^*(1)$. Let $L \to Y$ be a line bundle on some $Y \in \mathbf{Sm}_k$, $s : Y \to L$ a section transverse to the zero-section (in $\mathbf{Sm}_k$) and $i : Z \to Y$ the closed imbedding of the zero subscheme of s. Then
$$\tilde{c}_1(L)(1_Y) = i_*(1_Z).$$

Indeed, by proposition 1.4, A_* defines an oriented cohomology theory A^* on $\mathbf{Sm}_k$, and by lemma 1.5, (Sect*) holds for A^*. Since $\tilde{c}_1(L)(\eta) = c_1(L) \cup \eta$ for $L \to X$ a line bundle on $X \in \mathbf{Sm}_k$ and for $\eta \in A^*(X)$ (see the proof of proposition 1.4), it follows that (Sect) holds for A_*.

We note that, by [4, Remark 4.11, Lemma 10.1], A_* satisfies the axiom (Dim): Let Y be in $\mathbf{Sm}_k$. Then, given line bundles $L_1, \ldots, L_n$ on Y with $n > \dim_k Y$, we have

$$\tilde{c}_1(L_1) \circ \ldots \tilde{c}_1(L_n)(1_Y) = 0.$$

For (FGL), we need to show: there is a power series $F_A(u, v) \in A_*(k)[[u, v]]$ such that, given line bundles L and M on $Y \in \mathbf{Sm}_k$, we have

$$F_A(\tilde{c}_1(L), \tilde{c}_1(M))(1_Y) = \tilde{c}_1(L \otimes M)(1_Y).$$

(Note that, by the axiom (Dim), the left-hand side of this equation makes sense.) Since, by proposition 1.4, the oriented Borel-Moore homology theory A_*, restricted to $\mathbf{Sm}_k$, defines an oriented cohomology theory on $\mathbf{Sm}_k$, the axiom (FGL) follows from [4, Lemma 1, Introduction] and the identity $\tilde{c}_1(L)(\eta) = c_1(L) \cup \eta$ mentioned above. □

Remark 1.8. One consequence of propositions 1.4 and 1.7 is that an oriented cohomology theory A^* on $\mathbf{Sm}_k$ gives rise to an oriented Borel-Moore weak homology theory A_* on $\mathbf{Sm}_k$.

Definition 1.9. Given an oriented Borel-Moore homology theory A_* on some admisible $\mathcal{V} \subset \mathbf{Sch}_k$, the weak oriented Borel-Moore homology theory it defines is called the *underlying* one. Similarly, if A^* is an oriented cohomology theory on $\mathbf{Sm}_k$, the weak oriented Borel-Moore homology theory A_* it defines is called the underlying one.

Conversely, given an weak oriented Borel-Moore homology theory A_* on $\mathcal{V}$, we say that A_* admits a structure of an oriented Borel-Moore homology theory if there is such a theory $\tilde{A}_*$ whose underlying weak oriented Borel-Moore homology theory is A^*. We say that A_* admits a structure of an oriented cohomology theory if there is such a theory $\tilde{A}^*$ whose underlying weak oriented Borel-Moore homology theory is A_*.

Example 1.10. The *locally finite* singular homology and étale homology theories studied in [2] are examples of oriented Borel-Moore homology theories.

Example 1.11. The Chow groups functor $X \mapsto CH_*(X)$ on $\mathbf{Sch}'_k$. One can check, using the projective push-forwards, the pull-backs and their properties described in [1], that CH_* is indeed endowed with such a structure. In fact, there is one and only one structure of oriented Borel-Moore homology theory on CH_* whose underlying structure gives the usual one. This follows rather easily from [1].

Example 1.12. The functor $X \mapsto G_0(X)[\beta, \beta^{-1}]$ can be shown as well to admit a unique structure of oriented Borel-Moore homology theory on $\mathcal{V}$ whose underlying structure is the usual one.

Our main aim of this paper is to establish the following two results:

Theorem 1.13. *Let k be a field admitting resolution of singularities. Let $\mathcal{V}$ be an l.c.i.-closed admissible subcategory of $\mathbf{Sch}_k$. Then the oriented Borel-Moore weak homology theory*

$$X \mapsto \Omega_*(X)$$

on $\mathcal{V}$ admits one and only one structure of an oriented Borel-Moore homology theory on $\mathcal{V}$, which we still denote Ω_*.

Theorem 1.14. *Let k be a field admitting resolution of singularities. Then the oriented Borel-Moore weak homology theory Ω_* on $\mathbf{Sm}_k$ admits one and only one structure of an oriented cohomology theory Ω^* on $\mathbf{Sm}_k$.*

We will also show

Theorem 1.15. *Assume k admits resolution of singularities.*

1. *Let $\mathcal{V}$ be an l.c.i.-closed admissible subcategory of $\mathbf{Sch}_k$. Then algebraic cobordism, $X \mapsto \Omega_*(X)$, is the universal oriented Borel-Moore homology theory on $\mathcal{V}$.*
2. *Algebraic cobordism, considered as an oriented cohomology theory on $\mathbf{Sm}_k$, is the universal oriented cohomology theory on $\mathbf{Sm}_k$.*

Theorem 3 of [4], discussed in the introduction, is an immediate consequence of these results. Similarly, the results [4, Theorems 12 and 13] are proven in [4] only for the underlying Borel-Moore weak homology theories; theorem 1.15(2) completes the proofs of these results for the respective oriented cohomology theories on $\mathbf{Sm}_k$. For the sake of completeness, we state these results here, in their full generality:

Let $\mathbb{L}^* \to \mathbb{Z}$ and $\mathbb{L}^* \to \mathbb{Z}[\beta, \beta^{-1}]$ be the homomorphisms classifying the additive formal group law $F_a(u, v) = u + v$ and the multiplicative periodic formal group law $F_m(u, v) = u + v - \beta uv$.

Theorem 1.16. *Assume k admits resolution of singularities.*
1) The canonical morphism $\Omega^* \to K^0[\beta, \beta^{-1}]$ of oriented cohomology theories on $\mathbf{Sm}_k$ induces an isomorphism of such theories

$$\Omega_* \otimes_{\mathbb{L}^*} \mathbb{Z}[\beta, \beta^{-1}] \to K^0[\beta, \beta^{-1}].$$

2) Suppose that k has characteristic zero. Then the canonical morphism $\Omega^* \to \mathrm{CH}^*$ of oriented Borel-Moore homology theories on $\mathbf{Sch}_k$ induces an isomorphism of such theories

$$\Omega_* \otimes_{\mathbb{L}^*} \mathbb{Z} \to \mathrm{CH}^*.$$

In particular, the above isomorphism restricts to an isomorphism of oriented cohomology theories on $\mathbf{Sm}_k$.

Theorem 1 of [4] is a direct consequence of theorem 1.15(2) and theorem 1.16(2). Indeed, theorem 1.15(2) shows that $\Omega_* \otimes_{\mathbb{L}^*} \mathbb{Z}$ is the universal ordinary oriented cohomology theory on $\mathbf{Sm}_k$, and thus the isomorphism $\Omega_* \otimes_{\mathbb{L}^*} \mathbb{Z} \to \mathrm{CH}^*$ of theorem 1.16 yields Theorem 1 of [4].

1.5. **Fundamental classes.** Let A_* be an oriented Borel-Moore homology theory on an admissible subcategory $\mathcal{V}$ of $\mathbf{Sch}_S$.

Definition 1.17. Let $p_X : X \to S$ be an l.c.i. scheme over S. Define the *fundamental class* of X, $1_X \in A_*(X)$, by $1_X := p_X^*(1)$, where $1 \in A_0(k)$ is the unit element.

For $f : Y \to X$ a projective morphism in $\mathcal{V}$, with $Y \in \mathbf{Lci}_S$, we write $[f : Y \to X]_A$ for $f_*(1_Y)$. We omit the A in the notation if the context makes the meaning clear.

Remark 1.18. Take $S = \operatorname{Spec} k$, k a field. $X \to \operatorname{Spec} k$ is an l.c.i. morphism if and only X is a local complete intersection closed subscheme of a smooth quasi-projective P over k. In particular, an l.c.i. k-scheme X is a Cohen-Macaulay scheme, so X is unmixed (has no embedded components) and is locally equi-dimensional over k. Thus, for such a scheme, we may use cohomological notation: $A^n(X) = A_{d-n}(X)$ if X is connected of dimension d over k, and then extend to locally equi-dimensional X by taking the direct sum over the connected components. In particular, the fundamental class 1_X is in $A^0(X)$.

Remark 1.19. Let $f : Y \to X$ be an l.c.i. morphism of l.c.i. S-schemes. Then $f^*(1_X) = 1_Y$.

1.6. **Degree formulas.** Some of the degree formulas of [4, §13] can be improved, if one considers a Borel-Moore homology theory instead of a weak homology theory. For example, one can replace “smooth” with “l.c.i.” in [4, Theorem 13.6], yielding the following result (we refer the reader to [4, §13] for the various terms used in the statement):

Theorem 1.20. *Let k be a field. Let A_* be an oriented Borel-Moore homology theory on $\mathbf{Sch}_k$. Assume A_* is generically constant and has the localization property.*

Let X be a reduced finite type k-scheme. Assume that, for each closed integral subscheme $Z \subset X$, we are given a projective birational morphism $\tilde{Z} \to Z$ with $\tilde{Z}$ reduced and in $\mathbf{Lci}_k$. *Let $X_1, \ldots, X_r$ be the irreducible components of X, and let α be in $A_*(X)$. Then for each integral closed subscheme $Z \subset X$ with $\mathrm{codim}_X Z > 0$, there is an element $\omega_Z \in A_{*-\dim_k Z}(k)$, all but finitely many being zero, such that*

$$\alpha - \sum_{i=1}^{r} \deg_i(\alpha) \cdot [\tilde{X}_i \to X] = \sum_{Z,\ \mathrm{codim}_X Z > 0} \omega_Z [\tilde{Z} \to X].$$

Similarly, [4, Corollary 13.7] can be modified as follows:

Corollary 1.21. *With the assumptions as in theorem 1.20, suppose that each irreducible component X_i is in* $\mathbf{Lci}_k$.

1. *Let $f : Y \to X$ be a projective morphism with Y in* $\mathbf{Lci}_k$. *Then for each integral closed subscheme $Z \subset X$ with $\mathrm{codim}_X Z > 0$, there is an element $\omega_Z \in A_{*-\dim_k Z}(k)$, all but finitely many being zero, such that*

$$[f : Y \to X] - \sum_i \deg_i(f) \cdot [X_i \to X] = \sum_{Z,\ \mathrm{codim}_X Z > 0} \omega_Z [\tilde{Z} \to X].$$

2. *Let $f : Y \to X$ be a projective birational morphism, with Y in* $\mathbf{Lci}_k$. *Then, for each integral closed subscheme $Z \subset X$ with $\mathrm{codim}_X Z > 0$, there is an element $\omega_Z \in A_{*-\dim_k Z}(k)$, all but finitely many being zero, such that*

$$[f : Y \to X] = \sum_i [X_i \to X] + \sum_{Z,\ \mathrm{codim}_X Z > 0} \omega_Z \cdot [\tilde{Z} \to X].$$

Here, if Y is an l.c.i. k-scheme and $f : Y \to X$ is a projective morphism, we write $\deg_i(f)$ for $\deg_i([f : Y \to X])$.

For example, if k is perfect, X is in $\mathbf{Sm}_k$ and has dimension at most four over k, then, as one has resolution of singularities for finite type k-schemes of dimension at most two, and since each codimension one subvariety of X is an l.c.i. k-scheme, we may apply these two results without having resolution of singularities for arbitrary finite type k-schemes.

Also, suppose as in corollary 1.21 that X is a reduced finite type k-scheme such that the irreducible components $X_1, \ldots, X_s$ are all in $\mathbf{Lci}_k$. Let $d_i = \dim_k X_i$. We have the map $p_i^* : A_{*-d_i}(k) \to A_*(X)$ defined by

$$p_i(\alpha) = \alpha \cdot [X_i \to X].$$

Letting $\tilde{A}_*(X)$ be the kernel of the total degree map

$$\prod_i \deg_i : A_*(X) \to \oplus_{i=1}^s A_{*-d_i}(k),$$

the maps p_i define the splitting

$$A_*(X) = \tilde{A}_*(X) \oplus \oplus_{i=1}^s A_{*-d_i}(k). \tag{1.2}$$

1.7. **The axiom (CD).** We conclude this section by showing that the axiom (CD′) of remark 1.3 implies the axiom (CD) of definition 1.2, and that the axiom (CD′) is implied by a certain localization property. In this section, S is a noetherian separated scheme and $\mathcal{V}$ is an admissible subcategory of $\mathbf{Sch}_S$.

Lemma 1.22. *Suppose $\mathcal{V}$ contains $\mathbf{Lci}_S$. Suppose we are given the data from definition 1.2 (D1)-(D3) of an oriented Borel-Moore homology theory A_* on $\mathcal{V}$, satisfying the axioms of definition 1.2, with the possible exception of the axiom (CD), and suppose that A_* satisfies the axiom (CD′) of remark 1.3. Then A_* satisfies the axiom (CD).*

Proof. Let E be as in axiom (CD); Note that E is in $\mathbf{Lci}_S$, hence in $\mathcal{V}$. We may suppose that $n_1, \ldots, n_m$ are non-zero and $n_{m+1}, \ldots, n_r$ are all zero.

Let $X_0, \ldots, X_N$ be the standard homogeneous coordinates on $\mathbb{P}^N$, and let $X_j^i = p_i^*(X_j)$, where $p_i : W = (\mathbb{P}^N)^r \to \mathbb{P}^N$ is the ith projection.

We may thus write E_{red} as a strict normal crossing divisor $E_{\text{red}} = \sum_{i=1}^m E_i$, with

$$E_i = \mathbb{P}^N \times \ldots \times \mathbb{P}^{N-1} \times \ldots \times \mathbb{P}^N \subset (\mathbb{P}^N)^r,$$

with the $\mathbb{P}^{N-1}$ the linearly imbedded subspace of $\mathbb{P}^N$ defined by $X_N^i = 0$, $i = 1, \ldots, m$.

By proposition 1.4 and remark 1.6, A_* defines an oriented cohomology theory A^* on $\mathbf{Sm}_S$; we have also shown in the proof of proposition 1.4 that

$$\tilde{c}_1(L)(\eta) = c_1(L) \cup \eta$$

for $L \to Y$ a line bundle on $Y \in \mathbf{Sm}_S$ and $\eta \in A_*(Y) = A^*(Y)$. By lemma 1.5, the property (Sect*) is valid for A^*. Using these two properties, together with repeated applications of the axiom (PB), we see that $A_*(\mathbb{P}^{m_1} \times \ldots \times \mathbb{P}^{m_r})$ is a free $A_*(S)$-module with basis the classes $\hat{i}_*(1_{\mathbb{P}^{a_1} \times \ldots \times \mathbb{P}^{a_r}})$, where $0 \le a_j \le n_j$ for $1 \le j \le r$, and where

$$\hat{i} : \mathbb{P}^{a_1} \times \ldots \times \mathbb{P}^{a_r} \to \mathbb{P}^{m_1} \times \ldots \times \mathbb{P}^{m_r}$$

is the subscheme defined by the vanishing of $m_j - a_j$ coordinates X_l^j, $j = 1, \ldots, r$. It follows from the axiom (Sect*) that the class $\hat{i}_*(1_{\mathbb{P}^{a_1}\times\ldots\times\mathbb{P}^{a_r}})$ is independent of the particular $m_j - a_j$ coordinates chosen.

Using the standard cellular decomposition of each E_i, we have a filtration of $E_{\rm red}$ satisfying the conditions of axiom (CD′), and with each of the "closed cells" of the form of an imbedded product $\mathbb{P}^{a_1} \times \ldots \times \mathbb{P}^{a_r}$ in some E_i, defined by the vanishing of coordinates X_l^j as above. If a given product $\mathbb{P}^{a_1}\times\ldots\times\mathbb{P}^{a_r}$ should occur twice, say as a cell C in E_i and another cell C' in $E_{i'}$, then both $a_i \le N-1$ and $a_{i'} \le N-1$, so there is a cell C'' of the same type in $E_i \cap E_{i'}$. Thus $i_{C*}(1_C) = i_{C''*}(1_{C''}) = i_{C'*}(1_{C'})$ in $A_*(E)$, where $i_C : C \to E$, $i_{C'}$, $i_{C''}$ are the inclusions. Therefore, by axiom (CD′) and our description of $A_*(E_i)$ above, $A_*(E)$ is generated as an $A_*(S)$-module by the classes $\tilde{i}_*(1_{\mathbb{P}^{a_1}\times\ldots\times\mathbb{P}^{a_r}})$, where

$$\tilde{i} : \mathbb{P}^{a_1} \times \ldots \times \mathbb{P}^{a_r} \to E$$

is the composition of a map $\hat{i}$ followed by the inclusion $E_i \to E$, and the indices $(a_1, \ldots, a_r)$ run over all r-tuples with $0 \le a_j \le N$, $j = 1, \ldots, r$, with at least one index $a_i \le N-1$ for some $i \le m$. Comparing this with the description of $A_*((\mathbb{P}^N)^r)$ as a free $A_*(S)$-module with basis $\hat{i}_*(1_{\mathbb{P}^{a_1}\times\ldots\times\mathbb{P}^{a_r}})$, $0 \le a_i \le N$, $i = 1, \ldots, r$, we see that the $A_*(S)$-module generators for $A_*(E)$ described above are actually an $A_*(S)$-basis for $A_*(E)$, and therefore $A_*(E)$ is a summand of $A_*((\mathbb{P}^N)^r)$, which verifies the axiom (CD). □

Lemma 1.23. *Suppose we are given the data from definition 1.2 (D1)-(D3) of an oriented Borel-Moore homology theory A_* on $\mathbf{Sch}_S$, satisfying the axioms of definition 1.2, with the possible exception of the axiom (CD). Suppose that A_* satisfies the following* weak localization property*:*

> *Let $i : Z \to X$ be a closed imbedding of finite type k-schemes with complement $j : U \to X$. Then the sequence*
>
> $$A_*(Z) \xrightarrow{i_*} A_*(X) \xrightarrow{j^*} A_*(U)$$
>
> *is exact.*

Then A_* satisfies the axiom (CD′) of remark 1.3.

Proof. First take $Z = E_{\rm red}$. Then U is the empty scheme, hence $A_*(X_{\rm red}) \to A_*(X)$ is surjective. Thus, it suffices to prove the axiom (CD′) for reduced E.

We proceed by noetherian induction. Take a filtration of E by closed subschemes

$$\emptyset = E_0 \subset E_1 \subset \ldots \subset E_N = E$$

satisfying the conditions of axiom (CD′). By induction, the map

$$\oplus_{i\le N-1,j} A_*(E_{ij}) \to A_*(E_{N-1})$$

is surjective, where E_{ij} is the closure of the irreducible component E^0_{ij} of E_i, $i = 1, \dots, N-1$.

Let $E^0_{N1}, \dots, E^0_{Nm}$ be the irreducible components of $E_N \setminus E_{N-1}$, and let E_{Nj} be the closure of E^0_{Nj}. By assumption, each E^0_{Nj} is an affine space $\mathbb{A}^{N_j}$ over S and each E_{Nj} is smooth and quasi-projective over S. Let $\bar{p} : E_{Nj} \to S$ and $p : E^0_{Nj} \to S$ be the structure morphisms, and let $f : E^0_{Nj} \to E_{Nj}$ be the inclusion. We have the commutative diagram

$$\begin{array}{ccc} A_*(E_{Nj}) & \xrightarrow{f^*} & A_*(E^0_{Nj}) \\ & \nwarrow^{\bar{p}^*} & \uparrow{\scriptstyle p^*} \\ & & A_{*-N_j}(S) \end{array}$$

Since $p^* : A_{*-N_j}(S) \to A_*(E^0_{Nj})$ is an isomorphism by the homotopy property (H) of definition 1.2, it follows that $f^* : A_*(E_{Nj}) \to A_*(E^0_{Nj})$ is surjective. Letting $i : E_{Nj} \to E$ and $\tilde{f} : E^0_{Nj} \to E$ be the inclusions, we have $\tilde{f}^* i_* = f^*$, by axiom (BM2). Thus the restriction map

$$A_*(E) \to A_*(E \setminus E_N) = \oplus_{j=1}^m A_*(E^0_{Nj})$$

is surjective.

Adding these surjectivities to our weak localization property, we have the exact sequences

$$A_*(E_{N-1}) \to A_*(E_N) \to \oplus_{j=1}^m A_*(E^0_{Nj}) \to 0$$
$$A_*(E_{Nj} \cap E_{N-1}) \to A_*(E_{Nj}) \to A_*(E^0_{Nj}) \to 0$$

By an elementary diagram chase the map

$$\oplus_{i,j} A_*(E_{ij}) \to A_*(E_N)$$

is surjective, and the induction goes through. □

2. *Refined cobordism*

As mentioned in the introduction, our main goal is to construct functorial pull-back maps $f^* : \Omega^*(X) \to \Omega^*(Y)$ for l.c.i. morphisms $f : Y \to X$ in $\mathbf{Sch}_k$; the method of "deformation to the normal bundle" leads us to first consider the case of a divisor $i : D \to X$. In fact, a more flexible notion, due to Fulton [1], is that of a *pseudo-divisor*. In this section, we define a refined version $\Omega_*(X)_D$ of the cobordism group $\Omega_*(X)$, when one has the extra data of a pseudo-divisor D on

X; $\Omega_*(X)_D$ comes with a natural homomorphism $\Omega_*(X)_D \to \Omega_*(X)$. In the next section, we define intersection with a pseudo-divisor D

$$D(-) : \Omega_*(X)_D \to \Omega_*(|D|).$$

We go on to prove the required properties of this product, most notably the commutativity of intersection for two pseudo-divisors. For this, we will require the auxiliary construction of groups $\Omega_*(X)_{D|D'}$, which we also give in this section. We conclude the construction in §5 by proving a "moving lemma", showing that the homomorphism $\Omega_*(X)_D \to \Omega_*(X)$ is an isomorphism.

2.1. **Pseudo-divisors.** Let X be a finite type k-scheme. Following Fulton [1], a *pseudo-divisor* D on X is a triple $D := (Z, \mathcal{L}, s)$, where $Z \subset X$ is a closed subset, $\mathcal{L}$ is an invertible sheaf on X, and s is a section of $\mathcal{L}$ on X, such that the subscheme $s = 0$ has support contained in Z; we identify triples $(Z, \mathcal{L}, s)$, $(Z, \mathcal{L}', s')$ if there is an isomorphism $\phi : \mathcal{L} \to \mathcal{L}'$ with $s' = \phi(s)$. In particular, having fixed $\mathcal{L}$, the section s is determined exactly up to a global unit on X. If we have a morphism $f : Y \to X$, we define $f^*(Z, \mathcal{L}, s) := (f^{-1}(Z), f^*\mathcal{L}, f^*s)$; clearly $(fg)^*(D) = g^*(f^*D)$ for a pseudo-divisor D. Also, an effective Cartier divisor D on X uniquely determines a pseudo-divisor $(|D|, \mathcal{O}_X(D), s_D)$, where $s_D : \mathcal{O}_X \to \mathcal{O}_X(D)$ is the canonical section and $|D|$ is the support of D.

We call Z the *support* of a pseudo-divisor $D := (Z, \mathcal{L}, s)$, and write $Z = |D|$. Similarly, we call s the *defining equation* of D, and write $s = \mathrm{eq.}(D)$. We let $\mathrm{div}(D)$ denote the subscheme $s = 0$, and write $\mathcal{O}_X(D)$ for $\mathcal{L}$. If X is in $\mathbf{Sm}_k$, if $|D| = |\mathrm{div}D|$ and if this subset has codimension one on X, then we identify D with the Cartier divisor $\mathrm{div}D$.

The zero pseudo-divisor is $(\emptyset, \mathcal{O}_X, 1)$. If we have pseudo-divisors $D = (Z, \mathcal{L}, s)$ and $D' = (Z', \mathcal{L}', s')$, define $D + D' = (Z \cup Z', \mathcal{L} \otimes \mathcal{L}', s \otimes s')$.

2.2. **The group $\Omega_*(X)_D$.** Let X be a finite type k-scheme and D a pseudo-divisor on X. We define the series of groups

$$\mathcal{Z}_*(X)_D \to \underline{\mathcal{Z}}_*(X)_D \to \underline{\Omega}_*(X)_D \to \mathbb{L}_* \otimes \underline{\Omega}_*(X)_D \to \Omega_*(X)_D$$

analogous to the sequence

$$\mathcal{Z}_*(X) \to \underline{\mathcal{Z}}_*(X) \to \underline{\Omega}_*(X) \to \mathbb{L}_* \otimes \underline{\Omega}_*(X) \to \Omega_*(X)$$

used to define $\Omega_*(X)$ in [4, Part 1].

Recall that a *strict normal crossing divisor* E on a scheme $W \in \mathbf{Sm}_k$ is an effective divisor $E = \sum_{i=1}^r n_i E_i$ (with irreducible components

$E_1, \ldots, E_r$) such that, for each $J \subset \{1, \ldots, r\}$, the *face*

$$E_J := \cap_{j\in J} E_j$$

is smooth over k and has codimension $|J|$ on W. We call E *reduced* if all the $n_i = 1$.

We recall from [4, §1] that $\mathcal{M}(X)$ is the set of isomorphism classes of projective morphisms $f : Y \to X$, with Y in $\mathbf{Sm}_k$ (where "isomorphism" means isomorphism over X). $\mathcal{M}(X)$ is a monoid under disjoint union; we have the group completion $\mathcal{M}^+(X)$, which is the free abelian group on the isomorphism classes $f : Y \to X$ in $\mathcal{M}(X)$ with Y irreducible.

Let $\mathcal{M}(X)_D$ be the submonoid of $\mathcal{M}(X)$ generated by $f : Y \to X$, with Y irreducible, and with either $f(Y) \subset |D|$, or with $\mathrm{div} f^*D$ a strict normal crossing divisor on Y. We let $\mathcal{M}^+(X)_D$ be the group completion of $\mathcal{M}(X)_D$; $\mathcal{M}^+(X)_D$ is clearly a subgroup of $\mathcal{M}^+(X)$.

Recall from [4, Definition 1.6] the notion of a *cobordism cycle* over X, namely, a family $(f : Y \to X, L_1, \ldots, L_r)$ with $f : Y \to X$ in $\mathcal{M}(X)$, Y irreducible, and $L_1, \ldots, L_r$ line bundles on Y. One also defines a suitable notion of isomorphism of cobordism cycles, and we have the group $\mathcal{Z}_*(X)$, which is the free abelian group on the isomorphism classes of cobordism cycles, graded by giving $(f : Y \to X, L_1, \ldots, L_r)$ degree $\dim_k Y - r$. We also allow $r = 0$, which gives an inclusion $\mathcal{M}^+(X) \to \mathcal{Z}_*(X)$. We let $\mathcal{Z}_*(X)_D$ be the subgroup of $\mathcal{Z}_*(X)$ generated by the cobordism cycles $(f : Y \to X, L_1, \ldots, L_r)$ with $Y \to X$ in $\mathcal{M}(X)_D$.

Definition 2.1. Let X be in $\mathbf{Sch}_k$ and D a pseudo-divisor on X. Let $<\mathcal{R}_*^{Dim}>(X)_D$ be the subgroup of $\mathcal{Z}_*(X)_D$ generated by cobordism cycles of the form

$$(f : Y \to X, \pi^*(L_1), \ldots, \pi^*(L_r), M_1, \ldots, M_s),$$

where $\pi : Y \to Z$ is a smooth quasi-projective morphism, Z is in $\mathbf{Sm}_k$, $L_1, \ldots, L_r$ are line bundles on Z and $r > \dim_k Z$. We set

$$\underline{\mathcal{Z}}_*(X)_D := \mathcal{Z}_*(X)_D / <\mathcal{R}_*^{Dim}>(X)_D.$$

Just as for $\mathcal{M}(X)$, $\mathcal{Z}_*(X)$ and $\underline{\mathcal{Z}}_*(X)$, we have functoriality for smooth quasi-projective morphisms of relative dimension d, $f : X' \to X$:

$$f^* : \mathcal{M}(X)_D \to \mathcal{M}(X')_{f^*(D)},$$
$$f^* : \mathcal{Z}_*(X)_D \to \mathcal{Z}_{*+d}(X')_{f^*(D)},$$
$$f^* : \underline{\mathcal{Z}}_*(X)_D \to \underline{\mathcal{Z}}_{*+d}(X')_{f^*(D)},$$

and push-forward maps for projective morphisms $f : X' \to X$:

$$f_* : \mathcal{M}(X')_{f^*(D)} \to \mathcal{M}(X)_D,$$
$$f_* : \mathcal{Z}_*(X')_{f^*(D)} \to \mathcal{Z}_*(X)_D,$$
$$f_* : \underline{\mathcal{Z}}_*(X')_{f^*(D)} \to \underline{\mathcal{Z}}_*(X)_D.$$

Also, for $L \to X$ a line bundle on X, we have the Chern class endomorphism

$$\tilde{c}_1(L) : \mathcal{Z}_*(X)_D \to \mathcal{Z}_{*-}(X)_D,$$

defined as for $\mathcal{Z}_*(X)$ by sending $(f : Y \to X, L_1, \ldots, L_r)$ to $(f : Y \to X, L_1, \ldots, L_r, f^*L)$. This descends to the locally nilpotent endomorphism $\tilde{c}_1(L) : \underline{\mathcal{Z}}_*(X)_D \to \underline{\mathcal{Z}}_{*-1}(X)_D$. The operation of product over k defines external products

$$\times : \mathcal{Z}_*(X)_D \otimes \mathcal{Z}_*(X')_{D'} \to \mathcal{Z}_*(X \times_k X')_{D \times D'},$$

which descend to $\underline{\mathcal{Z}}_*(-)_-$, and have all the compatibilities with f_*, f^* and $\tilde{c}_1(L)$ as for $\mathcal{Z}_*(-)$ and $\underline{\mathcal{Z}}_*(-)$. All these operations are compatible with the corresponding ones defined for $\mathcal{Z}_*(-)$ and $\underline{\mathcal{Z}}_*(-)$, via the natural maps $\mathcal{Z}_*(X)_D \to \mathcal{Z}_*(X)$ and $\underline{\mathcal{Z}}_*(X)_D \to \underline{\mathcal{Z}}_*(X)$.

2.3. **Good position.** We define various notions of "good position" of a divisor E with respect to a pseudo-divisor D.

Definition 2.2. Let $f : W \to X$ be in $\mathcal{M}(X)$, with W irreducible, and let E be a strict normal crossing divisor on W. Let D be a pseudo-divisor on X. We say that E is *in good position with respect to* D if, for each face E_J of E, the composition $E_J \to W \to X$ is in $\mathcal{M}(X)_D$. We say that E is *in very good position with respect to* D if either $f(W) \subset |D|$, or, if not, $E + \mathrm{div} f^*D$ is a strict normal crossing divisor on W and E. Finally, we say that E is in *general position* with respect to D if E is in very good position with respect to D and in addition, in case $f(W) \not\subset |D|$, that E and $\mathrm{div} f^*D$ have no common components.

We extend these notions to W not necessarily irreducible by imposing the appropriate condition on each component of W.

Remarks 2.3. 1) It is easy to see that, for $f : W \to X$ in $\mathcal{M}(X)$ with strict normal crossing divisor E, if E is in very good position, it is in good position with respect to D.

2) If $f : W \to X$ is in $\mathcal{M}(X)_D$, and $\mathcal{L}$ is a very ample invertible sheaf on W, it follows from the Bertini theorem that, for a general section s of $\mathcal{L}$, the divisor of s is in general position with respect to D.

Definition 2.4. Let X be in $\mathbf{Sch}_k$ and D a pseudo-divisor on X. Let $<\mathcal{R}_*^{Sect}>(X)_D$ be the subgroup of $\underline{\mathcal{Z}}_*(X)_D$ generated by elements of the form

$$[f: Y \to X, L_1, \ldots, L_r] - [f \circ i: Z \to X, i^*(L_1), \ldots, i^*(L_{r-1})],$$

with $r > 0$, $[f: Y \to X, L_1, \ldots, L_r]$ a cobordism cycle in $\mathcal{Z}_*(X)_D$ and $i: Z \to Y$ the closed immersion of the subscheme defined by the vanishing of a transverse section $s: Y \to L_r$, such that Z is in very good position with respect to D.

We set

$$\underline{\Omega}_*(X)_D := \underline{\mathcal{Z}}_*(X)_D / <\mathcal{R}_*^{Sect}>(X)_D.$$

We have the evident natural map $\underline{\Omega}_*(X)_D \to \underline{\Omega}_*(X)$. The operations f^*, f_* and $\tilde{c}_1(L)$ descend to the quotient $\underline{\Omega}_*(-)_-$ of $\underline{\mathcal{Z}}_*(-)_-$, and are compatible with the corresponding operations on $\underline{\Omega}_*(-)$ via the natural maps $\underline{\Omega}_*(X)_D \to \underline{\Omega}_*(X)$. Similarly, the external products for $\underline{\mathcal{Z}}_*(-)_-$ descend to $\underline{\Omega}_*(-)_-$, and these external products are compatible with the external products on $\underline{\Omega}_*(-)$.

As in [4, Introduction], we let $(F_{\mathbb{L}}, \mathbb{L}_*)$ denote the universal formal group law; $\mathbb{L}_*$ is the *Lazard ring* and $F_{\mathbb{L}} = F_{\mathbb{L}}(u, v)$ is a power series with coefficients in $\mathbb{L}_*$.

If $T_1, T_2 : B \to B$ are commuting locally nilpotent operators on an abelian group B, and $F(u, v) = \sum_{i,j} a_{ij} u^i v^j$ is a power series with $\mathbb{L}_*$-coefficients, we have the well-defined $\mathbb{L}_*$-linear operator $F(T_1, T_2): \mathbb{L}_* \otimes B \to \mathbb{L}_* \otimes B$ defined by

$$F(T_1, T_2)(a \otimes b) := \sum_{i,j} a a_{ij} \otimes T_1^i(b) T_2^j(b).$$

Definition 2.5. For X in $\mathbf{Sch}_k$, let $<\mathbb{L}_* \mathcal{R}_*^{FGL}>(X)_D$ be the $\mathbb{L}_*$-submodule of $\mathbb{L}_* \otimes \underline{\Omega}_*(X)_D$ generated by elements of the form

$$(\mathrm{id} \otimes f_*)\big(F_{\mathbb{L}}(\tilde{c}_1(L), \tilde{c}_1(M))(\eta) - \tilde{c}_1(L \otimes M)(\eta)\big),$$

where $f: Y \to X$ is in $\mathcal{M}(X)_D$, L and M are line bundles on Y, and η is in $\underline{\Omega}_*(Y)_{f^*D}$. We set

$$\Omega_*(X)_D := \mathbb{L}_* \otimes \underline{\Omega}_*(X)_D / <\mathbb{L}_* \mathcal{R}_*^{FGL}>(X)_D.$$

The natural transformation $\underline{\Omega}_*(X)_D \to \underline{\Omega}_*(X)$ descends to a natural $\mathbb{L}_*$-linear transformation $\Omega_*(X)_D \to \Omega_*(X)$. The structures we have defined for $\underline{\Omega}_*(-)_-$: f^*, f_*, $\tilde{c}_1(L)$ and external products, all descend to $\Omega_*(-)_-$, and are compatible with the corresponding structures on $\Omega_*(-)$, via the natural transformation $\Omega_*(X)_D \to \Omega_*(X)$.

If $f: Y \to X$ is an X-scheme, and D is a pseudo-divisor on X, we will often write $\Omega_*(Y)_D$ for $\Omega_*(Y)_{f^*(D)}$, and similarly for $\underline{\Omega}_*(Y)_D$, etc.

2.4. **Refined divisor classes.** The operators

$$\tilde{c}_1(L) : \Omega_*(X)_D \to \Omega_{*-1}(X)_D$$

are locally nilpotent and commute with one another, thus, if we have line bundles $L_1, \ldots, L_r$ on X, and a power series $F(u_1, \ldots, u_r)$ with $\mathbb{L}_*$-coefficients, we have the $\mathbb{L}_*$-linear endomorphism

$$F(\tilde{c}_1(L_1), \ldots, \tilde{c}_1(L_r)) : \Omega_*(X)_D \to \Omega_{*-1}(X)_D.$$

If $f : W \to X$ is in $\mathcal{M}(X)_D$, we have the element $1_W^D = [\mathrm{id} : W \to W] \in \Omega_*(W)_D$. Given line bundles $L_1, \ldots, L_r$ on W we set

$$[F(L_1, \ldots, L_r)]_D := F(\tilde{c}_1(L_1), \ldots, \tilde{c}_1(L_r))(1_X^D) \in \Omega_*(W)_D.$$

We recall some notation from [4, §5]. Let $n_1,\ldots, n_r$ be positive integers. Let $G^{n_1,\ldots,n_r}$ be the power series with $\mathbb{L}_*$-coefficients giving the sum in the universal group law $(F_{\mathbb{L}}, \mathbb{L}_*)$:

$$G^{n_1,\ldots,n_r}(u_1, \ldots, u_r) = n_1 \cdot_F u_1 +_F \ldots +_F n_r \cdot_F u_r.$$

We have as well the canonical decomposition

$$G^{n_1,\ldots,n_r}(u_1, \ldots, u_r) = \sum_{||J||=1} u^J H_J^{n_1,\ldots,n_r}(u_1, \ldots, u_r),$$

defining the power series $H_J^{n_1,\ldots,n_r}$, where the sum is over multi-indices $J = (j_1, \ldots, j_r)$, $||J||$ is the maximum of the j_i, and $u^J = \prod_i u_i^{j_i}$; the $H_J^{n_1,\ldots,n_r}$ are characterized by the property that u_i does not occur in $H_J^{n_1,\ldots,n_r}$ if $j_i = 0$.

If $E = \sum_{i=1}^r n_i E_i$ is a strict normal crossing divisor on a scheme $W \in \mathbf{Sm}_k$, with support $|E| := \cup_{i=1}^r E_i$, and irreducible components $E_1, \ldots, E_r$, we have defined in [4, §5] the element $[E \to |E|]$ of $\Omega_*(|E|)$ by the formula

$$[E \to |E|] := \sum_{J,\ ||J||=1} \iota_*^J([H_J^{n_1,\ldots,n_r}(O_W(E_1)^J, \ldots, O_W(E_r)^J)]).$$

To explain the notation: For $J = (j_1, \ldots, j_r)$ with $||J| = 1$, we have the face $E^J = \cap_{i,\ j_i=1} E_i$, and the inclusion $\iota^J : E^J \to |E|$. $O_W(E_j)$ is the line bundle on W with sheaf of sections $\mathcal{O}_W(E_j)$, and $O_W(E_j)^J$ is the restriction of $O_W(E_j)$ to E^J.

Suppose now that we have $f : W \to X$ in $\mathcal{M}(X)$, and a strict normal crossing divisor E on W, such that E is in good position with respect to D. Write $E = \sum_{i=1}^m n_i E_i$, with the E_i irreducible.

Since the subscheme E^J is in good position with respect to D, the morphism $f \circ \iota^J : E^J \to X$ is in $\mathcal{M}(X)_D$, so we have the class

$$[H_J^{n_1,\ldots,n_r}(O_W(E_1)^J, \ldots, O_W(E_r)^J)]_D \in \Omega_*(E^J)_D,$$

giving the *refined divisor class*

$$[E \to |E|]_D := \sum_J \iota^J_*[H_J^{n_1,\ldots,n_r}(O_W(E_1)^J,\ldots,O_W(E_r)^J)]_D$$

in $\Omega_*(|E|)_D$.

The properties of $\tilde{c}_1(L)$, $[E \to |E|]$ and $\Omega_*(X)$ proved in [4, §4, §5] carry over without change to $\tilde{c}_1(L)$ (acting on $\Omega_*(X)_D$), $[E \to |E|]_D$ and $\Omega_*(X)_D$.

2.5. **A further refinement.** In order to discuss issues of functoriality, it will be necessary to make an extension of the above construction.

Definition 2.6. Let X be a finite type k-scheme, with pseudo-divisors D, D'. Let $i : |D| \to X$ be the inclusion. We let $\mathcal{M}(X)_{D|D'}$ be the submonoid of $\mathcal{M}(X)_D$ generated by those $f : Y \to X$, with Y irreducible, such that

1. If $f(Y) \subset |D|$, then $f : Y \to |D|$ is in $\mathcal{M}(|D|)_{D'}$.
2. If $f(Y) \not\subset |D|$, then, for each face F of $\mathrm{div} f^*D$, the map $f : F \to |D|$ is in $\mathcal{M}(|D|)_{D'}$.

Definition 2.7. Let $f : W \to X$ be in $\mathcal{M}(X)$, and let E be a strict normal crossing divisor on W. We say that E is *in good position with respect to* $D|D'$ if for each face E^J of E, the composition

$$E^J \xrightarrow{i^J} W \xrightarrow{f} X$$

is in $\mathcal{M}(X)_{D|D'}$.

For $f : W \to X$ in $\mathcal{M}(X)_{D|D'}$ with W irreducible, we say that E is in *very good position with respect to* $D|D'$ if E is in very good position with respect to D and

1. if $f(W) \subset |D|$, then E is in very good position with respect to D'
2. if $f(W) \not\subset |D|$, then for each face F of $\mathrm{div} f^*D$ not contained in E, the normal crossing divisor $F \cdot E$ on F is in very good position with respect to D'.

We say that E is in *general position with respect to* $D|D'$ if E is in very good position with respect to $D|D'$ and in general position with respect to D.

We extended these notions to reducible W by requiring the appropriate condition on each component of W.

Remark 2.8. Suppose that $W \to X$ is in $\mathcal{M}(X)_{D|D'}$. If E is a divisor on W, in very good position with respect to $D|D'$, and E^J is a face of E, then E^J is in $\mathcal{M}(X)_{D|D'}$. Indeed, if $f(W) \subset |D|$, this is evident, and if $f(W) \not\subset |D|$, then each face F_E of the normal crossing divisor $E^J \cdot \mathrm{div} f^*D$ on E^J is of the form $E^J \cdot F$ for F a face of $\mathrm{div} f^*D$. Thus F_E

is a face of the normal crossing divisor $F\cdot E$ on F, and hence $F_E\to|D|$ is in $\mathcal{M}(|D|)_{D'}$. This shows that $E^J\to X$ is in $\mathcal{M}(X)_{D|D'}$, as claimed. In other words, if E is in very good position with respect to $D|D'$, then E is in good position with respect to $D|D'$.

Making the evident modifications to the constructions of the previous section, we have the sequence of abelian groups

$$\mathcal{Z}_*(X)_{D|D'}\to\underline{\mathcal{Z}}_*(X)_{D|D'}\to\underline{\Omega}_*(X)_{D|D'}\to\Omega_*(X)_{D|D'}.$$

Explicitly, $\mathcal{Z}_*(X)_{D|D'}$ is the subgroup of $\mathcal{Z}_*(X)$ generated by the cobordism cycles $(Y\to X,L_1,\ldots,L_r)$ with $Y\to X$ in $\mathcal{M}(X)_{D|D'}$. Let $<\mathcal{R}^{Dim}_*>(X)_{D|D'}$ be the subgroup generated by cobordism cycles of the form

$$(f:Y\to X,\pi^*(L_1),\ldots,\pi^*(L_r),M_1,\ldots,M_s),$$

where $\pi:Y\to Z$ is a smooth quasi-projective morphism, Z is in $\mathbf{Sm}_k$, $L_1,\ldots,L_r$ are line bundles on Z and $r>\dim_k Z$. We let $\underline{\mathcal{Z}}_*(X)_{D|D'}$ be the quotient group $\mathcal{Z}_*(X)_{D|D'}/<\mathcal{R}^{Dim}_*>(X)_{D|D'}$. $\underline{\Omega}_*(X)_{D|D'}$ is the quotient of $\underline{\mathcal{Z}}_*(X)_{D|D'}$ by the subgroup $<\mathcal{R}^{Sect}_*>(X)_{D|D'}$ generated by elements of the form

$$(f:Y\to X,L_1,\ldots,L_r)-(fi:Z\to X,i^*(L_1),\ldots,i^*(L_{r-1})),$$

with $r>0$, $(f:Y\to X,L_1,\ldots,L_r)$ a cobordism cycle in $\mathcal{Z}_*(X)_{D|D'}$ and $i:Z\to Y$ the closed immersion of the subscheme defined by the vanishing of a transverse section $s:Y\to L_r$, such that Z is in very good position with respect to $D|D'$. $\Omega_*(X)_{D|D'}$ is the quotient of $\mathbb{L}_*\otimes\underline{\Omega}_*(X)_{D|D'}$ by the $\mathbb{L}_*$-submodule $<\mathbb{L}_*\mathcal{R}^{FGL}_*>(X)_{D|D'}$ generated by elements of the form

$$(\mathrm{id}\otimes f_*)\big(F_{\mathbb{L}}(\tilde{c}_1(L),\tilde{c}_1(M))(\eta)-\tilde{c}_1(L\otimes M)(\eta)\big),$$

where $f:Y\to X$ is in $\mathcal{M}(X)_{D|D'}$, L and M are line bundles on Y, and η is in $\underline{\Omega}_*(Y)_{f^*D|f^*D'}$.

Forgetting D' defines in an evident manner the natural transformations $\mathcal{Z}_*(X)_{D|D'}\to\mathcal{Z}_*(X)_D$, $\underline{\mathcal{Z}}_*(X)_{D|D'}\to\underline{\mathcal{Z}}_*(X)_D$, etc. The operations f^*, f_*, $\tilde{c}_1(L)$ and external products defined on $\mathcal{Z}_*(X)_D$, $\underline{\mathcal{Z}}_*(X)_D$, etc. have their evident refinements to $\mathcal{Z}_*(X)_{D|D'}$, $\underline{\mathcal{Z}}_*(X)_{D|D'}$, $\underline{\Omega}_*(X)_{D|D'}$ and $\Omega_*(X)_{D|D'}$, satisfying the same structural relations, and compatible with the operations on $\mathcal{Z}_*(X)_D$, $\underline{\mathcal{Z}}_*(X)_D$, etc. In particular, the operations $\tilde{c}_1(L)$ on $\underline{\mathcal{Z}}_*(X)_{D|D'}$ and $\underline{\Omega}_*(X)_{D|D'}$ are locally nilpotent and commute with one another, so the definition of $\Omega_*(X)_{D|D'}$ makes sense. Also, as for one pseudo-divisor, if we have an X-scheme $f:Y\to X$, we write $\mathcal{M}(Y)_{D|D'}$, $\Omega_*(Y)_{D|D'}$, etc., for $\mathcal{M}(Y)_{f^*D|f^*D'}$, $\Omega_*(Y)_{f^*D|f^*D'}$, etc.

For $f : W \to X$ in $\mathcal{M}(X)_{D|D'}$, the identity map $\mathrm{id}_W : W \to W$ is in $\mathcal{M}(W)_{D|D'}$, giving the identity class $1_W^{D|D'} \in \Omega_*(W)_{D|D'}$. Thus, if $F(u_1, \ldots, u_r)$ is a power series with $\mathbb{L}_*$-coefficients, and $L_1, \ldots, L_r$ are line bundles on W, we have the class

$$[F(L_1, \ldots, L_r)]_{D|D'} := F(\tilde{c}_1(L_1), \ldots, \tilde{c}_1(L_r))(1_W^{D|D'}) \in \Omega_*(W)_{D|D'}.$$

Therefore, given $f : W \to X$ in $\mathcal{M}(X)$ and a strict normal crossing divisor $E = \sum_{i=1}^r n_i E_i$ on W which is in good position with respect to $D|D'$, we have the class of E, $[E \to |E|]_{D|D'} \in \Omega_*(|E|)_{D|D'}$, defined by

$$[E \to |E|]_{D|D'} := \sum_{J,\ ||J||=1} \iota_*^J([H_J^{n_1,\ldots,n_r}(O_W(E_1)^J, \ldots, O_W(E_r)^J)]_{D|D'}).$$

We recover the groups $\mathcal{M}(X)_D$, $\underline{\Omega}_*(X)_D$, etc., as a special case by taking D' to be the 0 pseudo-divisor $(\emptyset, \mathcal{O}_X, 1)$. Similarly, we recover $\mathcal{M}(X)_{D'}$, $\underline{\Omega}_*(X)_{D'}$, etc., by taking $D = (X, \mathcal{O}_X, 0)$.

As above, the properties of $\underline{\Omega}_*(X)$, $\tilde{\Omega}_*(X)$, $\Omega_*(X)$, $\tilde{c}_1(L)$ and $[E \to |E|]$ discussed in [4, §4, §5] carry over without change to $\underline{\Omega}_*(X)_{D|D'}$, $\tilde{\Omega}_*(X)_{D|D'}$, $\Omega_*(X)_{D|D'}$, $\tilde{c}_1(L)$ and $[E \to |E|]_{D|D'}$. Also, as in [4], if E is a strict normal crossing divisor on some W, and $f : W \to X$ is in $\mathcal{M}(X)_{D|D'}$, we let $[E \to W]_{D|D'}$ denote $i_*([E \to |E|]_{D|D'})$, where $i : |E| \to W$ is the inclusion.

Remark 2.9. Let $f : Y \to X$ be in $\mathcal{M}(X)_{D|D'}$, with Y irreducible, and suppose that $f(Y) \not\subset |D|$. Then $\mathrm{div} f^*D$ is a strict normal crossing divisor, in good position with respect to D', so if E is an effective divisor on Y with $f(|E|) \subset |D|$, then E is in good position with respect to D', hence the class $[E \to |E|]_{D'}$ in $\Omega_*(|E|)_{D'}$ is defined.

Remark 2.10. Among the results of [4] that extend to $\Omega_*(X)_{D|D'}$, we note the analog of [4, Proposition 5.9]: Let E be a strict normal crossing divisor on some $W \in \mathbf{Sm}_k$. Let D, D' be pseudo-divisors on W. Suppose that E is in good position with respect to $D|D'$. Then

$$[E \to W]_{D|D'} = [O_W(E)]_{D|D'}.$$

The proof is exactly the same as for [4, Proposition 5.9].

2.6. **Some properties of $\Omega_*(X)_D$.** In this paragraph, we assume that k admits resolution of singularities.

Lemma 2.11. *Let X be in $\mathbf{Sch}_k$, and let D be a pseudo-divisor on X. Let $f : Y \to X$ be in $\mathcal{M}(X)_D$, let $i : F \to Y$ be a closed subscheme of Y and let $\mu : W \to Y$ be a projective birational morphism. Suppose* *that the exceptional divisor E of μ is in very good position with respect to D and that $\mu(|E|) \subset F$. Then there is an element $\alpha \in \Omega_*(F)_D$ with*

$$[f \circ \mu : W \to X]_D = [f : Y \to X]_D + (fi)_*(\alpha) \in \Omega_*(X)_D.$$

Proof. We may replace X with Y, so id_Y is in $\mathcal{M}(Y)_D$, and we may suppose that Y is irreducible. If $|D| = Y$, then $\Omega_*(Y)_D = \Omega_*(Y)$, $\Omega_*(F)_D = \Omega_*(F)$, and the result is [4, Proposition 6.4]. Suppose $|D| \neq Y$. Then D is a strict normal crossing divisor on Y. Since E is in very good position with respect to D, $E + D$ is a strict normal crossing divisor on W; in particular, $\mu : W \to Y$ is in $\mathcal{M}(Y)_D$, so $[\mu : W \to Y]_D$ is defined.

We consider the deformation to the normal bundle, as in [4, §6.1]. Let $Z \subset Y$ be a closed subscheme supported in F such that $W \to Y$ is the blow-up of Y along Z, and let $\rho' : T' \to Y \times \mathbb{P}^1$ be the blow-up of $Y \times \mathbb{P}^1$ along $Z \times 0$. Without loss of generality, we may assume that $F = Z_{\mathrm{red}}$. Let $<F \times \mathbb{P}^1> \subset T'$ be the proper transform of $F \times \mathbb{P}^1$, let $<Y \times 0>$ be the proper transform of $Y \times 0$, and let $\hat{E}'$ be the exceptional divisor of ρ'. Let $U = T' \setminus (<F \times \mathbb{P}^1> \cap \hat{E}')$. Then (by [4, Lemma 6.1]) U is smooth over k, U contains $<Y \times 0>$ and the induced morphism $<Y \times 0> \to Y$ is isomorphic to $W \to Y$. In addition (see the proof of [4, *loc. cit.*]) $U \to Y$ is locally isomorphic to $\mu \circ p_1 : W \times \mathbb{A}^1 \to Y$, and, in these coordinates, $\hat{E}' \cap U \to Y$ is locally isomorphic to $\mu \circ p_1 : E \times \mathbb{A}^1 \to Y$ and $<Y \times 0> \subset U$ is the subscheme $W \times 0$. Thus, $(\rho'^*(D \times \mathbb{P}^1) + \hat{E}' + <Y \times 0>) \cap U$ is a strict normal crossing divisor on U. Therefore, by resolution of singularities, there is a projective birational morphism $\pi : T \to T'$ which an isomorphism over U, such that the induced morphism $\rho : T \to Y \times \mathbb{P}^1$ satisfies:

1. $T \to Y \times \mathbb{P}^1$ is in $\mathcal{M}(Y \times \mathbb{P}^1)_D$,
2. Letting $\hat{E} \subset T$ be the exceptional divisor of ρ, $\rho^*(D \times \mathbb{P}^1) + \hat{E} + <Y \times 0>$ is a strict normal crossing divisor on T

Since $\rho : T \to Y \times \mathbb{P}^1$ is an isomorphism away from $Y \times 0$, $\rho^{-1}(Y \times 1)$ is isomorphic to Y. By (1) and (2) above, we have the classes $[\rho^*(Y \times 1) \to T]_D$ and $[\hat{E} + <Y \times 0> \to T]_D$. Since $\hat{E} + <Y \times 0> = \rho^*(Y \times 0)$, we have

(2.1)

$$[\rho^*(Y \times 1) \to T]_D = [(p_2 \circ \rho)^*(O_{\mathbb{P}^1}(1))]_D = [\hat{E} + <Y \times 0> \to T]_D$$

by remark 2.10. On the other hand, by definition of the divisor class $[\hat{E} + <Y \times 0> \to |\hat{E} + <Y \times 0>|]_D$, there is a class $\beta \in \Omega_*(|\hat{E}|)_D$ such that

$$[\hat{E} + <Y \times 0> \to T]_D = [<Y \times 0> \to T]_D + \hat{i}_*(\beta), \tag{2.2}$$

where $\hat{i} : |\hat{E}| \to T$ is the inclusion.

Let $p : T \to Y$ be $p_1 \circ \mu$, and let $p^F : |\hat{E}| \to F$ be the map induced by p. Let $\alpha = p^F_*(\beta)$. Applying the push-forward p_* to the identity (2.2) and using (2.1) yields

$$[\mathrm{id}_Y] = [f : W \to Y] + i_*(\alpha),$$

as desired. □

Lemma 2.12. *Let $f : X \to Z$ be a morphism in $\mathbf{Sch}_k$ with Z in $\mathbf{Sm}_k$, and let $L_1, \ldots, L_r$ be line bundles on Z with $r > \dim_k Z$. Let D be a pseudo-divisor on X. Then the operator $\tilde{c}_1(f^*L_1) \circ \ldots \circ \tilde{c}_1(f^*L_r)$ vanishes on $\Omega_*(X)_D$.*

Proof. We proceed by induction on $\dim_k Z$. Since the operators $\tilde{c}_1(L)$ are $\mathbb{L}_*$-linear and commute with each other, it suffices to show that the operator in question vanishes on elements $g : Y \to X$ of $\mathcal{M}(X)_D$. Using the projection formula reduces us to the case $g = \mathrm{id}_Y$, that is, it suffices to show that

$$(\mathrm{id}_Y, f^*L_1, \ldots, f^*L_r) = 0 \text{ in } \Omega_*(Y)_D,$$

assuming id_Y is in $\mathcal{M}(Y)_D$.

We may assume that Y is irreducible. Thus, either $Y \subset |D|$, or D is a strict normal crossing divisor on Y. We give the proof in the latter case; the proof in the former case is essentially the same, but easier, and is left to the reader.

We first reduce to the case of very ample line bundles. Since Z is quasi-projective, there are very ample line bundles $M_1, \ldots, M_r$ and $N_1, \ldots, N_r$ on Z such that $L_i \cong N_i \otimes M_i^{-1}$ for each i. Let $\chi(u)$ be the inverse for the group law $(F_{\mathbb{L}}, \mathbb{L}_*)$ and let $F_-(u, v) = F_{\mathbb{L}}(u, \chi(v))$. Using the relations in $<\mathcal{R}^{FGL}_*>_{D|D'}$, we see that

$$\begin{aligned}(\mathrm{id}_Y, f^*L_1, \ldots, f^*L_r)&\\ &= \tilde{c}_1(f^*L_1) \circ \ldots \circ \tilde{c}_1(f^*L_r)(1_Y^{D|D'})\\ &= F_-(\tilde{c}_1(f^*N_1), \tilde{c}_1(f^*M_1)) \circ \ldots\\ &\qquad \ldots \circ F_-(\tilde{c}_1(f^*N_r), \tilde{c}_1(f^*M_r))(1_Y^{D|D'}).\end{aligned}$$

Since $F_{\mathbb{L}}(u, v)$ and $\chi(u)$ both have non-zero constant terms, the expression $F_-(\tilde{c}_1(f^*N_1), \tilde{c}_1(f^*M_1)) \circ \ldots \circ F_-(\tilde{c}_1(f^*N_r), \tilde{c}_1(f^*M_r))$ is a sum of monomials in $\tilde{c}_1(f^*N_i)$, $\tilde{c}_1(f^*M_j)$, each of degree at least r, with coefficients in $\mathbb{L}_*$. Thus, it suffices to prove the result in case each L_i is very ample.

If $\dim_k Z = 0$, all the line bundles are trivial, hence have a nowhere vanishing section. We may then use the relations in $<\mathcal{R}_*^{Sect}>(Y)_{D|D'}$ (for the empty divisor) to conclude that $(\mathrm{id}_Y, f^*L_1, \ldots, f^*L_r) = 0$.

Suppose now that $\dim_k Z > 0$. Let s be a section of L_r, chosen so that f^*s is not identically zero on Y. We may also assume that $s = 0$ is a smooth divisor $\bar{i} : \bar{Z} \to Z$ on Z. Let H be the divisor of f^*s with inclusion $i : |H| \to Y$, and let $\bar{f} : |H| \to \bar{Z}$ be the induced morphism.

By resolution of singularities, there is a projective birational morphism $\mu : W \to Y$ such that $\mu^*(H + D)$ is a strict normal crossing divisor on W, and with μ an isomorphism over $Y \setminus |H|$. By lemma 2.11, there is a class $\alpha \in \Omega_*(H)_D$ with

$$[W \to Y] = [\mathrm{id}_Y] + i_*(\alpha).$$

Since

$$\tilde{c}_1(f^*L_1) \circ \ldots \tilde{c}_1(f^*L_r)(i_*(\alpha)) = i_*(\tilde{c}_1(\bar{f}^*(\bar{i}^*L_1)) \circ \ldots \tilde{c}_1(\bar{f}^*(\bar{i}^*L_r))(\alpha)),$$

our induction hypothesis implies that $\tilde{c}_1(f^*L_1) \circ \ldots \tilde{c}_1(f^*L_r)(i_*(\alpha)) = 0$. Thus, it suffices to show that $\tilde{c}_1(f^*L_1) \circ \ldots \tilde{c}_1(f^*L_r)([W \to Y]) = 0$. As this element is just the push-forward of $(W, (f\mu)^*L_1, \ldots, (f\mu)^*L_r)$ by μ, we may replace Y with W; changing notation, we may assume that $H + D$ is a strict normal crossing divisor on Y.

By remark 2.10, we have the identity in $\Omega_*(Y)_D$

$$[H \to Y]_D = [O_Y(H)]_D = [f^*L_r]_D.$$

Thus

$$\begin{aligned}(\mathrm{id}_Y, f^*L_1, \ldots, f^*L_r) \\ &= \tilde{c}_1(f^*L_1) \circ \ldots \circ \tilde{c}_1(f^*L_{r-1})([H \to Y]_D) \\ &= i_*(\tilde{c}_1(\bar{f}^*(\bar{i}^*L_1)) \circ \ldots \circ \tilde{c}_1(\bar{f}^*(\bar{i}^*L_r))([H \to |H|]_D)).\end{aligned}$$

As this last element is zero by our induction hypothesis, the lemma is proved. □

3. *Intersection with a pseudo-divisor*

In this section and for the remainder of the paper, we assume that k admits resolution of singularities, unless explicitly stated otherwise.

3.1. **The intersection map on cobordism cycles.** If L is a line bundle on a k-scheme X with sheaf of sections $\mathcal{L}$, we write $\tilde{c}_1(\mathcal{L})$ for $\tilde{c}_1(L)$.

Let $D = (|D|, \mathcal{O}_X(D), s)$ be a pseudo-divisor on X, let $f : Y \to X$ be in $\mathcal{M}(X)_D$ with Y irreducible, and consider a cobordism cycle $\eta := (Y \to X, L_1, \ldots, L_r)$ in $\mathcal{Z}_*(X)_D$. We define the element $D(\eta) \in \Omega_*(|D|)$ as follows: If $f(Y) \subset |D|$, let $f^D : Y \to |D|$ be the morphism induced by f. We have the element $\tilde{c}_1(f^*\mathcal{O}_X(D))(\eta)$ in $\Omega_*(Y)$; we then define

$$D(\eta) := f^D_*(\tilde{c}_1(f^*\mathcal{O}_X(D))(\mathrm{id}_Y, L_1, \ldots, L_r)) \in \Omega_*(|D|).$$

If $f(Y) \not\subset |D|$, then $\tilde{D} := \mathrm{div} f^*D$ is a strict normal crossing divisor on Y. We let $f^D : |\tilde{D}| \to |D|$ be the restriction of f, L_i^D the restriction of L_i to $|\tilde{D}|$, and define

$$D(\eta) := f^D_*(\tilde{c}_1(L_1^D) \circ \ldots \circ \tilde{c}_1(L_r^D)([\tilde{D} \to |\tilde{D}|])) \in \Omega_*(|D|).$$

We extend this operation to a homomorphism $D(-) : \mathcal{Z}_*(X)_D \to \Omega_*(|D|)$ by linearity.

Suppose we have a second pseudo-divisor D' on X. We refine the above construction to give, for each $\eta \in \mathcal{Z}_*(X)_{D|D'}$, a class $D(\eta)_{D'}$ in $\Omega_*(|D|)_{D'}$. For this, take $f : Y \to X$ in $\mathcal{M}(X)_{D|D'}$ with Y irreducible, and consider $\eta = (Y \to X, L_1, \ldots, L_r)$. If $f(Y) \subset |D|$, then $f^D : Y \to |D|$ is in $\mathcal{M}(|D|)_{D'}$. We may thus set

$$D(\eta)_{D'} := f^D_*\tilde{c}_1(f^*\mathcal{O}_X(D))((\mathrm{id}_Y, L_1, \ldots, L_r))$$

giving a well-defined class in $\Omega_*(|D|)_{D'}$. If $f(Y) \not\subset |D|$, then $\tilde{D} := \mathrm{div} f^*D$ is a strict normal crossing divisor contained in $f^{-1}(|D'|)$. The condition that f is in $\mathcal{M}(X)_{D|D'}$ implies that $\tilde{D}$ is in good position with respect to D', hence the refined divisor class $[\tilde{D} \to |\tilde{D}|]_{D'} \in \Omega_*(|\tilde{D}|)_{D'}$ is defined. Letting $f^D : |\tilde{D}| \to |D|$ be the map induced by f, we then set

$$D(\eta)_{D'} := f^D_*(\tilde{c}_1(L_1) \circ \ldots \tilde{c}_1(L_r)([\tilde{D} \to |\tilde{D}|]_{D'}))$$

in $\Omega_*(|D|)_{D'}$.

Extending by linearity defines $D(\eta)_{D'}$ in $\Omega_*(|D|)_{D'}$ for each $\eta \in \mathcal{Z}_*(X)_{D|D'}$. If we take $D' = 0$, then we recover the definitions for $D(f)$ given above. We sometimes omit the subscript D' from the notation if the context makes the meaning clear.

The next two results follow directly from the definitions:

Lemma 3.1. *Let X be a finite type k-scheme, with pseudo-divisors D, D'. Let $g : X' \to X$ be a morphism of finite type, and let $g_D : |g^*D| \to |D|$ be the restriction of g.*

1. *Suppose that g is projective and let η be in $\mathcal{Z}_*(X')_{D|D'}$. Then $g_*\eta$ is in $\mathcal{Z}_*(X)_{D|D'}$, and*
$$g_{D*}(g^*D(\eta)_{D'}) = D(g_*\eta)_{D'}.$$
2. *Suppose that g is smooth and quasi-projective. Take $\eta \in \mathcal{Z}_*(X)_D$. Then $g^*\eta$ is in $\mathcal{Z}_*(X')_D$, g_D is smooth and quasi-projective, and*
$$g_D^*(D(\eta)) = (g^*D)(g^*\eta).$$

Lemma 3.2. *Let X be a finite type k-scheme, with pseudo-divisors D, D'. Let η be in $\mathcal{Z}_*(X)_{D|D'}$, let L be a line bundle on X and let L^D be the restriction of L to D. Then*

$$\tilde{c}_1(L^D)(D(\eta)_{D'}) = D(\tilde{c}_1(L)(\eta))_{D'}.$$

We extend the operation $D(-)_{D'}$ to $\mathbb{L}_* \otimes \mathcal{Z}_*(X)_{D|D'}$ by $\mathbb{L}_*$-linearity. More generally, let $F(u_1, \ldots, u_r)$ be a power series with $\mathbb{L}_*$-coefficients, let $L_1, \ldots, L_r$ be line bundles on X, and let $f : Y \to X$ be in $\mathcal{M}(X)_{D|D'}$. We note that $f^*D(1_Y^{D|D'})_{D'}$ is an $\mathbb{L}_*$-linear combinationation of classes of the form $i : Z \to |f^*D|$, with $\dim_k Z \le \dim_k Z$. Thus, letting F_N denote the truncation of F after total degree N, we have, for all $N \ge \dim_k Y$ and all $m \ge 0$

$$\begin{aligned}
D(F_N(\tilde{c}_1(L_1), \ldots, \tilde{c}_1(L_m))([f]))_{D'} \\
&= f_{D*}(f^*D(F_N(\tilde{c}_1(f^*L_1), \ldots, \tilde{c}_1(f^*L_m))(1_Y^{D|D'})) \\
&= f_{D*}(F_N(\tilde{c}_1(f^*L_1), \ldots, \tilde{c}_1(f^*L_m))(f^*D(1_Y^{D|D'}))) \\
&= f_{D*}(F_{N+m}(\tilde{c}_1(f^*L_1), \ldots, \tilde{c}_1(f^*L_m))(f^*D(1_Y^{D|D'}))) \\
&= D(F_{N+m}(\tilde{c}_1(L_1), \ldots, \tilde{c}_1(L_m))([f]))_{D'}.
\end{aligned}$$

Thus, for $\eta \in \mathcal{Z}_*(X)_{D|D'}$, we may set

$$\begin{aligned}
D(F(\tilde{c}_1(L_1), \ldots, \tilde{c}_1(L_m))(\eta))_{D'} \\
&:= \lim_{N\to\infty} D(F_N(\tilde{c}_1(L_1), \ldots, \tilde{c}_1(L_m))(\eta))_{D'},
\end{aligned}$$

as the limit is eventually constant.

With this definition, we may extend lemma 3.2 to power series in the Chern class operators:

Lemma 3.3. *Let η be in $\mathcal{Z}_*(X)_{D|D'}$, let $f^D : |f^*D| \to |D|$ be the restriction of f and let $i : |f^*D| \to Y$ be the inclusion. Let $F(u_1, \ldots, u_r)$ be a power series with $\mathbb{L}_*$-coefficients, and let $L_1, \ldots, L_r$ be line bundles on Y. Then*

$$\begin{aligned}
D(f_*([F(L_1, \ldots, L_r)]_{D|D'})) \\
&= f_*^D(F(\tilde{c}_1(i^*L_1), \ldots, \tilde{c}_1(i^*L_r))(f^*D)(1_Y^{D|D'})).
\end{aligned}$$

The next result requires a bit more work.

Lemma 3.4. *Let $f : W \to X$ be in $\mathcal{M}(X)_{D|D'}$ and let $Y \to W$ be a smooth codimension one closed subscheme of W. Suppose that Y is in general position with respect to $D|D'$. Suppose further that W is* *irreducible and $f(W) \not\subset |D|$. Let $f^D : |f^*D| \to |D|$ be the morphism induced by f and let $i : |f^*D| \to W$ be the inclusion. Then*

$$D([Y \to X])_{D'} = f^D_*(\tilde{c}_1(i^*O_W(Y))([\mathrm{div} f^*D \to |f^*D|]_{D'})).$$

Proof. Write $\tilde{D}$ for $\mathrm{div} f^*D$. Let $i_Y : Y \to W$ denote the inclusion. Since Y is in general position with respect to $D|D'$, $Y \to X$ is in $\mathcal{M}(X)_{D|D'}$, $i_Y^*(\tilde{D})$ is a strict normal crossing divisor on Y and is in good position with respect to D'. Furthermore, $D([Y \to X])_{D'}$ is by definition $[i_Y^*(\tilde{D}) \to |D|]_{D'}$.

Write $\tilde{D} = \sum_{i=1}^m n_i\tilde{D}_i$, with each $\tilde{D}_i$ irreducible. We may write $[\tilde{D} \to |\tilde{D}|]_{D'}$ as a sum over the faces $\tilde{D}^J$ of $\tilde{D}$,

$$[\tilde{D} \to |\tilde{D}|]_{D'} = \sum_J \iota^J_*([H^{n_1,\ldots,n_m}_J(L^J_1, \ldots, L^J_m)]_{D'}),$$

where $\iota^J : \tilde{D}^J \to |\tilde{D}|$ is the inclusion, $L_i = O_W(\tilde{D}_i)$ and L^J_i is the restriction of L_i to $\tilde{D}^J$.

Since Y is in general position with respect to D, it follows that the intersection $Y^J := Y \cap \tilde{D}^J$ is tranverse; since Y is in very good position with respect to $D|D'$, the smooth codimension one subscheme $\iota^{YJ} : Y^J \to |\tilde{D}|$ of $\tilde{D}^J$ is in very good position with respect to D', for each index J. Thus, the relations in $<\mathcal{R}^{Sect}_*>(|\tilde{D}|)_{D'}$ imply that

$$\tilde{c}_1(i^*O_W(Y))([\tilde{D} \to |\tilde{D}|]_{D'}) = \sum_J \iota^{YJ}_*([H^{n_1,\ldots,n_m}_J(L^{YJ}_1, \ldots, L^{YJ}_m)]_{D'}),$$

where $\iota^{YJ} : Y^J \to |\tilde{D}|$ is the inclusion, and L^{YJ}_i is the restriction of L_i to Y^J. Letting $\bar{i} : |\tilde{D}| \cap Y \to |\tilde{D}|$ be the inclusion, the right-hand side above is clearly the same as the class $\bar{i}_*([i_Y^*(\tilde{D}) \to |\tilde{D}| \cap Y]_{D'})$. Pushing this identity forward via f^D gives

$$\begin{aligned} D([Y \to X])_{D'} &= [i_Y^*(\tilde{D}) \to |D|]_{D'} \\ &= f^D_*([i_Y^*(\tilde{D}) \to |\tilde{D}|]_{D'}) \\ &= f^D_*(\tilde{c}_1(i^*O_W(Y))([\tilde{D} \to |\tilde{D}|]_{D'})). \end{aligned}$$

□

3.2. **Descent to $\Omega_*(X)_{D|D'}$.** Let X be a finite type k-scheme with pseudo-divisors D and D'. We proceed to show that intersection with a pseudo-divisor D descends to a homomorphism $D(-)_{D'} : \Omega_*(X)_{D|D'} \to \Omega_{*-1}(|D|)_{D'}$.

Lemma 3.5. *Let $f : Y \to X$ a projective morphism in $\mathcal{M}(X)_{D|D'}$, and let $L_1, \ldots , L_r$ be line bundles on Y with $r \geq \dim_k Y$. Then $D((Y \to X, L_1, \ldots , L_r))_{D'} = 0$ in $\Omega_{*-1}(|D|)_{D'}$.*

Proof. We may suppose Y to be irreducible. Write $\tilde{D}$ for f^*D, and let $f^D : |\mathrm{div}\tilde{D}| \to |D|$ be the restriction of f. Using lemmas 3.1 and 3.2, we have

$$D((f : Y \to X, L_1, \ldots , L_r))_{D'} = f^D_*\big(\tilde{c}_1(L_1) \circ \ldots \circ \tilde{c}_1(L_r)(\tilde{D}(1_Y^{D|D'}))\big).$$

If $f(Y) \subset |D|$, then $\tilde{D}(1_Y^{D|D'}) = \tilde{c}_1(O_Y(\tilde{D}))(1_Y^{D|D'})$, and thus

$$\tilde{c}_1(L_1) \circ \ldots \circ \tilde{c}_1(L_r)(\tilde{D}(1_Y^{D|D'})) = (\mathrm{id}_Y, L_1, \ldots , L_r, O_Y(\tilde{D})) = 0$$

in $\underline{\mathcal{Z}}_*(Y)_{D'}$. If $f(Y) \not\subset |D|$, then $\tilde{D}(1_Y^{D|D'}))$ is a sum of term of the form $a \cdot \iota^J_*((\tilde{D}^J, M_1, \ldots , M_s))$, with $a \in \Omega_*(k)$, $\iota^J : \tilde{D}^J \to Y$ the inclusion of a face of $\tilde{D}^J$, and the M_i line bundles on $\tilde{D}^J$. Thus $\tilde{c}_1(L_1) \circ \ldots \circ \tilde{c}_1(L_r)(\tilde{D}(1_Y^{D|D'}))$ is a sum of terms of the form

$$a \cdot \iota^J_*((\tilde{D}^J, M_1, \ldots , M_s, \iota^{J*}L_1, \ldots , \iota^{J*}L_r))$$

Since each face $\tilde{D}^J$ has dimension $< \dim_k Y$, the terms

$$(\tilde{D}^J, M_1, \ldots , M_s, \iota^{J*}L_1, \ldots , \iota^{J*}L_r)$$

vanish in $\Omega_*(\tilde{D}^J)_{D'}$ (use the relations $<\mathcal{R}^{Dim}_*>(\tilde{D}^J)_{D'}$), whence the result. □

Lemma 3.6. *Suppose that X is in $\mathbf{Sm}_k$. Let Z_1, Z_2 be smooth disjoint divisors on X, both in very good position with respect to $D|D'$.*

1. *Let $i_J : D^J \to X$ be the inclusion of a face of D. Then*
$$\tilde{c}_1(i_J^*O_X(Z_1)) \circ \tilde{c}_1(i_J^*O_X(Z_2))(1^{D'}_{D^J}) = 0$$
in $\Omega_*(D^J)_{D'}$.
2. *Let $i : |D| \to X$ be the inclusion. Then*
$$\tilde{c}_1(i^*O_X(Z_1)) \circ \tilde{c}_1(i^*O_X(Z_2))([D \to |D|]_{D'}) = 0$$
in $\Omega_*(|D|)_{D'}$.

Proof. For each J, let $\iota^J : D^J \to |D|$ be the inclusion. Write $D = \sum_{i=1}^r n_iD_i$, and let $\eta_J = \iota^J_*(\tilde{c}_1(i_J^*O_X(Z_1)) \circ \tilde{c}_1(i_J^*O_X(Z_2))(1^{D'}_{D^J}))$. Then

$$\begin{aligned}\tilde{c}_1(i^*O_X(Z_1)) &\circ \tilde{c}_1(i^*O_X(Z_2))([D \to |D|]_{D'}) \\ &= \sum_J H^{n_1,\ldots,n_r}_J(\tilde{c}_1(i^*O_X(D_1)), \ldots , \tilde{c}_1(i^*O_X(D_r)))(\eta_J).\end{aligned}$$

Thus (2) follows from (1).

For (1), let D^J_α be an irreducible component of D^J with inclusion $i_{J\alpha}: D^J_\alpha \to X$. Either $D^J_\alpha \subset Z_2$ or $Z_2 \cap D^J_\alpha$ is smooth and codimension one on D^J_α (or is empty). In the former case, $Z_1 \cap D^J_\alpha = \emptyset$, so $i^*_{J\alpha} O_X(Z_1) \cong O_{D^J_\alpha}$ and

$$\begin{aligned}\tilde{c}_1(i^*_{J\alpha} O_X(Z_1)) \circ \tilde{c}_1(i^*_{J\alpha} O_X(Z_2))(1^{D'}_{D^J_\alpha}) \\ &= \tilde{c}_1(i^*_{J\alpha} O_X(Z_2)) \circ \tilde{c}_1(i^*_{J\alpha} O_X(Z_1))(1^{D'}_{D^J_\alpha}) \\ &= \tilde{c}_1(i^*_{J\alpha} O_X(Z_2))((\tilde{c}_1(O_{D^J_\alpha})(1^{D'}_{D^J_\alpha})) \\ &= 0,\end{aligned}$$

using the relations $<\mathcal{R}^{Sect}_*>(D^J)_{D|D'}$ in the case of an empty divisor. If $D^J_\alpha \not\subset Z_2$, then, using the relations $<\mathcal{R}^{Sect}_*>(D^J)_{D|D'}$ for the smooth divisor $Z_2 \cap D^J_\alpha$ on D^J_α, we see that, letting $g: Z_2 \cap D^J_\alpha \to D^J_\alpha$ be the inclusion,

$$\begin{aligned}\tilde{c}_1(i^*_{J\alpha} O_X(Z_1)) \circ \tilde{c}_1(i^*_{J\alpha} O_X(Z_2))(1^{D'}_{D^J_\alpha}) \\ &= \tilde{c}_1(i^*_{J\alpha} O_X(Z_1))(g_*(1^{D'}_{Z_2 \cap D^J_\alpha})) \\ &= g_*(\tilde{c}_1(g^* i^*_{J\alpha} O_X(Z_1))(1^{D'}_{Z_2 \cap D^J_\alpha})) \\ &= 0,\end{aligned}$$

because $g^* i^*_{J\alpha} O_X(Z_1) \cong O_{Z_2 \cap D^J_\alpha}$.

Letting $\iota_\alpha : D^J_\alpha \to D^J$ be the inclusion, we have $1^{D'}_{D^J} = \sum_\alpha \iota^J_{\alpha *}(1^{D'}_{D^J_\alpha})$. Thus

$$\tilde{c}_1(i^*_J O_X(Z_1)) \circ \tilde{c}_1(i^*_J O_X(Z_2))(1^{D'}_{D^J}) = 0,$$

as desired. □

Finally, we need an extension of lemma 3.4.

Lemma 3.7. *Let W be in $\mathbf{Sm}_k$ and irreducible. Let $i_Y : Y \to W$ be an irreducible codimension one closed subscheme, smooth over k. Let D, D' be pseudo-divisors on W such that $W \neq |D|$ and let $i_D : |D| \to W$ be the inclusion. Suppose that $Y + D$ is a strict normal crossing divisor on W, in very good position with respect to D'. Then*

$$D([Y \to W])_{D'} = \tilde{c}_1(i^*_D O_W(Y))([D \to |D|]_{D'}) \tag{3.1}$$

in $\Omega_*(|D|)_{D'}$.

Proof. The condition that $Y + D$ is a strict normal crossing divisor, in very good position with respect to D' implies that $i_Y : Y \to W$ is in $\mathcal{M}(W)_{D|D'}$, and that D is in good position with respect to D'. Thus, all the terms in (3.1) are defined.

In case Y is not a component of D, Y is in general position with respect to $D|D'$, so the result follows from lemma 3.4.

Now suppose that Y is a component of D. Write $D = \sum_{i=1}^m n_i D_i$, with $Y = D_1$. Let $\iota_Y : Y \to |D|$, $\eta^J : D^J \cap Y \to Y$, $\tau^J : D^J \cap Y \to D^J$, $i_Y^J : D^J \cap Y \to W$, $\iota^J : D^J \to |D|$ and $i^J : D^J \to W$ be the inclusions. Since

$$\begin{aligned} n_1 \cdot_F u_1 +_F + \ldots +_F n_m \cdot_F u_m &= G^{n_1,\ldots,n_m}(u_1, \ldots, u_m) \\ &= \sum_J u^J H_J^{n_1,\ldots,n_m}(u_1, \ldots, u_m), \end{aligned}$$

we have

$$\tilde{c}_1(O_W(D)) = \sum_J \tilde{c}_1(O_W(D_*))^J H_J(\tilde{c}_1(O_W(D_1)), \ldots, \tilde{c}_1(O_W(D_m))),$$

where $H_J := H_J^{n_1,\ldots,n_m}$ and if $J = (j_1, \ldots, j_m)$, then

$$\tilde{c}_1(O_W(D_*))^J = \tilde{c}_1(O_W(D_1))^{j_1} \circ \ldots \circ \tilde{c}_1(O_W(D_m))^{j_m}.$$

Since we therefore have

$$\begin{aligned} &D([Y \to W])_{D'} = \tilde{c}_1(i_D^* O_W(D))([Y \to |D|]) \\ &\quad = \sum_J \iota_{Y*}\big(\tilde{c}_1(i_Y^* O_W(D_*))^J([H_J(i_Y^* O_W(D_1), \ldots, i_Y^* O_W(D_m))]_{D'})\big); \\ &\tilde{c}_1(i_D^* O_W(Y))([D \to |D|]_{D'}) \\ &\quad = \sum_J \iota_*^J\big(\tilde{c}_1(i^{J*} O_W(Y))([H_J(i^{J*} O_W(D_1), \ldots, i^{J*} O_W(D_m))]_{D'})\big), \end{aligned}$$

it suffices to prove that

$$\begin{aligned} (3.2) \quad &\iota_*^J\big(\tilde{c}_1(i^{J*} O_W(Y))([H_J(i^{J*} O_W(D_1), \ldots, i^{J*} O_W(D_m))]_{D'})\big) \\ &\quad = \iota_{Y*}\big(\tilde{c}_1(i_Y^* O_W(D_*))^J([H_J(i_Y^* O_W(D_1), \ldots, i_Y^* O_W(D_m))]_{D'})\big) \end{aligned}$$

in $\Omega_*(|D|)_{D'}$, for each index J.

Suppose that D^J is not contained in Y. Since $Y + D$ is a strict normal crossing divisor, the intersection $Y \cap D^J$ is transverse. By symmetry, we may assume that $J = (1, \ldots, 1, 0, \ldots, 0)$, with say s 1's. Applying the relations $<\mathcal{R}_*^{Sect}>_{D'}$ repeatedly, we see that

$$\begin{aligned} &\tilde{c}_1(i_Y^* O_W(D_*))^J\big([H_J(i_Y^* O_W(D_1), \ldots, i_Y^* O_W(D_m))]_{D'}\big) \\ &\qquad = \eta_*^J[H_J(i_Y^{J*} O_W(D_1), \ldots, i_Y^{J*} O_W(D_m))]_{D'}. \end{aligned}$$

Applying the same relations to the divisor $Y \cap D^J$ on D^J, we have

$$\begin{aligned} &\tau_*^J[H_J(i_Y^{J*} O_W(D_1), \ldots, i_Y^{J*} O_W(D_m)]_{D'} \\ &\qquad = \tilde{c}_1(i^{J*} O_W(Y))\big([H_J(i^{J*} O_W(D_1), \ldots, i^{J*} O_W(D_m))]_{D'}\big). \end{aligned}$$

Since $\iota^J_* \circ \tau^J_* = \iota_{Y*} \circ \eta^J_*$, these two identities yield the equality (3.2) in this case.

In case D^J is contained in $Y = D_1$, then $J = (1, j_2, \dots, j_m)$. Letting $J' = (0, j_2, \dots, j_m)$, we have $D^J = Y \cap D^{J'}$, and $D^{J'}$ is not contained in Y. Suppose that $J' \neq (0, \dots, 0)$. By the same argument as above, we have

$$\begin{aligned} &\iota^{J'}_*\big(\tilde{c}_1(i^{J'*}O_W(Y))([H_J(i^{J'*}O_W(D_1), \dots, i^{J'*}O_W(D_m))]_{D'})\big) \\ &\qquad = \iota_{Y*}\big(\tilde{c}_1(i^*_Y O_W(D_*))^{J'}\big([H_J(i^*_Y O_W(D_1), \dots, i^*_Y O_W(D_m))]_{D'}\big)\big) \end{aligned}$$

in $\Omega_*(|D|)_{D'}$. Letting $\bar{i} : D^J \to D^{J'}$ be the inclusion, the relations $<\mathcal{R}^{Sect}_*>_{D'}$ yield the identity

$$\begin{aligned} &\tilde{c}_1(i^{J'*}O_W(Y))([H_J(i^{J'*}O_W(D_1), \dots, i^{J'*}O_W(D_m))]_{D'}) \\ &\qquad\qquad = \bar{i}_*[H_J(i^{J*}O_W(D_1), \dots, i^{J*}O_W(D_m))]_{D'} \end{aligned}$$

in $\Omega_*(D^{J'})_{D'}$. Thus,

$$\begin{aligned} &\iota^J_*\big(\tilde{c}_1(i^{J*}O_W(Y))([H_J(i^{J*}O_W(D_1), \dots)]_{D'})\big) \\ &\quad = \iota^{J'}_*\big(\tilde{c}_1(i^{J'*}O_W(Y))^2([H_J(i^{J'*}O_W(D_1), \dots)]_{D'})\big) \\ &\quad = \tilde{c}_1(i^*_D O_W(Y))\big(\iota^{J'}_*\big(\tilde{c}_1(i^{J'*}O_W(Y))([H_J(i^{J'*}O_W(D_1), \dots)]_{D'})\big)\big) \\ &\quad = \tilde{c}_1(i^*_D O_W(Y))\big(\iota_{Y*}(\tilde{c}_1(i^*_Y O_W(D_*))^{J'}([H_J(i^*_Y O_W(D_1), \dots)]_{D'}))\big) \\ &\quad = \iota_{Y*}\big((\tilde{c}_1(i^*_Y O_W(D_*))^J([H_J(i^*_Y O_W(D_1), \dots)]_{D'})\big), \end{aligned}$$

verifying (3.2). If $J' = (0, \dots, 0)$, then $J = (1, 0, \dots, 0)$, $D^J = D_1 = Y$ and $H_J(u_1, \dots, u_m) = n_1$. Thus in $\Omega_*(Y)_{D'}$, we have

$$\begin{aligned} &[H_J(\tilde{c}_1(i^*_Y O_W(D_1)), \dots, i^*_Y O_W(D_1)))]_{D'} \\ &\qquad\qquad = n_1 \cdot 1^{D'}_Y \\ &\qquad\qquad\quad = [H_J(i^{J*}O_W(D_1), \dots, i^{J*}O_W(D_m))]_{D'}. \end{aligned}$$

Similarly, $\tilde{c}_1(i^{J*}O_W(D_*))^J = \tilde{c}_1(i^*_Y O_W(Y))$, which yields (3.2). This finishes the proof. □

We now show that $D(-)_{D'}$ descends to $\Omega_*(X)_{D|D'}$ in a series of steps.

Step 1: the descent to $\underline{\mathcal{Z}}_*(X)_{D|D'}$. Let $\pi : Y \to Z$ be a smooth morphism with Z and Y in $\mathbf{Sm}_k$, $L_1, \dots, L_r$ line bundles on Z with $r > \dim_k Z$, and $f : Y \to X$ a projective morphism in $\mathcal{M}(X)_{D|D'}$, with Y irreducible. Using lemmas 3.1 and 3.2, it suffices to show that $(f^*D)(\mathrm{id}_Y, \pi^*L_1, \dots, \pi^*L_r)_{D'} = 0$ in $\Omega_*(|f^*D|)_{D'}$. Changing notation, we may assume that $X = Y$, and that either D is a strict normal crossing divisor on Y, or $|D| = Y$. We need to show that $D(\mathrm{id}_Y, \pi^*L_1, \dots, \pi^*L_r)_{D'} = 0$ in $\Omega_*(|D|)_{D'}$.

If $|D| = Y$, then

$$\begin{aligned} D(\mathrm{id}_Y, \pi^*L_1, \dots, \pi^*L_r)_{D'} &= \tilde{c}_1(O_Y(D))((\mathrm{id}_Y, \pi^*L_1, \dots, \pi^*L_r)) \\ &= (\mathrm{id}_Y, \pi^*L_1, \dots, \pi^*L_r, O_Y(D)), \end{aligned}$$

which is zero in $\Omega_*(|D|)_{D'} = \Omega_*(Y)_{D'}$ by the relations $<\mathcal{R}_*^{Dim}>(Y)_{D'}$.

If D is a strict normal crossing divisor on Y, with inclusion $i : |D| \to Y$, then

$$\begin{aligned} &D(\mathrm{id}_Y, \pi^*L_1, \dots, \pi^*L_r)_{D'} \\ &\qquad = \tilde{c}_1((\pi \circ i)^*L_1) \circ \dots \circ \tilde{c}_1((\pi \circ i)^*L_r)([D \to |D|]_{D'}). \end{aligned}$$

To see that this class vanishes, apply lemma 2.12 to $\pi \circ i : |D| \to Z$.

Step 2: the descent to $\underline{\Omega}_*(X)_{D|D'}$. Let $f : Y \to X$ be in $\mathcal{M}(X)_{D|D'}$, and let $Z \to Y$ be a codimension one smooth closed subscheme in very good position with respect to $D|D'$. Let $i : |D| \to Y$ be the inclusion. We may suppose that Y is irreducible. As in step I, we reduce to the case $X = Y$. Also, it suffices to show that

$$D([O_Y(Z)]_{D|D'})_{D'} = D([Z \to Y])_{D'}.$$

Write the divisor Z as a sum $Z = \sum_{i=1}^r Z_i$. Since Z is smooth, we have $Z_i \cap Z_j = \emptyset$ for $i \neq j$. Let $Z' = \sum_{i=2}^r Z_i$. Using lemma 3.2 and the relations $<\mathcal{R}_*^{FGL}>(|D|)_{D'}$, we have

$$\begin{aligned} D([O_Y(Z)]_{D|D'})_{D'} &= D(\tilde{c}_1(O_Y(Z))(1_Y^{D|D'}))_{D'} \\ &= \tilde{c}_1(i^*O_Y(Z))(D(1_Y^{D|D'})_{D'}) \\ &= F_{\mathbb{L}}(\tilde{c}_1(O_Y(Z_1)), \tilde{c}_1(O_Y(Z')))(D(1_Y^{D|D'})_{D'}). \end{aligned}$$

Since $F_{\mathbb{L}}(u, v) = u + v + \sum_{i,j \geq 1} a_{ij} u^i v^j$, it follows from lemma 3.6 that

$$\begin{aligned} &F_{\mathbb{L}}(\tilde{c}_1(i^*O_Y(Z_1)), \tilde{c}_1(i^*O_Y(Z')))(D(1_Y^{D|D'})_{D'}) \\ &\qquad = \tilde{c}_1(i^*O_Y(Z_1))(D(1_Y^{D|D'})_{D'}) + \tilde{c}_1(i^*O_Y(Z'))(D(1_Y^{D|D'})_{D'}) \\ &\qquad = D([O_Y(Z_1)]_{D|D'})_{D'} + D([O_Y(Z')]_{D|D'})_{D'}. \end{aligned}$$

Thus, by induction, we have

$$D([O_Y(Z)]_{D|D'})_{D'} = \sum_{i=1}^r D([O_Y(Z_i)]_{D|D'})_{D'}.$$

Since $[Z \to Y]_{D'} = \sum_i [Z_i \to Y]_{D'}$, we reduce to the case of an irreducible Z.

If $Z \not\subset |D|$, then Z is in general position with respect to D. Using lemma 3.2 and lemma 3.4, we have

$$\begin{aligned} D([O_Y(Z)]_{D|D'})_{D'} &= \tilde{c}_1(i^*O_Y(Z))(D(1_Y^{D|D'})) \\ &= \tilde{c}_1(i^*O_Y(Z))([D \to |D|]_{D'}) \\ &= D([Z \to Y])_{D'}, \end{aligned}$$

as desired. If $Z \subset |D|$, then the same argument, using lemma 3.7 in place of lemma 3.4, yields the desired identity.

Step 3: the descent to $\Omega_*(X)_{D|D'}$. Let $f : Y \to X$ be in $\mathcal{M}(X)_{D|D'}$ and let L and M be line bundles on Y. It suffices to show that

$$D([F_{\mathbb{L}}(L, M)])_{D'} = D([L \otimes M])_{D'}.$$

For this, let $\tilde{D} = f^*D$, let $f^D : |\tilde{D}| \to |D|$ be the morphism induced by f and let $i : |\tilde{D}| \to Y$ be the inclusion. Using the relations $<\mathcal{R}_*^{FGL}>(|f^*D|)_{D'}$ and lemma 3.3, we have

$$\begin{aligned} D([F_{\mathbb{L}}(L, M)])_{D'} &= D(F_{\mathbb{L}}(\tilde{c}_1(L), \tilde{c}_1(M))(1_Y^{D|D'}))_{D'} \\ &= f_*^D(F_{\mathbb{L}}(\tilde{c}_1(i^*L), \tilde{c}_1(i^*M))(\tilde{D}(1_Y^{D|D'})) \\ &= f_*^D(\tilde{c}_1(i^*L \otimes i^*M)(\tilde{D}(1_Y^{D|D'})) \\ &= D([L \otimes M])_{D'}, \end{aligned}$$

which finishes the descent to $\Omega_*(X)_{D|D'}$.

4. *Intersection with a divisor II*

We establish some of the basic properties of the operation $D(-)_{D'}$.

4.1. **Commutativity.**

The first important property is the commutativity of intersection. We begin with some preliminary results.

Lemma 4.1. *Let D and D' be pseudo-divisors on some finite type k-scheme X, and let $i : |D| \to X$ be the inclusion. Then, for $\eta \in \Omega_*(X)_{D|D'}$, $i_*(D(\eta)_{D'}) = \tilde{c}_1(O_X(D))(\eta)$ in $\Omega_*(X)_{D'}$.*

Proof. It suffices to consider the case of $\eta = [f : Y \to X]$ for some $f \in \mathcal{M}(X)_{D|D'}$, with Y irreducible. If $f(Y) \subset |D|$, then $D(f) = \tilde{c}_1(i^*O_X(D))([Y \to |D|]_{D'})$, from which the desired formula follows directly. If $f(Y) \not\subset |D|$, then $D(f) = f_*^D([f^*D \to |f^*D|]_{D'})$, where $f^D : |f^*D| \to |D|$ is the map induced by f. By remark 2.10, we have

$[f^*D \to Y]_{D'} = \tilde{c}_1(O_Y(f^*D))(1_Y^{D|D'})$. Thus

$$\begin{aligned} i_*(D(f)) &= f_*([f^*D \to Y]_{D'}) \\ &= f_*(\tilde{c}_1(f^*O_X(D))(1_Y^{D|D'})) \\ &= \tilde{c}_1(O_X(D))(f). \end{aligned}$$

□

Write $F_{\mathbb{L}}(u,v) = u + v + uvH_{11}(u,v)$. Let $F_{11}(u,v) = vH_{11}(u,v)$.

Lemma 4.2. *Let D, D' be pseudo-divisors on $W \in \mathbf{Sm}_k$. Suppose that D is a strict normal crossing divisor, in good position with respect to D'. Write $D = D_0 + D_1$, with $D_0 > 0$, $D_1 > 0$ and D_1 smooth. Let $i : |D| \to W$ be the inclusion.*

1. *We have the identity*

$$\begin{aligned} [D \to |D|]_{D'} = [D_0 \to |D|]_{D'} &+ [D_1 \to |D|]_{D'} \\ &+ F_{11}(\tilde{c}_1(i^*O_W(D_1)), \tilde{c}_1(i^*O_W(D_0)))([D_1 \to |D|]_{D'}). \end{aligned}$$

2. *Let $f : Y \to W$ be in $\mathcal{M}(W)_{D|D'} \cap \mathcal{M}(W)_{D_0|D'} \cap \mathcal{M}(W)_{D_1|D'}$. Let $i_0^Y : |f^*D_0| \to |f^*D|$, $i_1^Y : |f^*D_1| \to |f^*D|$ be the inclusions, and let $\bar{f} : |f^*D_1| \to W$ be the induced morphism. Suppose that Y is irreducible and either $f(Y) \subset |D_1|$ or f^*D_1 is a smooth divisor on Y. Then*

$$\begin{aligned} (f^*D)(1_Y^{D|D'})_{D'} = i_{0*}^Y((f^*D_0)(1_Y^{D_0|D'})_{D'}) &+ i_{1*}^Y((f^*D_1)(1_Y^{D_1|D'})_{D'}) + \\ i_{1*}^Y\big(F_{11}\big(\tilde{c}_1(\bar{f}^*O_W(D_1)), \tilde{c}_1(\bar{f}^*O_W(D_0))\big)&((f^*D_1)(1_Y^{D_1|D'})_{D'})\big) \end{aligned}$$

Proof. For the second assertion, suppose first that $f(Y) \subset |D|$. Let $i : |D| \to W$ be the inclusion. Then by definition

$$(f^*D)(1_Y^{D|D'})_{D'} = \tilde{c}_1(O_Y(f^*D))(1_Y^{D'}).$$

Since $D = D_0 + D_1$, we have

$$\begin{aligned} \tilde{c}_1(O_Y(f^*D)) &= F_{\mathbb{L}}(\tilde{c}_1(O_Y(f^*D_1)), \tilde{c}_1(O_Y(f^*D_0)) \\ &= \tilde{c}_1(O_Y(f^*D_1)) + \tilde{c}_1(O_Y(f^*D_0)) \\ &\quad + \tilde{c}_1(O_Y(f^*D_1))F_{11}(\tilde{c}_1(O_Y(f^*D_1)), \tilde{c}_1(O_Y(f^*D_0)). \end{aligned}$$

By lemma 4.1, $i_{j*}^Y((f^*D_j)(1_Y^{D|D'})_{D'}) = \tilde{c}_1(O_Y(f^*D_j))(1_Y^{D'})$, $j = 0, 1$. Thus

$$\begin{aligned} (f^*D)(1_Y^{D|D'})_{D'} = i_{0*}^Y((f^*D_0)(1_Y^{D|D'})_{D'}) &+ i_{1*}^Y((f^*D_1)(1_Y^{D|D'})_{D'}) \\ &+ \tilde{c}_1(O_Y(f^*D_1)) \circ (F_{11}(\tilde{c}_1(f^*O_W(D_1)), \tilde{c}_1(f^*O_W(D_0))(1_Y^{D'}). \end{aligned}$$

Using lemma 4.1 again, we have

$$\begin{aligned}
&\tilde{c}_1(f^*O_W(D_1)) \circ (F_{11}(\tilde{c}_1(f^*O_W(D_1)), \tilde{c}_1(f^*O_W(D_0))(1_Y^{D'})\\
&= F_{11}(\tilde{c}_1(f^*O_W(D_1)), \tilde{c}_1(f^*O_W(D_0)))\big(\tilde{c}_1(O_Y(f^*D_1))(1_Y^{D'})\big)\\
&= i^Y_{1*}\big(F_{11}(\tilde{c}_1(\bar{f}^*O_W(D_1)), \tilde{c}_1(\bar{f}^*O_W(D_0)))((f^*D_1)(1_Y^{D|D'}))\big).
\end{aligned}$$

This verfies the second assertion in this case.

If $f(Y) \not\subset |D|$, the second assertion is a consequence of the first. Indeed, in this case, f^*D is a strict normal crossing divisor on Y, in good position with respect to D', and thus $(f^*D)(1_Y^{D|D'})$ is $[f^*D \to |f^*D|]_{D'}$. Thus, applying the first assertion to $f^*D = f^*D_0 + f^*D_1$, we have

$$\begin{aligned}
&(f^*D)(1_Y^{D|D'}) = [f^*D \to |f^*D|]_{D'}\\
&= i^Y_{0*}[f^*D_0 \to |f^*D_0|]_{D'} + i^Y_{1*}[f^*D_1 \to |f^*D_1|]_{D'}\\
&\quad + i^Y_{1*}\big(F_{11}(\tilde{c}_1^{D'}(i_1^*O_Y(f^*D_1)), \tilde{c}_1(i_1^*O_Y(f^*D_0)))([f^*D_1 \to |f^*D_1|]_{D'})\big)\\
&= i^Y_{0*}(D_0(1_Y^{D_0|D'})_{D'}) + i^Y_{1*}(D_1(1_Y^{D_1|D'})_{D'})\\
&\quad + i_{1*}\big(F_{11}(\tilde{c}_1(\bar{f}^*O_W(D_1)), \tilde{c}_1(\bar{f}^*O_W(D_0)))(D_1(1_Y^{D_1|D'})_{D'})\big).
\end{aligned}$$

To prove (1), we first reduce to the case of irreducible D_1. Write $D_1 = E_1 + D_{11}$, with $E_1 > 0$ and D_{11} irreducible. Since D_{11} and E_1 are disjoint, $\tilde{c}_1(i^*_{D_{11}}O_W(E_1))([D_{11} \to |D_{11}|]_{D'}) = 0$. Thus

$$\begin{aligned}
&F_{11}(\tilde{c}_1(O_W(D_{11})), \tilde{c}_1(O_W(D_0 + E_1))([D_{11} \to |D_{11}|]_{D'})\\
&\quad = F_{11}(\tilde{c}_1(O_W(D_{11})), F_{\mathbb{L}}(\tilde{c}_1(O_W(D_0), \tilde{c}_1(E_1)))([D_{11} \to |D_{11}|]_{D'})\\
&\quad = F_{11}(\tilde{c}_1(O_W(D_{11})), \tilde{c}_1(O_W(D_0))([D_{11} \to |D_{11}|]_{D'})
\end{aligned}$$

(we omit the pull-back of the line bundles here and for the remainder of the argument to simplify the notation). Similarly,

$$\begin{aligned}
&F_{11}(\tilde{c}_1(O_W(E_1 + D_{11})), \tilde{c}_1(O_W(D_0)))([D_1 \to |D|]_{D'})\\
&\quad = F_{11}(\tilde{c}_1(O_W(E_1) + \tilde{c}_1(O_W(D_{11})), \tilde{c}_1(O_W(D_0)))([D_1 \to |D|]_{D'})\\
&\quad = F_{11}(\tilde{c}_1(O_W(E_1), \tilde{c}_1(O_W(D_0)))([E_1 \to |D|]_{D'})\\
&\qquad\quad + F_{11}(\tilde{c}_1(O_W(D_{11}), \tilde{c}_1(O_W(D_0)))([D_{11} \to |D|]_{D'}),
\end{aligned}$$

using lemma 3.6. With these formulas, one easily shows that (1) for the decompositions $E = D_0 + E_1$, $D = E + D_{11}$ and $D_1 = E_1 + D_{11}$ implies (1) for $D = D_0 + D_1$.

We now assume D_1 irreducible. Write $D = \sum_{i=1}^m n_i D_i$, with the D_i irreducible. For each face D^J of D properly contained in D_1, let $i_1^J : D^J \to D_1$ be the inclusion.

Let F_n denote the n-fold sum in the formal group $(F_{\mathbb{L}}, \mathbb{L}_*)$. The identity $F_{\mathbb{L}}(u_1, F_{n-1}(u_2, \ldots, u_n)) = F_n(u_1, \ldots, u_n)$ gives us the identity

$$\sum_J u^J H_J^{n_1,\ldots,n_m}(u_1, \ldots, u_m) = u_1 + V + u_1 V H_{11}(u_1, V)$$

where

$$V = \begin{cases} G^{n_1-1,\ldots,n_m}(u_1, \ldots, u_m) & \text{if } n_1 > 1, \\ G^{n_2,\ldots,n_m}(u_2, \ldots, u_m) & \text{if } n_1 = 1. \end{cases}$$

Thus

$$V = \begin{cases} \sum_{J'} u^{J'} H_{J'}^{n_1-1,n_2,\ldots,n_m}(u_1, u_2, \ldots, u_m) & \text{if } n_1 > 1, \\ \sum_{J'} u^{J'} H_{J'}^{n_2,\ldots,n_m}(u_2, \ldots, u_m) & \text{if } n_1 = 1, \end{cases}$$

where $\sum_{J'}$ is over all faces of D_0. We write $H^{0,n_2,\ldots,n_m}_{(0,j_2,\ldots,j_m)}(u_1, u_2, \ldots, u_m)$ for $H^{n_2,\ldots,n_m}_{(j_2,\ldots,j_m)}(u_2, \ldots, u_m)$ and set $H^{0,n_2,\ldots,n_m}_{(j_1,\ldots,j_m)} = 0$ if $j_1 \neq 0$.

Write $u_1 V H_{11}(u_1, V) = u_1 F_{11}(u_1, V)$ as the sum

$$u_1 V H_{11}(u_1, V) = \sum_K u^K H'_K(u_1, \ldots, u_m),$$

where the sum is over all faces K with $0 \le k_i \le 1$, $k_1 = 1$ (if $n_1 = 1$ $H'_K = 0$ unless $\sum_i k_i \ge 2$). For each index K of this form, we have

$$\begin{aligned} &H_K^{n_1,\ldots,n_m}(u_1, \ldots, u_m) \\ &\qquad = H_K^{n_1-1,n_2,\ldots,n_m}(u_1, u_2, \ldots, u_m) + H'_K(u_1, \ldots, u_m). \end{aligned}$$

Refering to the definition of $[D \to |D|]_{D'}$, we thus need to show

$$\begin{aligned} &\sum_{K=(1,k_2,\ldots,k_m)} \iota_*^K H'_K(\tilde{c}_1(O(D_1)), \ldots, \tilde{c}_1(O(D_m)))(1_{D^K}^{D'}) \\ &\qquad = F_{11}(\tilde{c}_1(O(D_1)), \tilde{c}_1(O(D_0)))([D_1 \to |D|]_{D'}). \end{aligned} \tag{4.1}$$

For each $K = (1, k_2, \ldots, k_m)$, let

$$\tilde{c}_1(O(D_*))^{K-1} = \tilde{c}_1(O(D_2))^{k_2} \circ \cdots \circ \tilde{c}_1(O(D_m))^{k_m}.$$

By repeated applications of the relations $<\mathcal{R}_*^{Sect}>_{D'}$, we have

$$\begin{aligned} &i_{1*}^K\big(H'_K(\tilde{c}_1(O(D_1)), \ldots, \tilde{c}_1(O(D_m)))(1_{D^K}^{D'})\big) \\ &\qquad = \big(H'_K(\tilde{c}_1(O(D_1)), \ldots, \tilde{c}_1(O(D_m))) \circ \tilde{c}_1(O(D_*))^{K-1}\big)(1_{D_1}^{D'}). \end{aligned}$$

Thus, as $\iota_*^K = i_{1*} \circ i_{1*}^K$, we have

$$\sum_K \iota_*^K \big(H'_K(\tilde{c}_1(O(D_1)), \ldots, \tilde{c}_1(O(D_m)))(1^{D'}_{D^K})\big)$$
$$= \sum_K i_{1*}\big(\tilde{c}_1(O(D_*))^{K-1} H'_K(\tilde{c}_1(O(D_1)), \ldots, \tilde{c}_1(O(D_m)))(1^{D'}_{D_1})\big)$$
$$= i_{1*}\big(F_{11}(\tilde{c}_1(O(D_1)), V(\tilde{c}_1(O(D_1)), \ldots, \tilde{c}_1(O(D_m))))(1^{D'}_{D_1})\big)$$
$$= F_{11}(\tilde{c}_1(O(D_1)), \tilde{c}_1(O(D_0)))([D_1 \to |D|]_{D'}).$$

This verifies the identity (4.1), completing the proof. □

Proposition 4.3 (Commutativity)**.** *Let D'' be a pseudo-divisor on a finite type k-scheme X. Let $f : T \to X$ be in $\mathcal{M}(X)$, and let D, D' be effective strict normal crossing divisors on T. Suppose that*

1. *$D + D'$ is a strict normal crossing divisor on T.*
2. *D is in good position with respect to $D'|D''$.*
3. *D' is in good position with respect to $D|D''$.*

Let $i_D : |D| \to T$ and $i_{D'} : |D'| \to T$ be the inclusions. Then

$$(i_{D'}^* D)([D' \to |D'|]_{D|D''})_{D''} = (i_D^* D')([D \to |D|]_{D'|D''})_{D''}$$

in $\Omega_*(|D| \cap |D'|)_{D''}$.

Proof. Write $D = \sum_i n_i D_i$ with each D_i irreducible, and similarly $D' = \sum_j n'_j D'_j$. We show more generally that

$$(i_{E'}^* E)([E' \to |E'|]_{D|D''})_{D''} = (i_E^* E')([E \to |E|]_{D'|D''})_{D''}$$

in $\Omega_*(|E| \cap |E'|)$ for all divisors $0 \le E \le D$, $0 \le E' \le D'$; if $E = \sum_i m_i D_i$ and $E' = \sum_j m'_j D'_j$, we may proceed by induction on $m := \sum_i m_i$ and $m' := \sum_i m'_i$.

Suppose $m = m' = 1$; we may suppose $E = D_1$ and $E' = D'_1$. If $D_1 \neq D'_1$, then $(i_{D_1}^* D'_1)([D_1 \to |D_1|]_{D'|D''})_{D''}$ is the element $1^{D''}_{D_1 \cap D'_1}$ in $\Omega_*(|D_1| \cap |D'_1|)_{D''}$, as is $(i_{D'_1}^* D_1)([D'_1 \to |D'_1|]_{D|D''})_{D''}$. If $D_1 = D'_1$, then $D_1([D'_1 \to |D'_1|]_{D|D''})_{D''}$ and $D'_1([D_1 \to |D_1|]_{D'|D''})_{D''}$ are obviously the same in $\Omega_*(|D_1| \cap |D'_1|)_{D''}$

In the general case, we may assume that D_1 is a component of E. Let $E_0 = E - D_1$. By symmetry, it suffices to induct on m and assume the result for the pairs E_0, E' and D_1, E'. Thus

$$(i_{E'}^* E_0)([E' \to |E'|]_{D|f^*D''}) = (i_{E_0}^* E')([E_0 \to |E_0|]_{D'|D''})_{D''}$$

in $\Omega_*(|E_0| \cap |E'|)_{D''}$ and

$$(i_{E'}^* D_1)([E' \to |E'|]_{D|D''})_{D''} = (i_{D_1}^* E')([D_1 \to |D_1|]_{D'|D''})_{D''}$$

in $\Omega_*(|D_1| \cap |E'|)_{D''}$.

The divisor class $[E' \to |E'|]_{D|D''}$ is an $\Omega_*(k)$-linear combination of maps of the form $g : Y \to |E'|$, with $g(Y) \subset |D_1|$, or $g^*(D_1)$ smooth, so we may apply lemma 4.2(2) to give

$$\begin{aligned}&(i^*_{E'}E)([E' \to |E'|]_{D|D''})_{D''} = \\ &i_{0*}\big((i^*_{E'}E_0)([E' \to |E'|]_{D|D''})_{D''}\big) + i_{1*}\big((i^*_{E'}D_1)([E' \to |E'|]_{D|D''})_{D''}\big) \\ &\ + i_{1*}\big(F_{11}(\tilde{c}_1(O_W(D_1)), \tilde{c}_1(O_W(E_0)))((i^*_{E'}D_1)([E' \to |E'|]_{D|D''})_{D''}\big),\end{aligned}$$

where $i_0 : |E_0| \cap |E'| \to |E| \cap |E'|$, $i_1 : |D_1| \cap |E'| \to |E| \cap |E'|$ are the inclusions. Using our induction hypothesis, together with lemma 3.3, we have

$$\begin{aligned}(i^*_{E'}&E)([E' \to |E'|]_{D|D''})_{D''} \\ &= (i^*_E E')([E_0 \to |E|]_{D'|D''})_{D''} + (i^*_E E')([D_1 \to |E|]_{D'|D''})_{D''} \\ &\quad + F_{11}(\tilde{c}_1(O_W(D_1)), \tilde{c}_1(O_W(E_0)))((i^*_E E')([D_1 \to |E|]_{D'|D''})_{D''}) \\ &= (i^*_E E')([E_0 \to |E|]_{D'|D''})_{D''} + (i^*_E E')([D_1 \to |E|]_{D'|D''})_{D''} \\ &\quad + (i^*_E E')(F_{11}(\tilde{c}_1(O_W(D_1)), \tilde{c}_1(O_W(E_0)))([D_1 \to |E|]_{D'|D''}))_{D''} \\ &= (i^*_E E')\big([E_0 \to |E|]_{D'|D''} + [D_1 \to |E|]_{D'|D''} \\ &\quad + F_{11}(\tilde{c}_1(O_W(D_1)), \tilde{c}_1(O_W(E_0)))([D_1 \to |E|]_{D'|D''})\big)_{D''} \\ &= (i^*_E E')([E \to |E|]_{D'|D''})_{D''},\end{aligned}$$

the last identity following from lemma 4.2(1). □

4.2. **Linear equivalent pseudo-divisors.**

We show how to relate the operations $D_0(-)_D$ and $D_1(-)_D$ for linearly equivalent pseudo-divisors D_0 and D_1.

Proposition 4.4. *Let W be in $\mathbf{Sm}_k$, with pseudo-divisors D_0, D_1 and D, such that $O_W(D_0) \cong O_W(D_1)$. Let $i_j : |D_j| \to W$ be the inclusion, $j = 0, 1$.*

1. *Let η be in $\mathcal{Z}_*(W)_{D_j|D}$, $j = 0, 1$. Then $i_{0*}(D_0(\eta)_D) = i_{1*}(D_1(\eta)_D)$ in $\Omega_*(W)_D$.*
2. *Let E be a strict normal crossing divisor on W, in good position with respect to $D_j|D$, $j = 0, 1$. Then*

$$i_{0*}(D_0([E \to |E|]_{D_0|D})_D) = i_{1*}(D_1([E \to |E|]_{D_1|D})_D) \in \Omega_*(W)_D.$$

Proof. Clearly (2) is a special case of (1). To prove (1), the operations $D_j(-)$ are compatible with the Chern class operators $\tilde{c}_1(L)$. Thus, it suffice to prove (1) for the case of $f : Y \to W$ in $\mathcal{M}(W)_{D_j|D}$, $j = 0, 1$. Using lemma 3.1, we reduce to the case $Y = W$, $f = \mathrm{id}$. But by lemma 4.1

$$i_{0*}(D_0(1_W^{D_0|D})_D) = [O_W(D_0)]_D = [O_W(D_1)]_D = i_{1*}(D_1(1_W^{D_1|D})_D).$$

□

5. *A moving lemma*

The next step is to show that the map $\Omega_*(X)_D \to \Omega_*(X)$ is an isomorphism.

5.1. **Distinguished liftings.** Given a finite type k-scheme X with a pseudo-divisor D, we give a method for lifting elements of $\mathcal{Z}_*(X)$ to $\Omega_*(X)_D$.

Lemma 5.1. *Let Y be in $\mathbf{Sm}_k$ and let $\tilde{D}$ be an effective divisor on Y. Then there is a projective birational morphism $\rho : W \to Y \times \mathbb{P}^1$, with $W \in \mathbf{Sm}_k$, such that*

(5.1)

1. *The fundamental locus of ρ is contained in $|\tilde{D}| \times 0$.*
2. *The proper transforms to W of $Y \times 0$ and $|\tilde{D}| \times \mathbb{P}^1$, denoted $<Y \times 0>$ and $<|\tilde{D}| \times \mathbb{P}^1>$ respectively, are disjoint.*
3. *$<Y \times 0>$ is smooth. Letting E be the exceptional divisor of ρ, $<Y \times 0> + E$ is a strict normal crossing divisor on W.*

Proof. We may assume that Y is irreducible. To construct such a ρ, first blow up $Y \times \mathbb{P}^1$ along the reduced subscheme $|\tilde{D}| \times 0$, forming the projective birational morphism

$$\rho_1 : W_1 \to Y \times \mathbb{P}^1.$$

Let $U = Y \times \mathbb{P}^1 \setminus |\tilde{D}| \times 0$. Since $|\tilde{D}| \times 0 = Y \times 0 \cap |\tilde{D}| \times \mathbb{P}^1$, the proper transforms of $Y \times 0$ and $|\tilde{D}| \times \mathbb{P}^1$ to W_1 are disjoint. By Hironaka [3, Main Theorem I*, pg.132], we may resolve the singularities of W_1 by a projective birational morphism $W_2 \to W_1$ which is an isomorphism over U. Let $\rho_2 : W_2 \to Y \times \mathbb{P}^1$ be the induced morphism, let E_2 be the exceptional divisor of ρ_2, and consider the Cartier divisor $D' := \rho_2^*(Y \times 0) + E_2$. Clearly $D' \cap \rho_2^{-1}(U)$ is a strict normal crossing divisor; by [4, Appendix, Corollary A.3], there is a projective birational morphism $\mu : W \to W_2$, with $W \in \mathbf{Sm}_k$, such that μ is an isomorphism over $\rho_2^{-1}(U)$, and with $\mu^* D' + E'$ a strict normal crossing divisor on W, where E' is the exceptional divisor of μ. Letting $\rho : W \to Y \times \mathbb{P}^1$ be the induced morphism, it is clear that ρ has all the necessary properties. □

Let X be a finite type k-scheme, and D a pseudo-divisor on X. Let $f : Y \to X$ be in $\mathcal{M}(X)$, with Y irreducible. Suppose that $f(Y) \not\subset |D|$. Then $\tilde{D} := \mathrm{div} f^* D$ is an effective Cartier divisor on Y. Take a projective birational morphism $\rho : W \to Y \times \mathbb{P}^1$ satisfying the conditions

(5.1). We claim that $\rho^*(Y\times 0)$ is in good position with respect to D. Indeed, E is supported in $|\rho^*p_1^*f^*D|$, and $\mathrm{div}\rho^*p_1^*D$ is supported in $|E|\cup <|p_1^*f^*D|>$. Since $<|p_1^*f^*D|>$ is disjoint from $<Y\times 0>$ and $<Y\times 0>+E$ is a strict normal crossing divisor, $<Y\times 0>+E$ is in good position with respect to D. Since $\rho^*(Y\times 0)$ has the same support as $<Y\times 0>+E$, $\rho^*(Y\times 0)$ is in good position with respect to D, as claimed.

Note that $\rho^*(Y\times 0)$ is linearly equivalent to $\rho^*(Y\times 1)\cong Y$, so $[f]=(p_1\circ\rho)_*([\rho^*(Y\times 0)\to W])$ in $\Omega_*(X)$. Thus $(p_1\circ\rho)_*([\rho^*(Y\times 0)\to W]_D)$ gives a lifting of f to an element of $\Omega_*(X)_D$.

Definition 5.2. (1) Given an element $f:Y\to X$ of $\mathcal{M}(X)$ with Y irreducible, $f(Y)\not\subset|D|$, and a birational morphism $\rho:W\to Y\times\mathbb{P}^1$ satisfying the conditions (5.1), we call the element

$$(f\circ p_1\circ\rho)_*([\rho^*(Y\times 0)\to W]_D)$$

of $\Omega_*(X)_D$ a *distinguished lifting* of $f\in\mathcal{M}(X)$. If $f(Y)\subset|D|$, a distinguished lifting of f is just $[f]\in\Omega_*(X)_D$.

(2) Let $\eta=(f:Y\to X,L_1,\ldots,L_r)$ be a cobordism cycle on X, with Y irreducible. Suppose that $f(Y)\not\subset|D|$. Choose $\rho:W\to Y\times\mathbb{P}^1$ as in (1), and let $\tilde{L}_i=(p_1\rho)^*L_i$. We call the element

$$(f\circ p_1\circ\rho)_*\big(\tilde{c}_1(\tilde{L}_1)\circ\ldots\circ\tilde{c}_1(\tilde{L}_r)([\rho^*(Y\times 0)\to W]_D)\big)$$

of $\Omega_*(X)_D$ a distinguished lifting of η. If $f(Y)\subset|D|$, then $[\eta]\in\Omega_*(X)_D$ is a distinguished lifting of η. We extend this notion to arbitrary elements of $\mathcal{Z}_*(X)$ by linearity.

Remark 5.3. Using the notation of definition 5.2(2), we can write a distinguished lifting of $(f:Y\to X,L_1,\ldots,L_r)$ as

$$f_*\big(\tilde{c}_1(L_1)\circ\ldots\circ\tilde{c}_1(L_r)((p_1\circ\rho)_*(([\rho^*(Y\times 0)\to W]_D)\big),$$

noting that $(p_1\circ\rho)_*([\rho^*(Y\times 0)\to W]_D)$ is in $\Omega_*(Y)_{f^*D}$.

Remark 5.4. Let T be a smooth projective k-scheme, and let $f:Y\to X$ be in $\mathcal{M}(X)$. If $\mu:W\to Y\times\mathbb{P}^1$ satisfies the conditions (5.1) for f^*D on Y, then clearly $\mathrm{id}_T\times\mu:T\times W\to T\times Y\times\mathbb{P}^1$ satisfies the conditions (5.1) for the divisor $(f\circ p_2)^*D$ on $T\times Y$. From this, it easily follows that, if $\tilde{\eta}$ is a distinguished lifting of some $\eta\in\mathcal{Z}_*(X)$, and α is in $\mathcal{Z}_*(k)$, then $\alpha\tilde{\eta}$ is a distinguished lifting of $\alpha\eta$.

Lemma 5.5. *Let η be in $\mathcal{Z}_*(X)$, and let η_1,η_2 be distinguished liftings of η. Then $\eta_1=\eta_2$ in $\Omega_*(X)_D$.*

Proof. First of all, we may assume that η is a cobordism cycle $(f:Y\to X,L_1,\ldots,L_r)$, with Y irreducible. Next, it follows from the formula in remark 5.3 that, if τ is a distinguished lifting of $(f, L_1, \dots, L_r)$, then there is a distinguished lifting $\tilde{\mathrm{id}}$ of the cobordism cycle $\mathrm{id}_Y \in \Omega_*(Y)_{f^*D}$ with

$$\tau = f_*(\tilde{c}_1(L_1) \circ \dots \circ \tilde{c}_1(L_r)(\tilde{\mathrm{id}}))$$

Thus, it suffices to consider the case of $X \in \mathbf{Sm}_k$, and to show that two distinguished liftings of $\mathrm{id}_X \in \mathcal{M}(X)$ agree in $\Omega_*(X)_D$.

We may assume that X is irreducible. If $|D| = X$, then $\mathcal{Z}_*(X) = \mathcal{Z}_*(X)_D$, $\Omega_*(X) = \Omega_*(X)_D$, and the unique distinguished lifting of id_X is the class of id_X in $\Omega_*(X)_D$. Thus, we may assume that D is a Cartier divisor on X. Let η_1 and η_2 be two distinguished liftings of id_X.

Suppose η_i is constructed via a birational morphism $\rho_i : W_i \to X \times \mathbb{P}^1$ satisfying (5.1), for $i = 1, 2$. Let Z_i be a subscheme of $X \times \mathbb{P}^1$, supported in $|D| \times 0$, such that W_i is the blow-up of $X \times \mathbb{P}^1$ along Z_i, $i = 1, 2$.

Let $T_1 \to X \times \mathbb{P}^1 \times \mathbb{P}^1$ be the blow-up along $Z_1 \times \mathbb{P}^1$, and let $<Z_2>$ denote the proper transform of $p_{13}^*(Z_2)$ to T_1. Let $T_2 \to T_1$ be the blow-up of T_1 along $<Z_2>$, with structure morphism $\phi : T_2 \to X \times \mathbb{P}^1 \times \mathbb{P}^1$.

We claim there is a blow-up of T_2 at a closed subscheme Z supported over $X \times 0 \times 0$, $T \to T_2$, such that T is smooth over k, and such that the divisor $X \times \mathbb{P}^1 \times 0 + X \times 0 \times \mathbb{P}^1$ pulls back to a strict normal crossing divisor on T, in good position with respect to D. To see this, let U be the open subscheme $T_2 \setminus \phi^{-1}(X \times 0 \times 0)$. We note that the pull-back of $X \times \mathbb{P}^1 \times 0 + X \times 0 \times \mathbb{P}^1$ to U is a strict normal crossing divisor, and the proper transform of $X \times \mathbb{P}^1 \times 0 + X \times 0 \times \mathbb{P}^1$ is disjoint from the proper transform of $D \times \mathbb{P}^1 \times \mathbb{P}^1$, after restricting to U. Arguing as in the proof of lemma 5.1, we construct a projective birational morphism $T \to T_2$, with T smooth over k and isomorphic to T_2 over U, such that $X \times \mathbb{P}^1 \times 0 + X \times 0 \times \mathbb{P}^1$ pulls back to a strict normal crossing divisor and the proper transform of $X \times \mathbb{P}^1 \times 0 + X \times 0 \times \mathbb{P}^1$ is disjoint from the proper transform of $D \times \mathbb{P}^1 \times \mathbb{P}^1$. This verifies our claim. We let $p : T \to X \times \mathbb{P}^1 \times \mathbb{P}^1$ be the induced morphism, and let $q : T \to X$ be p followed by the projection $p_X : X \times \mathbb{P}^1 \times \mathbb{P}^1 \to X$.

Clearly $p^{-1}(X \times \mathbb{P}^1 \times 1)$ is isomorphic to W_1, and $p^{-1}(X \times 1 \times \mathbb{P}^1)$ is isomorphic to W_2, as schemes over $X \times \mathbb{P}^1$. Let $D_0 = X \times \mathbb{P}^1 \times 0$ and $D_0' = X \times 0 \times \mathbb{P}^1$, $D_1 = X \times \mathbb{P}^1 \times 1$ and $D_1' = X \times 1 \times \mathbb{P}^1$. By our construction, we have the classes $[p^*D_0 \to T]_{D_i'|D}$, $[p^*D_0' \to T]_{D_i|D}$, $i = 0, 1$.

Let $i_j : |p^*D_j| \to T$, $i_j' : |p^*D_j'| \to T$, $j = 0, 1$, be the inclusions. We first note that

$$\eta_1 = (qi_1)_*\big((p^*D_1)([p^*D_0' \to T]_{D_1|D})_D\big).$$

Indeed, by proposition 4.3,

$$i_{1*}\big((p^*D_1)([p^*D'_0 \to T]_{D_1|D})_D\big) = i'_{0*}\big((p^*D'_0)([p^*D_1 \to T]_{D'_0|D})_D\big).$$

Let $j_1 : X \times 0 \to X \times \mathbb{P}^1$ be the inclusion. As $p^{-1}(X \times \mathbb{P}^1 \times 1)$ is isomorphic to W_1 over $X \times \mathbb{P}^1$, we have

$$\begin{aligned}(qi'_0)_*\big((p^*D'_0)([p^*D_1 \to T]_{D'_0|D})_D\big) &= (p_1j_1)_*\big((X \times 0)(W_1 \to X \times \mathbb{P}^1)_D\big)\\ &= \eta_1\end{aligned}$$

Similarly,

$$\eta_2 = (qi'_1)_*\big((p^*D'_1)([p^*D_0 \to T]_{D'_1|D})_D\big).$$

From proposition 4.4, we have

$$\begin{aligned}i_{0*}\big((p^*D_0)([p^*D'_0 \to T]_{D_0|D})_D\big) &= i_{1*}\big((p^*D_1)([p^*D'_0 \to T]_{D_1|D})_D\big)\\ i'_{0*}\big((p^*D'_0)([p^*D_0 \to T]_{D'_0|D})_D\big) &= i'_{1*}\big((p^*D'_1)([p^*D_0 \to T]_{D'_1|D})_D\big)\end{aligned}$$

in $\Omega_*(T)_D$. By proposition 4.3, we have

$$i'_{0*}\big((p^*D'_0)([p^*D_0 \to T]_{D'_0|D})_D\big) = i_{0*}\big((p^*D_0)([p^*D'_0 \to T]_{D_0|D})_D\big)$$

in $\Omega_*(T)_D$. Pushing forward to X, we have

$$\begin{aligned}\eta_1 &= (qi_1)_*\big((p^*D_1)([p^*D'_0 \to T]_{D_1|D})_D\big)\\ &= (qi'_1)_*\big((p^*D'_1)([p^*D_0 \to T]_{D'_1|D})_D\big)\\ &= \eta_2,\end{aligned}$$

as desired. □

Remark 5.6. Via this result, we may speak of *the* distinguished lifting of an element of $\mathcal{Z}_*(X)$ to $\Omega_*(X)_D$. We have the following properties of the distinguished lifting:

1. Sending $\eta \in \mathcal{Z}_*(X)$ to its distinguished lifting $\tilde\eta$ defines a $\mathcal{Z}_*(k)$-linear homomorphism $\mathcal{Z}_*(X) \to \Omega_*(X)_D$. The $\mathcal{Z}_*(k)$-linearity follows from remark 5.4.
2. Given $f : X' \to X$ projective, if $\tilde\eta \in \Omega_*(X')_{f^*D}$ is the distinguished lifting of $\eta \in \mathcal{Z}_*(X')$, then $f_*\tilde\eta \in \Omega_*(X)_D$ is the distinguished lifting of $f_*\eta$.
3. If $L_1, \ldots, L_r$ are line bundles on X, and $\tilde\eta$ is the distinguished lifting of $\eta \in \mathcal{Z}_*(X)$, then $\tilde c_1(L_1) \circ \ldots \circ \tilde c_1(L_r)(\tilde\eta)$ is the distinguished lifting of $\tilde c_1(L_1) \circ \ldots \circ \tilde c_1(L_r)(\tilde\eta)$.

These last two properties follows from the formula in remark 5.3.

We extend the distinguished lifting to $\mathbb{L}_* \otimes \mathcal{Z}_*(X)$ by linearity.

Lemma 5.7. *Let F be in $\mathbb{L}_*[[u_1, \dots, u_r]]$, let $f : W \to X$ be in $\mathcal{M}(X)$, and let $L_1, \dots, L_r$ be line bundles on W. Take an element $\eta \in \mathcal{Z}_*(W)$ and let $\tilde{\eta}$ be the distinguished lifting of η to $\Omega_*(W)_{f^*D}$. Let F_N denote the truncation of F after total degree N. Then, for all N sufficiently large, $f_*(F(\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_r))(\tilde{\eta}))$ is the distinguished lifting of $f_*(F_N(\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_r))(\eta))$.*

Proof. This follows from remark 5.6. □

Remark 5.8. As an application, consider the case of a strict normal crossing divisor $E = \sum_{j=1}^m n_j E_j$ on $Y \in \mathbf{Sm}_k$, with inclusion $i : |E| \to Y$, and a projective map $f : Y \to X$. The class $[E \to |E|] \in \Omega_*(|E|)$ is

$$\sum_J \iota^J_* H_J^{n_1,\dots,n_m}(\tilde{c}_1(O_Y(E_1)), \dots, \tilde{c}_1(O_Y(E_m)))(\mathrm{id}_{E^J}). \tag{5.2}$$

where $\iota^J : i^J : E^J \to Y$ is the inclusion. Let $\widetilde{[\mathrm{id}_{E^J}]}_D$ be the distinguished lifting of id_{E^J}, and let $[\widetilde{E \to |E|}]$ be a lifting of $[E \to |E|]$ to $\mathcal{Z}_*(Y)$ defined by truncating $H_J^{n_1,\dots,n_m}(u_1, \dots, u_m)$ after total degree N, for some $N > \dim_k E^J$. By remark 5.6 and lemma 5.7, the element

$$[\widetilde{E \to |E|}]_D := \sum_J \iota^J_* H_J^{n_1,\dots,n_m}(\tilde{c}_1(O_Y(E_1)), \dots, \tilde{c}_1(O_Y(E_m)))(\widetilde{[\mathrm{id}_{E^J}]}_D).$$

of $\Omega_*(|E|)_D$ is the distinguished lifting of $[\widetilde{E \to |E|}]$.

Lemma 5.9. *Let η be in $\mathcal{Z}_*(X)_D$, and let $\tilde{\eta}$ be the distinguished lifting of the image of η in $\mathcal{Z}_*(X)$. Then $\tilde{\eta}$ is the image of η in $\Omega_*(X)_D$ under the canonical homomorphism $\mathcal{Z}_*(X)_D \to \Omega_*(X)_D$.*

Proof. From the relations described in remark 5.6, we reduce to the case of $\eta = \mathrm{id}_X \in \mathcal{M}(X)_D$, with X irreducible and in $\mathbf{Sm}_k$. The case $|D| = X$ is evident, so assume that D is a strict normal crossing divisor on X. Let $\mu : W \to X \times \mathbb{P}^1$ be a projective birational morphism such that the conditions (5.1) are satisfied, and let $g = p_1 \circ \mu$. Then $g : W \to X$ is in $\mathcal{M}(X)_D$, and the divisors $\mu^*(X \times 1)$ and $\mu^*(X \times 0)$ are both in good position with respect to D. As both are divisors of sections of $\mu^*(O_{X \times \mathbb{P}^1}(1))$, we have

$$g_*([\mu^*(X \times 1) \to W]_D) = g_*([\mu^*(X \times 0) \to W]_D)$$

in $\Omega_*(X)_D$, by remark 2.10. As $g_*([\mu^*(X \times 1) \to W]_D) = \mathrm{id}_X$ and $g_*([\mu^*(X \times 0) \to W]_D)$ is the distinguished lifting of id_X, the result follows. □

5.2. **Lifting divisor classes.** Before proving the main moving lemma, we need some information on the distinguished lifting of divisor classes.

Let $f : Y \to X$ be in $\mathcal{M}(X)$, and let E be a strict normal crossing divisor on Y. Let D be a pseudo-divisor on X. Suppose that $f(Y) \not\subset |D|$, and that Y is irreducible. Let $\tilde{D} = \mathrm{div} f^*D$.

We apply the construction of §5.1: blow up $Y\times\mathbb{P}^1$ along a subscheme contained in $|\tilde{D}| \times 0$, forming the scheme $\rho : T \to Y \times \mathbb{P}^1$, satisfying the conditions (5.1). Note that $E \times \mathbb{P}^1 + Y \times 0$ is a strict normal crossing divisor on $Y \times \mathbb{P}^1$. By [4, corollary A.3], we may blow up T along smooth centers over $|\tilde{D}| \times 0$, forming the scheme $\tau : \tilde{T} \to Y \times \mathbb{P}^1$ which satisfies the conditions (5.1) and in addition has the property that $\tau^*(E \times \mathbb{P}^1 + Y \times 0)$ is a strict normal crossing divisor. We let $<Y \times 0>$ denote the proper transform of $Y \times 0$ and $<E \times \mathbb{P}^1>$ the proper transform of $E \times \mathbb{P}^1$. Let $\tilde{E} = \tau^*(E \times \mathbb{P}^1)$, $\tilde{Y} = \tau^*(Y \times 0)$, and let G be the exceptional divisor of τ.

Lemma 5.10. 1. *$\tilde{E} + \tilde{Y}$ is in good position with respect to D.*
2. *Let $q : |\tilde{Y}| \cap |\tilde{E}| \to X$ be the composition of the inclusion into $\tilde{Y}$ with $f \circ \tau$. Then $q_*(\tilde{Y}([\tilde{E} \to |\tilde{E}|]_{\tilde{Y}|D}))_D \in \Omega_*(X)_D$ is the distinguished lifting of $f_*([E \to Y])$, in the sense of remark 5.8.*

Proof. We we write $[E]$ for $[E \to Y]$, etc. (1) follows directly from the construction. For (2), we claim that

$$\tilde{Y}([\tilde{E}]_{\tilde{Y}|D})_D = \tilde{Y}([<E \times \mathbb{P}^1>]_{\tilde{Y}|D})_D \tag{5.3}$$

in $\Omega_*(|\tilde{Y}| \cap |\tilde{E}|)_D$. Indeed we may write

$$\tilde{E} = <E \times \mathbb{P}^1> + A,$$

where A is an effective divisor, supported in $|\tilde{E}| \cap G$. Since the exceptional divisor G is contained in $|\tilde{Y}|$, A is supported in $|\tilde{Y}|$. From the above decomposition of $\tilde{E}$, we have

$$[\tilde{E}]_{\tilde{Y}|D} = [<E \times \mathbb{P}^1>]_{\tilde{Y}|D} + i_*\alpha,$$

where α is a class in $\Omega_*(|A|)_{\tilde{Y}|D}$, and $i : |A| \to |\tilde{E}|$ is the inclusion. Then

$$\tilde{Y}([\tilde{E}]_{\tilde{Y}|D})_D = \tilde{Y}([<E \times \mathbb{P}^1>]_{\tilde{Y}|D})_D + i_*\tilde{c}_1(i^*O_{\tilde{T}}(\tilde{Y}))(\alpha).$$

But $\tilde{Y} = \tau^*(Y \times 0)$, hence $O_{\tilde{T}}(\tilde{Y}) \cong O_{\tilde{T}}$ in a neighborhood of $|A|$. Thus $\tilde{c}_1(i^*O_{\tilde{T}}(\tilde{Y}))(\alpha) = 0$, proving our claim.

We are thus reduced to showing that $q_*(\tilde{Y}([<E \times \mathbb{P}^1>]_{\tilde{Y}|D}))_D$ is the distinguished lifting of $f_*([E])$. For this, write $E = \sum_{j=1}^m n_jE_j$ with the E_j irreducible. Then $<E\times\mathbb{P}^1> = \sum_{j=1}^m n_j<E_j\times\mathbb{P}^1>$. In addition, over $\mathbb{P}^1 \setminus \{0\}$, we have $<E \times \mathbb{P}^1>^J = E^J \times \mathbb{P}^1$. Thus, the restriction of τ, $<E \times \mathbb{P}^1>^J \to E^J \times \mathbb{P}^1$, satisfies the conditions (5.1). Letting $\beta^J : <E \times \mathbb{P}^1>^J \to |<E \times \mathbb{P}^1>|$ the inclusion, it follows from lemma 5.7 that $\tilde{Y}(\beta^J_*[H_J^{n_1,\dots,n_m}(\beta^{J*}O_T(<E_1 \times \mathbb{P}^1>),\dots)]_{\tilde{Y}|D})_D$ is the distinguished lifting of $\iota^J_*[H_J^{n_1,\dots,n_m}(O_Y(E_1),\dots)])$.

On the other hand,

$$[<E \times \mathbb{P}^1>]_{\tilde{Y}|D} = \sum_J \beta^J_*[H_J(\beta^{J*}O_T(<E_1 \times \mathbb{P}^1>),\dots)]_{\tilde{Y}|D} + \gamma$$

where γ is a similar sum, involving the faces of $<E \times \mathbb{P}^1>$ which are contained in $|A|$. As above, we have $\tilde{Y}(\gamma) = 0$, so $\tilde{Y}([<E \times \mathbb{P}^1>]_{\tilde{Y}|D})_D$ is the distinguished lifting of $[E]$, as desired. □

5.3. **The proof of the moving lemma.** We are now ready to prove the main result of this section.

Theorem 5.11. *Let X be a finite type k-scheme, and D a pseudo-divisor on X. Then the canonical map $\vartheta_X : \Omega_*(X)_D \to \Omega_*(X)$ is an isomorphism.*

Proof. As noted in remark 5.6, taking the distinguished lifting defines a $\mathcal{Z}_*(k)$-linear homomorphism $\phi : \mathcal{Z}_*(X) \to \Omega_*(X)_D$, with $\vartheta_X(\phi(\eta))$ the image of η in $\Omega_*(X)$. In fact, by lemma 5.9, we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{Z}_*(X)_D & \longrightarrow & \Omega_*(X)_D \\
\downarrow{\scriptstyle \tilde{\vartheta}_X} & \nearrow_{\phi} & \downarrow{\scriptstyle \vartheta_X} \\
\mathcal{Z}_*(X) & \longrightarrow & \Omega_*(X)
\end{array}$$

where $\tilde{\vartheta}_X$ is the canonical map, and the horizontal arrows are the canonical maps.

Since $\mathcal{Z}_*(X) \to \Omega_*(X)$ is surjective [4, Lemma 4.13], the surjectivity of ϑ_X follows. To show that ϑ_X is injective, it suffices to show that the distinguished lifting homomorphism ϕ descends to an $\Omega_*(k)$-linear homomorphism $\bar{\phi} : \Omega_*(X) \to \Omega_*(X)_D$. Indeed, $\Omega_*(X)_D$ is generated as an $\mathbb{L}_*$-module by the image of $\mathcal{Z}_*(X)_D$, and thus $\Omega_*(X)_D$ is generated as an $\Omega_*(k)$-module by the image of $\mathcal{Z}_*(X)_D$. Therefore, $\bar{\phi}$ is surjective, and $\vartheta_X \circ \bar{\phi} = \mathrm{id}$, hence ϑ_X is injective. We proceed to show that ϕ descends to $\bar{\phi}$.

First, we show that ϕ descends to $\phi_1 : \underline{\mathcal{Z}}_*(X) \to \Omega_*(X)_D$. For this, take a generator $\eta := (f : Y \to X, \pi^*L_1, \dots, \pi^*L_r, M_1, \dots, M_s)$ of $<\mathcal{R}_*^{Dim}>(X)$ (see [4, Lemma 2.6]), which is the kernel of $\mathcal{Z}_*(X) \to \underline{\mathcal{Z}}_*(X)$. Here $f : Y \to X$ is in $\mathcal{M}(X)$, $\pi : Y \to Z$ is a smooth morphism to some $Z \in \mathbf{Sm}_k$, $L_1, \ldots, L_r$ are line bundles on Z, $M_1, \ldots, M_s$ are line bundles on Y, and $r > \dim_k Z$. Let $\mu : W \to Y \times \mathbb{P}^1$ be a projective birational morphism used to construct a distinguished lifting of $f : Y \to X$. Let $g = f \circ p_1 \circ \mu$, $\tau = \pi \circ p_1 \circ \mu$ and $\rho = p_1 \circ \mu$. Then the distinguished lifting of η is

$$g_*\big(\tilde{c}_1(\tau^* L_1) \circ \ldots \circ \tilde{c}_1(\tau^* L_r) \circ \tilde{c}_1(\rho^* M_1) \circ \ldots \\ \ldots \circ \tilde{c}_1(\rho^* M_s)([\mu^*(Y \times 0) \to W]_D)\big).$$

By lemma 2.12, the operator $\tilde{c}_1(\tau^* L_1) \circ \ldots \circ \tilde{c}_1(\tau^* L_r)$ is zero on $\Omega_*(W)_D$, so the distinguished lifting of η is zero, as desired.

Next, we check that ϕ_1 descends to $\phi_2 : \underline{\Omega}_*(X) \to \Omega_*(X)_D$. Since ϕ intertwines the operators $\tilde{c}_1(L)$ on $\mathcal{Z}_*(X)$ and $\Omega(X)_D$, it suffices by [4, Lemma 2.11] to check that ϕ_1 vanishes on elements of the form

$$[f : Y \to X, O_Y(S)] - [f \circ i : S \to X],$$

where f is in $\mathcal{M}(X)$, and $i : S \to Y$ is the inclusion of a smooth divisor.

Let $\eta \in \Omega_*(Y)_{f^*D}$ be the distinguished lifting of id_Y. By lemma 5.7, $f_*(\tilde{c}_1(O_Y(S))(\eta))$ is the distinguished lifting of $(f : Y \to X, O_Y(S)) = f_*(\tilde{c}_1(O_Y(S))(\mathrm{id}_Y))$.

On the other hand, let $\rho : W \to Y \times \mathbb{P}^1$ be a blow-up used to define the distinguished lifting η, so η is represented by $(p_1 \circ \rho)_*([\rho^*(Y \times 0) \to W]_D)$. Blowing up W further, and changing notation, we may assume that $\rho^*(S \times \mathbb{P}^1)$ is a normal crossing divisor on W, in good position with respect to $Y \times 0 | f^* D$. By lemma 5.10,

$$f_*\big((Y \times 0)((p_1 \circ \rho)_*[\rho^*(S \times \mathbb{P}^1) \to W]_{Y \times 0|D})_{f^*D}\big)$$

is the distinguished lifting of $f_*[S \to Y]$ to $\Omega_*(X)_D$. Let $\tilde{i}_S : |\rho^*(S \times \mathbb{P}^1)| \to W$ and $\tilde{i}_0 : |\rho^*(Y \times 0)| \to W$ be the inclusions. By lemma 4.1 and proposition 4.3, we have

$$\begin{aligned} f_*\big((Y \times 0)&((p_1 \circ \rho)_*[\rho^*(S \times \mathbb{P}^1) \to W]_{Y \times 0|f^*D})\big) \\ &= (f \circ p_1 \circ \rho)_*(\tilde{i}_{0*}(\rho^*(Y \times 0)([\rho^*(S \times \mathbb{P}^1) \to W]_{Y \times 0|D}))) \\ &= (f \circ p_1 \circ \rho)_*\big(\tilde{i}_{S*}(\rho^*(S \times \mathbb{P}^1)([\rho^*(Y \times 0) \to W]_{S \times \mathbb{P}^1|D}))\big) \\ &= (f \circ p_1 \circ \rho)_*\big(\tilde{c}_1((p_1 \circ \rho)^* O_Y(S))(\tilde{[}\rho^*(Y \times 0) \to W]_D)\big) \\ &= f_*(\tilde{c}_1(O_Y(S))(\eta)). \end{aligned}$$

Thus $f_*[S \to Y]$ and $f_*(\tilde{c}_1(O_Y(S))(1_Y))$ have the same distinugished lifting to $\Omega_*(X)_D$, that is $\phi_1([f : Y \to X, O_Y(S)] - [f \circ i : S \to X]) = 0$, as desired.

Finally, we check that ϕ_2 descends to $\bar{\phi} : \Omega_*(X) \to \Omega_*(X)_D$. For this we use the description of the kernel of the surjection $\underline{\Omega}_*(X) \to \Omega_*(X)$

given by [4, Proposition 4.19]. To describe this kernel, we consider the power series $F_\Omega(u,v) = u + v + \sum_{i,j\geq 1} a_{ij}u^iv^j$ giving the formal group law on $\Omega_*(k)$. We choose liftings $\alpha_{ij} \in \mathcal{M}^+(k)$ of $a_{ij} \in \Omega_*(k)$, and let $F(u,v) = u + v + \sum_{i,j\geq 1} \alpha_{ij}u^iv^j$ be the resulting lifting of F_Ω. Then the kernel of $\underline{\Omega}_*(X) \to \Omega_*(X)$ is generated by elements of the form

$$f_*\big(\tilde{c}_1(L_1) \circ \ldots \circ \tilde{c}_1(L_r)([F(L,M)] - [L \otimes M])\big),$$

where $f : Y \to X$ is in $\mathcal{M}(X)$, and $L_1, \ldots, L_r, L$ and M are line bundles on Y. Given such an element, it suffices to show that

$$\phi_2([F(L,M)] - [L \otimes M]) = 0$$

in $\Omega_*(Y)_D$, since ϕ_2 is compatible with f_*, and with the Chern class operators $\tilde{c}_1(L_i)$. Now, $[F(L,M)] = F(\tilde{c}_1(L), \tilde{c}_1(M))(\mathrm{id}_Y)$, and $[L \otimes M] = \tilde{c}_1(L \otimes M)(\mathrm{id}_Y)$. Thus, if $\eta \in \Omega_*(Y)_D$ is the distinguished lifting of id_Y, it follows from the definition of distinguished liftings that

$$\begin{aligned}\phi_2([F(L,M)]) &= F(\tilde{c}_1(L), \tilde{c}_1(M))(\eta),\\ \phi_2([L \otimes M]) &= \tilde{c}_1(L \otimes M)(\eta).\end{aligned}$$

Since $F(u,v)$ and $F_{\mathbb{L}}(u,v)$ both have image $F_\Omega(u,v)$ in $\Omega_*(k)[[u,v]]$, it follows that $F(\tilde{c}_1(L), \tilde{c}_1(M))(\eta) = F_{\mathbb{L}}(\tilde{c}_1(L), \tilde{c}_1(M))(\eta)$. Thus, using the relations $<\mathbb{L}_*\mathcal{R}_*^{FGL}>(Y)_{f^*D}$, we see that $\phi_2([F(L,M)] - [L \otimes M]) = 0$, which completes the descent, and the proof of the theorem. ☐

6. *Pull-back for l.c.i. morphisms*

Starting with the pull-back for a divisor, defined using the results of the previous sections, we use the deformation to the normal bundle to define the Gysin morphism for a regular imbedding of k-schemes. Combined with smooth pull-back, this gives us functorial pull-back maps for l.c.i. morphisms of k-schemes, in particular, for arbitrary morphisms in $\mathbf{Sm}_k$.

6.1. **Pull-back for divisors.** The results of §3-§5 allow us to make the following definition:

Definition 6.1. Let D be a pseudo-divisor on a finite type k-scheme X. We define the operation of "pull-back by D",

$$i_D^* : \Omega_*(X) \to \Omega_{*-1}(|D|),$$

to be the composition

$$\Omega_*(X) \xrightarrow{\vartheta_X^{-1}} \Omega_*(X)_D \xrightarrow{D(-)} \Omega_{*-1}(|D|).$$

6.2. **Algebraic cobordism of a blow-up.** Before defining the pull-back for a regular imbedding, we first prove some preliminarly results relating $\Omega_*(W)$ and $\Omega_*(W_F)$, where $W_F \to W$ is the blow-up of a smooth W along a smooth closed subscheme F.

Lemma 6.2. *Let X be a k-scheme of finite type, $\mathcal{E}$ a locally free coherent sheaf, and $q : \mathbb{P}(\mathcal{E}) \to X$ the associated projective space bundle. Then the map $q_* : \Omega_*(\mathbb{P}(\mathcal{E})) \to \Omega_*(X)$ is surjective.*

Proof. We proceed by noetherian induction. If $\mathcal{E}$ is a free $\mathcal{O}_X$-module of rank $n+1$, then $\mathbb{P}(\mathcal{E}) \cong X \times \mathbb{P}^n$. Since q admits a section in this case, q_* is surjective. In particular, if $\dim_k X = 0$, the result is true. In general, let $j : U \to X$ be a non-empty open subscheme over which $\mathcal{E}$ is free, and let $i : W \to X$ be the complement of U. We have the commutative diagram with exact rows [4, Theorem 6.7]

$$\begin{array}{ccccccc}
\Omega_*(\mathbb{P}(i^*\mathcal{E})) & \xrightarrow{\tilde{i}_*} & \Omega_*(\mathbb{P}(\mathcal{E})) & \xrightarrow{\tilde{j}^*} & \Omega_*(\mathbb{P}(j^*\mathcal{E})) & \longrightarrow & 0 \\
\downarrow{\scriptstyle q_{W*}} & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_{U*}} & & \\
\Omega_*(W) & \xrightarrow[i_*]{} & \Omega_*(X) & \xrightarrow[j^*]{} & \Omega_*(U) & \longrightarrow & 0
\end{array}$$

where q_W, q_U are the restrictions of q. By induction q_{W*} is surjective, and q_{U*} is surjective since $j^*\mathcal{E}$ is free, hence q_* is surjective. ☐

Lemma 6.3. *Let W be in $\mathbf{Sch}_k$, let $i_F : F \to W$ be a regularly imbedded closed subscheme, and let $\mu : W_F \to W$ be the blow-up of W along F. Let $i_E : E \to W_F$ be the exceptional divisor.*

1. *The map*

$$\mu_* : \Omega_*(W_F) \to \Omega_*(W)$$

is surjective.

2. *There is a subgroup A of $\Omega_*(E)$ such that* $\ker \mu_* = i_{E*}(A)$.

Proof. Let $j : U \to X$ be the complement of F. μ identifies U with the complement of E in W_F; let $j_F : U \to W_F$ be the inclusion. We have the commutative diagram with exact rows [4, *loc. cit.*]

$$\begin{array}{ccccccc}
\Omega_*(E) & \xrightarrow{i_{E*}} & \Omega_*(W_F) & \xrightarrow{j_F^*} & \Omega_*(U) & \longrightarrow & 0 \\
{\scriptstyle \mu_*}\downarrow & & {\scriptstyle \mu_*}\downarrow & & \| & & \\
\Omega_*(F) & \xrightarrow[i_{F*}]{} & \Omega_*(W) & \xrightarrow[j^*]{} & \Omega_*(U) & \longrightarrow & 0.
\end{array}$$

Since F is regularly imbedded the conormal sheaf $\mathcal{I}_F/\mathcal{I}_F^2$ is locally free; in addition, $E = \mathrm{Proj}_F(\mathcal{I}_F/\mathcal{I}_F^2)$. By lemma 6.2, the map $\mu_* : \Omega_*(E) \to$

$\Omega_*(F)$ is surjective. Thus, by the snake lemma, $\mu_* : \Omega_*(W_F) \to \Omega_*(W)$ is surjective, and we have

$$\ker(i_{E*}(\Omega_*(E)) \xrightarrow{\mu_*} i_{F*}(\Omega_*(F))) = \ker(\mu_* : \Omega_*(W_F) \to \Omega_*(W)).$$

Letting $A = i_{E*}^{-1}[\ker(i_{E*}(\Omega_*(E)) \xrightarrow{\mu_*} i_{F*}(\Omega_*(F)))]$ proves (2). □

Let $i : Z \to Y$ be a codimension d regular imbedding with Z and Y in $\mathbf{Sch}_k$. Let $\mu : W \to Y \times \mathbb{P}^1$ be the blow-up of $Y \times \mathbb{P}^1$ along $Z \times 0$, let E be the exceptional divisor and $<Y \times 0>$ the proper transform of $Y \times 0$. Let $V = E \setminus <Y \times 0>$. Letting $\mathcal{N}$ be the conormal sheaf of Z in Y, $\mu_E : E \to Z \times 0 = Z$ is the projective bundle $\mathbb{P}(\mathcal{N} \oplus \mathcal{O}_Z) \to Z$, and $\mu_V : V \to Z$ is the vector bundle $\mathrm{Spec}\,(\mathrm{Sym}^*(\mathcal{N})) \to Z$, i.e., the normal bundle of F in Y. Let $U = W \setminus <Y \times 0>$, and let $j : U \to W$, $i_E : E \to W$, $j_V : V \to E$ and $i_V : V \to U$ be the inclusions.

Lemma 6.4. *The map* $i_V^* \circ j^* \circ i_{E*} : \Omega_*(E) \to \Omega_{*-1}(V)$ *is the zero map*

Proof. We have $j^* \circ i_{E*} = i_{V*} \circ j_V^*$, so it suffices to show that $i_V^* \circ i_{V*} = 0$. If $f : T \to V$ is in $\Omega_*(V)$, then $i_V \circ f : T \to U$ is clearly in $\mathcal{M}(U)_V$, and $V(i_V \circ f) = f_*(\tilde{c}_1(f^* O_U(V))(\mathrm{id}_T))$. But on U, the divisor V is linearly equivalent to $\mu^*(Y \times 1)$, which is disjoint from V. Thus $f^* O_U(V) \cong O_T$ and hence $\tilde{c}_1(f^* O_U(V))(\mathrm{id}_T) = 0$. Thus $i_V^* i_{V*}(f) = V(i_V \circ f) = 0$, as desired. □

6.3. **The Gysin morphism.** Let $i_Z : Z \to Y$ be a regular imbedding of codimension $d > 0$. We proceed to define the map $i_Z^* : \Omega_*(Y) \to \Omega_{*-d}(Z)$. We have the diagram

$$\begin{array}{ccccc} \Omega_*(Y) & \xrightarrow{p_1^*} & \Omega_{*+1}(Y \times \mathbb{P}^1) & & \\ & & \uparrow{\scriptstyle \mu_*} & & \\ & & \Omega_{*+1}(W) & \xrightarrow{j^*} & \Omega_{*+1}(U) \\ & & & & \downarrow{\scriptstyle i_V^*} \\ & & \Omega_{*-d}(Z) & \xrightarrow[\mu_V^*]{} & \Omega_*(V), \end{array} \tag{6.1}$$

defined using the results and notations of the previous paragraph. We call (6.1) the *deformation diagram* for the regular imbedding i_Z. By lemma 6.3 and lemma 6.4, the composition

$$\Omega_{*+1}(Y \times \mathbb{P}^1) \xrightarrow{(\mu_*)^{-1}} \Omega_{*+1}(W) \xrightarrow{j^*} \Omega_{*+1}(U) \xrightarrow{i_V^*} \Omega_*(V)$$

gives a well-defined homomorphism $\psi_{Y,Z} : \Omega_{*+1}(Y \times \mathbb{P}^1) \to \Omega_*(V)$.

Since the map $\mu_V : V \to Z$ makes V into a vector bundle over Z, it follows from the the homotopy property for algebraic cobordism [4, Corollary 9.3] that the smooth pull-back

$$\mu_V^* : \Omega_*(Z) \to \Omega_{*+d}(V)$$

is an isomorphism.

Definition 6.5. The *Gysin morphism* $i_Z^* : \Omega_*(Y) \to \Omega_{*-d}(Z)$ is defined as the composition

$$\Omega_*(Y) \xrightarrow{p_1^*} \Omega_{*+1}(Y \times \mathbb{P}^1) \xrightarrow{\psi_{Y,Z}} \Omega_*(V) \xrightarrow{(\mu_V^*)^{-1}} \Omega_{*-d}(Z).$$

Remark 6.6. Let $p_0 : Y \times (\mathbb{P}^1 \setminus \{0\}) \to Y$ be the projection, let η be in $\Omega^*(Y)$, and let τ be an element of $\Omega^*(W)$ which restricts to $p_0^*(\eta)$ on $Y \times (\mathbb{P}^1 \setminus \{0\})$. Then $i_Z^*(\eta) = (\mu^*)^{-1}(i_V^* \circ j^*(\eta))$. Indeed, by comparing the localization sequences for $E \to W$ and $Z \times 0 \to Y \times \mathbb{P}^1$, and using the surjectivity result lemma 6.2, there is an element $\rho \in \Omega^{*-1}(E)$ such that $\mu_*(\tau + i_{E*}(\rho)) = p_1^*\eta$. By lemma 6.4, we have

$$i_Z^*(\eta) = (\mu^*)^{-1}(j^*(i_V^*(\tau))).$$

6.4. **Properties of the Gysin morphism.** Let $f : Y \to X$, $g : Z \to X$ be morphisms in $\mathbf{Sch}_k$. Recall that f and g are called *Tor-independent* if $\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z) = 0$ for $i > 0$.

Proposition 6.7. *Let $f : Y \to X$, $g : Z \to X$ be Tor-independent morphisms in $\mathbf{Sch}_k$, giving the cartesian diagram*

$$\begin{array}{ccc} Y \times_X Z & \xrightarrow{f'} & Z \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X \end{array}$$

1. *Suppose that g is a regular imbedding and f is projective. Then $g^* \circ f_* = f'_* \circ g'^*$.*
2. *Suppose that g is a regular imbedding and f is smooth. Then $f'^* \circ g^* = g'^* \circ f^*$.*

Proof. Since f and g are Tor-independent, the map g' is a regular imbedding, so g'^* is defined. Also, if we apply the functor $Y \times_X -$ to deformation diagram (6.1) for the regular imbedding $g : Z \to X$, we arrive at the deformation diagram for the regular imbedding $g' : Y \times_X Z \to Y$. Calling the first diagram $\mathcal{D}(g)$ and the second $\mathcal{D}(g')$, the projection $p_2 : Y \times_X ? \to ?$ gives the map of diagrams $p_{2*} : \mathcal{D}(g') \to \mathcal{D}(g)$ in case f is projective, and the map of diagrams $p_2^* : \mathcal{D}(g) \to \mathcal{D}(g')$ if f is smooth (with shift in the grading).

We note that the diagrams $p_{2*}:\mathcal{D}(g')\to\mathcal{D}(g)$ and $p_2^*:\mathcal{D}(g)\to\mathcal{D}(g')$ are commutative. Indeed, this follows from

1. projective push-forward commutes with smooth pull-back in cartesian diagrams [4, Definition 1.2(A3)]
2. pull-back by a divisor satisfies a projection formula with respect to projective push-forward (lemma 3.1(1)).
3. pull-back by a divisor commutes with smooth pull-back (lemma 3.1(2)).

This proves the lemma. □

Corollary 6.8. *Let Z and X be in $\mathbf{Sch}_k$.*

1. *Let $i:Z\to X$ be a regular imbedding, and let $f:Y\to X$ be in $\mathcal{M}(X)$. Suppose f and i are Tor-independent and that $Y\times_X Z$ is in $\mathbf{Sm}_k$. Then $i^*(f)$ is represented by $p_2:Y\times_X Z\to Z$.*
2. *Let $p:X\to Z$ be a smooth morphism with a section $i:Z\to X$. Then i is a regular imbedding and $i^*\circ p^*=\mathrm{id}$ on $\Omega_*(Z)$.*
3. *Let $p:X\to Z$ be a rank n vector bundle over Z, and let $i:Z\to X$ be a section. Then i is a regular imbedding and i^* is the inverse of p^*.*

Proof. For (3), the fact that p^* is an isomorphism if $p:X\to Z$ is a vector bundle over Z [4, Corollary 9.3] shows that (2) implies (3). Also, (2) follows from (1), since $p^*(f:Y\to Z)$ is represented by $p_2:Y\times_Z X\to X$. It remains to prove (1).

By proposition 6.7, we have $i^*f_*(\eta)=f'_*i'^*(\eta)$ for $\eta\in\Omega_*(Y)$, where $i':Y\times_X Z\to Y$, $f':Y\times_X Z\to Z$ are the projections. Thus, it suffices to show that $i'^*(\mathrm{id}_Y)=\mathrm{id}_{Y\times_X Z}$. Denoting $Y\times_X Z$ by Z', we form the deformation diagram (6.1) for i', letting $\mu:W\to Y\times\mathbb{P}^1$ be the blow-up of $Y\times\mathbb{P}^1$ along $Z'\times 0$ with exceptional divisor E, $j:U\to W$ the inclusion of $W\setminus <Y\times 0>$ into W, $i_V:V\to U$ the inclusion of the normal bundle of Z' in Y, and $\mu_{Z'}:V\to Z'$ the projection.

From the localization sequence for $E\to W$ and $Z'\times 0\to Y\times\mathbb{P}^1$, we see that there is an element $\eta\in\Omega_*(E)$ such that $\mu_*(\mathrm{id}_W+i_{E*}(\eta))=\mathrm{id}_{Y\times\mathbb{P}^1}$. Then

$$i'^*(\mathrm{id}_Y)=(\mu_{Z'}^*)^{-1}(i_V^*(j^*(\mathrm{id}_W+i_{E*}(\eta))).$$

By lemma 6.4, $i_V^*(j^*(i_{E*}(\eta))=0$, hence

$$i'^*(\mathrm{id}_Y)=(\mu_{Z'}^*)^{-1}(i_V^*(j^*(\mathrm{id}_W))=(\mu_Z^*)^{-1}(\mathrm{id}_V).$$

Since $\mu_{Z'}^*(\mathrm{id}_{Z'})=\mathrm{id}_V$, we have $i'^*(\mathrm{id}_Y)=\mathrm{id}_{Z'}$, as desired. □

Lemma 6.9. *Let $i:Z\to X$ be a regular imbedding, with* $\mathrm{codim}_X Z=1$*. Then, for $f:Y\to X$ in $\mathcal{M}(X)_Z$, we have $Z(f)=i^*(f)$ in $\Omega^*(Z)$, so $i^*=i_Z^*$.*

Proof. Let $\mu : W \to X \times \mathbb{P}^1$ be the blow-up of $X \times \mathbb{P}^1$ along $Z \times 0$, and let $i_E : E \to W$ be the exceptional divisor. Let $j : U \to W$ be the inclusion of $W \setminus <X \times 0>$, let $i_V : V \to U$ be the inclusion of the normal bundle of Z in X, and let $\mu_Z : V \to Z$ be the projection.

Since $Z \times 0$ is a divisor on $Z \times \mathbb{P}^1$, the proper transform $<Z \times \mathbb{P}^1>$ of $Z \times \mathbb{P}^1$ to W is isomorphic to $Z \times \mathbb{P}^1$ via μ.

Let $\rho : T \to Y \times \mathbb{P}^1$ be a blow-up of $Y \times \mathbb{P}^1$ along smooth centers lying over $f^{-1}(Z) \times 0$ so that the rational map $\mu^{-1} \circ (f \times \mathrm{id}) : Y \times \mathbb{P}^1 \to W$ defines a morphism $\phi : T \to W$. Blowing up further over $Y \times 0$, if necessary, we may assume that ϕ is in $\mathcal{M}(W)_{E|<Z \times \mathbb{P}^1>}$.

By considering the localization sequence for $E \to W$ and $Z \times 0 \to X \times \mathbb{P}^1$, we see there is an element η of $\Omega^*(E)$ such that $\mu_*(\phi + i_{E*}(\eta)) = f \times \mathrm{id}$ in $\Omega^*(Y \times \mathbb{P}^1)$. Thus, by definition of i^*, we have

$$i^*(f) = (\mu_Z)^{-1}(i_V^* \circ j^*(\phi + i_{E*}(\eta)))).$$

By lemma 6.4, we have

$$i^*(f) = (\mu_Z^*)^{-1}(i_V^* \circ j^*(\phi)).$$

Let $i_0 : Z \to V$ be the zero section, giving the divisor $i_0(Z)$ on V. Since $i_0(Z)(\mu_Z^* g) = g$ for $g \in \Omega^*(Z)$, the map $i_0(Z)(-) : \Omega^*(V) \to \Omega^*(Z)$ is inverse to μ^*. Thus

$$i^*(f) = i_0(Z)(i_V^*(j^*(\phi))) = i_0(Z)(V(j^*(\phi))) = i_0(Z)(\mu^*(X \times 0)(\phi)).$$

Let $i_1 : Z \to X \times 1$ be the evident inclusion.

Write D for $<Z \times \mathbb{P}^1>$, and let $i_D : |D| \to W$, $\iota_0 : |\mu^*(X \times 0)| \to W$ and $\iota_1 : |\mu^*(X \times 1)| \to W$ be the inclusions. Since $<Z \times \mathbb{P}^1> \cong Z \times \mathbb{P}^1$, we have the projection $p_1 : |D| \to Z$. Noting that $i_0(Z) = \iota_0^*(D)$, and using proposition 4.4, we have

$$\begin{aligned} i^*(f) &= i_0(Z)(\mu^*(X \times 0)(\phi)) = (\iota_0^* D)(\mu^*(X \times 0)(\phi)) \\ &= p_{1*}\big(D(\iota_{0*}(\mu^*(X \times 0)(\phi))\big) = p_{1*}\big(D(\iota_{1*}(\mu^*(X \times 1)(\phi))\big) \\ &= i_1(Z)(\mu^*(X \times 1)(\phi)) = Z(f). \end{aligned}$$

□

Lemma 6.10. *Let $i : Z \to X$ be a regular imbedding, let $p : Y \to X$ be a smooth quasi-projective morphism, and let $s : Z \to Y$ be a section of Y over Z. Then $s^* \circ p^* = i^*$.*

Proof. Form the deformation diagram for the inclusion $s : Z \to Y$, letting $\mu : W \to Y \times \mathbb{P}^1$ be the blow-up of $Y \times \mathbb{P}^1$ along $Z \times 0$, $i_E : E \to W$ the exceptional divisor, $j : U \to W$ the complement of $<Y \times 0>$, $j_V : V \to U$ the inclusion of the normal bundle $N_{Z/Y}$ and $p : V \to Z$ the projection. Let $i_X : <X \times \mathbb{P}^1> \to W$ be the proper transform of $X\times\mathbb{P}^1$ to W. Since Z is a subset of X, it follows that $<X\times\mathbb{P}^1>$ is isomorphic to the blow-up of $X\times\mathbb{P}^1$ along $Z\times 0$. Letting $j_X : U_X \to <X\times\mathbb{P}^1>$ be the inclusion of $<X\times\mathbb{P}^1> \setminus <X\times 0>$, and $i_{V_X} : V_X \to U_X$ the inclusion of the normal bundle $N_{Z/X}$. Then $V_X = <X\times\mathbb{P}^1>\cap V$, and the inclusion $V_X \to V$ is the natural inclusion $\tau : N_{Z/X} \to N_{Z/Y}$.

Checking in local coordinates, one sees that the projection $p : Y \to X$ extends to a smooth morphism $\tilde{p} : U \to U_X$, inducing the natural map $Np : N_{Z/Y} \to N_{Z/X}$ from V to V_X. Take an element $\eta \in \Omega^*(X)$, and let $\tilde{\eta}$ be a lifting of $p_1^*\eta$ to $\Omega^*(<X\times\mathbb{P}^1>)$. Then $\tilde{p}^*\tilde{\eta}$ is a lifting of $p_1^*p^*\eta$ to $\Omega^*(W)$. Since $\tilde{p}$ is smooth, we have

$$i_V^*\tilde{p}^*\tilde{\eta} = Np^* i_{V_X}^*\tilde{\eta}.$$

Letting $\mu_Z : V \to Z$, $\mu_{Z:X} : V_X \to Z$ be the projections, we have $\mu_Z^* = Np^* \circ \mu_{Z:X}^*$, hence, by proposition 6.7 and lemma 6.9,

$$s^*(p^*\eta) = (\mu_Z^*)^{-1}(i_V^*\tilde{p}^*\tilde{\eta}) = (\mu_{Z:X}^*)-1(i_{V_X}^*\tilde{\eta}) = i^*\eta.$$

□

Theorem 6.11. *Let $i : Z \to Z'$, $i' : Z' \to X$ be regular imbeddings. Then $(i'\circ i)^* = i^* \circ i'^*$.*

Proof. Form the deformation diagram for $i' : Z' \to X$ by blowing up $X\times\mathbb{P}^1$ along $Z'\times 0$, giving the birational morphism $\mu : W \to X\times\mathbb{P}^1$ with exceptional divisor E, let $j : U \to W$ be the complement of $<X\times 0>$, let $i_V : V \to U$ be the inclusion of the normal bundle $N_{Z'/X}$ and $\mu_{Z'} : V \to Z'$ the projection. We have the proper transform $<Z'\times\mathbb{P}^1>$ as a closed subscheme of W; since $Z'\times 0 \cap Z'\times\mathbb{P}^1$ is a Cartier divisor on $Z\times\mathbb{P}^1$, it follows that $\mu : <Z'\times\mathbb{P}^1> \to Z'\times\mathbb{P}^1$ is an isomorphism. It is easy to see that $<Z'\times\mathbb{P}^1> \cap <X\times 0> = \emptyset$, hence $<Z'\times\mathbb{P}^1>$ is contained in U. Also, the intersection $<Z'\times\mathbb{P}^1> \cap V$ is just the image of Z' in V by the zero section $s : Z' \to N_{Z'/X}$. Restricting to Z, we have the closed subscheme $<Z\times\mathbb{P}^1>$ of $<Z'\times\mathbb{P}^1>$, which is isomorphic to $Z\times\mathbb{P}^1$ via μ, and the restriction of s defines a section $s_0 : Z \to V$ over Z, with $s_0(Z) = V \cap <Z\times\mathbb{P}^1>$. Letting $s_1 : Z \to X\times 1 \subset W$ be the map $s_1(z) = (i'(i(z)), 1)$, we similarly have $s_1(Z) = X\times 1 \cap <Z\times\mathbb{P}^1>$.

Let η be an element of $\Omega^*(U)_{V+X\times 1}$. We claim that

$$s_0^*(V(\eta)) = s_1^*((X\times 1)(\eta)) \qquad (6.2)$$

in $\Omega^*(Z)$. Indeed, form the deformation diagram for the inclusion $i'' : <Z\times\mathbb{P}^1> \to U$: Let $\phi : T \to U\times\mathbb{P}^1$ be the blow-up of $U\times\mathbb{P}^1$ along $<Z\times\mathbb{P}^1>\times 0$, let $j_Z : U_Z \to T$ be open subscheme $T \setminus <U\times 0>$, let $i_{V_Z} : V_Z \to U_Z$ be the inclusion of the normal bundle of $N_{<Z\times\mathbb{P}^1>/U}$.

The structure morphism $W \to \mathbb{P}^1$ induces the morphism $\tau : U_Z \to \mathbb{P}^1$. Via τ the deformation diagram for i_{V_Z} is a diagram of schemes over $\mathbb{P}^1$, where the fiber over $0 \in \mathbb{P}^1$ is the deformation diagram for the inclusion $s_0 : Z \to V$, and the fiber over $1 \in \mathbb{P}^1$ is the deformation diagram for the inclusion $s_1 : Z \to X \times 1$. Let V_{Z0}, U_{Z0}, etc., denote the fibers over 0, and V_{Z1}, U_{Z1}, etc. the fibers over 1. Let $\rho_0 : V_{Z0} \to Z$, $\rho_1 : V_{Z1} \to Z$ and $\rho : V_Z \to Z \times \mathbb{P}^1$ denote the projections. Take an element $\tilde{\eta} \in \Omega^*(U_Z)_{U_{Z0}+U_{Z1}|V_Z}$ lifting $p_1^*(\eta)$. Then $U_{Z1}(\tilde{\eta})$ lifts $(p_1^*(X \times 1))(\eta)$ and $U_{Z0}(\tilde{\eta})$ lifts $(p_1^*V)(\eta)$, so

$$\begin{aligned} \rho_0^*(s_0^*(V(\eta))) &= V_{Z0}(U_{Z0}(\tilde{\eta})) \\ \rho_1^*(s_1^*((X \times 1)(\eta))) &= V_{Z1}(U_{Z1}(\tilde{\eta})). \end{aligned} \tag{6.3}$$

Here, we consider V_{Z0} as a pseudo-divisor on U_{Z0} and V_{Z1} as a pseudo-divisor on U_{Z1}.

Let $i_0 : U_{Z0} \to U_Z$, $i_0^V : V_{Z0} \to V_Z$ be the inclusions. We have $V_{Z0} = i_0^* V_Z$ (as pseudo-divisors), hence by lemma 3.1(1)

$$i_{0*}^V(V_{Z0}(U_{Z0}(\tilde{\eta}))) = i_{0*}^V(i_0^* V_Z(U_{Z0}(\tilde{\eta}))) = V_Z(i_{0*}(U_{Z0}(\tilde{\eta}))).$$

Similarly, letting $i_1^V : V_{Z1} \to V_Z$ be the inclusion, we have

$$i_{1*}^V(V_{Z1}(U_{Z1}(\tilde{\eta}))) = V_Z(i_{1*}(U_{Z1}(\tilde{\eta}))).$$

Also, by proposition 4.4, we have $i_{0*}(U_{Z0}(\tilde{\eta})) = i_{1*}(U_{Z1}(\tilde{\eta}))$ in $\Omega^*(U_Z)$. Thus, in $\Omega^*(V_Z)$ we have

$$\begin{aligned} i_{0*}^V(V_{Z0}(U_{Z0}(\tilde{\eta}))) &= V_Z(i_{0*}(U_{Z0}(\tilde{\eta}))) \\ &= V_Z(i_{1*}(U_{Z1}(\tilde{\eta}))) \\ &= i_{1*}^V(V_{Z1}(U_{Z1}(\tilde{\eta}))) \end{aligned} \tag{6.4}$$

Let $i_0^Z : Z \to Z \times \mathbb{P}^1$, $i_1^Z : Z \to Z \times \mathbb{P}^1$ be the 0 and 1 section, respectively. Since $\rho^* : \Omega^*(Z \times \mathbb{P}^1) \to \Omega^*(V_Z)$ is an isomorphism by the homotopy property [4, Corollary 9.3], (6.3) and (6.4) imply $i_{0*}^Z(s_0^*(V(\eta))) = i_{1*}^Z(s_1^*((X \times 1)(\eta)))$. Projectiving to $\Omega_*(Z)$ by p_{1*} shows that $s_0^*(V(\eta)) = s_1^*((X \times 1)(\eta))$ in $\Omega^*(Z)$, as claimed.

Now take an element x of $\Omega^*(X)$, and let η be a lifting of $p_1^* x$ to $\Omega^*(U)_{V+X\times 1}$. Then $(X \times 1)(\eta) = x$, so $s_1^*((X \times 1)(\eta)) = (i \circ i')^*(x)$. On the other hand, letting $\mu_{Z'} : V \to Z'$ be the projection, we have $V(\eta) = \mu_{Z'}^* i'^*(x)$. By lemma 6.10, we have

$$s_0^*(V(\eta)) = s_0^*(\mu_{Z'}^* i'^*(x)) = i^*(i'^*(x)),$$

hence $i^*(i'^*(x)) = (i' \circ i)^*(x)$. ◻

6.5. **L.c.i. pull-back.** Let $f : X \to Y$ be an l.c.i. morphism in $\mathbf{Sch}_k$. By definition, there is a factorization $f = q \circ i$, with $i : X \to P$ a regular imbedding and $q : P \to Y$ a smooth quasi-projective morphism.

Lemma 6.12. *Let $f : X \to Y$ be an l.c.i. morphism. If we have factorizations $f = q_1 \circ i_1 = q_2 \circ i_2$, with $i_j : X \to P_j$ regular imbeddings and $q_j : P_j \to Y$ smooth and quasi-projective, then*

$$i_1^* \circ q_1^* = i_2^* \circ q_2^*.$$

Proof. Form the diagonal imbedding $(i_1, i_2) : X \to P_1 \times_Y P_2$. By remark 1.1(2), (i_1, i_2) is a regular imbedding. Form the cartesian diagram $(j = 1, 2)$

$$\begin{array}{ccc} P^j & \xrightarrow{i'_j} & P_1 \times_Y P_2 \\ {\scriptstyle q_j}\downarrow & & \downarrow{\scriptstyle p_j} \\ X & \xrightarrow[i_j]{} & P_j \end{array}$$

Then $p_j : P_1 \times_Y P_2 \to P_j$ is smooth and quasi-projective, so by proposition 6.7(2), we have

$$q_j^* \circ i_j^* = i_j'^* \circ p_j^*; \quad j = 1, 2.$$

Also, the map (i_1, i_2) induces a section $s_j : X \to P^j$ to q_j. Applying lemma 6.10 and theorem 6.11 gives

$$\begin{aligned} i_j^* &= s_j^* \circ q_j^* \circ i_j^* \\ &= s_j^* \circ i_j'^* \circ p_j^* \\ &= (i_1, i_2)^* \circ p_j^*. \end{aligned}$$

Let $q : P_1 \times_Y P_2 \to Y$ be the map $q_1 p_1 = q_2 p_2$. Using the functoriality of smooth pull-back [4, §2], we have

$$\begin{aligned} i_j^* \circ q_j^* &= (i_1, i_2)^* \circ p_j^* \circ q_j^* \\ &= (i_1, i_2)^* \circ q^*. \end{aligned}$$

□

We may therefore make the following definition:

Definition 6.13. Let $f : X \to Y$ be an l.c.i. morphism in $\mathbf{Sch}_k$ of relative dimension d. Define $f^* : \Omega_*(Y) \to \Omega_{*+d}(X)$ as $i^* \circ q^*$, where $f = q \circ i$ is a factorization of f with i a regular imbedding and q smooth and quasi-projective.

Theorem 6.14. *Let $f_1 : X \to Y$, $f_2 : Y \to Z$ be l.c.i. morphisms in $\mathbf{Sch}_k$. Then $(f_2 \circ f_1)^* = f_1^* \circ f_2^*$.*

Proof. As in remark 1.1, we have a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i_1} & P_1 & \xrightarrow{i} & P \\ & \searrow^{f_1} & \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q} \\ & & Y & \xrightarrow[i_2]{} & P_2 \\ & & & \searrow^{f_2} & \downarrow{\scriptstyle q_2} \\ & & & & Z, \end{array}$$

with q_1, q_2 and q smooth and quasi-projective, i_1, i_2 and i regular imbeddings, and the square cartesian. Using the functoriality of smooth pull-back [4, §2], proposition 6.7(2) and theorem 6.11, we have

$$\begin{aligned}(f_2 \circ f_1)^* &= (ii_1)^* \circ (q_2 q)^* \\ &= i_1^* \circ i^* \circ q^* \circ q_2^* \\ &= i_1^* \circ q_1^* \circ i_2^* \circ q_2^* \\ &= f_1^* \circ f_2^*\end{aligned}$$

□

Theorem 6.15. *Let $f : X \to Z$, $g : Y \to Z$ be Tor-independent morphisms in $\mathbf{Sch}_k$, giving the cartesian diagram*

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{p_2} & Y \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z. \end{array}$$

Suppose that f is an l.c.i. morphism and that g projective. Then

$$f^* \circ g_* = p_{1*} \circ p_2^*.$$

Proof. Since f and g are Tor-independent, p_2 is an l.c.i. morphism, so the statement makes sense.

Write $f = q \circ i$, with $q : P \to Z$ smooth and quasi-projective, and $i : X \to P$ a regular imbedding. This gives us the diagram

$$\begin{array}{ccccc} X \times_Z Y & \xrightarrow{i \times \mathrm{id}} & P \times_Z Y & \xrightarrow{p_2} & Y \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[i]{} & P & \xrightarrow[q]{} & Z. \end{array}$$

with both squares cartesian. Using the functoriality of projective push-forward and thereom 6.14, it suffices to prove the case of f a regular imbedding, or f a smooth quasi-projective morphism. The first case is proposition 6.7(1), the second follows easily from the definition of algebraic cobordism [4, §2].

□

Proposition 6.16. *Let $f_i : X_i \to Y_i$, $i = 1, 2$ be l.c.i. morphisms in* $\mathbf{Sch}_k$. *Then for $\eta_i \in \Omega_*(Y_i)$, $i = 1, 2$, we have*

$$(f_1 \times f_2)^*(\eta_1 \times \eta_2) = f_1^*(\eta_1) \times f_2^*(\eta_2).$$

Proof. We first note that $f_1 \times f_2 : X_1 \times_k X_2 \to Y_1 \times Y_2$ is indeed an l.c.i. morphism: if $X_2 = Y_2$ and $f_2 = \mathrm{id}$, this is clear, and we have the factorization $f_1 \times f_2 = (f_1 \times \mathrm{id}) \circ (\mathrm{id} \times f_2)$. Similarly, it suffices to prove the case $X_2 = Y_2$, $f_2 = \mathrm{id}$.

We may assume η_2 is a cobordism cycle $(g : Z \to Y_2, L_1, \ldots, L_r)$. Since both smooth pull-back and the Gysin morphism are compatible with the Chern class operators $\tilde{c}_1(L)$, we may assume that $r = 0$.

Note that $\eta_2 = g_*(\mathrm{id}_Z)$, so $\eta_1 \times \eta_2 = (\mathrm{id} \times g)_*(\eta_1 \times \mathrm{id}_Z)$. Similarly, $f_1^*(\eta_1) \times \eta_2 = (\mathrm{id} \times g)_*(f_1^*(\eta_1) \times \mathrm{id}_{Z_2})$. Thus, using theorem 6.15, we may replace Y_2 with Z and η_2 with id_Z, so it suffices to prove the result with $Y_2 \in \mathbf{Sm}_k$ and $\eta_2 = \mathrm{id}_{Y_2}$.

In this case $\eta_1 \times \eta_2 = p^*(\eta_1)$, where $p : Y_1 \times Y_2 \to Y_1$ is the projection. Similarly, $f_1^*(\eta_1) \times \eta_2 = q^*(f_1^*(\eta_1))$, where $q : X_1 \times Y_2 \to X_1$ is the projection. Thus we need to show

$$(f_1 \times \mathrm{id})^*(p^*(\eta_1)) = q^*(f_1^*(\eta_1)).$$

This follows from the functoriality of l.c.i. pull-back, theorem 6.14. □

7. *Universality*

In this section, the main task is to prove the universality results of the introduction. We fix an l.c.i.-closed admissible subcategory $\mathcal{V}$ of $\mathbf{Sch}_k$ (see §1.1); in particular, $\mathcal{V}$ contains $\mathbf{Lci}_k$.

7.1. **Fundamental classes and Chern classes.**

We first consider an arbitrary oriented Borel-Moore homology theory A_* on $\mathcal{V}$.

Lemma 7.1. *Let X be an l.c.i. scheme over k, $L \to X$ a line bundle. Suppose we have two sections $s_j : X \to L$, $j = 1, 2$. Let $i_j : E_j \to X$ be the subscheme defined by $s_j = 0$, $j = 0, 1$. Suppose that each E_j has pure codimension one on X, and that $E_0 \cap E_1$ has pure codimension two on X. Then*

$$i_{0*}(1_{E_0}) = i_{1*}(1_{E_1})$$

in $A^1(X)$.

Proof. We first note that the E_j are l.c.i. schemes over k. Indeed, as X is an l.c.i. scheme over k, X is Cohen-Macaulay, and hence the sections s_j are local non-zero divisors on X. Thus, if X is a local complete intersection in a smooth quasi-projective k-scheme P, both E_1 and E_2 are also local complete intersections in P.

Let $i : E \to X \times \mathbb{A}^1$ be the subscheme of $X \times \mathbb{A}^1$ defined by the equation $ts_1 - (1-t)s_0 = 0$, where t is the standard coordinate on $\mathbb{A}^1$. By our assumptions on E_0 and E_1, $E \cap X \times x$ is a pure codimension one subscheme of $X \times x$ for each point x of $\mathbb{A}^1$; in particular, E is a pure codimension one closed subscheme of $X \times \mathbb{A}^1$. As in the preceeding paragraph, this implies that E is an l.c.i. scheme over k.

If $s_x : \operatorname{Spec} k \to \mathbb{A}^1$ is a k-rational point of $\mathbb{A}^1$, let $i_x : E_x \to X$ denote the pull-back of $i : E \to X \times \mathbb{A}^1$ via i_x, and let $\tilde{s}_x : E_x \to E$ be the inclusion. By axiom (BM2) of definition 1.2, we have

$$s_x^*(i_*(1_E)) = i_{x*}(\tilde{s}_x^*(1_E))$$

As the fundamental class is functorial (remark 1.19), we therefore have $s_x^*(i_*(1_E)) = i_{x*}(1_{E_x})$ for all k-points x of $\mathbb{A}^1$. On the other hand, by the homotopy property (H) of definition 1.2, it follows that $s_x^*(i_*(1_E)) = s_y^*(i_*(1_E))$ for any two k-points x and y. In particular,

$$i_{0*}(1_{E_0}) = s_0^*(i_*(1_E)) = s_1^*(i_*(1_E)) = i_{1*}(1_{E_1}).$$

□

Proposition 7.2. *Let X be an l.c.i. scheme over k, $p : L \to X$ a line bundle. Let $s : X \to L$ be a section of L such that the subscheme $i : E \to X$ defined by $s = 0$ has pure codimension one on X. Then*

$$i_*(1_E) = \tilde{c}_1(L)(1_X)$$

in $A^1(X)$.

Proof. Let $\tilde{i} : p^{-1}(E) \to L$ be the inclusion, and $p_E : p^{-1}(E) \to E$ the map induced by p. We then have

$$\begin{aligned}\tilde{i}_*(1_{p^{-1}(E)}) &= \tilde{i}_*(p_E^*(1_E)) \\ &= p^*(i_*(1_E))\end{aligned}$$

using the functoriality of the fundamental class and axiom (BM2) of definition 1.2. By the homotopy property for A_*, it therefore suffices to show that $\tilde{i}_*(1_{p^{-1}(E)}) = p^*\tilde{c}_1(L)(1_X)$.

For this, let $s_0 : X \to L$ be the zero-section. By definition we have $\tilde{c}_1(L)(1_X) = s_0^*(s_{0*}(1_X))$. Thus

$$\begin{aligned}p^*(\tilde{c}_1(L)(1_X)) &= p^*(s_0^*(s_{0*}(1_X))) \\ &= s_{0*}(1_X),\end{aligned}$$

as $p^* \circ s_{0*} = \mathrm{id}$.

On the other hand, let $\tilde{s} : L \to p^*L$ be the tautological section. Then $s_0 : X \to L$ is the subscheme of L defined by $\tilde{s} = 0$. Since $s_0(X) \cap p^{-1}(E) = E$, which has codimension two on L, we may use lemma 7.1 to conclude that

$$\tilde{i}_*(1_{p^{-1}(E)}) = s_{0*}(1_X),$$

completing the proof. □

7.2. **Divisor classes.** Let A_* be an oriented Borel-Moore homology theory on $\mathcal{V}$. Restricting A_* to $\mathbf{Sm}_k$, we have by proposition 1.4 the associated oriented cohomology theory A^* on $\mathbf{Sm}_k$. In particular, from [4, Introduction], there is a formal group law $(F_A, A^*(k))$, $F_A(u,v) \in A^*(k)[[u,v]]$, such that, for X in $\mathbf{Sm}_k$ and for line bundles L, M on X, we have $F_A(c_1(L), c_1(M)) = c_1(L \otimes M)$. Considering A_* as an oriented Borel-Moore homology theory, this gives the identity of endomorphisms of $A_*(X)$

$$F_A(\tilde{c}_1(L), \tilde{c}_1(M)) = \tilde{c}_1(L \otimes M).$$

Also, since $(F_A, A_*(k))$ is a formal group, there is a canonical ring homomorphism $\phi_A : \mathbb{L}_* \to A_*(k)$ with $\phi_A(F_{\mathbb{L}}) = F_A$.

In particular, given positive integers $n_1, \ldots, n_r$, we may form the power series with $A_*(k)$-coefficients

$$G^{n_1,\ldots,n_r}(u_1, \ldots, u_r)_A := n_1 \cdot_{F_A} u_1 +_{F_A} \ldots +_{F_A} n_r \cdot_{F_A} u_r,$$

and $H_J^{n_1,\ldots,n_r}(u_1, \ldots, u_r)_A$ with

$$G^{n_1,\ldots,n_r}(u_1, \ldots, u_r)_A = \sum_{J,\ ||J|| \le 1} u^J H_J^{n_1,\ldots,n_r}(u_1, \ldots, u_r)_A,$$

by the methods of [4, §5.2] (see also §2.4).

If $E = \sum_{i=1}^r n_i E_i$ is a strict normal crossing divisor on some $W \in \mathbf{Sm}_k$, let $i : |E| \to W$ denote the closed subscheme (not necessarily reduced) defined by E. We have the inclusions of the faces $\iota^J : E^J \to |E|$, and we may define the class $[E \to |E|]_A \in A_*(|E|)$ by

$$[E \to |E|]_A := \sum_{J,\ ||J|| \le 1} \iota^J_*([H_J^{n_1,\ldots,n_r}(\iota^{J*}O_W(E_1), \ldots, \iota^{J*}O_W(E_r)]_A),$$

where $[H_J^{n_1,\ldots,n_r}(\iota^{J*}O_W(E_1), \ldots, \iota^{J*}O_W(E_r)]_A$ denotes the element

$$H_J^{n_1,\ldots,n_r}(\tilde{c}_1(\iota^{J*}O_W(E_1)), \ldots, \tilde{c}_1(\iota^{J*}O_W(E_r))(1_{E^J}) \in A_*(E^J).$$

If $i : |E| \to W$ is the inclusion we write $[E \to W]_A$ for $i_*([E \to |E|]_A)$.

Lemma 7.3. $[E \to W]_A = \tilde{c}_1(O_W(E))(1_W)$.

Proof. For $A_* = \Omega_*$, this is proved in [4, Proposition 5.9]. One can either repeat the proof, replacing Ω_* with A_* throughout, or use the universality of Ω_* as an oriented Borel-Moore homology weak theory [4, Theorem 10.8]: if $\vartheta_A : \Omega_* \to A_*$ is the natural transformation of oriented Borel-Moore homology weak theories on $\mathbf{Sm}_k$, $\vartheta_A(f : Y \to X) = f_*(1_Y^A)$, then it is easy to check that $\vartheta_A([E \to W]_\Omega) = [E \to W]_A$. As ϑ_A interwines the respective Chern class operators, the formula for Ω_* implies the formula for A_*. □

Proposition 7.4. *Let A_* be an oriented Borel-Moore homology theory on $\mathcal{V}$. Let W be in $\mathbf{Sm}_k$, and let E be a strict normal crossing divisor on W. Then*

$$[E \to |E|]_A = 1_{|E|}$$

in $A_*(|E|)$.

Proof. Write $E = \sum_{i=1}^r n_i E_i$, with E_i irreducible. Note that $|E|$ is an l.c.i. scheme over k, so we have a fundamental class $1_{|E|}$. Using Jouanoulou's trick and the homotopy invariance of A_*, we may assume that W is an affine scheme. Thus, the line bundles $O_W(E_i)$ are all very ample, hence there are morphisms $f_i : W \to \mathbb{P}^N$ (for N sufficiently large) with $E_i = f_i^*(X_N = 0)$ (with $X_0, \ldots, X_N$ the standard homogeneous coordinates on $\mathbb{P}^N$). Let $f = (f_1, \ldots, f_r) : W \to (\mathbb{P}^N)^r := \tilde{W}$, and let $\tilde{E}_i$ be the subscheme $p_i^*(X_N = 0)$, where $p_i : (\mathbb{P}^N)^r \to \mathbb{P}^N$ is the projection on the ith factor. Let $\tilde{E} = \sum_{i=1}^r n_i \tilde{E}_i$. Then $f^{-1}(|\tilde{E}|) = |E|$ and $f^*(\tilde{E}) = E$. Letting $f_E : |E| \to |\tilde{E}|$ be the restriction of f, f_E is the Tor-independent pull-back of the l.c.i. morphism f by the regular imbedding $|\tilde{E}| \to (\mathbb{P}^N)^r$, hence f_E is an l.c.i. morphism. It follows from the functoriality of the fundamental class that $f_E^*(1_{|\tilde{E}|}) = 1_{|E|}$. Similarly, $1_{E^J} = f_E^*(1_{\tilde{E}^J})$ and

$$\begin{aligned}[][H_J^{n_1,\ldots,n_r}(\iota^{J*}O_W(E_1), \ldots, \iota^{J*}O_W(E_r)]_A \\ = f_E^*([H_J^{n_1,\ldots,n_r}(\iota^{J*}O_{\tilde{W}}(\tilde{E}_1), \ldots, \iota^{J*}O_{\tilde{W}}(\tilde{E}_r)]_A).\end{aligned}$$

Thus, $f_E^*([\tilde{E} \to |\tilde{E}|]_A) = [E \to |E|]_A$. Therefore, it suffices to prove the result for $W = (\mathbb{P}^N)^r$, $E = \sum_{i=1}^r n_i p_i^*(X_N = 0)$. In this case, applying the axiom (CD), the map $i_* : A_*(|E|) \to A_*(W)$ is injective, where $i : |E| \to W$ is the inclusion, so it suffices to show that

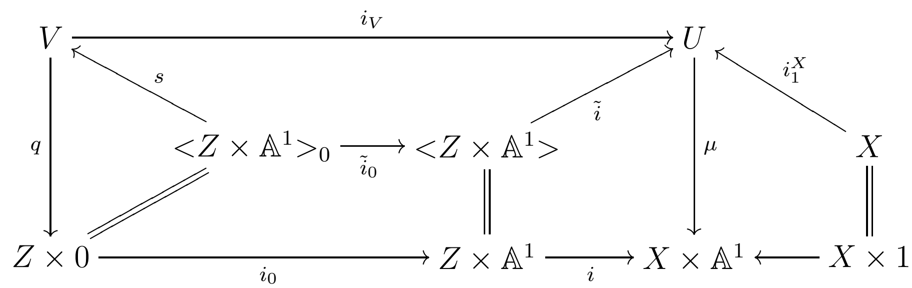
$$i_*([E \to |E|]_A) = i_*(1_{|E|}).$$

By proposition 7.2 and lemma 7.3, both sides are $\tilde{c}_1(O_W(E))(1_W)$, whence the result. □

7.3. **The deformation diagram revisited.** In this paragraph, we show that the method used to define the Gysin morphism for algebraic cobordism is compatible with the pull-back in an arbitrary oriented Borel-Moore homology theory A_*.

Let $i : Z \to X$ be a regular imbedding in $\mathcal{V}$. As in §6.3, let $\mu : W \to X \times \mathbb{A}^1$ be the blow-up of $X \times \mathbb{A}^1$ along $Z \times 0$, let $<X \times 0>$ and $<Z \times \mathbb{A}^1>$ denote the proper transforms of $X \times 0$ and $Z \times \mathbb{A}^1$ respectively, let E be the exceptional divisor of μ, let $V = E \setminus <X \times 0>$ and $U = W \setminus <X \times 0>$. This yields the commutative diagram

(7.1)



The equalities are isomorphisms induced by μ, q is the morphism induced by μ and $<Z \times \mathbb{A}^1>_0$ is the fiber of $<Z \times \mathbb{A}^1>$ over $0 \in \mathbb{A}^1$. We also have the identity $<Z \times \mathbb{A}^1>_0 = <Z \times \mathbb{A}^1> \cap V$, which gives the map s.

It follows from the next lemma and the condition (1.1)(4) that all the schemes in the above diagram are in $\mathcal{V}$.

Lemma 7.5. *1) The map $\tilde{i} : <Z \times \mathbb{A}^1> \to U$ is a regular imbedding.*

2) $\mu : W \to X \times \mathbb{A}^1$ is an l.c.i. morphism; more generally, if $i : T \to Y$ is a regular imbedding of finite type k-schemes, and if $\mu : W \to Y$ is the blow-up of Y along T, then μ is an l.c.i. morphism.

Proof. For (1), the assertion is local on X, so we may assume that X is affine, $X = \operatorname{Spec} R$, and that Z is a complete intersection, defined by a regular sequence $f_0, \ldots, f_N$.

Let $k[y_0, \ldots, y_N]$ be a polynomial ring, and consider the k-algebra homomorphism $\psi : k[y_0, \ldots, y_N] \to R$ defined by sending y_i to f_i. Since $f_0, \ldots, f_N$ is a regular sequence, ψ is a flat extension, at least after inverting some element $z \in R$ outside $(f_0, \ldots, f_N)$. Since a flat extension of a regular imbedding is still a regular imbedding, we see that it suffices to prove the result for $X = \operatorname{Spec} k[y_0, \ldots, y_N]$ and Z the subscheme defined by the maximal ideal $(y_0, \ldots, y_N)$. Letting t be the standard coordinate on $\mathbb{A}^1$, $Z \times 0$ is a complete intersection in $X \times \mathbb{A}^1$, defined by the regular sequence $y_0, \ldots, y_N, t$.

We therefore have the identification of W with the subscheme of $X \times \mathbb{A}^1 \times \mathbb{P}^{N+1}$ defined by the equations

$$\begin{aligned} X_i y_j - X_j y_i = 0,\ 0 \le i, j \le N \\ X_i t - X_{N+1} y_i = 0,\ 0 \le i \le N, \end{aligned}$$

where $X_0, \ldots, X_{N+1}$ are standard homogeneous coordinates on $\mathbb{P}^{N+1}$. Also, we have noted in [4, Lemma 6.1] that $<X \times 0>$ is the subscheme defined by $X_{N+1} = 0$. This gives the description of U as the subscheme of $\mathbb{A}^N \times \mathbb{A}^1 \times \mathbb{A}^{N+1} = \operatorname{Spec} k[y_0, \ldots, y_N, t, x_0 \ldots, x_N]$ defined by the equations

$$\begin{aligned} x_i y_j - x_j y_i = 0,\ 0 \le i, j \le N \\ x_i t - y_i = 0,\ 0 \le i \le N. \end{aligned}$$

Clearly, this gives an isomorphism $U \cong \operatorname{Spec} k[t, x_0, \ldots, x_N]$, and identifies $<Z \times \mathbb{A}^1>$ with the subscheme defined by the ideal $(x_0, \ldots, x_N)$, showing that $<Z \times \mathbb{A}^1> \to U$ is a regular imbedding.

For (2), $\mu : W \to Y$ is projective, so the assertion that μ is an l.c.i. -morphism is local on W. We may thus assume as above that Y is affine and T is a complete intersection, and, as $W \to Y$ is an isomorphism away from T, we may replace Y with any open neighborhood of T in Y. Since the pull-back of an l.c.i. morphism by a flat morphism is an l.c.i. morphism, we may thus assume as above that $Y = \operatorname{Spec} k[y_0, \ldots, y_N]$, and T is the subscheme defined by the ideal $(y_0, \ldots, y_N)$. In this case, W and Y are smooth over k, hence the map $W \to Y$ is an l.c.i. morphism, completing the proof. □

Proposition 7.6. *Let A_* be an oriented Borel-Moore homology theory on $\mathcal{V}$, and let $i_Z : Z \to X$ be a regular imbedding in $\mathcal{V}$. For the diagram* (**??**) *and take $\eta \in A_*(X)$. Suppose there is an element $\tilde\eta$ of $A_{*+1}(U)$ such that $i_1^{X*}(\tilde\eta) = \eta$. Then*

$$q^*(i_Z^*(\eta)) = i_V^*(\tilde\eta).$$

Proof. Let $\tilde{i}_0 : Z \to <Z \times \mathbb{A}^1>$ be the inclusion of the fiber over $0 \in \mathbb{A}^1$, and let $\tilde{i}_1 : Z \to <Z \times \mathbb{A}^1>$ be the inclusions of the fiber over $1 \in \mathbb{A}^1$. Let $\tau = \tilde{i}^*(\tilde\eta)$. Then $\tilde{i}_1^*(\tau) = i_Z^*(i_1^{X*}(\tilde\eta)) = i_Z^*(\eta)$. By the homotopy property (H) for A_*, we have $\tilde{i}_0^*(\tau) = \tilde{i}_1^*(\tau) = i_Z^*(\eta)$. But $\tilde{i}_0^*(\tau) = s^*(i_V^*(\tilde\eta))$ and $q^*s^* = \mathrm{id}$, so

$$\begin{aligned} q^*(i_Z^*(\eta)) &= q^*\tilde{i}_0^*(\tau) \\ &= q^*s^*(i_V^*(\tilde\eta)) \\ &= i_V^*(\tilde\eta). \end{aligned}$$

□

We can also use the deformation diagram to compute i_Z^* of a special type of element.

Lemma 7.7. *Let $i : Z \to X$ be a regular codimension one imbedding, and let $f : Y \to X$ be a projective morphism of finite type k-schemes. Suppose that $f(Y) \subset Z$ and that Y is l.c.i. over k. Then*

$$i^*(f_*(1_Y)) = \tilde{c}_1(i_Z^* O_X(Z))(f_*^Z(1_Y)),$$

where $f^Z : Y \to Z$ is the map induced by f.

Proof. We use the diagram (7.1). The map f^Z gives the map $f^Z \times \mathrm{id} : Y \times \mathbb{A}^1 \to Z \times \mathbb{A}^1$. Mapping $Z \times \mathbb{A}^1$ to U by identifying with $<Z \times \mathbb{A}^1>$, we have the map $\tilde{f} : Y \times \mathbb{A}^1 \to U$. Clearly $\tilde{f}_*(1_{Y \times \mathbb{A}^1})$ is an element $\tilde{\eta}$ of $A_*(U)$ with $i_1^{X*}(\tilde{\eta}) = f_*(1_Y)$. By proposition 7.6, we have

$$q^*(i_Z^*(f_*(1_Y))) = i_V^*(\tilde{\eta}).$$

Since $1_{Y \times \mathbb{A}^1} = p_1^*(1_Y)$, we have

$$i_V^*(\tilde{\eta}) = s_*(f_*(1_Y)).$$

Since s^* and q^* are inverse, we thus have

$$i_Z^*(f_*(1_Y)) = s^* s_*(f_*(1_Y)),$$

and the right-hand side is $\tilde{c}_1(i_Z^* O_X(Z))(f_*^Z(1_Y))$, by definition. □

7.4. **Algebraic cobordism.** We are now ready to prove the results announced in the introduction.

proof of theorem 1.13 and theorem 1.14. From [4, Theorem 10.8], Ω_* has the structure of an oriented Borel-Moore weak homology theory on $\mathcal{V}$ and is in fact the universal such theory. From the results of §6, we have pull-back maps $f^* : \Omega_*(X) \to \Omega_{*+d}(Y)$ for each l.c.i. morphism $f : Y \to X$ in $\mathbf{Sch}_k$, satisfying the axioms (BM1), (BM2) and (BM3). The axioms (PB) and (H) are already valid for an oriented Borel-Moore weak homology theory, so we need only verify the axiom (CD). This follows from lemma 1.23, as Ω_* satisfies the required localization property by [4, Theorem 6.7]. Thus, Ω_* defines an oriented Borel-Moore homology theory on $\mathbf{Sch}_k$. By proposition 1.4, the restriction of the oriented Borel-Moore homology theory Ω_* to $\mathbf{Sm}_k$ defines an extension of the oriented Borel-Moore weak homology theory Ω_* on $\mathbf{Sm}_k$ to an oriented cohomology theory Ω_* on $\mathbf{Sm}_k$.

The uniqueness of the extension of Ω_* to an an oriented Borel-Moore homology theory on $\mathcal{V}$, and to an oriented cohomology theory on $\mathbf{Sm}_k$ follow from the universality of Ω_* as a Borel-Moore homology theory on $\mathcal{V}$ and as an oriented cohomology theory on $\mathbf{Sm}_k$, respectively (theorem 1.15, which we prove below). Indeed, suppose the oriented

Borel-Moore weak homology theory Ω_* on $\mathbf{Sch}_k$ has a second extension $\hat{\Omega}_*$ to an oriented Borel-Moore homology theory on $\mathcal{V}$, with pull-back maps $\hat{f}^*$ for each l.c.i. morphism $f : Y \to X$ in $\mathbf{Sch}_k$. By the universality of Ω_* as an oriented Borel-Moore homology theory, there is a unique natural transformation of oriented Borel-Moore homology theories on $\mathcal{V}$, $\vartheta : \Omega_* \to \hat{\Omega}_*$. But the underlying oriented Borel-Moore weak homology theory Ω_* on $\mathcal{V}$ is also universal, and both Ω_* and $\hat{\Omega}_*$ agree as weak homology theories, so ϑ is the identity transformation, forcing $f^* = \hat{f}^*$ for all l.c.i. morphisms f. The uniqueness of Ω_* as an oriented cohomology theory on $\mathbf{Sm}_k$ is proved the same way. □

proof of theorem 1.15. Let A_* be an oriented Borel-Moore homology theory on $\mathbf{Sch}_k$. By [4, Theorem 10.8], Ω_* is the universal oriented Borel-Moore weak homology theory on $\mathbf{Sch}_k$, so there is a unique natural transformation

$$\vartheta_A : \Omega_* \to A_*$$

of the underlying oriented Borel-Moore weak homology theories, i.e., ϑ_A is compatible with projective push-forward, smooth pull-back, Chern class operators and external products. It thus suffices to show that ϑ_A is compatible with pull-backs for l.c.i. morphisms. We use the notation f^*_A, f^A_*, etc., to indicate which theory we are using.

As ϑ_A is already compatible with smooth pull-back,, we need only check compatibility with respect to regular imbeddings $i_Z : Z \to X$.

It suffices to check that, for $f : Y \to X$ in $\mathcal{M}(X)$, we have

$$i^*_{ZA}(\vartheta_A([f : Y \to X])) = \vartheta_A(i^*_{Z\Omega}([f : Y \to X])).$$

Note that, as $p : Y \to \operatorname{Spec} k$ is smooth and quasi-projective, we have

$$\vartheta_A(1^\Omega_Y) = \vartheta_A(p^*_\Omega(1_\Omega)) = p^*_A(1_A) = 1^A_Y.$$

Thus

$$\begin{aligned}\vartheta_A([f : Y \to X]) &= \vartheta_A(f^\Omega_*(1^\Omega_Y)) \\ &= f^A_*(1^A_Y).\end{aligned}$$

Therefore, we need to show

$$i^*_{ZA}(f^A_*(1^A_Y)) = \vartheta_A(i^*_{Z\Omega}(f^\Omega_*(1^\Omega_Y))).$$

We first reduce to the case of a codimension one regular imbedding by using the deformation diagram (7.1); we retain the notation of §7.3. Let $\eta = f^A_*(1^A_Y)$ and let $\eta_\Omega = f^\Omega_*(1^\Omega_Y)$.

As the map $\mu : U \to X \times \mathbb{A}^1$ is an isomorphism over $X \times \mathbb{A}^1 \setminus Z \times 0$, we can lift $p_1^*\eta_\Omega \in \Omega_{*+1}(X \times \mathbb{A}^1)$ to an element $\eta'_\Omega \in \Omega_{*+1}(U \setminus V)$. In particular, $i_1^{X*}(\eta'_\Omega) = \eta_\Omega$. Since the (smooth) restriction map

$$j^* : \Omega_{*+1}(U) \to \Omega_{*+1}(U \setminus V)$$

is surjective, we can lift η'_Ω to an element $\tilde{\eta}_\Omega \in \Omega_{*+1}(U)$. Let $\tilde{\eta} = \vartheta_A(\tilde{\eta}_\Omega)$. Then, as $i_{1A}^{X*} \circ p_{1A}^* = \mathrm{id}$, we have

$$i_{1A}^{X*}(\tilde{\eta}) = \eta.$$

We may therefore apply proposition 7.6, giving

$$q_A^*(i_{ZA}^*(\vartheta_A(f_*^\Omega(1_Y^\Omega)))) = i_{VA}^*(\vartheta_A(\tilde{\eta}_\Omega)).$$

On the Ω side, we similarly have

$$q_\Omega^*(i_{Z\Omega}^*(f_*^\Omega(1_Y^\Omega))) = i_{V\Omega}^*(\tilde{\eta}_\Omega).$$

Since q is smooth, and q_A^*, q_Ω^* are isomorphisms, this reduces us to the case of a codimension one regular imbedding $i_Z : |Z| \to X$, where Z is a Cartier divisor on X.

If $f(Y) \subset |Z|$, let $f^Z : Y \to |Z|$ be the map induced by f. By lemma 7.7, we have

$$i_{ZA}^*(f_*^A(1_Y^A)) = \tilde{c}_1^A(i_Z^* O_X(Z))(f_*^{ZA}(1_Y^A))$$
$$i_{Z\Omega}^*(f_*^\Omega(1_Y^\Omega)) = \tilde{c}_1^\Omega(i_Z^* O_X(Z))(f_*^{Z\Omega}(1_Y^\Omega))$$

Since ϑ_A is compatible with Chern class operators, push-forward and units 1_Y for $Y \in \mathbf{Sm}_k$, we have the desired compatibility in this case. Thus, we have

$$i_{ZA}^*(\vartheta_A(i_{Z*}^\Omega(\rho))) = \vartheta_A(i_{Z\Omega}^*(i_{Z*}^\Omega(\rho)))$$

for all $\rho \in \Omega_*(Z)$.

If $f(Y) \not\subset |Z|$, then there is a projective birational map $\tau : Y' \to Y$, which is an isomorphism over $X \setminus |Z|$, such that $(f\tau)^*(Z)$ is a strict normal crossing divisor on Y'. As $f - (f\tau)$ vanishes in $\Omega_*(X \setminus |Z|)$, it follows from the localization sequence

$$\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X) \to \Omega_*(X \setminus |Z|) \to 0$$

that $f = (f\tau) + i_*(\rho)$ for some $\rho \in \Omega_*(|Z|)$, Thus, we may assume that f^*Z is a strict normal crossing divisor on Y with associated codimension one subscheme $i_{f^*Z} : |f^*Z| \to Y$.

Let $f^Z : |f^*Z| \to |Z|$ be the map induced by f. Since Y is smooth, the maps $f : Y \to X$ and $i_Z : |Z| \to X$ are Tor-independent, hence transverse in $\mathbf{Sch}_k$. But, as the diagram

$$\begin{array}{ccc} |f^*Z| & \xrightarrow{i_{f^*Z}} & Y \\ {\scriptstyle f^Z}\downarrow & & \downarrow{\scriptstyle f} \\ |Z| & \xrightarrow[i_Z]{} & X \end{array}$$

is cartesian, it follows from axiom (BM2) that, for both the theory Ω_* and the theory A_*, we have

$$\begin{aligned} i_Z^*(f_*(1_Y)) &= f_*^Z(i_{f^*Z}^*(1_Y)) \\ &= f_*^Z(1_{|f^*Z|}). \end{aligned}$$

By proposition 7.4, we have (in both theories)

$$1_{|f^*Z|} = [f^*Z \to |f^*Z|].$$

Thus

$$\begin{aligned} i_{ZA}^*(f_*^A(1_Y^A)) &= f_*^{ZA}([f^*Z \to |f^*Z|]_A), \\ i_{Z\Omega}^*(f_*^\Omega(1_Y^\Omega)) &= f_*^{Z\Omega}([f^*Z \to |f^*Z|]_\Omega). \end{aligned} \tag{7.2}$$

Let E be a Cartier divisor on some $W \in \mathbf{Sm}_k$. Since the divisor class $[E \to |E|]_A \in A_*(|E|)$ depends only on the weak homology theory underlying A_*, we have

$$\vartheta_A([E \to |E|]_\Omega) = [E \to |E|]_A$$

This, together with (7.2), shows that

$$\begin{aligned} i_{ZA}^*(f_*^A(1_Y^A)) &= f_*^{ZA}(\vartheta_A([f^*Z \to |f^*Z|]_\Omega)) \\ &= \vartheta_A(f_*^{Z\Omega}([f^*Z \to |f^*Z|]_\Omega)) \\ &= \vartheta_A(i_{Z\Omega}^*(f_*^\Omega(1_Y^\Omega))). \end{aligned}$$

This completes the proof of the universality of Ω_* as an oriented Borel-Moore homology theory on $\mathbf{Sch}_k$.

To show that Ω_* is the universal oriented Borel-Moore homology theory on $\mathbf{Sm}_k$, we will need the following result:

Lemma 7.8 ([4, Lemma 7.1]). *Let X be in $\mathbf{Sm}_k$ and let $i : Z \to X$ be a smooth closed subscheme. Then $\Omega_*(X)$ is generated by standard cobordism cycles of the form $[f : Y \to X]$, with f transverse to i.*

Let A_* be an oriented Borel-Moore homology theory on $\mathbf{Sm}_k$. As above, we have the unique natural transformation $\vartheta_A : \Omega_* \to A_*$ of weak homology theories on $\mathbf{Sm}_k$, and we need to show that ϑ_A intertwines the pull-back maps g^*_Ω and g^*_A for each morphism $g : X' \to X$ in $\mathbf{Sm}_k$.

It suffices to prove the result for g a regular imbedding $i : Z \to X$, and for elements of $\Omega_*(X)$ of the form $[f : Y \to X]$, $f \in \mathcal{M}(X)$. By the lemma, we need only consider maps f which are transverse to i in $\mathbf{Sm}_k$. Form the cartesian diagram

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{i}} & Y \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[i]{} & X \end{array}$$

Using axiom (BM3), we have

$$\begin{aligned} i^*_\Omega(f : Y \to X) &= i^*_\Omega(f^\Omega_*(1^\Omega_Y)) \\ &= \tilde{f}^\Omega_*(\tilde{i}^*_\Omega(1^\Omega_Y)) \\ &= \tilde{f}^\Omega_*(1_{\tilde{Y}}). \end{aligned}$$

Similarly, as $\vartheta_A(f : Y \to X) = f^A_*(1^A_Y)$, we have

$$i^*_A(\vartheta_A(f)) = \tilde{f}^A_*(1^A_{\tilde{Y}}).$$

Thus, as ϑ_A is compatible with push-forward and units, we have

$$\vartheta_A(i^*_\Omega(f)) = i^*_A(\vartheta_A(f)),$$

as desired. □

8. *Some applications*

Having extended Ω_* to an oriented Borel-Moore homology theory on $\mathbf{Sch}_k$ (assuming k admits resolution of singularities), we are able to extend some of the main applications of the theory from smooth varieties to l.c.i. schemes over k. For most of this section, k will be a field of characteristic zero, although the paragraphs §8.1 and §8.2 are valid over an arbitrary field.

8.1. **Chern classes and Conner-Floyd classes.** Let A_* be an oriented Borel-Moore weak homology theory on an admissible $\mathcal{V} \subset \mathbf{Sch}_k$. In [4, Lemma 10.12] we showed how to define Chern class operators $\tilde{c}_i(E) : A_*(X) \to A_{*-i}(X)$, $i = 0, \ldots, n$, for each rank n vector bundle $E \to X$, $X \in \mathcal{V}$, satisfying the "standard" properties. Also, given a sequence $\tau = (\tau_i) \in \prod_{i=0}^\infty A_i(k)$ with $\tau_0 = 1$, we defined in [4, Lemma 10.14], for each vector bundle $E \to X$, $X \in \mathcal{V}$, a degree zero endomorphism $\tilde{c}_\tau : A_*(X) \to A_*(X)$, with properties listed in that lemma.

Now let A_* be an oriented Borel-Moore homology theory on an admissible $\mathcal{V} \subset \mathbf{Sch}_k$. Exactly the same construction gives the Chern class operators $\tilde{c}_i(E) : A_*(X) \to A_{*-i}(X)$ for $E \to X$ a vector bundle on $X \in \mathcal{V}$, and, given a sequence τ as above, the degree-zero endomorphisms $c_\tau(E)$ of $A_*(X)$. These satisfy exactly the same properties as in the case of the weak homology theory, with the added property of functoriality with respect to l.c.i. pull-back. For the reader's convenience, we list these properties here:

Lemma 8.1. *Let A_* be an oriented Borel-Moore homology theory on $\mathcal{V}$. Then the Chern classes satisfy the following properties:*

(0) *Given vector bundles $E \to X$ and $F \to X$ on $X \in \mathcal{V}$ one has*

$$\tilde{c}_i(E) \circ \tilde{c}_j(F) = \tilde{c}_j(F) \circ \tilde{c}_i(E)$$

for any (i, j).

(1) *For any line bundle L, $\tilde{c}_1(L)$ agrees with the one given in axiom (PB) of definition 1.2, applied to A_*.*

(2) *For any l.c.i. morphism $Y \to X \in \mathcal{V}$, and any vector bundle $E \to X$ over X one has*

$$\tilde{c}_i(f^*E) \circ f^* = f^* \circ \tilde{c}_i(E).$$

(3) *If $0 \to E' \to E \to E'' \to 0$ is an exact sequence of vector bundles over X, then for each integer $n \geq 0$ one has the following equation in End$(A_*(X))$:*

$$\tilde{c}_n(E) = \sum_{i=0}^{n} \tilde{c}_i(E')\, \tilde{c}_{n-i}(E'').$$

(4) *For any projective morphism $Y \to X$ in $\mathcal{V}$ and any vector bundle $E \to X$ over X, one has*

$$f_* \circ \tilde{c}_i(f^*E) = \tilde{c}_i(E) \circ f_*.$$

Moreover, the Chern class operators are characterized by the properties (0)-(3).

Lemma 8.2. *Let A_* be an oriented Borel-Moore homology theory on $\mathcal{V}$ and let $\tau = (\tau_i) \in \Pi_{i=0}^{\infty} A_i(k)$, with $\tau_0 = 1$. Then one can define in a unique way, for each $X \in \mathcal{V}$ and each vector bundle E on X, an endomorphism (of degree zero)*

$$\tilde{c}_\tau(E) : A_*(X) \to A_*(X)$$

such that the following holds:

(0) *Given vector bundles $E \to X$ and $F \to X$ one has*

$$\tilde{c}_\tau(E) \circ \tilde{c}_\tau(F) = \tilde{c}_\tau(F) \circ \tilde{c}_\tau(E).$$

(1) *For a line bundle L one has:*

$$\tilde{c}_\tau(L) = \sum_{i=0}^{\infty} \tilde{c}_1(L)^i \, \tau_i.$$

(2) *For any l.c.i. morphism $Y \to X$ in $\mathcal{V}$, and any vector bundle $E \to X$ over X one has*

$$\tilde{c}_\tau(f^*E) \circ f^* = f^* \circ \tilde{c}_\tau(E).$$

(3) *If $0 \to E' \to E \to E'' \to 0$ is an exact sequence of vector bundles over X, then one has:*

$$\tilde{c}_\tau(E) = \tilde{c}_\tau(E') \circ \tilde{c}_\tau(E'').$$

(4) *For any projective morphism $Y \to X$ in $\mathcal{V}$ and any vector bundle $E \to X$ over X, one has*

$$f_* \circ \tilde{c}_\tau(f^*E) = \tilde{c}_\tau(E) \circ f_*.$$

Remark 8.3. If $E \to X$ is a vector bundle on an l.c.i. k-scheme X in $\mathcal{V}$, we may evaluate the operators $\tilde{c}_i(E)$ or $\tilde{c}_\tau(E)$ on 1_X, yielding the Chern classes $c_i(E) \in A^i(X)$ and the total Conner-Floyd class $c_\tau(E) \in A^0(X)$.

It follows directly from the two lemmas above that the Chern class operators $\tilde{c}_i(E)$ and the total Conner-Floyd operator $\tilde{c}_\tau(E)$ depend only on the class of E in $K^0(X)$.

As in [4, §10.5, Example 10.16], we consider the "universal" example: Let A_* be a Borel-Moore homology theory on $\mathcal{V}$, let $\mathbb{Z}[\mathbf{t}] := \mathbb{Z}[t_1, t_2, \ldots, t_n, \ldots]$, with t_i having degree i, and consider the Borel-Moore homology theory $X \mapsto A_*(X)[\mathbf{t}] := A_*(X) \otimes \mathbb{Z}[\mathbf{t}]$. Let $\mathbf{t}$ be the family $(1, t_1, t_2, \ldots)$. For each line bundle $L \to X$, $X \in \mathcal{V}$, we thus have the automorphism

$$\tilde{c}_{\mathbf{t}}(L) = \sum_{i=0}^{\infty} \tilde{c}_1(L)^i t_i : A_*(X)[\mathbf{t}] \to A_*(X)[\mathbf{t}]$$

and for each vector bundle $E \to X$ the automorphism $\tilde{c}_{\mathbf{t}}(E)$, which we expand as

$$\tilde{c}_{\mathbf{t}}(E) = \sum_{(n_1, n_2, \ldots)} \tilde{c}_{n_1, n_2, \ldots, n_r}(E) t_1^{n_1} \cdot \ldots \cdot t_r^{n_r}.$$

As in [4], the endomorphisms $\tilde{c}_{n_1, n_2, \ldots, n_r}(E)$ are called the *Conner-Floyd* endomorphisms for E. For X an l.c.i. k-scheme, we have as well the Conner-Floyd classes $c_{n_1, n_2, \ldots, n_r}(E) = \tilde{c}_{n_1, n_2, \ldots, n_r}(E)(1_X) \in A_{d - \sum_i i n_i}(X)$, $d = \dim_k X$.

We write $\tilde{c}^A_{n_1, n_2, \ldots, n_r}(E)$ or $c^A_{n_1, n_2, \ldots, n_r}(E)$ if we need to specify A.

8.2. **Twisting a Borel-Moore homology theory.** We extend the constructions of [4, §10.5] from oriented Borel-Moore weak homology theories to oriented Borel-Moore homology theories, and from smooth k-schemes to l.c.i. k-schemes.

Let $f : Y \to X$ be an l.c.i. morphism. Choose a factorization of f as $f = qi$, with $i : Y \to P$ a regular imbedding and $q : P \to X$ a smooth morphism. We have the *relative tangent bundle* $T_q \to P$, defined as the vector bundle whose dual has sheaf of sections the relative differentials $\Omega^1_{Y/X}$. Letting $\mathcal{I}$ be the ideal sheaf of Y in P, we let $N_i \to Y$ be the bundle whose dual has sheaf of sections $\mathcal{I}/\mathcal{I}^2$. We let $[N_f] \in K^0(Y)$ be the class $N_i - i^*T_q$. It is easy to see that $[N_f]$ is independent of the choice of the factorization of f.

If $X = \operatorname{Spec} k$, we write $[N_Y]$ for $[N_f]$, and set $[T_Y] := -[N_Y]$. If $f : Y \to X$ is an arbitrary morphism of l.c.i. k-schemes, define the *virtual tangent bundle* of f by $[T_f] := [T_Y] - f^*[T_X] \in K^0(Y)$.

Given a Borel-Moore homology theory A_* on $\mathcal{V}$, and a family τ as in the previous paragraph, we may twist A_* by τ, forming the Borel-Moore homology theory $A_*^{(\tau)}$ with the same push-forward maps as A_*, and with

$$f^*_{(\tau)} = \tilde{c}_\tau([N_f]) \circ f^*$$

for an l.c.i. morphism f. For a line bundle $L \to X$, one has the Chern class operator

$$\tilde{c}_1^{(\tau)}(L) = \tilde{c}_\tau(L) \circ \tilde{c}_1(L).$$

One easily checks that this does define an oriented Borel-Moore homology theory on $\mathcal{V}$.

If $\mathcal{V} \subset \mathbf{Lci}$, we can extend the construction given in [4, §10.5] for $\mathcal{V} = \mathbf{Sm}_k$. Define the Borel-Moore homology theory A_*^τ on $\mathcal{V}$ as having the same l.c.i. pull-backs as A_*, with push-forward

$$f_*^\tau := f_* \circ \tilde{c}_\tau([T_f])$$

for $f : Y \to X$ projective. The Chern class operators are given by

$$\tilde{c}_1^\tau(L) = \tilde{c}_1(L) \circ \tilde{c}_\tau(-L)$$

for each line bundle $L \to X$, $X \in \mathcal{V}$.

For a family $\tau \in \prod_{i=0}^\infty A_i(k)$ as above, we let $\tau^{(-1)}$ be the family with

$$\sum_{i=0}^\infty \tau_i^{(-1)} u^i = \frac{1}{\sum_{i=0}^\infty \tau_i u^i}.$$

As in [4, Lemma 10.17], the isomorphisms

$$\tilde{c}_\tau([N_X]) : A_*^{(\tau)}(X) \cong A_*^{\tau^{(-1)}}(X)$$

determine an isomorphism of oriented Borel-Moore homology theories $A_*^{(\tau)} \to A_*^{\tau^{(-1)}}$ on $\mathcal{V} \subset \mathbf{Lci}$.

8.3. **Operations on Ω_*.** For the remainder of this section, we assume that k admits resolution of singularities.

Take $\mathcal{V} = \mathbf{Lci}_k$, and consider the universal twisting $\Omega_*[\mathbf{t}]^{\mathbf{t}}$ of Ω_*. By the universality of Ω_*, we have a canonical transformation of Borel-Moore homology theories

$$\vartheta^{LN} : \Omega_* \to \Omega_*[\mathbf{t}]^{\mathbf{t}},$$

which we expand as

$$\vartheta^{LN} = \sum_{I=(n_1,\dots,n_r,\dots)} s_I t^I.$$

The individual terms $s_{(n_1,\dots,n_r)} : \Omega_* \to \Omega_{*-n}$, $n = \sum_i in_i$, are the Landweber-Novikov operations defined in [4, Example 10.19]; our extension of ϑ^{LN} to the setting of Borel-Moore homology theories has just verified that the Landweber-Novikov operations are natural with respect to l.c.i. pull-back.

Taking $\mathcal{V} = \mathbf{Sch}_k$ and using the other twisting $\Omega_*[\mathbf{t}]^{(\mathbf{t})}$, we have the natural transformation $\vartheta^{LN'} : \Omega_* \to \Omega_*[\mathbf{t}]^{(\mathbf{t})}$ of homology theories on $\mathbf{Sch}_k$. Using the canonical transformation $\Omega_* \to \mathrm{CH}_*$, we have the natural transformation

$$\vartheta^{CF} : \Omega_* \to \mathrm{CH}_*[\mathbf{t}]^{(\mathbf{t})}.$$

Expanding ϑ^{CF} as

$$\vartheta^{CF} = \sum_I c_I^{CF} t^I$$

defines the tranformations $c^{CF}_{n_1,\dots,n_r} : \Omega_* \to \mathrm{CH}_{*-n}$, $n = \sum_i in_i$.

Explicitly, we have

Lemma 8.4. *Let $\pi : Y \to \operatorname{Spec} k$ be an l.c.i. k-scheme. For $f : Y \to X$ a projective morphism with $X \in \mathbf{Sch}_k$, we have*

$$c^{CF}_{n_1,\dots,n_r}([f : Y \to X]) = f_*^{\mathrm{CH}}(c^{\mathrm{CH}}_{n_1,\dots,n_r}([N_Y])).$$

Proof. Note that $1_Y = \pi^*(1)$ by definition, and $[f : Y \to X] = f_*(1_Y)$. Thus, since ϑ^{CF} is a natural transformation of oriented Borel-Moore homology theories, we have

$$\begin{aligned}\vartheta^{CF}([f:Y\to X]) &= \vartheta^{CF}(f_*^{\Omega}(\pi_\Omega^*(1)))\\ &= f_*^{\mathrm{CH}_*[\mathbf{t}]^{(\mathbf{t})}}(\pi^*_{\mathrm{CH}_*[\mathbf{t}]^{(\mathbf{t})}}(1))\\ &= \sum_I f_*^{\mathrm{CH}}\big(\tilde{c}_I^{\mathrm{CH}}([N_Y])(\pi^*_{\mathrm{CH}}(1))\big)t^I\\ &= \sum_I f_*^{\mathrm{CH}}(c_I^{\mathrm{CH}}([N_Y]))t^I.\end{aligned}$$

Equating the coefficients of $t_1^{n_1}\cdot\ldots\cdot t_r^{n_r}$ yields the result. □

The same proof shows that, for l.c.i. k-schemes X and Y and a projective morphism $f:Y\to X$, we have

$$s_{n_1,\ldots,n_r}([f:Y\to X]) = f_*^{\Omega}(c^{\Omega}_{n_1,\ldots,n_r}([T_f]))\in\Omega_*(X). \tag{8.1}$$

Let $P(x_1,\ldots,x_d)$ be a degree d weighted-homogeneous polynomial with coefficients in a commutative ring R, where we give x_i degree i. Let $\pi:X\to\operatorname{Spec}k$ be a projective l.c.i. k-scheme of dimension d over k. Define $P(X)\in R$ by

$$P(X):=\deg(P(c_1^{\mathrm{CH}},\ldots,c_d^{\mathrm{CH}})([N_X])).$$

Here $\deg:\mathrm{CH}_0(X)\otimes R\to R$ is map induced by the composition of the push-forward $\pi_*:\mathrm{CH}_0(X)\to\mathrm{CH}_0(k)$ followed by the canonical isomorphism $\mathrm{CH}_0(k)\cong\mathbb{Z}$.

Using lemma 8.4, the following result has the same proof as [4, Lemma 13.19].

Proposition 8.5. *Let R be a commutative ring, and let $P(x_1,\ldots,x_d)$ be a degree d weighted-homogeneous polynomial with coefficients in R. There is a unique homomorphism*

$$P_\Omega:\Omega_d\to R$$

with $P_\Omega(\pi_*(1_X)) = P(X)$ for each projective l.c.i. k-scheme $\pi:X\to$ $\operatorname{Spec}k$ of dimension d over k.

8.4. **Degree formulas for Ω_*.** For the remainder of this section, k will be a field of characteristic zero. We can now extend many of the results of [4, §13.3-.4] from smooth k-schemes to l.c.i. k-schemes. For example, the extensions theorem 1.20 and corollary 1.21 of [4, Theorem 13.6 and Corollary 13.7] apply to Ω_*.

Let $\pi:X\to\operatorname{Spec}k$ be a projective l.c.i. k-scheme of dimension d over k. We write $[X]\in\Omega_d(k)$ for $\pi_*(1_X)$. For X smooth over k, this agrees with the terminology used in [4].

We recall from [4, §13.2] that, for X a projective purely d dimensional k-scheme, we have the ideal $M(X)$ of $\Omega_*(k)$, defined as the ideal generated by the classes $[Y] \in \Omega_*(k)$, for those Y smooth and projective over k, of dimension $< d$, for which there is a morphism $f : Y \to X$ over k. More generally, if X is locally equi-dimensional over k, define $M(X)$ as the ideal generated by the $M(X_i)$, as X_i runs over the connected components of X.

With these notations, [4, Theorem 13.15] generalizes to

Theorem 8.6. *Let k be a field of characteristic zero. Let X and Y be reduced projective l.c.i. k-schemes and let $f : Y \to X$ be a morphism. Suppose that X is irreducible. Then one has*

$$[Y] - \deg(f) \cdot [X] \in M(X).$$

The proof is exactly the same as for [4, Theorem 13.15].

Remark 8.7. Let X be a finite type k-scheme of dimension d over k, and let η be an arbitrary element of $\Omega_*(X)$. It follows from the generalized degree formula [4, Theorem 13.8] applied to $A_* = \Omega_*$ that $\Omega_*(X)$ is generated as an $\Omega_*(k)$-module by classes of the form $f : Y \to X \in \mathcal{M}(X)$, with $\dim_k Y \leq d$. Thus $M(X)$ is the ideal in $\Omega_*(k)$ generated by the classes $[Y]$, where Y is a projective l.c.i. k-scheme of dimension $< d$ for which there is a morphism $f : Y \to X$.

Recall from [4, §13.3] the $\mathbb{Z}$-valued characteristic class s_d and the $\mathbb{F}_p$-valued characteristic classes $t_{d,r}$. By lemma 8.4 and proposition 8.5, these characteristic classes extend uniquely to projective l.c.i. k-schemes Y by the formulas

$$s_d(Y) := \deg(s_d(c_1, \ldots, c_d)([N_Y])),$$
$$t_{d,r}(Y) := \deg(t_{d,r}(c_1, \ldots, c_{dr})([N_Y])),$$

and these functions uniquely define homomorphisms

$$s_d^\Omega : \Omega_d(k) \to \mathbb{Z},$$
$$t_{d,r}^\Omega : \Omega_{dr}(k) \to \mathbb{F}_p,$$

with $s_d(Y) = s_d^\Omega([Y])$ and $t_{d,r}(Y) = t_{d,r}^\Omega([Y])$ for all projective l.c.i. k-schemes Y.

Noting these remarks, the proofs of [4, Theorems 13.23 and 13.24] yield the following extensions of these results to l.c.i. schemes:

Theorem 8.8. *Let $f : Y \to X$ be a morphism of reduced projective l.c.i. k-schemes, both of dimension d, with X irreducible. Suppose that*

$d = p^n - 1$ for some prime p and some integer $n > 0$. Then there is a zero-cycle η on X such that

$$s_d(Y) - (\deg f)s_d(X) = p \cdot \deg(\eta).$$

Theorem 8.9. *Let $f : Y \to X$ be a morphism of reduced projective l.c.i. k-schemes, with X irreducible. Suppose both X and Y have dimension rd over k, where $r > 0$ is an integer and $d = p^n - 1$ for some prime p and some integer $n > 0$. Suppose in addition that X admits a sequence of surjective morphisms to reduced finite type k-schemes*

$$X = X_0 \to X_1 \to \ldots \to X_{r-1} \to X_r = \operatorname{Spec} k$$

such that:

1. *Each X_i is in $\mathbf{Lci}_k$ and $\dim_k X_i = d(r - i)$.*
2. *Let η be a zero-cycle on $X_i \times_{X_{i+1}} \operatorname{Spec} k(X_{i+1})$. Then $p | \deg(\eta)$.*

Then

$$t_{d,r}(Y) = \deg(f)t_{d,r}(X).$$

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