ON MORAVA K -THEORIES OF AN S -ARITHMETIC GROUP

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ABSTRACT. We completely describe the Morava K -theories with respect to the prime p for the étale model of the classifying space of $GL_m(\mathbb{Z}[\sqrt[p]{1}, 1/p])$ when p is an odd regular prime. For $p = 3$ and $m = 2$ (and conjecturally for $m = \infty$) these cohomologies are the same as those of the classifying space itself.

1. INTRODUCTION

By using an Eilenberg-Moore type spectral sequence, Tanabe calculated the Morava K -theories for the classifying spaces of certain Chevalley groups. In particular, if $K(n)$ is the n -th Morava K -theory with the ring of coefficients

$$K(n)^*(pt) = \mathbb{F}_p[v_n, v_n^{-1}]$$

where p is a prime and v_n has degree $2(p^n - 1)$, and if q is a power of a prime different from p , then [12]

$$(1.1) \quad K(n)^*BGL_m(\mathbb{F}_q) \approx K(n)^*BGL_m(\bar{\mathbb{F}}_q)_{\psi^q} \approx \frac{K(n)^*(pt)[[c_1, \dots, c_m]]}{(c_1 - \psi^q c_1, \dots, c_m - \psi^q c_m)}$$

i.e. a ring of formal power series in certain "Chern classes" $c_1, \dots, c_m$ modulo an ideal given in terms of generators. Here ψ^q is the "Adams operation" induced from the Frobenius automorphism $x \mapsto x^q$ of the algebraic closure $\bar{\mathbb{F}}_q$ of the field $\mathbb{F}_q$ with q elements. The same formula (1.1) holds for the p -adic version $\hat{K}(n)$ of $K(n)$ obtained by replacing $K(n)^*(pt)$ with $\hat{K}(n)^*(pt) = \mathbb{Z}_p[v_n, v_n^{-1}]$ where $\mathbb{Z}_p$ denotes the ring of p -adic integers [12].

On the other hand, if $A = \mathbb{Z}[\sqrt[p]{1}, 1/p]$ and p is a *regular* prime in the sense of number theory, then Dwyer and Friedlander [5, 6] calculated the mod p cohomology of a space $BGL_m(A_{\text{ét}})$ which is naturally associated to the classifying space $BGL_m(A)$ of the S -arithmetic group $GL_m(A)$. We call the space $BGL_m(A_{\text{ét}})$ the *étale model at p* for the classifying space $BGL_m(A)$ and recall that it is endowed with a natural map [4, 2.5]

$$(1.2) \quad f_A : BGL_m(A) \rightarrow BGL_m(A_{\text{ét}})$$

The goal of this article is to show how we can use these two calculations in order to completely describe the Morava K -theories with respect to the prime p of the étale model above. The main result is

Theorem 1.1. *If $A = \mathbb{Z}[\sqrt[p]{1}, 1/p]$ with p an odd regular prime, then the n -th Morava K -theory with respect to the prime p of the étale model $BGL_m(A_{\text{ét}})$ is an*

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exterior algebra given by the formula

$$K(n)^*BGL_m(A_{\text{ét}}) \approx K(n)^*BGL_m(\mathbb{F}_q)\langle\sigma_1, \dots, \sigma_m\rangle^{\otimes(p-1)/2}$$

where q is a prime $\equiv 1 \pmod{p}$ but $\not\equiv 1 \pmod{p^2}$, the tensor product is over the ring $K(n)^*BGL_m(\mathbb{F}_q)$, and σ_i has degree $2i - 3$ ($1 \leq i \leq m$). Moreover, the same formula holds for the p -adic version $\hat{K}(n)$.

In particular, if $m = \infty$ and $n = 1$ (and conjecturally for $n > 1$) the above theorem and (1.1) give $K(n)$ and $\hat{K}(n)$ theories of the classifying space $BGL_\infty(A)$ itself for p odd and regular, according to [7]. Here GL_∞ denotes the union of all GL_n for $n \geq 1$ with respect to the block inclusions.

Also, if $p = 3$ and $m = 2$, then we showed that the natural map (1.2) is a mod p equivalence [1]. Hence we deduce the following

Corollary 1.2. *The n -th Morava K -theory at the prime 3 of the S -arithmetic group $GL_2(\mathbb{Z}[\sqrt[3]{1}, 1/3])$ is given by*

$$K(n)^*BGL_2(\mathbb{Z}[\sqrt[3]{1}, 1/3]) \approx \frac{\mathbb{F}_3[v_n, v_n^{-1}][[a, c_2]]}{(a^{7^n}, c_2^{(7^n+1)/2} \pmod{a})} \langle\sigma_1, \sigma_2\rangle$$

where the degrees of the generators are $|v_n| = 2(3^n - 1)$, $|a| = 2$, $|c_2| = 4$, $|\sigma_1| = -1$, $|\sigma_2| = 1$, and the second generator of the ideal is up to an indeterminacy $\pmod{a}$. Moreover, a similar formula holds for the 3-adic version $\hat{K}(n)$.

Notation 1.3. In what follows p is an odd regular prime when not otherwise stated and $A = \mathbb{Z}[\zeta_p, 1/p]$ where $\zeta_p = \exp(2\pi i/p)$ is a prescribed p -th root of unity in the field $\mathbb{C}$ of complex numbers.

2. ÉTALE MODELS FOR CLASSIFYING SPACES

2.1. The original definition. Let $R = \mathbb{Z}[1/p]$, G a group scheme over $\text{Spec}(R)$, and BG the classifying simplicial scheme obtained by a bar construction as in [8, 1.2]. Then the classifying space $BG(D)$ of the group $G(D)$ of the D -points of G where D is any finitely generated R -algebra can be thought of as the connected component of a simplicial function complex [6, 1.4]

$$(2.1) \quad BG(D) = \text{Map}^0(\text{Spec}(D), BG)_{\text{Spec}(R)}$$

containing the natural base point induced by the unit map $\text{Spec}(R) \rightarrow G$ of G over $\text{Spec}(R)$. We recall that $\text{Map}(X, Y)_Z$ is a simplicial set given in dimension i by the set of simplicial scheme maps $X \otimes \Delta[i] \rightarrow Y$ over Z where X and Y are simplicial schemes over Z (a scheme is regarded as a constant simplicial scheme) and $\Delta[i]$ is the standard simplicial i -simplex. The tensor product between a simplicial scheme and a simplicial set is defined in [8, 1.1].

Also, we recall that the étale topological type $X_{\text{ét}}$ in the sense of Friedlander [8, 4.4] is a pro-space (i.e. inverse system of simplicial sets) which is naturally associated to a noetherian simplicial scheme X and reflects the étale cohomology of X . For any finitely generated R -algebra D , let $D_{\text{ét}}$ denote the étale topological type $\text{Spec}(D)_{\text{ét}}$. By replacing $\text{Spec}(D)$, BG , and $\text{Spec}(R)$ in (2.1) by their étale topological types $D_{\text{ét}}$, $(BG)_{\text{ét}}$, and $R_{\text{ét}}$, the space $BG(D_{\text{ét}})$ is defined in [6, 1.2] as the connected component of the simplicial complex of p -adic functions over $R_{\text{ét}}$

$$(2.2) \quad BG(D_{\text{ét}}) = \text{Hom}_p^0(D_{\text{ét}}, (BG)_{\text{ét}})_{R_{\text{ét}}}$$

containing the corresponding natural base point. This construction is similar to (2.1) and we can associate with each i -simplex of $BG(D)$ an i -simplex of $BG(D_{\text{ét}})$ regarded by definition as a map of pro-spaces over $R_{\text{ét}}$ from $D_{\text{ét}} \times \Delta[i]$ to the fibrewise p -adic completion of $(BG)_{\text{ét}}$ over $R_{\text{ét}}$ denoted by $(\mathbb{Z}/p)^{\bullet}(BG)_{\text{ét}}$ [4, 2.4]. This assignment is natural in both G and D and gives a map [4, 2.5]

$$f_D^G : BG(D) \rightarrow BG(D_{\text{ét}})$$

from the classifying space of the group $G(D)$ to its étale model $BG(D_{\text{ét}})$ at p . In the case when $G = GL_m$ is the group scheme over $\text{Spec} R$ corresponding to the general linear group $G(R) = GL_m(R)$ and $D = A$, we obtain the map (1.2). These definitions actually hold for *any* prime p .

2.2. A model structure definition. For convenience we will give an alternative way of thinking of (2.2) pointed out by Isaksen and based on its model structure. Namely, if $\text{pro-}\mathcal{SS}$ is the category of pro-spaces then there is a proper simplicial model structure on $\text{pro-}\mathcal{SS}$ introduced in [9]. This means that there are three classes of morphisms in $\text{pro-}\mathcal{SS}$ called *weak equivalences*, *cofibrations*, and *fibrations* subject to various axioms. Also there is a notion of *simplicial function complex* i.e. a natural assignment to each two pro-spaces X and Y of a simplicial set $\text{Map}(X, Y)$ interacting appropriately with the model structure [9, 16.2].

For the purpose of this paper we will use the induced proper simplicial model structure on the over-category $\text{pro-}\mathcal{SS}_V$ of pro-spaces over a fixed pro-space V . With respect to this model structure there is a *relative* simplicial function complex $\text{Map}(X, Y)_V$ naturally associated with every pair of objects X, Y in $\text{pro-}\mathcal{SS}_V$. Keeping the same notations as in the previous subsection we have the following

Proposition 2.1. *For any finitely generated R -algebra D , the space $BG(D_{\text{ét}})$ is weakly equivalent to the connected component of the natural base point of the simplicial function complex $\text{Map}(D_{\text{ét}}, T_p(BG)_{\text{ét}})_{R_{\text{ét}}}$ in the over-category of pro-spaces over $R_{\text{ét}}$,*

$$BG(D_{\text{ét}}) \simeq \text{Map}^0(D_{\text{ét}}, T_p(BG)_{\text{ét}})_{R_{\text{ét}}}$$

Here $T_p(BG)_{\text{ét}}$ is a fibrant replacement of $(\mathbb{Z}/p)^{\bullet}(BG)_{\text{ét}}$ over $R_{\text{ét}}$ in the sense of the simplicial model structure of [9].

Proof. Let $X = D_{\text{ét}} = \{X_{\alpha}\}$, $Y = (BG)_{\text{ét}}$, and $V = R_{\text{ét}}$. Then $Y \rightarrow V$ is a (strict) map of pro-spaces and let $T'_p(Y)$ be the level-space Moore-Postnikov tower naturally associated to the fibrewise p -adic completion of Y over V . Then we can think of $T'_p(Y) = \{T'_p(Y)_{\delta}\}$ as a pro-space over $V = \{V_{\delta}\}$ and by definition [4, 2.3]

$$(2.3) \quad \text{Hom}_p(X, Y)_V = \text{holim}_{\delta} \text{colim}_{\alpha} \text{Map}(X_{\alpha}, T'_p(Y)_{\delta})_{V_{\delta}}$$

where Map is the usual relative simplicial function complex of simplicial sets and holim denotes the homotopy inverse functor from pro-spaces to spaces [2, §6]. By [9, 10.6], the pro-space $T_p(Y)$ is the fibrant replacement of $T'_p(Y)$ in the model structure of Edwards-Hastings. By standard arguments, the space (2.3) is weakly equivalent to

$$\text{Map}(X, T_p(Y))_V = \text{lim}_{\delta} \text{colim}_{\alpha} \text{Map}(X_{\alpha}, T_p(Y)_{\delta})_{V_{\delta}}$$

and the conclusion follows from (2.2). $\square$

3. A HOMOTOPY FIBRE SQUARE

3.1. Preliminaries. We collect here a couple of known facts which will be used in the construction of a computable model for $BGL_m(A_{\text{ét}})$ given in the next subsection. This model is naturally associated to the action of $\pi_1(R_{\text{ét}})$ on the p -primary roots of unity.

Let D be a finitely generated normal R -algebra and $pt : \text{Spec}(k) \rightarrow \text{Spec}(D)$ a geometric point corresponding to a homomorphism from D to a separable closed field k . Then pt determines a base point of $D_{\text{ét}}$ and we recall that $\pi_1(D_{\text{ét}}, pt)$ is the pro-finite Grothendieck fundamental group of D pointed by pt [8, §5]. This group classifies finite étale covering spaces of $\text{Spec}(D)$.

Let μ_{p^ν} be the set of all complex numbers z such that $z^{p^\nu} = 1$ and μ_{p^∞} the union of all μ_{p^ν} for $\nu \geq 0$. Let R_∞ denote the ring obtained from R by adjoining the set μ_{p^∞} of all p -primary roots of unity,

$$R_\infty = R[\sqrt[p^\infty]{1}] = \mathbb{Z}[1/p, \mu_{p^\infty}],$$

and Γ the Galois (pro-)group

$$\Gamma = \text{Gal}(R_\infty, R) = \{\text{Aut}(\mu_{p^\nu}), \nu \geq 1\} \approx \{(\mathbb{Z}/p^\nu)^*, \nu \geq 1\}$$

In this context, observe that $\pi_1(R_{\text{ét}})$ is the Galois group of the maximal unramified extension of R and let

$$(3.1) \quad \theta : \pi_1(R_{\text{ét}}) \rightarrow \Gamma$$

be the homomorphism given by the action of this Galois group on the p -primary roots of unity. In other words, $R_{\text{ét}}$ is provided with the natural structure map $R_{\text{ét}} \rightarrow K(\Gamma, 1)$ which "classifies" the finite étale extensions $R \rightarrow R[\mu_{p^\nu}]$. Also, $A_{\text{ét}}$ is provided with a natural structure map

$$A_{\text{ét}} \rightarrow R_{\text{ét}} \rightarrow K(\Gamma, 1)$$

If k is a field, then $k_{\text{ét}}$ is a pro-space of type $K(\pi, 1)$, where π is the Galois group over k of the separable algebraic closure of k . In particular, $\mathbb{C}_{\text{ét}}$ is contractible and $(\mathbb{F}_q)_{\text{ét}}$ is equivalent to the pro-finite completion of a circle. If $R \rightarrow \mathbb{F}_q$ is a residue field map, then $(\mathbb{F}_q)_{\text{ét}}$ is provided with a natural structure map

$$(\mathbb{F}_q)_{\text{ét}} \rightarrow R_{\text{ét}} \rightarrow K(\Gamma, 1)$$

as well. This structure map sends the Frobenius element of the Galois group of $\bar{\mathbb{F}}_q$ over $\mathbb{F}_q$ identified with $\pi_1((\mathbb{F}_q)_{\text{ét}})$ to $q \in \text{Aut}(\mu_{p^\nu}) \cong (\mathbb{Z}/p^\nu)^*$ in Γ [6, 3.2].

3.2. A homotopy fibre square. Let U_m be the Lie group of $m \times m$ unitary matrices and $\hat{B}U_m$ the p -completion of its classifying space. The following proposition is the unstable analogue of [5, 4.5] and its proof is almost the same. For convenience, we review here the main arguments.

Proposition 3.1. *Let p be an odd regular prime, $A = \mathbb{Z}[\zeta_p, 1/p]$, and q a rational prime $\equiv 1 \pmod{p}$ but $\not\equiv 1 \pmod{p^2}$. Then there is a homotopy fibre square*

$$\begin{array}{ccc} BGL_m(A_{\text{ét}}) & \longrightarrow & \hat{B}U_m^W \\ \downarrow & & \downarrow \\ \hat{B}GL_m(\mathbb{F}_q) & \longrightarrow & \hat{B}U_m \end{array}$$

where W is the wedge of $(p-1)/2$ circles, $\hat{BU}_m^W$ denotes the simplicial function complex of unpointed maps from W to $\hat{BU}_m$, and the right-hand vertical map is the evaluation at the base-point.

Proof. As in [5, p. 145] we construct a map

$$(\mathbb{F}_q)_{\text{ét}} \vee W \rightarrow K(\Gamma, 1)$$

by sending the first summand via the natural structure map and mapping the other summand trivially. By a class-field argument (assuming the properties of q from hypothesis), there exists a map

$$g : (\mathbb{F}_q)_{\text{ét}} \vee W \rightarrow A_{\text{ét}}$$

over $K(\Gamma, 1)$ which is a mod p cohomology equivalence [5, p. 145]. In other words g is a "good mod p model" for A in the sense of [6, 1.9]. This means that by a spectral sequence argument [4, 2.11] and using 2.1 for $G = GL_m$ the map g induces a homotopy equivalence

$$\text{Map}^0(A_{\text{ét}}, T_p(BGL_m)_{\text{ét}})_{R_{\text{ét}}} \simeq \text{Map}^0((\mathbb{F}_q)_{\text{ét}} \vee W, T_p(BGL_m)_{\text{ét}})_{R_{\text{ét}}}$$

which can be reformulated by saying that we get a homotopy fibre square

$$\begin{array}{ccc} BGL_m(A_{\text{ét}}) & \longrightarrow & \text{Map}^0(W, T_p(BGL_m)_{\text{ét}})_{R_{\text{ét}}} \\ \downarrow & & \downarrow \\ BGL_m((\mathbb{F}_q)_{\text{ét}}) & \longrightarrow & \text{Map}^0(pt, T_p(BGL_m)_{\text{ét}})_{R_{\text{ét}}} \end{array}$$

where the right-hand vertical map is the evaluation at the base-point (the map $pt \rightarrow R_{\text{ét}}$ is induced from $R \subset \mathbb{C}$ recalling that $\mathbb{C}_{\text{ét}}$ is contractible). To finish the proof, we need only to identify the appropriate corners of this square.

For the two right-hand corners, we start with the fibration sequence [3, 2.3]

$$(3.2) \quad \{(\mathbb{Z}/p)_s(BGL_{m, \mathbb{F}_q})_{\text{ét}}\}_s \rightarrow (\mathbb{Z}/p)^\bullet(BGL_m)_{\text{ét}} \rightarrow R_{\text{ét}}$$

where $\{(\mathbb{Z}/p)_s(-)\}_s$ denotes the Bousfield-Kan p -completion tower and $BGL_{m, \mathbb{F}_q}$ is the classifying object of GL_m over $\mathbb{F}_q$. Hence we get that

$$\text{Map}^0(pt, T_p(BG)_{\text{ét}})_{R_{\text{ét}}} \simeq \text{holim}\{(\mathbb{Z}/p)_s BGL_{m, \mathbb{F}_q}\}_{\text{ét}} \simeq \hat{BU}_m$$

where the last equivalence is proved in [8, 8.8]. Because the composite map

$$\pi_1(W) \rightarrow \pi_1(A_{\text{ét}}) \rightarrow \pi_1(R_{\text{ét}}) \xrightarrow{\theta} \Gamma$$

is trivial by construction, as in [4, p. 146] we get a homotopy equivalence

$$\text{Map}^0(W, T_p(BGL_m)_{\text{ét}})_{R_{\text{ét}}} \simeq \hat{BU}_m^W$$

where $\hat{BU}_m^W$ denotes the function complex of unpointed maps from W to $\hat{BU}_m$ (basically $\pi_1(R_{\text{ét}})$ acts on the fibre of (3.2) via θ).

Finally, for the lower left-hand corner, there is a homotopy equivalence

$$\hat{BGL}_m(\mathbb{F}_q) \simeq BGL_m((\mathbb{F}_q)_{\text{ét}})$$

given in [3, 2.11] by exploiting the action of the Frobenius element on the fibre of (3.2) via the composite

$$\pi_1((\mathbb{F}_q)_{\text{ét}}) \rightarrow \pi_1(R_{\text{ét}}) \xrightarrow{\theta} \Gamma$$

and Quillen's homotopy fix point description of $\hat{BGL}_m(\mathbb{F}_q)$ [10]. $\square$

4. THE PROOF OF THE MAIN THEOREM AND ITS COROLLARY

4.1. **Proof of 1.1.** The proof of the main theorem is based on Strickland's analysis of unitary bundles in [11] applied to the homotopy fibre square 3.1.

Let V be a complex vector bundle over a space X and write PV for the associated bundle of projective spaces and $U(V)$ for the associated bundle of unitary groups

$$U(V) = \{(x, g) | x \in X \text{ and } g \in U(V_x)\}$$

Let $EU(V)$ denote the geometric realization of the simplicial space $\{U(V)^{n+1}\}_{n \geq 0}$ and put $BU(V) = EU(V)/U(V)$ the usual simplicial model for the classifying space of $U(V)$.

Let E^* be an even periodic cohomology theory with complex orientation $x \in \tilde{E}^0 \mathbb{C}P^\infty$. We are interested in describing $E^*U(V)$ as a Hopf algebra over E^*X (using the group structure on $U(V)$). The main result involves the exterior algebra over the ring E^*X generated by the module E^*PV which we denote by $\lambda_{E^*X}^* E^{*-1}PV$ and which is a Hopf algebra over E^*X by declaring E^*PV to be primitive.

Proposition 4.1 ([11, 4.4]). *There is a natural isomorphism of Hopf algebras over E^*X*

$$\lambda_{E^*X}^* E^{*-1}PV \approx E^*U(V)$$

We apply this proposition to the tautological bundle

$$V = \gamma_m = EU_m \times_{U_m} \mathbb{C}^m$$

over $X = BU_m$. In this case, we have

$$E^*P\gamma_m \approx \frac{E^*BU_m[x]}{(x^m + c_1x^{m-1} + \dots + c_m)} \approx E^*BU_m\{1, \dots, x^{m-1}\}$$

where c_i is the i -th Chern class of γ_m and the last isomorphism indicates that $E^*P\gamma_m$ is a free module over E^*BU_m with basis $1, x, \dots, x^{m-1}$. In particular,

$$(4.1) \quad E^*U(\gamma_m) \approx \lambda_{E^*BU_m}^* E^{*-1}P\gamma_m \approx E^*BU_m\langle\sigma(1), \dots, \sigma(x^{m-1})\rangle$$

where σ lowers the degree by 1.

Going back to the homotopy fibre square 3.1, we observe that $\hat{B}U_m^W$ is the $(p-1)/2$ -fold fibre product of $\hat{B}U_m^{S^1} = \hat{U}(\gamma_m)$ over $\hat{B}U_m$ and for $E^* = K(n)$ or $\hat{K}(n)$ and any space X we have $E^*\hat{X} = E^*X$. In this case, if we apply E^* to $\hat{B}U_m^W$ and use (4.1) we obtain

$$\begin{aligned} E^*(\hat{B}U_m^W) &\approx (\lambda_{E^*BU_m}^* E^{*-1}P\gamma_m)^{\otimes (p-1)/2} \\ &\approx E^*BU_m\langle\sigma(1), \dots, \sigma(x^{m-1})\rangle^{\otimes (p-1)/2} \end{aligned}$$

where the tensor product is over E^*BU_m . In particular, $E^*(\hat{B}U_m^W)$ is a free module over $E^*\hat{B}U_m = E^*BU_m$ and therefore 1.1 follows from the above formula by a base-change induced from 3.1:

$$E^*BGL_m(A_{\text{ét}}) \approx E^*\hat{B}GL_m(\mathbb{F}_q) \otimes_{E^*\hat{B}U_m} E^*(\hat{B}U_m^W)$$

where q can be always chosen with the prescribed properties (by Dirichlet's density theorem for instance).

4.2. Proof of 1.2. This has been already explained in the Introduction, except for the analysis of the formula (1.1) in the case $p = 3$, $m = 2$, and $q = 7$. The goal of this subsection is to complete this analysis.

Proposition 4.2. *Let p be an odd rational prime and $\mathbb{F}_q$ a finite field with q elements such that $q \equiv 1 \pmod{p^r}$ but $\not\equiv 1 \pmod{p^{r+1}}$ for some integer $r > 0$. Then*

$$K(n)^* BGL_2(\mathbb{F}_q) \approx \frac{K(n)^*(pt)[[a, c_2]]}{(a^{p^{nr}}, c_2^{(p^{nr}+1)/2} \pmod{a})}$$

Proof. According to Tanabe's formula

$$(4.2) \quad K(n)^* BGL_2(\mathbb{F}_q) \approx (K(n)^* BU_2)_\psi$$

where the co-invariants are calculated with respect to the q -th Adams operation ψ [12]. Recall that

$$(4.3) \quad K(n)^* BU_2 \approx K(n)^*(pt)[[c_1, c_2]]$$

where $c_1 = x + y$ and $c_2 = xy$ are expressed in terms of the generators of the ring

$$K(n)^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \approx K(n)^*(pt)[[x, y]]$$

which are induced by a complex orientation on $K(n)^*(\mathbb{C}P^\infty)$ [12, 2.12]. It is easy to see that we can replace c_1 in (4.3) by the formal group sum $a = x + {}^{K(n)}y$ of x and y (induced from the tensor product of complex line bundles). Then the proposition follows from (4.2) and the following lemmas

Lemma 4.3. $a - \psi(a) = (unit) \times a^{p^{nr}}$

Lemma 4.4. $c_2 - \psi(c_2) \equiv (unit) \times c_2^{(p^{nr}+1)/2} \pmod{a}$

where "unit" means invertible element in (4.3). □

Proof of 4.3. Let us expand q in the ring $\mathbb{Z}_p$ of p -adic integers as

$$q = \sum_{k=0}^{\infty} \alpha_k p^k$$

where the coefficients $\alpha_k \in \mathbb{Z}$ are subject to $0 \leq \alpha_k < p$, $\alpha_0 = 1$, $\alpha_r \neq 0$, and $\alpha_k = 0$ for $0 < k < r$. Then for $t = x$ or y we have

$$\psi(t) = [q](t) = \sum {}^{K(n)}[\alpha_k](t^{p^{nk}}) = t + \alpha_r t^{p^{nr}} + \dots$$

where $[q](t)$ means the formal group q -multiple of t . Hence,

$$\psi(a) = \psi(x) + {}^{K(n)}\psi(y) = \sum {}^{K(n)}[\alpha_k](a^{p^{nk}}) = a + \alpha_r a^{p^{nr}} + \dots$$

and the conclusion follows. □

Proof of 4.4. With the same notations as in the previous proof, we have

$$x \equiv [-1](y) \equiv -y \pmod{a}$$

and hence

$$c_2 - \psi(c_2) \equiv -x^2 + \psi(x)\psi(y) \equiv (unit) \times x^{p^{nr}+1} \equiv (unit) \times c_2^{(p^{nr}+1)/2} \pmod{a}$$

□

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