$\mathbf{A}^1$-REPRESENTABILITY OF HERMITIAN K-THEORY AND WITT GROUPS

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Abstract

We show that hermitian K-theory and Witt groups are representable both in the unstable and in the stable $\mathbf{A}^1$-homotopy category of Morel and Voevodsky. In particular, Balmer Witt groups can be nicely expressed as homotopy groups of a topological space. The proof and its consequences include other new results related to Nisnevish-Mayer-Vietoris squares, Laurent polynomials, the projective line, blow ups and homotopy purity. The results will hopefully become part of a proof of Morel's conjecture on certain $\mathbf{A}^1$-homotopy groups of spheres.

Introduction

Hermitian K-theory is the natural framework to describe objects in an exact category equipped with a non-degenerate symmetric bilinear form by the standard methods of algebraic K-theory. If our exact category are the finitely generated projective modules $P(A)$ over some fixed base ring A in which 2 is invertible, then the hermitian K-groups K_*^h are the the homotopy groups of $K_0^h(A) \times BO(A)^+$ where $K_0^h(A) = GW(A)$ is just the *Grothendieck-Witt group* (take the Grothendieck group of the monoid of isomorphism classes of objects in $P(A)$ equipped with a symmetric bilinear non-degenerate form). In the theory of quadratic or symmetric bilinear forms, an even more classical object of study is the Witt group $W(A)$. It is the quotient of $GW(A)$ when identifying the hyperbolic objects with 0. Recently, Balmer introduced a graded 4-periodic generalization W_B^* of Witt groups. This is defined for triangulated categories with duality in general. When applied to the bounded derived category $D^b(P(A))$ of finitely generated projective modules over a given ring, he rediscovers the classical Witt group in degree 0 [Ba1]. His theory allows him to prove powerful theorems, see [Ba2] and many others. We will show in this paper that there is a space (and in fact a spectrum) whose homotopy groups coincide with W_B^*.

Both real and complex topological K-theory are representable in the classical stable homotopy category (*i.e.*, the homotopy category of spectra) as are cohomology theories in general. Morel and Voevodsky proved that algebraic K-theory is representable in the unstable [MV] and stable [Vo] $\mathbf{A}^1$-homotopy category. In this article, we prove a similar result for hermitian K-theory and Balmer Witt groups: They are representable in the unstable and in the stable $\mathbf{A}^1$-homotopy category as well. Our motivation is at least twofold: On the one hand side, this implies that we have theorems on hermitian K-theory and Balmer Witt groups not only for elementary distinguished (*i.e.*, Nisnevish) squares but also for blow-ups and Gysin morphisms (*cf.* Corollaries 6.2 and 6.3) resulting from the corresponding triangles in the $\mathbf{A}^1$-homotopy category [MV, 3.2.23, 3.2.29],[Vo, 4.11, 4.12, 4.13]. On the other side, we hope that this result is a first step in proving the following conjecture of Fabien Morel [Mo1], [Mo2] on stable $\mathbf{A}^1$-homotopy groups of spheres:

0.1 Conjecture. *Let k be a field, $n > 0$, $\mathcal{SH}(k)$ be the stable $\mathbf{A}^1$-homotopy category as defined in [Vo] and S^0 the sphere spectrum over Spec(k). Then there are isomorphisms*

$$Hom_{\mathcal{SH}(k)}(S^0, S^0) \cong K_0^h(k)$$

and

$$Hom_{\mathcal{SH}(k)}(\mathbf{G}_m^{\wedge n}, S^0) \cong W(k) \ .$$

Recall that the sphere spectrum is given by $(S^0)_n = (\mathbf{P}_k^1)^{\wedge n}$ and $\mathbf{P}_k^1$ is homotopy equivalent to the smash product of the simplicial circle and the multiplicative group [MV, p.110-112]. We know that $K_0^h(k) = GW(k)$ is generated by the elements $< u >$, $u \in k^*$ which are subject to the relations $< u > = < uv^2 >$ and $< u > + < v > = < u+v > + < (u+v)uv > \ \forall\ u+v \neq 0$ [Sc, p.66].

Let us denote the unstable $\mathbf{A}^1$-homotopy category of [MV] by $\mathcal{H}(k)$ and the morphism $\mathbf{P}_k^1 \to \mathbf{P}_k^1$ mapping $[x, y]$ to $[ux, y]$ by μ_u. We have a stabilization map $\Sigma_{\mathbf{P}^1}^{\infty} : Hom_{\mathcal{H}(k)}(\mathbf{P}_k^1, \mathbf{P}_k^1) \to Hom_{\mathcal{SH}(k)}(S^0, S^0)$. Lannes and Morel show that the morphism $K_0^h(k) \to Hom_{\mathcal{SH}(k)}(S^0, S^0)$ given by mapping $< u >$ to $\Sigma_{\mathbf{P}^1}^{\infty}\mu_u$ is well defined. The isomorphism in Conjecture 0.1 above should be given by this morphism. The main evidence for the part of conjecture about $K_0^h(k)$ is the the fact that the topological Adams spectral sequences has a motivic counterpart converging to a certain completion of $K_0^h(k)$, see [Mo1] for the details.

To conclude that ordinary algebraic K-theory is representable not only in $\mathcal{H}(k)$, but also in $\mathcal{SH}(k)$, Voevodsky proves a periodicity theorem [Vo, Theorem 6.8] which is essentially Quillen's projective bundle theorem [Q1, 8.2]. This is more or less the algebraic counterpart of Bott periodicity in complex topological K-theory.

Our strategy to prove stable representability for hermitian K-theory and Witt groups in section 5 relies on the study of $K_*^h(R[t, t^{-1}])$ and $W_B^*(R[t, t^{-1}])$ instead. Our results also give a variant of Voevodsky's proof [Vo] for the representability of algebraic K-theory (see Remark 5.8). We obtain a periodicity theorem for hermitian K-theory corresponding to the one in real topological K-theory (see Corollary 5.3) and an $\Omega_{\mathbf{P}^1}$-spectrum **KO** which follows the same periodicity pattern as real topological K-theory. The representability of hermitian K-theory yields a morphism $Hom_{\mathcal{SH}(k)}(S^0, S^0) \to Hom_{\mathcal{SH}(k)}(S^0, \mathbf{KO}) \cong K_0^h(k)$ which conjecturally - following Morel - should be the inverse of the morphism constructed above. The details are explained in section 6.

After some discussions with Paul Balmer and Fabien Morel, it became clear that one can deduce unstable representability of algebraic K-theory from two geometric facts: Nisnevish-Mayer-Vietoris [MV, "Brown-Gersten property" 3.1.13] and homotopy invariance. Then one can work with any fibrant replacement of the K-theory presheaf instead of using the explicit construction of Thomason [Th] as done in [MV, Proposition 4.3.9]. The situation is just as in topology where the behaviour of a cohomology theory (fulfilling homotopy invariance and Mayer-Vietoris) for cofibrant spaces (e.g., CW-complexes) is determined by its values on a point. The proof of the representability of hermitian K-theory we give in section 3 also follows from these two properties of hermitian K-theory. It works in a much more general setting, see Theorem 3.1 for this general $\mathbf{A}^1$-representability theorem.

It follows a more detailed overview of this article. We prove homotopy invariance and Nisnevish-Mayer-Vietoris for hermitian K-theory of regular affine schemes in section 1, using some techniques of my joint work with Marco Schlichting [HS] and the corresponding results for K-theory and Balmer Witt groups. But hermitian K-theory K^h as defined in [Ho] is not sufficiently well developed yet to prove such results for non-affine schemes. In section 2, we will extend the definition of hermitian K-theory from affine schemes to non-affine schemes using techniques of [Jo],[TT],[We] in a way that this new theory KO automatically fulfills Nisnevish-Mayer-Vietoris and homotopy invariance. We do not prove that $K^h(X) \simeq KO(X)$ for non-affine X, although we conjecture that this is true if X is regular.

In fact, once we have a fibrant presheaf on affine schemes fulfilling Nisnevish-Mayer-Vietoris and homotopy invariance, we can always extend it to non-affine schemes as done in section 2 and prove representability by some fibrant replacement of its sheafification as in Theorem 3.1.

Balmer Witt groups are defined purely algebraically. In order to prove unstable representability of Balmer Witt groups W_B^*, we first show that they are isomorphic to the homotopy groups of the homotopy colimit of the Karoubi tower (Lemma 4.6). The proof that the simplicial sheaf obtained this way represents W_B^* is then similar to the proof of the analogous result for hermitian K-theory. That is, we need the Nisnevish-Mayer-Vietoris property for W_B^* to show representability in the simplicial homotopy category of Joyal and Jardine [Ja1] and homotopy invariance to show representability in the unstable $\mathbf{A}^1$-homotopy category of Morel and Voevodsky [MV]. All this is carried out in section 4.

In section 5, we see that computations of the hermitian K-theory and the Witt groups of $R[t, t^{-1}]$ when combined with Karoubi's Fundamental Theorem can be reinterpreted as periodicity theorems in the $\mathbf{A}^1$-setting. We then construct $\Omega_{\mathbf{P}^1}$-spectra representing hermitian K-theory and (Balmer) Witt groups in the stable homotopy category $\mathcal{SH}(k)$.

In section 6, we compute the hermitian K-theory of the projective line and prove the Thom isomorphism (also called *homotopy purity*) and the blow up isomorphism both for Witt groups and hermitian K-theory. We then discuss the relation of our representability theorems and Morel's conjecture on stable $\mathbf{A}^1$-homotopy groups of spheres. Comparing this to a topological version of such a periodicity theorem for K-theory of spaces with involution (*cf.* the work of Atiyah [At] on *Real K-theory*), we conjecture a link with the Hopf map also in the algebraic setting (Conjecture 6.5).

The appendix (section 7) contains a dictionary between L-theory, Balmer Witt groups and hermitian K-theory.

Throughout the whole article, we assume that 2 is invertible in all our rings R if not stated explicitely otherwise. Moreover, *scheme* means a separated noetherian (thus quasi-compact) scheme of finite type over a fixed base field k in which 2 is invertible. Working with a noetherian regular ring in which 2 is invertible as a base wouldn't change anything.

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1 Nisnevish-Mayer-Vietoris and homotopy invariance for hermitian K-theory of regular rings

Throughout this article, we will always assume that A is a unitary ring and that 2 is invertible in A. Theorem 1.11 and Corollary 1.12 are not really new results, but an immediate consequence of the techniques developed in [HS]. Recall that a category is called idempotent complete if any idempotent map $p = p^2$ has an image.

1.1 Definition. A *category with duality* is a triple $(\mathcal{C}, *, \eta)$ with

1. $\mathcal{C}$ a category,
2. $* : \mathcal{C} \to \mathcal{C}^{op}$ a functor and
3. $\eta : id_{\mathcal{C}} => **$ a natural equivalence such that
4. for all objects A of $\mathcal{C}$ we have $1_{A^*} = \eta_A^* \circ \eta_{A^*}$.

1.2 Definition. Given a category with duality $(\mathcal{C}, *, \eta)$, its associated hermitian category $\mathcal{C}_h$ is defined as follows: An object (M, ϕ) is an isomorphism $\phi : M \xrightarrow{\cong} M^*$ such that $\phi = \phi^* \eta$. A morphism $\alpha : (M, \phi) \to (N, \psi)$ is a morphism $\alpha : M \to N$ in $\mathcal{C}$ such that $\alpha^* \psi \alpha = \phi$.

1.3. Examples 1) Let A be a ring with unit, and let $^- : A \to A^{op}$ be an involution, *i.e.*, $\overline{a+b} = \bar{a} + \bar{b}$, $\overline{ba} = \bar{a}\bar{b}$ and $\bar{\bar{a}} = a$. Let $P(A)$ be the category of finitely generated projective right A-modules. We define a duality on $P(A)$ by setting $M^* = \{f \in Hom_{\mathbf{Z}}(M, A) | f(ma) = \bar{a} f(m)\}$ which is an A-module via $fa(m) = f(m)a$. For example, on the group ring $A = RG$, we have the involution $rg \mapsto rg^{-1}$. Then $(P(A), Hom_A(\ \ , A), \eta)$ is an additive category with duality, where $\eta(m)(f) = \overline{f(m)}$ for all $m \in M$ and f in M^*.

2) Let X be a scheme. Then the category of locally free $\mathcal{O}_X$-sheaves of finite rank $Vect(X)$ is an exact category with $\underline{Hom}_{\mathcal{O}_X}(\ \ , \mathcal{O}_X)$ as duality functor. This generalizes Example 1 for commutative rings considered as affine schemes.

1.4. For any category $\mathcal{C}$ we write $B\mathcal{C} := |N\mathcal{C}|$ for the topological space given by the geometric realization of the nerve of $\mathcal{C}$. We write $i\mathcal{C}$ for the category which has the same objects as $\mathcal{C}$ and whose morphisms are the isomorphisms of $\mathcal{C}$. If $\mathcal{C}$ is a symmetric monoidal category such that $\mathcal{C} = i\mathcal{C}$, Quillen [Q2] constructs a new category $\mathcal{C}^{-1}\mathcal{C}$, which we abbreviate by $\mathcal{C}^+$, and a functor $\mathcal{C} \to \mathcal{C}^{-1}\mathcal{C}$ such that $B\mathcal{C} \to B\mathcal{C}^{-1}\mathcal{C}$ is a group completion under very mild hypotheses [Q2, p.222] which are always satisfied for all categories constructed in this article. For an additive category with duality $(\mathcal{A}, *)$, we observe that the orthogonal sum $(A, \alpha) \perp (B, \beta) := (A \oplus B, \alpha \oplus \beta)$ makes $(\mathcal{A}_h, \oplus)$ into a symmetric monoidal category. Hence we can use Quillen's $S^{-1}S$-construction [Q2] to define its K-theory:

1.5 Definition. Let $(\mathcal{A}, *, \eta)$ be an additive category with duality. Then its hermitian K-theory space is defined by

$$K^h(\mathcal{A}) := B(i\mathcal{A}_h)^+$$

and its hermitian K-groups are defined by

$$K_n^h(\mathcal{A}) := \pi_n K^h(\mathcal{A}), n \geq 0.$$

Using explicit deloopings, we can also define negative hermitian K-groups [HS, section 2] just as one defines ordinary negative K-groups (*cf.* for example [PW]).

1.6 Definition. An object (M, ϕ) is called *hyperbolic* if there is an object L in $\mathcal{A}$ together with an isomorphism $(M, \phi) \cong (L \oplus L^*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) =: (H(L), \mu_L)$.

1.7. If 2 is invertible, there is an isometry $(M, \phi) \oplus (M, -\phi) \xrightarrow{\cong} (H(M), \mu_M)$ given by $\begin{pmatrix} 1 & -1 \\ \frac{1}{2}\phi & \frac{1}{2}\phi \end{pmatrix}$ for any (M, ϕ) in $\mathcal{C}_h$. This implies that the full subcategory of hyperbolic objects is cofinal in $\mathcal{A}_h$. Moreover, if $\mathcal{A} = P(A)$ as in Example 1.3.1, the free hyperbolic modules are cofinal in $P(A)_h$ and hence the connected component of 0 of $K^h(P(A))$ is homotopy equivalent to the plus construction applied to $BO(A) = Bcolim_n \operatorname{Aut} H(R^n)$ [Ka1, Théorème 1.6]. We will often write $K^h(A)$ and $K^h_n(A)$ instead of $K^h(P(A))$ and $K^h_n(P(A))$.

1.8. Recall the following definition of Witt groups suggested by Balmer [Ba1] for triangulated categories $(\mathcal{T}, T)$ with a duality functor $*$ (which has to preserve the exact, *i.e.*, distinguished, triangles). He defines $W^n_B(\mathcal{T})$ to be the monoid of isomorphism classes of objects of $\mathcal{T}_h$ relative to the duality functor $T^n \circ *$, divided by the *metabolic objects*. Balmer proves [Ba1, Theorem 4.3] that there is a natural isomorphism with the classical Witt group $W(\mathcal{A}) \to W^0_B(D^b(\mathcal{A}))$ if $\mathcal{A}$ is an idempotent complete additive category in which 2 is invertible with duality and $D^b(\mathcal{A})$ is the homotopy category of bounded chain complexes with its standard triangulation. These groups are of period four, *i.e.*, $W^n_B = W^{n+4}_B$. If $\mathcal{A}$ is idempotent complete, we write $W^n_B(\mathcal{A})$ for $W^n_B(D^b(\mathcal{A}))$.

We now recall Balmer's definition of Witt groups W^*_B for a triangulated category with duality. Keep in mind that our main example is $\mathcal{T} = D^b(Vect(X))$ with T the shift of chain complexes and duality $*$ induced by $\underline{Hom}_{\mathcal{O}_X}(\ \ , \mathcal{O}_X)$. This generalizes the classical Witt group as we assume 2 to be invertible (see [Ba1]).

1.9 Definition. Given a triangulated category with duality $(\mathcal{T}, T, *, \eta)$, a symmetric bilinear nondegenerate object (M, ϕ) is called metabolic if it posseses a Lagrangian (L, α, z). This means by definition that we have an exact triangle $T^{-1}(L^*) \xrightarrow{z} L \xrightarrow{\alpha} M \xrightarrow{\alpha^*\phi} L^*$ and that $T^{-1}(z^*) = \eta \circ z$ (in other words, z is symmetric with respect to $T^{-1} \circ *$).

1.10 Definition. For any triangulated category with duality $(\mathcal{T}, T, *, \eta)$, we define its *nth derived Witt group* $W^n_B(\mathcal{T})$ to be the set of isomorphism classes of symmetric bilinear nondegenerate objects in $\mathcal{T}$ with respect to the duality T^n*, divided by the metabolic objects (see Definition 1.9 above). The orthogonal sum makes $W^n_B(\mathcal{T})$ into a monoid. As any object is a direct summand of a metabolic object, $W^n_B(\mathcal{T})$ is actually a group.

The following results are known only for additive categories, but not for general exact categories (having possibly not split short exact sequences). In [Ho], we gave a definition of hermitian K-theory K^h of an exact category $\mathcal{E}$ with duality which generalizes the Definition 1.5 if all short exact sequences in $\mathcal{E}$ split. But we can't prove Corollaries 1.13 and 1.15 below for hermitian K-theory of schemes defined this way. Hence we will give another variant of hermitian K-theory for schemes (*cf.* Definitions 2.2 and 2.4), called KO, which will allow us to generalize these Corollaries to non-affine schemes.

1.11 Theorem. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between idempotent complete categories with duality. Assume that $K_{-n}(\mathcal{A}) \cong 0 \cong K_{-n}(\mathcal{B})$ for $n = 1, 2$. Assume furthermore that $K_n(\mathcal{A}) \overset{\cong}{\to} K_n(\mathcal{B})$ for all $n \geq 0$ and $W^n_B(\mathcal{A}) \overset{\cong}{\to} W^n_B(\mathcal{B})$ for all $n \in \mathbf{Z}$. Then we also have isomorphisms $K^h_n(\mathcal{A}) \overset{\cong}{\to} K^h_n(\mathcal{B})$ for all $n \geq -2$.*

Proof. This is a special case of Karoubi's induction priciple [HS, Lemma 3.6, Proposition 3.9]. □

1.12 Corollary. *Consider a commutative square*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{g} & \mathcal{D} \end{array}$$

of additive idempotent complete categories with dualities which becomes homotopy cartesian after applying K, and all negative K-groups vanish. Assume moreover that $D^b(f)$ and $D^b(g)$ are localizations of triangulated categories with dualities and equivalent kernel categories $\mathcal{T}$. Then the square becomes homotopy cartesian after applying K^h.

Proof. Let $C(f)$ and $C(g)$ be the push-out of additive categories of $C\mathcal{A} \leftarrow A \to \mathcal{B}$ and $C\mathcal{C} \leftarrow \mathcal{C} \to \mathcal{D}$, resp. as defined in [HS, section 5]. By [HS, Lemma 3.6, Lemma 3.9], it suffices to show that the natural map $C(f) \to C(g)$ induces an isomorphism on Balmer Witt groups W^* to deduce a homotopy equivalence $K^h(C(f)) \to K^h(C(g))$ and thus [HS, 2.20] the desired homotopy cartesian square. We know that the maps $D^b(\mathcal{A}) \to D^b(C\mathcal{A})$ and $D^b(\mathcal{B}) \to D^b(C(f))$ are full inclusions with equivalent cokernels. This follows as by [HS, section 5] $\mathcal{B} \to C(f)$ is also filtering, hence the quotients $C\mathcal{A}/\mathcal{A}$ and $C(f)/\mathcal{B}$ are equivalent additive categories and we can apply [Sch, Theorem 10.1] to get two short exact sequences of triangulated categories. The same argument applies to $g : \mathcal{C} \to \mathcal{D}$. We will prove that the natural maps $D^b(C\mathcal{A})/\mathcal{T} \to D^b(C(f))$ and $D^b(C\mathcal{C})/\mathcal{T} \to D^b(C(g))$ induce isomorphism on W^*_B which suffices by the five Lemma as $W^*(C\mathcal{A}) \cong 0 \cong W^*(C\mathcal{C})$ by the usual Eilenberg swindle. But the map $W^*(D^b(C\mathcal{A})/\mathcal{T}) \to W^*(D^b(C(f)))$ fits into a commutative (check this!) latter of long exact sequences of Witt groups associated to the two short exact sequences of triangulated categories $D^b(\mathcal{B}) \to D^b(C(f)) \to D^b(S\mathcal{A})$ and $\mathcal{T} \to D^b(\mathcal{A}) \to D^b(\mathcal{B})$. Hence the claim follows by the five lemma and the identical argument for $W^*(D^b(C\mathcal{C})/\mathcal{T}) \to W^*(D^b(C(g)))$. □

Remark. The category of finitely generated projective modules over a given ring $P(A)$ is idem-complete, and it fulfills the above hypothesis $K_n(P(A)) = 0$ for all $n < 0$ if A is regular [Bas, p.685].

Theorem 1.11 allows us to prove homotopy invariance:

1.13 Corollary. *Let A be a regular ring. Then we have a homotopy equivalence*

$$K^h(A) \overset{\simeq}{\to} K^h(A[t]) \ .$$

Proof. Use [Q1, p.122],[Ba2, Theorem 3.1] and Theorem 1.11. □

More generally, any map of regular rings yielding an affine vector bundle torsor as discussed in the next section induces a K^h-equivalence by Theorem 1.11 and the corresponding result for K-theory [Q1, p.128] and Witt groups [Gi2]. Homotopy invariance could probably have been proved in a more classical way, but there seems to be no reference for this.

Next, we establish Nisnevish-Mayer-Vietoris applying Corollary 1.12. First we recall the definition of an elementary distinguished square [MV, Definition 3.1.3]:

1.14 Definition. An *elementary distinguished square* (or a *Nisnevish square* for short) is given by a commutative diagram of schemes

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{j} & X \end{array}$$

such that p is étale, j is an open embedding and $p^{-1}(X - U) \to X - U$ is an isomorphism of the associated reduced schemes.

Recall that a presheaf F is a Nisnevish sheaf if and only if for any elementary square applying F yields a cartesian square of sets [MV, 3.1.4].

We now show that the presheaf KO fulfills the Nisnevish-Mayer-Vietoris property (also called "Brown-Gersten property" or "Nisnevish excision"):

1.15 Corollary. *For any elementary distinguished square as in Definition 1.14 consisting of regular affine schemes, applying K^h yields a homotopy cartesian square.*

Proof. For K-theory, this follows from excision [TT, Proposition 3.19] and localization [TT, Theorem 7.4]. For D^b (and hence Balmer Witt groups) this is excision and localization as well [Ba2, 1.6, 2.3]. Now apply Corollary 1.12. □

2 The definition of KO

For the rest of the article, *scheme* means a separated noetherian (thus quasi-compact) scheme of finite type over a fixed base field k in which 2 is invertible. We first extend the definition of the presheaf on affine schemes K^h to a presheaf KO on quasi-compact schemes using Jouanolou's device (*cf.* Definition 2.2). Then we observe that we can define KO using affine covers as well (see Definition 2.4, Lemma 2.5) following essentially Thomason [TT],[We]. All this works just as well for any other fibrant presheaf on affine schemes, provided that it fulfills homotopy invariance and Nisnevish-Mayer-Vietoris.

First, we recall the following result of Jouanolou.

2.1 Lemma. *Let X be a regular scheme. Then there existe an affine vector bundle torsor W over X.*

Proof. [Jo, Lemme 1.5] or [We, Proposition 4.3], recall that any regular noetherian scheme has an ample family of line bundles ([SGA6, II, 2.2.7.1] or [TT, p.284]). □

Moreover, taking the fiber product of two affine torsors W and W' over X as in [Jo, Proposition 1.6], we see that the homotopy type of $K^h(W)$ is independent of the choice of the torsor W. As K^h is homotopy invariant, this justifies the following definition:

2.2 Definition. Let X be a regular scheme. Then we define $KO(X) := K^h(W)$ where W is an affine torsor over X as in Lemma 2.1.

2.3. Remark. Observe that this is a definition only *up to homotopy.* See [We, Appendix] for techniques how to make it functorial considering all torsors simultaneously, and observe that [We, Lemma A.2] carries over as K^h commutes with colimits. Hence we may assume from now on that KO is a simplicial presheaf on Sm/k which will be necessary in the sequel. More generally, given any presheaf P on affine schemes (commuting with colimits) fulfilling Mayer-Vietoris and homotopy invariance for vector bundle torsors, the techniques of [We] allow us to extend it to a presheaf on Sm/k fulfilling these two properties for non-affine schemes as well. So in this section the reader might replace KO by a preshaef P fullfilling the properties mentioned above as this is all the proofs we will give in this section require (except the Nisnevish-Mayer-Vietoris property which requires the corresponding statement for affine Nisnevish-Mayer-Vietoris squares).

2.4 Definition. Let X be a scheme and $\mathcal{U} = (U_i)_{i \in I}$ an affine cover of X. Then we define $KO(X,\mathcal{U}) := \mathit{holim}\, KO(U_1 \cap ... \cap U_j)$ where the *holim* is taken over the poset over all finite intersections.

In the notation of Thomason [Th, Definition 1.9], one would write $\check{\mathbf{H}}^\bullet(\mathcal{U}, KO)$ instead of $KO(X,\mathcal{U})$.

2.5 Lemma. *Let X be a regular scheme and $\mathcal{U}$, $\mathcal{V}$ be two affine covers of X. Then we have a homotopy equivalence $KO(X,\mathcal{U}) \simeq KO(X,\mathcal{V})$. In particular, $K^h(X) \simeq KO(X) \simeq KO(X,\{X\})$ for X affine where $\{X\}$ is the trivial cover of X.*

Proof. The arguments of Thomason-Weibel carry over to KO: We can first prove Mayer-Vietoris for quasi-compact separated schemes as in [We, Theorem 5.1] using Corollary 1.15. Then we deduce *Čech descent* from [We, Theorem 6.3]. From this, the Lemma follows similar to [We, Proposition 6.6]. □

2.6. Remark. One might try to define KO directly from K^h via this *holim*-construction without Jouanolou's device. But the problem (besides functoriality) is that we need Čech descent and thus the Mayer-Vietoris property of KO for regular schemes to prove Lemma 2.5, and proving the general Mayer-Vietoris property from the one for affine schemes requires Jouanolou's device. Nevertheless, once Mayer-Vietoris is established, we can deduce theorems on KO for non-affine schemes once we know they are true for affine schemes. Compare [TT, section 9] and [Jo] for possible applications. We can also define $KO(X)$ for X not quasi-compact this way.

2.7 Proposition. *For any regular scheme X, we have a homotopy equivalence*

$$KO(X) \simeq KO(X \times \mathbf{A}^1)\ .$$

Proof. This follows from Definition 2.2. □

2.8 Theorem. *For any Nisnevish square as in Definition 1.14, the following square*

$$\begin{array}{ccc} KO(X) & \xrightarrow{KO(j)} & KO(U) \\ \downarrow{\scriptstyle KO(p)} & & \downarrow \\ KO(V) & \longrightarrow & KO(U \times_X V) \end{array}$$

is homotopy cartesian.

Proof. For ordinary Mayer-Vietoris, the proof is the same as in [We, Theorem 5.1]. Considering Nisnevish squares in general, we proceed applying Corollary 1.15, Definition 2.4 and Lemma 2.5. For this, observe that any Nisnevish square can be covered by affine Nisnevish squares. More precisely, given a Nisnevish square as in Definition 1.14, we choose an affine cover of $X - U$ and an affine cover of U. By [Mi, Theorem I.3.14], we then may assume that there are compatible affine covers of U and $V \times_X U$ such that p is covered by affine étale morphisms. □

3 Unstable $\mathbf{A}^1$-representability of hermitian K-theory

We first prove a general representability theorem for a certain class of functors from smooth schemes to graded abelian groups. For any presheaf of simplicial sets P on the big Nisnevish site $(Sm/k)_{Nis}$, we denote by $aP \in \Delta^{op}Shv$ its sheafification (with respect to the Nisnevish topology). If F is a sheaf, we write F_f for a fibrant replacement with respect to the simplicial model structure of [Ja1, Corollary 2.7]. Recall that in this model structure, a map $F \to G$ is a weak equivalence if and only if it induces a weak equivalence of simplicial sets for all stalks. It is a cofibration if $F(U) \to G(U)$ is a monomorphism for any scheme U, hence any object is cofibrant. Recall further that this simplicial structure yields the (pointed) simplicial homotopy category $\mathcal{H}^s_\bullet(k)$ which then leads to the (pointed) $\mathbf{A}^1$-homotopy category $\mathcal{H}(k)$ of [MV] by inverting the $\mathbf{A}^1$-equivalences. In particular, the stalks of F and F_f are weakly equivalent simplicial sets. We denote by X_+ the scheme X with an added disjoint base point. The Yoneda embedding sends any (pointed) scheme to a (pointed) presheaf which is an étale sheaf [SGA4, p. 347]. We will often consider it as a simplicially constant sheaf and still denote it by X (resp. X_+).

The model category of simplicial sheafs $\Delta^{op}Shv(Sm/k)_{Nis}$ is actually a simplicial model category [MV, Remark 1.9]. In particular, we have a bifunctor $\mathbf{Hom}_{\Delta^{op}Shv(Sm/k)_{Nis}} : \Delta^{op}Shv(Sm/k)_{Nis}^{op} \times \Delta^{op}Shv(Sm/k)_{Nis} \to \Delta^{op}Sets$.

We can now establish our general $\mathbf{A}^1$-representability result:

3.1 Theorem. *For any presheaf $P : (Sm/k)_{Nis} \to \Delta^{op}Sets$ fulfilling the Nisnevish-Mayer-Vietoris property and homotopy invariance and for any regular scheme X, we have a natural isomorphism*

$$\pi_n(P(X)) \cong Hom_{\mathcal{H}(k)}(S^n \wedge X_+, aP_f).$$

Proof. We have $\pi_n(P(X)) \cong \pi_n(aP_f(X))$ by applying [MV, 3.1.18] to $\mathcal{A} =$ *all regular noetherian schemes.* Furthermore, we have $\pi_n(aP_f(X)) \cong \pi_n(\mathbf{Hom}_{\Delta^{op}Shv(Sm/k)_{Nis}}(X, aP_f))$ because by

Yoneda we have $\mathbf{Hom}_{\Delta^{op}Shv(Sm/k)_{Nis}}(X,F)_k := Hom_{\Delta^{op}Shv(Sm/k)_{Nis}}(X \times \Delta^k, F) \cong F(X)_k$ and hence $\mathbf{Hom}_{\Delta^{op}Shv(Sm/k)_{Nis}}(X,F) \cong F(X)$ for any simplicial sheaf F. By [Ja1, p. 73], $\pi_n(\mathbf{Hom}_{\Delta^{op}Shv(Sm/k)_{Nis}}(X, aP_f))$ is isomorphic to $Hom_{\Delta^{op}Shv}(X, \Omega^n aP_f)/\sim$ where $\sim$ stands for the smallest equivalence relation generated by simplicial equivalence. We write $Hom_\bullet$ for pointed morphisms. As adding a base point is left adjoint to forgetting the base points, we get $Hom_{\Delta^{op}Shv}(X, \Omega^n aP_f)/\sim \; \cong Hom_{\Delta^{op}Shv_\bullet}(X_+, \Omega^n aP_f)/\sim$. As aP_f is fibrant, it follows [Ja1, p. 73] that $\Omega^n aP_f$ is also fibrant and hence [Ja1, p. 72] that $Hom_{\Delta^{op}Shv_\bullet}(X_+, \Omega^n aP_f)/\sim \; \cong Hom_{\mathcal{H}^s_\bullet(k)}(S^n \wedge X_+, aP_f)$. To show that the latter one is isomorphic to $Hom_{\mathcal{H}(k)}(S^n \wedge X_+, aP_f)$, we have to check [MV, 2.2.5] that aP_f is $\mathbf{A}^1$-local. By [MV, 2.3.19], a fibrant simplicial sheaf is $\mathbf{A}^1$-local if and only if it is homotopy invariant. We also know [MV, 3.1.18] that a map of presheaves that is stalkwise a weak equivalence (hence a weak equivalence of the associated sheaves) is a weak equivalence for any section provided both presheaves fulfill the Brown-Gersten property. Applying this to $P \to aP_f$ yields the desired result as P and hence aP_f is homotopy invariant. □

3.2. Remark. The part of the proof showing that $\pi_n(P(X)) \cong Hom_{\mathcal{H}^s_\bullet(k)}(S^n \wedge X_+, aP_f)$ is not discussed in the corresponding proof of Morel and Voevodsky for the representability of algebraic K-theory [MV, Proposition 3.9] because they consider it to be "formal". Hopefully, the above details allow some more people to understand this part of their proof. More precisely, our proof allows us to slightly simplify their original proof, replacing Thomason's hypercohomology spectrum $\mathbf{H}^\bullet_{Nis}(\quad, K^B)$ and his Nisnevish descent result [TT, Theorem 10.8] by an arbitrary fibrant replacement aK_f and the Brown-Gersten property.

The above result strongly emphasizes the analogy with classical homology theories for CW-complexes: The behaviour of the homotopy groups of any presheaf P satisfying Nisnevish-Mayer-Vietoris and homotopy invariance (read: an Eilenberg-Steenrod cohomology theory) is determined by their behaviour on a point. This is not a full analogue of Brown's representability theorem [Br] as we already assumed our cohomology theory to be given as homotopy groups of a simplicial presheaf instead of starting with just a presheaf with values in graded abelian groups. This is why our proof for the $\mathbf{A}^1$-representability of Balmer Witt groups and in particular the classical Witt group (*cf.* section 4) is slightly more complicated.

3.3 Corollary. *For any regular scheme X, we have a natural isomorphism*

$$KO_n(X) \cong Hom_{\mathcal{H}(k)}(S^n \wedge X_+, aKO_f).$$

Proof. Apply Theorem 3.1, Theorem 2.8 and Proposition 2.7. □

This allows us to extend the definition of KO from Sm/k to any simplicial sheaf:

3.4 Definition. For any object F of $\Delta^{op}Shv(Sm/k)_{Nis}$, we set

$$aKO_f(F) := \mathbf{Hom}_\bullet(F_+, aKO_f) \cong \mathbf{Hom}(F, aKO_f)$$

and

$$KO_n(F) := Hom_{\mathcal{H}}(S^n \wedge F_+, aKO_f).$$

For any object with base point F of $\Delta^{op}Shv(Sm/k)_{Nis\bullet}$, we set

$$aKO_f(F) := \mathbf{Hom}(F, aKO_f)$$

and

$$KO_n(F) := Hom_{\mathcal{H}}(S^n \wedge F, aKO_f).$$

The object aP_f does not look very explicit, so one might look out for other descriptions of aKO_f. Let $\pi : (Sm/k)_{et} \to (Sm/k)_{Nis}$ be the obvious morphism from the big étale site to the big Nisnevish site (compare [MV, p. 130]). Then we have a pair of adjoint functors $\pi^* : \Delta^{op}Shv(Sm/k)_{Nis} \to \Delta^{op}Shv(Sm/k)_{et}$ and $\pi_* : \Delta^{op}Shv(Sm/k)_{et} \to \Delta^{op}Shv(Sm/k)_{Nis}$ In fact, π_* is just the forgetful functor, having the sheafification functor π^* as a right adjoint. Define a model structure on $\Delta^{op}Shv(Sm/k)_{et}$ as in [Ja1], that is the weak equivalences being the (étale) stalkwise weak equivalences of simplicial sets and every object is cofibrant. Writing $\mathcal{H}^s_{et}(k)$ for its homotopy category, following [MV, Proposition 1.47] we obtain a pair of (Quillen) adjoint functors $\mathbf{L}\pi^*$ and $\mathbf{R}\pi_*$ between $\mathcal{H}^s(k)$ and $\mathcal{H}^s_{et}(k)$. The functor $\mathbf{L}\pi^*$ is induced by π^* and $\mathbf{R}\pi_*$ by first choosing a fibrant resolution and then applying π_*.

One reason for considering the étale topology is the following.

3.5 Definition. Let A be a ring and A^n be equipped with the diagonal form $1_n :=< 1, ..., 1 >$. Then we set $O_n(A) = Aut_{P(A)_h}(A^n, 1_n)$.

3.6 Lemma. *Let X be a scheme. Then in the étale topology, any object (E, ϕ) of $Vect(X)_h$ is locally isomorphic to $(A^n, 1_n)$ for some $n \in \mathbf{N}$.*

Proof. In the Zariski topology, any vector bundle E is locally free, and there are elements $a_1, ...a_n$ in the local ring A such that locally $(E, \phi) \cong (A^n, < a_1, ..., a_n >)$ where n is the rank of E (*cf.* [Sc, Theorem 1.6.4]). To construct an isomorphism $(A^n, < a_1, ..., a_n >) \cong (A^n, < 1, ..., 1 >)$, we need to add the square roots of the a_i to A which yields an étale neighbourhood as 2 is invertible [Ray, Proposition VI.1] (remember that we assume our forms to be non-degenerate). □

3.7. As in [MV, p.123], we denote by $\mathbf{R}\Omega$ the total right derived functor of Ω which is right adjoint to $S^1\wedge$ in $\mathcal{H}^s_\bullet$. It is given by first taking a fibrant resolution and then taking loops, thus always yields a fibrant object. Finally, we define $B_{et}O_n := \mathbf{R}\pi_*\pi^*BO_n$, where $\pi : (Sm/k)_{et} \to (Sm/k)_{Nis}$ is the obvious morphism of sites and $BO(Spec(A))_n$ is the classifying space (*i.e.*, the nerve) of the group $O_n(A)$. The Lemma 3.6 above implies that π^*aKO_f and $\pi^*\mathbf{R}\Omega B\amalg BO_n$ are locally isomorphic in the étale topology. To study the relationship between aKO_f and $\mathbf{R}\Omega B\amalg_{n\geq 0}B_{et}O_n$ in $\mathcal{H}(k)$ remains an open problem. The fact that we can't use orthogonal Grassmannians to represent hermitian K-theory as we can use ordinary Grassmannians to represent algebraic K-theory turns out to be an obstruction to construct spectra representing KO_* and W^*_B as we will see in section 5.

4 The Karoubi tower and unstable $\mathbf{A}^1$-representability of Witt groups

In this section, we will show (Corollary 4.9) that Balmer Witt groups and in particular the classical Witt group are representable in the unstable $\mathbf{A}^1$-homotopy category $\mathcal{H}(k)$. In order to do so, we first need to construct in a functorial way for each regular ring a topological space whose homotopy groups are the purely algebraically defined Witt groups (Lemma 4.6). This result is obvoiusly interesting on its own.

We first define U- and V-theory and Karoubi Witt groups W^K.

4.1 Definition. For any ring A with involution, we define the hyperbolic functor by

$$H : iP(A) \to iP(A)_h$$
$$M \mapsto (H(M), \mu_M)$$
$$\alpha \mapsto \alpha \oplus \alpha^{*\,-1}$$

and the forgetful functor by

$$F : P(A)_h \to P(A)$$
$$(N, \psi) \mapsto N$$
$$\beta \mapsto \beta\ .$$

We further set

$$U(A) := hofib(K(A) \xrightarrow{H} K^h(A))$$
$$V(A) := hofib(K^h(A) \xrightarrow{F} K(A))$$
$$U_n(A) := \pi_n(U(A))$$
$$V_n(A) := \pi_n(V(A))$$
$$W_n^K(A) := coker(K_n(A) \xrightarrow{H} K_n^h(A))$$

where the homotopy fiber and homotopy groups are always with respect to the basepoint given by the zero object.

Observe that W_0^K is nothing else but the classical Witt group. We write ${}_{-}K^h$ for the hermitian K-theory of antisymmetric forms, *i.e.*, when replacing η by $-\eta$, and similar for U- and V-theory.

The following Theorem is due to Karoubi. He calls it "Fundamental Theorem", and you should think about it as a generalization of Bott periodicity to algebraic K-theory (compare also the remarks at the end of this section):

4.2 Theorem. *For any ring with involution A in which 2 is a unit, we have a natural homotopy equivalence*

$$\Omega_{-}U(A) \simeq V(A)\ .$$

Proof. See [Ka3]. □

4.3. In the proof of the Fundamental Theorem 4.2, the following construction of the rings U_A and V_A [Ka3, 1.4] is crucial: For any ring A, one defines CA as being the ring of those infinite matrices with coefficients in A having only a finite number of coefficients different from zero in each row and each column. Then one defines $SA := CA/A$ and shows that one gets a homotopy fibration $K^h(A) \to K^h(CA) \to K^h(SA)$. In particular, as K and K^h of CA vanish by the Eilenberg swindle, we see that $\Omega K^h(SA) \simeq K^h(A)$ which allows us to consider hermitian K-theory as a non-connective Ω-spectrum. Everything above including Karoubi's Fundamental Theorem 4.2 carrries over to these non-connective Ω-spectra. In particular, we have negative K^h-,U- and W^K-groups.

In fact, the above construction can be carried out on the level of additive categories with duality in general [HS, section 2]. Now we define $U_A := lim(CA \to SA \leftarrow SA \times SA^{op})$ and $V_A := lim(CA \times CA^{op} \to SA \times SA^{op} \leftarrow SA)$. We deduce that $\Omega K^h(U_A) \simeq U(A)$, $\Omega U(U_A) \simeq_- K^h(A)$ and $\Omega K^h(V_A) \simeq V(A)$. See [Ka3] for more details.

Iterating this construction, we get the so-called Karoubi tower:

4.4 Definition. For any ring A with involution, its Karoubi tower is by definition the sequence of maps

$$K^h(A) \to K^h(U_A) \to K^h(U_A^2) \to K^h(U_A^3) \to ...$$

We also set $KT(A) := hocolim(K^h(A) \to K^h(U_A) \to K^h(U_A^2) \to ...)$

The idea of considering this tower is due to B. Williams. The properties of this tower have been studied by Kobal [Ko]. An easy observation is the following:

4.5 Lemma. *The homotopy groups of $KT(A)$ are 4-periodic:*

$$\pi_n(KT(A)) \cong \pi_{n+4}(KT(A)) \ .$$

Proof. We first observe that $\Omega^4 K^h(U_A^4) \simeq K^h(A)$ by the above constructions and the Fundamental Theorem. Hence we have $\pi_n(KT(A)) \cong \pi_n(hocolim\, K^h(U_A^{4i})) \cong \pi_n(hocolim\, \Omega^4 K^h(U_A^{4i})) \cong \pi_n(colim\, \Omega^4 K^h(U_A^{4i}))$ where the last isomorphism follows from the mapping telescope (*i.e.*, the *hocolim* is by definition the *colim* of the diagram replacing every map by a cofibration. Replacing every simplicial set by a fibrant one, and using the explicit description of π_n for fibrant simplicial sets, we get $\cong colim\, \pi_n(\Omega^4 K^h(U_A^{4i})) \cong colim\, \pi_{n+4}(K^h(U_A^{4i})) \cong \pi_{n+4}(KT(A))$. □

For regular rings, we actually discover the 4-periodic Balmer Witt groups (see 1.10 for the definition):

4.6 Lemma. *If A is a regular ring, we have natural isomorphisms*

$$\pi_k(KT(A)) \cong \pi_k(K^h(U_A^{k+1})) \cong W_B^{-k}(A) \ .$$

Proof. As the negative K-theory of regular rings vanishes [Bas, p. 685], the map $K^h(U_A^n) \to K^h(U_A^{n+1})$ is an isomorphism for π_k, $k < n$. Thus $\pi_k(KT(A)) \cong \pi_k(K^h(U_A^n))\, \forall n > k$. Use Proposition 7.4 to identify these homotopy groups with Balmer Witt groups. □

See 7.6, 7.7 for a statement about non-regular rings.

4.7. As all the above constructions can be done in the category of simplicial sets rather than in **Top**, we can consider KT as a simplicial presheaf on regular affine schemes. Using the functorial version of Jouanolou's device ([Jo, Lemme 1.5] or [We, Proposition 4.3] and Remark 2.3), we can extend KT to a presheaf on regular schemes which we still denote by KT similar to the way we extended K^h to KO in section 2. As S. Gille [Gi2] has recently shown that Balmer Witt groups fulfill strong homotopy invariance (*i.e.*, for bundles having affine spaces as fibers), we know that this still coincides with Balmer's original definition: $\pi_n(KT(X)) \cong W_B^{-n}(X)$.

Next, we show that the Nisnevish-Mayer-Vietoris property holds for KT.

4.8 Proposition. *Applying KT to a Nisnevish square as in Definition 1.14 yields a homotopy cartesian square.*

Proof. It suffices to prove this for affine Nisnevish squares. We then can deduce the general statement using affine covers and proceed exactly as for KO in section 2. The affine case follows from our homotopy cartesian square when applying K^h to an elementary distinguished square (see Corollary 1.15) and the following more general argument. Assume that we have a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of rings (or more generally of idempotent complete additive categories) inducing a homotopy fibration for non-connective Ω-spectra when applying K or K^h. Then we also have homotopy fibrations of non-connective Ω-spectra when applying K^h to $SA \to SB \to SC$ and to $SA \times SA^{op} \to SB \times SB^{op} \to SC \times SC^{op}$. Looking at the homotopy fibers, we get a homotopy fibration of non-connective Ω-spectra

$$K^h(U_A) \to K^h(U_B) \to K^h(U_C)$$

and inductively for all the higher $K^h(U^n\ \ )$. It remains to show that $KT(A) \simeq F$ where $F := hofib\,(KT(B) \to KT(C))$. Defining $F_B^n := hofib\,(K^h(U_B^n) \to KT(B))$, $F_C^n := hofib\,(K^h(U_C^n) \to KT(C))$ and $X_n := hofib\,(F_B^n \to F_C^n)$, this yields $\pi_i(F_B^n) \cong \pi_i(F_C^n)$ for $n > i$. Hence $\pi_i(X_n) = 0$ and consequently $\pi_i(K^h(U_A^n)) \cong \pi_i(F)$ for $n > i$, and the proposition follows. The argument for a commutative square is exactly the same, setting $X_n = holim(F_C^n \to F_D^n \leftarrow F_B^n)$. □

Denote by aKT the sheafification of KT with respect to the Nisnevish topology. Next we choose a fibrant replacement aKT_f with respect to the model structure of [Ja1, Corollary 2.7].

We will now prove that the simplicial sheaf aKT_f on $(Sm/k)_{Nis}$ represents Balmer Witt groups in the unstable $\mathbf{A}^1$-homotopy category $\mathcal{H}(k)$. As usual, we denote by X_+ the scheme X with an added disjoint base point. Then the main result of this section is the following:

4.9 Corollary. *For any regular scheme X, we have a natural isomorphism*

$$W_B^{-n}(X) \cong Hom_{\mathcal{H}(k)}(S^n \wedge X_+, aKT_f).$$

Proof. As Balmer Witt groups fulfill the Nisnevish-Mayer-Vietoris property (Proposition 4.8) and homotopy invariance ([Ba2]), we can apply Theorem 3.1. □

As for KO, this allows us to extend the definition of W_B^* from Sm/k to any simplicial sheaf:

4.10 Definition. For any object F of $\Delta^{op}Shv(Sm/k)_{Nis}$, we set

$$W_B^{-n}(F) := Hom_{\mathcal{H}(k)}(S^n \wedge F_+, aKT_f).$$

For any object with base point F of $\Delta^{op}Shv(Sm/k)_{Nis\bullet}$, we set

$$W_B^{-n}(F) := Hom_{\mathcal{H}(k)}(S^n \wedge F, aKT_f).$$

5 Stable Representability of KO and W

Recall [Vo] that the stable $\mathbf{A}^1$-homotopy category $\mathcal{SH}(k)$ is the homotopy category of $\mathbf{P}^1$-spectra. The structure maps $E_n \wedge \mathbf{P}^1 \to E_{n+1}$ have adjoint maps $t_n : E_n \to \Omega_{\mathbf{P}^1} E_{n+1}$, and as in topology a spectrum is called an $\Omega_{\mathbf{P}^1}$-spectrum if all the t_n are weak equivalences (*i.e.*, isomorphisms in $\mathcal{H}(k)$). Recall that we have an isomorphism $\mathbf{P}^1 \cong S^1 \wedge \mathbf{G}_m$ in $\mathcal{H}(k)$ (see [MV, Lemma 2.15 and Corollary 2.18]). More details on stable $\mathbf{A}^1$-homotopy can also be found in [Ja2] and [Hu] where the $\mathbf{A}^1$-analogues of symmetric spectra resp. S-modules are considered.

In this section, we will construct $\Omega_{\mathbf{P}^1}$-spectra $\mathbf{KO}$ and $\mathbf{KT}$ with $\mathbf{KO}_0 = aKO_f$ and $\mathbf{KT}_0 = aKT_f$ which represent hermitian K-theory KO_* and Balmer Witt groups W_B^* of regular schemes in $\mathcal{SH}(k)$. In order to do so, we need to study $KO_*(R[t,t^{-1}])$ and $W_B^*(R[t,t^{-1}])$ for a regular ring R (in which 2 is invertible). We are only interested in the trivial involution on $P(R[t,t^{-1}])$. Using the involution that sends t to t^{-1} yields different results, of course.

5.1 Proposition. *For any regular ring R, we have a homotopy fibration*

$$KO(R) \to KO(R[t,t^{-1}]) \to K^h(U_R)\ .$$

Proof. The first map $f : KO(R) \to KO(R[t,t^{-1}])$ is induced by the composition $R \to R[t] \to R[t,t^{-1}]$. Let $C(f)$ be the "mapping cone" of this map as defined in [HS, 2.20]. Then the map $KO(C(f)) \to KO(U_R)$ is an isomorphism for π_1 by [Ka1, Corollaires 3.12, 3.13] and [Ho, Lemma 4.2]. It is also an isomorphism for π_0 because both $K_0(U_R) \cong K_{-1}(R)$ and $K_0(C(f))$ are 0 (R is regular) and $W_0^K(C(f)) \to W_0^K(U_R)$ is an isomorphism (this is a special case of Proposition 5.5 below) . The Proposition now follows by Karoubi's induction principle [HS, Proposition 3.9] and the corresponding Theorem of Quillen for K-theory [Q1, p. 122]. □

5.2 Corollary. *For any regular ring R, there are natural isomorphisms*

$$KO_n(R[t,t^{-1}]) \cong KO_n(R) \oplus U_{n-1}(R)$$

and

$$U_n(R[t,t^{-1}]) \cong U_n(R) \oplus_- KO_{n-1}(R)\ .$$

Proof. Recall that $\Omega K(U_R) \simeq K(R)$ and $\Omega K^h(U_R) \simeq U(R)$. By Quillen [Q1, p. 122], we also have a homotopy fibration $K(R) \to K(R[t,t^{-1}]) \to K(U_R)$. Now use Karoubi's Fundamental Theorem 4.2 and observe that we have splittings similar to [Q1, p. 122]. □

We will now combine Proposition 5.1 with Karoubi's Fundamental Theorem and deduce the desired Periodicity Theorem which then will yield to the definition of the spectrum **KO**.

For an unpointed scheme (or simplicial sheaf) X and the pointed scheme $\mathbf{G}_m$, consider the cofiber sequence in $\Delta^{op}Shv(Sm/k)_{Nis}$

$$X_+ \vee \mathbf{G}_m \to X_+ \times \mathbf{G}_m \to X_+ \wedge \mathbf{G}_m \ .$$

We define $U(X) := hofibK(X) \xrightarrow{H} KO(X)$ also for non-affine regular X. Writing $_-KO$ for the hermitian K-theory of antisymmetric forms as before, we get the following.

5.3 Corollary. *For any regular scheme X, there are homotopy equivalences*

$$\Omega aKO_f(X_+ \wedge \mathbf{G}_m) \simeq aU_f(X)$$

and

$$\Omega aU_f(X_+ \wedge \mathbf{G}_m) \simeq a_-KO_f(X) \ .$$

Proof. As aKO_f is fibrant, we obtain a homotopy fibration [GJ, Proposition II.3.2]

$$aKO_f(X_+ \wedge \mathbf{G}_m) \to aKO_f(X_+ \times \mathbf{G}_m) \to aKO_f(X_+ \vee \mathbf{G}_m)$$

and similar for aK_f, aU_f and aKT_f. Using the fact that $X_+ \times \mathbf{G}_m = X \times \mathbf{G}_m \amalg \mathbf{G}_m$ and $X_+ \vee \mathbf{G}_m = X \amalg \mathbf{G}_m$, we obtain the homotopy fibration

$$aKO_f(X_+ \wedge \mathbf{G}_m) \to aKO_f(X \times \mathbf{G}_m) \to aKO_f(X) \ .$$

Comparing this homotopy fibration with the split one we just constructed in Proposition 5.1 (and reducing to X affine by Jouanolou's device), we obtain the first homotopy equivalence using that $\Omega KO(U_R) \simeq U(R)$. For the second homotopy equivalence, look at the corresponding homotopy fibrations for U and use that $\Omega U(U_R) \simeq_- KO(R)$. □

Now we will construct the spectrum **KO**. For all $k \in \mathbf{N}$, we set $\mathbf{KO}_{4k} := aKO_f$, $\mathbf{KO}_{4k+1} := a_-U_f$, $\mathbf{KO}_{4k+2} := a_-KO_f$ and $\mathbf{KO}_{4k+3} := aU_f$. To define the adjoints of the structure maps $\mathbf{KO}_n \to \Omega_{\mathbf{P}^1}\mathbf{KO}_{n+1}$, we first recall (see [Du, Proposition 2.8] or [MV, Lemma 1.1.16]) that any object F in $\Delta^{op}Shv(Sm/k)_{Nis}$ can be written as a homotopy colimit of objects of of (Sm/k). More precisely, one can construct a simplicial sheaf QF which is the realization of a diagram of coproducts of schemes, that is $QF = hocolim_I X_i$ with X_i an object of Sm/k for all $i \in I$, together with a map $QF \to F$ which is a weak equivalence of simplicial sets for each section. In particular $hocolim_I X_i \simeq QF \simeq F$ with respect to the model structures [Ja1],[MV] we care about.

Next, we define the adjoints of the structure maps and show that they are weak equivalences. For $F \simeq hocolim_I X_i$ as above, we have

$\mathbf{Hom}_\bullet(F_+, aU_f) \simeq \mathbf{Hom}_\bullet(hocolim X_{i+}, aU_f) \cong holim \mathbf{Hom}_\bullet(X_{i+}, aU_f)$ by [MV, Lemma 1.1.19] and the fact that all objects are cofibrant. By Corollary 5.3 and the simplicial Yoneda Lemma we already used in the proof of Theorem 3.1, we have $\mathbf{Hom}_\bullet(X_{i+}, aU_f) \cong aU_f(X_{i+}) \simeq \Omega aKO_f(X_{i+} \wedge \mathbf{G}_m) \cong \mathbf{Hom}_\bullet(X_{i+} \wedge \mathbf{G}_m, \Omega aKO_f) \cong \mathbf{Hom}_\bullet(X_{i+} \wedge \mathbf{G}_m \wedge S^1, aKO_f) \cong$

$\mathbf{Hom_\bullet}(X_{i+}, \Omega_{\mathbf{G}_m \wedge S^1} aKO_f)$ and thus $\mathbf{Hom_\bullet}(F_+, aU_f) \simeq holim\mathbf{Hom_\bullet}(X_{i+}, \Omega_{\mathbf{G}_m \wedge S^1} aKO_f) \cong \mathbf{Hom_\bullet}(F_+, \Omega_{\mathbf{G}_m \wedge S^1} aKO_f)$.

Applying π_0, the homotopy class of the idendity of aU_f then corresponds to a homotopy class of a map t_{4k+3} from aU_f to $\Omega_{\mathbf{P}^1} aKO_f$ via the natural isomorphisms $Hom_{\mathcal{H}(k)}(aU_f, aU_f) \cong Hom_{\mathcal{H}(k)}(aU_f, \Omega_{\mathbf{G}_m \wedge S^1} aKO_f)$. This $t_{4k+3} : \mathbf{KO}_{4k+3} \to \Omega_{\mathbf{P}^1} \mathbf{KO}_{4(k+1)}$ is one of our four structure maps. The above computation for $F = X$ being already a scheme shows that α is stalkwise a weak equivalence. The three other structure maps are defined in a similar way.

5.4 Theorem. *The spectrum* **KO** *constructed above is an* $\Omega_{\mathbf{P}^1}$*-spectrum. For any regular scheme* X*, we have a natural isomorphism*

$$KO_n(X) \cong Hom_{\mathcal{SH}(k)}(S^n \wedge X_+, \mathbf{KO}).$$

Proof. Follows from the above construction and discussion of **KO** and Corollary 3.3. □

Now let us look at Balmer Witt groups.

5.5 Proposition. *For any regular ring* R*, there are natural homotopy equivalences*

$$KT(R) \simeq \Omega^2_- KT(R) \simeq \Omega^4 KT(R)$$

and

$$KT(R[t, t^{-1}]) \simeq KT(R) \times KT(R) \ .$$

Proof. The first claim follows from Lemma 4.6 and the fact that Balmer Witt groups are 4-periodic [Ba1]. The second claim follows from Lemma 4.6 and Gille's [Gi1] computation $W^n_B(R[t, t^{-1}]) \cong W^n_B(R) \oplus W^n_B(R)$. □

5.6 Corollary. *For any regular scheme* X*, there is a homotopy equivalence*

$$aKT_f(X_+ \wedge \mathbf{G}_m) \simeq aKT_f(X) \ .$$

Proof. Using Proposition 5.5, the proof is similar to Corollary 5.3. □

We now define the spectrum **KT** by setting $\mathbf{KT}_{4k} := aKT_f, \mathbf{KT}_{4k+1} := \Omega a_- KT_f, \mathbf{KT}_{4k+2} := a_- KT_f$ and $\mathbf{KT}_{4k+3} := \Omega aKT_f$. The structure maps are defined similar to the ones of **KO**. For example, the natural weak equivalences $\mathbf{Hom_\bullet}(F_+, aKTf) \simeq holim\mathbf{Hom_\bullet}(X_{i+} \wedge \mathbf{G}_m, \Omega\Omega^3 aKTf) \simeq holim\mathbf{Hom_\bullet}(X_{i+}, \Omega_{\mathbf{G}_m \wedge S^1} \Omega a_- KTf) \simeq \mathbf{Hom_\bullet}(F, \Omega_{\mathbf{G}_m \wedge S^1} \Omega a_- KTf)$ we have by Proposition 5.5 and Corollary 5.6 yield the structure map t_{4k} which is a weak equivalence.

5.7 Theorem. *The spectrum* **KT** *we just constructed is an* $\Omega_{\mathbf{P}^1}$*-spectrum. For any regular scheme* X*, we have a natural isomorphism*

$$W_B^{-n}(X) \cong Hom_{\mathcal{SH}(k)}(S^n \wedge X_+, \mathbf{KT}).$$

Proof. Follows from the construction of **KT** and Corollary 4.9. □

5.8. Remark. By the Fundamental Theorem for regular rings [Q1, p.122], we have $K(R[t,t^{-1}]) \simeq K(R) \times K(SR)$ where $\Omega K(SR) \simeq K(R)$ and S stands for the algebraic suspension of a ring [Wag]. Hence we can also use our method above to construct a $(2,1)$-periodic spectrum which represents algebraic K-theory in $\mathcal{SH}(k)$. This gives a variant of the proof of Voevodsky [Vo, section 6.2] who uses Quillen's projective bundle theorem [Q1, Proposition 8.2], instead. The fact that one can use a colimit of Grassmannians in order to represent K-theory in $\mathcal{H}(k)$ [MV, Theorem 3.13] and a colimit argument [Vo, Lemma 6.7] allows him to to construct the $\Omega_{\mathbf{P}^1}$-spectrum **BGL** that represents algebraic K-theory in $\mathcal{SH}(k)$. As discussed at the end of section 3, orthogonal Grassmanians do not behave as well with respect to hermitian K-theory. Hence Voevodsky's approach to construct the structure maps does not carry over to KO and KT. Roughly spoken, we replace his particular colimit argument involving Grassmannians by the more general one of [Du]. Voevodsky [Vo, section 6.2] doesn't explicitly say why the spectrum **BGL** he constructs is an $\Omega_{\mathbf{P}^1}$-spectrum and hence represents algebraic K-theory in $\mathcal{SH}(k)$. The material of this section provides you with the necessary techniques to fill in the missing details in [Vo].

6 Applications

We first compute the hermitian K-theory of the projective line:

6.1 Proposition. *For any regular ring R, we have a homotopy fibration*

$$U(R) \to KO(\mathbf{P}^1_R) \to KO(R) \ .$$

Proof. We have a covering of $\mathbf{P}^1_R$ by two copies of $R[t]$ with intersection $R[t,t^{-1}]$. The claim now follows from Proposition 5.1, Corollary 1.13 and Theorem 2.8. □

The computation of the Witt groups can be done in a similar way, but is already known by [Gi1, Theorem 5.4].

Next, we deduce the Thom isomorphism and the blow-up isomorphism. If E is a vector bundle over X, we define the Thom space to be the pointed sheaf $Th(E) := E/(E - i(X))$ (compare [MV, Definition 3.2.16]) where $i : X \to E$ is the zero section. Here and in the sequel, quotients are formed in the category of simplicial Nisnevish sheaves.

6.2 Corollary. *Let $j : Z \to X$ be a closed embedding of smooth schemes with $N_{X,Z}$ as normal bundle. Then we have natural homotopy equivalences $KO(X/(X - j(Z))) \simeq KO(Th(N_{X,Z}))$, $KT(X/(X - j(Z))) \simeq KT(Th(N_{X,Z}))$ and exact triangles*

$$... \to KO_{n+1}(X - j(Z)) \to KO_n(Th(N_{X,Z})) \to KO_n(X) \to KO_n(X - j(Z)) \to ...$$

and

$$... \to W_B^{n-1}(X - j(Z)) \to W_B^n(Th(N_{X,Z})) \to W_B^n(X) \to W_B^n(X - j(Z)) \to ...$$

in $\mathcal{SH}(k)$.

Proof. Apply Corollaries 3.3, 4.9, Theorems 5.4, 5.7 and [MV, Theorem 3.2.23], [Vo, Proposition 4.12]. □

6.3 Corollary. *Let $j : Z \to X$ be a closed embedding of smooth schemes, $p : X_Z \to X$ be the blow-up of $j(Z)$ in X and $U = X - j(Z) = X_Z - p^{-1}(j(Z))$. Then we have natural homotopy equivalences $KO((X_Z/U)\amalg_{p^{-1}(Z)}Z) \simeq KO(X/U)$, $KT((X_Z/U)\amalg_{p^{-1}(Z)}Z) \simeq KT(X/U)$ and exact triangles*

$$... \to KO_n(X) \to KO_n(Z) \oplus KO_n(X_Z) \to KO_n(p^{-1}(Z)) \to KO_{n-1}(X) \to ...$$

and

$$... \to W^n_B(X) \to W^n_B(Z) \oplus W^n_B(X_Z) \to W^n_B(p^{-1}(Z)) \to W^{n+1}_B(X) \to ...$$

in $\mathcal{SH}(k)$.

Proof. Apply Corollaries 3.3, 4.9, Theorems 5.4, 5.7 and [MV, Theorem 3.2.29], [Vo, Proposition 4.13]. □

6.4. Now consider the Hopf map $\mathbf{P}^1_k \wedge \mathbf{G}_m \simeq \mathbf{A}^2 - 0 \to \mathbf{P}^1_k$. It induces a stable map $\eta : \mathbf{G}_m \to S^0$. Morel conjectured (January 2001) that W^*_B is represented by $\mathbf{KO}[\eta^{-1}] := hocolim(\mathbf{KO} \xrightarrow{\eta} \mathbf{KO} \wedge \mathbf{G}_m^{-1} \xrightarrow{\eta} \mathbf{KO} \wedge \mathbf{G}_m^{-2} ...)$ where $\mathbf{KO}$ stands for an $\Omega_{\mathbf{P}^1}$-spectrum representing hermitian K-theory. In fact, his conjectural description of $\mathbf{KO}$ looked more geometric than ours, involving $B_{et}O$, $B_{et}Sp$ etc., similar to real topological K-theory (compare 3.7). We remark that there is also some work of J. Barge and J. Lannes (not yet published) which allows them to prove some periodicity theorem with respect to a *naive* notion of homotopy of algebraic varieties. In particular, they seem to have a more geometric proof of Karoubi's Fundamental Theorem (Theorem 4.2) for regular rings.

Morel also conjectured that there is an exact triangle $\mathbf{KO} \wedge \mathbf{G}_m \xrightarrow{\eta} \mathbf{KO} \to \mathbf{BGL}$ in $\mathcal{SH}(k)$. This does not only look similar to Karoubi's Fundamental Theorem, but even more to Atiyah's work on *Real K-theory* where the Hopf map appears precisely in the analogous topological situation, see [At, Propositions 3.2, 3.3].

Comparing this to the Karoubi tower and knowing that its *hocolim* KT represents W^*_B by Theorem 5.7, this leads us to the following:

6.5 Conjecture. *The stabilization of the map in the Karoubi tower $\mathbf{KO}(\quad) \to \mathbf{KO}(U \quad)$ coincides with the Hopf map $\mathbf{KO} \xrightarrow{\eta} \mathbf{KO} \wedge \mathbf{G}_m^{-1}$ in $\mathcal{SH}(k)$.*

Next, recall the following conjecture of F. Morel [Mo1],[Mo2] about stable $\mathbf{A}^1$-homotopy groups of spheres:

6.6 Conjecture. *For any field k of characteristic different from 2 and for all $n > 0$, there is are isomorphisms*

$$Hom_{\mathcal{SH}(k)}(\mathbf{G}_m^{\wedge n}, S^0) \cong W^0_B(k)$$

$$Hom_{\mathcal{SH}(k)}(S^0, S^0) \cong K^h_0(k)$$

where $\mathcal{SH}(k)$ denotes the stable $\mathbf{A}^1$-homotopy category of [Vo].

Introducing *Milnor-Witt K-theory* [Mo2], this conjecture can be extended to $Hom_{\mathcal{SH}(k)}(S^0, \mathbf{G}_m^{\wedge n})$. In order to better understand how these different conjectures are related and what this has to do with $\mathbf{A}^1$-representability of Balmer Witt groups and hermitian K-theory, consider the following diagram:

$$\begin{array}{ccccccc}
K_0^h(k) & \xrightarrow{\cong} & Hom_{\mathcal{H}(k)}(Spec(k)_+, aKO_f) & \xrightarrow{\Sigma^\infty_{\mathbf{P}^1_k}} & Hom_{\mathcal{SH}(k)}(Spec(k)_+, \mathbf{KO}) & \xleftarrow[i]{} & Hom_{\mathcal{SH}(k)}(S^0, S^0) \\
\downarrow p & & \downarrow \tilde{\alpha} & & \downarrow \tilde{\alpha} & & \downarrow \eta \\
W_B^0(k) & \xrightarrow{\cong} & Hom_{\mathcal{H}(k)}(Spec(k)_+, aKT_f) & \xrightarrow{\Sigma^\infty_{\mathbf{P}^1_k}} & Hom_{\mathcal{SH}(k)}(Spec(k)_+, \mathbf{KT}) & \xleftarrow[j]{} & Hom_{\mathcal{SH}(k)}(S^0, \mathbf{G}_m^{\wedge -1})
\end{array}$$

Here p is the projection, $\alpha : K^h(\ \ ) \to K^h(U\ \ )$ is the map from the Karoubi tower, $\tilde{\alpha}$ are the induced maps and i is given by the fact that the sphere spectrum is a unit. More precisely, we choose the map $Hom_{\mathcal{H}(k)}(S^0, aKO_f) \cong K_0^h(k)$ represented by the form $< 1 >$. The left hand side and the middle square commute. The conjecture is that there is a map j such that the right hand side square also commutes and that all the horizontal maps are isomorphisms. If all the above conjectures hold, then j is induced by $\mathbf{G}_m^{\wedge -1} \stackrel{i\wedge id}{\to} \mathbf{KO} \wedge \mathbf{G}_m^{\wedge -1} \to \mathbf{KO}[\eta^{-1}] \simeq \mathbf{KT}$.

7 Appendix: On the relationship of hermitian K-theory, L-theory and Balmer Witt groups

People study symmetric bilinear forms (and quadratic forms) for different reasons. Depending on their motivation and their background, they end up with different theories.

L-theory has been extensively studied by topologists because quadratic L-groups contain *surgery obstruction classes*, and *higher signatures* have values in symmetric L-groups. The rings involved are integral group rings of the fundamental group of some topological space. Hence they are often non-commutative, and assuming 2 to be invertible is a rather strong restriction.

People interested in the theory of quadratic forms in itself work in particular over fields and commutative rings, and more and more also over non-affine schemes. Many of them think that Balmer Witt groups are a very convenient generalization of the classical Witt group, allowing to prove strong theorems which were not available before.

Finally, from a purely K-theoretical point of view, it seems most natural to study hermitian K-theory. In fact, this is essentially the mother theory from which all other theories can be deduced, as we have seen in section 2. One important difference between hermitian K-theory and the other theories is that hermitian K-groups are defined as being the homotopy groups of a space (or spectrum). This tends to make concrete calculations more difficult, but is much more convenient to establish $\mathbf{A}^1$-representability.

Unfortunately, many mathematicians are only comfortable with one of the three theories. We now give a couple of both old and new comparison results (and conjectures) between these three theories.

Historically, quadratic L-groups have been defined first for group rings by Wall (depending on a fixed subgroup of the Whitehead group) and then for rings R with involution in general (depending on a fixed subgroup of $K_1(R)$). Symmetric L-groups were introduced by Mischenko. Later Ranicki gave a definition of L-groups depending on a fix subgroup X of $K_0(R)$. There is always a quadratic and a symmetric version of L-groups, denoted by L_*^X and by L_X^*. The case

$X = K_0(R)$ is often denoted by L^p_* (and also by U_* in some older articles, not to confuse with our U-groups).

Quadratic L-theory is always 4-periodic. Symmetric L-theory coincides with quadratic L-theory only if 2 is invertible in our ring. Otherwise, symmetric L-theory need not to be 4-periodic. It becomes 4-periodic after passing to the colimit which is good enough for most geometric applications. It also becomes 4-periodic if we neglect the 2-torsion (*i.e.*, after applying $\otimes_{\mathbf{Z}}\mathbf{Z}[1/2]$) because quadratic and symmetric L-theory coincide up to 2-torsion. When we say a certain L-theory is 4-periodic, actually more is true: we have $L^n \cong {}_{-}L^{n+2}$ where ${}_{-}L$ denotes the theory of antisymmetric forms. The same periodicity pattern holds for Balmer Witt groups. Neglecting the 2-torsion, it also holds for Karoubi Witt groups as defined in 4.1 (see [Ka2]). Hence it suffices in general to consider only degrees 0 and 1.

Ranicki first gave a definition of L-theory in terms of *forms* and *formations* and later in terms of *algebraic Poincaré complexes*. See [Ra2, section 5] for a proof that both definitions coincide.

We will now investigate symmetric L-theory of a ring R in which 2 is invertible with respect to $X = K_0(R)$, which we denote by $L^n(R)$ from now on.

7.1 Proposition. *There is a natural isomorphism*

$$W_n^K(R) \otimes_{\mathbf{Z}} \mathbf{Z}[1/2] \cong L^n(R) \otimes_{\mathbf{Z}} \mathbf{Z}[1/2] \ .$$

Proof. This is already stated (without proof) in [Lo, p. 321]. For $n = 0$, it is true by definition (use forms instead of algebraic Poincaré complexes to describe L^0). For $n = 1$, first observe [Wall, p.286] that $W_1^K(R) = L_1^{\widetilde{K_1(R)}}(R) \cong L_1^{0\subset\widetilde{K_0(R)}}(R)$ [Ra1, p.14] and hence also $L_1^{K_1(R)}(R) \cong L_1^{0\subset K_0(R)}(R)$ as they differ by the same Tate-cohomology group [HTW, p.61],[HRT, p.138]. By the Rothenberg sequence [Ra2, Proposition 9.1], $L_1^{0\subset K_0(R)}(R)$ and $L_1^{K_0(R)}(R) = L_1^p(R)$ are isomorphic after tensoring with $\mathbf{Z}[1/2]$. The proposition now follows from the periodicity of L-theory and Karoubi's 12-term exact sequence [Ka2],[Ka3]. □

Balmer Witt groups follow the same periodicity pattern as L-theory: $W_B^n \cong_{-} W_B^{n+2} \cong W^{n+4}$. The following comparison statement has been conjectured by many people, but nobody has written down a proof so far:

7.2 Conjecture. *There is a natural isomorphism*

$$L^n(R) \cong W_B^{-n}(R) \ .$$

By periodicity, we have $L^n(R) = colim\, L^{n+4k}(R)$ is 2 is invertible, otherwise we take this as a definition. As $P(R)$ is split exact and idempotent complete, a map of complexes is a quasi-isomorphism if and only if it is a homotopy equivalence. Both $L^n(R)$ and $W_B^{-n}(R)$ are generated by elements in $D^b(P(A))$ equipped with a non-degenerate symmetric bilinear form with respect to the duality $Ext^n_R(\ \ , R)$. It remains to compare the cobordism relation for L-groups with the metabolic objects for W_B^*. Ch. Walter and independently M. Varisco claim to have a proof of Conjecture 7.2. This will hopefully appear soon.

Recall from section 2 that for a scheme X we can extend K^h from rings to schemes using Jouanolou's device or *holims* of affine covers and get by definition $KO(X)$. We conjecture, but we cannot prove that the theory KO obtained this way coincides with K^h as defined in [Ho] for non-affine schemes. We also define $WO_n(X) := coker(K_n(X) \to KO_n(X))$. Hence we obtain the following comparison result between Balmer and Karoubi Witt groups:

7.3 Corollary. *Assume that Conjecture 7.2 is true. Then*
i) there is a natural isomorphism

$$W_n^K(R) \otimes_{\mathbf{Z}} \mathbf{Z}[1/2] \cong W_B^{-n}(R) \otimes_{\mathbf{Z}} \mathbf{Z}[1/2] \ ;$$

ii) for any regular scheme X, there is a natural isomorphism

$$WO_n(X) \otimes_{\mathbf{Z}} \mathbf{Z}[1/2] \cong W_B^{-n}(X) \otimes_{\mathbf{Z}} \mathbf{Z}[1/2] \ .$$

Proof. i) follows from Proposition 7.1 and Conjecture 7.2 (and is hence true even if R is not regular). To prove ii), use strong homotopy invariance of K, KO and W_B^*.

Observe that Corollary 7.3 is false even for fields k if we do not tensor with $\mathbf{Z}[1/2]$. In this case, we have $W_1^K(k) = \mathbf{Z}/2$, but $W_B^{-1}(k) = 0$. On the other hand, we have the following comparison result for regular rings including the 2-torsion if we look at negative hermitian K-theory:

7.4 Proposition. *For any regular ring R, there are natural isomorphisms for $n > 0$*

$$W_B^n(R) \cong W_{-n}^K(R) \cong K_{-n}^h(R) \cong U_{-n-1}(R) \ .$$

Proof. See [HS, Lemma 3.6] and remember that the negative K-theory of a regular ring is trivial. □

7.5 Corollary. *For any regular scheme X, there are natural isomorphisms for $n > 0$*

$$KO_{-n}(X) \cong WO_{-n}(X) \cong W_B^n(X) \ .$$

Proof. Follows from Proposition 7.4 and strong homotopy invariance. □

The following is probably rather a theorem of Bruce Williams than a conjecture of his. He has outlined a proof to me and hopefully, this will be written down soon [Wi].

7.6 Conjecture. *(B. Williams) For any (not necessarily regular) ring R in which 2 is invertible, we have a natural isomorphism*

$$\pi_n(KT(R)) \cong L^n(R) \ .$$

7.7 Corollary. *If Conjecture 7.6 is true, we have a natural isomorphism*

$$L^n(R) \cong colim\, K_n^h(U_R^i) \ .$$

Proof. Observe that $\pi_n(KT(R)) \cong colim\, \pi_n(K^h(U_R^i))$. □

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