BIRATIONAL MOTIVES, I

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Preliminary version

Contents

Introduction

In this article, we try to conciliate two important ideas in algebraic geometry: motives and birational geometry.

There are many reasons to do this; the one which motivated us originally was to understand unramified cohomology from a motivic point of view. Although this is not yet achieved in this paper (but see (2) at the end of this introduction), it led us to rather large developments and surprising structure theorems. For an application of an elementary part of our theory to function fields over finite fields, see [18].

In order to give the reader a brief resumé of our results, we assume familiarity with Voevodsky's construction of his triangulated categories of motives and recall the naturally commutative diagram of categories

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[32, 34], where F is a perfect field

$$\begin{CD}
\operatorname{Sm}(F) @>>> \operatorname{SmCor}(F) @>>> DM^{\text{eff}}_{\text{gm}}(F) @>{ff}>> DM^{\text{eff}}_{-}(F)\\
@A{ff}AA @A{ff}AA @A{ff}AA \\
\operatorname{Sm}_{\text{proj}}(F) @>>> \operatorname{SmCor}_{\text{proj}}(F) @>>> \operatorname{Chow}^{\text{eff}}(F)
\end{CD}$$

in which ff means fully faithful. Here $\operatorname{Sm}(F)$ is the category of smooth connected F-varieties, $\operatorname{Sm}_{\text{proj}}(F)$ is its full subcategory consisting of smooth projective varieties, $\operatorname{SmCor}(F)$ is the category of smooth varieties with morphisms finite correspondences, $\operatorname{SmCor}_{\text{proj}}(F)$ its full subcategory consisting of smooth projective varieties, $\operatorname{Chow}^{\text{eff}}(F)$ the category of effective Chow motives, $DM^{\text{eff}}_{\text{gm}}(F)$ the triangulated category of effective geometrical motives and $DM^{\text{eff}}_{-}(F)$ the triangulated category of effective motivic complexes (for the Nisnevich topology). Then, for F of characteristic 0, we construct a diagram to which the former maps naturally

$$\begin{CD}
S_r^{-1}\operatorname{Sm}(F) @>>> S_r^{-1}\operatorname{SmCor}(F)^\natural @>{ff}>> DM^{\text{o}}_{\text{gm}}(F) @>{ff}>> DM^{\text{o}}_{-}(F)\\
@A{\wr}AA @A{\wr}AA @A{ff}AA \\
S_r^{-1}\operatorname{Sm}_{\text{proj}}(F) @>>> S_r^{-1}\operatorname{SmCor}_{\text{proj}}(F)^\natural @>{\sim}>> \operatorname{Chow}^{\text{o}}(F)\\
@AAA @. @V{\text{Alb}}VV \\
T^{-1}\operatorname{place}(F)^{\text{op}} @. @. \operatorname{AbS}(F)
\end{CD}$$

Here $\sim$ means an equivalence of categories, $^\natural$ denotes karoubian envelope (or pseudo-abelian envelope, or idempotent completion), S_r^{-1} and T^{-1} denote localisation with respect to certain sets of morphisms S_r and T, $\operatorname{place}(F)$ is the category of function fields over F with morphisms given by F-places, $\operatorname{Chow}^{\text{o}}(F)$ is the category of *birational Chow motives*, $DM^{\text{o}}_{\text{gm}}(F)$ the triangulated category of *birational geometrical motives*, $DM^{\text{o}}_{-}(F)$ the triangulated category of *rationally invariant motivic complexes* and $\operatorname{AbS}(F)$ the category of *locally abelian schemes*.

As a very special case, this diagram shows that any function field has a birational Chow motive which is natural with respect to F-places. It also shows that the situation is strikingly simpler in the birational case than in the "regular" case. In particular, the equivalence between a localisation of the category of finite correspondences on smooth projective varieties with that of birational Chow motives, although not one of the most difficult results of this paper, is crucial in our proof that the functor $\operatorname{Chow}^{\text{o}}(F) \to DM^{\text{o}}_{\text{gm}}(F)$ is fully faithful.

Let us now give a few more details on the various categories introduced above and some intuition of the situation. First, if one wants to make sense of a "birational category", one is confronted at the outset with two different ideas: use the notion of place of Zariski-Samuel [35] or use the geometric idea of a birational map. It turns out that both ideas work but give rather different answers.

The first idea gives the category $\operatorname{place}(F)$, that we like to call the *coarse birational category*. For the second idea, one has to be a little careful: the naïve attempt at taking as objects smooth varieties and as morphisms birational maps does not work because, as was pointed out to us by Hélène Esnault, one cannot compose birational maps in general. On the other hand, one can certainly start from the category $\operatorname{Sm}(F)$ and localise it with respect to the multiplicative set S_b of birational isomorphisms. We like to call the resulting category $S_b^{-1}\operatorname{Sm}(F)$ the *fine birational category*. By hindsight, the problem mentioned just above can be understood as a problem of calculus of fractions in $S_b^{-1}\operatorname{Sm}(F)$.

These definitions raise the following issues. First, in $\operatorname{place}(F)$, the Hom sets are very big. Second, in $\operatorname{Sm}(F)$, the set S_b does not admit calculus of left or right fractions in the sense of Gabriel-Zisman [12]. And finally, there is no obvious comparison functor between the coarse and the fine birational categories at this stage.

In order to answer these issues at least to an extent, we introduce an "incidence category" $\operatorname{SmP}(F)$, whose objects are smooth connected F-varieties and morphisms from X to Y are given by pairs (f, v), where f is a morphism $X \to Y$, v is a place $F(Y) \dashrightarrow F(X)$ and f, v are *compatible* in an obvious sense (see Definition 1.4 below). This category maps to both $\operatorname{place}(F)^{\mathrm{op}}$ and $\operatorname{Sm}(F)$ by obvious forgetful functors. Denote by $\operatorname{Sm}_{\mathrm{proj}}\operatorname{P}(F)$ the full subcategory of $\operatorname{SmP}(F)$ consisting of smooth projective varieties. Note that the set S_b lifts naturally to $\operatorname{SmP}(F)$ and restricts to $\operatorname{Sm}_{\mathrm{proj}}(F)$ and $\operatorname{Sm}_{\mathrm{proj}}\operatorname{P}(F)$ (same notation). Then:

Theorem 1 (*cf.* Theorem 3.8)**.** *Assume F of characteristic* 0*. Then localisation with respect to S_b yields a naturally commutative diagram of categories, in which vertical functors are equivalences of categories* *while horizontal functors are full and essentially surjective:*

$$\begin{array}{ccc}
S_b^{-1}\operatorname{Sm}_{\mathrm{proj}}\mathrm{P}(F) & \twoheadrightarrow & S_b^{-1}\operatorname{Sm}_{\mathrm{proj}}(F)\\
\downarrow\wr & & \downarrow\wr\\
S_b^{-1}\operatorname{SmP}(F) & \twoheadrightarrow & S_b^{-1}\operatorname{Sm}(F)\\
\downarrow\wr & & \\
\operatorname{place}(F)^{\mathrm{op}}. & &
\end{array}$$

In particular, we get a full and essentially surjective functor $\operatorname{place}(F)^{\mathrm{op}} \to S_b^{-1}\operatorname{Sm}(F)$, justifying the terminology of "coarse" and "fine" birational categories. In the course of the proof, we get that, at least, the first axiom of calculus of right fractions ("Ore condition") holds for S_b, which is by no means obvious *a priori* (see Proposition 3.2). Hence in $S_b^{-1}\operatorname{Sm}(F)$ any morphism may be written as a single fraction fs^{-1}, with f a regular map and $s \in S_b$. On the other hand, the second axiom definitely does not hold.

If X, Y are two connected smooth projective varieties over F, we get in this way a surjective map from the set of F-places from $F(Y)$ to $F(X)$ to the set of morphisms from X to Y in $S_b^{-1}\operatorname{Sm}_{\mathrm{proj}}(F)$. This defines a natural equivalence relation on the first set, which is stable under composition of places. Unfortunately we don't know how to describe this equivalence relation in concrete terms, and this appears to be a very challenging question.

If we now define T to be the multiplicative set of (trivial) places in $\operatorname{place}(F)$ given by purely transcendental extensions of function fields, and S_r as the multiplicative set of *rational morphisms* in $\operatorname{Sm}(F)$, *i.e.* dominant morphisms such that the corresponding extension of function fields is purely transcendental, then we may localise further the diagram of Theorem 1, getting part of the second diagram in this introduction (see Theorem 3.8). We call the corresponding categories the *coarse and fine stable birational categories*.

Having the above categories at hand is well and good but, as for usual algebraic geometry, it is very difficult to compute with them. So we apply the classical idea to enlarge morphisms by adding algebraic correspondences and making the categories additive.

In this programme, having Voevodsky's categories in mind, the first logical step (even if for us it came rather late in the development) is to extend part of Theorem 1 to finite correspondences, *i.e.* to study the functor

$$S_b^{-1}\operatorname{SmCor}_{\mathrm{proj}}(F) \to S_b^{-1}\operatorname{SmCor}(F)$$

where S_b denotes the image of S_b under the natural functor $\mathrm{Sm}(F) \to \mathrm{SmCor}(F)$, *i.e.* the multiplicative set of graphs of birational morphisms. In Proposition 4.1, we prove that this functor is an equivalence of categories if F has characteristic zero. (The issue is faithfulness.) Although the idea of the proof is rather simple (interpret finite correspondences as maps to symmetric powers as in [29]), details are not really straightforward and the proof takes several pages. It uses the category $\mathrm{SmP}(F)$ at a crucial step.

Although, in view of the above, the most natural definition of birational Chow motives would be to localise effective Chow motives with respect to S_r, this is not the way we proceed and we go through a rather more convoluted way, even though in the end our definition coincides with the one just outlined (see Corollary 7.3). We define the category $\mathrm{Chow}^o(F)$ as the pseudo-abelian envelope of the quotient of $\mathrm{Chow}^{\mathrm{eff}}(F)$ by the ideal I characterised as follows: for X, Y two smooth projective varieties, $I(h(X), h(Y))$ is the subgroup of $CH^{\dim Y}(X \times Y)$ formed of those correspondences whose restriction to $U \times Y$ is 0 for some dense open subset U of X. The fact that I is an ideal, *i.e.* is stable under left and right composition by any correspondence, is not obvious and amounts to a generalisation of the argument in [11, Ex. 16.1.11]. In fact, I is even a monoidal ideal, see Lemma 5.3, which means that the tensor structure of $\mathrm{Chow}^{\mathrm{eff}}(F)$ passes to $\mathrm{Chow}^o(F)$. In characteristic 0, I can be described more concretely as the set of those morphisms factoring through an object of the form $M \otimes L$, where L is the Lefschetz motive; in characteristic p, this remains true at least if one tensors all morphisms with $\mathbf{Q}$ (Lemma 5.4). Hence, to obtain birational Chow motives, we do something orthogonal to what is done habitually: instead of inverting the Lefschetz motive, we kill it!

This rather surprising picture becomes a little more natural if we consider the parallel one for Voeovdsky's triangulated motives. To get $DM^o_{\mathrm{gm}}(F)$, we simply invert open immersions (or, equivalently, the image of S_b in $DM^{\mathrm{eff}}_{\mathrm{gm}}(F)$) and add projectors. The resulting category is a tensor triangulated category. It is easy to see, only assuming F perfect, that $DM^o_{\mathrm{gm}}(F) = DM^{\mathrm{eff}}_{\mathrm{gm}}(F)/DM^{\mathrm{eff}}_{\mathrm{gm}}(F)(1)$ (Proposition 5.2 b)); this is made intuitive by thinking of the Gysin exact triangles. Then one easily sees that the functor $\mathrm{Chow}^{\mathrm{eff}}(F) \to DM^{\mathrm{eff}}_{\mathrm{gm}}(F)$ of Voevodsky induces a functor $\mathrm{Chow}^o(F) \to DM^o_{\mathrm{gm}}(F)$.

The main result of this paper is the computation of $Hom(\bar{M}(X), \bar{M}(Y)[i])$ for two smooth projective varieties X, Y and $i \in \mathbf{Z}$, where $\bar{M}(X)$ and $\bar{M}(Y)$ denote their motives in $DM^o_{\mathrm{gm}}(F)$. It may be stated as follows:

Theorem 2 (*cf.* Cor. 7.9)**.** *a) The functor* $\mathrm{Chow}^{\mathrm{o}}(F) \to DM^{\mathrm{o}}_{\mathrm{gm}}(F)$ *is fully faithful.*
b) For X, Y, i *as above, we have*

$$Hom(\bar{M}(X), \bar{M}(Y)[i]) = \begin{cases} CH_0(Y_{F(X)}) & \text{if } i = 0 \\ 0 & \text{if } i \neq 0. \end{cases}$$

The proof of this theorem is rather intricate. We follow the method Voevodsky used to compute Hom groups in $DM^{\mathrm{eff}}_{\mathrm{gm}}(F)$: we introduce another category $DM^{\mathrm{o}}_{-}(F)$ of "rationally invariant motivic complexes" and construct a functor $DM^{\mathrm{o}}_{\mathrm{gm}}(F) \to DM^{\mathrm{o}}_{-}(F)$ that we show to be fully faithful. In fact the definition of $DM^{\mathrm{o}}_{-}(F)$ is simple: it is the full subcategory of $DM^{\mathrm{eff}}_{-}(F)$ consisting of those objects C such that $\mathbb{H}^i_{\mathrm{Nis}}(X, C) \xrightarrow{\sim} \mathbb{H}^i_{\mathrm{Nis}}(U, C)$ for any dense open immersion of smooth schemes $U \to X$. So $DM^{\mathrm{o}}_{-}(F)$ is defined naturally as a subcategory of $DM^{\mathrm{eff}}_{-}(F)$, while $DM^{\mathrm{o}}_{\mathrm{gm}}(F)$ is defined as a quotient of $DM^{\mathrm{eff}}_{\mathrm{gm}}(F)$. This is not surprising, as $DM^{\mathrm{eff}}_{-}(F)$ is a subcategory of the category of functors from $DM^{\mathrm{eff}}_{\mathrm{gm}}(F)$ to abelian groups. However, we have to show that the embedding

$$i : DM^{\mathrm{o}}_{-}(F) \to DM^{\mathrm{eff}}_{-}(F) \tag{1}$$

has a *left adjoint* $\nu_{\leq 0}$. This is done fairly easily in Lemma 6.3. The fact that Voevodsky's full embedding $DM^{\mathrm{eff}}_{\mathrm{gm}}(F) \to DM^{\mathrm{eff}}_{-}(F)$ descends to a full embedding $DM^{\mathrm{o}}_{\mathrm{gm}}(F) \to DM^{\mathrm{o}}_{-}(F)$ is then formal (Theorem 6.4).

At this point, it remains to compute the group $Hom(\bar{M}(X), \bar{M}(Y)[i])$ within $DM^{\mathrm{o}}_{-}(F)$. This turns out to be very delicate and we can only refer the reader to Section 7 for the proof.

We end this paper by relating the previous constructions to more classical objects. We define a tensor additive category $\mathrm{AbS}(F)$ of *locally abelian schemes*, whose objects are those F-group schemes that are extensions of a lattice (*i.e.* locally isomorphic for the étale topology to a free finitely generated abelian group) by an abelian variety. We then show that the classical construction of the Albanese variety of a smooth projective variety extends to a tensor functor

$$\mathrm{Alb} : \mathrm{Chow}^{\mathrm{o}}(F) \to \mathrm{AbS}(F)$$

which becomes full and essentially surjective after tensoring morphisms by $\mathbf{Q}$ (Proposition 9.2). So, one could say that $\mathrm{AbS}(F)$ is the *representable part* of $\mathrm{Chow}^{\mathrm{o}}(F)$. We also show that (after tensoring with $\mathbf{Q}$ Alb has a right adjoint-right inverse, which identifies $\mathrm{AbS}(F) \otimes \mathbf{Q}$

with the thick subcategory of $\mathrm{Chow}^{\mathrm{o}}(F) \otimes \mathbf{Q}$ generated by motives of varieties of dimension ≤ 1.

This work is only the beginning of our investigations on birational motives. Let us just mention for the moment future lines of research:

(1) Structure of $DM^{\mathrm{o}}_{\mathrm{gm}}(F)$. In the course of the proof of theorem 2, we realise $\mathrm{Chow}^{\mathrm{o}}(F)$ as the subcategory of compact objects in the abelian category $HI^{o}(F)$ of *rationally invariant Nisnevich sheaves with transfers* (Propositions 7.2 and 7.4). In particular, this describes $\mathrm{Chow}^{\mathrm{o}}(F)$ as an exact subcategory of an abelian category. What is the relationship between $DM^{\mathrm{o}}_{\mathrm{gm}}(F)$ and the bounded derived category of $\mathrm{Chow}^{\mathrm{o}}(F)$?

(2) Where does unramified cohomology enter this picture? Expected answer: it should "be" a right adjoint to the functor i of (1).

(3) We hope to use this formalism to study the unramified cohomology of BG, where G is a linear algebraic group.

(4) Is it true that $i(DM^{\mathrm{o}}_{\mathrm{gm}}(F)) \subset DM^{\mathrm{eff}}_{\mathrm{gm}}(F)$ and that $i(\mathrm{Chow}^{\mathrm{o}}(F)) \subset \mathrm{Chow}^{\mathrm{eff}}(F)$? This question was suggested by Luca Barbieri-Viale and is expected to be difficult to answer: it is closely related to a conjecture of Voevodsky [31, Conj. 0.0.11]. We expect this to be true after tensoring with $\mathbf{Q}$. At the very least, the second statement becomes true (and easy) if one replaces Chow motives by numerical motives, *cf.* [18, Prop. 1].

(5) The localisation of the Morel-Voevodsky $\mathbf{A}^1$-homotopy category of schemes $\mathcal{H}(F)$ [21] with respect to S_r should be studied, as well as that of Voevodsky's effective stable $\mathbf{A}^1$-homotopy category of schemes. For the latter, this is very likely any "chunk" of the slice filtration of [33]; this analogy was pointed out to us by Chuck Weibel.

(6) The relationship between these ideas and the proofs that the Bloch-Kato conjecture implies the Beilinson-Lichtenbaum conjecture [30, 13, 14, 17] should definitely be investigated: a start is given in [17, Def. 2.14 and Lemma 2.15]. The best context for this seems to be in the previous item.

(7) Equally the relationship with Déglise's category of generic motives [6, 7]: at this point it is not completely clear what this relationship is.

(8) Finally, the exact relationship between the category $\mathrm{Chow}^{\mathrm{o}}(F)$ and Beilinson's "correspondences at the generic point" [3] should be investigated as well.

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Notation. F is the base field. All varieties are F-varieties and all morphisms are F-morphisms. If X is irreducible, η_X denotes its generic point.

1. Places and morphisms

1.1. **Definition.** Let K/F and L/F be two extensions. An *F-place* from K to L is a pair v formed of a valuation ring $\mathcal{O}$ of K/F (*i.e.* $F \subseteq \mathcal{O} \subseteq K$) and an F-homomorphism $M \to L$, where M is the residue field of $\mathcal{O}$. We write $\mathcal{O} = \mathcal{O}_v$ and $M = F(v)$.
Composition of places defines the category place(F) with objects finitely generated extensions of F and morphisms F-places.

1.2. **Remark.** If $v : K \dashrightarrow L$ is a morphism in place(F), then its residue field $F(v)$ is finitely generated over F, as a subfield of the finitely generated field L. On the other hand, given a finitely generated extension K/F, there exist valuation rings of K/F with infinitely generated residue fields as soon as $\operatorname{trdeg}(K/F) > 1$, *cf.* [35, Ch. VI, §15, Ex. 4].

1.3. **Lemma.** *Let $f, g : X \to Y$ be two morphisms, with X integral and Y separated. Then $f = g$ if and only if $f(\eta_X) = g(\eta_X) =: y$ and f, g induce the same map $F(y) \to F(X)$ on the residue fields.*

Proof. Let $\varphi : X \to Y \times_F Y$ be given by (f, g). Then the diagonal Δ_Y is closed, so $\operatorname{Ker}(f, g) = \varphi^{-1}(\Delta_Y)$ is closed in X and contains η_X (compare [EGA, Ch. I, Prop. 5.1.5 and Cor. 5.1.6]). □

1.4. **Definition.** Let X, Y be two integral varieties, with Y separated, $f : X \to Y$ a morphism and $v : F(Y) \dashrightarrow F(X)$ a place. We say that f and v are *compatible* if

- v is finite on Y (*i.e.* has a centre in Y).
- The corresponding diagram

$$\begin{array}{ccc} \eta_X & \xrightarrow{v^*} & \operatorname{Spec} \mathcal{O}_v \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

commutes.

1.5. **Remark.** Any morphism which is a birational isomorphism is compatible with the identical place. This applies in particular to open immersions.

1.6. **Proposition.** *Let X, Y, v be as in Definition 1.4. Suppose that v is finite on Y, and let $y \in Y$ be its centre. Then a morphism $f : X \to Y$ is compatible with v if and only if*

- *$y = f(\eta_X)$ and*
- *the diagram of fields*

$$\begin{array}{ccc} F(v) & & \\ \uparrow & \searrow^{v} & \\ F(y) & \xrightarrow{f^*} & F(X) \end{array}$$

commutes.

In particular, there is at most one such f.

Proof. Suppose v and f compatible. Then $y = f(\eta_X)$ because $v^*(\eta_X)$ is the closed point of $\operatorname{Spec} \mathcal{O}_v$. The commutation of the diagram then follows from the one in Definition 1.4. Conversely, if f verifies the two conditions, then it is obviously compatible with v. The last assertion follows from Lemma 1.3. □

1.7. **Corollary.** *a) Let Y be an integral variety, and let $\mathcal{O}$ be a valuation ring of $F(Y)/F$ with residue field K and centre $y \in Y$. Assume that $F(y) \xrightarrow{\sim} K$. Then, for any morphism $f : X \to Y$ with X integral, such that $f(\eta_X) = y$, there exists a unique place $v : F(Y) \dashrightarrow F(X)$ with valuation ring $\mathcal{O}$ which is compatible with f.*
b) If f is an immersion, the condition $F(y) \xrightarrow{\sim} K$ is also necessary for the existence of v. □

The following lemma generalises Remark 1.5:

1.8. **Lemma.** *Let $f : X \to Y$ be dominant. Then f is compatible with the trivial place $F(Y) \hookrightarrow F(X)$, and this place is the only one with which f is compatible.*

Proof. This follows immediately from Proposition 1.6. □

1.9. **Proposition.** *Let $f : X \to Y$, $g : Y \to Z$ be two morphisms of integral separated varieties. Let $v : F(Y) \dashrightarrow F(X)$ and $w : F(Z) \dashrightarrow F(Y)$ be two places. Suppose that f and v are compatible and that g and w are compatible. Then $g \circ f$ and $v \circ w$ are compatible.*

Proof. We first show that $v \circ w$ is finite on Z. By definition, the diagram

$$\begin{array}{ccc} \eta_Y & \xrightarrow{w^*} & \operatorname{Spec} \mathcal{O}_w \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathcal{O}_v & \longrightarrow & \operatorname{Spec} \mathcal{O}_{v\circ w} \end{array}$$

is cocartesian. Since the two compositions

$$\eta_Y \xrightarrow{w^*} \operatorname{Spec} \mathcal{O}_w \to Z$$

and

$$\eta_Y \to \operatorname{Spec} \mathcal{O}_v \to Y \xrightarrow{g} Z$$

coincide (by the compatibility of g and w), there is a unique induced (dominant) map $\operatorname{Spec} \mathcal{O}_{v\circ w} \to Z$. In the diagram

$$\begin{array}{ccccc} \eta_X & \xrightarrow{v^*} & \operatorname{Spec} \mathcal{O}_v & \longrightarrow & \operatorname{Spec} \mathcal{O}_{v\circ w} \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

the left square commutes by compatibility of f and v, and the right square commutes by construction. Therefore the big rectangle commutes, which means that $g \circ f$ and $v \circ w$ are compatible. □

1.10. **Definition.** We denote by $\mathrm{SmP}(F)$ the following category:

- Objects are smooth F-schemes of finite type.
- Let $X, Y \in \mathrm{SmP}(F)$. A morphism $\varphi \in \mathrm{SmP}(X, Y)$ is a pair (v, f) with $f : X \to Y$, $v : F(Y) \dashrightarrow F(X)$ and v, f compatible.
- The composition of morphisms is given by Proposition 1.9.

We denote by $\mathrm{Sm}_{\mathrm{proj}} \mathrm{P}(F)$ the full subcategories of $\mathrm{SmP}(F)$ consisting of smooth projective varieties.

1.11. **Lemma.** *Let $f : X \to Y$ be a morphism from an integral variety to a regular variety. Then there is a place $v : F(Y) \dashrightarrow F(X)$ compatible with f.*

Proof. Let $y = f(\eta_X)$. The local ring $A = \mathcal{O}_{Y,y}$ is regular: By Corollary 1.7 a), it is sufficient to produce a valuation ring $\mathcal{O}$ containing A and with the same residue field as A.

The following construction is certainly classical. Let $\mathfrak{m}$ be the maximal ideal of A and let $(x_1, \dots, x_d)$ be a regular sequence generating $\mathfrak{m}$, with $d = \dim A = \operatorname{codim}_Y y$. For $0 \le i < j \le d+1$, let $A_{i,j} = A[x_1^{-1}, \dots, x_i^{-1}]/(x_j, \dots, x_d)$ (for $i = 0$ we invert no x_k, and for $j = d+1$ we mod out no x_k). Then, for any (i, j), $A_{i,j}$ is a regular local ring of dimension $j-i-1$. In particular, $F_i = A_{i,i+1}$ is the residue field of $A_{i,j}$ for any $j \geq i+1$. We have $A_{0,d+1} = A$ and there are obvious maps

$$\begin{aligned} A_{i,j} &\to A_{i+1,j} \quad \text{(injective)} \\ A_{i,j} &\to A_{i,j-1} \quad \text{(surjective).} \end{aligned}$$

Consider the discrete valuation v_i associated to the discrete valuation ring $A_{i,i+2}$: it defines a place, still denoted by v_i, from F_{i+1} to F_i. The composition of these places is a place v from $F_d = F(Y)$ to $F_0 = F(y)$, whose valuation ring dominates A and whose residue field is clearly $F(y)$. □

1.12**. Lemma.** *Suppose F perfect. Let* place(F) *be the category of function fields over F with morphisms the F-places, and let* Sm(F) *be the category of integral separated F-schemes of finite type. There are forgetful essentially surjective functors*

$$\begin{array}{ccc} \mathrm{SmP}(F) & \xrightarrow{\Phi_1} & \mathrm{Sm}(F) \\ {\scriptstyle\Phi_2}\big\downarrow & & \\ \mathrm{place}(F)^{\mathrm{op}} & & \end{array}$$

with Φ_1 full and Φ_2 faithful. The restriction of Φ_2 to $\mathrm{Sm}_{\mathrm{proj}}\,\mathrm{P}(F)$ *is essentially surjective when F is of characteristic* 0*.*

Proof. The definitions and essential surjectivity of Φ_1 and Φ_2 are obvious. The restricted case of essential surjectivity for Φ_2 is clear by Hironaka's resolution of singularities. The fullness of Φ_1 follows from Lemma 1.11 and the faithfulness of Φ_2 follows from Proposition 1.6. □

1.13**. Lemma.** *Let Z, Z' be two models of a function field K, with Z' separated, and v a place of K with centres z, z' respectively on Z and Z'. Assume that there is a morphism $g : Z \to Z'$ which is a birational isomorphism. Then $g(z) = z'$.*

Proof. Let $f : \operatorname{Spec} \mathcal{O}_v \to Z$ be the dominant map determined by z. Then $f' = g \circ f$ is a dominant map $\operatorname{Spec} \mathcal{O}_v \to Z'$. By the valuative criterion of separatedness, it must correspond to z'. □

1.14. **Lemma.** *Consider a diagram*

$$\begin{array}{ccc} & & Z \\ & \overset{f}{\nearrow} & \\ X & & \big\downarrow g \\ & \underset{f'}{\searrow} & \\ & & Z' \end{array}$$

with g a birational isomorphism. Let $K = F(X)$, $L = F(Z) = F(Z')$ and suppose given a place $v : L \dashrightarrow K$ compatible both with f and f'. Then $f' = g \circ f$.

Proof. This follows from Proposition 1.6 and Lemma 1.13. □

1.15. **Lemma.** *Let X, Y, v, y be as in Proposition 1.6. Then there exists an open subset $U \subseteq X$ and a morphism $f : U \to Y$ compatible with v.*

Proof. Let $V = \operatorname{Spec} R$ be an affine neighbourhood of y in Y, so that $R \subset \mathcal{O}_v$, and let S be the image of R in $F(v)$. Choose a finitely generated F-subalgebra T of $F(X)$ containing S, with quotient field $F(X)$. Then $X' = \operatorname{Spec} T$ is an affine model of $F(X)/F$. The composition $X' \to \operatorname{Spec} S \to V \to Y$ is then compatible with v. Its restriction to a common open subset U of X and X' defines the desired map f. □

1.16. **Definition.** A projective birational isomorphism $f : X \to Y$ of smooth varieties will be called an *abstract blow-up*. The *isomorphism locus* of f is the largest open subset U of Y such that $f : f^{-1}(U) \to U$ is an isomorphism. The complement Z of U is the *centre* of f: it is considered as a reduced subscheme of X.

The following proposition strengthens Lemma 1.15.

1.17. **Proposition.** *a) Let $v : L \dashrightarrow K$ be a morphism in* place(F)*, and let Y be a projective model of L. Then there exists a normal projective model X of K and a morphism $f : X \to Y$ compatible with v. If F is perfect, we may choose X smooth affine instead. If* $\operatorname{char} F = 0$*, we may choose X smooth projective.*
b) Suppose that $\operatorname{char} F = 0$*, and let X_0 be a smooth projective model of K. Then there exists another smooth projective model X_1 of K, an abstract blow-up $u : X_1 \to X_0$ and a map $f_1 : X_1 \to Y$ compatible with v.*

Proof. a) By the valuative criterion of properness, v has a centre y in Y. Let $N = F(y)$ be the residue field of y: it maps into K by v.

Let $Z = \overline{\{y\}}$: this is a projective model of N. Choose a projective model X_0 of K/N and spread $X_0 \to \operatorname{Spec} N$ to a projective map $X_1 \to Z$. Then X_1 is projective. Let X be the normalisation of X_1. The composition

$$X \to X_1 \to Z \to Y$$

is clearly compatible with v.

If F is perfect, we may replace X by a smooth dense open subset. Finally, if $\operatorname{char} F = 0$, we may take X_0 projective and end by desingularising X_1 instead of normalising it.

b) Let U be a common open subset to X and X_0 (where X has been constructed in a)). We may view U as diagonally embedded into $X \times X_0$. Let X_1 be a desingularisation of the closure of U in $X \times X_0$: we have a diagram of abstract blow-ups

$$\begin{array}{ccc} X_1 & \xrightarrow{g} & X \\ {\scriptstyle u}\downarrow & & \\ X_0. & & \end{array}$$

By Proposition 1.9, $f_1 = f \circ g$ is compatible with v. □

2. Calculus of fractions

Let C be a category and S a set of morphisms of C. Recall from [12, I.1] the category $C[S^{-1}]$ and the canonical functor $P_S : C \to C[S^{-1}]$: P_S is universal among functors from C that render all arrows of S invertible.

We shall denote by $\langle S\rangle$ the set of morphisms s of C such that $P_S(s)$ is invertible: this is the *saturation* of S.

Recall (*loc. cit.*, I.2) that *S admits a calculus of left fractions* when it verifies the following conditions:

(1) The identities are in S; S is stable under composition.
(2) Each diagram

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle s}\downarrow & & \\ X' & & \end{array} \tag{2.1}$$

where $s \in S$ can be completed in a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ X' & \xrightarrow{u'} & Y' \end{array}$$

with $s' \in S$.

(3) If $f, g : X \rightrightarrows Y$ are two morphisms and if $s : X' \to X$ is a morphism of S such that $fs = gs$, there exists a morphism $t : Y \to Y'$ of S such that $tf = tg$:

$$X' \xrightarrow{s} X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{t} Y.$$

Under calculus of left fractions, the category $C[S^{-1}]$ has a very nice description $S^{-1}C$ (loc. cit. I.2.3).

In this article, we shall encounter situations where Condition 2 is verified only for certain pairs (u, s) and where Condition 3 is not verified. This leads us to give more careful definitions.

2.1. **Definition.** A pair (u, s) as in Condition 2 above *admits a calculus of left fractions within* S if there exists a pair (u', s') as in the said condition. It *admits cocartesian calculus of left fractions within* S if the pushout of (2.1) exists and provides such a pair (u', s').

We won't repeat the dual definitions for calculus of right fractions.

3. Equivalences of categories

In this section, we assume that $\operatorname{char} F = 0$. We shall work with several multiplicative subsets of $\mathrm{Sm}(F)$ and $\mathrm{Sm}_{\mathrm{proj}}(F)$:

- $S_o = \{$open immersions$\}$.
- $S_b = \{$birational isomorphisms$\}$.
- $S_b^p = \{$projective morphisms in $S_b\}$.
- S_h = the multiplicative subset generated by morphisms of the form $X \times \mathbf{A}^1 \xrightarrow{pr_1} X$.
- $S_r = \{$rational isomorphisms$\}$ (a morphism is a *rational isomorphism* if the corresponding extension of function fields is purely transcendental).
- $S_r^p = \{$projective morphisms in $S_r\}$.

For S of this form, we also write S for the multiplicative subset of $\mathrm{SmP}(F)$ or $\mathrm{Sm}_{\mathrm{proj}}\,\mathrm{P}(F)$ formed by the pairs (v, f) with $f \in S$.

We shall write $S^{-1}C$ for the localisation of a category instead of the heavier notation $C[S^{-1}]$, even when there is no calculus of fractions.

3.1. **Lemma.** *a) We have the inclusions $S_o \subseteq S_b \subseteq \langle S_o \rangle$ and $S_b \subseteq S_r$. In particular, $S_o^{-1}\,\mathrm{Sm}(F) = S_b^{-1}\,\mathrm{Sm}(F)$ and $S_o^{-1}\,\mathrm{SmP}(F) = S_b^{-1}\,\mathrm{SmP}(F)$.*
b) Suppose $\operatorname{char} F = 0$. *In $\mathrm{SmP}(F)$, any morphism of S_r^p can be covered by the composition of a morphism in S_b^p and morphisms of the form $X \times \mathbf{P}^1 \xrightarrow{pr_1} X$.*

Proof. a) is obvious. For b), we can forget about places thanks to Lemma 1.8. Let $s : X \to Y \in S^p_r$. By assumption, X is birationally equivalent to $X' = Y \times (\mathbf{P}^1)^n$ for some $n \geq 0$.

We proceed as usual: let U be a common open subset of X and X' and X'' the closure of the diagonal embedding of U in $X \times_Y X'$. Then X'' is also a model of $F(X)$. Since s and $pr_1 : Y \times (\mathbf{P}^1)^n \to Y$ are projective, so is the composition $X'' \to X \times_Y X' \to Y$. It remains to resolve the singularities of X'' via a succession of blow-ups. □

3.2. **Proposition.** *Let*

$$\begin{array}{ccc} & & Y' \\ & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{u} & Y \end{array}$$

be a diagram in $\mathrm{SmP}(F)$, *with* $s \in S^p_b$ *(resp.* $s \in S^p_r$*). Then* (u, s) *admits calculus of right fractions within* S^p_b *(resp. within* S^p_r*). The same holds in* $\mathrm{Sm}(F)$.

Proof. Note that the statement for $\mathrm{Sm}(F)$ follows from the case of $\mathrm{SmP}(F)$, thanks to Lemma 1.11. Moreover, the case of S^p_r follows from that of S^p_b by Lemma 3.1 b), since calculus of fractions is obvious for a morphism of the form $X \times \mathbf{P}^1 \to X$.

Here is the proof for $S^p_b \subset \mathrm{SmP}(F)$. Let $v : F(Y) \dashrightarrow F(X)$ be the place compatible with u which is implicit in the statement. By assumption v has a centre z on Y. Since s is proper, v therefore has also a centre z' on Y'. By Lemma 1.13, $s(z') = z$. Hence we have inclusions of fields

$$F(z) \hookrightarrow F(z') \hookrightarrow F(v) \hookrightarrow F(X). \tag{3.1}$$

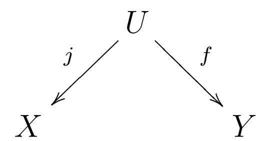
Let $Z = \overline{\{z\}}$ and $Z' = \overline{\{z'\}}$. The map u factors through a map $\bar{u} : X \to Z$. The chain (3.1) shows that $\bar{u}$ lifts to a rational map from X to Z'. Blowing up X suitably, we get a commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{\bar{u}'} & Z' & \hookrightarrow & Y' \\ {\scriptstyle s'}\downarrow & & \downarrow & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{\bar{u}} & Z & \hookrightarrow & Y \end{array}$$

in which s' is an abstract blow-up; moreover, the composition $u' : X' \to Y'$ is compatible with v (compare Corollary 1.7 b)). □

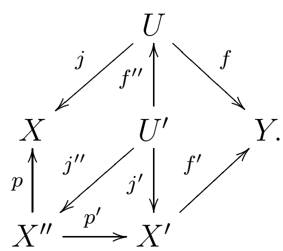
3.3. **Lemma.** *Any morphism in* $S_b^{-1}\,\mathrm{SmP}(F)$ *of the form* fj^{-1} *with* j *an open immersion is equal to a morphism of the form* $a^{-1}gp^{-1}b$*, where* p *is an abstract blow-up and* a, b *are open immersions.*

Proof. Consider a diagram



where j is an open immersion. Choose open embeddings $a : Y \to \bar{Y}$ and $b : X \to \bar{X}$, with $\bar{X}, \bar{Y}$ smooth projective. This reduces us to the case where X and Y are projective.

By proposition 1.17, choose a smooth projective model X' of $F(X)$ and a morphism $f' : X' \to Y$ compatible with the underlying place v. Let X'' be a third smooth projective model of $F(X)$ mapping both to X and X', and let U' be a common open subset of X, X' and X'' contained in U:



By Lemma 1.14, all paths in this diagram commute. Hence we find

$$fj^{-1} = f'p'p^{-1}.$$

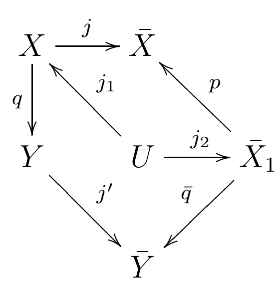
□

3.4. **Lemma.** *In $S_b^{-1}\operatorname{SmP}(F)$, any morphism $q \in S_r$ can be written in the form*

$$q = {j'}^{-1}\bar{q}p^{-1}j$$

with $j, j' \in S_o$, $p \in S_b^p$ and $\bar{q} \in S_b^p$. Same statement within $S_r^{-1}\operatorname{SmP}(F)$ for $q \in S_r$, with $\bar{q} \in S_r^p$.

Proof. We don't need to take care of the places, thanks to Lemma 1.8. Let us prove the first statement. By resolution of singularities, we can find a commutative diagram



where j, j', j_1, j_2 are open immersions, $\bar{X}, \bar{X}_1$ and $\bar{Y}$ are smooth projective, p is an abstract blow-up and $\bar{q} \in S_r^p$. Lemma 3.4 now follows from a small computation. The proof of the second statement is exactly the same. □

3.5. **Proposition.** *Any morphism in* $S_b^{-1}\operatorname{SmP}(F)$ *can be written as* $j^{-1}fq^{-1}$, *with* $j \in S_o$ *and* $q \in S_b^p$. *Same statement for* $S_r^{-1}\operatorname{SmP}(F)$, *with* $q \in S_r^p$.

Proof. We first show that any morphism in either category can be written as a composition of morphisms of the form $j^{-1}fq^{-1}$ as in the statement of Proposition 3.5. It is sufficient to prove this for a morphism of the form fq^{-1}, with $q \in S_b$ (*resp.* $q \in S_r$).

Write $q = j'^{-1}\bar{q}p^{-1}j$ as in Lemma 3.4. Then we have

$$fq^{-1} = fj^{-1}p\bar{q}^{-1}j'$$

with $j, j' \in S_o$, $p \in S_b^p$ and $\bar{q} \in S_b^p$ (*resp.* $\bar{q} \in S_r^p$).

Applying now Lemma 3.3 to fj^{-1}, we get

$$fq^{-1} = a^{-1}gp'^{-1}bp\bar{q}^{-1}j' = (a^{-1}gp'^{-1})(bp\bar{q}^{-1})j'$$

with $a, b \in S_o$ and $p' \in S_b^p$.

It now suffices to prove that the composition $j_1^{-1}f_1q_1^{-1}j_2^{-1}f_2q_2^{-1}$ of two morphisms of this form is still of this form. By Lemma 3.4, write $j_2q_1 = j'^{-1}q_3p^{-1}j_3$, with $j_3, j' \in S_o$, $p \in S_b^p$ and $q_3 \in S_b^p$, so that

$$f_1q_1^{-1}j_2^{-1} = f_1j_3^{-1}pq_3^{-1}j'.$$

By Lemma 3.3, write $f_1j_3^{-1} = a^{-1}gp_1^{-1}b$, where $p_1 \in S_b^p$ and $a, b \in S_o$, so that

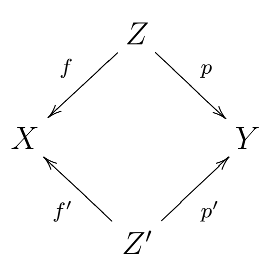
$$f_1j_3^{-1}pq_3^{-1}j' = a^{-1}gp_1^{-1}bpq_3^{-1}j'$$

and

$$j_1^{-1}f_1q_1^{-1}j_2^{-1}f_2q_2^{-1} = j_1^{-1}a^{-1}gp_1^{-1}bpq_3^{-1}j'f_2q_2^{-1}.$$

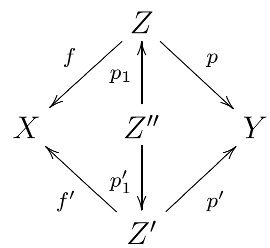
It now suffices to apply Proposition 3.2 twice. □

3.6. **Proposition.** *Consider a diagram in* $\operatorname{Sm}_{\operatorname{proj}}\operatorname{P}(F)$



where p and p' are abstract blow-ups. Let $K = F(Z) = F(Z') = F(Y)$, $L = F(X)$ and suppose given a place $v : L \dashrightarrow K$ compatible both with f and f'. Then $(v, fp^{-1}) = (v, f'p'^{-1})$ in $S_b^{-1}\operatorname{Sm}_{\mathrm{proj}}\mathrm{P}(F)$.

Proof. Complete the diagram as follows:



where p_1 and p'_1 are abstract blow-ups. Then we have

$$pp_1 = p'p'_1, \quad fp_1 = f'p'_1$$

(the latter by Lemma 1.14), hence the claim. □

3.7. **Definition.** We extend the functor Φ_2 of Lemma 1.12 to functors

$$\Phi_2^b : S_b^{-1}\operatorname{SmP}(F) \to \operatorname{place}(F)^{\mathrm{op}}$$
$$\Phi_2^r : S_r^{-1}\operatorname{SmP}(F) \to T^{-1}\operatorname{place}(F)^{\mathrm{op}}$$

via Remark 1.5, where T is the multiplicative set of morphisms in place(F) given by rational extensions.

Let also

$$JP : \operatorname{Sm}_{\mathrm{proj}}\mathrm{P}(F) \to \operatorname{SmP}(F)$$
$$J : \operatorname{Sm}_{\mathrm{proj}}(F) \to \operatorname{Sm}(F)$$

denote the inclusion functors. Finally, we recall the forgetful functor

$$\Phi_1 : \operatorname{SmP}(F) \to \operatorname{Sm}(F)$$

and its restriction to $\operatorname{Sm}_{\mathrm{proj}}\mathrm{P}(F)$, denoted by the same letter.

3.8. **Theorem.** *The functors Φ_2^b, $S_b^{-1}JP$, $\Phi_2^b \circ S_b^{-1}JP$ and $S_b^{-1}J$ are equivalences of categories, while the two functors $S_b^{-1}\Phi_1$ are full and* *essentially surjective:*

$$\begin{array}{ccc}
S_b^{-1}\operatorname{Sm}_{\mathrm{proj}}\mathrm{P}(F) & \xrightarrow{S_b^{-1}\Phi_1} & S_b^{-1}\operatorname{Sm}_{\mathrm{proj}}(F)\\
{\scriptstyle S_b^{-1}JP}\downarrow\wr & & {\scriptstyle S_b^{-1}J}\downarrow\wr\\
S_b^{-1}\operatorname{SmP}(F) & \xrightarrow{S_b^{-1}\Phi_1} & S_b^{-1}\operatorname{Sm}(F)\\
{\scriptstyle \Phi_2^b}\downarrow\wr & & \\
\operatorname{place}(F)^{\mathrm{op}}. & &
\end{array}$$

The same holds by replacing Φ_2^b by Φ_2^r and S_b by S_r:

$$\begin{array}{ccc}
S_r^{-1}\operatorname{Sm}_{\mathrm{proj}}\mathrm{P}(F) & \xrightarrow{S_r^{-1}\Phi_1} & S_r^{-1}\operatorname{Sm}_{\mathrm{proj}}(F)\\
{\scriptstyle S_r^{-1}JP}\downarrow\wr & & {\scriptstyle S_r^{-1}J}\downarrow\wr\\
S_r^{-1}\operatorname{SmP}(F) & \xrightarrow{S_r^{-1}\Phi_1} & S_r^{-1}\operatorname{Sm}(F)\\
{\scriptstyle \Phi_2^r}\downarrow\wr & & \\
T^{-1}\operatorname{place}(F)^{\mathrm{op}}. & &
\end{array}$$

Proof. It is enough to prove the first assertion: the second one follows by localising the first with respect to S_r.

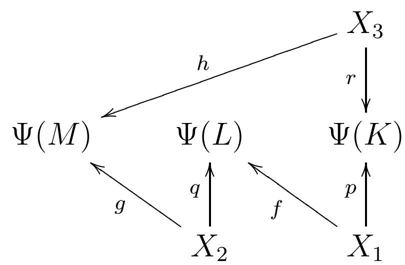
A) We first prove that $\Phi_2^b \circ S_b^{-1}JP$ is an equivalence of categories. By Lemma 1.12, this functor is essentially surjective. Proposition 1.17 shows that it is full. It remains to see that it is faithful.

Let us construct a functor $\Psi : \operatorname{place}(F) \to S_b^{-1}\operatorname{Sm}_{\mathrm{proj}}\mathrm{P}(F)^{\mathrm{o}}$ as follows:

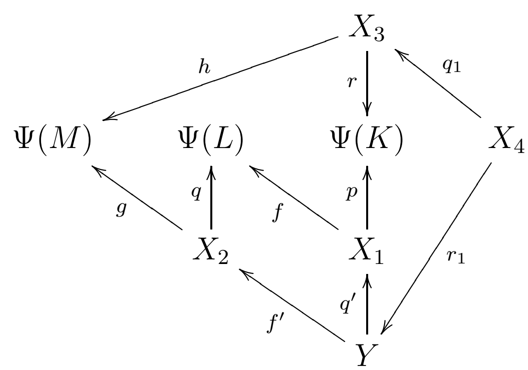
- To a function field K we associate a smooth projective model $\Psi(K)$, chosen once and for all.
- Let $v : L \dashrightarrow K$ be a place. By Proposition 1.17, there exists a pair (f,p) with $f : X_1 \to \Psi(L)$ compatible with v and $p : X_1 \to \Psi(K)$ an abstract blow-up. We define $\Psi(v)$ as (v, fp^{-1}).

The map $v \mapsto \Psi(v)$ is well-defined by Proposition 3.6. To prove that it respects composition, consider two composable places $K \xrightarrow{v} L \xrightarrow{w}$

M and the following diagram



with $(v, fp^{-1}) = \Psi(v)$, $(w, gq^{-1}) = \Psi(w)$ and $(wv, hr^{-1}) = \Psi(wv)$. We may complete it into a commutative diagram



where $q', q_1, r_1 \in S_b^p$ (the existence of (f', q') comes from Proposition 3.2). The equality $\Psi(wv) = \Psi(v)\Psi(w)$ now easily follows from this diagram.

The equality $\Phi_2^b S_r^{-1} JP\Psi = Id$ is obvious and Ψ is clearly full. Therefore $\Phi_2^b S_b^{-1} JP$ is faithful, hence an equivalence of categories (with quasi-inverse Ψ).

B) We now prove that $S_b^{-1} JP$ is an equivalence of categories. The above shows that it is faithful, and it is also essentially surjective thanks to resolution of singularities. It remains to see that it is full.

By Proposition 3.5, for two objects $X, Y \in S_b^{-1}\operatorname{SmP}(F)$, any morphism $\varphi : X \to Y$ is of the form $j^{-1}fp^{-1}$ with j an open immersion and p a projective rational map. If Y is projective, j is necessarily an isomorphism. If furthermore X is projective, the source of p is projective too. Hence φ comes from $S_b^{-1}\operatorname{Sm}_{\text{proj}}\operatorname{P}(F)$, as desired.

C) Now the functor $\Phi_1 : \operatorname{SmP}(F) \to \operatorname{Sm}(F)$ is full and essentially surjective by Lemma 1.12, and this is preserved by localisation. Similarly for its restriction to $\operatorname{Sm}_{\text{proj}}\operatorname{P}(F)$. As a consequence, $S_b^{-1} J$ is full (and essentially surjective).

D) It remains to show that $S_b^{-1}J$ is faithful. For this, we proceed exactly as in A): define a functor $\Pi : \mathrm{Sm}(F) \to S_b^{-1}\mathrm{Sm}_{\mathrm{proj}}(F)$ by sending a smooth connected variety X to a smooth compactification $\Pi(X)$ chosen once and for all, and a map $f : X \to Y$ to a map $\Pi(f) = s^{-1}\widetilde{f}$, where $s : \widetilde{X} \to \Pi(X)$ is a suitable abstract blow-up with centre disjoint from X and $\widetilde{f} : \widetilde{X} \to \Pi(Y)$ restricts to f on $s^{-1}(X) = X$. By an analogous statement to Proposition 3.6, $\Pi(f)$ is well-defined. That Π is indeed a functor now is proven exactly as in A), by using Proposition 3.2, and Π is obviously full. Then Π factors through $S_b^{-1}\mathrm{Sm}(F)$ into a full functor, which is a quasi-inverse to $S_b^{-1}J$. □

3.9**. Definition.** We call place(F) the *coarse birational category of F* and $S_b^{-1}\mathrm{Sm}_{\mathrm{proj}}(F)$ the *fine birational category of F*. Similarly, we call $T^{-1}\,\mathrm{place}(F)$ the *coarse stable birational category of F* and $S_r^{-1}\mathrm{Sm}_{\mathrm{proj}}(F)$ the *fine stable birational category of F*.

By Theorem 3.8, we have a commutative diagram of categories and functors

$$\begin{array}{ccc}
\mathrm{place}(F)^{\mathrm{op}} & \xrightarrow{\bar{\Phi}_1} & S_b^{-1}\mathrm{Sm}_{\mathrm{proj}}(F) \\
\downarrow & & \downarrow \\
T^{-1}\,\mathrm{place}(F)^{\mathrm{op}} & \xrightarrow{T^{-1}\bar{\Phi}_1} & S_r^{-1}\mathrm{Sm}_{\mathrm{proj}}(F)
\end{array}$$

where the horizontal functors are full and essentially surjective.

4. Birational finite correspondences

We shall need the following proposition, which is an additive analogue of part of Theorem 3.8, in Section 6:

4.1**. Proposition.** *Let* $\mathrm{SmCor}(F)$ *be Voevodsky's category of finite correspondences on smooth varieties* [32], *and let* $\mathrm{SmCor}_{\mathrm{proj}}(F)$ *be its full subcategory consisting of smooth projective varieties. Let J denote the inclusion functor* $\mathrm{SmCor}_{\mathrm{proj}}(F) \to \mathrm{SmCor}(F)$*. Let further* $S_b \subset \mathrm{SmCor}(F)$ *be the set of [graphs of] morphisms that are birational isomorphisms, as well as its restriction to* $\mathrm{SmCor}_{\mathrm{proj}}(F)$*. Then the equivalence of categories $S_b^{-1}J$ of Theorem 3.8 extends to an equivalence of categories*

$$S_b^{-1}J : S_b^{-1}\mathrm{SmCor}_{\mathrm{proj}}(F) \xrightarrow{\sim} S_b^{-1}\mathrm{SmCor}(F)$$

via the canonical functors $\mathrm{Sm}(F) \to \mathrm{SmCor}(F)$ *and* $\mathrm{Sm}_{\mathrm{proj}}(F) \to \mathrm{SmCor}_{\mathrm{proj}}(F)$*. The same holds when replacing S_b by S_r.*

The proof will occupy the entire section.

Proof. We limit ourselves to S_b, the proofs for S_r being exactly the same. The functor $S_b^{-1}J$ is clearly full and essentially surjective. We prove its faithfulness by constructing a quasi-inverse. In order to do this, we proceed using the notation and arguments as in part D) of the proof of Theorem 3.8. Thus we first extend the functor Π of *loc. cit.*to an additive functor

$$\Pi : \mathrm{SmCor}(F) \to S_b^{-1}\,\mathrm{SmCor}_{\mathrm{proj}}(F).$$

Given two smooth varieties X, Y, we need to define a homomorphism

$$\Pi : c(X, Y) \to S_b^{-1}\,\mathrm{SmCor}_{\mathrm{proj}}(F)(\Pi(X), \Pi(Y)).$$

Since the left-hand side is by definition the free abelian group on the set $b(X, Y)$ of closed integral subschemes of $X \times Y$ which are finite and surjective over a connected component of X, it would be sufficient to define a map $\Pi : b(X, Y) \to S_b^{-1}\,\mathrm{SmCor}_{\mathrm{proj}}(F)(\Pi(X), \Pi(Y))$. To check functoriality, however, it will be more convenient to define Π directly on the monoid of positive correspondences[1] $c^+(X, Y) = \mathbf{N}b(X, Y)$, and to check that it is additive.

We shall use the idea of Suslin-Voevodsky [29]: a cycle $Z \in b(X, Y)$ of generic degree d over X defines a map

$$[Z] : X \to \mathbf{S}^d(Y).$$

This rule extends to a homomorphism of abelian monoids

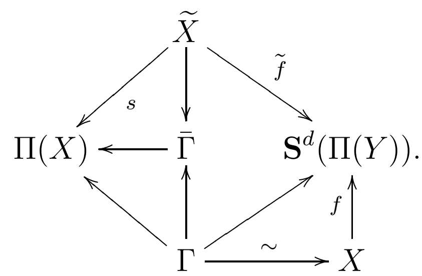
$$c^+(X, Y) \to \coprod_{n \geq 0} Map_F(X, \mathbf{S}^n(Y)) \tag{4.1}$$

which is an isomorphism by [29, Th. 6.8] (because F is of characteristic 0).

Let then $Z \in c^+(X, Y)$, and $[Z] : X \to \mathbf{S}^d(Y)$ the corresponding map. Composing with the open immersion $Y \to \Pi(Y)$, we get a map $X \to \mathbf{S}^d(\Pi(Y))$. Consider its graph Γ in $X \times \mathbf{S}^d(\Pi(Y))$ and its closure $\bar{\Gamma}$ in $\Pi(X) \times \mathbf{S}^d(\Pi(Y))$. Clearly $\bar{\Gamma}$ is projective and birational to X (since the projection $\Gamma \to X$ is an isomorphism). Desingularising it, we get $s : \widetilde{X} \to \Pi(X)$ with $s \in S_b$ and a map $\widetilde{f} : \widetilde{X} \to \mathbf{S}^d(\Pi(Y))$

[1] We prefer to use the term *positive* rather than *effective*, in order not to create a possible confusion with effective motives later in this text.

extending f on some open subset of X:

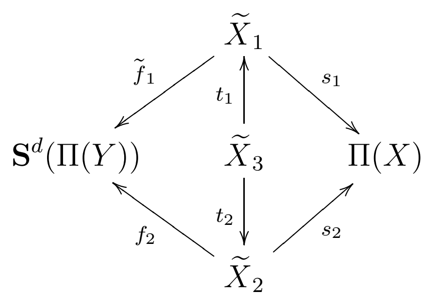


On the other hand, by the bijectivity of (4.1), $\widetilde{f}$ corresponds to a unique effective cycle $c \in c^+(\widetilde{X}, \Pi(Y))$. We may now define $\Pi(Z) = c\Gamma_s^{-1}$.

We now proceed to check successively various compatibilities:

1) If Z is the graph Γ_f of a morphism $f : X \to Y$, then $d = 1$ and $[Z] = f$, so we get exactly the same construction as in part D) of the proof of Theorem 3.8.

2) We check that $c\Gamma_s^{-1}$ is independent of the choice of $\widetilde{X}$. This follows from an argument analogous to that in Proposition 3.6. Indeed, if $\widetilde{X}_1$ and $\widetilde{X}_2$ are two different abstract blow-ups which give maps $\widetilde{f}_1$ and $\widetilde{f}_2$ into $\mathbf{S}^d(\Pi(Y))$, then we have a diagram as follows where t_1, t_2 are abstract blow-ups and hence $t_1, t_2 \in S_b$:



If c_1, c_2 are the effective cycles corresponding to $\widetilde{f}_1$ and $\widetilde{f}_2$, then we have $c_1\Gamma_{t_1} = c_2\Gamma_{t_2}$ in $\mathrm{SmCor}(F)$, and also $s_1t_1 = s_2t_2$, hence $c_1\Gamma_{s_1}^{-1} = c_2\Gamma_{s_2}^{-1}$.

3) We now check that $\Pi(Z + Z') = \Pi(Z) + \Pi(Z')$ for two effective finite correspondences Z and Z'. Let $f : X \to \mathbf{S}^d(Y)$ and $f' : X \to \mathbf{S}^{d'}(Y)$ be the two corresponding maps. Then to $Z + Z'$ is associated the composite

$$f'' : X \xrightarrow{(f,f')} \mathbf{S}^d(Y) \times \mathbf{S}^{d'}(Y) \to \mathbf{S}^{d+d'}(Y).$$

Let $s : \widetilde{X} \to \Pi(X)$ and $s' : \widetilde{X}' \to \Pi(X)$ be two birational isomorphisms chosen as above respectively for f and for f', and let $t : \widetilde{X}'' \to \widetilde{X}$, $t' : \widetilde{X}'' \to \widetilde{X}'$ be two abstract blow-ups ($\widetilde{X}''$ smooth projective), defining two equal compositions $s'' = s \circ t = s' \circ t' : \widetilde{X}'' \to \Pi(X)$. Then we get a diagram

$$\begin{array}{ccccc}
X & \xrightarrow{(f,f')} & \mathbf{S}^d(Y) \times \mathbf{S}^{d'}(Y) & \longrightarrow & \mathbf{S}^{d+d'}(Y) \\
\downarrow & & \downarrow & & \downarrow \\
\Pi(X) & & \mathbf{S}^d(\Pi(Y)) \times \mathbf{S}^{d'}(\Pi(Y)) & \longrightarrow & \mathbf{S}^{d+d'}(\Pi(Y)) \\
{\scriptstyle s''}\uparrow & \nearrow{\scriptstyle (ft,f't')} & & & \\
\widetilde{X}'' & & & &
\end{array}$$

Let $\widetilde{f}'' : \widetilde{X}'' \to \mathbf{S}^{d+d'}(\Pi(Y))$ be the composition from this diagram. Then, by 2), $\Pi(Z + Z')$ corresponds to $\widetilde{f}'' s''^{-1}$, and the diagram concludes the check of 3).

4) We check that Π is a functor. Let $\alpha \in c^+(X, Y)$ and $\beta \in c^+(Y, Z)$ be effective correspondences, with $f : X \to \mathbf{S}^d(Y)$ (where $d = \sum n_i[F(U_i) : F(X)]$ if $\alpha = \sum n_i\alpha_i$ with α_i irreducible with supports U_i in $X \times Y$), and $g : Y \to \mathbf{S}^k(Z)$ the corresponding maps coming from the Suslin-Voevodsky isomorphism.

4.2. **Lemma.** *The composition $\beta \circ \alpha \in c^+(X, Z)$ maps under (4.1) to the composite*

$$X \xrightarrow{f} \mathbf{S}^d(Y) \xrightarrow{\mathbf{S}^d(g)} \mathbf{S}^d(\mathbf{S}^k(Z)) \to \mathbf{S}^{dk}(Z).$$

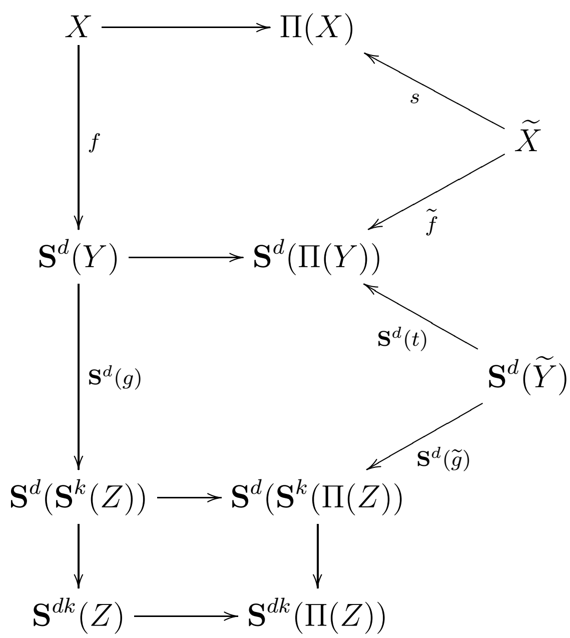
Proof. (Friedlander and Voevodsky) We give it using that F is of characteristic 0, although this assumption is probably not necessary (Fabien Morel has indicated us that he has a proof based on the results of [29], at least after inverting the exponential characteristic). We have to show that the image of $\beta \circ \alpha$ in $Map_F(X, \mathbf{S}^{dk}(Z))$ equals $\mathbf{S}^d(g) \circ f$. We may assume X connected. Let $\eta = \operatorname{Spec} K$ be its generic point. Since

$$\begin{aligned} Map_F(X, \mathbf{S}^{dk}(Z/F)) &\hookrightarrow Map_F(\eta, \mathbf{S}^{dk}(Z/F)) \\ &= Map_K(\operatorname{Spec} K, \mathbf{S}^{dk}(Z_K/K)), \end{aligned}$$

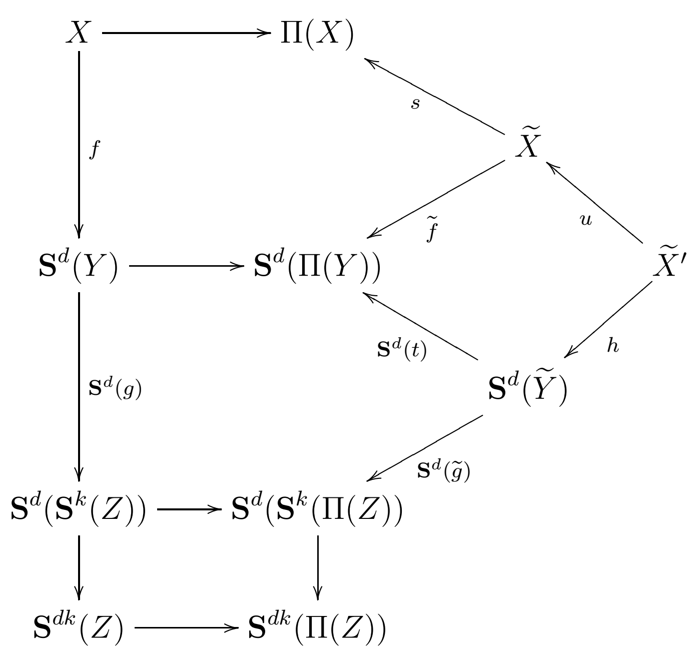
we may assume that $X = \operatorname{Spec} F$. Then $Map_F(\operatorname{Spec} F, \mathbf{S}^{dk}(Z))$ is the set of effective zero-cycles of degree dk on Z. Since F is of characteristic 0, this set injects into $Map_{\bar{F}}(\operatorname{Spec} \bar{F}, \mathbf{S}^{dk}(Z_{\bar{F}}))$ and we further reduce to the case where F is algebraically closed and finally where α

is irreducible. Then it corresponds to a rational point of Y, $d = 1$ and the claim is obvious. □

With notation as above, we now have a (not very) commutative diagram:



and the proof will be finished if we can complete it into a (rather more) commutative diagram



where $\widetilde{X}'$ is smooth and u is a birational isomorphism. (The commutativity means that $\mathbf{S}^d(t)h = \widetilde{f}u$, or equivalently $\mathbf{S}^d(t)^{-1}\widetilde{f} = hu^{-1}$, which implies $\mathbf{S}^d(\widetilde{g})\,\mathbf{S}^d(t)^{-1}\widetilde{f}s^{-1} = \mathbf{S}^d(\widetilde{g})hu^{-1}s^{-1}$.) This means extending Proposition 3.2 to a case (with its notation) where X and Y are not smooth.

Examining the proof of this proposition, still with its notation, smoothness is irrelevant everywhere provided we have a place v compatible with u. Referring to Lemma 1.11, we do use that Y is regular. However, all we use is that $u(\eta_X)$ is a regular point of Y. Therefore, coming back to the above notation, the proof of Proposition 4.1 will be complete if we can prove that $\widetilde{f}(\eta_{\tilde{X}})$ is a smooth point of $\mathbf{S}^d(\Pi(Y))$. But this point is the image of $f(\eta_X)$ by the open immersion $\mathbf{S}^d(Y) \to \mathbf{S}^d(\Pi(Y))$, so it suffices to know that $f(\eta_X)$ is a smooth point of $\mathbf{S}^d(Y)$.

For any scheme T (over some base), let $\widetilde{\mathbf{U}}^d(T)$ be the open subset of T^d defined by the conditions that all coordinates are distinct, and let $\mathbf{U}^d(T)$ be the image of $\widetilde{\mathbf{U}}^d(T)$ in $\mathbf{S}^d(T)$: the projection $\widetilde{\mathbf{U}}^d(T) \to \mathbf{U}^d(T)$ is finite étale. If T is smooth, $\mathbf{U}^d(T)$ is smooth as well. Note that if $i : S \to T$ is an immersion, then the immersion $\mathbf{S}^d(i)$ carries $\mathbf{U}^d(S)$ into $\mathbf{U}^d(T)$. Therefore, to check 4) we are reduced to proving the following

4.3. **Lemma.** *Let $U \to X$ be a finite surjective morphism of schemes, with U integral and X normal connected. Let $d = [\kappa(U) : \kappa(X)]$, and assume the extension $\kappa(X)/\kappa(U)$ separable. Then the Suslin-Voevodsky map* [29, p. 81]

$$X \to \mathbf{S}^d(U)$$

sends the generic point η_X in $\mathbf{U}^d(U)$.

Note that the assumption $\operatorname{char} F = 0$ is needed to be able to apply Lemma 4.3!

Proof. We have a commutative diagram

$$\begin{array}{ccc} \eta_X & \longrightarrow & \mathbf{S}^d(\eta_U) \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbf{S}^d(U). \end{array}$$

Since the vertical maps are immersions, we are reduced to the case where X, hence U, is the spectrum of a field, say $X = \operatorname{Spec} K$, $U = \operatorname{Spec} L$. We shall prove the lemma in the slightly more general case where L is an étale K-algebra.

Suppose first that $L = K^d$. Let $x_1, \dots, x_d$ be the d rational points $\operatorname{Spec} K \to \operatorname{Spec} L$. Then the image of $\operatorname{Spec} K$ in $\mathbf{S}^d(\operatorname{Spec} L)$

is $(x_1, \dots, x_d)$, which is clearly in $\mathbf{U}^d(\operatorname{Spec} L)$. In general, we reduce to this case by passing to a separable closure of K. □

5) Finally we need to check that Π induces a functor $S_b^{-1} \operatorname{SmCor}(F) \to S_b^{-1} \operatorname{SmCor}_{\mathrm{proj}}(F)$, *i.e.* that $\Pi(\Gamma_s)$ is invertible if $s \in S_b$. This follows immediately from 1). The fact that Π is a quasi-inverse to $S_b^{-1}J$ is immediate. □

5. Triangulated category of birational motives

In this section, we construct a triangulated category of birational geometric motives and an additive category of birational Chow motives, and construct a functor from the second to the first.

We assume that the reader is familiar with Voevodsky's triangulated categories of motives [32]; for a smooth variety X over F, we simplify his notation $M_{\mathrm{gm}}(X)$ into $M(X)$ for the motive of X in $DM_{\mathrm{gm}}^{\mathrm{eff}}(F)$.

5.1**. Definition.** We denote by $DM_{\mathrm{gm}}^{\mathrm{o}}(F)$ the pseudo-abelian envelope of the quotient of $DM_{\mathrm{gm}}^{\mathrm{eff}}(F)$ [32] by the thick triangulated subcategory $\mathcal{C}$ generated by the cones of maps

$$M(U) \xrightarrow{j_*} M(X)$$

where X is a smooth variety and j is an open immersion. We denote by $\bar{M}(X)$ the image of $M(X)$ in $DM_{\mathrm{gm}}^{\mathrm{o}}(F)$.

By [2], $DM_{\mathrm{gm}}^{\mathrm{o}}(F)$ is still triangulated.

5.2**. Proposition.** *a) The tensor structure of* $DM_{\mathrm{gm}}^{\mathrm{eff}}(F)$ *passes to* $DM_{\mathrm{gm}}^{\mathrm{o}}(F)$.
b) Suppose F perfect. Then $\mathcal{C}$ consists of those motives of the form $M(1)$ and the functor $DM_{\mathrm{gm}}^{\mathrm{eff}}(F) \to \mathcal{C}$ given by $M \mapsto M(1)$ is an equivalence of categories.

Proof. a) Recall that the category $DM_{\mathrm{gm}}^{\mathrm{eff}}(F)$ is generated by the $M(Y)$ (Y smooth) and there is a canonical isomorphism $M(X_1 \times X_2) \simeq M(X_1) \otimes M(X_2)$. Clearly the tensor structure is compatible on further localising with respect to morphisms $M(U) \to M(X)$ where $U \hookrightarrow X$ is an open immersion.

b) Since $\mathbf{Z} \oplus \mathbf{Z}(1)[2] = M(\mathbf{P}^1) = M(\mathbf{A}^1) = \mathbf{Z}$ in $DM_{\mathrm{gm}}^{\mathrm{o}}(F)$, $\mathbf{Z}(1) = 0$ in $DM_{\mathrm{gm}}^{\mathrm{o}}(F)$. By a), $DM_{\mathrm{gm}}^{\mathrm{o}}(F)$ is therefore a localisation of $DM_{\mathrm{gm}}^{\mathrm{eff}}(F)/\mathcal{C}$. To see conversely that $\operatorname{Ker}(DM_{\mathrm{gm}}^{\mathrm{eff}}(F) \to DM_{\mathrm{gm}}^{\mathrm{o}}(F)) \subseteq \mathcal{C}$, we have to prove that $\operatorname{cone}(M(U) \xrightarrow{j_*} M(X))$ is in $\mathcal{C}$ for any open immersion j. We argue by Noetherian induction on the (reduced) closed complement Z in a standard way. If Z is smooth of pure codimension

c, then the cone is isomorphic to $M(Z)(c)[2c]$ [32, Prop. 3.5.4]. In general, let $Z_{sing} \subset Z$ be the singular locus of Z. Then $Z - Z_{sing}$ is smooth in $X - Z_{sing}$. Let C and C' denote respectively cones of the maps $M(U) \to M(X)$ and $M(X - Z_{sing}) \to M(X)$. By the axioms of triangulated categories, we may complete the diagram

$$\begin{array}{ccccc}
M(U) & = & M(U) & & \\
\downarrow & & \downarrow & & \\
M(X - Z_{sing}) & \longrightarrow & M(X) & \longrightarrow & C' \\
\downarrow & & \downarrow & & \\
M(Z - Z_{sing})(c)[2c] & & C & & \\
\downarrow & & \downarrow & & \\
M(U)[1] & = & M(U)[1] & &
\end{array}$$

into a commutative diagram of exact triangles

$$\begin{array}{ccccc}
M(U) & = & M(U) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
M(X - Z_{sing}) & \longrightarrow & M(X) & \longrightarrow & C' \\
\downarrow & & \downarrow & & \downarrow \\
M(Z - Z_{sing})(c)[2c] & \longrightarrow & C & \longrightarrow & C'' \\
\downarrow & & \downarrow & & \downarrow \\
M(U)[1] & = & M(U)[1] & \longrightarrow & 0
\end{array}$$

where C'' is by definition a cone of the map $M(Z - Z_{sing})(c)[2c] \longrightarrow C$. Hence we get an exact triangle

$$C'[-1] \longrightarrow M(Z - Z_{sing})(c)[2c] \longrightarrow C \longrightarrow C'.$$

Since F is perfect, Z_{sing} is strictly smaller than Z and we have $C' \in \mathcal{C}$ by Noetherian induction, hence also $C \in \mathcal{C}$.

To conclude, we have to show that, if $M, N \in DM^{\text{eff}}_{\text{gm}}(F)$ and $f \in Hom(M(1), N(1))$, then the cone of f is of the form $P(1)$. This follows from the cancellation theorem [34], which is now valid over any perfect field. (We are indebted to Fabien Morel for pointing out this issue.) The last assertion similarly follows from the cancellation theorem. $\square$

Let $\mathrm{Cor}^{\mathrm{eff}}(F)$ denote the category of effective correspondences, i.e. the category whose objects are smooth projective varieties and morphisms are Chow correspondences. The category $\mathrm{Chow}^{\mathrm{eff}}(F)$ of effective Chow motives is the pseudo-abelian envelope of $\mathrm{Cor}^{\mathrm{eff}}(F)$. Recall the functor [32, Prop. 2.1.4]

$$\mathrm{Chow}^{\mathrm{eff}}(F) \to DM_{\mathrm{gm}}^{\mathrm{eff}}(F) \tag{5.1}$$

from the category of effective Chow motives (viewed as a homological category: the natural functor $\mathrm{Sm}_{\mathrm{proj}}(F) \to \mathrm{Chow}^{\mathrm{eff}}(F)$ is covariant). We would like to describe a similar functor to $DM_{\mathrm{gm}}^{\mathrm{o}}(F)$. To this effect we quotient $\mathrm{Chow}^{\mathrm{eff}}(F)$ in a parallel way.

In $\mathrm{Chow}^{\mathrm{eff}}(F)$, we write $h(X)$ for the Chow motive associated to a smooth projective variety X, $\mathbf{1}$ for $h(\mathrm{Spec}\, F)$ and L for the Lefschetz motive, defined by $h(\mathbf{P}^1) = \mathbf{1} \oplus L$. Then $L \mapsto \mathbf{Z}(1)[2]$ under (5.1). For $M \in \mathrm{Chow}^{\mathrm{eff}}(F)$, we write $M(n)$ for $M \otimes L^{\otimes n}$, rather than the traditional notation $M(-n)$.

5.3. **Lemma.** *For two smooth projective varieties X, Y, let $I(X,Y)$ be the subgroup of $CH_{\dim X}(X \times Y)$ consisting of those classes vanishing in $CH_{\dim X}(U \times Y)$ for some open subset U of X. Then I is a monoidal ideal in* $\mathrm{Cor}^{\mathrm{eff}}(F)$ *(i.e. is closed with respect to tensor products on the left and right). In the factor category* $\mathrm{Cor}^{\mathrm{o}}(F)$*, one has the formula*

$$Hom(\bar{h}(X), \bar{h}(Y)) = CH_0(Y_{F(X)})$$

where $\bar{h}$ denotes the natural composite functor $\mathrm{Sm}_{\mathrm{proj}}(F) \xrightarrow{h} \mathrm{Cor}^{\mathrm{eff}}(F) \to \mathrm{Cor}^{\mathrm{o}}(F)$.

Proof. Let X, Y, Z be 3 smooth projective varieties. If U is an open subset of X, it is clear that the usual formula defines a composition of correspondences

$$CH_{\dim X}(U \times Y) \times CH_{\dim Y}(Y \times Z) \to CH_{\dim X}(U \times Z)$$

and that this composition commutes with restriction to smaller and smaller open subsets. Passing to the limit on U, we get a composition

$$CH^{\dim Y}(Y_{F(X)}) \times CH_{\dim Y}(Y \times Z) \to CH^{\dim Z}(Z_{F(X)})$$

or

$$CH_0(Y_{F(X)}) \times CH_{\dim Y}(Y \times Z) \to CH_0(Z_{F(X)}).$$

This pairing is actually nothing else than the action of correspondences on Chow groups of 0-cycles (it extends to an action of $CH_{\dim Y}(Y_{F(X)} \times Z_{F(X)})$). We now need to prove that this action factors through an action of $CH_{\dim Y}(V \times Z)$ for any open subset V of

Y. Without loss of generality, we may pass to $F(X)$ and hence assume that $X = \operatorname{Spec} F$.

The proof is basically a generalisation of Fulton's proof of the Colliot-Thélène–Coray theorem that CH_0 is a birational invariant of smooth projective varieties [5], [11, Ex. 16.1.11]. Let M be a proper closed subset of Y, and $i : M \to Y$ be the corresponding closed immersion. We have to prove that for any $\alpha \in CH_0(Y)$ and $\beta \in CH_{\dim Y}(M \times Z)$,

$$(i \times 1_Z)_*(\beta)(\alpha) := (p_2)_*((i \times 1_Z)_*\beta \cdot p_1^*\alpha) = 0$$

where p_1 and p_2 are respectively the first and second projections on $Y \times Z$.

We shall actually prove that $(i \times 1_Z)_*\beta \cdot p_1^*\alpha = 0$. For this, we may assume that α is represented by a point $y \in Y_{(0)}$ and β by some integral variety $W \subseteq M \times Z$. Then $(i \times 1_Z)_*\beta \cdot p_1^*\alpha$ has support in $(i \times 1_Z)(W) \cap (\{y\} \times Z) \subset (M \times Z) \cap (\{y\} \times Z)$. If $y \notin M$, this subset is empty and we are done. Otherwise, up to linear equivalence we may replace y by a 0-cycle disjoint from M (*cf.* [25]), and we are back to the previous case.

This shows that I is an ideal of $\operatorname{Cor}^{\text{eff}}(F)$. The fact that it is a monoidal ideal is essentially obvious. □

We extend the ideal I from the category of effective Chow correspondences to the category of effective Chow motives (its pseudo-abelian envelope) in the obvious way, keeping the same notation. Let us define

$$\operatorname{Chow}^{\text{o}}(F) = (\operatorname{Chow}^{\text{eff}}(F)/I)^\natural$$

where $\natural$ denotes pseudo-abelian envelope.

5.4. **Lemma.** *In* $\operatorname{Chow}^{\text{eff}}(F)$*,* I *contains the morphisms factoring through an object of the form* $M(1)$*. If* $\operatorname{char} F = 0$*, any morphism in* I *is of this form. If* $\operatorname{char} F > 0$*, this is true at least after tensoring morphisms with* $\mathbf{Q}$*.*

Proof. Consider the Lefschetz motive L. As it is defined by the projector $\infty \times \mathbf{P}^1 \in CH^1(\mathbf{P}^1 \times \mathbf{P}^1)$, $L \mapsto 0 \in \operatorname{Chow}^{\text{o}}(F)$. Since I is monoidal, the first statement is clear. For the converse, it is enough to handle morphisms $f \in I(h(X), h(Y))$ for two smooth projective varieties X, Y. Assume that the cycle $f \in CH_{\dim X}(X \times Y)$ is of the form $(i \times 1_Y)_*g$ for some closed immersion $i : Z \to X$, where $g \in CH_{\dim X}(Z \times Y)$. If Z is smooth of codimension c, then f factors through $h(Z)(c)$. Without loss of generality, we may assume that Z is an integral divisor. We may further assume that g is the class of an integral variety $W \subseteq Z \times Y$. Moreover, let Z' be the singular locus of

Z: we may reduce to the case where $W \not\subseteq Z' \times Y$ by moving W, up to rational equivalence (*cf.* [25]).

Suppose first $\operatorname{char} F = 0$. Let $\pi : \widetilde{Z} \to Z$ be a desingularisation of Z. If we show that g is of the form $(\pi \times 1_Y)_*(\widetilde{g})$ for $\widetilde{g} \in CH_{\dim X}(\widetilde{Z} \times Y)$, we will have factored f as

$$h(X) \xrightarrow{(i\circ\pi)^!} h(\widetilde{Z})(1) \xrightarrow{\widetilde{g}} h(Y).$$

In view of the assumption on W, a suitable irreducible component $\widetilde{W}$ of $\pi^{-1}(W)$ has the same dimension as W and $\widetilde{W} \to W$ is a birational isomorphism. We take $\widetilde{g} = [\widetilde{W}]$.

Suppose now $\operatorname{char} F > 0$. By de Jong's theorem [8, Th. 4.1], rather than a desingularisation of Z we may at least find an alteration $\pi : \widetilde{Z} \to Z$ with $\widetilde{Z}$ smooth projective. Then an irreducible component $\widetilde{W}$ of $\pi^{-1}(W)$ has the same dimension as W and $\widetilde{W} \to W$ is an alteration. If d is its generic degree, we take $\widetilde{g} = \frac{1}{d}[\widetilde{W}]$. □

5.5. **Lemma.** *If* $\operatorname{char} F = 0$, *the functor* (5.1) *induces a commutative diagram of tensor functors*

$$\begin{array}{ccc} \mathrm{Chow}^{\mathrm{eff}}(F) & \longrightarrow & DM^{\mathrm{eff}}_{\mathrm{gm}}(F) \\ \downarrow & & \downarrow \\ \mathrm{Chow}^{\mathrm{o}}(F) & \longrightarrow & DM^{\mathrm{o}}_{\mathrm{gm}}(F). \end{array}$$

Proof. This follows from Proposition 5.2 and Lemma 5.4. □

For a smooth projective variety X, denote as above by $\bar{h}(X)$ the image of $h(X)$ in $\mathrm{Chow}^{\mathrm{o}}(F)$.

5.6. **Lemma.** *Suppose* $\operatorname{char} F = 0$, *and let* $s \in S_r$ *be a morphism between two smooth projective varieties* X *and* Y. *Then* $\bar{h}(s) : \bar{h}(X) \to \bar{h}(Y)$ *is an isomorphism. Therefore the commutative diagram of Lemma 5.5 may be completed into a commutative diagram*

$$\begin{array}{ccccccc} & & \mathrm{Sm}_{\mathrm{proj}}(F) & \longrightarrow & \mathrm{Chow}^{\mathrm{eff}}(F) & \longrightarrow & DM^{\mathrm{eff}}_{\mathrm{gm}}(F) \\ & & \downarrow & & \downarrow & & \downarrow \\ T^{-1}\,\mathrm{place}(F)^{\mathrm{op}} & \xrightarrow{T^{-1}\bar{\Phi}_1} & S_r^{-1}\,\mathrm{Sm}_{\mathrm{proj}}(F) & \longrightarrow & \mathrm{Chow}^{\mathrm{o}}(F) & \longrightarrow & DM^{\mathrm{o}}_{\mathrm{gm}}(F). \end{array}$$

Proof. By lemma 3.1 b), it suffices to deal with the cases where s is a projection $Y \times \mathbf{P}^1 \to Y$ and where $s \in S_b$. In the first case, we have $h(Y \times \mathbf{P}^1) = h(Y) \oplus h(Y)(1)$, so the result follows from Lemma 5.4.

In the second case, by resolution of singularities it suffices to prove it when s is a [concrete] blow-up with smooth centre Z. Then

$$h(X) = h(Y) \oplus \bigoplus_{i=1}^{d} h(Z)(i)$$

with $d = \operatorname{codim}_Y Z$, and the result again follows from Lemma 5.4. □

5.7. **Question.** In Lemma 5.5, are the vertical functors surjective?

6. Birational motivic complexes

6.1. **Definition.** We denote by $DM_-^{o}(F)$ the full subcategory of $DM_-^{\mathrm{eff}}(F)$ consisting of those objects C such that, for any open immersion $j : U \to X$ of smooth schemes, the map

$$H^i_{\mathrm{Nis}}(X, C) \to H^i_{\mathrm{Nis}}(U, C)$$

is an isomorphism for any i.

(Note that we write $H^i_{\mathrm{Nis}}(X, C)$ for the hypercohomology of C, which is sometimes written $\mathbb{H}^i_{\mathrm{Nis}}(X, C)$.)

6.2. **Lemma.** $DM_-^{o}(F) \subseteq \{C \in DM_-^{\mathrm{eff}}(F) \mid \underline{\mathrm{Hom}}(\mathbf{Z}(1), C) = 0\}$. *If F is perfect, this inclusion is an equality.*

Proof. Recall that, for any $X \in \mathrm{Sm}(F)$ and any $i \in \mathbf{Z}$, we have an isomorphism

$$H^i_{\mathrm{Nis}}(X, C) \simeq Hom(M(X), C[i]).$$

Let $C \in DM_-^{o}(F)$. For any $X \in \mathrm{Sm}(F)$, we have an open immersion $j : X \times \mathbf{A}^1 \hookrightarrow X \times \mathbf{P}^1$, yielding an exact triangle

$$M(X \times \mathbf{A}^1) \to M(X \times \mathbf{P}^1) \to M(X)(1)[2] \to M(X \times \mathbf{A}^1)[1]$$

hence a long exact sequence

$$\cdots \to H^{i+1}_{\mathrm{Nis}}(X \times \mathbf{P}^1, C) \to H^{i+1}_{\mathrm{Nis}}(X \times \mathbf{A}^1, C) \to Hom(M(X)(1), C[i]) \\ \to H^{i+2}_{\mathrm{Nis}}(X \times \mathbf{P}^1, C) \to H^{i+2}_{\mathrm{Nis}}(X \times \mathbf{A}^1, C) \to \dots$$

The condition on C then translates as $Hom(M(X)(1), C[i]) = 0$ for all i, or [32, §3.2] $Hom(M(X), \underline{\mathrm{Hom}}(\mathbf{Z}(1), C)[i]) = 0$ for all i. Since the $M(X)$ generate a dense triangulated subcategory of $DM_-^{\mathrm{eff}}(F)$, it follows that $Hom(D, \underline{\mathrm{Hom}}(\mathbf{Z}(1), C)) = 0$ for any $D \in DM_-^{\mathrm{eff}}(F)$, hence that $\underline{\mathrm{Hom}}(\mathbf{Z}(1), C) = 0$.

Conversely, suppose F perfect and let C verify the condition of the lemma. Let $j : U \to X$ be an open immersion with (reduced) closed complement Z. If Z is smooth of pure codimension $c > 0$, then the

Gysin exact triangle yields that $H^*_{\mathrm{Nis}}(j, C)$ is an isomorphism. In general, we filter Z by suitable closed subvarieties (starting with its singular locus) so as to reduce to this case. □

To go further, we need to recall a result from [15]. Let $DM_-^{\mathrm{eff}}(F)(1)$ denote the full subcategory of $DM_-^{\mathrm{eff}}(F)$ consisting of motives of the form $M(1)$. As in the proof of Proposition 5.2, we see by the cancellation theorem of [34], or rather its extension to $DM_-^{\mathrm{eff}}(F)$, that $DM_-^{\mathrm{eff}}(F)(1)$ is a thick triangulated subcategory of $DM_-^{\mathrm{eff}}(F)$.

For $M \in DM_-^{\mathrm{eff}}(F)$, let

$$\nu^{\geq 1}(M) = \underline{\mathrm{Hom}}_{\mathrm{eff}}(\mathbf{Z}(1), M)(1)$$

where $\underline{\mathrm{Hom}}_{\mathrm{eff}}$ is the [partially defined] internal Hom of $DM_-^{\mathrm{eff}}(F)$ [32, Prop. 3.2.8]. There is a tautological morphism

$$\nu^{\geq 1} M \to M. \tag{6.1}$$

It is proven in [15] that for any $M \in DM_-^{\mathrm{eff}}(F)$, the cone $\nu_{\leq 0} M$ of (6.1) is well-defined up to *unique* isomorphism, and that this defines a triangulated functor

$$\nu_{\leq 0} : DM_-^{\mathrm{eff}}(F) \to DM_-^{\mathrm{eff}}(F).$$

Let us briefly recall the argument: using the cancellation theorem [34], we get

$$Hom_{DM_-^{\mathrm{eff}}(F)}(DM_-^{\mathrm{eff}}(F)(1), \nu_{\leq 0} M) = 0 \text{ for any } M \in DM_-^{\mathrm{eff}}(F) \tag{6.2}$$

and the two claims easily follow from this by a classical triangle argument.

The following lemma constructs a left adjoint/left inverse for the inclusion functor.

6.3. **Lemma.** *Suppose F perfect. Then the functor $\nu_{\leq 0}$ takes its values in $DM_-^{\mathrm{o}}(F)$ and is left adjoint/left inverse to the inclusion $i : DM_-^{\mathrm{o}}(F) \to DM_-^{\mathrm{eff}}(F)$.*

Proof. For the first claim, we have to see that for any $X \in Sm/F$ and any dense open subset U, the homomorphisms

$$Hom_{DM_-^{\mathrm{eff}}(F)}(M(X), \nu_{\leq 0} M[i]) \to Hom_{DM_-^{\mathrm{eff}}(F)}(M(U), \nu_{\leq 0} M[i])$$

are isomorphisms for any $i \in \mathbf{Z}$. This follows from (6.2) and Proposition 5.2 b). For left adjointness, we have to see that the map

$$Hom_{DM_-^{\mathrm{eff}}(F)}(\nu_{\leq 0} M, N) \to Hom_{DM_-^{\mathrm{eff}}(F)}(M, N)$$

is bijective for any $M \in DM_-^{\mathrm{eff}}(F)$ and any $N \in DM_-^{\mathrm{o}}(F)$. This is equivalent to seeing that $Hom_{DM_-^{\mathrm{eff}}(F)}(\nu^{\geq 1} M, N) = 0$ for any such pair

(M,N), or that $Hom_{DM_-^{\mathrm{eff}}(F)}(P(1),N)=0$ for any $P\in DM_-^{\mathrm{eff}}(F)$ and any $N\in DM_-^{\mathrm{o}}(F)$. Since $DM_{\mathrm{gm}}^{\mathrm{eff}}(F)$ is dense in $DM_-^{\mathrm{eff}}(F)$, we may assume that $P\in DM_{\mathrm{gm}}^{\mathrm{eff}}(F)$). Then it follows again from Proposition 5.2 b).

Finally, to see that $\nu_{\leq 0}$ is left inverse to i, it suffices to see that $\nu^{\geq 1}\circ i=0$, which is proven in the same way as above. □

6.4. **Theorem.** *Suppose F perfect. Then*
a) The functor $\nu_{\leq 0}$ identifies $DM_-^{\mathrm{o}}(F)$ with the localisation of $DM_-^{\mathrm{eff}}(F)$ with respect to $DM_-^{\mathrm{eff}}(F)(1)$.
b) $DM_-^{\mathrm{o}}(F)$ inherits a canonical tensor structure.
c) The diagram of Lemma 5.5 extends into a commutative diagram of tensor functors

$$\begin{array}{ccccc}
\mathrm{Chow}^{\mathrm{eff}}(F) & \longrightarrow & DM_{\mathrm{gm}}^{\mathrm{eff}}(F) & \longrightarrow & DM_-^{\mathrm{eff}}(F)\\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \nu_{\leq 0}}\\
\mathrm{Chow}^{\mathrm{o}}(F) & \longrightarrow & DM_{\mathrm{gm}}^{\mathrm{o}}(F) & \longrightarrow & DM_-^{\mathrm{o}}(F).
\end{array}$$

The functor $DM_{\mathrm{gm}}^{\mathrm{o}}(F)\longrightarrow DM_-^{\mathrm{o}}(F)$ is fully faithful with dense image, and the image of any object of $DM_{\mathrm{gm}}^{\mathrm{o}}(F)$ is compact in $DM_-^{\mathrm{o}}(F)$.

Proof. It follows from the definition of $\nu_{\leq 0}$ that its kernel is $DM_-^{\mathrm{eff}}(F)(1)$. Together with Lemma 6.3, this shows a). It implies that the tensor structure of $DM_-^{\mathrm{eff}}(F)$ descends to a tensor structure on $DM_-^{\mathrm{o}}(F)$ via $\nu_{\leq 0}$, hence b). This also shows that the composition $DM_{\mathrm{gm}}^{\mathrm{eff}}(F)\to DM_-^{\mathrm{eff}}(F)\to DM_-^{\mathrm{o}}(F)$ factors uniquely through $DM_{\mathrm{gm}}^{\mathrm{o}}(F)$. By construction, the resulting functor $DM_{\mathrm{gm}}^{\mathrm{o}}(F)\longrightarrow DM_-^{\mathrm{o}}(F)$ is monoidal.

The other claims of c) follow from Lemma A.1: the only thing to observe is that objects of $DM_{\mathrm{gm}}^{\mathrm{eff}}(F)$ are compact in $DM_-^{\mathrm{eff}}(F)$. This is well-known: one immediately reduces to an object of the form $M(X)$, with X smooth. Then, as used in the proof of Lemma 6.2, $Hom(M(X),-)$ is Nisnevich cohomology of X, which commutes with arbitrary direct sums of sheaves. (We are indebted to Fabien Morel for pointing out this fact and the argument.) □

7. Main theorem

In this section, we assume that $\mathrm{char}\,F=0$**.** For the reader's convenience, we include a proof of the following lemma, which is in [32, proof of Prop. 2.1.4] for X projective:

7.1. **Lemma.** *Let X, Y be two smooth F-varieties, with Y projective. Then the sequence*

$$c(X \times \mathbf{A}^1, Y) \xrightarrow{s_0^* - s_1^*} c(X, Y) \to CH_{\dim X}(X \times Y) \to 0 \tag{7.1}$$

is exact.

Proof. Let $L(Y)$ and $L^c(Y)$ be the presheaves with transfers defined in [32, §4.1]. Then the cokernel of $s_0^* - s_1^*$ is clearly isomorphic to $h_0(C_*(L(Y))(X))$. On the other hand, since Y is projective, the morphism of presheaves $L(Y) \to L^c(Y)$ is an isomorphism. The latter presheaf is canonically isomorphic to $z_{\mathrm{equi}}(Y, 0)$ (compare [32, §4.2]). The group $CH_{\dim X}(X \times Y)$, in its turn, is canonically isomorphic to $h_0(z_{\mathrm{equi}}(X \times Y, \dim X))(\operatorname{Spec} F)$. We therefore have to see that the natural map

$$h_0(z_{\mathrm{equi}}(Y, 0))(X) \to h_0(z_{\mathrm{equi}}(X \times Y, \dim X))(\operatorname{Spec} F)$$

is an isomorphism. But the left hand side may be further rewritten

$$h_0(z_{\mathrm{equi}}(Y, 0))(X) = h_0(z_{\mathrm{equi}}(X, Y, 0))(\operatorname{Spec} F)$$

(*cf.* [10, bottom p. 142]). The result now follows from [10, Th. 7.4]. (If X is projective, we may just use [10, Th. 7.1], which does not require resolution of singularities.) □

By Lemma 5.6 and Theorem 3.8, we have a canonical functor

$$\widetilde{h}^o : S_r^{-1} \operatorname{Sm}(F) \to \operatorname{Chow}^o(F).$$

7.2. **Proposition.** *The functor $\widetilde{h}^o$ factors through an equivalence of categories*

$$h^o : S_r^{-1} \operatorname{SmCor}(F)^\natural \xrightarrow{\sim} \operatorname{Chow}^o(F)$$

where $S_r^{-1} \operatorname{SmCor}(F)^\natural$ is the pseudo-abelian envelope of $S_r^{-1} \operatorname{SmCor}(F)$.

Proof. Recall the full subcategory $\operatorname{SmCor}_{\mathrm{proj}}(F)$ of $\operatorname{SmCor}(F)$ given by smooth projective varieties. The proof of [32, Prop. 2.1.4] in effect defines a functor $\operatorname{SmCor}_{\mathrm{proj}}(F) \to \operatorname{Chow}^{\mathrm{eff}}(F)$, induced by the obvious homomorphisms $c(X, Y) \to CH_{\dim X}(X \times Y)$ for two smooth projective varieties X, Y. By Lemma 5.6, this functor fits into a commutative diagram

$$\begin{array}{ccccc}
\operatorname{SmCor}(F) & \longleftarrow & \operatorname{SmCor}_{\mathrm{proj}}(F) & \longrightarrow & \operatorname{Chow}^{\mathrm{eff}}(F) \\
\downarrow & & \downarrow & & \downarrow \\
S_r^{-1} \operatorname{SmCor}(F)^\natural & \longleftarrow & S_r^{-1} \operatorname{SmCor}_{\mathrm{proj}}(F)^\natural & \longrightarrow & \operatorname{Chow}^o(F).
\end{array}$$

In view of Proposition 4.1, the bottom left horizontal functor is an equivalence of categories: this extends $\widetilde{h}^o$ to a functor h^o as said. To see that $h^{\rm o}$ is an equivalence of categories, note that the exactness of (7.1) at $CH_{\dim X}(X \times Y)$ already implies that this functor is full. On the other hand, its exactness at $c(X, Y)$ and Proposition 4.1 imply that the obvious functor $T : \mathrm{SmCor}_{\mathrm{proj}}(F) \to S_r^{-1}\,\mathrm{SmCor}_{\mathrm{proj}}(F)^\natural$ factors through $\mathrm{Chow}^{\mathrm{eff}}(F)$ (since the two morphisms s_0 and s_1 become equal in $S_r^{-1}\,\mathrm{SmCor}(F)$ by homotopy invariance). Using now the exactness of (7.1) for the open subsets U of X, a small diagram chase implies that T further factors through $\mathrm{Chow}^{\rm o}(F)$ (because $S_r^{-1}c(X,Y) \xrightarrow{\sim} S_r^{-1}c(U,Y)$ for any such U), which constructs an inverse to $h^{\rm o}$. □

7.3. **Corollary.** *The projection functor* $\mathrm{Chow}^{\mathrm{eff}}(F) \to \mathrm{Chow}^{\rm o}(F)$ *induces equivalences of categories*

$$S_b^{-1}\,\mathrm{Chow}^{\mathrm{eff}}(F)^\natural \xrightarrow{\sim} S_r^{-1}\,\mathrm{Chow}^{\mathrm{eff}}(F)^\natural \xrightarrow{\sim} \mathrm{Chow}^{\rm o}(F).$$

Proof. The existence of the functors follows from Lemma 5.6. The second functor is clearly full. In the commutative diagram

$$\begin{array}{ccc} S_r^{-1}\,\mathrm{Chow}^{\mathrm{eff}}(F)^\natural & \longrightarrow & \mathrm{Chow}^{\rm o}(F) \\ \uparrow & \nearrow_{\simeq} & \\ S_r^{-1}\,\mathrm{SmCor}(F)^\natural & & \end{array}$$

the vertical functor is full by Lemma 7.1 and the diagonal functor is an equivalence of categories by Proposition 7.2. It follows that all functors are equivalences of categories.

To see that the composite functor in Corollary 7.3 is an equivalence of categories, we construct a quasi-inverse by showing that the functor $\mathrm{Chow}^{\mathrm{eff}} \to S_b^{-1}\,\mathrm{Chow}^{\mathrm{eff}}(F)^\natural$ factors through $\mathrm{Chow}^{\rm o}$. For this, it suffices by Lemma 5.4 to show that $L \mapsto 0$. In $S_b^{-1}\,\mathrm{Chow}^{\mathrm{eff}}(F)$, we have $h(\mathbf{P}^2) \simeq h(\mathbf{P}^1 \times \mathbf{P}^1)$, or

$$\mathbf{1} \oplus L \oplus L^{\otimes 2} \simeq (\mathbf{1} \oplus L)^{\otimes 2} \simeq \mathbf{1} \oplus L^{\oplus 2} \oplus L^{\otimes 2}$$

which implies $L \simeq 0$, as requested. □

7.4. **Proposition.** *Let $HI(F)$ be the category of homotopy invariant Nisnevich sheaves with transfers (compare* [32, Prop. 3.1.13]*), and let $HI^{\rm o}(F) = HI(F) \cap DM^{\rm o}_-(F)$. Then*
a) The obvious functor

$$HI^{\rm o}(F) \to S_r^{-1}\,\mathrm{SmCor}(F)\text{–}Mod = S_r^{-1}\,\mathrm{SmCor}(F)^\natural\text{–}Mod$$

is an isomorphism of categories, where $\mathcal{A}$–$Mod = Ab(\mathcal{A}, Ab)$ denotes the abelian dual of an additive category $\mathcal{A}$ (also called category of left $\mathcal{A}$-modules).

b) For any $\mathcal{F} \in HI^o(F)$ and any $X \in \mathrm{Sm}/F$, one has $H^i_{\mathrm{Nis}}(X, \mathcal{F}) = 0$ for all $i \neq 0$.

Proof. a) This amounts to seeing that any presheaf with transfers which is homotopy invariant and birationally invariant is a sheaf in the Nisnevich topology. But such a presheaf is locally constant for the Zariski topology; the conclusion readily follows, by using [21, p. 96, Prop. 1.4].

b) Since $\mathcal{F} \in HI(F)$, one has isomorphisms $H^i_{\mathrm{Zar}}(X, \mathcal{F}) \xrightarrow{\sim} H^i_{\mathrm{Nis}}(X, \mathcal{F})$ by [32, Th. 3.1.12]. Moreover, since $\mathcal{F}$ is locally constant for the Zariski topology, it is flasque. □

7.5. **Corollary.** *For any smooth projective X, the functor*

$$\bar{h}_0(X) : Y \mapsto CH_0(X_{F(Y)})$$

is a homotopy invariant and birationally invariant Nisnevich sheaf with transfers.

Proof. This follows from Propositions 7.2 and 7.4 a). □

We are now ready to compute the object $\nu_{\leq 0} M(X) \in DM^o_-(F)$ for any smooth projective variety X, and some Hom groups in $DM^o_{\mathrm{gm}}(F)$. Recall that, for any smooth X, $L(X)$ denotes the Nisnevich sheaf with transfers $Y \mapsto c(Y, X)$.

7.6. **Proposition.** *Let X be a smooth projective variety and, for any $Y \in \mathrm{SmCor}(F)$*

$$L(X)(Y) = c(Y, X) \to CH_0(X_{F(Y)}) = \bar{h}_0(X)(Y)$$

the morphism induced by the composite functor $\mathrm{SmCor}(F) \to S_r^{-1}\mathrm{SmCor}(F) \to \mathrm{Chow}^o(F)$. Then the corresponding morphism of Nisnevich sheaves with transfers $L(X) \to \bar{h}_0(X)$ (Corollary 7.5) factors through a morphism in $DM^o_-(F)$

$$\Phi_X : \nu_{\leq 0} M(X) \to \bar{h}_0(X)[0]$$

where $\bar{h}_0(X)[0]$ denotes the complex of Nisnevich sheaves with transfers defined by $\bar{h}_0(X)$ placed in degree 0.

Proof. Recall that $M(X) \in DM^{\mathrm{eff}}_-(F)$ is defined by the composite functor

$$\mathrm{SmCor}(F) \xrightarrow{L} Shv_{\mathrm{Nis}}(\mathrm{SmCor}(F)) \longrightarrow D^-(Shv_{\mathrm{Nis}}(\mathrm{SmCor}(F))) \xrightarrow{RC} DM^{\mathrm{eff}}_-(F)$$

where RC, left adjoint to the inclusion $DM^{\mathrm{eff}}_-(F) \to D^-(Shv_{\mathrm{Nis}}(\mathrm{SmCor}(F)))$, is induced by the Suslin-Voevodsky complex

C_* that "makes homotopy invariant" [32, §3.2]. Since $\bar h_0(X)\in HI^o(F)$, the claim now follows from a double adjunction (use Lemma 6.3). □

7.7. **Theorem.** *For any smooth projective X, the morphism Φ_X of Proposition 7.6 is an isomorphism.*

Proof. We first define a morphism in the other direction. Let A denote the composite functor

$$\mathrm{Chow}^o(F)\to DM^o_{\mathrm{gm}}(F)\to DM^o_-(F)$$

from Theorem 6.4 c). For any smooth variety Y, A induces a homomorphism

$$\begin{aligned}\bar h_0(X)(Y)&=Hom_{\mathrm{Chow}^o(F)}(h^o(Y),h^o(X))\\&\to Hom_{DM^o_-(F)}(Ah^o(Y),Ah^o(X))\\&=Hom_{DM^o_-(F)}(\nu_{\le 0}M(Y),\nu_{\le 0}M(X))\text{ (by Theorem 6.4 c))}\\&=Hom_{DM^{\mathrm{eff}}_-(F)}(M(Y),i\nu_{\le 0}M(X))\text{ (by Lemma 6.3)}\\&=H^0_{\mathrm{Nis}}(Y,i\nu_{\le 0}M(X))\end{aligned}$$

hence a morphism in $DM^o_-(F)$

$$\Psi_X:\bar h_0(X)[0]\to\nu_{\le 0}M(X).$$

It is clear from the construction of Φ_X and Ψ_X that $\Psi_X\Phi_X=Id$. In particular, $\nu_{\le 0}M(X)$ is concentrated in degree 0 and $H^0(\Phi_X)$ is a split monomorphism of sheaves. On the other hand, since $L(X)(Y)\to\bar h_0(X)(Y)$ is surjective for any X, the morphism of sheaves $L(X)\to\bar h_0(X)$ is an epimorphism, and so is $H^0(\Phi_X):H^0(\nu_{\le 0}M(X))\to\bar h_0(X)$, hence $H^0(\Phi_X)$ is an isomorphism of sheaves and Φ_X and Ψ_X are inverse to each other. □

7.8. **Corollary.** *Let B be the composite functor*

$$S_r^{-1}\,\mathrm{SmCor}(F)^\natural\to\mathrm{Chow}^o(F)\to DM^o_{\mathrm{gm}}(F)\to DM^o_-(F)$$

from Theorem 6.4 c). Then $X\mapsto\Phi_X$ induces a natural isomorphism from B to the composite functor

$$C:S_r^{-1}\,\mathrm{SmCor}(F)^\natural\to(S_r^{-1}\,\mathrm{SmCor}(F))^\vee\to HI^o(F)\to DM^o_-(F)$$

where the first functor is the Yoneda embedding, the second one is the inverse of the functor in Proposition 7.4 a) and the third is the natural inclusion.

Proof. If we restrict B and C to $S_r^{-1}\operatorname{SmCor}(F)$, this is just a reformulation of Theorem 7.7. Then the natural transformation Φ automatically extends to $S_r^{-1}\operatorname{SmCor}(F)^\natural$. □

7.9. **Corollary.** *a) The functor* $\operatorname{Chow}^o(F) \to DM^o_{\mathrm{gm}}(F)$ *of Lemma 5.5 is fully faithful (hence, by Theorem 6.4 c), the composite functor* $\operatorname{Chow}^o(F) \to DM^o_{\mathrm{gm}}(F) \to DM^o_-(F)$ *is fully faithful).*
b) The image of the category $\operatorname{Chow}^o(F)$ *in* $HI^o(F)$ *consists of all compact objects of* $HI^o(F)$*, and all its objects are projective. It is an exact subcategory of* $HI^o(F)$*, and all exact sequences are split. In particular, for any smooth projective* X*,* $\nu_{\le 0}M(X)$ *is a compact projective object of* $HI^o(F)$*.*
c) In $DM^o_{\mathrm{gm}}(F)$*, one has*

$$Hom(\bar{M}(X), \bar{M}(Y)[i]) = \begin{cases} CH_0(Y_{F(X)}) & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

for two smooth varieties X, Y*, with* Y *projective, and* $i \in \mathbf{Z}$*.*

Proof. a) This follows from Theorem 6.4 c), Proposition 7.2, Proposition 7.4 a) and Theorem 7.7.

b) The statements are classical in view of Proposition 7.4 a), *cf.* for example [1, prop. 1.3.6 f)]. The last claim follows from Theorem 7.7.

c) This follows from Theorem 6.4 c), Theorem 7.7 and Proposition 7.4 b). □

8. Abelian schemes

In this section, F is perfect.

8.1. **Definition.** a) Let X be a smooth F-scheme (not necessarily of finite type). For each connected component X_i of X, let E_i be its field of constants, that is, the algebraic closure of F into $F(X_i)$. We define

$$\pi_0(X) = \coprod_i \operatorname{Spec} E_i.$$

There is a canonical F-morphism $X \to \pi_0(X)$; $\pi_0(X)$ is called the *scheme of constants* of X.
b) If $\dim X = 0$ (equivalently $X \xrightarrow{\sim} \pi_0(X)$), we write $\mathbf{Z}[X]$ for the 0-dimensional group scheme representing the étale sheaf $f_*\mathbf{Z}$, where $f : X \to \operatorname{Spec} F$ is the structural morphism.

For any smooth projective F-variety X, let $\mathcal{A}_{X/F} = \mathcal{A}_X$ be the Albanese scheme of X over F [24]: there is a canonical morphism

$$\varphi_X : X \to \mathcal{A}_X \tag{8.1}$$

which is universal for morphisms from X to group schemes locally of finite type whose neutral component is an Abelian variety. There is an exact sequence of group schemes

$$0 \to \mathcal{A}^0_X \to \mathcal{A}_X \to \mathbf{Z}[\pi_0(X)] \to 0$$

where $\mathcal{A}^0_X$ is the Albanese variety of X (an abelian variety) and $\pi_0(X)$ has been defined above.

The aim of this section is to organise the $\mathcal{A}_X$ into a pseudo-abelian symmetric monoidal additive category (see Propositions 8.10 and 9.2).

Let us recall from [24] a description of $\mathcal{A}_X$. Let $\mathbf{Z}[X]$ be the "free" presheaf on F-schemes defined by $\mathbf{Z}[X](Y) = \mathbf{Z}[X(Y)]$ and $\mathcal{Z}_{X/F} = \mathcal{Z}_X$ the associated sheaf on the big fppf site of $\operatorname{Spec} F$. Then $\mathcal{A}_X$ is the universal representable quotient of $\mathcal{Z}_X$. In other terms, there is a homomorphism

$$\mathcal{Z}_X \to \mathcal{A}_X$$

where $\mathcal{A}_X$ is considered as a representable sheaf, which is universal for homomorphisms from $\mathcal{Z}_X$ to abelian sheaves representable by an abelian F-scheme.

Let us also denote by P_X the universal torsor under $\mathcal{A}^0_X$ constructed by Serre [27]. There is a map $X \xrightarrow{\tilde\varphi_X} P_X$ which is universal for maps from X to torsors under abelian varieties. The torsor P_X and the group scheme $\mathcal{A}_X$ have the same class in $Ext^1_{(Sch/F)_{\text{ét}}}(\pi_0(\mathcal{A}_X), \mathcal{A}^0_X) = H^1_{\text{ét}}(\pi_0(X), \mathcal{A}^0_X)$ (here we identify $\mathcal{A}^0_X$ with the corresponding representable étale sheaf over the big étale site of $\operatorname{Spec} F$). A beautiful concrete description of this correspondence is given in [24, 1.2]. The map $\tilde\varphi_X$ induces an isomorphism

$$\mathcal{A}_{P_X} \xrightarrow{\sim} \mathcal{A}_X.$$

We repeat some properties of $\mathcal{A}_X$ as taken from [24, Prop. 1.6 and Cor. 1.12] and add one.

8.2. **Proposition.** *a) $X \mapsto \mathcal{A}_X$ is covariant for arbitrary [proper] maps.*
b) Let K/F be an extension. Then the natural map $\mathcal{A}_{X_K/K} \to \mathcal{A}_{X/F} \otimes_F K$ stemming from the universal property is an isomorphism.
c) If $X = Y \coprod Z$, then the natural map $\mathcal{A}_{Y/F} \oplus \mathcal{A}_{Z/F} \to \mathcal{A}_{X/F}$ is an isomorphism.
d) Let E/F be a finite extension. For any E-scheme S, let $S_{(F)}$ denote the (ordinary) restriction of scalars of S, i.e. we view S as an F-scheme. Then there is a natural isomorphism

$$\mathcal{A}_{X_{(F)}/F} \to R_{E/F}\mathcal{A}_{X/E}$$

where $R_{E/F}$ denotes Weil's restriction of scalars.

Proof. The only thing which is not in [24] is d). We shall construct the isomorphism by descent from c), using b).

Let $f : \operatorname{Spec} E \to \operatorname{Spec} F$ be the structural morphism. Recall that, for any sheaf $\mathcal{G}$ on $(Sch/E)_{\text{ét}}$, the trace map defines an isomorphism

$$f_*\mathcal{G} \xrightarrow{\sim} f_!\mathcal{G}$$

where $f_!$ (*resp.* f_*) is the left (*resp.* right) adjoint of the restriction functor f^*. This isomorphism is natural in $\mathcal{G}$.

This being said, the additive version of Yoneda's lemma immediately yields

$$f_!\mathcal{Z}_{X/E} = \mathcal{Z}_{X_{(F)}/F}$$

hence a composition of homomorphisms of sheaves

$$f_*\mathcal{Z}_{X/E} \xrightarrow{\sim} \mathcal{Z}_{X_{(F)}/F} \to Shv(\mathcal{A}_{X_{(F)}/F}) \tag{8.2}$$

where, for clarity, $Shv(\mathcal{A}_{X_{(F)}/F})$ denotes the sheaf associated to the group scheme $\mathcal{A}_{X_{(F)}/F}$. We also have a chain of homomorphisms

$$f_*\mathcal{Z}_{X/E} \to f_*Shv(\mathcal{A}_{X/E}) \xrightarrow{\sim} Shv(R_{E/F}\mathcal{A}_{X/E}) \tag{8.3}$$

where the last isomorphism is formal. If we can prove that (8.2) factors through (8.3) into an isomorphism, we are done by Yoneda.

In order to do this, we may assume via b) that F is algebraically closed, hence that f is completely split. Then the claim follows from c). □

We record here similar properties for the torsor $P_X = P_{X/F}$ (proofs are similar):

8.3. **Proposition.** *a) $X \mapsto P_X$ is a functor.*
b) Let K/F be an extension. Then the natural map $P_{X_K/K} \to P_{X/F} \otimes_F K$ stemming from the universal property is an isomorphism.
c) If $X = Y \coprod Z$, then there is an isomorphism $P_{Y/F} \times P_{Z/F} \xrightarrow{\sim} P_{X/F}$ which is natural in (Y, Z).
d) Let E/F be a finite extension. Then there is a natural isomorphism

$$P_{X_{(F)}/F} \to R_{E/F}P_{X/E}.\square$$

(In c), the map stems from the fact that coproducts correspond to scheme-theoretic products in an appropriate category of torsors.)

8.4. **Definition.** a) For an F-group scheme G, we denote by G^0 the kernel of the canonical map $G \to \pi_0(G)$ of Definition 8.1: this is the *neutral component* of G.
b) An F-group scheme G is called a *lattice* if $G^0 = \{1\}$ and the geometric fibre of $\pi_0(G)(= G)$ is a free finitely generated abelian group.

8.5. **Definition.** We denote by $\mathrm{AbS}(F)$ the full subcategory of the category of abelian F-group schemes consisting of those objects $\mathcal{A}$ such that

- $\pi_0(\mathcal{A})$ is a lattice;
- $\mathcal{A}^0$ is an abelian variety.

Objects of $\mathrm{AbS}(F)$ will be called *abelian F-schemes*.

Recall the Yoneda full embedding $Shv : \mathrm{AbS}(F) \to Ab((Sch/F)_{\text{ét}})$, where the latter is the category of sheaves of abelian groups over the big étale site of $\operatorname{Spec} F$.

8.6. **Lemma.** *a) If a sheaf $\mathcal{F}$ in $Ab((\operatorname{Spec} F)_{\text{ét}})$ is an extension of a lattice L by an abelian variety A, then it is represented by an object of $\mathrm{AbS}(F)$.*
b) Let A be an abelian variety and L a lattice. Then the étale sheaf $B = A \otimes L$ is represented by an abelian variety.

Proof. a) If L is constant, then the choice of a basis of L determines a section of the projection $\mathcal{F} \to Shv(L)$, hence an isomorphism $\mathcal{F} \simeq Shv(A) \oplus Shv(L)$. Then $\mathcal{F}$ is represented by $\coprod_{l\in L} A$. In general, L becomes constant on some finite extension E/F, hence $\mathcal{F}_E$ is representable. By full faithfulness, the descent data of $\mathcal{F}_E$ are morphisms of schemes; then we may apply [28, Cor. V.4.2 a) or b)].
b) Same method as in a). □

8.7. **Example.** If $L = \mathbf{Z}[\operatorname{Spec} E]$, where E is an étale F-algebra, then $A \otimes L = R_{E/F} A_E$.

Let $\mathcal{A}, \mathcal{B} \in \mathrm{AbS}(F)$. Viewing them as étale sheaves, we may consider their tensor product $\mathcal{A} \otimes_{shv} \mathcal{B}$. This tensor product contains the subsheaf $\mathcal{A}^0 \otimes_{shv} \mathcal{B}^0$, which is clearly not representable. We define

$$\mathcal{A} \otimes_{\text{rep}} \mathcal{B} = \mathcal{A} \otimes_{shv} \mathcal{B} / \mathcal{A}^0 \otimes_{shv} \mathcal{B}^0.$$

8.8. **Proposition.** *a) $\mathcal{A} \otimes_{\text{rep}} \mathcal{B}$ is representable by an object of* $\mathrm{AbS}(F)$.
b) For $X, Y \in \mathrm{Sm}_{\text{proj}}(F)$, the natural map

$$\mathcal{Z}_X \otimes_{shv} \mathcal{Z}_Y = \mathcal{Z}_{X\times Y} \to \mathcal{A}_{X\times Y}$$

factors into an isomorphism

$$\mathcal{A}_X \otimes_{\text{rep}} \mathcal{A}_Y \xrightarrow{\sim} \mathcal{A}_{X\times Y}.$$

(This corrects [24, Cor. 1.12 (vi)]*.)*

Proof.
a) We have a short exact sequence

$$0 \to \mathcal{A}^0 \otimes \pi_0(\mathcal{B}) \oplus \mathcal{B}^0 \otimes \pi_0(\mathcal{A}) \to \mathcal{A} \otimes_{\text{rep}} \mathcal{B} \to \pi_0(\mathcal{A}) \otimes \pi_0(\mathcal{B}) \to 0.$$

By Lemma 8.6 b), the left hand side is representable by an abelian variety, and the right hand side is clearly a lattice. We conclude by Lemma 8.6 a).

b) It is enough to show that this holds over the algebraic closure of F. Using Proposition 8.2 c) (and the similar statement for $\mathcal{Z}$), we may assume that X and Y are connected. We shall show more generally that, for any abelian scheme $\mathcal{B}$ and any map $X \times Y \to \mathcal{B}$, the induced sheaf-theoretic map

$$\mathcal{Z}_X \otimes_{shv} \mathcal{Z}_Y \to \mathcal{B} \tag{8.4}$$

factors through $\mathcal{A}_X \otimes_{\mathrm{rep}} \mathcal{A}_Y$. By a), this will show that the latter has the universal property of $\mathcal{A}_{X\times Y}$.

For $n \in \mathbf{Z}$, we shall denote by $\mathcal{Z}_X^n$ or $\mathcal{A}_X^n$ the inverse image of n under the augmentation map $\mathcal{Z}_X \to \mathbf{Z}$ or $\mathcal{A}_X \to \mathbf{Z}$ stemming from the structural morphism $X \to \operatorname{Spec} F$. It is a subsheaf of $\mathcal{Z}_X$ or $\mathcal{A}_X$, and $\mathcal{A}_X^n$ is clearly representable (by a variety isomorphic to the abelian variety $\mathcal{A}_X^0$). We shall also identify varieties with representable sheaves: this should create no confusion in view of Yoneda's lemma.

We first show that (8.4) factors through $\mathcal{A}_X \otimes_{shv} \mathcal{A}_Y$. It suffices to show that the composition

$$\mathcal{Z}_X \times Y \to \mathcal{Z}_X \otimes \mathcal{Z}_Y \to \mathcal{B}$$

factors through $\mathcal{A}_X \times Y$, and to conclude by symmetry. But $X \times Y$ is connected, so its image in $\mathcal{B}$ falls in some connected component $\mathcal{B}^t$ of $\mathcal{B}$, which is a torsor under $\mathcal{B}^0$; applying the "Variation en fonction d'un paramètre" statement in [27, p. 10-05], we see that it extends into a morphism $\mathcal{A}_X^1 \times Y \to \mathcal{B}^t$. Including $\mathcal{B}^t$ into $\mathcal{B}$, we get a commutative diagram

$$\begin{array}{ccc} \mathcal{A}_X^1 \times Y & \longrightarrow & \mathcal{B} \\ \uparrow & & \uparrow \\ \mathcal{Z}_X^1 \times Y & \longrightarrow & \mathcal{Z}_X \times Y. \end{array}$$

Let $\mathcal{K} = \operatorname{Ker}(\mathcal{Z}_X \to \mathcal{A}_X) = \operatorname{Ker}(\mathcal{Z}_X^0 \to \mathcal{A}_X^0)$. The diagram shows that the following diagram

$$\begin{array}{ccc} \mathcal{K} \times \mathcal{Z}_X^1 \times Y & \longrightarrow & \mathcal{Z}_X^1 \times Y \\ \downarrow & & \downarrow \\ \mathcal{Z}_X^1 \times Y & \longrightarrow & \mathcal{B} \end{array}$$

commutes, where the top horizontal map is given by the action of $\mathcal{K}$ on $\mathcal{Z}_X^1$ by left translation and the left vertical map is given by $(k, z, y) \mapsto$

(z, y). Since $\mathcal{Z}_X \times Y \to \mathcal{B}$ is a homomorphism in the first variable, this implies the desired factorisation.

We now show that the composition

$$\mathcal{A}_X^0 \otimes_{shv} \mathcal{A}_Y^0 \to \mathcal{A}_X \otimes \mathcal{A}_Y \to \mathcal{B}$$

is 0. It is sufficient to show that the composition of this map with the inclusion $\mathcal{A}_X^0 \times \mathcal{A}_Y^0 \to \mathcal{A}_X^0 \otimes \mathcal{A}_Y^0$ is 0. But $\mathcal{A}_X^0 \times \mathcal{A}_Y^0$ is connected, hence its image falls in some connected component, in fact in $\mathcal{B}^0$. This is a map from a product of irreducible varieties to an abelian variety which is 0 at $(0, 0)$: it has to be 0 by [9, Th. 2] (see also [20, Th. 3.4]). □

8.9**. Corollary.** $P_{X\times Y} = R_{\pi_0(X)/F}(P_Y \times_F \pi_0(X)) \times R_{\pi_0(Y)/F}(P_X \times_F \pi_0(Y))$.

Proposition 8.8 a) endows $\mathrm{AbS}(F)$ with a symmetric monoidal structure compatible with its additive structure.

8.10**. Proposition.** *The category* $\mathrm{AbS}(F)$ *is symmetric monoidal (for* $\otimes_{\mathrm{rep}}$*) and pseudo-abelian. Its Kelly radical* $\mathcal{R}$ *is monoidal and has square* 0*. After tensoring with* $\mathbf{Q}$*,* $\mathrm{AbS}(F)/\mathcal{R}$ *becomes isomorphic to the semi-simple category product of the category of abelian varieties up to isogenies and the category of* G_F*-*$\mathbf{Q}$*-lattices.*

Recall that the *Kelly radical* $\mathcal{R}$ of an additive catgegory $\mathcal{A}$ is defined by

$$\mathcal{R}(A, B) = \{f \in \mathcal{A}(A, B) \mid \forall g \in \mathcal{A}(B, A)\ 1_A - gf \text{ is invertible}\}$$

and that it is a [two-sided] ideal of $\mathcal{A}$ [19].

Proof. For the first claim, we just observe that kernels exist in the category of commutative F-group schemes, and that a direct summand of an abelian variety (*resp.* of a lattice) is an abelian variety (*resp.* a lattice). For the second claim, consider the functor

$$T : \mathrm{AbS}(F) \to \mathrm{Ab}(F) \times \mathrm{Lat}(F)$$
$$\mathcal{A} \mapsto (\mathcal{A}^0, \pi_0(\mathcal{A}))$$

where $\mathrm{Ab}(F)$ and $\mathrm{Lat}(F)$ are respectively the category of abelian varieties and the category of lattices over F (viewed, for example, as full subcategories of the category of étale sheaves over Sm/F). This functor is obviously essentially surjective. After tensoring with $\mathbf{Q}$, it becomes full, because any extension

$$0 \to \mathcal{A}^0 \to \mathcal{A} \to \pi_0(\mathcal{A}) \to 0$$

is rationally split. Now the collections of sets

$$\mathcal{I}(\mathcal{A}, \mathcal{B}) = \{f : \mathcal{A} \to \mathcal{B} \mid T(f) = 0\}$$

defines an ideal $\mathcal{I}$ of AbS(F). If $f \in \mathcal{I}(\mathcal{A}, \mathcal{B})$, then f induces a map

$$\bar{f} : \pi_0(\mathcal{A}) \to \mathcal{B}^0$$

and this gives a description of $\mathcal{I}$. From this description, it follows immediately that $\mathcal{I}^2 = 0$. In particular, $\mathcal{I} \subseteq \mathcal{R}$.

If we tensor with $\mathbf{Q}$, then Ab(F) $\times$ Lat(F) becomes semi-simple; since AbS(F)$/\mathcal{I} \otimes \mathbf{Q}$ is semi-simple and $\mathcal{I} \otimes \mathbf{Q}$ is nilpotent, it follows that $\mathcal{I} \otimes \mathbf{Q} = \mathcal{R} \otimes \mathbf{Q}$. In other terms, $\mathcal{R}/\mathcal{I}$ is torsion.

Let $f \in \mathcal{R}(\mathcal{A}, \mathcal{B})$. There exists $n > 0$ such that $nf(\mathcal{A}^0) = 0$. But $f(\mathcal{A}^0)$ is an abelian subvariety of $\mathcal{B}^0$, hence $f(\mathcal{A}^0) = 0$ and $f \in \mathcal{I}(\mathcal{A}, \mathcal{B})$. So $\mathcal{R} = \mathcal{I}$.

If we endow the category Ab(F) $\times$ Lat(F) with the tensor structure

$$(A, L) \otimes (B, M) = (A \otimes M \oplus B \otimes L, L \otimes M)$$

then T becomes a monoidal functor, which shows that $\mathcal{R} = \mathcal{I}$ is monoidal. This completes the proof of Proposition 8.10. □

8.11. **Remarks.** a) The morphisms in AbS(F) are best represented in matrix form:

$$Hom(\mathcal{A}, \mathcal{B}) = \begin{pmatrix} Hom(\mathcal{A}_0, \mathcal{B}_0) & Hom(\pi_0(\mathcal{A}), \mathcal{B}_0) \\ 0 & Hom(\pi_0(\mathcal{A}), \pi_0(\mathcal{B})) \end{pmatrix}$$

(note that $Hom(\mathcal{A}_0, \pi_0(\mathcal{B})) = 0$). This clarifies the arguments in the proof of Proposition 8.10 somewhat.

b) The Hom groups of Ab(F) $\times$ Lat(F) are finitely generated $\mathbf{Z}$-modules. It follows from the proof of Proposition 8.10 that, for $\mathcal{A}, \mathcal{B} \in$ AbS(F), $T(Hom(\mathcal{A}, \mathcal{B}))$ has finite index in $Hom(T(\mathcal{A}), T(\mathcal{B}))$. In particular, for any $\mathcal{A} \in$ AbS(F), $End(\mathcal{A})$ is an extension of an order in a semi-simple $\mathbf{Q}$-algebra by an ideal of square 0.

c) The functor T has the explicit section

$$(A, L) \mapsto A \oplus L.$$

This section is symmetric monoidal.

9. Birational motives and locally abelian schemes

For any smooth projective variety X, there is a canonical map

$$CH_0(X) \xrightarrow{\mathrm{Alb}_X^F} \mathcal{A}_X(F). \tag{9.1}$$

Recall the construction of Alb_X: the map φ_X of (8.1) defines for any extension E/F a map $X(E) \to \mathcal{A}_X(E)$, still denoted by φ_X. When E/F is finite, viewing $\mathcal{A}_X$ as an étale sheaf, we have a trace map $Tr_{E/F} : \mathcal{A}_X(E) \to \mathcal{A}_X(F)$. Then Alb_X maps the class of a closed point $x \in X$ with residue field E to $Tr_{E/F}\, \varphi_X(x)$.

The map Alb_X is injective for $\dim X = 1$ and surjective if F is algebraically closed. For a curve, this map corresponds to the isomorphism $\mathrm{Pic}_X \simeq \mathcal{A}_X$, where Pic_X is the Picard scheme of X; we then also have $\mathcal{A}_X^0 \simeq J_X$, where J_X is the Jacobian variety of X.

The functoriality of $\mathcal{A}$ shows that there is a chain of isomorphisms

$$\Phi_{X,Y} : Hom(\mathcal{A}_X, \mathcal{A}_Y) \xrightarrow{\sim} Mor(X, \mathcal{A}_Y) \xrightarrow{\sim} \mathcal{A}_Y(F(X)) \tag{9.2}$$

(the latter by Weil's theorem on extensions of morphisms to abelian varieties [20, Th. 3.1]), hence a canonical map

$$CH_0(Y_{F(X)}) \xrightarrow{\mathrm{Alb}_{X,Y}} Hom(\mathcal{A}_X, \mathcal{A}_Y) \tag{9.3}$$

which generalises (9.1); more precisely, we have

$$\Phi_{X,Y} \circ \mathrm{Alb}_{X,Y} = \mathrm{Alb}_Y^{F(X)}. \tag{9.4}$$

On the other hand, there is an exact sequence

$$\begin{aligned} 0 \to \mathcal{A}_Y(\pi_0(X)) = Hom(\mathbf{Z}[\pi_0(X)], \mathcal{A}_Y) \to Hom(\mathcal{A}_X, \mathcal{A}_Y) \\ \to Hom(\mathcal{A}_X^0, \mathcal{A}_Y) \to Ext^1(\mathbf{Z}[\pi_0(X)], \mathcal{A}_Y) = H^1(\pi_0(X), \mathcal{A}_Y) \end{aligned}$$

and the map $Hom(\mathcal{A}_X^0, \mathcal{A}_Y^0) \to Hom(\mathcal{A}_X^0, \mathcal{A}_Y)$ is an isomorphism. From this we get a zero sequence

$$0 \to CH_0(Y) \to CH_0(Y_{F(X)}) \to Hom(\mathcal{A}_X^0, \mathcal{A}_Y^0) \to 0. \tag{9.5}$$

9.1. **Lemma.** *Let Y, Z be two smooth projective varieties and $\beta \in CH_0(Z_{F(Y)})$. Then the following diagram commutes:*

$$\begin{array}{ccc} CH_0(Y) & \xrightarrow{\beta_*} & CH_0(Z) \\ {\scriptstyle \mathrm{Alb}_Y^F}\downarrow & & {\scriptstyle \mathrm{Alb}_Z^F}\downarrow \\ \mathcal{A}_Y(F) & \xrightarrow{\mathrm{Alb}_{Y,Z}(\beta)_*} & \mathcal{A}_Z(F). \end{array}$$

Proof. Without loss of generality, we may assume that β is given by an integral subscheme W in $Y \times Z$. Then the composite $f = p_Y i_W$ is a proper surjective generically finite morphism, where p_Y denotes the projection and i_W is the inclusion of W in $Y \times Z$.

Let V be an affine dense open subset of Y such that $f_{|f^{-1}(V)}$ is finite. Any element of $CH_0(Y)$ may be represented by a zero-cycle with support in V (*cf.* [25]), so it is enough to check the commutativity of the diagram on zero-cycles on Y of the form y, where $y \in V_{(0)}$. For such a y, we have $\beta_* y = p_*(f^{-1}(y))$, where $p = p_Z i_W$.

On the other hand, the composition $\mathrm{Alb}_{Y,Z}(\beta)_* \circ (\mathrm{Alb}_Y^F)_{|V}$ may be described as follows: let d be the degree of $f_{|f^{-1}(V)}$, $f^{-1}(V)^{[d]}$ the d-fold symmetric power of $f^{-1}(V)$ and $f^* : V \to f^{-1}(V)^{[d]}$ the map

$x \mapsto f^{-1}(x)$. Then

$$\mathrm{Alb}_{Y,Z}(\beta)_* \circ (\mathrm{Alb}_Y^F)_{|V} = \Sigma_d \circ (\varphi_Z)^{[d]} \circ p_*^{[d]} \circ f^*$$

where $\Sigma_d : \mathcal{A}_Z^{[d]} \to \mathcal{A}_Z$ is the summation map. The commutation of the diagram is now clear. □

9.2. **Proposition.** *The assignment $X \mapsto \mathcal{A}_X$ defines via* (9.3) *a symmetric monoidal additive functor*

$$\mathrm{Alb} : \mathrm{Chow}^o(F) \to \mathrm{AbS}(F)$$

which becomes full and essentially surjective after tensoring with $\mathbf{Q}$.

Proof. Since $\mathrm{AbS}(F)$ is pseudo-abelian, it suffices to construct the functor on $\mathrm{Cor}^o(F)$. Let $\alpha \in CH_0(Y_{F(X)})$ and $\beta \in CH_0(Z_{F(Y)})$. We want to show that $\mathrm{Alb}_{X,Z}(\beta \circ \alpha) = \mathrm{Alb}_{Y,Z}(\beta) \circ \mathrm{Alb}_{X,Y}(\alpha)$. But β induces a map

$$\beta_* : CH_0(Y_{F(X)}) \to CH_0(Z_{F(X)}),$$

and we have the equality $\beta_*\alpha = \beta \circ \alpha$ (*cf.* proof of Lemma 5.3). Hence, applying Lemma 9.1 by replacing F by $F(X)$, we get

$$\mathrm{Alb}_Z^{F(X)}(\beta \circ \alpha) = \mathrm{Alb}_Z^{F(X)}(\beta_*\alpha) = \mathrm{Alb}_{Y,Z}(\beta)_*(\mathrm{Alb}_Y^{F(X)}(\alpha)).$$

Applying now (9.4), we get

$$\Phi_{X,Z} \circ \mathrm{Alb}_{X,Z}(\beta \circ \alpha) = \mathrm{Alb}_{Y,Z}(\beta)_*(\Phi_{X,Y} \circ \mathrm{Alb}_{X,Y}(\alpha)).$$

On the other hand, the diagram

$$\begin{array}{ccc} \mathcal{A}_Y(F(X)) & \xrightarrow{\mathrm{Alb}_{Y,Z}(\beta)_*} & \mathcal{A}_Z(F(X)) \\ {\scriptstyle \Phi_{X,Y}}\uparrow\wr & & {\scriptstyle \Phi_{X,Z}}\uparrow\wr \\ Hom(\mathcal{A}_X, \mathcal{A}_Y) & \xrightarrow{\mathrm{Alb}_{Y,Z}(\beta)_*} & Hom(\mathcal{A}_X, \mathcal{A}_Y) \end{array}$$

obviously commutes, which concludes the proof that Alb is a functor.

Compatibility with the monoidal structures follows from Proposition 8.8 b). It remains to show the assertions on fullness and essential surjectivity.

Fullness: for any Y, the map $\mathrm{Alb}_Y^F \otimes \mathbf{Q}$ is surjective. This follows from the case where F is algebraically closed (in which case Alb_Y^F itself is surjective) by a transfer argument. Replacing the ground field F by $F(X)$ for some other X, we get that $\mathrm{Alb}_{X,Y} \otimes \mathbf{Q}$ is surjective. This shows that the restriction of $\mathrm{Alb} \otimes \mathbf{Q}$ to $\mathrm{Cor}^o(F) \otimes \mathbf{Q}$ is full; but the pseudo-abelianisation of a full functor is evidently full (a direct summand of a surjective homomorphism of abelian groups is surjective).

Essential surjectivity: we first note that, after tensoring with $\mathbf{Q}$, the extension

$$0 \to \mathcal{A}^0 \to \mathcal{A} \to \pi_0(\mathcal{A}) \to 0$$

becomes split for any $\mathcal{A} \in \mathrm{AbS}(F)$. Indeed the extension class belongs to $Ext^1_F(\pi_0(\mathcal{A}), \mathcal{A}^0)$; this group sits in an exact sequence (coming from an Ext spectral sequence)

$$0 \to H^1(F, Hom_{\bar F}(\pi_0(\mathcal{A})_{|\bar F}, \mathcal{A}^0_{|\bar F})) \to Ext^1_F(\pi_0(\mathcal{A}), \mathcal{A}^0) \\ \to H^0(F, Ext^1_{\bar F}(\pi_0(\mathcal{A})_{|\bar F}, \mathcal{A}^0_{|\bar F})).$$

Since the restriction $\pi_0(\mathcal{A})_{|\bar F}$ is a constant sheaf of free finitely generated abelian groups, the group $Ext^1_{\bar F}(\pi_0(\mathcal{A})_{|\bar F}, \mathcal{A}^0_{|\bar F})$ is 0, while the left group is torsion as a Galois cohomology group. It is now sufficient to show separately that L and A are in the essential image of $\mathrm{Alb} \otimes \mathbf{Q}$, where L (*resp.* A) is a lattice (*resp.* an abelian variety).

A lattice L corresponds to a continuous integral representation ρ of G_F. But it is well-known that $\rho \otimes \mathbf{Q}$ is of the form $\theta \otimes \mathbf{Q}$, where θ is a permutation representation of G_F. If E is the corresponding étale algebra, we therefore have an isomorphism $L \simeq (\mathrm{Alb} \otimes \mathbf{Q})(E)$.

Given an abelian variety A, we simply note that

$$A = \mathrm{Alb}(\widetilde{h}(A))$$

where $\widetilde{h}(A)$ is the reduced motive of A: $h(A) = \mathbf{1} \oplus \widetilde{h}(A)$, where the splitting is given by the rational point $0 \in A(F)$. □

9.3**. Remark.** Let $\mathcal{R}$ be the Kelly radical of $\mathrm{AbS}(F)$ (cf. Proposition 8.10). If F is a finitely generated field, the groups $\mathcal{R}(\mathcal{A}, \mathcal{B})$ are finitely generated by the Mordell-Weil-Néron theorem. To see this, note that if L is a lattice and A an abelian variety, then

$$Hom(L, A) \xrightarrow{\sim} Hom(L_{|\bar F}, A_{|\bar F})^{G_F}$$

and that the right term may be rewritten as $B(F)$, where $B = L^* \otimes A$ (compare Lemma 8.6). Hence the Hom groups in $\mathrm{AbS}(F)$ are finitely generated as well. In this case, Proposition 9.2 implies that, for any $M, N \in \mathrm{Chow}^o(F)$, *the image of the map* $\mathrm{Alb}_{M,N}$ *has finite index in the group* $Hom(\mathrm{Alb}(M), \mathrm{Alb}(N))$.

9.4**. Lemma.** *Suppose that Y is a curve. Then the map* (9.3) *inserts into an exact sequence*

$$0 \to CH_0(Y_{F(X)}) \xrightarrow{\mathrm{Alb}_{X,Y}} Hom(\mathcal{A}_X, \mathcal{A}_Y) \\ \to Br(F(X)) \to Br(F(X \times Y))$$

where Br denotes the Brauer group. In particular, (9.3)$\otimes\mathbf{Q}$ *is an isomorphism.*

Proof. In view of the construction of (9.3), we may assume that X is a point; then (9.3) reduces to (9.1). Suppose first that F is separably closed. Then (9.1) is bijective (see comments at the beginning of this section). In the general case, let F_s be a separable closure of F, and $G = Gal(F_s/F)$. Since $\mathcal{A}_Y$ is a sheaf for the étale topology, we get a commutative diagram

$$\begin{array}{ccc} CH_0(Y_s)^G & \xrightarrow[\sim]{\mathrm{Alb}_Y^{F_s}} & \mathcal{A}_Y(F_s)^G \\ \uparrow & & \wr\uparrow \\ CH_0(Y) & \xrightarrow{\mathrm{Alb}_Y^F} & \mathcal{A}_Y(F) \end{array}$$

where $Y_s = Y \times_F F_s$ and the top horizontal and right vertical maps are bijective. The lemma then follows from the classical exact sequence

$$0 \to CH_0(Y) \to CH_0(Y_s)^G \to Br(F(X)) \to Br(F(X \times Y)).$$

$\square$

9.5. **Proposition.** *Let* $\mathrm{Chow}^{\mathrm{o}}_{\leq 1}(F)$ *denote the thick subcategory of* $\mathrm{Chow}^{\mathrm{o}}(F)$ *generated by motives of curves, and let* $\iota : \mathrm{Chow}^{\mathrm{o}}_{\leq 1}(F) \to \mathrm{Chow}^{\mathrm{o}}(F)$ *be the canonical inclusion. Then*
a) After tensoring morphisms by $\mathbf{Q}$*,* $\mathrm{Alb}\circ\iota : \mathrm{Chow}^{\mathrm{o}}_{\leq 1}(F) \to \mathrm{AbS}(F)$ *becomes an equivalence of categories.*
b) Let j be a quasi-inverse. Then $\iota \circ j$ is right adjoint to Alb.

Proof. a) The full faithfulness follows from Lemma 9.4. For the essential surjectivity, we may reduce as in the proof of Proposition 9.2 to proving that lattices and abelian varieties are in the essential image. For lattices, this is proven in *loc. cit.*. For an abelian variety A, use the fact that A is isogenous to a sub-abelian variety of the Jacobian of a curve.

b) Let $(M, \mathcal{A}) \in \mathrm{Chow}^{\mathrm{o}}_{\leq 1}(F) \times \mathrm{AbS}(F)$. To produce a natural isomorphism $\mathrm{Chow}^{\mathrm{o}}_{\leq 1}(F)(M, \iota j(\mathcal{A})) \simeq \mathrm{AbS}(F)(\mathrm{Alb}(M), \mathcal{A}))$, it is sufficient by a) to handle the case $M = \bar{h}(X), \mathcal{A} = \mathcal{A}_Y$ for some smooth projective varieties X, Y with $\dim Y \leq 1$. Then the isomorphism follows from the adjunctions (9.2) and from Lemma 9.4. $\square$

9.6. **Remarks.** a) Of course the functor $\iota \circ j$ is not a tensor functor (since its image is not closed under tensor product).
b) In particular, the inclusion functor ι has the left-adjoint-left inverse

$j \circ \mathrm{Alb}$. This is a birational version of Murre's results for effective Chow motives ([22], [23, §2.1], see also [26, §4], and in the triangulated context [32, §3.4]). Beware however that we have taken the opposite to the usual convention for the variance of Chow motives (our functor $X \mapsto h(X)$ is covariant rather than contravariant), so the direction of arrows has to be reversed with respect to Murre's work.

Appendix A. A triangulated lemma

In this appendix, it will be convenient to use the notation $\mathcal{C}(X, Y)$ for the Hom group between two objects X, Y of a category $\mathcal{C}$.

Recall that an object X of a category $\mathcal{C}$ is *compact* if $\mathcal{C}(X, -)$ commutes with arbitrary direct limits (representable in $\mathcal{C}$). In a triangulated category, this amounts to commuting with representable direct sums. On the other hand, a triangulated functor $T : \mathcal{C} \to \mathcal{C}'$ between two triangulated categories is *dense* if the image of T generates $\mathcal{C}'$ in the sense that any object of $\mathcal{C}'$ is the third vertex of a triangle where the two other vertices are (possibly infinite) direct sums of objects of the form $T(X)$. Finally, a triangulated subcategory of $\mathcal{C}'$ is *thick* (*resp. localising*) if it is stable under direct summands (*resp.* and under representable direct sums).

A.1. **Lemma.** *Let $T : \mathcal{C} \to \mathcal{C}'$ be a triangulated functor between two triangulated categories. Assume that T is fully faithful, dense and that each object $X \in \mathcal{C}$ is compact in $\mathcal{C}'$. Let $\mathcal{D}$ be a thick subcategory of $\mathcal{C}$ and $\mathcal{D}'$ the localising subcategory of $\mathcal{C}'$ generated by $T(\mathcal{D})$. Then the induced functor*

$$\bar{T} : \mathcal{C}/\mathcal{D} \to \mathcal{C}'/\mathcal{D}'$$

on localised categories is fully faithful with dense image; moreover, the objects of $\mathcal{C}/\mathcal{D}$ are compact in $\mathcal{C}'/\mathcal{D}'$.

Proof. Recall that localisation with respect to thick subcategories of triangulated categories enjoys calculus of left and right fractions. Let $X, Y \in \mathcal{C}$, and denote their images in $\mathcal{C}/\mathcal{D}$ still by X and Y.

Faithfulness: Let $\bar{f} \in (\mathcal{C}/\mathcal{D})(X, Y)$. Then $\bar{f}$ is represented by a diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ & & \uparrow{\scriptstyle s} \\ & & Y \end{array}$$

with $cone(s) \in \mathcal{D}$. Suppose that $\bar{T}(\bar{f}) = 0$. Then there exists $t : U \to T(X)$ with $C = cone(t) \in \mathcal{C}'$ such that $T(f)t = 0$. Then $T(f)$ factors through C:

$$\begin{array}{ccccc} & & & C & \\ & & {}^{f'}\nearrow & & \searrow \\ U & \xrightarrow{t} & T(X) & \xrightarrow{T(f)} & T(Z) \\ & & & & \uparrow{\scriptstyle T(s)} \\ & & & & T(Y). \end{array}$$

Write $C = \bigoplus_{i\in I} T(C_i)$, with $C_i \in \mathcal{D}$. Since $T(X)$ is compact in $\mathcal{C}'$, there exists a finite subset $J \subseteq I$ such that f' factors through $\bigoplus_{i\in J} T(C_i)$; by the full faithfulness of T, there exist $f_i : X \to C_i$ and $g_i : C_i \to Z$ $(i \in J)$ such that the diagram

$$\begin{array}{ccccc} & & \bigoplus_{i\in J} C_i & & \\ & {}^{(f_i)}\nearrow & & \searrow^{(g_i)} & \\ X & & \xrightarrow{\quad f \quad} & & Z \end{array}$$

commutes. Recall that if $f : A \to B$ is a morphism in a triangulated category, then the *fibre* of $f := cone(f)[-1]$[2]. There are exact triangles $fibre(f) \to A \to B$ and $A \to B \to cone(f)$. Let U' be the fibre of (f_i), and $s' : U' \to X$ the corresponding map: then $fs' = 0$, hence $\bar{f} = 0$.

Fullness: let $\bar{f} \in (\mathcal{C}'/\mathcal{D}')(\bar{T}(X), \bar{T}(Y))$. Then $\bar{f}$ is represented by a diagram

$$\begin{array}{ccc} T(X) & \xrightarrow{f} & Z \\ & & \uparrow{\scriptstyle s} \\ & & T(Y) \end{array}$$

with $C = cone(s) \in \mathcal{D}'$. Write $C = \bigoplus_{i\in I} T(C_i)$ with $C_i \in \mathcal{D}$. Since $T(X)$ is compact in $\mathcal{C}'$ and T is fully faithful, there exists $J \subseteq I$ finite and a commutative diagram

$$\begin{array}{ccc} \bigoplus_{i\in J} T(C_i) & \longrightarrow & \bigoplus_{i\in I} T(C_i) \\ {\scriptstyle (T(h_i))}\uparrow & & \uparrow{\scriptstyle g} \\ T(X) & \xrightarrow{f} & Z. \end{array}$$

Using again the full faithfulness of T, there exists a map $k : \bigoplus_{i\in J} C_i[-1] \to Y$ such that the diagram

[2] Of course this is only defined up to non-unique isomorphism in general.

$$\begin{array}{ccc} & & T(Y) \\ & \overset{T(k)}{\nearrow} & \uparrow \\ \bigoplus_{i\in J} T(C_i)[-1] & \longrightarrow & \bigoplus_{i\in I} T(C_i)[-1] \end{array}$$

commutes, where the vertical map comes from the triangle associated to s and g.

Let U be the cone of k and $s' : Y \to U$ the corresponding map. Since in the commutative diagram

$$\begin{array}{ccc} & & T(Y) \\ & \overset{T(k)}{\nearrow} & \uparrow \\ \bigoplus_{i\in J} T(C_i)[-1] & \longrightarrow & \bigoplus_{i\in I} T(C_i)[-1] \\ {\scriptstyle T((h_i))}\uparrow & & {\scriptstyle g}\uparrow \\ T(X) & \xrightarrow{f} & Z \end{array}$$

the paths from $T(X)$ to $T(Y)[1]$ are 0, $(T(h_i))$ factors through $T(U)$ and hence (h_i) factors through U:

$$\begin{array}{ccc} \bigoplus_{i\in J} C_i & & \\ \uparrow & \overset{(h_i)}{\nwarrow} & \\ U & \xleftarrow{f'} & X \\ {\scriptstyle s'}\uparrow & & \\ Y. & & \end{array}$$

By an axiom of triangulated categories, there exists $t : T(U) \to Z$ such that the diagram of triangles

$$\begin{array}{ccc} \bigoplus_{i\in J} T(C_i) & \longrightarrow & \bigoplus_{i\in I} T(C_i) \\ {\scriptstyle h'}\uparrow & & {\scriptstyle g}\uparrow \\ T(U) & \xrightarrow{t} & Z \\ {\scriptstyle T(s')}\uparrow & & {\scriptstyle s}\uparrow \\ T(Y) & = & T(Y) \end{array}$$

commutes. Moreover, the cone of t is isomorphic to $\bigoplus_{i\in I-J} T(C_i)$, hence belongs to $\mathcal{C}'$. Since $g(f - tT(f')) = 0$, there exists $a : X \to Y$ such that $f - tT(f') = sT(a)$. Now, computing in $\mathcal{C}'/\mathcal{D}'$:

$$s^{-1}f = T(s')^{-1}t^{-1}f = T(s')^{-1}T(f') + \bar{T}(a) = \bar{T}({s'}^{-1}f' + a) \\ \in \bar{T}((\mathcal{C}/\mathcal{D})(X,Y)).$$

Dense image: this is obvious.

Compactness: let $X \in \mathcal{C}$, viewed as an object of $\mathcal{C}/\mathcal{D}$, $(Y_i)_{i\in I}$ a family of objects of $\mathcal{C}'$, viewed as objects of $\mathcal{C}'/\mathcal{D}'$, and $\bar{f} \in (\mathcal{C}'/\mathcal{D}'(\bar{T}(X), \bigoplus_{i\in I} Y_i)$. Then f is represented by a diagram

$$\begin{array}{ccc} T(X) & \xrightarrow{f} & Z \\ & & \uparrow{\scriptstyle s} \\ & & \bigoplus\limits_{i\in I} Y_i \end{array}$$

with $C = \mathit{cone}(s) \in \mathcal{D}'$. We start as in the proof of fullness. Let $g : Z \to C$ be the corresponding map, and write $C = \bigoplus_{j\in J} T(C_j)$, with $C_j \in \mathcal{D}$. By compactness, gf factors through a finite subset of J:

$$\begin{array}{ccc} & & \bigoplus\limits_{i\in I} Y_i[1] \\ & & \uparrow{\scriptstyle \beta} \\ \bigoplus\limits_{j\in K} T(C_j) & \xrightarrow{\alpha} & \bigoplus\limits_{j\in J} T(C_j) \\ \uparrow{\scriptstyle T(h)} & & \uparrow{\scriptstyle g} \\ T(X) & \xrightarrow{f} & Z. \end{array}$$

Here α is the inclusion. Using compactness again, $\beta\alpha$ factors through a finite subset of I:

$$\begin{array}{ccc} \bigoplus\limits_{i\in L} Y_i[1] & \xrightarrow{\iota} & \bigoplus\limits_{i\in I} Y_i[1] \\ \uparrow{\scriptstyle \beta'} & & \uparrow{\scriptstyle \beta} \\ \bigoplus\limits_{j\in K} T(C_j) & \xrightarrow{\alpha} & \bigoplus\limits_{j\in J} T(C_j) \\ \uparrow{\scriptstyle T(h)} & & \uparrow{\scriptstyle g} \\ T(X) & \xrightarrow{f} & Z. \end{array}$$

Here ι is again the inclusion. Since $\iota\beta' T(h) = 0$ and ι is split, $\beta' T(h) = 0$ and $T(h)$ factors through the fibre Z of β':

$$\begin{array}{ccccc}
\bigoplus_{j\in K} T(C_j) & & \xrightarrow{\alpha} & & \bigoplus_{j\in J} T(C_j) \\
\uparrow{\scriptstyle g'} & & & & \uparrow{\scriptstyle g} \\
Z' & \xleftarrow{f'} & T(X) & \xrightarrow{f} & Z \\
\uparrow{\scriptstyle s'} & & & & \uparrow{\scriptstyle s} \\
\bigoplus_{i\in L} Y_i & & \xrightarrow{\iota} & & \bigoplus_{i\in I} Y_i.
\end{array}$$

On the other hand, there exists a map $k : Z' \to Z$ defining (together with α and ι) a map from the left vertical triangle to the right vertical triangle in the above diagram. We have

$$gf = \alpha g' f' = gkf'$$

or $g(f - kf') = 0$, hence $f - kf' = sa$, with $a \in \mathcal{C}'(T(X), \bigoplus_{i\in I} Y_i)$. Applying compactness a third time, $a = \iota' a'$, with $a' : T(X) \to \bigoplus_{i\in L'} Y_i$, where L' is finite and (without loss of generality) contains L. Now, computing in $\mathcal{C}'/\mathcal{D}'$:

$$s^{-1} f = s^{-1} k f' + \iota' a' = \iota {s'}^{-1} f' + \iota' a'$$

and $\bar{T}(X)$ is indeed compact in $\mathcal{C}'/\mathcal{D}'$. □

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