ON THE GRAYSON SPECTRAL SEQUENCE

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Introduction

The main purpose of these notes is to show that Grayson's motivic cohomology coincides with the usual definition of motivic cohomology - see [V2, S-V] for example and hence Grayson's spectral sequence [Gr] for a smooth semilocal scheme X essentially of finite type over a field F takes the form

$$E_2^{pq} = H^{p-q}(X, \mathbb{Z}(-q)) \Longrightarrow K_{-p-q}(X).$$

One can use next the globalization machinery developed in [S-F] to get a similar looking spectral sequence for any smooth scheme of finite type over a field. Moreover, it's not hard to see that the resulting spectral sequence coincides with the one constructed in [F-S]. What is nice, however, with this approach is that it avoids completely the use of the paper of Bloch and Lichtenbaum [B-L], which many people still find doubtful, still is not published and possibly never will be.

The main result of the paper says that the canonical homomorphism of complexes of sheaves $\mathbb{Z}^{Gr}(n) \to \mathbb{Z}(n)$ is a quasi-isomorphism. There are essentially three reasons behind this quasi-isomorphism. First, cohomology sheaves of the complex $\mathbb{Z}^{Gr}(n)$ are homotopy invariant $K_0^{\oplus}$-sheaves and hence homotopy invariant pretheories - M. Walker [W]. Second, the complex $\mathbb{Z}^{Gr}(n)$ is defined by a rationally contractible presheaf, as is the complex $\mathbb{Z}(n)$; see Proposition 2.2 below, which implies vanishing of certain polyrelative cohomology groups in the semilocal case as in Theorem 2.7 below. Finally, just as the usual motivic cohomology does, Grayson's motivic cohomology satisfies cohomology purity, i.e., if $Z \subset Y$ is a smooth subscheme of pure codimension d, then we have canonical Gysin isomorphisms

$$H^p_Z(Y, \mathbb{Z}^{Gr}(n)) = H^{p-2d}(Z, \mathbb{Z}^{Gr}(n-d)).$$

Given these three properties the proof of the comparison theorem is immediate. In view of the first point and the usual properties of homotopy invariant pretheories (see [V1]) it suffices to establish that for any field extension E/F the corresponding map in motivic cohomology

$$H^p(E, \mathbb{Z}^{Gr}(n)) \to H^p(E, \mathbb{Z}(n)) \qquad (0.0)$$

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is an isomorphism. Making a base change we may even assume that $E = F$. Following the method developed in [S-V], we proceed to show that the map (0.0) is an isomorphism by induction on the weight n. In case $n = 0$ there is nothing to prove. For any n both sides are zero for $p > n$. It's easy to check by a direct computation that the map in question is an isomorphism in degree $p = n$, but we won't use this computation. Instead we note that for any p and any m we have a degree shift isomorphism

$$H^p(F, \mathbb{Z}^{Gr}(n)) = H^{m+p}(\Delta^m, \{\Delta_i^{m-1}\}_{i=0}^m; \mathbb{Z}^{Gr}(n))$$

where on the right we have polyrelative cohomology of the m-simplex with respect to all its faces. Let $\mathcal{Z}$ be the family of supports on Δ^m, consisting of all closed subschemes $Z \subset \Delta^m$ containing no vertices. Consider the long exact sequence for polyrelative cohomology with supports

$$H^{m+p-1}(\hat{\Delta}^m, \{\hat{\Delta}_i^{m-1}\}_{i=0}^m; \mathbb{Z}^{Gr}(n)) \to H_{\mathcal{Z}}^{m+p}(\Delta^m, \{\Delta_i^{m-1}\}_{i=0}^m; \mathbb{Z}^{Gr}(n)) \to$$
$$\to H^{m+p}(\Delta^m, \{\Delta_i^{m-1}\}_{i=0}^m; \mathbb{Z}^{Gr}(n)) \to H^{m+p}(\hat{\Delta}^m, \{\hat{\Delta}_i^{m-1}\}_{i=0}^m; \mathbb{Z}^{Gr}(n))$$

and a natural homomorphism from this exact sequence to a similar sequence for polyrelative cohomology with supports with coefficients in $\mathbb{Z}(n)$. Vanishing of polyrelative cohomology in the semilocal case implies that

$$H^i(\hat{\Delta}^m, \{\hat{\Delta}_i^{m-1}\}_{i=0}^m; \mathbb{Z}^{Gr}(n)) = H^i(\hat{\Delta}^m, \{\hat{\Delta}_i^{m-1}\}_{i=0}^m; \mathbb{Z}(n)) = 0 \text{ for } i > n.$$

Thus taking m large enough we see that the natural map

$$H_{\mathcal{Z}}^{m+p}(\Delta^m, \{\Delta_i^{m-1}\}_{i=0}^m; \mathbb{Z}^{Gr}(n)) \to H^{m+p}(\Delta^m, \{\Delta_i^{m-1}\}_{i=0}^m; \mathbb{Z}^{Gr}(n)) =$$
$$= H^p(F, \mathbb{Z}^{Gr}(n))$$

(and a similar map for $\mathbb{Z}(n)$-cohomology) is an isomorphism. Finally the map in cohomology with supports

$$H_{\mathcal{Z}}^{m+p}(\Delta^m, \{\Delta_i^{m-1}\}_{i=0}^m; \mathbb{Z}^{Gr}(n)) \to H_{\mathcal{Z}}^{m+p}(\Delta^m, \{\Delta_i^{m-1}\}_{i=0}^m; \mathbb{Z}(n))$$

is an isomorphism in view of the cohomology purity and the inductive assumption.

Of the three main properties of Grayson's motivic cohomology the most subtle one undoubtedly is the cohomology purity. My original plan was to use the fibrations produced by Grayson in [Gr] and first prove cohomology purity for cohomology with coefficients in the spectra constituting Grayson's tower. This can be done by first comparing these cohomology theories to the cohomology theories provided by the tower constructed in [F-S] and then backtracking the construction used in [F-S] and returning to the spectra corresponding to the topological filtration on the K-theory of $X \times \Delta^\bullet$. The latter cohomology theories were investigated by M. Levine in [Le], who proved, in particular, a localization theorem for them that is equivalent to, or rather slightly stronger than, the cohomological purity theorem, since it does not require Z to be smooth. This plan is clearly workable but rather lengthy and technical. Fortunately the recent paper of Voevodsky [V3] makes this plan unnecessary. In this paper Voevodsky provides an incredibly short and absolutely ingenious proof of a cancellation theorem for motivic cohomology that is easily adaptable to the Grayson motivic cohomology.

Here are the contents of the paper by section. In § 1 we discuss various general facts about $K_0^{\oplus}$-presheaves, K_0-presheaves and presheaves with transfers, recalling, in particular, some results from M. Walker's thesis. The main result of § 2, Theorem 2.7, gives vanishing of certain polyrelative cohomology groups in the semilocal case. In § 3 we recall the definition of Grayson's cohomology and establish some of its basic elementary properties. Section 4 contains our adaptation of Voevodsky's proof of the Cancellation Theorem. In § 5 we discuss at some length Grayson's cohomology with supports and in § 6 we give the proof of the comparison theorem.

We borrowed the idea to use polyrelative cohomology of the semilocal scheme $\hat{\Delta}^n$ with respect to its faces from the paper of Geisser and Levine [G-L], [G-L1], who showed that these polyrelative cohomology groups can be used as an effective tool in dealing with various problems of the motivic cohomology theory. One of the main achievements of the paper [G-L] was the proof of the fact that Bloch-Kato Conjecture implies Beilinson-Lichtenbaum Conjecture over a field of any characteristic provided one replaces motivic cohomology with higher Chow groups. In § 7 we show how Theorem 2.7 can be used to significantly simplify (as we hope) the arguments used in [S-V] and get rid of the assumption about resolution of singularities. This new version of the proof of the fact that Bloch-Kato Conjecture implies Beilinson-Lichtenbaum Conjecture is shorter, clearer and more to the point (in my opinion) than the versions presented in [S-V] and [G-L].

Throughout the paper we denote by Sm/F the category of smooth separated schemes of finite type over the base field F and by Sch/F the category of all separated schemes of finite type over F.

§ 1. $K_0^{\oplus}$-presheaves, K_0-presheaves and presheaves with transfers.

For any $X, Y \in Sm/F$ set

$\mathcal{P}(X,Y) =$ The category of coherent $\mathcal{O}_{X\times Y}$-modules P such that $\operatorname{Supp} P$ is finite over X and the coherent $\mathcal{O}_X$-module $(p_X)_*(P)$ is locally free.

Note that the second condition imposed on the sheaf P above is equivalent to saying that P is flat over X. To abbreviate the language we'll refer to sheaves $P \in \mathcal{P}(X,Y)$ as sheaves finite and flat over X.

Note that the category $\mathcal{P}(X,Y)$ is closed under extensions in the abelian category of all coherent $\mathcal{O}_{X\times Y}$-modules and hence has a natural structure of an exact category. Moreover, the functor

$$(p_X)_* : \mathcal{P}(X,Y) \to \text{Locally free } \mathcal{O}_X\text{-modules}$$

is obviously exact. Consider the following abelian groups

$K_0^{\oplus}(X,Y) = K_0^{\oplus}(\mathcal{P}(X,Y))$, $K_0(X,Y) = K_0(\mathcal{P}(X,Y))$, $\mathrm{Cor}(X,Y) =$ Free abelian group generated by closed integral subschemes $Z \subset X \times Y$ finite and surjective over a component of X

These groups are related by canonical homomorphisms

$$K_0^{\oplus}(X,Y) \to K_0(X,Y) \to \mathrm{Cor}(X,Y)$$

Here the first map is the obvious surjective homomorphism and the second one takes the class $[P]$ of the coherent sheaf $P \in \mathcal{P}(X,Y)$ to $\sum_Z \ell_{\mathcal{O}_{X\times Y,z}} P_z \cdot [Z]$, where the sum is taken over all closed integral subschemes $Z \subset X \times Y$ finite and surjective over a component of X and z denotes the generic point of the corresponding scheme Z.

Let $X, Y, U \in Sm/F$ be three smooth schemes. In this case we have a natural bifunctor

$$\mathcal{P}(X,Y) \times \mathcal{P}(Y,U) \to \mathcal{P}(X,U) \quad P \times Q \mapsto (p_{X,U})_*(p_{X,Y}^*(P) \otimes_{\mathcal{O}_{X\times Y\times U}} p_{Y,U}^*(Q))$$

One checks easily that the sheaf on the right really belongs to $\mathcal{P}(X,U)$ and moreover the above bifunctor is biexact. Thus we get a natural composition law

$$K_0(X,Y) \otimes K_0(Y,U) \to K_0(X,Y)$$

In a similar way one defines composition laws

$$K_0^{\oplus}(X,Y) \otimes K_0^{\oplus}(Y,U) \to K_0^{\oplus}(X,U)$$
$$\mathrm{Cor}(X,Y) \otimes \mathrm{Cor}(Y,U) \to \mathrm{Cor}(X,U)$$

All these composition laws are associative, and that allows us to make Sm/F into an additive category in three different ways: taking $Hom(X,Y)$ to be either $K_0^{\oplus}(X,Y)$, $K_0(X,Y)$ or $\mathrm{Cor}(X,Y)$ respectively. We denote the resulting additive categories $K_0^{\oplus}(Sm/F)$, $K_0(Sm/F)$ and $SmCor/F$ respectively.

Definition 1.1. *A $K_0^{\oplus}$-presheaf (resp. K_0-presheaf, presheaf with transfers) on the category Sm/F is an additive contravariant functor $\mathcal{F} : K_0^{\oplus}(Sm/F) \to Ab$ (resp. $K_0(Sm/F) \to Ab, SmCor/F \to Ab$).*

Remark 1.1.1. *An advanced theory of presheaves with transfers was developed by V. Voevodsky* [V1], [V2]. *$K_0^{\oplus}$-presheaves and K_0-presheaves were introduced and studied by M. Walker* [W].

According to what was said at the beginning of this section we have canonical functors

$$K_0^{\oplus}(Sm/F) \to K_0(Sm/F) \to SmCor/F$$

This remark implies that every presheaf with transfers determines a K_0-presheaf and every K_0-presheaf determines a $K_0^{\oplus}$-presheaf. Note also that we have a canonical functor

$$Sm/F \longrightarrow K_0^{\oplus}(Sm/F)$$

that associates to every morphism $f : X \longrightarrow Y$ of smooth schemes the class $\mathcal{O}_{\Gamma_f} \in K_0^{\oplus}(X, Y)$ of its graph $\Gamma_f \subset X \times Y$. Quite often we use the same notation f for the $K_0^{\oplus}$-morphism $X \to Y$ determined by a morphism $f : X \to Y$ of schemes and sometimes (when it can not lead to a confusion) even for the coherent sheaf $\mathcal{O}_{\Gamma_f}$ that represents this morphism.

There is another useful construction present in all three categories $K_0^{\oplus}(Sm/F)$, $K_0(Sm/F)$ and $SmCor/F$. We'll discuss it only in case of $K_0^{\oplus}(Sm/F)$ leaving the obvious modifications needed for two other categories to the reader. Let X, Y, X', Y' be four smooth schemes. Let further $P \in \mathcal{P}(X, Y)$, $P' \in \mathcal{P}(X', Y')$ be sheaves finite and flat over X and X' respectively. In this case the external tensor product $P \boxtimes P'$ is obviously finite and flat over $X \times X'$. Thus we get a bifunctor

$$\mathcal{P}(X, Y) \times \mathcal{P}(X', Y') \xrightarrow{\boxtimes} \mathcal{P}(X \times X', Y \times Y')$$

which is obviously additive and biexact. This gives a canonical operation – external tensor product

$$K_0^{\oplus}(X, Y) \otimes K_0^{\oplus}(X', Y') \xrightarrow{\boxtimes} K_0^{\oplus}(X \times X', Y \times Y')$$

According to what was said above every $K_0^{\oplus}$-presheaf determines a presheaf $Sm/F \longrightarrow Ab$ in the usual sense. This remark allows us to consider presheaves with transfers (resp. K_0-presheaves, $K_0^{\oplus}$-presheaves) as presheaves in the usual sense provided with an appropriate additional data. In this section we'll be mostly dealing with the following questions

(1) What happens to a $K_0^{\oplus}$-presheaf structure (resp. K_0-presheaf structure, presheaf with transfers structure) on a presheaf $\mathcal{F}$ when we sheafify it in Zariski or Nisnevich topologies.
(2) Find conditions that assure that the given $K_0^{\oplus}$-presheaf structure on $\mathcal{F}$ (resp. K_0-presheaf structure) may be descended to a K_0-presheaf structure (resp. presheaf with transfers structure).

Most part of the results presented below are taken from [V1, V2], [W], so unless the result is new we just give the formulation and indicate the reference, sometimes giving the sketch of the proof (if it's not too long).

In the computations we are about to perform the following formula computing the composition of the usual morphisms with ones given by $K_0^{\oplus}(X, Y)$ is often used. The verification of this formula is quite straightforward and we leave it as an easy exercise to the reader.

Lemma 1.2.0. *For any $P \in \mathcal{P}(X, Y)$ and any morphism $f : X' \to X$, $g : Y \to Y'$ of smooth schemes we have the following identities*

$$[P] \circ f = [(f \times 1_Y)^*(P)] \in K_0^{\oplus}(X', Y)$$
$$g \circ [P] = [(1_X \times g)_*(P)] \in K_0^{\oplus}(X, Y')$$

Recall that a presheaf $\mathcal{F} : Sm/F \longrightarrow Ab$ is said to be homotopy invariant provided that for any $X \in Sm/F$ the canonical homomorphism $(p_X)^* : \mathcal{F}(X) \longrightarrow \mathcal{F}(X \times \mathbb{A}^1)$ is an isomorphism.

Lemma 1.2 (M.Walker [W]**).** *Let $\mathcal{F}$ be a homotopy invariant $K_0^{\oplus}$-presheaf on the category Sm/F. Then the $K_0^{\oplus}$-presheaf structure on $\mathcal{F}$ descends uniquely to a K_0-presheaf structure.*

Proof. Uniqueness of the K_0-presheaf structure on $\mathcal{F}$ compatible with the original $K_0^{\oplus}$-presheaf structure is obvious. To prove its existence we have to verify that for any $X, Y \in Sm/F$ and any short exact sequence $0 \to P' \to P \to P'' \to 0$ in $\mathcal{P}(X, Y)$ the homomorphism

$$\mathcal{F}(Y) \xrightarrow{[P]^*-[P']^*-[P'']^*} \mathcal{F}(X)$$

given by the $K_0^{\oplus}$-presheaf structure on $\mathcal{F}$ is trivial. To do so we consider the following commutative diagram with exact rows in $\mathcal{P}(X \times \mathbb{A}^1, Y)$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P'[T] & \longrightarrow & P[T] & \longrightarrow & P''[T] & \longrightarrow & 0 \\ & & =\big\uparrow & & \big\uparrow & & T\big\uparrow & & \\ 0 & \longrightarrow & P'[T] & \longrightarrow & Q & \longrightarrow & P''[T] & \longrightarrow & 0 \end{array} \tag{1.2.1}$$

where the right-hand square is a pull-back diagram. Set $\beta = [Q]-[P'[T]]-[P''[T]] \in K_0^{\oplus}(X \times \mathbb{A}^1, Y)$ and consider the following diagram

$$\mathcal{F}(Y) \xrightarrow{\beta^*} \mathcal{F}(X \times \mathbb{A}^1)) \underset{i_1^*}{\overset{i_0^*}{\rightrightarrows}} \mathcal{F}(X)$$

Since $\mathcal{F}$ is homotopy invariant the homomorphisms i_0^* and i_1^* are equal. On the other hand Lemma 1.2.0 allows us to conclude that $i_1^*\beta^* = (\beta \circ i_1)^* = ([P]-[P']-[P''])^*$, whereas $i_0^*\beta^* = (\beta \circ i_0)^* = 0$, since the bottom line of the diagram (1.2.1) splits being specialized at $T = 0$.

For the future use we write down explicitly the properties of the sheaf Q constructed above.

Lemma 1.2.2. *Starting with any short exact sequence $0 \to P' \to P \to P'' \to 0$ in $\mathcal{P}(X, Y)$ the above construction gives us a sheaf $Q \in \mathcal{P}(X \times \mathbb{A}^1, Y)$ with the following properties:*

(1) $Q_0 \cong P' \oplus P''$; $Q_1 \cong P$
(2) $[Q_1] - [Q_0] = [P] - [P'] - [P''] \in K_0^{\oplus}(X, Y)$.

Lemma 1.3 (V.Voevodsky). *Let $\mathcal{F}$ be a homotopy invariant presheaf with transfers on the category Sm/F. Then the associated Zariski sheaf $\mathcal{F}_{Zar}$ has a unique* *structure of a presheaf with transfers for which the canonical homomorphism* $\mathcal{F} \xrightarrow{\phi} \mathcal{F}_{Zar}$ *is compatible with transfers. Moreover,* $\mathcal{F}_{Zar}$ *is also homotopy invariant.*

Proof. Every presheaf with transfers is obviously a pretheory in the sense of Voevodsky [V1]. Furthermore, Voevodsky proved in [V1] that the Zariski sheaf associated to a homotopy invariant pretheory is again a homotopy invariant pretheory. This implies immediately that $\mathcal{F}_{Zar}$ is homotopy invariant. To show that $\mathcal{F}_{Zar}$ is actually a presheaf with transfers amounts to the verification of the following two points:

(1) Let $X, Y \in Sm/F$ be two smooth schemes, let further $Z \subset X \times Y$ be a closed integral subscheme finite and surjective over a component of X and let $a \in \mathcal{F}(Y)$ be a section that dies in $\mathcal{F}_{Zar}(Y)$, then $Z^*(a) \in \mathcal{F}(X)$ dies in $\mathcal{F}_{Zar}(X)$.
(2) In the same situation as above let $a \in \mathcal{F}_{Zar}(Y)$ be any section of $\mathcal{F}_{Zar}$ over Y. Then there exists an open covering $X = \cup X_i$ and for each i an open $Y_i \subset Y$ and a section $a_i \in \mathcal{F}(Y_i)$ such that

$$Z \cap X_i \times Y \subset X_i \times Y_i, \qquad a_{|Y_i} = \phi(a_i)$$

In both cases it suffices to consider the case when the scheme X is local with the closed point $x_0 \in X$. Let $z_0, \ldots, z_n$ be all points of Z over x_0 and let $y_0, \ldots, y_n$ be their images in Y. Note that for any open $U \subset Y$, containing all y_i the scheme Z is contained in $X \times U$ and hence $Z^*(a) = Z^*(a_{|U})$. Consider the semilocalization Y_y of Y at the set of points $y_0, \ldots, y_n$. According to another theorem of Voevodsky [V1], Proposition 4.24 for any pretheory $\mathcal{F}$ the sections of $\mathcal{F}$ and $\mathcal{F}_{Zar}$ over any smooth semilocal scheme Y_y coincide, i.e.,

$$\varinjlim_{U \ni y_0, \ldots, y_n} \mathcal{F}(U) = \varinjlim_{U \ni y_0, \ldots, y_n} \mathcal{F}_{Zar}(U) \tag{1.3.1}$$

This formula readily implies that there exists $U \ni y_0, \ldots, y_n$ such that $a_{|U} = 0$ and thus $Z^*(a) = Z^*(a_{|U}) = 0$. Second statement also follows from the formula (1.3.1).

Lemma 1.4 (M. Walker [W]**).** *Let* $\mathcal{F}$ *be a homotopy invariant* K_0*-presheaf. Then the associated Zariski sheaf* $\mathcal{F}_{Zar}$ *has a unique structure of a* K_0*-presheaf for which the canonical homomorphism* $\mathcal{F} \xrightarrow{\phi} \mathcal{F}_{Zar}$ *is a homomorphism of* K_0*-presheaves. The presheaf* $\mathcal{F}_{Zar}$ *is homotopy invariant and moreover has a canonical structure of a homotopy invariant pretheory.*

Proof. Walker verified in [W] that homotopy invariant K_0-presheaves satisfy many properties enjoyed by homotopy invariant pretheories. In particular, they satisfy the formula (1.3.1). To show that $\mathcal{F}_{Zar}$ has a unique structure of a K_0-sheaf it suffices now to repeat the previous argument replacing everywhere the closed subscheme Z by a coherent sheaf $P \in \mathcal{P}(X, Y)$ (finite and flat over X). The remaining facts are not to hard to prove either – see [W] for details.

Remark 1.4.1. *With some more work one can show that in conditions of Lemma 1.4 the Zariski sheaf $\mathcal{F}_{Zar}$ actually has a canonical structure of a Zariski sheaf with transfers. Since for most practical purposes pretheories are as good as sheaves with transfers we don't try to give (a rather lengthy) proof of this fact here.*

Lemma 1.5. *Let $\mathcal{F}$ be a $K_0^{\oplus}$-presheaf (resp. K_0-presheaf, presheaf with transfers). Then the associated Nisnevich sheaf $\mathcal{F}_{Nis}$ has a unique structure of $K_0^{\oplus}$-presheaf (resp. K_0-presheaf, presheaf with transfers) for which the canonical homomorphism $\mathcal{F} \xrightarrow{\phi} \mathcal{F}_{Nis}$ is a homomorphism of $K_0^{\oplus}$-presheaves (resp. of K_0-presheaves, presheaves with transfers).*

Proof. In case of presheaves with transfers this fact is proved in [V2] Lemma 3.1.6. In case of $K_0^{\oplus}$-presheaves and K_0-presheaves the proof is essentially the same as for presheaves with transfers, so we just sketch it briefly. Once again we have to verify two things

(1) Let $X, Y \in Sm/F$ be two smooth schemes, let further $P \in \mathcal{P}(X, Y)$ be a coherent sheaf finite and flat over X. Let finally $a \in \mathcal{F}(Y)$ be a section that dies in $\mathcal{F}_{Nis}(Y)$. Then $P^*(a) \in \mathcal{F}(X)$ dies in $\mathcal{F}_{Nis}(X)$.
(2) In the same situation as above let $a \in \mathcal{F}_{Nis}(Y)$ be any section of $\mathcal{F}_{Nis}$ over Y. Then there exists a Nisnevich covering $\{X_i \xrightarrow{f_i} X\}_i$ and for each i a scheme $Y_i \xrightarrow{g_i} Y$ over Y, a sheaf $P_i \in \mathcal{P}(X_i, Y_i)$ and a section $a_i \in \mathcal{F}(Y_i)$ such that
$$P \circ f_i = g_i \circ P_i, \quad a_{|Y_i} = \phi(a_i)$$

This time, since we are working now in Nisnevich topology, we may assume X to be a Henselian local scheme. The closed subscheme $\operatorname{Supp} P = \operatorname{Spec} \mathcal{O}_{X \times Y} / \operatorname{Ann} P \subset X \times Y$ being finite over X is a disjoint sum of local Henselian schemes Supp_j. This decomposition of $\operatorname{Supp} P$ determines a canonical direct sum decomposition $P = \oplus_j P_j$ with support of P_j being equal to Supp_j. It suffices clearly to treat the case $P = P_j$, i.e., we may assume that $\operatorname{Supp} P$ is a local Henselian scheme with closed point $z \in \operatorname{Supp} P$. Let y be the image of z in Y. By the assumption there exists an étale neighborhood $(U, u) \xrightarrow{s} (Y, y)$ such that $a_{|U} = 0$. The scheme $(\operatorname{Supp} P, z)$ being Henselian we conclude from the definitions that the morphism $p_Y : (\operatorname{Supp} P, z) \to (Y, y)$ factors uniquely through (U, u). The corresponding morphism $\operatorname{Supp} P \to X \times U$ is obviously a closed embedding. This allows us to define a coherent $\mathcal{O}_{X \times U}$-module Q, whose support identifies with that of P under the above closed embedding. Obviously $Q \in \mathcal{P}(X, U)$ and $(1_X \times s)_*(Q) = P$, i.e., $[P] = s \circ [Q] \in K_0^{\oplus}(X, Y)$. Finally $P^*(a) = Q^*(s^*(a)) = 0$.

We end this section with an application of the previous results to polyrelative cohomology. For any presheaf $\mathcal{F} : Sm/F \to Ab$ we denote by $C_n(\mathcal{F})$ a new presheaf on the category Sm/F defined by the formula $C_n(\mathcal{F})(X) = \mathcal{F}(X \times \Delta^n)$. Thus $C_\bullet(\mathcal{F})$ is a simplicial presheaf and we use the notation $C_*(\mathcal{F})$ for the corresponding non-negative complex of sheaves with differential (of degree -1) equal to the alternating sum of face operations. We use the notation $C^*(\mathcal{F})$ for the same complex reindexed cohomologically (i.e., $C^i = C_{-i}$).

Let $X \in Sm/F$ be a smooth scheme and let $\{X_i\}_{i=0}^n$ be a family of subschemes in X such that all intersections $X_{i_0} \cap \ldots \cap X_{i_k}$ are smooth. We denote by $\mathbb{Z}_{Zar}(X; \{X_i\}_{i=0}^n)$ the following complex (with differential of degree $+1$) of Zariski sheaves:

$$\ldots \to \oplus_{i_1 < \ldots < i_k} \mathbb{Z}_{Zar}(X_{i_1} \cap \ldots \cap X_{i_k}) \to \ \ldots \ \to \oplus_i \mathbb{Z}_{Zar}(X_i) \to \mathbb{Z}_{Zar}(X)$$

Here $\mathbb{Z}_{Zar}(X)$ stands in degree zero, $\oplus_{i_1 < \ldots < i_k} \mathbb{Z}_{Zar}(X_{i_1} \cap \ldots \cap X_{i_k})$ stands in degree $-k$ and the differential is the alternating sum of maps induced by inclusions. We use the notation $\mathbb{Z}_{Nis}(X; \{X_i\}_{i=0}^n)$ for the obvious counterpart of the above complex with Zariski topology replaced by the Nisnevich topology. If $\mathcal{F}^*$ is any complex of Zariski sheaves we define polyrelative cohomology of X with respect to $\{X_i\}_{i=0}^n$ as (hyper)Ext-groups for complexes of sheaves:

$$H^p_{Zar}(X, \{X_i\}_{i=0}^n; \mathcal{F}^*) = \mathrm{Ext}^p(\mathbb{Z}_{Zar}(X; \{X_i\}_{i=0}^n), \mathcal{F}^*)$$

In a similar way one defines polyrelative Nisnevich cohomology with coefficients in a complex of Nisnevich sheaves. The following Lemma summarizes some of the standard (and obvious) properties of polyrelative cohomology. We do not specify the topology in the formulation of this Lemma since it works for both the Zariski and the Nisnevich topologies.

Lemma 1.6. *a. We have a natural long exact sequence*

$$H^p(X, \{X_i\}_{i=0}^n; \mathcal{F}^*) \to H^p(X, \{X_i\}_{i=0}^{n-1}; \mathcal{F}^*) \to H^p(X_n, \{X_i \cap X_n\}_{i=0}^{n-1}; \mathcal{F}^*) \xrightarrow{\delta}$$
$$\xrightarrow{\delta} H^{p+1}(X, \{X_i\}_{i=0}^n; \mathcal{F}^*)$$

b. We have a natural spectral sequence

$$E^1_{pq} = \oplus_{i_1 < \ldots < i_q} H^p(X_{i_1} \cap \ldots \cap X_{i_q}, \mathcal{F}^*) \Longrightarrow H^{p+q}(X, \{X_i\}_{i=0}^n; \mathcal{F}^*)$$

c. Letting $\mathcal{F}^* \to I^*$ be an injective (or flasque) resolution of the complex $\mathcal{F}^*$, the polyrelative cohomology $H^*(X, \{X_i\}_{i=0}^n; \mathcal{F}^*)$ coincides with cohomology of the bicomplex

$$(1.6.1) \quad I^*(X) \to \oplus_i I^*(X_i) \to \ \ldots \ \to \oplus_{i_1 < \ldots < i_k} I^*(X_{i_1} \cap \ldots \cap I^*(X_{i_k}) \to \ \ldots$$

For any complex of presheaves I^* and any smooth scheme X, provided with a family of closed subschemes $\{X_i\}_{i=0}^n$ such that all intersections $X_{i_0} \cap \ldots \cap X_{i_k}$ are smooth we use the notation $I^*(X, \{X_i\}_{i=0}^n)$ for the bicomplex (1.6.1). Thus the last statement of Lemma 1.6 may be rephrased by saying that for a flasque resolution I^* of $\mathcal{F}^*$ we have a natural identification $H^*(X, \{X_i\}_{i=0}^n; \mathcal{F}^*) = H^*(I^*(X, \{X_i\}_{i=0}^n))$. This remark shows, in particular, that for any complex of sheaves $\mathcal{F}^*$ we have canonical homomorphisms

$$H^*(\mathcal{F}^*(X, \{X_i\}_{i=0}^n)) \to H^*(I^*(X, \{X_i\}_{i=0}^n)) = H^*(X, \{X_i\}_{i=0}^n; \mathcal{F}^*).$$

Proposition 1.7. *Assume that $\mathcal{F}$ is a $K_0^{\oplus}$-presheaf. Assume further that X is a smooth semilocal scheme provided with the family $\{X_i\}_{i=0}^n$ of closed subschemes such that all intersections $X_{i_0} \cap \ldots \cap X_{i_k}$ are smooth (semilocal) schemes. Then the canonical map*

$$H^*(C^*(\mathcal{F})(X, \{X_i\}_{i=0}^n)) \to H^*(C^*(\mathcal{F})_{Zar}(X, \{X_i\}_{i=0}^n)) \to \\ \to H^*_{Zar}(X, \{X_i\}_{i=0}^n; C^*(\mathcal{F})_{Zar})$$

is an isomorphism.

Proof. The standard spectral sequence argument shows that it suffices to deal with the absolute case, i.e., to show that canonical homomorphisms $C^*(\mathcal{F})(X) \to H^*_{Zar}(X, C^*(\mathcal{F})_{Zar})$ is an isomorphism. Denote by $\mathcal{H}^q$ the q-th cohomology presheaf of $C^*(\mathcal{F})$. Then $\mathcal{H}^q$ is a homotopy invariant $K_0^{\oplus}$-presheaf and hence a homotopy invariant K_0-presheaf (see Lemma 1.2). Consider next the hypercohomology spectral sequence

$$E_2^{pq} = H^p_{Zar}(X, \mathcal{H}^q_{Zar}) \Longrightarrow H^{p+q}_{Zar}(X, C^*(\mathcal{F})_{Zar})$$

According to Lemma 1.4 the sheaves $\mathcal{H}^q_{Zar}$ are homotopy invariant pretheories and hence $H^p_{Zar}(X, \mathcal{H}^q_{Zar}) = 0$ for $p > 0$ – see [V1], Lemma 4.28. This implies that the above hypercohomology spectral sequence degenerates and provides isomorphisms $H^q_{Zar}(X, C^*(\mathcal{F})_{Zar}) = \Gamma(X, \mathcal{H}^q_{Zar})$. Finally, as was mentioned in the proof of Lemma 1.4, M. Walker has shown that for any homotopy invariant K_0-presheaf $\mathcal{H}^q$ and any smooth semilocal scheme X the natural map $\Gamma(X, \mathcal{H}^q) \to \Gamma(X, \mathcal{H}^q_{Zar})$ is an isomorphism.

Remark 1.7.1. *For any $K_0^{\oplus}$-presheaf $\mathcal{F}$ and any smooth scheme X, provided with the family $\{X_i\}_{i=0}^n$ of closed subschemes such that all intersections $X_{i_0} \cap \ldots \cap X_{i_k}$ are smooth, the canonical map*

$$H^*_{Zar}(X, \{X_i\}_{i=0}^n; C^*(\mathcal{F})_{Zar}) \to H^*_{Nis}(X, \{X_i\}_{i=0}^n; C^*(\mathcal{F})_{Nis})$$

is an isomorphism. Thus one can replace Zariski cohomology in 1.7 with Nisnevich cohomology.

Proof. Once again it suffices to treat the absolute case. Since cohomology sheaves of the complex $C^*(\mathcal{F})_{Zar}$ are homotopy invariant pretheories our statement follows immediately from general properties of such sheaves – see [V1].

§ 2. Rationally contractible presheaves.

In this section we prove one of the main new results of this paper (Theorem 2.7), which gives vanishing of certain polyrelative cohomology groups in the semilocal case. It should be mentioned that this Theorem is a motivic cohomology version of a result proved by Geisser and Levine in [G-L1] for higher Chow groups. The present formulation is however much more general and the proof we hope is much more understandable.

For any presheaf $\mathcal{F} : Sm/F \to Ab$ let $\widetilde{C}_1\mathcal{F}$ denote the following presheaf

$$\widetilde{C}_1\mathcal{F}(X) = \varinjlim_{X\times\{0,1\}\subset U\subset X\times\mathbb{A}^1} \mathcal{F}(U)$$

Note that there are two obvious presheaf homomorphisms (restrictions to $X \times 0$ and $X \times 1$ respectively) $i_0^*, i_1^* : \widetilde{C}_1\mathcal{F} \to \mathcal{F}$.

Definition 2.1 ([S-V] § **9**). *A presheaf $\mathcal{F}$ is called rationally (or generically) contractible if there exists a presheaf homomorphism $s : \mathcal{F} \to \widetilde{C}_1\mathcal{F}$, such that $i_0^*s = 0, i_1^*s = 1_{\mathcal{F}}$.*

The following result, which is a minor generalization of [S-V] (9.6), serves as a main supply of rationally contractible sheaves.

Proposition 2.2. *Let $X \in Sm/F$ be a smooth connected scheme and let $x_0 \in X$ be a rational point of X. Assume that there exists an open subscheme $W \subset X \times \mathbb{A}^1$, containing $X \times \{0,1\} \cup x_0 \times \mathbb{A}^1$ and a morphism of schemes $f : W \to X$ such that $f|_{X\times 0} = x_0, f|_{X\times 1} = 1_X, f|_{x_0\times\mathbb{A}^1} = x_0$. Then the following presheaves are rationally contractible:*

(1) $\mathbb{Z}(X)/\mathbb{Z}(x_0)$
(2) $\mathbb{Z}_{tr}(X)/\mathbb{Z}_{tr}(x_0)$
(3) $U \mapsto K_0(U,X)/K_0(U,x_0)$
(4) $U \mapsto K_0^{\oplus}(U,X)/K_0^{\oplus}(U,x_0)$

Proof. We start with the easiest case, that of the presheaf $\mathbb{Z}(X)/\mathbb{Z}(x_0)$. Let $g : Y \to X$ be a morphism of schemes over F. Consider the morphism $g \times 1_{\mathbb{A}^1} : Y \times \mathbb{A}^1 \to X\times\mathbb{A}^1$ and let U be the inverse image of W under this morphism. Thus $U \subset Y\times\mathbb{A}^1$ is an open subscheme, containing $Y \times \{0,1\}$, and the composition $f \circ (g \times 1_{\mathbb{A}^1})$ defines a morphism $s(g) : U \to X$. Thus associating to $g \in \mathbb{Z}(X)(Y)$ the section $s(g) \in \mathbb{Z}(X)(U)$ we get a homomorphism of presheaves $s : \mathbb{Z}(X) \to \widetilde{C}_1\mathbb{Z}(X)$, such that $i_1^* \circ s = 1_{\mathbb{Z}(X)}, i_0^* \circ s = x_0$. Moreover, the prescribed behavior of f with respect to x_0 implies readily that the homomorphism s takes $\mathbb{Z}(x_0)$ to $\widetilde{C}_1\mathbb{Z}(x_0)$ and hence defines the rational homotopy operator $s : \mathbb{Z}(X)/\mathbb{Z}(x_0) \to \widetilde{C}_1(\mathbb{Z}(X)/\mathbb{Z}(x_0))$ with the required properties.

In the remaining cases the proof is essentially the same; one should interpret the sections of the above presheaves as morphisms in an appropriate sense and make sure that the constructions performed above make sense for these morphisms. We give details concerning the last two presheaves and leave the second presheaf (which was treated in [S-V], (9.6)) to the reader. Start with a coherent sheaf $\mathcal{P}$ on $Y \times X$ finite and flat over Y and denote by $Z \subset Y \times X$ its support. Note that the sheaf $\mathcal{P} \boxtimes 1_{\mathbb{A}^1}$ is a coherent $\mathcal{O}_{Y\times\mathbb{A}^1\times X\times\mathbb{A}^1}$-module finite and flat over $Y \times \mathbb{A}^1$. Moreover, the support of this sheaf coincides with $Z \times \Delta_{\mathbb{A}^1}$. Let further $T \subset X \times \mathbb{A}^1$ be the closed subscheme – complement of W. The intersection $(Z\times\Delta_{\mathbb{A}^1})\cap[(Y\times\mathbb{A}^1)\times T] \subset (Y \times \mathbb{A}^1) \times (X \times \mathbb{A}^1)$ does not contain points where the second $\mathbb{A}^1$-coordinate is equal to 0 or 1, on the other hand both $\mathbb{A}^1$-coordinates on this intersection coincide and hence it does not contain points where the first $\mathbb{A}^1$ coordinate is equal to 0 or 1. Moreover, this intersection is obviously finite over $Y \times \mathbb{A}^1$. The above remarks imply that $p_{Y\times\mathbb{A}^1}((Z \times \Delta_{\mathbb{A}^1}) \cap [(Y \times \mathbb{A}^1) \times T]) \subset Y \times \mathbb{A}^1$ is a closed subscheme not intersecting $Y \times \{0,1\}$. Denote by $U = U_{\mathcal{P}} \subset Y \times \mathbb{A}^1$ the complementary open subscheme. By the very construction we have the following inclusion:

$$[U \times (X \times \mathbb{A}^1)] \cap (Z \times \Delta_{\mathbb{A}^1}) \subset U \times W$$

The restriction of the sheaf $\mathcal{P} \boxtimes 1_{\mathbb{A}^1}$ to $U \times (X \times \mathbb{A}^1)$ is finite and flat over U and its support is contained in $U \times W$, which implies readily that the restriction

$$(\mathcal{P} \boxtimes 1_{\mathbb{A}^1})|_{U\times W}$$

is still finite and flat over U. Finally we set

$$Q = (1_U \times f)_*((\mathcal{P} \boxtimes 1_{\mathbb{A}^1})|_{U\times W}).$$

The sheaf Q is a coherent $\mathcal{O}_{U\times X}$-module, finite and flat over U. One checks easily that associating to $\mathcal{P}$ the class of Q in $\varinjlim_{Y\times\{0,1\}\subset U\subset Y\times\mathbb{A}^1} K_0(U,X)$ (resp. in $\varinjlim_{Y\times\{0,1\}\subset U\subset Y\times\mathbb{A}^1} K_0^{\oplus}(U,X)$ defines a function on the category $\mathcal{P}(Y,X)$ which is additive with respect to all short exact sequences (resp. to split short exact sequences only) and hence defines a homomorphism

$$s : K_0(Y,X) \to \varinjlim_{Y\times\{0,1\}\subset U\subset Y\times\mathbb{A}^1} K_0(U,X)$$

(resp. a homomorphism

$$s : K_0^{\oplus}(Y,X) \to \varinjlim_{Y\times\{0,1\}\subset U\subset Y\times\mathbb{A}^1} K_0^{\oplus}(U,X).$$

Finally one checks easily that s is a homomorphism of presheaves and has all the required properties.

Remark 2.3. *Note that a direct summand in a rationally contractible presheaf is obviously also rationally contractible. Thus Grayson's presheaf* $U \mapsto K_0^{\oplus}(U, \mathbb{G}_m^{\wedge n})$ *being a direct summand in* $U \mapsto K_0^{\oplus}(U, \mathbb{G}_m^{\times n})/K_0^{\oplus}(U, 1 \times ... \times 1)$ *is rationally contractible.*

Lemma 2.4. *Let* $\mathcal{F}$ *be a rationally contractible presheaf . Then the presheaf* $C_n(\mathcal{F})$ *is equally rationally contractible.*

Proof. The choice of a vertex $v \in \Delta^n$ defines a splitting of the presheaf $C_n(\mathcal{F})$ into a direct sum of $\mathcal{F}$ and the kernel $C_n^1(\mathcal{F})$ of the restriction map defined by v: $\partial_v : C_n(\mathcal{F}) \to \mathcal{F}$. Moreover, the sheaf $C_n^1(\mathcal{F})$ is contractible (see [S-V] Lemma 1.9) and hence rationally contractible.

For any $n \geq 0$ denote by $\hat{\Delta}^n$ the semilocalization of Δ^n at the set of its vertices $v_0, ..., v_n \in \Delta^n$. Since all the structure maps of the cosimplicial scheme $\Delta^\bullet$ take vertices to vertices we conclude that $\hat{\Delta}^\bullet$ is a cosimplicial semilocal scheme. Applying a presheaf $\mathcal{F}$ to this cosimplicial scheme we get a simplicial abelian group, i.e., a complex. The following elementary property of rationally contractible sheaves is the basis for all further applications.

Proposition 2.5. *Assume that the presheaf $\mathcal{F}$ is rationally contractible. Then the complex $\mathcal{F}(\hat{\Delta}^\bullet)$ is contractible and hence acyclic.*

Proof. For each $n \geq 0$ the homomorphism s defines a map

$$s_{\hat{\Delta}^n} : \mathcal{F}(\hat{\Delta}^n) \to \widetilde{C}_1\mathcal{F}(\hat{\Delta}^n) = \mathcal{F}(\Delta^n \hat{\times} \mathbb{A}^1)$$

where the latter hat indicates that the scheme $\Delta^n \times \mathbb{A}^1$ is semilocalized with respect to the set of its vertices, i.e., points $v_i \times 0, v_i \times 1$. Consider finally the usual triangularization of $\Delta^n \times \mathbb{A}^1$, defined by the family of maps $\psi_i : \Delta^{n+1} \to \Delta^n \times \mathbb{A}^1$ $(0 \leq i \leq n)$, where ψ_i is the linear isomorphism taking the vertex v_j to $v_j \times 0$ if $j \leq i$ and to $v_{j-1} \times 1$ if $j > i$. Once again ψ_i take vertices to vertices and hence define maps on the corresponding semilocalizations. The contracting homotopy operator for the complex $\mathcal{F}(\hat{\Delta}^\bullet)$ is given by the formula

$$\sigma(u) = \sum_{i=0}^{n} (-1)^i (\psi_i)^*(s(u))$$

Consider a smooth semilocal scheme $\hat{\Delta}^n$ and the family of its closed subschemes $\{\hat{\Delta}_i^{n-1}\}_{i=0}^n$. Since all intersections are obviously smooth we may consider (for any presheaf $\mathcal{F}$) the corresponding non-negative complex $\mathcal{F}(\hat{\Delta}^n, \{\hat{\Delta}_i^{n-1}\}_{i=0}^n)$.

Corollary 2.6. *Assume that the presheaf $\mathcal{F}$ is rationally contractible. Then the non-negative complex $\mathcal{F}(\hat{\Delta}^n, \{\hat{\Delta}_i^{n-1}\}_{i=0}^n)$ is acyclic in positive degrees, and its zero-dimensional homology group coincides with the intersection of the kernels of all the face maps $\partial_i : \mathcal{F}(\hat{\Delta}^n) \to \mathcal{F}(\hat{\Delta}^{n-1})$:*

$$H^0(\mathcal{F}(\hat{\Delta}^n, \{\hat{\Delta}_i^{n-1}\}_{i=0}^n) = \mathcal{F}(\hat{\Delta}^n)^{ker} \overset{\text{def}}{=} \cap_{i=0}^n Ker(\partial_i : \mathcal{F}(\hat{\Delta}^n) \to \mathcal{F}(\hat{\Delta}^{n-1}).$$

Proof. Reindex the complex $\mathcal{F}(\hat{\Delta}^n, \{\hat{\Delta}_i^{n-1}\}_{i=0}^n)$ homologically and then shift it so that $\mathcal{F}(\hat{\Delta}^n)$ stands in degree n. It was proved in [F-S] Theorem 1.2, that the resulting homological complex is canonically quasi-isomorphic to $(\sigma_{\leq n}\mathcal{M}(\mathcal{F}(\hat{\Delta}^\bullet))$. Here $\sigma_{\leq n}$ denotes the operation of stupid truncation at level n and $\mathcal{M}(\mathcal{F}(\hat{\Delta}^\bullet))$ is the Moore complex corresponding to the simplicial abelian group $\mathcal{F}(\hat{\Delta}^\bullet)$. Proposition 2.5 shows that the complex $\mathcal{F}(\hat{\Delta}^\bullet)$ is acyclic and hence $\sigma_{\leq n}\mathcal{M}(\mathcal{F}(\hat{\Delta}^\bullet)$ is acyclic in degrees $\neq n$. Returning to our original cohomological complex we conclude that the complex $\mathcal{F}(\hat{\Delta}^n, \{\hat{\Delta}_i^{n-1}\}_{i=0}^n$ is acyclic in degrees $\neq 0$. Finally the statement concerning its zero-dimensional homology is obvious.

Theorem 2.7. *Let $\mathcal{F}$ be a rationally contractible $K_0^\oplus$-presheaf. Then the canonical embedding $C^*(\mathcal{F})(\hat{\Delta}^n)^{ker} \hookrightarrow C^*(\mathcal{F})(\hat{\Delta}^n, \{\hat{\Delta}_i^{n-1}\}_{i=0}^n)$ is a quasi-isomorphism. In particular, the complex $C^*(\mathcal{F})(\hat{\Delta}^n, \{\hat{\Delta}_i^{n-1}\}_{i=0}^n)$ is acyclic in positive degrees and hence*

$$H^p_{Zar}(\hat{\Delta}^n, \{\hat{\Delta}_i^{n-1}\}_{i=0}^n; C^*(\mathcal{F})_{Zar}) = H^p_{Nis}(\hat{\Delta}^n, \{\hat{\Delta}_i^{n-1}\}_{i=0}^n; C^*(\mathcal{F})_{Nis}) = 0$$

for $p > 0$.

Proof. The first statement follows immediately from Corollary 2.6 since presheaves $C^n(\mathcal{F})$ are rationally contractible according to Lemma 2.4. The second statement follows from the first one Proposition 1.7 and Remark 1.7.1.

§ 3. Grayson's Cohomology.

For any $X \in Sm/F$ the category $\mathcal{P}(X, \mathbb{G}_m^n)$ may be identified with the category of vector bundles P over X equipped with an n-tuple $(\alpha_1, ..., \alpha_n)$ of commuting automorphisms. We denote by $K_0^{\oplus}(\mathbb{G}_m^n)$ the presheaf of abelian groups on Sm/F given by the formula $K_0^{\oplus}(\mathbb{G}_m^n)(X) = K_0^{\oplus}(\mathcal{P}(X, \mathbb{G}_m^n))$. Canonical embeddings

$$\mathbb{G}_m^{n-1} \xrightarrow{i_k} \mathbb{G}_m^n \quad (\alpha_1, ..., \alpha_{n-1}) \mapsto (\alpha_1, ..., \underset{k}{1}, ..., \alpha_{n-1}) \quad k = 1, ..., n$$

define embeddings of presheaves $K_0^{\oplus}(\mathbb{G}_m^{n-1}) \xrightarrow{(i_k)_*} K_0^{\oplus}(\mathbb{G}_m^n)$. Moreover, one checks easily that the sum of images of these embeddings is a canonical direct summand in $K_0^{\oplus}(\mathbb{G}_m^n)$ – see for example [S-V], § 0. We define the presheaf $K_0^{\oplus}(\mathbb{G}_m^{\wedge n})$ as the quotient of $K_0^{\oplus}(\mathbb{G}_m^n)$ modulo $\sum_{k=0}^n (i_k)_*(K_0^{\oplus}(\mathbb{G}_m^{n-1}))$. We define the complex of Zariski sheaves $\mathbb{Z}^{Gr}(n)$ via the formula

$$\mathbb{Z}^{Gr}(n) = C^*(K_0^{\oplus}(\mathbb{G}_m^{\wedge n}))_{Zar}[-n]$$

Finally we define Grayson's motivic cohomology of X as the Zariski hypercohomology of X with coefficients in the complex of sheaves $\mathbb{Z}^{Gr}(n)$.

Proposition 3.1. *a) The complex $\mathbb{Z}^{Gr}(0)$ coincides with $\mathbb{Z}$.*

b) For any n the cohomology sheaves of $\mathbb{Z}^{Gr}(n)$ are homotopy invariant pretheories and, in particular, are strictly homotopy invariant. Hence for any $X \in Sm/F$ we have canonical isomorphisms

$$H^*_{Zar}(X, \mathbb{Z}^{Gr}(n)) = H^*_{Zar}(X, \mathbb{Z}^{Gr}(n)_{Nis}) = H^*_{Nis}(X, \mathbb{Z}^{Gr}(n)_{Nis})$$
$$H^*_{Zar}(X, \mathbb{Z}^{Gr}(n)) = H^*_{Zar}(X \times \mathbb{A}^1, \mathbb{Z}^{Gr}(n))$$

Proof. The first statement is obvious. The second statement follows from Lemma 1.2, Lemma 1.4 and the standard properties of homotopy invariant pretheories – see [V1].

To simplify notation we keep the same notation $\mathbb{Z}^{Gr}(n)$ for the Nisnevich sheafification $\mathbb{Z}^{Gr}(n)_{Nis}$ of the complex $\mathbb{Z}^{Gr}(n)$. Proposition 3.1 shows that this does not lead to any kind of confusion.

For any schemes $X, Y \in Sm/F$ and any integers $m, n \geq 0$ we have canonical pairings – see § 1

$$K_0^{\oplus}(\mathbb{G}_m^{\wedge n})(X) \otimes K_0^{\oplus}(\mathbb{G}_m^{\wedge m})(Y) \to K_0^{\oplus}(\mathbb{G}_m^{\wedge (n+m)})(X \times Y)$$

These pairings define further a canonical homomorphism of complexes

$$C^*(K_0^{\oplus}(\mathbb{G}_m^{\wedge n}))(X)\otimes C^*(K_0^{\oplus}(\mathbb{G}_m^{\wedge m}))(Y)\to \operatorname{Tot} C^{*,*}(K_0^{\oplus}(\mathbb{G}_m^{\wedge(n+m)}))(X\times Y)\xrightarrow{\nabla}$$
$$\xrightarrow{\nabla} C^*(K_0^{\oplus}(\mathbb{G}_m^{\wedge(n+m)}))(X\times Y)$$

Here $C^{p,q}(K_0^{\oplus}(\mathbb{G}_m^{\wedge(n+m)}))(X\times Y) = K_0^{\oplus}(\mathbb{G}_m^{\wedge(n+m)})(X\times Y\times\Delta^p\times\Delta^q)$ and the second arrow is the shuffle map – see e.g., [S-V] § 0. Specializing now to the case $X=Y$ and composing the resulting homomorphism of complexes with the map $C^*(K_0^{\oplus}(\mathbb{G}_m^{\wedge(n+m)}))(X\times X)\xrightarrow{(\Delta_X)^*} C^*(K_0^{\oplus}(\mathbb{G}_m^{\wedge(n+m)}))(X)$ we get a homomorphism of complexes of abelian presheaves, which upon sheafification and shift defines canonical homomorphisms of complexes of sheaves $\mathbb{Z}^{Gr}(n)\otimes\mathbb{Z}^{Gr}(m)\xrightarrow{\mu_{n,m}}\mathbb{Z}^{Gr}(n+m)$. It's well-known (and easy to verify) that the product maps $\mu_{n,m}$ are strictly associative and commutative. This observation implies readily the following result.

Proposition 3.2. *For any $X\in Sm/F$*

$$\bigoplus_{n=0}^{\infty} H^*(X,\mathbb{Z}^{Gr}(n))$$

is a bigraded associative ring with identity, which is graded commutative with respect to the cohomological degree.

Proposition 3.3. *The presheaf $K_0^{\oplus}(\mathbb{G}_m^{\wedge n})$ is rationally contractible and hence $\forall\, n,m\geq 0$*

$$H^p(\hat{\Delta}^m,\{\hat{\Delta}_i^{m-1}\}_{i=0}^m;\mathbb{Z}^{Gr}(n))=0\quad \textit{for } p>n$$

Proof. The first statement follows from Proposition 2.2 and Remark 2.3 and the second statement follows from Theorem 2.7.

Recall that motivic complexes $\mathbb{Z}(n)$ are defined in a way very similar to the one we used above to define complexes $\mathbb{Z}^{Gr}(n)$ – see [V2, S-V]. Denote by $\mathbb{Z}_{tr}(\mathbb{G}_m^n)$ the presheaf of abelian groups on Sm/F given by the formula $\mathbb{Z}_{tr}(\mathbb{G}_m^n)(X) = Cor(X,\mathbb{G}_m^n)$. It's easy to see that this presheaf is actually a sheaf in the Nisnevich topology so that this time no sheafification is needed. A straightforward verification shows that the sum of the images of the homomorphisms $\mathbb{Z}_{tr}(\mathbb{G}_m^{n-1})\to\mathbb{Z}_{tr}(\mathbb{G}_m^n)$ induced by the canonical embeddings $i_k:\mathbb{G}_m^{n-1}\to\mathbb{G}_m^n\quad(1\leq k\leq n)$ is a direct summand in $\mathbb{Z}_{tr}(\mathbb{G}_m^n)$ (see [S-V] § 0). One defines the Nisnevich sheaf with transfers $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})$ as the quotient of $\mathbb{Z}_{tr}(\mathbb{G}_m^n)$ modulo $\sum_{k=0}^n (i_k)_*(\mathbb{Z}_{tr}(\mathbb{G}_m^{n-1}))$. Finally one defines the complex of Nisnevich sheaves with transfers $\mathbb{Z}(n)$ via the formula

$$\mathbb{Z}(n)=C^*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n}))[-n]$$

As was explained at the beginning of § 1 we have canonical homomorphisms of presheaves $K_0^{\oplus}(\mathbb{G}_m^n)\to\mathbb{Z}_{tr}(\mathbb{G}_m^n)$. These homomorphisms induce obviously homomorphisms $K_0^{\oplus}(\mathbb{G}_m^{\wedge n})\to\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})$, from which we derive in an obvious way homomorphisms of complexes of Nisnevich sheaves $f_n:\mathbb{Z}^{Gr}(n)\to\mathbb{Z}(n)$.

Lemma 3.4. *The homomorphisms $f_n : \mathbb{Z}^{Gr}(n) \longrightarrow \mathbb{Z}(n)$ are compatible with products. Moreover the homomorphism f_0 is an isomorphism of complexes.*

Proof. The second statement is trivial since both presheaves $K_0^{\oplus}(Spec\, F)$ and $\mathbb{Z}_{tr}(Spec\, F)$ coincide with the constant sheaf $\mathbb{Z}$. The first statement is also clear since product maps $\mathbb{Z}(m) \otimes \mathbb{Z}(n) \longrightarrow \mathbb{Z}(m+n)$ may be defined in exactly the same way as product maps $\mathbb{Z}^{Gr}(m) \otimes \mathbb{Z}^{Gr}(n) \longrightarrow \mathbb{Z}^{Gr}(m+n)$ were defined above – cf. [S-V] § 2, [F-S] § 12.

§ 4. The Cancellation Theorem for Grayson's Cohomology.

In this section we repeat, essentially verbatim, Voevodsky's proof of the Cancellation Theorem for motivic cohomology, replacing everywhere motivic cohomology by Grayson's cohomology. The only difference is that we have to modify slightly Voevodsky's argument to make sure that it works in $K_0^{\oplus}$-context. We also replace the final part of the Voevodsky proof with an explicit construction of a homotopy operator.

For any scheme $X \in Sm/F$ and any abelian presheaf $\mathcal{F} : Sm/F \to Ab$ the group $\mathcal{F}(X)$ identifies canonically as a direct summand in $\mathcal{F}(X \times \mathbb{G}_m)$. We use the notation $\mathcal{F}(X \wedge \mathbb{G}_m)$ for the complementary direct summand, i.e., $\mathcal{F}(X \wedge \mathbb{G}_m) = \mathrm{Ker}(\mathcal{F}(X \times \mathbb{G}_m) \xrightarrow{res} \mathcal{F}(X \times e) = \mathcal{F}(X))$, where $e \in \mathbb{G}_m$ is the identity element.

Fix smooth schemes X and Y. For our applications only the affine case is of interest, so we'll be assuming that both X and Y are affine. This assumption makes many constructions more transparent, however, as the reader will see, it is irrelevant for the validity of results proved below. For this reason we'll formulate results in complete generality but will make verifications in the affine case only.

Let P be a coherent sheaf on $X \times \mathbb{G}_m \times Y \times \mathbb{G}_m$ finite and flat over $X \times \mathbb{G}_m$. Denote by f_1 and f_2 the two invertible functions on $X \times \mathbb{G}_m \times Y \times \mathbb{G}_m$ corresponding to the two projections onto $\mathbb{G}_m$.

Proposition 4.1. *a) For any $n \geq 0$ the sheaf $P/(f_1^{n+1} - 1)P$ is finite and flat over X.*

b) There exists an integer N such that for any $n \geq N$ the sheaf $P/(f_1^{n+1} - f_2)P$ is finite and flat over X.

Proof. Let $A = F[X]$ be the coordinate ring of the affine scheme X. The first part of our statement is clear: we know that P is a finitely generated projective module over the Laurent polynomial ring $A[f_1, f_1^{-1}]$ and claim that in this case $P/(f_1^{n+1} - 1)P$ is a finitely generated projective A-module, which is perfectly obvious.

To prove the second part we note that multiplication by f_2 determines an automorphism of the $A[f_1, f_1^{-1}]$-module P, which we denote by α. Our claim concerns the properties of the module $\mathrm{Coker}(f_1^{n+1} - \alpha)$. To show that this module is finitely generated over A we note that, considered as an $A[f_1, f_1^{-1}]$-module, it is killed by $\det(f_1^{n+1} - \alpha)$. Denote by $\chi = \chi_\alpha$ the characteristic polynomial of α. Thus $\chi = \chi(T)$ is a monic polynomial in T of degree $d = \mathrm{rank}\, P$:

$$\chi(T) = T^d + b_1 T^{d-1} + \ldots + b_d.$$

Here $b_i \in A[f_1, f_1^{-1}]$ are Laurent polynomials in f_1. The constant term b_d coincides with $\det(\alpha)$ and hence is invertible in $A[f_1, f_1^{-1}]$. Thus it is of the form $b_d = a \cdot f_1^s$ for some $s \in \mathbb{Z}$ and some $a \in A^*$. Note that

$$\det(f_1^{n+1} - \alpha) = \chi(f_1^{n+1}) = f_1^{d(n+1)} + b_1 \cdot (f_1)^{(d-1)(n+1)} + \ldots + b_{d-1} \cdot f_1^{n+1} + a \cdot f_1^s$$

A straightforward examination shows that if we take n large enough then $\det(f_1^{n+1} - \alpha) \cdot f_1^{-s}$ is a monic polynomial in f_1 of degree $d(n+1) - s$ and with invertible constant term (equal to a). This readily implies that $\mathrm{Coker}(f_1^{n+1} - \alpha)$ is a finitely generated module over $A[f_1, f_1^{-1}]/\det(f_1^{n+1} - \alpha) \cdot f_1^{-s} = A[f_1]/\det(f_1^{n+1} - \alpha) \cdot f_1^{-s}$ and hence is a finitely generated A-module. Moreover, under the same conditions on n, we see that the homomorphism $f_1^{n+1} - \alpha : P \to P$ is injective and hence $\mathrm{Coker}(f_1^{n+1} - \alpha)$ has a two-term flat resolution over A, which may be used to compute the Tor-groups. To show that the finitely generated A-module $\mathrm{Coker}(f_1^{n+1} - \alpha)$ is projective it suffices to show that for any prime ideal $\mu \subset A$ the group $\mathrm{Tor}_1^A(P/(f_1^{n+1} - \alpha)P, k(\mu))$ is trivial. However, this Tor_1-group coincides with the kernel of the homomorphism

$$f_1^{n+1} - \alpha(\mu) : P \otimes_A k(\mu) \to P \otimes_A k(\mu)$$

and the kernel of this endomorphism is trivial since this kernel is a submodule in a finitely generated projective $k(\mu)[f_1, f_1^{-1}]$-module and is killed by the monic polynomial $\det(f_1^{n+1} - \alpha(\mu)) \cdot f_1^{-s}$.

For any coherent sheaf P on $X \times \mathbb{G}_m \times Y \times \mathbb{G}_m$ that is finite and flat over $X \times \mathbb{G}_m$ we denote by $\rho_n^+(P)$ the direct image of $P/(f_1^{n+1} - 1)P$ under the projection $X \times \mathbb{G}_m \times Y \times \mathbb{G}_m \to X \times Y$. According to Proposition 4.1 the sheaf $\rho_n^+(P)$ is finite and flat over X. We'll say that $\rho_n^-(P)$ is defined provided that the sheaf $P/(f_1^{n+1} - f_2)P$ is finite and flat over X, in which case we define $\rho_n^-(P)$ to be the direct image of $P/(f_1^{n+1} - f_2)P$ under the projection $X \times \mathbb{G}_m \times Y \times \mathbb{G}_m \to X \times Y$. Proposition 4.1 shows that $\rho_n^-(P)$ is defined for all n large enough, and when defined $\rho_n^-(P)$ is a coherent sheaf on $X \times Y$ finite and flat over X. Finally we'll say that $\rho_n(P)$ is defined provided that $\rho_n^-(P)$ is defined, in which case we set $\rho_n(P) = [\rho_n^+(P)] - [\rho_n^-(P)] \in K_0^{\oplus}(X, Y)$

Lemma 4.2. *a) For any P_1, P_2 and any n, there is an isomorphism $\rho_n^+(P_1 \oplus P_2) \cong \rho_n^+(P_1) \oplus \rho_n^+(P_2)$.*

b) Assume that $\rho_n^-(P_1)$ and $\rho_n^-(P_2)$ are defined. Then $\rho_n^-(P_1 \oplus P_2)$ is defined as well and $\rho_n^-(P_1 \oplus P_2) \cong \rho_n^-(P_1) \oplus \rho_n^-(P_2)$. In this case we also have the identity

$$\rho_n(P_1 \oplus P_2) = \rho_n(P_1) + \rho_n(P_2) \in K_0^{\oplus}(X, Y)$$

Lemma 4.3. *Let Q be a coherent sheaf on $X \times Y$ finite and flat over X.*

(1) *The sheaf $\rho_n^-(Q \boxtimes 1_{\mathbb{G}_m})$ is defined for all $n \geq 0$. Moreover,*

$$\rho_n^+(Q \boxtimes 1_{\mathbb{G}_m}) \cong Q^{n+1}$$
$$\rho_n^-(Q \boxtimes 1_{\mathbb{G}_m}) \cong Q^n$$

and hence $\rho_n(Q \boxtimes 1_{\mathbb{G}_m}) = [Q]$.

(2) *The sheaf* $\rho_n^-(Q \boxtimes e_{\mathbb{G}_m})$ *is defined for all* $n \geq 0$ *(here* $e : \mathbb{G}_m \to \mathbb{G}_m$ *is the trivial morphism that takes* $\mathbb{G}_m$ *to the unit* $e \in \mathbb{G}_m$*). Moreover,*

$$\rho_n^+(Q \boxtimes e_{\mathbb{G}_m}) \cong Q^{n+1}$$
$$\rho_n^-(Q \boxtimes e_{\mathbb{G}_m}) \cong Q^{n+1}$$

and hence $\rho_n(Q \boxtimes e_{\mathbb{G}_m}) = 0$

Proof. Let $A = F[X], B = F[Y]$ be the corresponding coordinate rings. The $A \otimes B[f_1, f_1^{-1}, f_2, f_2^{-1}]$-module $Q \boxtimes 1_{\mathbb{G}_m}/f_1^{n+1} - f_2$ coincides with $Q \otimes_F F[f_1, f_1^{-1}, f_2, f_2^{-1}]$ $/(f_1 - f_2, f_1^{n+1} - f_2) = Q \otimes_F F[f_1, f_1^{-1}]/(f_1^{n+1} - f_1)$. Considered as an A-module this module is obviously finitely generated projective and moreover is isomorphic to Q^n. Similarly $Q \boxtimes e_{\mathbb{G}_m}/f_1^{n+1} - f_2$ coincides with $Q \otimes_F F[f_1, f_1^{-1}, f_2, f_2^{-1}]/(f_2 - 1, f_1^{n+1} - f_2) = Q \otimes_F F[f_1, f_1^{-1}]/(f_1^{n+1} - 1)$. This module is finitely generated and projective over A and is isomorphic to Q^{n+1}. The computations concerning ρ_n^+ are performed similarly.

Lemma 4.4. *Let* P *be a coherent sheaf on* $X \times \mathbb{G}_m \times Y \times \mathbb{G}_m$ *finite and flat over* $X \times \mathbb{G}_m$. *Assume that* $\rho_n^-(P)$ *is defined.*

(1) *For any morphism* $g : X' \to X$ *of smooth schemes the element* $\rho_n^-((g \times 1_{\mathbb{G}_m \times Y \times \mathbb{G}_m})^* P)$ *is also defined and moreover we have the following natural isomorphisms*

$$\rho_n^+((g \times 1_{\mathbb{G}_m \times Y \times \mathbb{G}_m})^* P) = (g \times 1_Y)^*(\rho_n^+(P))$$
$$\rho_n^-((g \times 1_{\mathbb{G}_m \times Y \times \mathbb{G}_m})^* P) = (g \times 1_Y)^*(\rho_n^-(P))$$

(2) *For any morphism* $g : Y \to Y'$ *of smooth schemes the element* $\rho_n^-((1_{X \times \mathbb{G}_m} \times g \times 1_{\mathbb{G}_m})_* P)$ *is also defined and moreover we have the following natural isomorphisms*

$$\rho_n^+((1_{X \times \mathbb{G}_m} \times g \times 1_{\mathbb{G}_m})_* P) = (1_X \times g)_*(\rho_n^+(P))$$
$$\rho_n^-((1_{X \times \mathbb{G}_m} \times g \times 1_{\mathbb{G}_m})_* P) = (1_X \times g)_*(\rho_n^-(P))$$

Proof. Set $B = F[Y], A = F[X], A' = F[X']$. The $(A' \otimes B)[f_1, f_1^{-1}, f_2, f_2^{-1}]$-module $(g \times 1_{\mathbb{G}_m \times Y \times \mathbb{G}_m})^* P)/(f_1^{n+1} - f_2)$ coincides with $A' \otimes_A P/(f_1^{n+1} - f_2)$. This module is obviously finitely generated and projective over A' and moreover it coincides also with $(g \times 1_Y)^*(\rho_n^-(P))$. The part concerning ρ_n^+ is established similarly. The second part is even easier since in this case the corresponding module does not change at all.

Here is one more result of the same character established in the same absolutely straightforward fashion. We skip the obvious details.

Lemma 4.5. *Let P be a coherent sheaf on $X \times \mathbb{G}_m \times Y \times \mathbb{G}_m$ finite and flat over $X \times \mathbb{G}_m$. Assume that $\rho_n^-(P)$ is defined. Then for any scheme $X' \in Sm/F$ $\rho_n^-(P \boxtimes 1_{X'})$ is also defined. Moreover, we have the following identifications:*

$$\rho_n^+(P \boxtimes 1_{X'}) = \rho_n^+(P) \boxtimes 1_{X'}$$
$$\rho_n^-(P \boxtimes 1_{X'}) = \rho_n^-(P) \boxtimes 1_{X'}$$

Next we have to construct, following [S-V] § 3 certain standard homotopies.

Proposition 4.6. *There exists a coherent sheaf H on $(\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}^1) \times (\mathbb{G}_m \times \mathbb{G}_m)$ finite and flat over $(\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}^1)$ and such that*

$$[H_0] - [H_1] = [\sigma] - [1_{\mathbb{G}_m} \times i] + [(a,e)] - [(b,e)] - [(e,a)] - [(e,b)] + 2 \cdot [(e,e)]$$

where $\sigma : \mathbb{G}_m \times \mathbb{G}_m \longrightarrow \mathbb{G}_m \times \mathbb{G}_m$ is the permutation of coordinates morphism, $i : \mathbb{G}_m \to \mathbb{G}_m$ is the inversion morphism $i(x) = x^{-1}$, a and b are the first and the second coordinate functions on $\mathbb{G}_m \times \mathbb{G}_m$ respectively and $e = 1 \in \mathbb{G}_m$ is the identity element.

We start with a few auxiliary homotopies.

Lemma 4.6.1. *Let a and b be two invertible functions on a smooth scheme S . Then there exists a coherent sheaf P on $S \times \mathbb{A}^1 \times \mathbb{G}_m$, finite and flat over $S \times \mathbb{A}^1$ and such that $[P_0] - [P_1] = [ab] - [a] - [b] + [1] \in K_0^{\oplus}(S, \mathbb{G}_m)$*

Proof. Let X denote the $\mathbb{G}_m$-coordinate function and T denote the $\mathbb{A}^1$-coordinate function. Consider the closed subscheme $Y \subset S \times \mathbb{A}^1 \times \mathbb{G}_m$ given by the equation

$$X^2 - (T(a+b) + (1-T)(1+ab))X + ab = 0$$

An immediate verification shows that this scheme is finite and flat over $S \times \mathbb{A}^1$. The corresponding subschemes $Y_0, Y_1 \subset S \times \mathbb{G}_m$ are given by the equations $(X-1)(X-ab) = 0$ and $(X-a)(X-b) = 0$ respectively. Denoting the coordinate ring of S by A we note further that we have short exact sequences of $A[X, X^{-1}]$-modules

$$0 \to \frac{A[X]}{X-ab} \xrightarrow{X-1} \frac{A[X]}{(X-ab)(X-1)} \to \frac{A[X]}{X-1} \to 0$$
$$0 \to \frac{A[X]}{X-a} \xrightarrow{X-b} \frac{A[X]}{(X-a)(X-b)} \to \frac{A[X]}{X-b} \to 0$$

From these short exact sequences of $A[X, X^{-1}]$-modules we derive (see (1.2.2)) coherent sheaves P^1 and P^2 on $S \times \mathbb{A}^1 \times \mathbb{G}_m$ finite and flat over $S \times \mathbb{A}^1$ for which

$$P_0^1 = \frac{A[X]}{X-ab} \oplus \frac{A[X]}{X-1}, \quad P_1^1 = \frac{A[X]}{(X-ab)(X-1)}$$
$$P_0^2 = \frac{A[X]}{(X-a)(X-b)}, \quad P_1^2 = \frac{A[X]}{X-a} \oplus \frac{A[X]}{X-b}$$

Now it suffices to take $P = \mathcal{O}_Y \oplus P^1 \oplus P^2$.

Corollary 4.6.2. *Let a, b and c be three invertible functions on a smooth scheme S . Then there exists a coherent sheaf Q on $S\times\mathbb{A}^1\times\mathbb{G}_m\times\mathbb{G}_m$, finite and flat over $S\times\mathbb{A}^1$ and such that $[Q_0]-[Q_1]=[(ab,c)]-[(a,c)]-[(b,c)]+[(1,c)]$.*

Proof. Consider the closed embedding $i_c: S\times\mathbb{A}^1\times\mathbb{G}_m\longrightarrow S\times\mathbb{A}^1\times\mathbb{G}_m\times\mathbb{G}_m$, given by the formula $i_c(s,t,x)=(s,t,x,c(s))$ and take $Q=(i_c)_*(P)$.

Corollary 4.6.3. *Let a and b be two invertible functions on a smooth scheme S. Then there exists a coherent sheaf R on $S\times\mathbb{A}^1\times\mathbb{G}_m\times\mathbb{G}_m$, finite and flat over $S\times\mathbb{A}^1$ and such that $[R_0]-[R_1]=[(ab,-ab)]-[(a,-a)]-[(b,-b)]+[(1,-1)]$*

Proof. Consider the closed embedding $i: S\times\mathbb{A}^1\times\mathbb{G}_m\longrightarrow S\times\mathbb{A}^1\times\mathbb{G}_m\times\mathbb{G}_m$, given by the formula $i(s,t,x)=(s,t,x,-x)$ and take $R=i_*(P)$.

Proof of Proposition 4.6 Take $S=\mathbb{G}_m\times\mathbb{G}_m$ and denote by a and b the first and the second coordinate functions respectively. According to Corollaries 4.6.2 and 4.6.3 we have the following homotopies between $K_0^{\oplus}$-morphisms from $\mathbb{G}_m\times\mathbb{G}_m$ to itself:

$$[(ab,-ab)]\sim[(a,-a)]+[(b,-b)]-[(1,-1)]$$
$$[(ab,-ab)]\sim[(a,-ab)]+[(b,-ab)]-[(1,-ab)]\sim[(a,-a)]+[(a,b)]-[(a,1)]+$$
$$+[(b,-b)]+[(b,a)]-[(b,1)]-[(1,-ab)]$$

These relations combine to give the following homotopy:

$$[(a,b)]+[(b,a)]\sim[(a,1)]+[(b,1)]+[(1,-ab)]-[(1,-1)]$$

Furthermore, $[(a,b)]+[(a,b^{-1})]\sim 2[(a,1)]$ and hence $[\sigma]=[(b,a)]\sim[(a,b^{-1})]-[(a,1)]+[(b,1)]+[(1,-ab)]-[(1,-1)]$. Finally Corollary 4.6.2 implies that we have the following relation:

$$[(1,-ab)]-[(1,-1)]\sim[(1,ab)]-[(1,1)]\sim[(1,a)]+[(1,b)]-2\cdot[(1,1)].$$

Start with an element $v\in K_0^{\oplus}(X\times\mathbb{G}_m,Y\times\mathbb{G}_m)$ and consider the element $v\boxtimes(1_{\mathbb{G}_m}-e)\in K_0^{\oplus}(X\times\mathbb{G}_m\times\mathbb{G}_m,Y\times\mathbb{G}_m\times\mathbb{G}_m)$. Denote further by $v\boxtimes^{\sigma}(1_{\mathbb{G}_m}-e)$ the element $(1_Y\times\sigma)\circ(v\boxtimes(1_{\mathbb{G}_m}-e))\circ(1_X\times\sigma)$ obtained from $v\boxtimes(1_{\mathbb{G}_m}-e)$ by permuting two copies of $\mathbb{G}_m$ both in $X\times\mathbb{G}_m\times\mathbb{G}_m$ and in $Y\times\mathbb{G}_m\times\mathbb{G}_m$. Consider finally the following homotopy

$$\phi=\phi_v=(1_Y\boxtimes H)\circ(((v\boxtimes(1_{\mathbb{G}_m}-e))\circ(1_{X\times\mathbb{G}_m}\times i))\boxtimes 1_{\mathbb{A}^1})+(1_Y\times\sigma)\circ$$
$$\circ(v\boxtimes(1_{\mathbb{G}_m}-e))\circ(1_X\boxtimes H)\in K_0^{\oplus}(X\times\mathbb{G}_m\times\mathbb{G}_m\times\mathbb{A}^1,Y\times\mathbb{G}_m\times\mathbb{G}_m)$$

Lemma 4.7.0. *Assume that $(1_Y\times e)\circ v=v\circ(1_X\times e)=0$. Then $\phi_0-\phi_1=v\boxtimes^{\sigma}(1_{\mathbb{G}_m}-e)-v\boxtimes(1_{\mathbb{G}_m}-e)$.*

Proof. This follows immediately from the properties of the homotopy H and the fact that $v\boxtimes(1_{\mathbb{G}_m}-e)$ is killed by $1_Y\times(a,e)$; $1_Y\times(b,e)$; $1_Y\times(e,a)$; $1_Y\times(e,b)$; $1_Y\times(e,e)$ on the left and by similar elements with 1_Y replaced by 1_X on the right.

Theorem 4.7. *For any $X, Y \in Sm/F$ the natural homomorphism*

$$C_*K_0^{\oplus}(Y)(X) \to C_*K_0^{\oplus}(Y \wedge \mathbb{G}_m)(X \wedge \mathbb{G}_m): \quad u \mapsto u \boxtimes (1_{\mathbb{G}_m} - e_{\mathbb{G}_m})$$

is a quasi-isomorphism of complexes.

Proof. Note that $C_nK_0^{\oplus}(Y \wedge \mathbb{G}_m)(X \wedge \mathbb{G}_m)$ is a direct summand in $K_0^{\oplus}(X \times \Delta^n \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ consisting of all those elements $v \in K_0^{\oplus}(X \times \Delta^n \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ for which $v \circ (1_{X \times \Delta^n} \times e) = (1_Y \times e) \circ v = 0$. A straightforward verification shows that the elements $u \boxtimes (1_{\mathbb{G}_m} - e_{\mathbb{G}_m})$ have both required properties and hence the map in question is well-defined.

Denote by $L(X, Y)$ the free abelian group generated by isomorphism classes of coherent sheaves P on $X \times Y$ finite and flat over X. Thus we have a canonical surjective homomorphism $L(X, Y) \to K_0^{\oplus}(X, Y)$. For an element $V \in L(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ we'll say that the operation $\rho_n(V)$ is defined if $\rho_n(P)$ is defined for every P appearing in V with $\neq 0$ coefficient. Proposition 4.1 shows that for any $V \in L(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ the operation $\rho_n(V)$ is defined for all n large enough. Moreover, Lemma 4.2 implies immediately the following fact.

(4.7.1) If the image of V in $K_0^{\oplus}(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ is trivial then $\rho_n(V) = 0$ for all n large enough.

We first show that the homomorphism $C_*(K_0^{\oplus}(Y))(X) \to C_*(K_0^{\oplus}(Y \wedge \mathbb{G}_m))(X \wedge \mathbb{G}_m)$ induces injective maps on homology. Since both sides are simplicial abelian groups we may work with the corresponding Moore complexes. Thus assume that $u \in K_0^{\oplus}(X \times \Delta^n, Y)$ is a Moore cycle for which $u \boxtimes (1_{\mathbb{G}_m} - e_{\mathbb{G}_m})$ is a boundary, i.e., there exists $v \in K_0^{\oplus}(X \times \Delta^{n+1} \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ such that $\partial_{n+1}(v) = u \boxtimes (1_{\mathbb{G}_m} - e_{\mathbb{G}_m})$, $\partial_i(v) = 0$ for $0 \leq i \leq n$. Represent u and v by appropriate elements $U \in L(X \times \Delta^n, Y), V \in L(X \times \Delta^{n+1} \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ and find N such that $\rho_N(V)$ is defined. Lemma 4.4 shows that $\rho_N(\partial_i(V))$ is also defined and moreover $\partial_i(\rho_N(V)) = \rho_N(\partial_i(V))$. Fact (4.7.1) shows next that upon increasing N we will get the following identities:

$$\partial_i(\rho_N(V)) = \rho_N(\partial_i(V)) = 0 \text{ for } 0 \leq i \leq n$$
$$\partial_{n+1}(\rho_N(V)) = \rho_N(\partial_{n+1}(V)) = \rho_N(U \boxtimes (1_{\mathbb{G}_m} - e_{\mathbb{G}_m})) = u$$

where the latter identity follows from Lemma 4.3. This implies that u itself is a boundary and concludes the proof of the injectivity statement.

We next prove the surjectivity of the induced map in homology. Let $v \in C_n(K_0^{\oplus}(Y \wedge \mathbb{G}_m))(X \wedge \mathbb{G}_m)$ be a Moore cycle. Thus $v \in K_0^{\oplus}(X \times \Delta^n \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ has the following properties:

$$v \circ (1_{X \times \Delta^n} \times e) = (1_Y \times e) \circ v = 0, \quad \partial_i(v) = 0 \text{ for } 0 \leq i \leq n$$

Choose a representative $V \in L(X \times \Delta^n \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ for v, consider the homotopy $\phi = \phi_v \in K_0^{\oplus}(X \times \Delta^n \times \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m \times \mathbb{G}_m)$ and choose a representative $\Phi \in L(X \times \Delta^n \times \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m \times \mathbb{G}_m)$ for ϕ. According to Lemma 4.7.0 we have

$$[\Phi_0] - [\Phi_1] = [V \boxtimes^{\sigma} (1_{\mathbb{G}_m} - e)] - [V \boxtimes (1_{\mathbb{G}_m} - e)] \tag{4.7.2}$$

In the computations we are about to perform we'll be using the operations ρ_N with respect to the newly adjoined copy of $\mathbb{G}_m$. Note that, according to (4.3), (4.5) and (4.4.2) we have the following formulae.

$$\rho_N(V \boxtimes (1_{\mathbb{G}_m} - e)) = [V] = v \text{ for all } N \geq 0$$
$$\rho_N(V \boxtimes^\sigma (1_{\mathbb{G}_m} - e)) = [\rho_N(V)] \boxtimes (1_{\mathbb{G}_m} - e) \text{ for all } N \text{ large enough}$$

Consider finally the element $\psi_N = \rho_N(\Phi) \in K_0^\oplus(X \times \Delta^n \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m)$ defined for all sufficiently large N. According to (4.7.1) and (4.7.2) for all N large enough we have the following identities:

$$(\psi_N)_0 - (\psi_N)_1 = \rho_N(\Phi_0 - \Phi_1) = v - [\rho_N(V)] \boxtimes (1_{\mathbb{G}_m} - e)$$

Moreover, according to (4.4.1) and (4.7.1) for all N large enough we get: $\partial_i(\psi_N) = \rho_N(\partial_i \Phi) = 0$, since $[\partial_i \Phi] = \phi_{\partial_i v} = 0$. The latter relations show that for N large enough ψ_N is a Moore cycle of the simplicial abelian group $C_*(K_0^\oplus(Y \times \mathbb{G}_m))(X \times \mathbb{G}_m \times \mathbb{A}^1)$. Since the two restrictions $C_*(K_0^\oplus(Y \times \mathbb{G}_m))(X \times \mathbb{G}_m \times \mathbb{A}^1) \longrightarrow C_*(K_0^\oplus(Y \times \mathbb{G}_m))(X \times \mathbb{G}_m)$ defined by rational points $0, 1 \in \mathbb{A}^1$ define the same map in homology we conclude that $(\psi_N)_0 - (\psi_N)_1 = v - [\rho_N(V)] \boxtimes (1_{\mathbb{G}_m} - e)$ is a boundary in $C_*(K_0^\oplus(Y \times \mathbb{G}_m))(X \times \mathbb{G}_m)$. Finally the complex $C_*(K_0^\oplus(Y \wedge \mathbb{G}_m))(X \wedge \mathbb{G}_m)$ is a direct summand in $C_*(K_0^\oplus(Y \times \mathbb{G}_m))(X \times \mathbb{G}_m)$ and hence $v - [\rho_N(V)] \boxtimes (1_{\mathbb{G}_m} - e)$ is a boundary in $C_*(K_0^\oplus(Y \wedge \mathbb{G}_m))(X \wedge \mathbb{G}_m)$ as well.

Theorem 4.8. *For any schemes $X, Y \in Sm/F$ the natural homomorphism*

$$H^*_{Zar}(X, C^*(K_0^\oplus(Y))_{Zar}) \to H^*_{Zar}(X \wedge \mathbb{G}_m, C^*(K_0^\oplus(Y \wedge \mathbb{G}_m))_{Zar})$$

defined as the external multiplication by $1_{\mathbb{G}_m} - e \in H^0(\mathbb{G}_m^{\wedge 1}, C^*(K_0^\oplus(\mathbb{G}_m^{\wedge 1}))_{Zar})$ is an isomorphism.

Proof. We first compute $H^*_{Zar}(X \times \mathbb{G}_m, C^*(K_0^\oplus(Y \wedge \mathbb{G}_m))_{Zar})$ using Leray spectral sequence

$$E_2^{pq} = H^p(X, R^q\pi_*(C^*(K_0^\oplus(Y \wedge \mathbb{G}_m))_{Zar}|_{X \times \mathbb{G}_m})) \Longrightarrow$$
$$\Longrightarrow H^{p+q}(X \times \mathbb{G}_m, C^*(K_0^\oplus(Y \wedge \mathbb{G}_m))_{Zar})$$

where $\pi : X \times \mathbb{G}_M \longrightarrow X$ is the obvious projection and we use the notation $|_X$ to denote the restriction of a sheaf or a complex of sheaves onto the small Zariski site of X. Denote by $\mathcal{H}^q$ the q-th cohomology presheaf of the complex $C^*(K_0^\oplus(Y \wedge \mathbb{G}_m))$ and by $H^q = \mathcal{H}^q_{Zar}$ denote the q-th cohomology sheaf of the complex $C^*(K_0^\oplus(Y \wedge \mathbb{G}_m))_{Zar}$. To compute the complex $R\pi_*(C^*(K_0^\oplus(Y \wedge \mathbb{G}_m))_{Zar}|_{X \times \mathbb{G}_m})$ we use the hypercohomology spectral sequence

$$E_2^{pq} = R^p\pi_*(H^q|_{X \times \mathbb{G}_m}) \Longrightarrow H^{p+q}(R\pi_*(C^*(K_0^\oplus(Y \wedge \mathbb{G}_m))_{Zar}|_{X \times \mathbb{G}_m}))$$

The stalks $(R^p\pi_*(H^q|_{X \times \mathbb{G}_m}))_x = H^p(X_x \times \mathbb{G}_m, H^q)$ are trivial for $p > 0$ since H^q is a homotopy invariant pretheory - see [V1] Lemma 3.35. Thus $E_2^{pq} = 0$ for $p \neq 0$ and hence the spectral sequence above degenerates, providing canonical isomorphisms $H^q(R\pi_*(C^*(K_0^{\oplus}(Y\wedge\mathbb{G}_m))_{Zar}|_{X\times\mathbb{G}_m}))=\pi_*(H^q|_{X\times\mathbb{G}_m})$. Furthermore, the stalks $\pi_*(H^q|_{X\times\mathbb{G}_m})_x=H^0(X_x\times\mathbb{G}_m,H^q)$ split canonically into direct sum, where we use Voevodsky [V1] notation $\mathcal{F}_{-1}(X)=\mathrm{Ker}(\mathcal{F}(X\times\mathbb{G}_m)\xrightarrow{res}\mathcal{F}(X\times e))$:

$$H^0(X_x\times\mathbb{G}_m,H^q)=H^q(X_x)\oplus H^q_{-1}(X_x)$$

Denote further by $\widetilde{\mathcal{H}}^q$ the q-th cohomology presheaf of the complex $C^*(K_0^{\oplus}(Y))$. Theorem 4.7 shows that $\mathcal{H}^q_{-1}=\widetilde{\mathcal{H}}^q$. Another theorem of Voevodsky ([V1], Proposition 3.34) shows that for a homotopy invariant pretheory $\mathcal{H}$ the following relation holds:

$$(\mathcal{H}_{Zar})_{-1}=(\mathcal{H}_{-1})_{Zar}$$

Thus if $\mathcal{H}^q$ were a homotopy invariant pretheory we would conclude immediately that

$$H^q(R\pi_*(C^*(Y\wedge\mathbb{G}_m)_{Zar}|_{X\times\mathbb{G}_m}))=H^q(C^*(Y\wedge\mathbb{G}_m)_{Zar}|_X)\oplus H^q(C^*(Y)_{Zar}|_X)$$

i.e., the canonical homomorphism of complexes

$$C^*(K_0^{\oplus}(Y\wedge\mathbb{G}_m))_{Zar}|_X\oplus C^*(K_0^{\oplus}(Y))_{Zar}|_X\to R\pi_*(C^*(K_0^{\oplus}(Y\wedge\mathbb{G}_m))_{Zar}|_{X\times\mathbb{G}_m})$$

is a quasi-isomorphism and hence $H^*(X\times\mathbb{G}_m,C^*(K_0^{\oplus}(Y\wedge\mathbb{G}_m))_{Zar})=$ $=H^*(X,C^*(K_0^{\oplus}(Y\wedge\mathbb{G}_m))_{Zar})\oplus H^*(X,C^*(K_0^{\oplus}(Y))_{Zar})$ as stated. The presheaf $\mathcal{H}^q$ is a homotopy invariant K_0-presheaf and it's not too hard to check that the argument used in the proof of [V1] Proposition 3.34 works for such presheaves. The basic properties of homotopy invariant pretheories used in the proof of Proposition 3.34 [V1] are as follows:

(1) For any smooth semilocal scheme X with field of functions K the natural map $\mathcal{H}(X)\xrightarrow{res}\mathcal{H}(K)$ is injective.
(2) $\mathcal{H}|_{\mathbb{A}^1}$ is a sheaf in Zariski topology

To complete the proof of the Theorem it suffices now to show that the same properties are valid for homotopy invariant K_0-presheaves. The first part follows from the results of M. Walker [W]. In fact we mentioned already in § 1 that Walker established that for any homotopy invariant K_0-presheaf $\mathcal{H}$ its sections over any semilocal scheme X coincide with the sections of $\mathcal{H}_{Zar}$ over X and furthermore $\mathcal{H}_{Zar}$ is a homotopy invariant pretheory. The proof of the second point, given in [V1] is based on the existence and properties of maps $Tr_Z:\mathcal{H}(U)\to\mathcal{H}(V)$ $(U,V\subset\mathbb{A}^1$ - open subsets) defined by effective Cartier divisors $Z\subset V\times U$ finite and surjective over V. We still have such maps for K_0-presheaves (since every such Z is finite and flat over V). The only problem that does not allow us to conclude in general that a homotopy invariant K_0-presheaf is a homotopy invariant pretheory is that usually it is not true that $Tr_{Z\cdot Z'}=Tr_Z+Tr_{Z'}$. This property, however, holds at least in the case when one of the divisors (let's say Z) is principal: $I_Z=f\cdot\mathcal{O}_{V\times U}$, as one sees from the short exact sequence of coherent sheaves on $V \times U$ finite and flat over V:

$$0 \to \mathcal{O}_{Z'} \xrightarrow{f} \mathcal{O}_{Z \cdot Z'} \to \mathcal{O}_Z \to 0$$

Since every Cartier divisor on an open subscheme of $\mathbb{A}^2$ is principal the statement follows.

Corollary 4.9. *For any scheme $X \in Sm/F$ we have a canonical isomorphism*

$$H^{s-1}_{Nis}(X, \mathbb{Z}^{Gr}(n-1)) \xrightarrow{\sim} H^s_{Nis}(X \wedge \mathbb{G}_m, \mathbb{Z}^{Gr}(n))$$

defined as external multiplication by the element $\lambda = 1_{\mathbb{G}_m} - e \in H^1(\mathbb{G}_m^{\wedge 1}, \mathbb{Z}^{Gr}(1))$.

§ 5 Cohomology with supports.

In this section we recall the definition and basic properties of Nisnevich cohomology with supports and prove the Cohomology Purity Theorem for Grayson's Cohomology (which is an easy application of the Cancellation Theorem). All cohomology groups below are taken with respect to the Nisnevich topology.

Let $Z \subset X$ be a closed subscheme of a smooth scheme X with the complementary open subscheme $U \overset{i}{\hookrightarrow} X$. For any complex C^* of Nisnevich sheaves define Nisnevich cohomology of X with supports in Z (and coefficients in C^*) via the formula

$$H^p_Z(X, C^*) = \mathrm{Ext}^p(\mathbb{Z}_Z(X), C^*) = \mathrm{Hom}_{D(\mathcal{A})}(\mathbb{Z}_Z(X), C^*[p])$$

where $D(\mathcal{A})$ is the derived category of complexes of Nisnevich sheaves on Sm/F and $\mathbb{Z}_Z(X)$ is the following complex of sheaves

$$\mathbb{Z}_Z(X) = Cone(\mathbb{Z}_{Nis}(U) \xrightarrow{\mathbb{Z}(i)} \mathbb{Z}_{Nis}(X)) = \quad \ldots 0 \to \underset{-1}{\mathbb{Z}_{Nis}(U)} \to \underset{0}{\mathbb{Z}_{Nis}(X)}$$

The distinguished triangle $\mathbb{Z}_{Nis}(U) \to \mathbb{Z}_{Nis}(X) \to \mathbb{Z}_Z(X) \to \mathbb{Z}_{Nis}(U)[1]$ gives a long exact cohomology sequence
(5.0.0)

$$\ldots \to H^p_Z(X, C^*) \to H^p(X, C^*) \to H^p(U, C^*) \xrightarrow{\delta_{X;Z}} H^{p+1}_Z(X, C^*) \to \ldots$$

Let $f : X' \to X$ be a morphism of smooth schemes. Set $Z' = f^{-1}(Z)$. Then we have canonical homomorphisms $f^* : H^*_Z(X) \to H^*_{Z'}(X')$ and moreover f^* induces a homomorphism from the exact sequence (5.0.0) corresponding to the pair X, Z to the same sequence corresponding to the pair X', Z'.

Lemma 5.0.1 (Nisnevich Excision). *Let $f : X' \to X$ be an étale map such that $f|_{Z'} : Z' \to Z$ is an isomorphism. Then the corresponding homomorphism of complexes of sheaves $\mathbb{Z}_{Z'}(X') \to \mathbb{Z}_Z(X)$ is a quasi-isomorphism and hence for any C^* the corresponding map in cohomology with supports $H^*_Z(X, C^*) \xrightarrow{\sim} H^*_{Z'}(X', C^*)$ is an isomorphism.*

Corollary 5.0.2 (Zariski Excision). *Assume that we have a chain of subschemes* $Z \subset X' \subset X$, *where* X' *is an open subscheme in* X *and* Z *is a closed subscheme in* X. *Then for any* C^* *the corresponding map* $H^*_Z(X, C^*) \to H^*_Z(X', C^*)$ *is an isomorphism.*

Lemma 5.0.3. *Let* $Z' \subset Z \subset X$ *be a chain of closed subschemes. In this case we get a natural distinguished triangle*

$$\mathbb{Z}_{Z\setminus Z'}(X \setminus Z') \to \mathbb{Z}_Z(X) \to \mathbb{Z}_{Z'}(X) \to \mathbb{Z}_{Z\setminus Z'}(X \setminus Z')[1] \to \ \dots$$

and hence for any complex C^* *we get a long exact sequence (in which all cohomology groups are taken with coefficients in* C^**)*

$$(5.0.4) \quad \dots \ \to H^p_{Z'}(X) \to H^p_Z(X) \to H^p_{Z\setminus Z'}(X \setminus Z') \xrightarrow{\delta_{X;Z,Z'}} H^{p+1}_{Z'}(X) \to \ \dots$$

The connecting homomorphism $\delta_{X;Z}$ *is a special case of* $\delta_{X;Z,Z'}$, *corresponding to the case* $Z = X$.

To define products in cohomology with supports we note that the complex $\mathbb{Z}_Z(X) \otimes^L \mathbb{Z}_T(Y) = \mathbb{Z}_Z(X) \otimes \mathbb{Z}_T(Y)$ is canonically quasi-isomorphic to $\mathbb{Z}_{Z\times T}(X \times Y)$. Hence for any complexes C^* and D^* we get canonical pairings

$$H^p_Z(X, C^*) \otimes H^q_T(Y, D^*) \xrightarrow{\times} H^{p+q}_{Z\times T}(X \times Y, C^* \otimes^L D^*)$$

Combining these external cohomology products with the map in cohomology induced by the diagonal embedding we get next $\cup$-products in cohomology with supports.

$$H^p_Z(X, C^*) \otimes H^q_T(X, D^*) \xrightarrow{\cup} H^{p+q}_{Z\cap T}(X, C^* \otimes^L D^*)$$

Let $Z' \subset Z \subset X$ be a chain of closed subschemes of a smooth scheme X and let further $T \subset Y$ be a closed subscheme of a smooth scheme Y. Starting with cohomology classes $a \in H^p_{Z\setminus Z'}(X \setminus Z'), b \in H^q_T(Y)$ we can obtain the following cohomology classes

$$a \times b \in H^{p+q}_{Z\times T\setminus Z'\times T}(X \times Y \setminus Z' \times Y) \xrightarrow{ex^{-1}} H^{p+q}_{Z\times T\setminus Z'\times T}(X \times Y \setminus Z' \times T)$$
$$\delta_{X\times Y;Z\times T,Z'\times T}(ex^{-1}(a \times b)) \in H^{p+q+1}_{Z'\times T}(X \times Y)$$
$$\delta_{X;Z,Z'}(a) \times b \in H^{p+q+1}_{Z'\times T}(X \times Y)$$

Lemma 5.0.4 (Leibniz Rule).

$$\delta_{X\times Y;Z\times T,Z'\times T}(ex^{-1}(a \times b)) = \delta_{X;Z,Z'}(a) \times b \in H^{p+q+1}_{Z'\times T}(X \times Y)$$

(5.1.0) Assume now that we are given a sequence of complexes $C^*(i) \quad 0 \le i \le \infty$ of Nisnevich sheaves ($C^*(i) = \mathbb{Z}^{Gr}(i)$ is our main application, but we need a few other cases as well) with the following properties

(1) The complex $C^*(0)$ is canonically quasi-isomorphic to the constant sheaf Λ, positioned in degree zero, where $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{Z}/m$ for a certain integer m.

(2) All cohomology sheaves $\mathcal{H}^s(C^*(i))$ are strictly homotopy invariant and hence for any scheme X the pull-back maps

$$p_X^* : H^*_{Nis}(X, C^*(i)) \to H^*_{Nis}(X \times \mathbb{A}^1, C^*(i))$$

are isomorphisms.

(3) For all $i, j \geq 0$ we are given pairings in the derived category $C^*(i) \otimes^L C^*(j) \xrightarrow{\mu_{i,j}} C^*(i+j)$ that are associative and commutative. Moreover for any $j \geq 0$ the following diagram commutes in the derived category:

$$\begin{array}{ccc} \mathbb{Z} \otimes^L C^*(j) & \xrightarrow{=} & C^*(j) \\ \downarrow & & {\scriptstyle =}\downarrow \\ \Lambda \otimes^L C^*(j) & \xrightarrow{\mu_{0,j}} & C^*(j). \end{array}$$

These pairings define in the usual way products in cohomology

$$H^p_{Nis}(X, C^*(i)) \otimes H^q_{Nis}(Y, C^*(j)) \xrightarrow{\times} H^{p+q}_{Nis}(X \times Y, C^*(i+j))$$

and according to the above property the element $1 \in H^0_{Nis}(F, C^*(0)) = H^0_{Nis}(F, \Lambda) = \Lambda$ is a two-sided identity for these products.

(4) We have a canonical cohomology class $\lambda \in H^1(\mathbb{G}_m^{\wedge 1}, C^*(1))$ such that for any smooth scheme X external multiplication by λ defines isomorphisms

$$H^{p-1}(X, C^*(n-1)) \xrightarrow{\times\lambda} H^p(X \wedge \mathbb{G}_m, C^*(n))$$

Recall that according to our definitions $H^*(X \wedge \mathbb{G}_m, C^*(n))$ is a direct summand in $H^*(X \times \mathbb{G}_m, C^*(n))$ described as the kernel of the restriction map

$$H^*(X \times \mathbb{G}_m, C^*(n)) \xrightarrow{(1_X \times e)^*} H^*(X, C^*(n))$$

so that, in particular, $H^1(\mathbb{G}_m^{\wedge 1}, C^*(1))$ coincides with the kernel of the restriction map $H^1(\mathbb{G}_m, C^*(1)) \xrightarrow{e^*} H^*(F, C^*(1))$. Thus the last property above may be rephrased by saying that we have a split short exact sequence

$$0 \to H^{p-1}(X, C^*(n-1)) \xrightarrow{\times\lambda} H^p(X \times \mathbb{G}_m, C^*(n)) \xrightarrow{(1_X \times e)*} H^p(X, C^*(n)) \to 0$$

Lemma 5.1. *Denote by $\mu : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ the multiplication map: $\mu(s,t) = s \cdot t$. Then we have the following identity in the group $H^1(\mathbb{G}_m \times \mathbb{G}_m, C^*(1))$*

$$\mu^*(\lambda) = p_1^*(\lambda) + p_2^*(\lambda) = \lambda \times 1 + 1 \times \lambda.$$

Proof. The property (4) of the cohomology theory $H^*(-, C^*)$ implies that

$$H^1(\mathbb{G}_m \times \mathbb{G}_m, C^*(1)) = H^1(F, C^*(1)) \oplus \Lambda \cdot (\lambda \times 1) \oplus \Lambda \cdot (1 \times \lambda).$$

Thus $\mu^*(\lambda) = \lambda_0 + a(\lambda \times 1) + b(1 \times \lambda)$ for appropriate $\lambda_0 \in H^1(F, C^*(1)), a, b \in \Lambda$. Restricting both sides to $e \times \mathbb{G}_m$ and $\mathbb{G}_m \times e$ we immediately conclude that $\lambda_0 = 0, a = b = 1$.

Consider now an exact sequence of cohomology with supports:

$$H^{p-1}(X \times \mathbb{A}^1, C^*(n)) \to H^{p-1}(X \times \mathbb{G}_m, C^*(n)) \xrightarrow{\delta} H^p_{X\times 0}(X \times \mathbb{A}^1, C^*(n))$$

Note that $H^{p-1}(X \times \mathbb{A}^1, C^*(n))$ coincides with $H^{p-1}(X, C^*(n))$ and the corresponding map $H^{p-1}(X, C^*(n)) \to H^{p-1}(X \times \mathbb{G}_m, C^*(n))$ is split injective and has $H^{p-1}(X \wedge \mathbb{G}_m, C^*(n))$ as cokernel. Thus we have the following chain of isomorphisms:

$$H^{p-2}(X, C^*(n-1)) \xrightarrow{\times\lambda} H^{p-1}(X \wedge \mathbb{G}_m, C^*(n)) \xrightarrow{\delta_{X\times\mathbb{A}^1;X\times 0}} H^p_{X\times 0}(X \times \mathbb{A}^1, C^*(n))$$

Moreover, according to the Leibniz Rule, the composite isomorphism is given by external product with the cohomology class $\tau = \delta_{\mathbb{A}^1;0}(\lambda) \in H^2_0(\mathbb{A}^1, C^*(1))$.

Next we consider cohomology of $X \times \mathbb{A}^m$ with supports in $X \times 0$ for all m. To do so we consider m coordinate projections $p_i : \mathbb{A}^m \to \mathbb{A}^1 \quad (1 \le i \le m)$. Pulling back τ via these projections we get cohomology classes $p_i^*(\tau) \in H^2_{\mathbb{A}_i^{m-1}}(\mathbb{A}^m, C^*(1))$, where $\mathbb{A}_i^{m-1} \subset \mathbb{A}^m$ is the hyperplane section defined by the equation $T_i = 0$. Taking the product of the above cohomology classes we get a class

$$\tau_m = p_1^*(\tau) \cup ... \cup p_m^*(\tau) \in H^{2m}_0(\mathbb{A}^m, C^*(m))$$

Proposition 5.2. *For any smooth scheme X external multiplication by τ_m defines an isomorphism*

$$H^{p-2m}(X, C^*(n-m)) \to H^p_{X\times 0}(X \times \mathbb{A}^m, C^*(n))$$

Proof. We proceed by induction on m. The case $m = 1$ was discussed previously. In the general case we consider the affine line $\mathbb{A}^1 \subset \mathbb{A}^m$, defined by the equations $T_2 = ... = T_m = 0$ and the corresponding long exact sequence in cohomology with supports:

$$H^{p-1}_{X\times 0}(X \times \mathbb{A}^m, C^*(n)) \to H^{p-1}_{X\times\mathbb{A}^1}(X \times \mathbb{A}^m, C^*(n)) \to$$
$$\to H^{p-1}_{X\times(\mathbb{A}^1\setminus 0)}(X \times (\mathbb{A}^m \setminus 0), C^*(n)) \xrightarrow{\delta_{\mathbb{A}^m;\mathbb{A}^1,0}} H^p_{X\times 0}(X \times \mathbb{A}^m, C^*(n))$$

Our inductive assumption implies that we have the following identifications

$$H^{p-2m+1}(X, C^*(n-m+1)) = H^{p-2m+1}(X \times \mathbb{A}^1, C^*(n-m+1)) \xrightarrow{\sim}$$
$$\xrightarrow[\times\tau_{m-1}]{\sim} H^{p-1}_{X\times\mathbb{A}^1}(X \times \mathbb{A}^m, C^*(n))$$
$$H^{p-2m+1}(X \times (\mathbb{A}^1 \setminus 0), C^*(n-m+1)) \xrightarrow[\times\tau_{m-1}]{\sim} H^{p-1}_{X\times(\mathbb{A}^1\setminus 0)}(X \times (\mathbb{A}^1 \setminus 0)\times$$
$$\times \mathbb{A}^{m-1}, C^*(n)) \xrightarrow{ex^{-1}} H^{p-1}_{X\times(\mathbb{A}^1\setminus 0)}(X \times (\mathbb{A}^m \setminus 0), C^*(n))$$

Furthermore, the homomorphism

$$H^{p-2m+1}(X, C^*(n-m+1)) = H^{p-2m+1}(X \times \mathbb{A}^1, C^*(n-m+1)) \to \\ \to H^{p-2m+1}(X \times \mathbb{G}_m, C^*(n-m+1))$$

is split injective and its cokernel coincides with $H^{p-2m+1}(X \wedge \mathbb{G}_m, C^*(n-m+1))$ Thus we get a chain of isomorphisms

$$H^{p-2m}(X, C^*(n-m)) \xrightarrow{\times\lambda} H^{p-2m+1}(X \wedge \mathbb{G}_m, C^*(n-m+1)) \xrightarrow{\times\tau_{m-1}} \\ \frac{H^{p-1}_{X\times\mathbb{G}_m}(X \times (\mathbb{A}^m \setminus 0), C^*(n))}{H^{p-1}_{X\times\mathbb{A}^1}(X \times \mathbb{A}^m, C^*(n))} \xrightarrow{\delta} H^p_{X\times 0}(X \times \mathbb{A}^m, C^*(n))$$

Applying once again the Leibniz Rule we conclude that the composition of the above isomorphisms coincides with the external multiplication by τ_m.

Thom Class and Thom Isomorphism.

Let $p : E \to X$ be a vector bundle over X of (constant) rank m. We identify X with a closed subscheme of E defined by the zero section.

Theorem 5.3. *There exists a unique class $\tau = \tau_{E/X} \in H^{2m}_X(E, C^*(m))$ such that for any choice of a trivialization of $E|_U$*

$$f : E|_U \xrightarrow{\sim} U \times \mathbb{A}^m$$

$\tau|_U = f^*(1_U \times \tau_m)$. *Moreover, multiplication by $\tau_{E/X}$ defines isomorphisms*

$$H^{s-2m}(X, C^*(n-m)) \xrightarrow{u \mapsto p^*(u)\cup\tau} H^s_X(E, C^*(n))$$

Proof. We first verify that if $f : U \times \mathbb{A}^m \to U \times \mathbb{A}^m$ is an automorphism of a trivial bundle then $f^*(1_U \times \tau_m) = 1_U \times \tau_m$. We start with a special case $m = 1$, $U = \mathbb{G}_m$ and f is an automorphism of the trivial rank one bundle over $\mathbb{G}_m$ defined by the distinguished invertible function T on $\mathbb{G}_m$, i.e. $f(t, x) = (t, tx)$. Since $\tau = \delta_{\mathbb{A}^1,0}(\lambda)$ and δ commutes with pull-backs it suffices to show that $\delta_{U\times\mathbb{A}^1,U\times 0}(f^*(1_U \times \lambda) - 1_U \times \lambda) = 0$. However Lemma 5.1 shows that $f^*(1_U \times \lambda) = \mu^*(\lambda) = 1_U \times \lambda + \lambda \times 1_{\mathbb{G}_m}$ and the element $\lambda \times 1_{\mathbb{G}_m}$ comes from $H^1(U \times \mathbb{A}^1)$ and hence is killed by $\delta_{U\times\mathbb{A}^1,U\times 0}$.

Since every automorphism of the trivial rank one bundle is given by an invertible function and every invertible function is a pull-back of the distinguished invertible function T on $\mathbb{G}_m$ under a uniquely defined morphism $U \to \mathbb{G}_m$ we conclude by naturality that our statement is true for (trivial) rank one bundles over an arbitrary base U.

In the general case we note that according to Proposition 5.2 the group $H^{2m}_U(U \times \mathbb{A}^m, C^*(m))$ is equal to $H^0(U, \Lambda)$. Assuming for simplicity that U is connected we conclude that $f^*(1_U \times \tau_m) = c \cdot (1_U \times \tau_m)$ for a certain $c \in \Lambda^*$. To show that the corresponding element c is always equal to 1 it suffices to work out the case when

$U = \operatorname{Spec} \mathcal{O}$ is a local scheme. Automorphisms of the trivial bundle are in one to one correspondence with matrices $\alpha \in GL_m(\mathcal{O})$. Thus associating to each matrix the corresponding element $c \in \Lambda^*$ we get a group homomorphism $GL_m(\mathcal{O}) \to \Lambda^*$. In view of the homotopy invariance of the cohomology theory $H^*(-, C^*(-))$ this homomorphism kills all elementary matrices. Since every invertible matrix over $\mathcal{O}$ is a product of an elementary matrix and a diagonal matrix we conclude that it suffices to work out the case of the diagonal matrix $diag(a, 1, ..., 1)$, which follows from the special case $m = 1$ treated above.

We next show that the classes $f^*(1_U \times \tau_m) \in H^{2m}_U(E|_U, C^*(m))$ patch canonically to a global section $\tau \in H^{2m}_X(E|_X, C^*(m))$. This follows immediately from the Mayer-Vietoris exact sequence, taking into account that the cohomology groups $H^{2m-1}_{U \cap V}(E|_{U \cap V}, C^*(m)) = H^{-1}(U \cap V, \Lambda)$ vanish.

The last part follows immediately from Proposition 5.2 and the Mayer-Vietoris exact sequence.

Gisin Isomorphisms.

Finally we discuss briefly the deformation to the normal cone construction and Gisin isomorphisms. Let $Z \subset X$ be a smooth closed subscheme everywhere of codimension m with normal bundle N. Denote by $p : B_{Z\times 0}(X \times \mathbb{A}^1) \longrightarrow X \times \mathbb{A}^1$ the blow-up of $X \times \mathbb{A}^1$ with the smooth center $Z \times 0$. The proper inverse image $\mathcal{Z} = p^!(Z \times \mathbb{A}^1)$ of $Z \times \mathbb{A}^1$ under p identifies canonically (by means of p) with $Z \times \mathbb{A}^1$. The exceptional divisor $E = p^{-1}(Z \times 0)$ may be identified with $\mathbb{P}(N \oplus \mathcal{O}_Z)$ and the intersection $\mathcal{Z} \cap p^{-1}(Z \times 0)$ coincides with the zero section of the vector bundle $N = \mathbb{P}(N \oplus \mathcal{O}_Z) \setminus \mathbb{P}(N) \subset \mathbb{P}(N \oplus \mathcal{O}_Z)$. The inverse image scheme $p^{-1}(X \times 0)$ consists of two components, one of them being the exceptional divisor E, while the other – the proper inverse image $p^!(X \times 0)$, identifies with $B_{Z\times 0}(X \times 0)$. The intersection of these two components coincides with the exceptional divisor $\mathbb{P}(N)$ of $B_{Z\times 0}(X \times 0)$. Denote by $\mathcal{X}$ the open subscheme $\mathcal{X} = B_{Z\times 0}(X \times \mathbb{A}^1) \setminus p^!(X \times 0)$. As was explained above $\mathcal{Z}$ is a closed subscheme of $\mathcal{X}$ and the fiber of the pair $(\mathcal{X}, \mathcal{Z})$ over the point $0 \in \mathbb{A}^1$ (resp. over the point $1 \in \mathbb{A}^1$) coincides with the pair (N, Z) (resp. with the pair (X, Z).

Theorem 5.4. *Let $Z \subset X$ be a smooth closed subscheme everywhere of codimension m with normal bundle N. Then the restriction maps*

$$H^s_{\mathcal{Z}}(\mathcal{X}, C^*(n)) \longrightarrow H^s_{\mathcal{Z}_0}(\mathcal{X}_0, C^*(n)) = H^s_Z(N, C^*(n))$$
$$H^s_{\mathcal{Z}}(\mathcal{X}, C^*(n)) \longrightarrow H^s_{\mathcal{Z}_1}(\mathcal{X}_1, C^*(n)) = H^s_Z(X, C^*(n))$$

are isomorphisms and hence we get a canonical "deformation to the normal cone" isomorphism

$$H^s_Z(N, C^*(n)) \xrightarrow{\sim} H^s_{\mathcal{Z}}(\mathcal{X}, C^*(n)) \xrightarrow{\sim} H^s_Z(X, C^*(n))$$

Proof. This is a special case of a Theorem proved by Panin and Smirnov [P-S], who has shown (following the work of Morel and Voevodsky [M-V]) that more generally the deformation to the normal cone isomorphisms hold for any generalized cohomology theory with supports which is homotopy invariant, satisfies Nisnevich excision and has Mayer-Vietoris exact sequences.

Corollary 5.4.1. *In conditions and notations of Theorem 5.4 we have canonical Gisin isomorphisms*

$$H^{s-2m}(Z, C^*(n-m)) \xrightarrow{\sim} H^s_Z(N, C^*(n)) \xrightarrow{\sim} H^s_Z(X, C^*(n))$$

In particular we have Gisin isomorphisms for Grayson's cohomology

$$H^{s-2m}(Z, \mathbb{Z}^{Gr}(n-m)) \xrightarrow{\sim} H^s_Z(N, \mathbb{Z}^{Gr}(n)) \xrightarrow{\sim} H^s_Z(X, \mathbb{Z}^{Gr}(n))$$

Let $C'^*(n)$ $(0 \leq n < \infty)$ be another sequence of complexes of Nisnevich sheaves with the same properties as above and let $\{f_n : C^*(n) \to C'^*(n)\}_{n=0}^{\infty}$ be a sequence of maps in the derived category, which commute with products, take identity element $1 \in H^0(F, C^*(0))$ to the identity element $1 \in H^0(F, C'^*(0))$ and take the distinguished element $\lambda \in H^1_{Nis}(\mathbb{G}_m, C^*(1))$ to the distinguished element $\lambda' \in H^1_{Nis}(\mathbb{G}_m, C'^*(1))$. The naturality of the constructions described above implies immediately the following result.

Proposition 5.5. *Let $Z \subset X$ be a smooth closed subscheme of a smooth scheme X everywhere of codimension m. Then the following diagram involving the Gisin isomorphisms commutes*

$$\begin{array}{ccc} H^{s-2m}_{Nis}(Z, C^*(n-m)) & \xrightarrow{\sim} & H^s_Z(X, C^*(n)) \\ {\scriptstyle (f_{n-m})_*}\downarrow & & {\scriptstyle (f_n)_*}\downarrow \\ H^{s-2m}_{Nis}(Z, C'^*(n-m)) & \xrightarrow{\sim} & H^s_Z(X, C'^*(n)) \end{array}$$

Corollary 5.5.1. *Let $Z \subset X$ be a smooth closed subscheme of a smooth scheme X everywhere of codimension m. Then the following diagram involving the Gisin isomorphisms commutes*

$$\begin{array}{ccc} H^{s-2m}_{Nis}(Z, \mathbb{Z}^{Gr}(n-m)) & \xrightarrow{\sim} & H^s_Z(X, \mathbb{Z}^{Gr}(n)) \\ {\scriptstyle (f_{n-m})_*}\downarrow & & {\scriptstyle (f_n)_*}\downarrow \\ H^{s-2m}_{Nis}(Z, \mathbb{Z}(n-m)) & \xrightarrow{\sim} & H^s_Z(X, \mathbb{Z}(n)) \end{array}$$

Proof. This follows immediately from Proposition 5.5 and Lemma 3.4.

§ 6. Proof of the Main Theorem.

In this section we give the proof of the main theorem, comparing $\mathbb{Z}^{Gr}(n)$-cohomology with $\mathbb{Z}(n)$-cohomology. All cohomology groups below are taken with respect to the Nisnevich topology.

Theorem 6.1. *For any $n \geq 0$ the canonical homomorphism of complexes of Nisnevich sheaves*

$$f_n : \mathbb{Z}^{Gr}(n) \to \mathbb{Z}(n)$$

is a quasi-isomorphism. Hence for any smooth scheme $X \in Sm/F$ Grayson's cohomology $H^*(X, \mathbb{Z}^{Gr}(n))$ coincide with motivic cohomology $H^*(X, \mathbb{Z}(n))$. Hence Grayson's spectral sequence [Gr] *in case of a smooth semilocal scheme essentially of finite type over a field F takes the form*

$$E_2^{pq} = H^{p-q}(X, \mathbb{Z}(-q)) \Longrightarrow K_{-p-q}(X).$$

Proof. We proceed by induction on weight n. The case $n = 0$ is trivial and was treated previously in Lemma 3.4. Assume now that the Theorem has been proved already in weights $< n$. Assume first that the base field F is perfect. Our primary goal is to show that the canonical map in hypercohomology $H^*_{Nis}(X, \mathbb{Z}^{Gr}(n)) \to H^*_{Nis}(X, \mathbb{Z}(n))$ is an isomorphism for any $X \in Sm/F$. We first look on cohomology with supports.

Lemma 6.1.1. *Let E/F be a finitely generated field extension. Let further $X \in Sm/E$ be a smooth irreducible scheme. Then for any closed subscheme $Z \subsetneq X$ the corresponding map in cohomology with supports*

$$(f_n)_* : H^*_Z(X, \mathbb{Z}^{Gr}(n)) \to H^*_Z(X, \mathbb{Z}(n))$$

is an isomorphism.

Proof. Assume first that $E = F$ is a perfect field. In case Z is smooth our statement follows from Corollary 5.5.1 and the inductive assumption. In general we proceed by induction on $\dim Z$. If $\dim Z = 0$ then Z is smooth and we are done. In general let $Z' \subset Z$ be the singular locus of Z. Since the base field E is perfect every reduced scheme of finite type is smooth in the neighborhood of the generic point. Thus $\dim Z' < \dim Z$ and $Z \setminus Z'$ is smooth so that our statement follows from the comparison of the long exact sequences (in which cohomology groups are taken either with coefficients in $\mathbb{Z}^{Gr}(n)$ or in $\mathbb{Z}(n)$):

$$H^p_{Z'}(X) \to H^p_Z(X) \to H^p_{Z \setminus Z'}(X \setminus Z') \xrightarrow{\delta} H^{p+1}_{Z'}(X).$$

In the general case the field E may be identified with the field of rational functions $E = F(T)$ of an appropriate smooth scheme of finite type $T \in Sm/F$. Moreover the scheme X coincides with the generic fiber of an appropriate smooth morphism $p : \mathcal{X} \to T$ and the scheme Z coincides with the generic fiber of an appropriate closed subscheme $\mathcal{Z} \subset \mathcal{X}$:

$$X = \mathcal{X} \times_T Spec\, E, \qquad Z = \mathcal{Z} \times_T Spec\, E$$

It suffices to note now that

$$H^*_Z(X) = \varinjlim_{U \subset T} H^*_{\mathcal{Z}_U}(\mathcal{X}_U),$$

where the limit is taken over all open subschemes $U \subset T$ and for any open subscheme $U \subset T$ we have, according to what was proved above, equality of cohomology with supports

$$(f_n)_* : H^*_{\mathcal{Z}_U}(\mathcal{X}_U, \mathbb{Z}^{Gr}(n)) \xrightarrow{\sim} H^*_{\mathcal{Z}_U}(\mathcal{X}_U, \mathbb{Z}(n))$$

We also need a minor generalization of (6.1.1) to polyrelative cohomology with supports. So let $X \in Sm/F$ be a smooth scheme provided with a family$\{X_i\}_{i=0}^n$ of closed subschemes such that all intersections $X_{i_0} \cap ... \cap X_{i_k}$ are smooth. Let further Z be a closed subscheme of X with open complement U. We denote by $\mathbb{Z}_Z(X; \{X_i\}_{i=0}^n)$ the cone of the canonical homomorphism $\mathbb{Z}_{Nis}(U; \{U \cap X_i\}_{i=0}^n) \to \mathbb{Z}_{Nis}(X; \{X_i\}_{i=0}^n)$. For any complex of Nisnevich sheaves $\mathcal{F}^*$ we define the polyrelative cohomology with supports $H^p_Z(X, \{X_i\}_{i=0}^n; \mathcal{F}^*)$ as Ext-groups $\mathrm{Ext}^p(\mathbb{Z}_Z(X; \{X_i\}_{i=0}^n), \mathcal{F}^*)$. The spectral sequence (1.6.b) generalizes immediately to the case of polyrelative cohomology with supports and takes the form

(6.1.2.0)

$$E^1_{pq} = \oplus_{i_1 < ... < i_q} H^p_{Z \cap X_{i_1} \cap ... \cap X_{i_q}}(X_{i_1} \cap ... \cap X_{i_q}, \mathcal{F}^*) \Longrightarrow H^{p+q}_Z(X, \{X_i\}_{i=0}^n; \mathcal{F}^*)$$

Corollary 6.1.2. *Let E/F be a finitely generated field extension. Let further $X \in Sm/E$ be a smooth scheme provided with a family $\{X_i\}_{i=0}^m$ of closed subschemes such that all intersections $X_{i_0} \cap ... \cap X_{i_k}$ are smooth. Let finally Z be a closed subscheme in X which does not contain any component of any intersection $X_{i_0} \cap ... \cap X_{i_k}$. Then the natural map*

$$H^*_Z(X, \{X_i\}_{i=0}^m; \mathbb{Z}^{Gr}(n))) \to H^*_Z(X, \{X_i\}_{i=0}^m; \mathbb{Z}(n)))$$

is an isomorphism.

Proof. This follows immediately from Lemma 6.1.1, taking into account the spectral sequence (6.1.2.0).

Let $\mathcal{H}^q_{Gr}$ (resp. $\mathcal{H}^q$) denote the q-th cohomology sheaf of the complex $\mathbb{Z}^{Gr}(n)$ (resp. $\mathbb{Z}(n)$). The statement of Theorem 6.1 amounts to showing that the canonical map $\mathcal{H}^q_{Gr}(X) \to \mathcal{H}^q(X)$ is an isomorphism whenever X is a smooth local scheme. However, both sheaves are homotopy invariant pretheories, and a homomorphism from one homotopy invariant pretheory to the other is an isomorphism for all smooth local schemes if and only if it is an isomorphism for all finitely generated field extensions E/F – see [V1]. Thus it suffices to show that for any finitely generated field extension E/F the map

$$H^q_{Nis}(E, \mathbb{Z}^{Gr}(n)) \to H^q_{Nis}(E, \mathbb{Z}(n))$$

is an isomorphism for all q. To prove this statement we consider the polyrelative cohomology groups $H^*(\Delta^m_E, \{\Delta^{m-1}_{i,E}\}_{i=0}^m; \mathbb{Z}^{Gr}(n))$ and the corresponding polyrelative cohomology groups for $\mathbb{Z}(n)$. Using the spectral sequence (1.6.b) and the homotopy invariance of the corresponding motivic cohomology groups one concludes easily that there is a canonical degree shift isomorphism

$$H^p(E, \mathbb{Z}^{Gr}(n)) = H^{m+p}(\Delta^m_E, \{\Delta^{m-1}_{i,E}\}_{i=0}^m; \mathbb{Z}^{Gr}(n))$$

and a similar degree shift isomorphism for $\mathbb{Z}(n)$-cohomology. Let $\mathcal{Z}$ denote the family of supports on Δ^m_E, consisting of all closed subschemes $Z \subset \Delta^m_E$ containing no vertices. Consider the long exact sequence for polyrelative cohomology with supports (in which $\hat{\Delta}^m_E$ denotes the semilocalization of Δ^m_E with respect to the vertices)

$$H^{m+p-1}(\hat{\Delta}^m_E, \{\hat{\Delta}^{m-1}_{i,E}\}^m_{i=0}; \mathbb{Z}^{Gr}(n)) \to H^{m+p}_{\mathcal{Z}}(\Delta^m_E, \{\Delta^{m-1}_{i,E}\}^m_{i=0}; \mathbb{Z}^{Gr}(n)) \to$$
$$\to H^{m+p}(\Delta^m_E, \{\Delta^{m-1}_{i,E}\}^m_{i=0}; \mathbb{Z}^{Gr}(n)) \to H^{m+p}(\hat{\Delta}^m_E, \{\hat{\Delta}^{m-1}_{i,E}\}^m_{i=0}; \mathbb{Z}^{Gr}(n))$$

and a natural homomorphism from this exact sequence to a similar sequence for polyrelative cohomology with supports with coefficients in $\mathbb{Z}(n)$. Vanishing of polyrelative cohomology in the semilocal case (see - Proposition 3.3) implies that

$$H^i(\hat{\Delta}^m_E, \{\hat{\Delta}^{m-1}_{i,E}\}^m_{i=0}; \mathbb{Z}^{Gr}(n)) = H^i(\hat{\Delta}^m_E, \{\hat{\Delta}^{m-1}_{i,E}\}^m_{i=0}; \mathbb{Z}(n)) = 0 \text{ for } i > n.$$

Thus taking m large enough we see that the natural map

$$H^{m+p}_{\mathcal{Z}}(\Delta^m_E, \{\Delta^{m-1}_{i,E}\}^m_{i=0}; \mathbb{Z}^{Gr}(n)) \to H^{m+p}(\Delta^m_E, \{\Delta^{m-1}_{i,E}\}^m_{i=0}; \mathbb{Z}^{Gr}(n)) =$$
$$= H^p(E, \mathbb{Z}^{Gr}(n))$$

(and a similar map for $\mathbb{Z}(n)$-cohomology) is an isomorphism. Finally the map in cohomology with supports

$$H^{m+p}_{\mathcal{Z}}(\Delta^m_E, \{\Delta^{m-1}_{i,E}\}^m_{i=0}; \mathbb{Z}^{Gr}(n)) \to H^{m+p}_{\mathcal{Z}}(\Delta^m_E, \{\Delta^{m-1}_{i,E}\}^m_{i=0}; \mathbb{Z}(n))$$

is an isomorphism in view of (6.1.2)

This ends up the proof in the case of the perfect base field F. In the general case we note first of all that once again it suffices to establish that the canonical map

$$(f_n)_* : H^*(E, \mathbb{Z}^{Gr}(n)) \to H^*(E, \mathbb{Z}(n))$$

is an isomorphism for any finitely generated field extension E/F. However the cohomology groups $H^*(E, \mathbb{Z}^{Gr}(n)), H^*(E, \mathbb{Z}(n))$ are defined intrinsically in terms of the field E and are independent of the choice of the base field F. Assuming that $char\, E = p > 0$ we conclude from the first part of the proof that the map in question is an isomorphism provided that the field E is finitely generated over the prime field $\mathbb{Z}/p$. However any field of characteristic p may be written as a direct limit of fields finitely generated over $\mathbb{Z}/p$ and the above cohomology groups obviously commute with direct limits.

§ 7. Bloch-Kato Conjecture versus Beilinson-Lichtenbaum Conjecture.

In this section we show how the machinery developed in § 2 (specifically Theorem 2.7) can be used to simplify significantly the proof of the main theorem of [S-V], allowing simultaneously to get rid of the assumption about resolution of singularities. The present approach is the combination of the approaches developed in [S-V] and [G-L] and sheds (as we hope) additional light on the situation.

For a not necessarily smooth scheme X (of finite type over the base field F) we define a Nisnevich sheaf $\mathbb{Z}_{Sm/F}(X)$ on the category Sm/F as a sheaf associated to the presheaf $U \mapsto \mathbb{Z}[Hom_{Sch/F}(U,X)]$. It's not difficult to see that in case X is separated (which we always assume in the sequel) the presheaf $U \mapsto \mathbb{Z}[Hom_{Sch/F}(U,X)]$ is actually a sheaf in Nisnevich topology, so that no sheafification is needed. For a complex $\mathcal{F}^*$ of Nisnevich sheaves on the category Sm/F we define cohomology groups $H^p_{Nis}(X,\mathcal{F}^*)$ as appropriate Ext-groups:

$$H^p_{Nis}(X,\mathcal{F}^*) = \mathrm{Ext}^p(\mathbb{Z}_{Sm/F}(X),\mathcal{F}^*) = \mathrm{Hom}_{D(Sm/F)}(\mathbb{Z}_{Sm/F}(X),\mathcal{F}^*[p]),$$

where $D(Sm/F)$ denotes the derived category of complexes of Nisnevich sheaves on Sm/F. In case $X \in Sm/F$ the sheaf $\mathbb{Z}_{Sm/F}(X)$ coincides obviously with the free Nisnevich sheaf $\mathbb{Z}_{Nis}(X)$ generated by X and the cohomology groups $H^*_{Nis}(X,\mathcal{F}^*)$ coincide with the usual (hyper-)cohomology groups of X with coefficients in $\mathcal{F}^*$. In case resolution of singularities holds over F one can identify (for any $X \in Sch/F$ and any bounded above complex $\mathcal{F}^*$ of Nisnevich sheaves with transfers)) the cohomology groups $H^*_{Nis}(X,\mathcal{F}^*)$ with the usual hypercohomology of X with coefficients in the complex of cdh sheaves $\mathcal{F}^*_{cdh}$ on the *cdh*-site Sch/F of all separated schemes of finite type over F, see [S-V], § 5. Independently of resolution of singularities the above cohomology groups satisfy Mayer-Vietoris both for open and closed coverings.

Lemma 7.1. *Let $X = \cup_{i=0}^n X_i$ be an open (resp. closed) covering of a scheme X. In this case we have the following resolution of the sheaf $\mathbb{Z}_{Sm/F}(X)$:*

$$\dots \to \bigoplus_{i_0<\dots<i_k} \mathbb{Z}_{Sm/F}(X_{i_0}\cap\dots\cap X_{i_k}) \to \ \dots \ \to \bigoplus_i \mathbb{Z}_{Sm/F}(X_i) \to \mathbb{Z}_{Sm/F}(X)$$

Here $\bigoplus_{i_0<\dots<i_k}\mathbb{Z}_{Sm/F}(X_{i_0}\cap\dots\cap X_{i_k})$ stands in degree $-k$ and the differential is the alternating sum of maps induced by inclusions. Thus for any complex of Nisnevich sheaves $\mathcal{F}^*$ on Sm/F we get a Mayer-Vietoris spectral sequence

$$E_2^{pq} = \bigoplus_{i_0<\dots<i_q} H^p_{Nis}(X_{i_0}\cap\dots\cap X_{i_q},\mathcal{F}^*) \Longrightarrow H^{p+q}_{Nis}(X,\mathcal{F}^*)$$

Proof. In both cases the sheaf $\mathbb{Z}_{Sm/F}(X)$ coincides with the sum of its subsheaves $\mathbb{Z}_{Sm/F}(X_i)$ and the intersection $\mathbb{Z}_{Sm/F}(X_{i_0})\cap\dots\cap\mathbb{Z}_{Sm/F}(X_{i_q})$ coincides obviously with $\mathbb{Z}_{Sm/F}(X_{i_0}\cap\dots\cap X_{i_q})$.

Assume now that $X \in Sm/F$ is a smooth scheme provided with a family $\{X_i\}_{i=0}^n$ of closed subschemes such that all intersections $X_{i_0}\cap\dots\cap X_{i_k}$ are smooth. In this case the polyrelative complex $\mathbb{Z}_{Nis}(X;\{X_i\}_{i=0}^n)$ coincides, according to Lemma 7.1, with the cone of the obvious homomorphism $\mathbb{Z}_{Sm/F}(\cup_{i=0}^n X_i) \to \mathbb{Z}_{Nis}(X)$. This remark together with Theorem 2.7 prove immediately the following result.

Proposition 7.2. *Let $\mathcal{F}$ be a rationally contractible $K_0^{\oplus}$-presheaf. Then*

$$H^p_{Nis}(\partial(\hat{\Delta}^n), C^*(\mathcal{F})_{Nis}) = 0 \quad \text{for } p > 0,$$

where $\partial(\hat{\Delta}^n) = \cup_{i=0}^n \hat{\Delta}_i^{n-1}$ is the boundary of the semilocal n-simplex $\hat{\Delta}^n$.

Proof. The remark preceding the formulation of the Proposition shows that we have a long exact sequence

$$\begin{aligned} \ldots \to & H^p_{Nis}(\hat{\Delta}^n, \{\hat{\Delta}_i^{n-1}\}_{i=0}^n; C^*(\mathcal{F})_{Nis}) \to H^p_{Nis}(\hat{\Delta}^n, C^*(\mathcal{F})_{Nis}) \to \\ & \to H^p_{Nis}(\partial(\hat{\Delta}^n), C^*(\mathcal{F})_{Nis}) \to H^{p+1}_{Nis}(\hat{\Delta}^n, \{\hat{\Delta}_i^{n-1}\}_{i=0}^n; C^*(\mathcal{F})_{Nis}) \to \ \ldots \end{aligned}$$

and hence our statement follows from Theorem 2.7 and Proposition 1.7.

Applying Proposition 7.2 to the rationally contractible sheaf $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge m})/l$ and noting that the corresponding complex $C^*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge m})/l)$ coincides with $\mathbb{Z}/l(m)[m]$ we get the following Corollary.

Corollary 7.3. *For any $n, m \geq 0$ we have the following vanishing result for motivic cohomology of the non-smooth semilocal scheme $\partial(\hat{\Delta}^n)$:*

$$H^p_{Nis}(\partial(\hat{\Delta}^n), \mathbb{Z}/l(m)) = 0 \quad \text{for } p > m.$$

We will need also the following vanishing result.

Lemma 7.4. *Let $\mathcal{F}^*$ be a complex of Nisnevich sheaves acyclic in degrees $i \leq m$. Then for any separated scheme X of finite type over F we have:*

$$H^p_{Nis}(X, \mathcal{F}^*) = 0 \qquad \text{for} \quad p \leq m$$

Proof. Note that any element $\alpha \in H^p_{Nis}(X, \mathcal{F}^*) = \mathrm{Hom}_{D(Sm/F)}(\mathbb{Z}_{Nis}(X), \mathcal{F}^*[p])$ may be written as a composition $\alpha = \beta^{-1} \circ \gamma$, where $\beta : \mathcal{F}^*[p] \to \widetilde{\mathcal{F}}^*$ is a quasi-isomorphism and $\gamma : \mathbb{Z}_{Nis}(X) \to \widetilde{\mathcal{F}}^*$ is a homomorphism of complexes. It suffices to note now, that according to our assumptions the complex $\widetilde{\mathcal{F}}^*$ is acyclic in degrees $i \leq 0$ and hence the canonical homomorphism $\widetilde{\mathcal{F}}^* \xrightarrow{p} \tau_{>0}\widetilde{\mathcal{F}}^*$ is a quasi-isomorphism. On the other hand the composition $\mathbb{Z}_{Nis}(X) \xrightarrow{\gamma} \widetilde{\mathcal{F}}^* \xrightarrow{p} \tau_{>0}\widetilde{\mathcal{F}}^*$ is obviously a zero map.

Recall the complex of Nisnevich sheaves with transfers $B_l(m)$ defined as $B_l(m) = \tau_{\leq m} R\pi_*(\mu_l^{\otimes m})$ (see [S-V] § 6), where $\tau_{\leq m}$ denotes the level m truncation functor and $\pi : (Sch/F)_h \to (Sm/F)_{Nis}$ is the obvious morphism of sites.

Corollary 7.5. *For any scheme X of finite type over F and any $p \le m$ we have natural isomorphisms*

$$H^p_{Nis}(X, B_l(m)) = H^p_{Nis}(X, R\pi_*(\mu_l^{\otimes m})) = H^p_h(X, \mu_l^{\otimes m}) = \\ = H^p_{et}(X, \mu_l^{\otimes m})$$

Proof. The first isomorphism follows from Lemma 7.4, applied to the complex of sheaves $\tau_{>m} R\pi_*(\mu_l^{\otimes m})$. The second isomorphism is obvious (taking into account that the functor π^* is exact - see [S-V], § 6) and the last isomorphism follows from the coincidence of h-cohomology and etale cohomology - see [S-V 1], § 10.

Theorem 7.6. *Let F be a perfect field and let l be a prime different from char F. Assume that for any finitely generated field extension E/F the norm-residue homomorphism $K_i^M(E)/l \to H^i_{et}(E, \mu_l^{\otimes i})$ is an epimorphism in weights $i \le m$. Then the Beilinson-Lichtenbaum Conjecture holds in weights $i \le m$, i.e. the natural homomorphism of complexes of Nisnevich sheaves with transfers $\alpha_i : \mathbb{Z}/l(i) \to B_l(i) = \tau_{\le i} R\pi_*(\mu_l^{\otimes i})$ is a quasi-isomorphism for all $i \le m$.*

Proof. We proceed by induction on m. Induction hypothesis implies that the homomorphism of complexes $\alpha_i : \mathbb{Z}/l(i) \to B_l(i) = \tau_{\le i} R\pi_*(\mu_l^{\otimes i})$ is a quasi-isomorphism for $i \le m-1$. To show that α_m is a quasi-isomorphism we first consider the induced maps in cohomology with supports. Recall that according to the Voevodsky Cancellation Theorem [V3] for any smooth scheme X external multiplication by the canonical generator $\lambda = [1_{\mathbb{G}_m}] - [e] \in H^1_{Nis}(\mathbb{G}_m^{\wedge 1}, \mathbb{Z}(1))$ defines isomorphisms

$$H^{p-1}_{Nis}(X, \mathbb{Z}(i-1)) \xrightarrow{\times \lambda} H^p_{Nis}(X \wedge \mathbb{G}_m, \mathbb{Z}(i)).$$

Keeping the same notation λ for the image of λ in $H^1_{Nis}(\mathbb{G}_m^{\wedge 1}, \mathbb{Z}/l(1))$ we conclude immediately that external multiplication by λ defines isomorphisms

$$H^{p-1}_{Nis}(X, \mathbb{Z}/l(i-1)) \xrightarrow{\times \lambda} H^p_{Nis}(X \wedge \mathbb{G}_m, \mathbb{Z}/l(i))$$

Denote further by $\tilde{\lambda}$ the image of λ in $H^1(\mathbb{G}_m^{\wedge 1}, B_l(1))$. A well-known and easy argument - cf. [S-V], § 7 shows that for any smooth scheme X the external multiplication by $\tilde{\lambda}$ defines isomorphisms

$$H^{p-1}_{Nis}(X, B_l(i-1)) \xrightarrow{\times \tilde{\lambda}} H^p_{Nis}(X \wedge \mathbb{G}_m, B_l(i))$$

Applying finally to the cohomology theories $H^*_{Nis}(-, \mathbb{Z}/l(*))$ and $H^*_{Nis}(-, B_l(*))$ the standard routine, which we reminded in section § 5 we get for any smooth scheme X and its smooth subscheme $Z \subset X$ (everywhere) of codimension d Gisin isomorphisms

$$H^{p-2d}_{Nis}(Z, \mathbb{Z}/l(m-d)) \xrightarrow{\sim} H^p_Z(X, \mathbb{Z}/l(m))$$
$$H^{p-2d}_{Nis}(Z, B_l(m-d)) \xrightarrow{\sim} H^p_Z(X, B_l(m))$$

Moreover the following diagram commutes (according to Proposition 5.5 and [S-V] Corollary 7.1.1)

$$\begin{array}{ccc} H^{p-2d}_{Nis}(Z, \mathbb{Z}/l(m-d)) & \xrightarrow{\sim} & H^p_Z(X, \mathbb{Z}/l(m)) \\ {\scriptstyle (\alpha_{m-d})_*}\downarrow & & {\scriptstyle (\alpha_m)_*}\downarrow \\ H^{p-2d}_{Nis}(Z, B_l(m-d)) & \xrightarrow{\sim} & H^p_Z(X, B_l(m)) \end{array}$$

Lemma 7.6.1. *Let E/F be a finitely generated field extension. Let further $X \in Sm/E$ be a smooth irreducible scheme and let $Z \neq X$ be a proper closed subscheme of X. Then the induced map in cohomology with supports $H^*_Z(X, \mathbb{Z}/l(m)) \xrightarrow{\alpha_m} H^*_Z(X, B_l(m))$ is an isomorphism.*

Proof. The proof is identical to that of Lemma 6.1.1.

Corollary 7.6.2. *Let E/F be a finitely generated field extension. Let further Z be a closed subscheme of $\partial(\Delta^n_E)$, containing no vertices of Δ^n_E. Then the induced map in cohomology with supports*

$$H^*_Z(\partial(\Delta^n_E), \mathbb{Z}/l(m)) \xrightarrow{(\alpha_m)_*} H^*_Z(\partial(\Delta^n_E), B_l(m))$$

is an isomorphism.

Proof. Applying Lemma 7.1 to the closed covering $\partial(\Delta^n_E) = \cup_{i=0}^n (\Delta^{n-1}_i)_E$ and to the induced closed covering of $U = \partial(\Delta^n_E)\backslash Z$ we get immediately spectral sequences

$$E_2^{pq} = \bigoplus_{I=\{i_0<\ldots<i_q\}} H^p_{Z\cap(\Delta_I^{n-q-1})_E}((\Delta_I^{n-q-1})_E, \mathbb{Z}/l(m)) \Longrightarrow H^{p+q}_Z(\partial(\Delta^n_E), \mathbb{Z}/l(m))$$

$$E_2^{pq} = \bigoplus_{I=\{i_0<\ldots<i_q\}} H^p_{Z\cap(\Delta_I^{n-q-1})_E}((\Delta_I^{n-q-1})_E, B_l(m)) \Longrightarrow H^{p+q}_Z(\partial(\Delta^n_E), B_l(m))$$

and a homomorphism from the top spectral sequence to the bottom one. Lemma 7.6.1 shows that the map on the E_2-terms is an isomorphism and hence the corresponding map on the limits is an isomorphism as well.

Another major idea in the proof of the Theorem 7.6, developed in [S-V], is the use of the Theorem of O. Gabber [Ga], which allows to extend the assumed surjectivity of the homomorphism $\alpha_m : H^m_{Nis}(S, \mathbb{Z}/l(m)) \to H^m_{Nis}(S, B_l(m))$ from the case of field extensions E/F to the case of not necessarily smooth semilocal schemes.

Lemma 7.6.3 ([S-V], **Proposition 7.6).** *Let E/F be a finitely generated field extension and let S be a semilocal scheme essentially of finite type over E. Then the associated homomorphism*

$$(\alpha_m)_* : H^m_{Nis}(S, \mathbb{Z}/l(m)) \to H^m_{Nis}(S, B_l(m))$$

is an epimorphism.

Proof. In case S is smooth this follows immediately from the general properties of homotopy invariant presheaves with transfers: any homomorphism of homotopy invariant presheaves with transfers which is an epimorphism for fields is equally an epimorphism for all smooth semilocal schemes essentially of finite type over a field - [V1]. Alternatively one can use Gersten resolutions for $H^m_{Nis}(-,\mathbb{Z}/l(m))$ and $H^m_{Nis}(-,B_l(m))$ and inductive assumption.

For a general semilocal S the theorem of O.Gabber [Ga] shows that every cohomology class $x \in H^m_{Nis}(S,B_l(m)) = H^m_{et}(S,\mu_l^{\otimes m})$ may be obtained as a pull-back $x = f^*(x')$ of a cohomology class $x' \in H^m_{Nis}(S',B_l(m)) = H^m_{et}(S',\mu_l^{\otimes m})$ for an appropriate smooth semilocal scheme S' essentially of finite type over E and an appropriate morphism of schemes $f : S \to S'$ over E. Since the scheme S' is smooth we conclude that x' is in the image of $H^m_{Nis}(S',\mathbb{Z}/l(m))$ and hence x is in the image of $H^m_{Nis}(S,\mathbb{Z}/l(m))$.

Proof of the Theorem 7.6. In view of the general properties of homotopy invariant presheaves with transfers, to show that the homomorphism of complexes $\alpha_m : \mathbb{Z}/l(m) \to B_l(m)$ is a quasi-isomorphism it suffices to show that for any finitely generated field extension E/F the induced maps in motivic cohomology $H^p_{Nis}(E,\mathbb{Z}/l(m)) \to H^p_{Nis}(E,B_l(m))$ are isomorphisms. Both sides are trivial in degrees $p > m$, thus we may assume in the sequel that $p \leq m$. As was mentioned in § 6 we have canonical degree shift isomorphisms

$$H^p(E,\mathbb{Z}/l(m)) = H^{p+n}(\Delta^n_E, \{(\Delta^{n-1}_i)_E\}^n_{i=0};\mathbb{Z}/l(m))$$
$$H^p(E,B_l(m)) = H^{p+n}(\Delta^n_E, \{(\Delta^{n-1}_i)_E\}^n_{i=0};B_l(m))$$

Using next the distinguished triangle

$$\mathbb{Z}_{Nis}(\partial(\Delta^n_E)) \to \mathbb{Z}_{Nis}(\Delta^n_E) \to \mathbb{Z}_{Nis}(\Delta^n_E;\{(\Delta^{n-1}_i)_E\}^n_{i=0}) \to \mathbb{Z}_{Nis}(\partial(\Delta^n_E))[1]$$

we get a long exact sequence

$$\begin{aligned} \dots \to & H^p(E,\mathbb{Z}/l(m)) = H^p(\Delta^n_E,\mathbb{Z}/l(m)) \to H^p(\partial\Delta^n_E,\mathbb{Z}/l(m)) \to \\ & \to H^{p+1}((\Delta^n_E,\{(\Delta^{n-1}_i)_E\}^n_{i=0};\mathbb{Z}/l(m)) = H^{p+1-n}(E,\mathbb{Z}/l(m)) \to \dots \end{aligned}$$

and a similar exact sequence for $B_l(m)$-cohomology. The homomorphism

$$H^p(E,\mathbb{Z}/l(m)) \to H^p(\partial\Delta^n_E,\mathbb{Z}/l(m))$$

is a split monomorphism (splitting is defined by a choice of a vertex of the simplex Δ^n). Thus we get canonical direct sum decompositions

$$H^p_{Nis}(\partial\Delta^n_E, \mathbb{Z}/l(m)) = H^p_{Nis}(E, \mathbb{Z}/l(m)) \oplus H^{p+1-n}_{Nis}(E, \mathbb{Z}/l(m))$$
$$H^p_{Nis}(\partial\Delta^n_E, B_l(m)) = H^p_{Nis}(E, \mathbb{Z}/l(m)) \oplus H^{p+1-n}_{Nis}(E, B_l(m))$$

Take first $n = m+1-p$ and denote by $\mathcal{Z}$ the family of supports on $\partial\Delta^n_E$ consisting of all closed subschemes containing no vertices. Applying the five-Lemma to the commutative diagram with exact rows (in which we skipped the coefficients in the last two cohomology groups to save the space)

$$\begin{array}{cccccccc} H^m_{\mathcal{Z}}(\partial\Delta^n_E, \mathbb{Z}/l(m)) & \to & H^m(\partial\Delta^n_E, \mathbb{Z}/l(m)) & \to & H^m(\partial\hat{\Delta}^n_E) & \to & H^{m+1}_{\mathcal{Z}}(\partial\Delta^n_E) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^m_{\mathcal{Z}}(\partial\Delta^n_E, B_l(m)) & \to & H^m(\partial\Delta^n_E, B_l(m)) & \to & H^m(\partial\hat{\Delta}^n_E) & \to & H^{m+1}_{\mathcal{Z}}(\partial\Delta^n_E), \end{array}$$

in which the first and the last maps are isomorphisms according to Corollary 7.6.2 and the third map is an epimorphism according to Lemma 7.6.3, we conclude that the homomorphism

$$\begin{aligned} H^m(\partial\Delta^n_E, \mathbb{Z}/l(m)) =& H^m(E, \mathbb{Z}/l(m)) \oplus H^p(E, \mathbb{Z}/l(m)) \to \\ & \to H^m(\partial\Delta^n_E, B_l(m)) = H^m(E, B_l(m)) \oplus H^p(E, B_l(m)) \end{aligned}$$

is an epimorphism and hence $H^p(E, \mathbb{Z}/l(m)) \to H^p(E, \mathbb{Z}/l(m))$ is an epimorphism as well.

Taking next $n = m + 2 - p$ and applying the five-Lemma to the commutative diagram with exact rows

$$\begin{array}{cccccccc} H^m(\partial\hat{\Delta}^n_E) & \to & H^{m+1}_{\mathcal{Z}}(\partial\Delta^n_E, \mathbb{Z}/l(m)) & \to & H^{m+1}(\partial\Delta^n_E, \mathbb{Z}/l(m)) & \to & H^{m+1}(\partial\hat{\Delta}^n_E) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^m(\partial\hat{\Delta}^n_E) & \to & H^{m+1}_{\mathcal{Z}}(\partial\Delta^n_E, B_l(m)) & \to & H^{m+1}(\partial\Delta^n_E, B_l(m)) & \to & H^{m+1}(\partial\hat{\Delta}^n_E) \end{array}$$

in which the first map is surjective according to Lemma 7.6.3, the second map is an isomorphism according to Corollary 7.6.2 and the last map is injective since $H^{m+1}(\partial\hat{\Delta}^n_E, \mathbb{Z}/l(m)) = 0$ according to Corollary 7.3, we conclude that the homomorphism

$$H^{m+1}(\partial\Delta^n_E, \mathbb{Z}/l(m)) = H^p(E, \mathbb{Z}/l(m)) \to H^{m+1}(\partial\Delta^n_E, B_l(m)) = H^p(E, B_l(m))$$

is injective.

We end up this section by showing that the main result of [S-V], stating that vanishing of Bockstein homomorphisms alone implies the validity of the Beilinson-Lichtenbaum Conjecture, also holds without assumption concerning resolution of singularities. The proof given in [S-V] goes essentially without changes (modulo the changes made above in the proof of the Theorem 7.6) so we just remind the main steps. We start with the following version of Theorem 7.6, the proof of which is identical to that of Theorem 7.6. Define complexes of Nisnevich sheaves with transfers $B_{l^k}(m), B_{l^\infty}(m)$ via the formulae:

$$B_{l^k}(m) = \tau_{\leq m} R\pi_*(\mu_{l^k}^{\otimes m}) \quad B_{l^\infty}(m) = \tau_{\leq m} R\pi_*(\mathbb{Q}_l/\mathbb{Z}_l(m)) = \varinjlim_{k\geq 0} B_{l^k}(m)$$

For each k we have a canonical homomorphism of complexes (of Nisnevich sheaves with transfers) $\mathbb{Z}/l^k(m) \xrightarrow{\alpha_m} B_{l^k}(m)$ - see [S-V], § 6. We also have the corresponding homomorphism in case $k = \infty$

$$\mathbb{Q}_l/\mathbb{Z}_l(m) \xrightarrow{\alpha_m} B_{l^\infty}(m)$$

Proposition 7.6.4. *Let F be a perfect field and let l be a prime different from $\operatorname{char} F$. Assume that for any finitely generated field extension E/F the norm-residue homomorphism*

$$K_i^M(E) \otimes \mathbb{Q}_l/\mathbb{Z}_l = H^i_{Nis}(E, \mathbb{Q}_l/\mathbb{Z}_l(i)) \to H^i_{Nis}(E, B_{l^\infty}(i)) = H^i_{et}(E, \mathbb{Q}_l/\mathbb{Z}_l(i))$$

is an epimorphism in weights $i \leq m$. Then the natural homomorphism of complexes of Nisnevich sheaves with transfers $\mathbb{Q}_l/\mathbb{Z}_l(i) \xrightarrow{\alpha_i} B_{l^\infty}(i)$ is a quasi-isomorphism for all $i \leq m$.

We mention here for the future use also the following elementary general statement concerning the Beilinson-Lichtenbaum Conjecture (cf. [S-V], § 6).

Lemma 7.6.5. *Assume that the Beilinson-Lichtenbaum Conjecture (modulo l) holds in weight m. Then the canonical homomorphism*

$$\alpha_m : \mathbb{Z}/l^k(m) \longrightarrow B_{l^k}(m) = \tau_{\leq m} R\pi_*(\mu_{l^k}^{\otimes m})$$

is a quasi-isomorphism for all $k \geq 1$ and the corresponding homomorphism

$$\mathbb{Q}_l/\mathbb{Z}_l(m) \xrightarrow{\alpha_m} B_{l^\infty}(m)$$

is a quasi-isomorphism as well.

Theorem 7.7. *Let F be a perfect field. Assume that for any finitely generated field extension E/F the Bockstein homomorphisms*

$$\beta^i_{1,k} : H^i_{et}(E, \mu_l^{\otimes i}) \to H^{i+1}_{et}(E, \mu_{l^k}^{\otimes i})$$

are trivial for all k and all $i \leq m$. Then the Beilinson-Lichtenbaum Conjecture (modulo l) holds over F in weights $i \leq m$, i.e. the natural homomorphism of complexes of Nisnevich sheaves with transfers $\mathbb{Z}/l(i) \to B_l(i)$ is a quasi-isomorphism for all $i \leq m$.

Proof. Proceeding by induction we may assume that the Beilinson-Lichtenbaum Conjecture is already known to be true in weights $i < m$. Passing to the maximal algebraic extension of F of degree prime to l and using transfer maps we easily reduce the general case to the situation when the field F is infinite and contains a primitive l-th root of unity ξ. We start with the following elementary observation.

Lemma 7.7.1. *Under the assumptions of the Theorem 7.7 for any finitely generated field extension E/F the group $H^m_{et}(E, \mathbb{Q}_l/\mathbb{Z}_l(m))$ is l-divisible. Hence we have a canonical distinguished triangle*

$$\dots \to B_l(m) \to B_{l^\infty}(m) \xrightarrow{l} B_{l^\infty}(m) \to B_l(m)[1] \to \dots$$

Proof. The first statement is straightforward from the definition of the Bockstein homomorphisms - see [S-V], Lemma 11.1. To prove the second one we note that we have an obvious quasi-isomorphism

$$R\pi_*(\mu_l^{\otimes m}) \xrightarrow{\sim} Cone(R\pi_*(\mathbb{Q}_l/\mathbb{Z}_l(m)) \xrightarrow{l} R\pi_*(\mathbb{Q}_l/\mathbb{Z}_l(m)))[-1]$$

Applying the operation $\tau_{\leq m}$ to all complexes we get a canonical homomorphism

$$B_l(m) \to Cone(B_{l^\infty}(m) \xrightarrow{l} B_{l^\infty}(m))[-1]$$

Moreover this homomorphism induces isomorphisms on all cohomology sheaves except possibly $\mathcal{H}^{m+1}$. Finally for any finitely generated field extension E/F we have a canonical identification $\mathcal{H}^{m+1}(Cone)(E) = H^m_{et}(E, \mathbb{Q}_l/\mathbb{Z}_l(m))/l = 0$, according to the first statement of the Lemma. Since $\mathcal{H}^{m+1}(Cone)$ is a homotopy invariant Nisnevich sheaf with transfers we conclude immediately that this sheaf is trivial (see [V1]) and hence the above homomorphism of complexes is a quasi-isomorphism.

Corollary 7.7.2. *Under assumptions of Theorem 7.7 for any finitely generated field extension E/F and any smooth semilocal scheme S essentially of finite type over E the group $H^m_{et}(S, \mathbb{Q}_l/\mathbb{Z}_l(m))$ is l-divisible.*

Proof. Once again this follows immediately from the general properties of homotopy invariant presheaves with transfers: every endomorphism (multiplication by l in this case) of a homotopy invariant presheaf with transfers $H^m_{et}(-, \mathbb{Q}_l/\mathbb{Z}_l(m))$ which is an epimorphism for fields is an epimorphism for smooth semilocal schemes essentially of finite type over a field as well - see [V1].

We show next that vanishing of the Bockstein homomorphisms implies that for any finitely generated field extension E/F the induced map in motivic cohomology

$$H^m_{Nis}(E, \mathbb{Q}_l/\mathbb{Z}_l(m)) \xrightarrow{(\alpha_m)_*} H^m_{Nis}(E, B_{l^\infty}(m)) = H^m_{et}(E, \mathbb{Q}_l/\mathbb{Z}_l(m))$$

is surjective. To do so we note that both l-primary groups are l-divisible: the one on the left coincides with $K^M_m(E) \otimes \mathbb{Q}_l/\mathbb{Z}_l$ and the divisibility of the group on the right was proved in Lemma 7.7.1. Thus it suffices to show that l-torsion subgroup of $H^m_{et}(E, \mathbb{Q}_l/\mathbb{Z}_l(m))$ lies in the image of $H^m_{Nis}(E, \mathbb{Q}_l/\mathbb{Z}_l(m))$. Moreover this l-torsion subgroup coincides obviously with the image of $H^m_{et}(E, \mu_l^{\otimes m})$ in $H^m_{et}(E, \mathbb{Q}_l/\mathbb{Z}_l(m))$. The basic tool in showing that the image of any element $s_0 \in H^m_{et}(E, \mu_l^{\otimes m})$ in $H^m_{et}(E, \mathbb{Q}_l/\mathbb{Z}_l(m))$ may be lifted to $H^m_{Nis}(E, \mathbb{Q}_l/\mathbb{Z}_l(m))$ is provided by the following Lemma.

Lemma 7.7.3 ([S-V], Lemma 11.2). *Let E be an infinite field containing a primitive l-th root of unity. Then for any cohomology class $s_0 \in H^m_{et}(E, \mu_l^{\otimes m})$ there exists an integer N, a non empty open subscheme $U \subset \mathbb{A}^N_E$, an etale Galois covering $p : U \to V$ of degree N, a cohomology class $s \in H^m_{et}(V, \mu_l^{\otimes m})$ and two E-rational points $v_0, v_1 \in V$ such that*

1. *v_0 admits a lifting to an E-rational point $u_0 \in U$*
2. *$s(v_0) = 0, \quad s(v_1) = s_0$*

Proposition 7.7.4. *Under assumptions of Theorem 7.7 for any finitely generated field extension E/F the natural homomorphism*

$$H^m_{Nis}(E, \mathbb{Q}_l/\mathbb{Z}_l(m)) \xrightarrow{(\alpha_m)_*} H^m_{Nis}(E, B_{l^\infty}(m)) = H^m_{et}(E, \mathbb{Q}_l/\mathbb{Z}_l(m))$$

is surjective.

Proof. According to remarks made above it suffices to show that for any cohomology class $s_0 \in H^m_{et}(E, \mu_l^{\otimes m})$ its image in $H^m_{et}(E, \mathbb{Q}_l/\mathbb{Z}_l(m))$ may be lifted to $H^m_{Nis}(E, \mathbb{Q}_l/\mathbb{Z}_l(m))$. Find N, etale Galois covering $p : U \to V$ of degree N, the cohomology class $s \in H^m_{et}(V, \mu_l^{\otimes m})$ and two E-rational points $v_0, v_1 \in V$ as in Lemma 7.7.3. Factor next N in the form $N = l^k \cdot N_0$, where N_0 is prime to l. Applying Corollary 7.7.2 to the semilocalization of V at $\{v_0, v_1\}$ we see that shrinking V if necessary we may assume that the image of s in $H^m_{et}(V, \mathbb{Q}_l/\mathbb{Z}_l(m))$ is divisible by l^k i.e. $s = l^k \cdot t$ for an appropriate $t \in H^m_{et}(V, \mathbb{Q}_l/\mathbb{Z}_l(m))$. Consider an etale cohomology class $p^*(t) - p^*(t)(u_0) \in H^m_{et}(U/u_0, \mathbb{Q}_l/\mathbb{Z}_l(m))$, where $H^m_{et}(U/u_0, \mathbb{Q}_l/\mathbb{Z}_l(m))$ stands for the subgroup of $H^m_{et}(U, \mathbb{Q}_l/\mathbb{Z}_l(m))$ consisting of cohomology classes vanishing at u_0.

Lemma 7.7.5. *Let $U \subset \mathbb{A}^N_E$ be an open subscheme and let $u_0 \in U$ be an E-rational point. Then the natural homomorphism*

$$H^m_{Nis}(U/u_0, \mathbb{Q}_l/\mathbb{Z}_l(m)) \xrightarrow{(\alpha_m)_*} H^m_{Nis}(U/u_0, B_{l^\infty}(m)) = H^m_{et}(U/u_0, \mathbb{Q}_l/\mathbb{Z}_l(m))$$

is an isomorphism.

Proof. Denote by $Z = \mathbb{A}^N_E \setminus U$ the complement of U. The exact sequence

$$H^m_{Nis}(E) = H^m_{Nis}(\mathbb{A}^N_E) \to H^m_{Nis}(U) \to H^{m+1}_Z(\mathbb{A}^N_E) \to H^{m+1}_{Nis}(\mathbb{A}^N_E) \to \ \ldots$$

implies readily that

$$\begin{aligned} H^m_{Nis}(U/u_0, \mathbb{Q}_l/\mathbb{Z}_l(m)) &= H^{m+1}_Z(\mathbb{A}^N_E, \mathbb{Q}_l/\mathbb{Z}_l(m)) \\ H^m_{Nis}(U/u_0, B_{l^\infty}(m)) &= H^{m+1}_Z(\mathbb{A}^N_E, B_{l^\infty}(m)), \end{aligned}$$

so that our statement follows from Lemma 7.6.5 and Lemma 7.6.1 or rather its l^∞-version.

Lemma 7.7.5 gives us a cohomology class $r \in H^m_{Nis}(U/u_0, \mathbb{Q}_l/\mathbb{Z}_l(m))$ such that $\alpha_m(r) = p^*(t) - p^*(t)(u_0)$. Set $q = p_*(r) \in H^m_{Nis}(V, \mathbb{Q}_l/\mathbb{Z}_l(m))$. The image of q in $H^m_{Nis}(V, B_{l^\infty}(m)) = H^m_{et}(V, \mathbb{Q}_l/\mathbb{Z}_l(m))$ coincides with $N \cdot (t - t(v_0)) = N_0 \cdot (s - s(v_0)) = N_0 \cdot s$. This shows that s_0 may be lifted to $H^m_{Nis}(E, \mathbb{Q}_l/\mathbb{Z}_l(m))$ to a cohomology class $q_0 = N_1 \cdot q(v_1)$, where $N_1 = N_0^{-1} \in \mathbb{Z}_l$.

End of the Proof of the Theorem 7.7. Proposition 7.7.4 shows that for any finitely generated field extension E/F the natural homomorphism

$$H^m_{Nis}(E, \mathbb{Q}_l/\mathbb{Z}_l(m)) \xrightarrow{(\alpha_m)_*} H^m_{Nis}(E, B_{l^\infty}(m)) = H^m_{et}(E, \mathbb{Q}_l/\mathbb{Z}_l(m))$$

is surjective. Thus we may apply Proposition 7.6.4 and conclude that the canonical homomorphism of complexes of Nisnevich sheaves with transfers $\mathbb{Q}_l/\mathbb{Z}_l(m) \xrightarrow{\alpha_m} B_{l^\infty}(m)$ is a quasi-isomorphism. Finally to show that the Beilinson-Lichtenbaum Conjecture modulo l holds in weight m we use the Bockstein distinguished triangles - cf. Lemma 7.7.1

$$\begin{array}{ccccccc}
\mathbb{Z}/l(m) & \longrightarrow & \mathbb{Q}_l/\mathbb{Z}_l(m) & \xrightarrow{l} & \mathbb{Q}_l/\mathbb{Z}_l(m) & \longrightarrow & \mathbb{Z}_l(m)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B_l(m) & \longrightarrow & B_{l^\infty}(m) & \xrightarrow{l} & B_{l^\infty}(m) & \longrightarrow & B_l(m)[1]
\end{array}$$

Remark 7.8. *Using the same tricks as in § 6 it's not hard to get rid of the assumption of perfectness of the field F in the formulation of Theorems 7.6 and 7.7, however the present formulation is apparently sufficient for all practical purposes. The usual way to prove the Beilinson-Lichtenbaum Conjecture goes as follows. One proves the Conjecture (using Theorems 7.6 and 7.7) over the (perfect) prime field F_0 (equal to $\mathbb{Q}$ or $\mathbb{Z}/p$ depending on the characteristics). Using the limit argument one concludes next that the canonical map $H^*_{Nis}(E, \mathbb{Z}/l(m)) \to H^*_{Nis}(E, B_l(m))$ is an isomorphism for any field E. Now the usual properties of homotopy invariant Nisnevich sheaves with transfers imply readily that the conjecture holds over any field F.*

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